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On the optimal solution of interval linear complementarity problems

1. The linear complementarity problem

The linear complementarity problem, abbreviated LCP, is to find a vector z such that

$$q + Mz \geq o, \quad z \geq o, \quad (q + Mz)^T z = 0, \quad (1)$$

or to show that no such vector z exists. The inequalities appearing in (1) and in the sequel are meant componentwise and o denotes the zero vector. For a detailed introduction to the LCP we refer to [3].

The present paper is concerned with the case, where the given matrix M and the given vector q are not exactly known but can be enclosed in an interval matrix $[M]$ and an interval vector $[q]$, respectively. This generalization arises, for example, from free boundary problems by discretizing the problem without neglecting the discretization error ([4]).

2. Interval linear complementarity problems

We consider compact intervals $[\underline{a}, \bar{a}] := \{x \in \mathbb{R} : \underline{a} \leq x \leq \bar{a}\}$ and denote the set of all such intervals by \mathbf{IR} . We also write $[a]$ instead of $[\underline{a}, \bar{a}]$. Furthermore, we consider matrices with an interval in each of its elements; i.e., $[\underline{A}, \bar{A}] = ([a_{ij}]) = ([\underline{a}_{ij}, \bar{a}_{ij}])$. We also write $[\underline{A}, \bar{A}] := \{A \in \mathbb{R}^{n \times n} : \underline{A} \leq A \leq \bar{A}\}$. By $\mathbf{IR}^{n \times n}$ we denote the set of all these so-called interval matrices. We also write $[A]$ instead of $[\underline{A}, \bar{A}]$. The set of interval vectors with n components is constructed in the same way and denoted by \mathbf{IR}^n . For an introduction to interval computations we refer to [2].

Let $[q] \in \mathbf{IR}^n$ and $[M] \in \mathbf{IR}^{n \times n}$ be given. Then, we are interested in the set

$$\Sigma([M], [q]) := \left\{ z \in \mathbb{R}^n : z \geq o, q + Mz \geq o, (q + Mz)^T z = 0, M \in [M], q \in [q] \right\}. \quad (2)$$

To describe $\Sigma([M], [q])$ we consider the auxiliary set

$$AUX([M], [q]) := \left\{ \begin{pmatrix} \omega \\ z \end{pmatrix} \in \mathbb{R}^{2n} : \omega - Mz = q, \omega \geq o, z \geq o, \omega^T z = 0, M \in [M], q \in [q] \right\}. \quad (3)$$

Lemma 1 ([5]). *Let $[M] \in \mathbf{IR}^{n \times n}$ and $[q] \in \mathbf{IR}^n$. Then, $z \in \Sigma([M], [q])$ iff there exists $\omega \in \mathbb{R}^n$ such that $\begin{pmatrix} \omega \\ z \end{pmatrix} \in AUX([M], [q])$.*

Theorem 1 ([5]). *Let $[M] \in \mathbf{IR}^{n \times n}$, I the $n \times n$ identity matrix, $[q] \in \mathbf{IR}^n$. Then,*

$$\begin{pmatrix} \omega \\ z \end{pmatrix} \in AUX([M], [q]) \Leftrightarrow (I : -[M]) \cdot \begin{pmatrix} \omega \\ z \end{pmatrix} \cap [q] \neq \emptyset \text{ and } \omega^T z = 0, \omega \geq o, z \geq o.$$

We mention the following equivalence:

$$[a] \cap [b] \neq \emptyset \Leftrightarrow \underline{a} \leq \bar{b} \text{ and } \underline{b} \leq \bar{a}, \quad \text{if } [a], [b] \in \mathbf{IR}. \quad (4)$$

Let $\begin{pmatrix} \omega \\ z \end{pmatrix} \in AUX([M], [q])$. Then we have via Theorem 1 and (4) (used componentwise)

$$\omega_i - \sum_{j=1}^n \bar{m}_{ij} \cdot z_j \leq \bar{q}_i \text{ and } \underline{q}_i \leq \omega_i - \sum_{j=1}^n \underline{m}_{ij} \cdot z_j, \quad \text{for all } i \in \{1, \dots, n\}, \quad (5)$$

since $z_j \geq 0$. Due to the complementarity we have to consider 2^n cases then.

Example 1. Let

$$[M] = \begin{pmatrix} [\frac{1}{8}, 1] & [-\frac{1}{4}, -\frac{1}{5}] \\ [-\frac{1}{4}, -\frac{1}{10}] & 1 \end{pmatrix} \text{ and } [q] = \begin{pmatrix} [-3, -1] \\ [1, 2] \end{pmatrix}.$$

(5) leads to four inequalities:

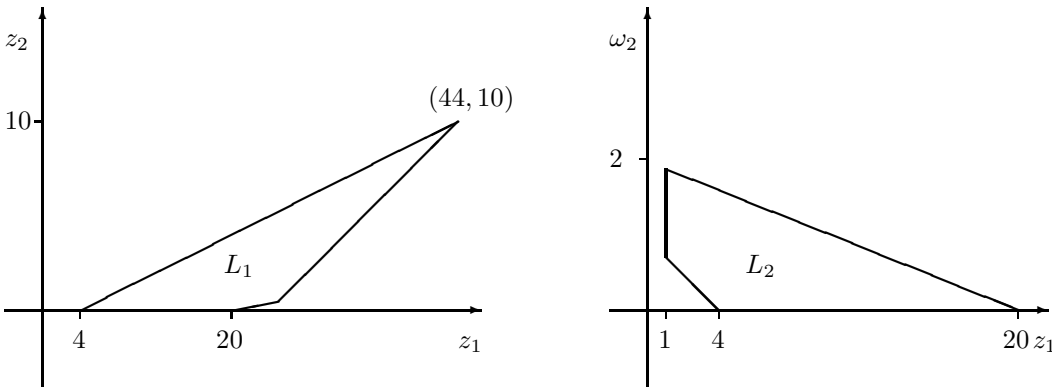
$$\begin{aligned} \text{(I)} \quad & \omega_1 - (z_1 - \frac{1}{5} z_2) \leq -1; & \text{(II)} \quad & -3 \leq \omega_1 - (\frac{1}{8} z_1 - \frac{1}{4} z_2); \\ \text{(III)} \quad & \omega_2 - (-\frac{1}{10} z_1 + z_2) \leq 2; & \text{(IV)} \quad & 1 \leq \omega_2 - (-\frac{1}{4} z_1 + z_2). \end{aligned}$$

We consider four cases: 1. $\omega_1 = 0, \omega_2 = 0$; 2. $\omega_1 = 0, z_2 = 0$; 3. $z_1 = 0, \omega_2 = 0$; 4. $z_1 = 0, z_2 = 0$.

The cases 3. and 4. cannot give a contribution to $AUX([M], [q])$ due to (I). Considering the cases 1. and 2. we get

$$AUX([M], [q]) = \left\{ \begin{pmatrix} 0 \\ 0 \\ z_1 \\ z_2 \end{pmatrix} : (z_1, z_2) \in L_1 \right\} \cup \left\{ \begin{pmatrix} 0 \\ \omega_2 \\ z_1 \\ 0 \end{pmatrix} : (z_1, \omega_2) \in L_2 \right\},$$

where L_1 and L_2 are illustrated in the following figure.



By Lemma 1 we get $\Sigma([M], [q]) = L_1 \cup \begin{pmatrix} [1, 20] \\ 0 \end{pmatrix}$.

3. The optimal solution of an interval linear complementarity problem

The optimal solution of an interval linear complementarity problem is the narrowest interval vector that includes $\Sigma([M], [q])$. Concerning Example 1 the optimal solution is $\begin{pmatrix} [1, 44] \\ [0, 10] \end{pmatrix}$. In [1], we have shown that the total step method, the single step method and the symmetric single step method are convergent to the optimal solution for the case that all $M \in [M]$ are M(inkowski) matrices. For the case that all $M \in [M]$ are H-matrices with positive diagonal entries the above mentioned methods are also convergent to an interval vector that includes $\Sigma([M], [q])$, but this inclusion is not necessarily optimal.

4. References

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