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On the modulus algorithm for the linear complementarity problem

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1 Introduction

Given a real $n \times n$ matrix M and a real n-dimensional vector q, the linear complementarity problem, abbreviated by LCP, is to find two vectors $\omega,\ z$ such that

$$\omega = q + Mz, \quad \omega \geq o, \quad z \geq o, \quad \omega^{\mathrm{T}} z = 0, \tag{1}$$

or to conclude that no such vectors $\omega,\ z$ exist. The inequalities appearing in (1) and in the sequel are understood componentwise and o denotes the zero vector. Many applications and solution methods for (1) can be found in [3] and [4], respectively.

In [8] (see also Section 9.2 in [4]), the so-called modulus algorithm was developed for solving the LCP: Let I denote the identity and with $x \in \mathbf{R^n}$ we define

$$|x| := \begin{pmatrix} |x_1| \\ \vdots \\ |x_n| \end{pmatrix} \in \mathbf{R^n}.$$

If $I + M$ is nonsingular, then the LCP defined by $M \in \mathbf{R^{n\times n}}$ and $q \in \mathbf{R^n}$ is equivalent to the fixed point problem of determining $x \in \mathbf{R^n}$ satisfying

$$x = f(x) := (I+M)^{-1}(I-M)|x| - (I+M)^{-1}q. \tag{2}$$

More precisely (see the proof of Theorem 9.1 in [4]), if x is a solution of (2), then

$$\omega := |x| - x, \quad z := |x| + x$$

define a solution of (1). On the other hand, if $\omega,\ z$ solve (1), then $x := \frac{1}{2}(z - \omega)$ is a solution of (2). The modulus algorithm is then defined as an iterative method concerning (2):

$$\left.\begin{array}{rcl} x^0 & \in & \mathbf{R^n} \text{ arbitrary}, \\ x^{k+1} & := & f(x^k) = (I+M)^{-1}(I-M)|x^k| - (I+M)^{-1}q. \end{array}\right\} \tag{3}$$

For the case that M is symmetric positive definite, and for the case that M is a so-called H-matrix with positive diagonal entries, it is guaranteed that (3) is convergent to a unique solution. See Section 9.2 in [4] and Theorem 2.3 in [7], respectively.

In the following section we present another situation where (3) is convergent to a unique solution.

2 Extreme vectors of the solution set of systems of linear interval equations

We consider a family of matrices and vectors

$$[A] := [\underline{A}, \overline{A}] := \{A \in \mathbf{R^{n\times n}} : \underline{\mathbf{A}} \leq \mathbf{A} \leq \overline{\mathbf{A}}\}, \quad [\mathbf{b}] := [\underline{\mathbf{b}}, \overline{\mathbf{b}}] := \{\mathbf{b} \in \mathbf{R^n} : \underline{\mathbf{b}} \leq \mathbf{b} \leq \overline{\mathbf{b}}\}.$$

If all $A \in [A]$ are regular, we are interested in finding an interval vector $[x]$ that includes the solution set

$$\Sigma([A],[b]) := \{x \in \mathbf{R^n} : \mathbf{Ax} = \mathbf{b},\ \mathbf{A} \in [\mathbf{A}],\ \mathbf{b} \in [\mathbf{b}]\}, \quad \text{see [1].}$$

The narrowest interval vector that includes $\Sigma([A],[b])$ is defined by its extreme vectors.

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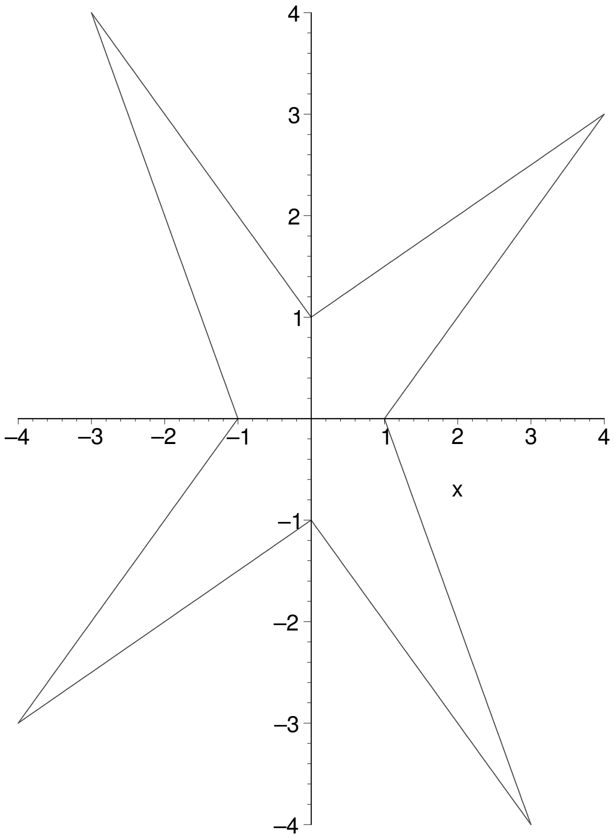

Fig. 1 The shape of $\Sigma([A],[b])$ concerning Example 2.1.

Example 2.1 *Let*

$$[A] = \begin{pmatrix} [2,4] & [-2,1] \\ [-1,2] & [2,4] \end{pmatrix}, \quad [b] = \begin{pmatrix} [-2,2] \\ [-2,2] \end{pmatrix}.$$

Then $\Sigma([A],[b])$ *is not an interval vector (see* [2]*). It can be described as depicted in Figure 1. The extreme vectors of* $\Sigma([A],[b])$ *are*

$$\left\{ \begin{pmatrix} -3 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \end{pmatrix}, \begin{pmatrix} -4 \\ -3 \end{pmatrix} \right\}.$$

So, the narrowest interval vector that includes $\Sigma([A],[b])$ *is* $\begin{pmatrix} [-4,4] \\ [-4,4] \end{pmatrix}$.

In [5], it was shown that the extreme vectors of $\Sigma([A],[b])$ can be calculated via solutions of LCPs. The arising matrices are so-called P-matrices which guarantee the unique solvability of the LCPs. However, the matrices are neither necessarily H-matrices nor positive definite matrices (see [6]). As a consequence, it is not clear if the modulus algorithm can be applied. However, under slight additional assumptions on $[A]$ the convergence of the modulus algorithm can be guaranteed. For details we refer to [7].

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