Journal of Computational and Applied Mechanics, Vol. 4., No. 2., (2003), pp. 175–188

PENALTY-REGULARIZATION OF A DISSIPATIVE VIBRO-IMPACTING SYSTEM

Ingolf Müller and Peter Vielsack
Institute of Mechanics, University of Karlsruhe
D-76131 Karlsruhe, Germany
mechanik@ifm.uni-karlsruhe.de

[Received: June 18, 2002]

Abstract. The so-called penalty method in FE-calculation regularises the strong contact conditions by introducing contact stiffnesses and damping in order to reduce the mathematical effort. The problem, however, lies in an appropriate choice of the values of parameters for these artificially introduced springs and dampers. The principal problem of regularisation, however, can be studied for simple rigid body systems. As an example, two neighbouring physical pendulums with different natural frequencies are treated. During the motion sudden impacts and states of permanent contact interchange with states of separated motions of the two pendulums.
The first step in the consideration comprises the calculation of a semi-analytical reference to classify the properties of the motion with regard to the main features of the non-linear system's response. The results are verified by experimental investigations in the next step. Finally, the system is regularised by the penalty method and integrated by NEWMARK's method. This procedure needs three unknown numbers, two regularisation parameters and a time step. Their correct choice depends on detailed information from the experimental results for each type of motion.

Keywords: impact, penalty method, non-linear oscillation

1. Introduction

The presence of damage in sandwich materials, in particular delaminations between adjacent laminae, degrade severely the mechanical behaviour of a structure. A vibration-based non-destructive damage identification needs a suitable model to capture the non-linear phenomena of the oscillation [3]. Experimental investigations show that oscillations of delaminated structures are dominated by impacts [4]. They occur when separated parts of the structure come into contact during the motion. Each contact gives rise to an impact, which leads to energy dissipation. The actual available mechanical model with minimal DOF is based on an elastic beam with lumped masses and a simple law of impact [3]. The integration of this non-smooth dynamic system leads to a sequence of smooth systems, whose analytical solutions are known. They must be patched together at those times when irregularities due to contact occur [6]. This simple model captures the main oscillation phenomena and allows a discussion in principle of the influence of the internal dissipation due to the impacts on the non-linear system's response and the evolution of the impacts near resonance.

An improvement of the mechanical description can be expected by the utilisation of the finite element method. In order to reduce the numerical effort the regularisation of the strong contact conditions is required. The penalty method introduces contact stiffnesses and damping for regularisation [1]. Despite the fact that FE-calculations lead to oscillations with multi-degrees of freedom, the fundamental problem of an appropriate choice of the values for these artificially introduced springs and dampers can be discussed for simple rigid body systems.

As an example the forced vibrations of two neighbouring pendulums will be considered. The first step comprises the consideration of a semi-analytical reference to classify the properties of the oscillations with regard to the main features of the non-linear system's response. These results will be verified by experimental investigations in the next step. Finally, the validity of two different mechanical models for the contact, namely the classical theory of impact and the regularisation-technique for impacts, is compared.

2. **The investigated system and its semi-analytical description**

As an example, let us consider two neighbouring physical pendulums with different natural frequencies and different damping. The pendulums touch each other with a vanishing contact force in the equilibrium state. Vibrations are induced by a harmonic base excitation. This non-smooth dynamic system gives a first approximation for a delaminated sandwich beam [4]. The mechanical description is based on the model

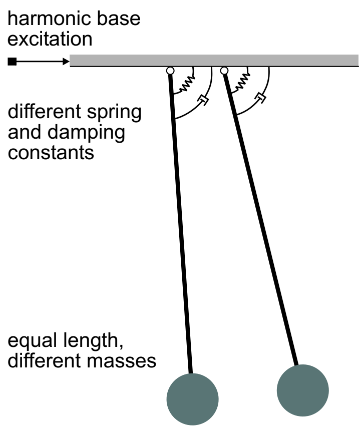

Figure 1. Mechanical model

shown in Figure 1. It consists of two rigid bodies with different masses and different elastic suspensions and dampers at the top.

Firstly the semi-analytical procedure for integration of the non-smooth dynamic system is considered. In this case, only the coefficient of restitution e has to be determined from experiments. Exciting the system, discontinuities of the motion due to impacts occur. This leads to sudden changes in the system's behavior at unknown separation times. The only numerical task is to find these separation times. Between two successive separation times the system is a linear one and the solutions of the equations of motion are known explicitly. Three different states must be considered

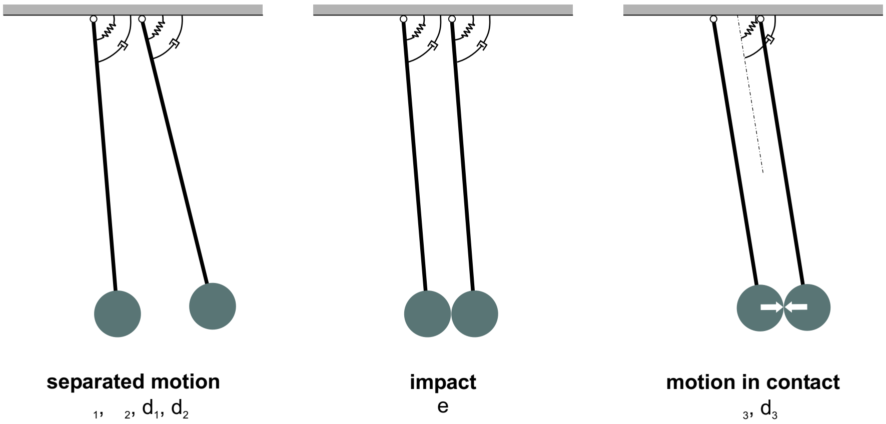

Figure 2. Possible states of motion

(see Figure 2). In the case of a separated motion, both pendulums move independently of each other, characterised by the natural frequencies ω_1, ω_2 and the damping constants d_1, d_2, respectively. When the two pendulums come into contact, an impact occurs. In this second state the sudden impact is modelled by NEWTON's assumption with a coefficient of restitution $e = 0.5$. A third possible state is a motion in permanent contact, where the two pendulums behave as a single one with a frequency ω_3 and a damping constant d_3. All constants can be found in Table 1. They came from the real physical system under experimental investigation, considered later on. The calculation procedure is described in [4]. A detailed discussion is therefore omitted. For a better understanding only some hints are needed. All results are given in a non-dimensional representation. The non-dimensional time $\tau = \frac{t}{\omega_1}$ is refers the lowest natural frequency. The values ξ_1 and ξ_2 are non-dimensional displacements of the end masses of the pendulums (a motion in permanent contact gives $\xi_1 = \xi_2$). The corresponding velocities are ${\xi_1}'$ and ${\xi_2}'$. A transition from a separated motion to a motion in permanent contact theoretically leads to a sequence of infinite numbers of impacts with time intervals tending to zero. The beginning of a motion in permanent contact is therefore defined by a small threshold ${\xi_2}' - {\xi_1}' \leq 0.002$ to avoid numerical problems. The frequency ratio $\eta = \frac{\Omega}{\omega_1}$ indicates the frequency of excitation. In the following only stationary system's responses are considered. Depending on the frequency of excitation η the system's response shows a broad variety of bifurcated motions. The POINCARÉ-section method is used to collect samples of stationary

	Natural frequency ω_i [1/s]	Damping d_i [-]
Pendulum 1 (separate)	$\omega_1 = 1.00$	$d_1 = 0.0033$
Pendulum 2 (separate)	$\omega_2 = 2.41$	$d_2 = 0.0120$
Two pendulums (fixed connection)	$\omega_3 = 1.80$	$d_3 = 0.0330$

Table 1. Parameters for natural frequencies and viscous damping

responses of the displacements ξ_i, which can be assembled into a bifurcation diagram (Figure 3). The typical feature of the bifurcation diagram is an alternation of regions

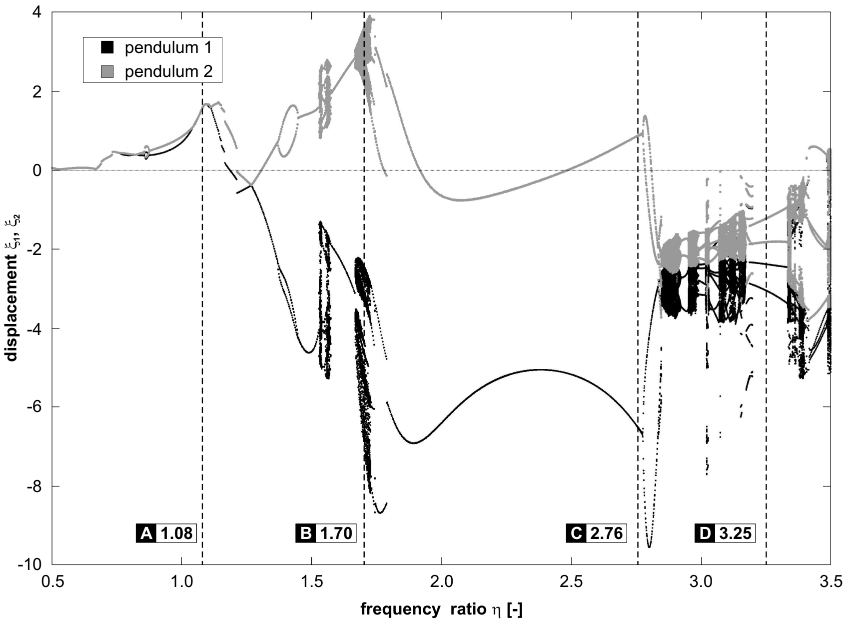

Figure 3. Bifurcation diagram

of irregularity and windows of periodic responses. As an example, only four typical kinds of motion will be considered (Figure 3, sections [A] - [D]) by their phase plots in Figure 4. In the vicinity of the frequency $\eta = 1.08$ (case [A]) the oscillation is non-bifurcated. As can be seen in Figure 4 [A], this type of motion contains multiple impacts in one period and a phase of permanent contact. Section [B] (Figure 3 [B]), taken at a frequency of excitation $\eta = 1.70$, shows a quasi-periodic motion. The case of quasi-periodic motions can be seen in the bifurcation diagram (Figure 3, section [B]) as widening of the lines to stripes of different widths. Increasing η to the range of $\eta = 2.76$, the system's response changes to a non-bifurcated one (Figure 3 [C]). The corresponding phase plot (Figure 4 [C]) shows one impact in one period. Finally, in the region of $\eta = 3.25$ (Figure 3, [D]) a period-doubling exists.

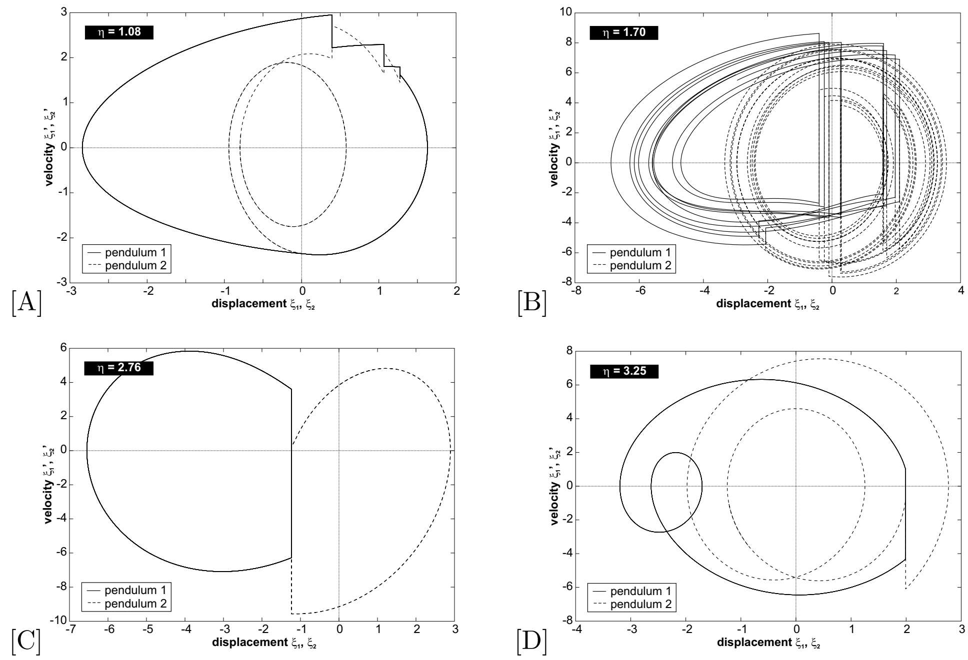

Figure 4. Phase plots of four typical kinds of oscillations

Despite this broad variety of motions, the following treatment will be restricted from now on to two typical kinds of oscillations, which are of major interest in the investigation of a delaminated beam. These are a motion caused by an excitation $\eta = 1.08$ leading to multiple impacts and permanent contact (Figure 4 [A]) and a motion caused by $\eta = 2.76$ leading to one sudden impact in one response period (Figure 4 [C]).

3. Experimental confirmation of the semi-analytical results

The experimental equipment is shown in Figure 5. It consists of two physical pendulums of length 618 mm with the vibrational parameters given in Table 1. A shaker induces vibrations as an adjustable harmonic base excitation. The amplitude of excitation is kept constant at $1.07mm$. The above mentioned excitations $\Omega = 1.08\frac{1}{s}$ and $\Omega = 2.76\frac{1}{s}$ (Figure 4, [A] and [C]) are chosen for an experimental verification of the semi-analytical results. Opto-electronical displacement sensors give the absolute positions x_i characterising the response of the system. The frequency of excitation Ω can be monitored. A contact sensor controls the opening and closing of an electric circuit and gives information about contact or no contact. Considering the stationary system's response in form of time-displacement plots of about two excitation periods, the chosen cases of the frequency of harmonic base excitation ($\eta = 1.08$ and $\eta = 2.76$) show an excellent agreement between experimental and numerical results (Figure 6, upper pictures). It must be noticed, however, that the experimental time-displacement

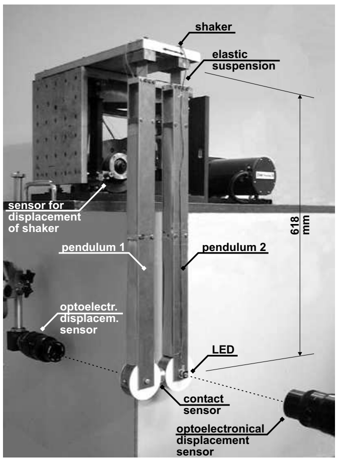

Figure 5. Experimental equipment

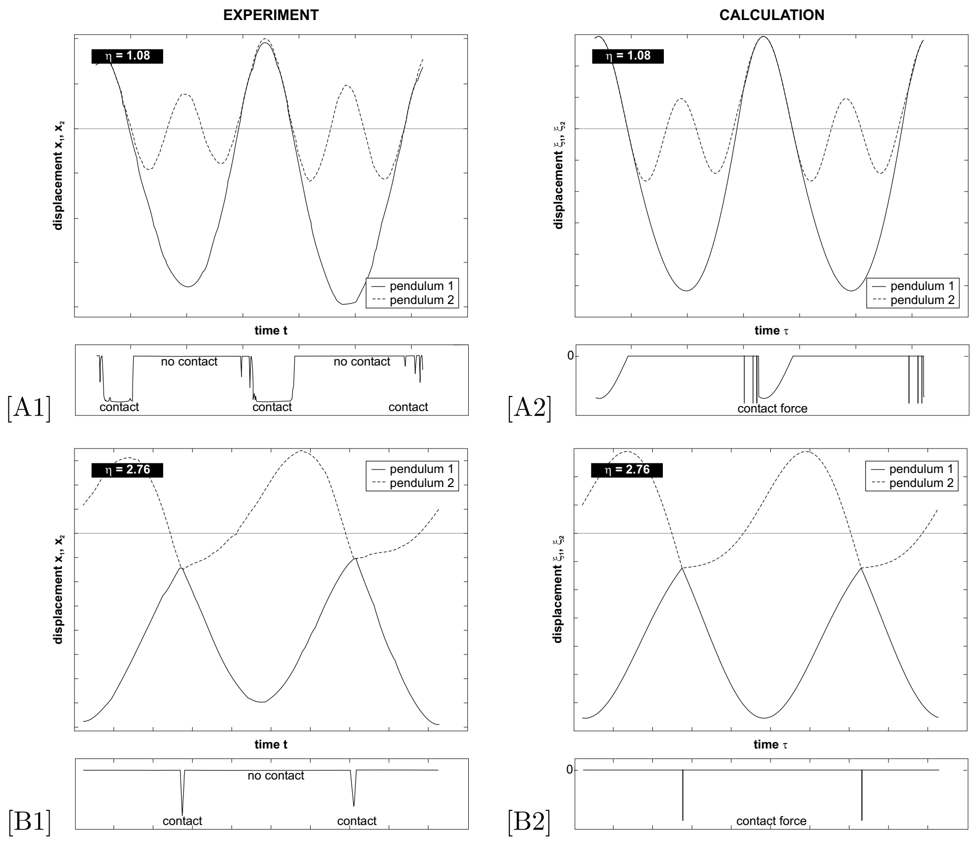

Figure 6. Comparison of stationary displacement and contact force versus time from experiment [A1], [B1] and calculation [A2], [B2]

he numerical results in a non-dimensional form. In addition, the change of the contact force in the same time domain (Figure 6, lower pictures) confirms the correctness of all calculations. Here, the experiments only give information about contact and no contact, whereas the numerical result shows the course of the contact force. The case $\eta = 1.08$ clearly shows multiple impacts with decreasing time intervals leading to a motion with permanent contact.

4. **Penalty regularisation**

Regularisation of the strong contact conditions leads to a smoothing of the points of discontinuity. In contrast to the semi-analytical procedure the number of DOF does not change in the regularized system in all partial states. This allows a fast numerical integration.

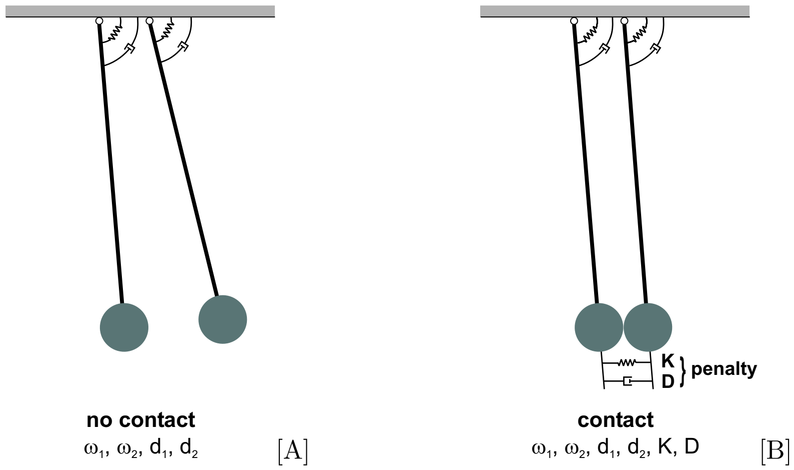

Figure 7. Mechanical systems of the different partial states
[A] no contact, [B] contact

As illustrated in Figure 7, only two states exist, namely a motion with or without contact. The state of motion without contact (Figure 7 [A]) is kept unaltered (cp. Figure 1) compared to the preceding system. In the case of contact, which means a vanishing or negative relative displacement $\xi_2 - \xi_1$, a contact spring with stiffness K is added to the basic system (Figure 7 [B]). An additionally introduced viscous damper D captures the dissipation of impact, comparable to the coefficient of restitution e. Introducing the ratio $\kappa = \frac{\omega_2}{\omega_1}$, a non-dimensional representation of the equations of motion for both states is given in Figure 8. Starting at the state without contact the mathematical description consists of two non-coupled equations. If contact occurs, the equations are linked by penalty stiffness K and damping D. It is obvious that the non-linearity of the regularized system only consists on the mutual change of the system from a free motion of both pendulums to a common motion in contact and vice versa. In addition to the fact that the number of DOF is constant in time, the second advantage is the simplification of the switching conditions. Only a control of the relative displacement decides about a transition from one state to the next. The semi-analytical procedure controls the contact force in a state of common motion.

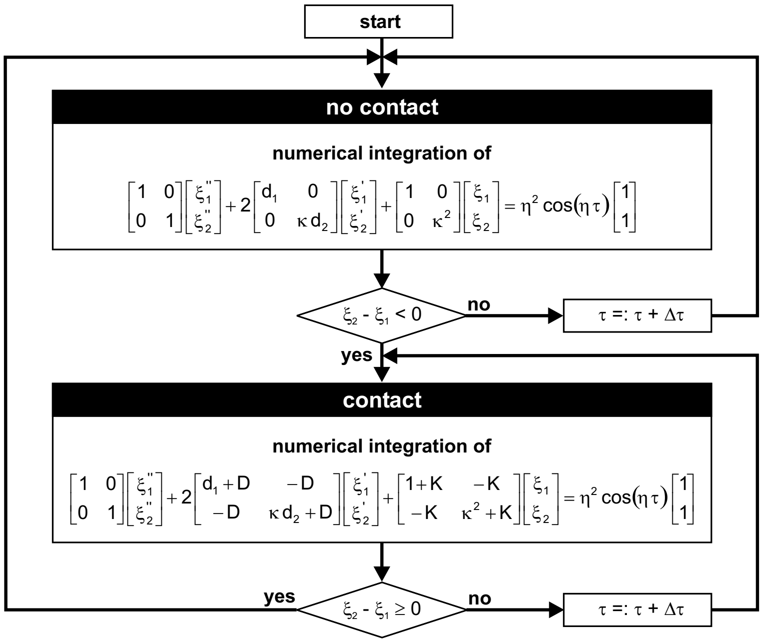

Figure 8. Scheme for switching from one state to the other

The fundamental disadvantage, however, lies in an appropriate choice of the artificially introduced constants K and D for a certain type of motion, because these values do not represent real physical or mechanical parameters. As will be shown in the following, a correct choice of K and D needs a reference. This can be achieved by matching the input data with experimental information. In the present case, the solutions of the semi-analytical procedure can be taken. In general, low values of K give wrong results caused by the poorly satisfied contact condition, followed by a strong penetration of the subsystems. The opposite case of a large contact stiffness K gives rise to a stiff set of equations leading to problems of integration. Furthermore, the choice of the parameters K and D depends on each other and requires a correct adjustment.

The numerical integration needs a time step $\Delta\tau$. NEWMARK's method, commonly used in FE-method, is applied taking $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{4}$. Therefore, the time step is constant. This fact can lead to severe errors and even totally wrong responses [5]. The reason lies in the inaccurate determination of the transition points. Therefore, the time step should be as small as possible. In the following, two kinds of motions with excitation $\eta = 1.08$ and $\eta = 2.76$ (Figure 6) will be investigated to show the problems in choosing the three numbers K, D and $\Delta\tau$.

4.1. **Motion with a state of permanent contact:** $\eta = 1.08$**.** The time step $\Delta\tau$ refers the non-dimensional periodical time $T = \frac{2\pi}{\eta}$. A time step $\Delta\tau = \frac{T}{2000}$ is taken and kept constant. Now, only contact stiffness K and damping D can be chosen freely. Figure 9 compares the phase plots, which are obtained by regularization using the sets of parameters ([A] $K = 2500$, $D = 8$ and [B] $K = 100$,$D = 50$), with the semi-analytical result [C]. As evident from Figure 9, the set of penalty parameters in case [A] captures the phenomena of oscillation given by the exact solution (case [C]). The choice of $K = 100$, $D = 50$ (case [B]) yields a completely different system behavior, which is far from reality.

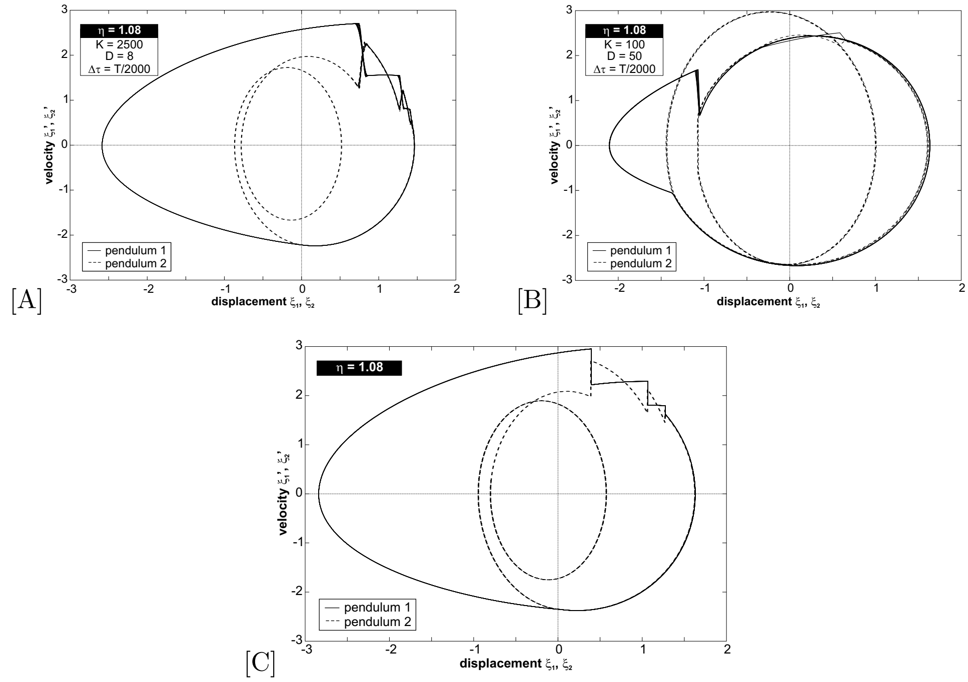

Figure 9. Comparison of numerical results gained by using different penalty parameters [A], [B] and the correct (semi-analytical) result [C]

As a conclusion it is evident that detailed information is needed with regard to the expected type of motion to determine the parameters K, D and $\Delta\tau$. The basic conditions are the properties of the response (bifurcated / non-bifurcated, periodic motion / quasiperiodic motion), number of impacts and instants of impacts in a response period. Remembering the broad variety of different kinds of oscillations shown in the bifurcation diagram (Figure 3), it must be emphasized here that the solution in Figure 9 [B] could be considered the correct one, if no information existed.

4.2. **Separated motion with one impact:** $\eta = 2.76$**.** At the beginning of the investigation the same time step $\Delta\tau = \frac{T}{2000}$ as before is taken. The simplest information which is needed to determine K and D is the non-existence of bifurcations. The

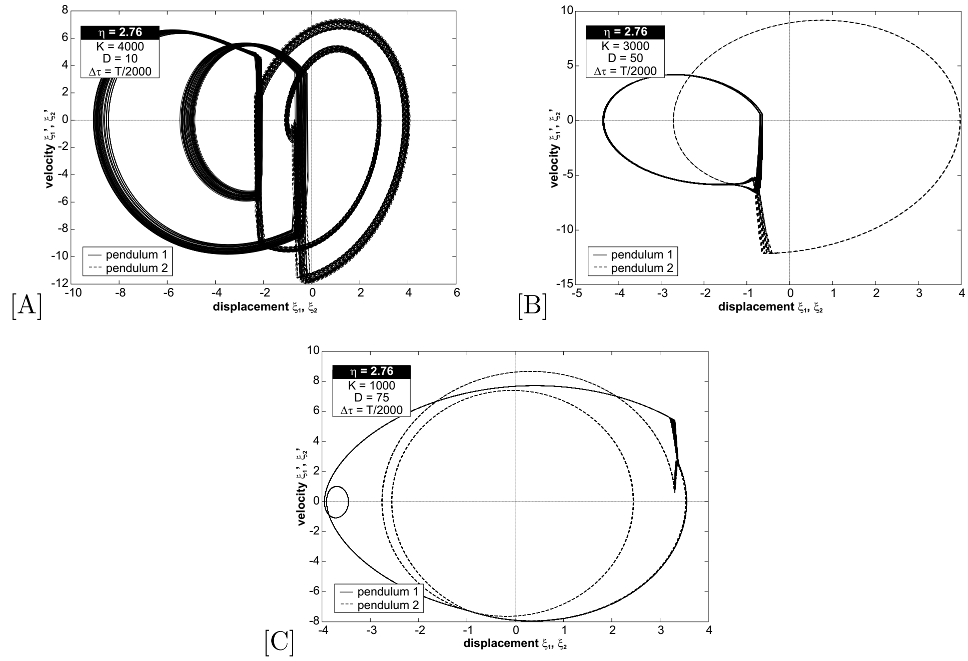

Figure 10. Examples of phase plots for bifurcated and non-bifurcated motions

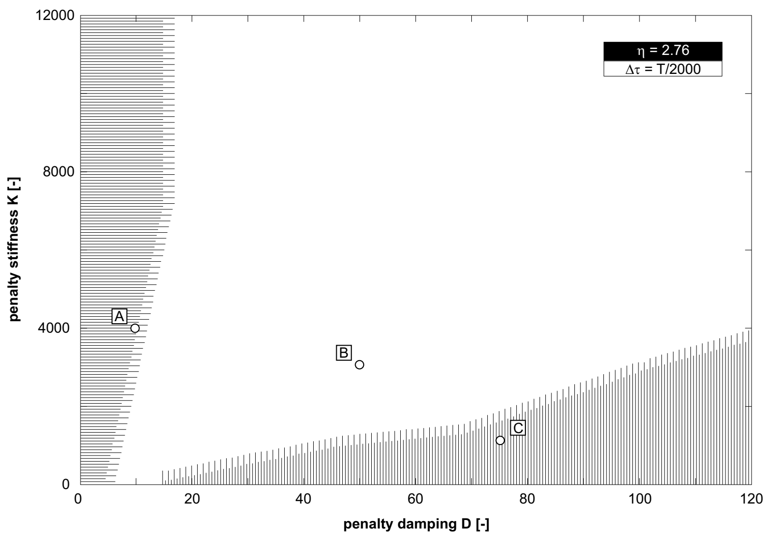

Figure 11. Parametric plane for sets K and D showing regions of bifurcated and non-bifurcated oscillations

responses for three different sets of parameters are shown in Figure 10. Two of them (Figure 10, [A] and [C]) exhibit a bifurcated motion. A systematical variation of K and D excludes the sets of K and D leading to bifurcations. This allows us to construct a parametric plane, as can be seen in Figure 11. The dashed regions in Figure 11 are out of interest. The examples of Figure 10 give three points [A], [B] and [C].

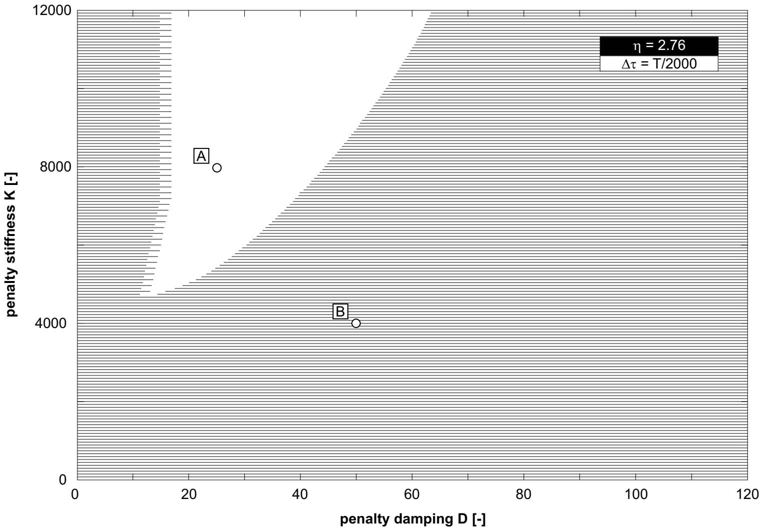

Figure 12. Updated parametric plane for sets K and D

Two examples are given in Figure 13. They correspond to the points [A] and [B] in Figure 12.

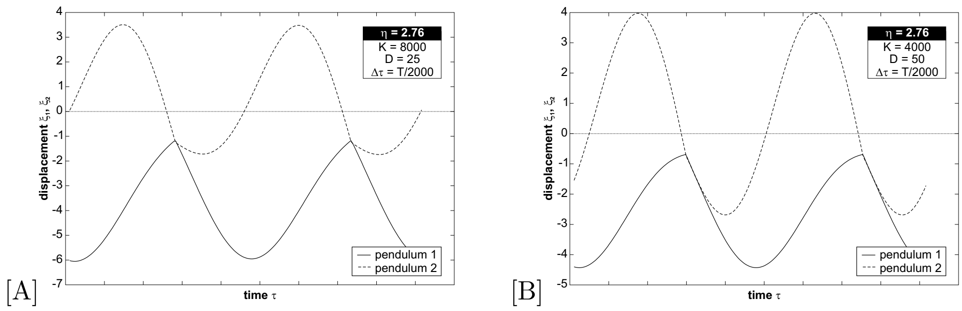

Figure 13. Time-displacement plots for responses with a [A] "sudden" impact and a [B] state of permanent contact

More detailed information is the knowledge about the number of impacts and the states of permanent contact in a period. In the present example ($\eta = 2.76$) one sudden impact occurs in one response period. The regularisation by the penalty method does not allow the reproduction of sudden impacts. That means that sudden impacts are modelled by a short interval of permanent contact. As an example, less than 30 time steps $\Delta\tau = \frac{T}{2000}$ are assumed to describe a "sudden" impact. This assumption diminishes the region of possible values K and D in an updated parametric plane (Figure 12).

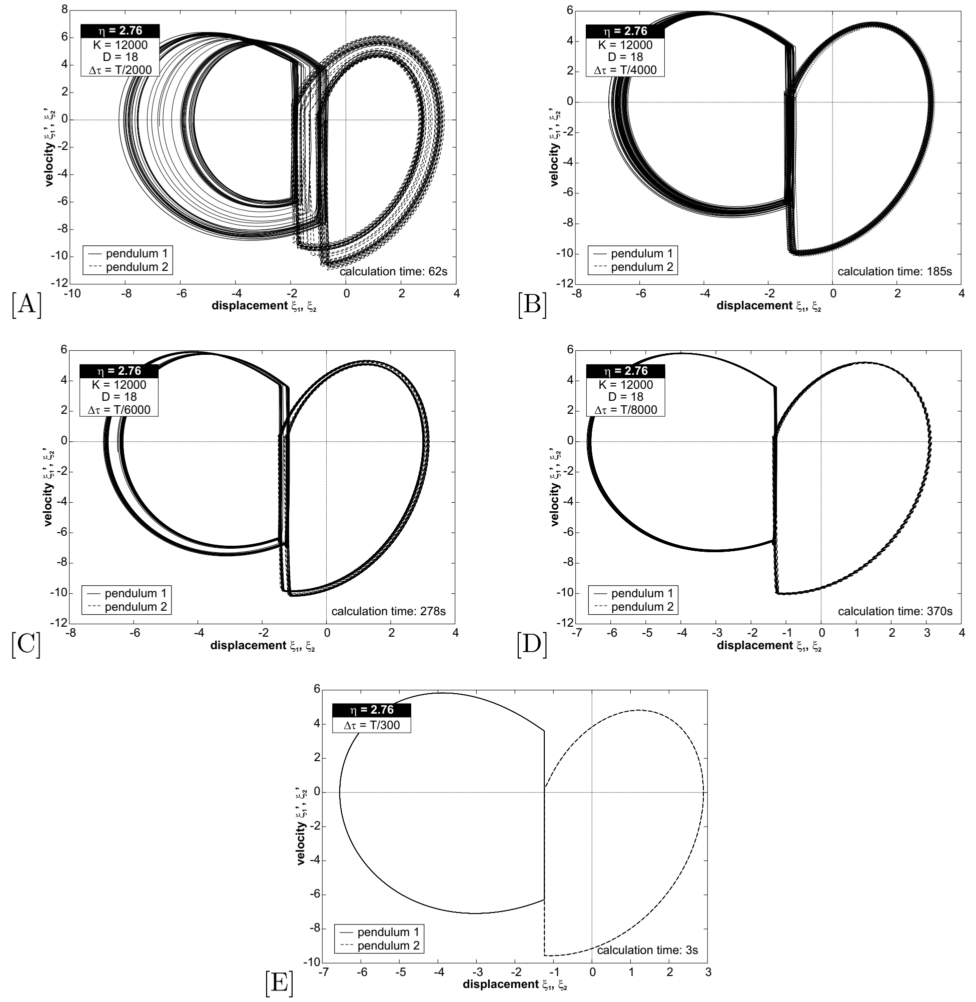

Figure 14. Stationary phase plots for various time steps $\Delta\tau$ [A]-[D] compared to the semi-analytical result [E]

A further improvement of the values for K and D can be achieved by consideration of the instant of impact during a response period. This procedure demands a comparative work and is not executed here.

Finally, the problem of the choice of a sufficiently small step $\Delta\tau$ for an orbital stable solution [6] must be considered. Assuming now $K = 12000.0$ and $D = 18.0$ according to the previous investigations, stationary phase plots are computed with different time steps $\Delta\tau$. Figure 14 contains the results for time steps $\Delta\tau = \frac{T}{2000}$, $\Delta\tau = \frac{T}{4000}$, $\Delta\tau = \frac{T}{6000}$ and $\Delta\tau = \frac{T}{8000}$. In all cases 2050 excitation periods are calculated, but only the last 50 are plotted. As predicted, the larger values of $\Delta\tau$ cannot capture the transition times with sufficient accuracy. Permanent numerical disturbances due to this systematic error give rise to quasi-periodic responses. As already mentioned, such kinds of motion are possible when remembering the bifurcation diagram (Figure 3). Only the known reference solution qualify them to be wrong.

Summarizing the facts, the required three numbers for the regularization are given by a penalty stiffness $K = 12000.0$, a penalty damping $D = 18.0$ and time steps $\Delta\tau = \frac{T}{8000}$.

5. Conclusions

The semi-analytical procedure for integration of non-smooth dynamic contact problems leads to a sequence of smooth systems, whose solutions must be patched together at times when irregularities due to contact occur. In order to reduce the extensive mathematical effort, the penalty method regularizes the strong contact conditions by introducing contact stiffness and contact damping.

The regularized system keeps a constant number of DOF's in all partial states and allows a fast numerical integration by the usual methods. A smoothing of the points of discontinuity is obtained. Additionally, the regularization by the penalty method leads to a simplification of switching conditions for the transition to another state of motion.

The problem, however, lies in the appropriate choice of the values for penalty parameters K and D for each type of motion. For a correct determination of K and D a reference is required, which is given in experimental investigations or semi-analytical results. Without information about the expected motion, a decision is not possible, whether the chosen numbers for K and D are right or wrong. The reference results contain the information needed for the choice of the penalty parameters, which can be obtained by consideration of the motion properties - bifurcated or non-bifurcated motion, number and instant of impacts. Treating a new type of motion, a new validation of the values K and D by the reference is required.

Recapitulating the results of the influence of time steps $\Delta\tau$ shows that the choice of K and D is not independent of steps $\Delta\tau$. A high precision of the results needs an immense numerical effort.

REFERENCES

1. Engleder, T., Vielsack, P. and Schweizerhof, K.: *F.E.-regularisation of non-smooth vibrations due to friction and impacts*, Computational Mechanics, **28**, (2002), 162-168.
2. Holmes P. J.: *The dynamics of repeated impacts with a sinussoidally vibrating table*, Journal of Sound and Vibration, **84**, (1982), 173-189.
3. Müller, I., Schmieg, H. and Vielsack, P.: *Non-smooth forced oscillations of a delaminated sandwich beam*, PAMM - Proceedings in Applied Mathematics and Mechanics, **2**, (2003), 286-287.
4. Vielsack, P.: *A vibro-impacting model for the detection of delamination*, Journal of Sound and Vibration, **253**(2), (2002), 347-358.

5. Vielsack, P.: *Orbitale Stabilität nichtglatter Bewegungen bei permanenten numerischen Störungen*, ZAMM, **79**(1), (1999), 105-108.
6. Vielsack, P. and Hartung, A.: *An example for the orbital stability of permanently disturbed non-smooth motions*, ZAMM, **79**(6), (1999), 389-397.