Stable analysis of long duration motions of FE-discretized structures in central force fields

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Received January 2003
Revised September 2003
Accepted October 2003

Keywords *Finite element analysis, Force, Motion*

Abstract *The computation of structures moving in central force fields generally requires long-time integration including geometrically nonlinear behavior (large rotations) as such, e.g. satellite structures move for a long time. To achieve a numerically stable computation the energy momentum method which fulfills linear and angular momentum as well as energy conservation within the time step is chosen for the time integration. The focus in the contribution is on Hamiltonian systems. A formulation for the gravitational force in a central force field as external force on a rigid or flexible satellite is given. The presented formulation enables the computation of the exact spatial distribution of the gravitational forces acting on a structure using the FE-discretization which is necessary to analyze, e.g. the orientation of a satellite in a gravitational field. The fulfillment of the conservation laws within the time step is proved. The necessity for considering the spatial distribution of the gravitational forces is discussed based on numerical examples.*

1. Introduction

The computation of structures moving in central force fields such as satellites generally requires long-time integration taking the geometrically nonlinear behavior (large rotations) into account as a large overall long duration motion has to be considered. Therefore, the major requirements concerning the time integration scheme are energy conservation as well as high numerical stability. Among the time integration schemes actually discussed in the literature (Betsch and Steinmann, 2000a, b; Kuhl and Crisfield, 1999; Kuhl and Ramm, 1996) the focus will be on the so-called one-step schemes. Based on their high numerical stability for structural dynamics, both the implicit midpoint rule and the energy momentum method (Simo and Tarnow, 1992; Gonzalez, 2000) appear to be well suited for the solution. However, due to the symplectic behavior of the implicit midpoint rule, the method is not energy conserving. Thus our focus will be mainly on the energy momentum method with its property of conserving linear and angular momentum as well as energy within the time step. The method was originally proposed by Simo and Tarnow (1992) for flexible structures. Rigid bodies can be considered using an algorithm proposed by Simo and Wong (1991) for the midpoint rule or the energy momentum method. A general extension considering holonomic constraints on mechanical systems is given by Gonzalez (1999).

For gravitational fields based on the Kepler potential, Gonzalez and Simo (1996) compared both methods with respect to their numerical stability. Greenspan (1995) proposed an energy-conserving formulation according to the energy momentum method for the computation of N-body systems within potential fields caused by these bodies.

Engineering Computations
Vol. 21 No. 7, 2004
pp. 708-717

0264-4401
DOI 10.1108/02644400410548350

The focus of the current contribution is on the correct consideration of the gravitational force as an external position dependent force acting on a satellite, thus only Hamiltonian systems are considered. It is assumed that the structure itself does not have an influence on the central force field. Of special interest is the consideration of the spatial extension of the body, such that the orientation of the body within space can be analyzed. Therefore, a formulation is presented that enables the computation of the exact spatial distribution of gravitational forces acting on structures using a FE-discretization.

First a special definition of central force fields related to the chosen time integration scheme is proposed. Then the proof of the conservation laws is given followed by the discussion of the spatial discretization and the influence of the time integration scheme on the matrix form. Finally numerical examples with emphasis on the necessity of a spatial discretization of the satellite are presented followed by some concluding remarks.

2. Definition of a central force field

The considered central force field is stationary and is not affected by, e.g. a satellite occurring in this field. With a given value of the gravitational acceleration g_{ref} in a reference distance r_{ref}, the vector of the gravitational acceleration acting on a point i inside the field is described by – as it is well known for such central force fields:

$$\mathbf{g}_i = g_{\mathrm{ref}} \frac{r_{\mathrm{ref}}^2}{r_i^2} \mathbf{e}_i^r \quad \text{with} \quad \mathbf{e}_i^r = \frac{\mathbf{r}_i}{r_i}. \tag{1}$$

According to Figure 1 the vector from point i to the center of attraction Z of the field is given by the expression

$$\mathbf{r}_i = \mathbf{X}_Z - \mathbf{X}_i - \mathbf{u}_i = \mathbf{X}_Z - \mathbf{x}_i. \tag{2}$$

$\mathbf{X}$ is the position vector in the reference configuration and $\mathbf{x} = \mathbf{X} + \mathbf{u}$ is the position vector in the actual configuration with the displacement vector $\mathbf{u}$. With the density ρ and assuming a constant vector of gravitational acceleration $\mathbf{g}_{\mathrm{em}}$ within a time step, gravitation leads to the volume force $\mathbf{p} = \varrho \mathbf{g}_{\mathrm{em}}$. In order to achieve energy and momentum conservation, a special interpolation for the gravitational acceleration within a time step is needed for the energy momentum method.

The weak form of an otherwise unloaded structure within a central force field for the time step $t_n \rightarrow t_{n+1}$ takes the form

$$\delta\Pi = \delta\Pi^M + \delta\Pi^E - \int_V \varrho \mathbf{g}_{\mathrm{em}} \cdot \delta\mathbf{u} \, \mathrm{d}V = \mathbf{0}, \tag{3}$$

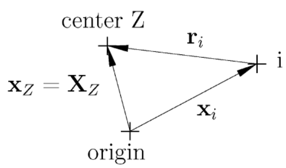

Figure 1.
Definition of position vectors

with the two other parts of virtual work $\delta\Pi^M$ and $\delta\Pi^E$ caused by inertia of rigid and flexible parts, respectively, strains of elastic structural parts. As central force fields on mechanical structures can be derived from a potential, for the energy momentum method as well as for the implicit midpoint rule the expression for the velocities $\dot{\mathbf{u}}$ and displacements $\mathbf{u}$

$$\dot{\mathbf{u}}_{n+\frac{1}{2}} = \frac{1}{2}(\dot{\mathbf{u}}_n + \dot{\mathbf{u}}_{n+1}) = \frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{\Delta t_n} \quad \text{with} \quad \Delta t_n = t_{n+1} - t_n \tag{4}$$

holds within the time step with n and n+1 being the index marking the beginning, respectively, end of the time step. Thus the velocity $\dot{\mathbf{u}}_{n+\frac{1}{2}}$ is assumed to be constant within the time step.

For the average gravitational acceleration $\mathbf{g}_{\mathrm{em}}$ within the time step the following special interpolation is introduced:

$$\mathbf{g}_{\mathrm{em}} = g_{\mathrm{ref}} \frac{r_{\mathrm{ref}}^2}{r_n r_{n+1}} \quad \frac{\mathbf{r}_n + \mathbf{r}_{n+1}}{r_n + r_{n+1}}. \tag{5}$$

The major difference to equation (1) is in the expression in the denominator ($\mathbf{r}_n$+$\mathbf{r}_{n+1}$) which is due to the conservation conditions as it will be proven later in this paper.

It is clear that the gravitational acceleration vector is a function of the actual position of the structure and thus the displacements. The effects of the chosen approach concerning the exact simulation of stationary central force fields will be discussed in the following sections.

3. Verification of the conservation laws

It is obvious from Newtonian mechanics that within a central force field for the considered Hamiltonian systems energy conservation as well as linear and angular momentum conservation must hold. On the other hand, it is also well known that using the energy momentum method as time integration scheme, conservation of linear and angular momentum as well as energy within a time step according to equation (3) holds for rigid and flexible systems outside of a central force field. Formulations for flexible structural parts are given by Simo and Tarnow (1992), for rigid bodies see Simo and Wong (1991). The coupling of flexible and rigid parts within the energy momentum method is discussed by Chen (1998), Ibrahimbegovic *et al.* (2000) and Ibrahimbegovic and Mamouri (2000). Two very general derivations are given by Gonzalez (1999, 2000).

Concerning the virtual work of the gravitational forces in equation (3), the following conditions have to be fulfilled for a verification of the conservation laws. This approach is preferred over the derivation from a potential, as it allows to compare the EM-scheme with the midpoint rule.

3.1 Conservation of angular momentum with respect to the center of attraction

Conservation of angular momentum with respect to the center of the gravitational field leads on the basis of a constant velocity within the time step to the condition

$$\int_V \mathbf{r}_{n+\frac{1}{2}} \times \varrho\, \mathbf{g}_{\mathrm{em}}\, \mathrm{d}V = 0 \tag{6}$$

with

$$\mathbf{r}_{n+\frac{1}{2}} = \frac{1}{2}(\mathbf{r}_n + \mathbf{r}_{n+1}). \tag{7}$$

Equation (6) is obviously fulfilled, as the vectors $\mathbf{g}_{\mathrm{em}}$ (5) and $\mathbf{r}_{n+\frac{1}{2}}$ (7) are per definition in parallel as it is also shown in Figure 2.

It should be noted that a formulation using the midpoint approximation

$$\mathbf{g}_{n+\frac{1}{2}} = g_{\mathrm{ref}} \frac{r_{\mathrm{ref}}^2}{r_{n+\frac{1}{2}}^2} \frac{\mathbf{r}_{n+\frac{1}{2}}}{r_{n+\frac{1}{2}}}$$

is also in parallel and also conserves angular momentum.

3.2 Energy conservation

The work of the gravitational forces within the time step is given as

$$W_{\mathrm{ext}}^{\Delta t} = \Delta t \int_V \varrho\, \mathbf{g}_{\mathrm{em}} \cdot \dot{\mathbf{u}}_{n+\frac{1}{2}} \,\mathrm{d}V. \tag{8}$$

Equations (4) and (5) lead to

$$\begin{aligned} \mathbf{g}_{\mathrm{em}} \cdot \dot{\mathbf{u}}_{n+\frac{1}{2}} &= g_{\mathrm{ref}} \frac{r_{\mathrm{ref}}^2}{\Delta t} \frac{(\mathbf{r}_n + \mathbf{r}_{n+1}) \cdot (\mathbf{r}_n - \mathbf{r}_{n+1})}{(r_n + r_{n+1}) r_n r_{n+1}} \\ &= g_{\mathrm{ref}} \frac{r_{\mathrm{ref}}^2}{\Delta t} \frac{r_n^2 - r_{n+1}^2}{(r_n + r_{n+1}) r_n r_{n+1}} \\ &= g_{\mathrm{ref}} \frac{r_{\mathrm{ref}}^2}{\Delta t} \frac{r_n - r_{n+1}}{r_n r_{n+1}}. \end{aligned} \tag{9}$$

As gravitational forces can be considered as general external forces on the system, energy conservation within the time step is not a direct property of equation (3), though – as is well known – a potential exists. Thus it has to be proven, that the work W_{ext} of the gravitational forces within the time step is identical to the loss of potential W_{a} in the central force field.

Inserting (9) into (8) gives

$$W_{\mathrm{ext}}^{\Delta t} = \int_V \varrho g_{\mathrm{ref}} r_{\mathrm{ref}}^2 \frac{r_n - r_{n+1}}{r_n r_{n+1}} \,\mathrm{d}V = g_{\mathrm{ref}} r_{\mathrm{ref}}^2 \int_V \varrho \frac{r_n - r_{n+1}}{r_n r_{n+1}} \,\mathrm{d}V. \tag{10}$$

The difference in the potential between the beginning and the end of the time step is

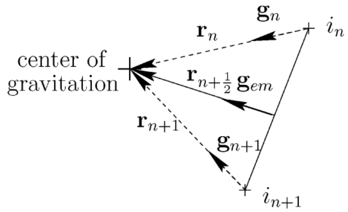

Figure 2.
Vectors in the central gravitational field

$$
\begin{aligned}
W_{\mathrm{a}}^{\Delta t} &= \int_{B_0} \varrho(\mathbf{g}_{n+1}\cdot\mathbf{r}_{n+1} - \mathbf{g}_n\cdot\mathbf{r}_n)\,\mathrm{d}V \\
&= \int_{B_0} g_{\mathrm{ref}} r_{\mathrm{ref}}^2\,\varrho\left[\frac{\mathbf{e}_{n+1}^r}{r_{n+1}^2}\cdot(\mathbf{e}_{n+1}^r r_{n+1}) - \frac{\mathbf{e}_n^r}{r_n^2}\cdot(\mathbf{e}_n^r r_n)\right]\mathrm{d}V \quad \text{with (1)} \\
&= g_{\mathrm{ref}}\, r_{\mathrm{ref}}^2 \int_{B_0} \varrho\left(\frac{1}{r_{n+1}} - \frac{1}{r_n}\right)\mathrm{d}V \\
&= g_{\mathrm{ref}}\, r_{\mathrm{ref}}^2 \int_{V} \varrho\frac{r_n - r_{n+1}}{r_n r_{n+1}}\,\mathrm{d}V \Rightarrow W_{\mathrm{a}}^{\Delta t} = W_{\mathrm{ext}}^{\Delta t}.
\end{aligned} \tag{11}
$$

It again has to be pointed out that only with the specific expression in equation (5) for the average acceleration the satisfaction of both conservation laws is guaranteed, whereas in contrast the midpoint approximation does not show energy conservation.

4. FE-discretization in space

Using displacement based elements with the matrix of shape functions as $\mathbf{N}_{\mathrm{e}} = [\mathbf{I}_{3\times3}N_1, \ldots, \mathbf{I}_{3\times3}N_{\mathrm{nen}}]$, the virtual displacements within the element are given by the standard expression

$$\delta\mathbf{u}_{\mathrm{e}} = \mathbf{N}_{\mathrm{e}}\delta\mathbf{d}_{\mathrm{e}}, \tag{12}$$

with the nodal displacement vector $\mathbf{d}_{\mathrm{e}}$. The residual of the volume forces $\mathbf{f}_{\mathrm{e}}$ results from the weak form equation (3)

$$\mathbf{f}_{\mathrm{e}}\cdot\delta\mathbf{d}_{\mathrm{e}} = \int_{V_{\mathrm{e}}} \varrho\mathbf{g}_{\mathrm{em}}^{\mathrm{T}}(\mathbf{r})\mathbf{N}_{\mathrm{e}}^{\mathrm{T}}\,\mathrm{d}V\cdot\delta\mathbf{d}_{\mathrm{e}}. \tag{13}$$

The linearization within a Newton Raphson scheme for the solution leads to nonsymmetric so-called "load parts" for the effective stiffness matrix according to

$$\mathbf{K}_{\mathrm{e}} = \int_{V_{\mathrm{e}}} \varrho\mathbf{N}_{\mathrm{e}}^{\mathrm{T}}\mathbf{K}_l\mathbf{N}_{\mathrm{e}}\,\mathrm{d}V \tag{14}$$

with a non-symmetric kernel $\mathbf{K}_l$

$$\mathbf{K}_l = g_{\mathrm{ref}}\, r_{\mathrm{ref}}^2\left[\left(\frac{1}{r_n r_{n+1}^2} + C_r\right)\mathbf{e}_{\mathrm{em}}^{*}\mathbf{e}_{n+1}^{\mathrm{T}} - C_r\mathbf{I}_{3\times3}\right] \tag{15}$$

$$C_r = \frac{1}{r_n r_{n+1}(r_n + r_{n+1})}; \quad \mathbf{e}_{\mathrm{em}}^{*} = \frac{\mathbf{r}_n + \mathbf{r}_{n+1}}{r_n + r_{n+1}}. \tag{16}$$

Although the underlying problem is conservative, which normally leads to symmetric matrices (Bufler, 1984; Schweizerhof and Ramm, 1984), the special definition of the gravitational acceleration within the energy momentum approach results in an nonsymmetric load part for the effective stiffness matrix. This is a typical effect of the energy momentum method proposed by Simo and Tarnow (1992) known from applications to elastic systems. However, with physically realistic gravitational fields, thus large radii, and time step sizes which are adapted to the time of circulation, these nonsymmetric terms are very small compared to the other terms in the effective stiffness matrix of the algorithm and can normally be neglected in the solution scheme. This has been tested on various numerical examples and convergence was not affected.

An alternative for an effective algorithm, the symmetrization of $\mathbf{K}_l$ by averaging the nonsymmetric part

$$\mathbf{e}^*_{\mathrm{em}}\mathbf{e}^{\mathrm{T}}_{n+1} \quad \text{with} \quad \frac{1}{2}\left(\mathbf{e}^*_{\mathrm{em}}\mathbf{e}^{\mathrm{T}}_{n+1} + \mathbf{e}^*_{n+1}\mathbf{e}^{\mathrm{T}}_{\mathrm{em}}\right)$$

did not have any effect on the convergence for the examples considered. This is certainly entirely different for small distances as may occur in electric fields.

For the calculation of the gravitational forces the rigid bodies are discretized by "FE-like" parts. Using such a discretization, mass, center of mass and the inertia tensor of such structures can be easily computed for the reference configuration. It is recommended to select the center of mass as reference point for the translational and rotational degrees of freedom.

To compute the orientation of a satellite in a central force field and not only the position of its center of mass, the volume integration of the gravitational forces in the actual configuration is absolutely mandatory for the satellites independently, if they are considered as flexible or rigid.

When satellites are simplified as rigid bodies, all nodal forces resulting from the volume integration have also to be transformed to the six dofs of the center of mass, using constraint conditions. In an exact computation the resulting load parts to the stiffness matrix (equation 14) have also to be transformed with these constrained conditions to the effective stiffness matrix of the dofs of the rigid body. As mentioned earlier, these parts can normally be neglected to improve the efficiency for examples with large radii.

With a sufficiently fine spatial discretization the approximation of equation (13) by

$$\mathbf{f}_{\mathrm{e}} \cdot \delta\mathbf{d}_{\mathrm{e}} = \left(\mathbf{M}^l_{\mathrm{e}}\,\hat{\mathbf{g}}_{\mathrm{em}}\right) \cdot \delta\mathbf{d}_{\mathrm{e}}, \tag{17}$$

with a diagonal matrix consisting of the lumped mass terms times the corresponding nodal values of the gravitational acceleration $\hat{\mathbf{g}}_{\mathrm{em}}$ leads to negligible errors, but to a large reduction of the computational effort.

5. Numerical example: rigid body in circular orbit in radial position

The goal is to simulate the motion of a satellite with a rectangular shape on a circular orbit within the gravitation field of the earth. The satellite is assumed as a rigid body and the gravitational field is simplified as a central force field. As the orientation of the satellite is of interest, the action of the gravitational force is not modelled by resultant forces acting on the center of mass, but the real distribution over the continuum is taken into account. This distribution is computed by evaluation of equation (13). The gradient of the force within the structure is rather small. As a consequence the load terms in the effective stiffness matrix can be neglected in order to increase computational efficiency. However, it has to be pointed out that this small gradient has to be considered for the force vector correctly in order to compute the correct orientation of the satellite.

The gravitation field is defined by the reference acceleration of $g_{\mathrm{ref}} = 9.8100\,\mathrm{m/s^2}$ on the earth's surface ($r_{\mathrm{ref}} = 6,370\,\mathrm{km}$). The satellite is assumed to move on a physically realistic circular orbit 400 km above the surface. It has constant mass density and hexahedral geometry with $l = 2.0\,\mathrm{m}$ and a cross section of $0.2\,\mathrm{m} \times 0.2\,\mathrm{m}$. In the starting position ($t = 0$), the longitudinal axis of the satellite points to the center of gravitation as shown in Figure 3. The starting velocities referring to the center of mass

$$\mathbf{v}_0 = \left[0.0,\ 0.0,\ \ -\frac{2r_{\mathrm{s}}\pi}{\mathrm{T}}\right]^{\mathrm{T}} \qquad \omega_0 = \left[0.0,\ \frac{2\pi}{T},\ 0.0\right]^{\mathrm{T}} \tag{18}$$

lead to a stable motion on the circular orbit in radial position. A time step size of $\Delta t = 20\,\mathrm{s}$ is taken, while the duration of one circulation is $T = 5547.4\,\mathrm{s}$. The position during the first circulation is shown in Figure 4. As this motion is stable, the orientation of the satellite should not be affected by small disturbances, for example occurring from numerical errors within the computation. This has been checked numerically in a long time control simulation of about 1,000 circulations with $\Delta t = 10\,\mathrm{s}$. The reason for the stable motion is the fact, that once a nonaligned position is assumed the larger gravitational forces close to the center of gravitation cause a

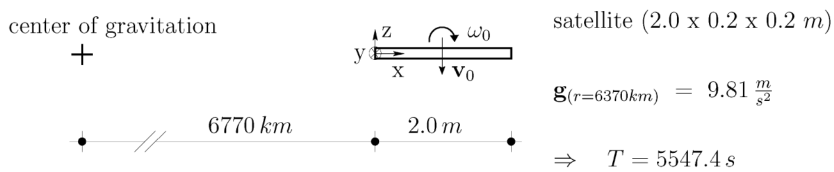

Figure 3.
Starting conditions for the satellite (rigid body) on a circular orbit

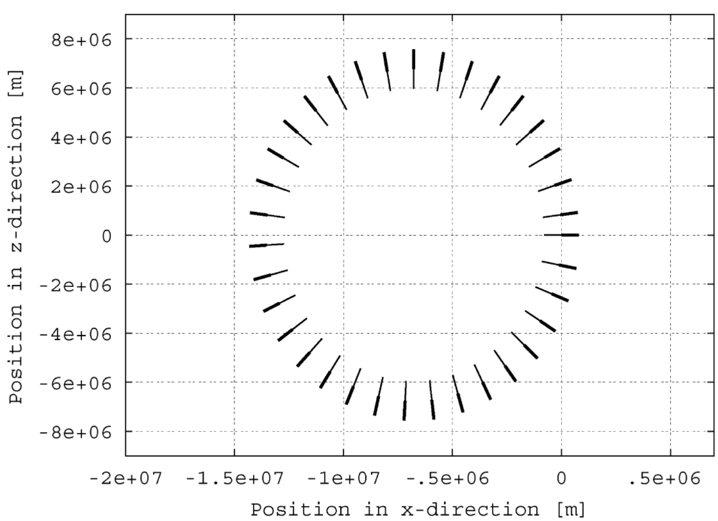

Figure 4.
Orientation of the satellite in the first circulation with starting conditions as described in equation (18)

moment that acts against the perturbation. As energy is conserved, small perturbations lead only to a pendulum type motion around the ideal radial orientation.

To analyze the effect of major perturbations, the starting conditions are modified considerably by setting the angular velocity to zero.

$$\mathbf{v}_0 = \left[0.0,\ \ 0.0,\ -\frac{2r_{\mathrm{s}}\pi}{T}\right]^{\mathrm{T}} \qquad \omega_0 = [0.0,\ \ 0.0,\ \ 0.0]^{\mathrm{T}}. \tag{19}$$

The numerical simulation with the same time step size as above shows that the missing angular velocity around the y-axis leads to a harmonic pendulum motion. The orientation of the satellite during the first circulation is shown in Figure 5. The harmonic rotation with respect to the y-axis during the first five circulations is given in Figure 6.

It is clearly visible that the correct consideration of the gradient of the gravitational force as well as the full shape of the satellite is necessary to compute its real position. In addition, neglecting the nonsymmetric terms in the effective stiffness matrix due to the EM-scheme did not affect the quadratic convergence in the examples considered.

6. Conclusions

Focusing on Hamiltonian systems a formulation is given for the correct algorithmic consideration of a stationary central force field acting as external forces on a satellite. With the energy momentum method as time integration scheme all effects of the modelling of such a gravitational field are shown for rigid and flexible continua. Of special interest is the computation of the orientation of satellites within a central force field.

The numerical examples are restricted to rigid bodies, because a significant relation between deformation and attraction forces would only occur for satellites with extremely large deformations or for satellites that are very close to the center of the

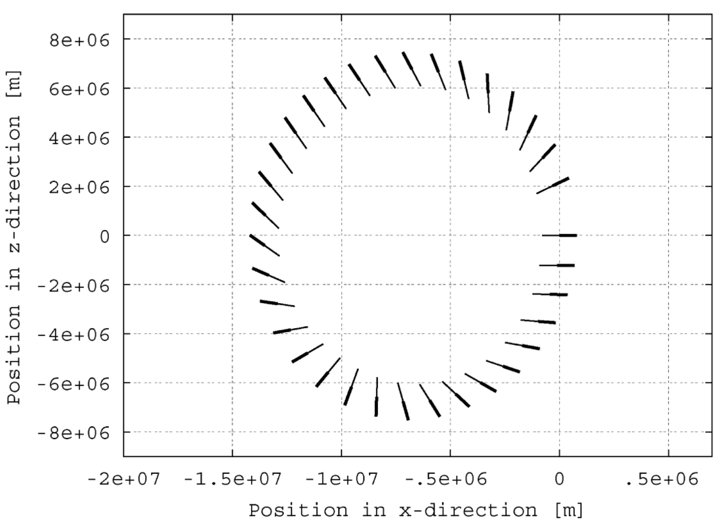

Figure 5.
Orientation of the satellite in the first circulation with starting conditions as described in equation (19)

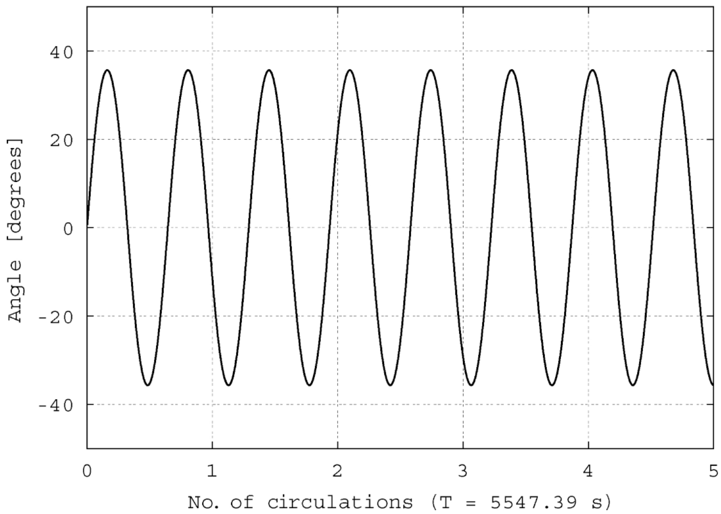

Figure 6.
Angle between the ideal radial orientation and the actual orientation of the satellite caused by starting condition as described in equation (19)

force field. In addition, in such situations the numerical simulation of flexible structures requires very small time steps and high numerical effort; the latter in particular due to the nonsymmetric effective matrices arising from the time integration scheme for energy and momentum conservation. Thus for the satellites with rather large distances to the center of gravitation it can be recommended to reduce the model to a rigid body concerning the effects of the gravitational forces. The presented formulation is based on the high numerical stability of the underlying energy momentum method, which is well suited for long duration simulations.

References

Betsch, P. and Steinmann, P. (2000a), "Conservation properties of a time FE method. Part I: time-stepping schemes for n-body problems", *International Journal for Numerical Methods in Engineering*, Vol. 49, pp. 599-638.

Betsch, P. and Steinmann, P. (2000b), "Inherently energy conserving time finite elements for classical mechanics", *Journal of Computational Physics*, Vol. 160, pp. 88-116.

Bufler, H. (1984), "Pressure loaded structures under large deformations", *ZAMM Zeitung fuer angewandte Mathematik und Mechanik*, Vol. 7, pp. 287-95.

Chen, A. (1998), "Energy-momentum conserving methods for three dimensional dynamic nonlinear multibody systems", PhD thesis, Division of mechanics and computation, Stanford University.

Gonzalez, O. (1999), "Mechanical systems subject to holonomic constraints: differential-algebraic formulations and conservative integration", *Physica D*, Vol. 132, pp. 165-74.

Gonzalez, O. (2000), "Exact energy and momentum conserving algorithms for general models in nonlinear elasticity", *Computers and Mathematics with Applications*, Vol. 190, pp. 1763-83.

Gonzalez, O. and Simo, J. (1996), "On the stability of symplectic and energy-momentum algorithms for non-linear hamiltonian systems with symmetry", *Computer Methods in Applied Mechanics and Engineering*, Vol. 134, pp. 197-222.

Greenspan, D. (1995), "Completely conservative, covariant numerical methodology", *Computers and Mathematics with Applications*, Vol. 29 No. 4, pp. 37-43.

Ibrahimbegovic, A. and Mamouri, S. (2000), "On rigid components and joint constraints in nonlinear dynamics of flexible multibody systems employing 3d geometrically exact beam model", *Computer Methods in Applied Mechanics and Engineering*, Vol. 188, pp. 805-32.

Ibrahimbegovic, A., Mamouri, S., Taylor, R. and Chen, A. (2000), "Finite element method in dynamics of flexible multibody systems: modeling of holonomic constraints and energy conserving integration schemes", *Multibody System Dynamics*, Vol. 4, pp. 195-224.

Kuhl, D. and Crisfield, M. (1999), "Energy-conserving and decaying algorithms in non-linear structural dynamics", *International Journal for Numerical Methods in Engineering*, Vol. 45, pp. 569-99.

Kuhl, D. and Ramm, E. (1996), "Constraint energy momentum algorithm and its application to non-linear dynamics of shells", *Computer Methods in Applied Mechanics and Engineering*, Vol. 136, pp. 293-315.

Schweizerhof, K. and Ramm, E. (1984), "Displacement dependent pressure loads in nonlinear finite element analysis", *Computer and Structures*, Vol. 18, pp. 1099-114.

Simo, J. and Tarnow, N. (1992), "The discrete energy-momentum method. Conserving algorithms for nonlinear elastodynamics", *ZAMP, Zeitschrift fur angewandte Mathematik und Physik*, Vol. 43, pp. 757-93.

Simo, J. and Wong, K. (1991), "Unconditionally stable algorithms for rigid body dynamics that exactly preserve energy and momentum", *International Journal for Numerical Methods in Engineering*, Vol. 31, pp. 19-52.