Measuring non-Gaussian fluctuations through incoherent Cooper pair current

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We study a Josephson junction (JJ) in the regime of incoherent Cooper pair tunneling, capacitively coupled to a nonequilibrium noise source. The current-voltage (I-V) characteristics of the JJ are sensitive to the excess voltage fluctuations in the source, and can thus be used for wide-band noise detection. Under weak driving, the odd part of the I-V can be related to the second cumulant of noise, whereas the even part is due to the third cumulant. After calibration, one can measure the Fano factors for the noise source, and get information about the frequency dependence of the noise.

The current in electric circuits fluctuates in time, even when driven with a constant voltage. At equilibrium or in large conductors, this current noise can be quantified using the fluctuation-dissipation theorem (FDT), which relates the magnitude of the fluctuations to the temperature T and the impedance of the circuit. Moreover, in large wires the current statistics is described by a Gaussian probability density which has only two nonzero cumulants, the average current and noise power. This situation changes for small, mesoscopic-scale resistors exhibiting shot noise [1, 2]: The noise power at low frequencies is proportional to the average current. Further, the statistics of the transmitted charge is no longer Gaussian: higher cumulants are finite, and the probability density is "skew", i.e., odd cumulants do not vanish.

For small samples, the frequency scale for the shot noise is given by the voltage, $eV/\hbar$. Shot noise has been measured at low frequencies in many types of mesoscopic structures (see the references in [1, 2]). However, there are only a few direct measurements of shot noise at high frequencies $\omega \sim eV/\hbar$ [3], and only one of the higher (than second) cumulants [4] (at $\omega \ll eV/\hbar$). One of the main reasons for the shortage of such measurements is the difficulty to couple the fluctuations to the detector at high frequencies, or to devise wide-band detection, as required for the third and higher cumulants [4, 5].

In this Letter, we analyze an on-chip detector of voltage fluctuations, based on capacitively coupling a noise source to a small JJ in Coulomb blockade [6]. There, the current can flow only if the environmental fluctuations provide the necessary energy to cross the blockade. In this way, the current through the small JJ provides detailed information of the voltage fluctuations in the source over a wide bandwidth. This information includes effects of a non-Gaussian ("skew") environment on a quantum system. For the measurement of the second cumulant, its characteristics compare well with the other suggested on-chip detectors [7, 8, 9, 10, 11], based on various mesoscopic devices and techniques. The detectors proposed in [10, 11] detect the non-Gaussian character of the noise, but mapping the output back to the different cumulants has not been carried out. The on-chip scheme presented here is the first to directly measure the third cumulant of fluctuations. In the Gaussian regime, our analysis of the noise detection resembles that of Ref. [8], but probes the noise at low measurement voltages.
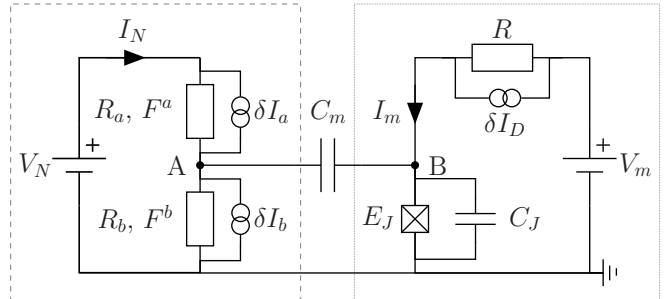

FIG. 1: Noise measurement circuit. The voltage noise at point A, driven with V_N, induces voltage noise at point B through the capacitor C_m and in this way affects JJ. JJ is in a highly resistive environment, $R > R_Q = h/(4e^2)$. $F_n^{a,b}$ are the Fano factors for the n'th cumulants in the noise source. The noise is read by investigating $I_m(V_m)$, as explained in the text.

We investigate the system depicted in Fig. 1. The part indicated by the dashed lines represents the noise source, and the other part is the detector. The excess voltage noise at point A, induced by driving the source ($V_N \neq 0$), gives rise to voltage noise at point B, and the latter can be read by examining the current $I_m(V_m)$. There is also another source of noise: the equilibrium fluctuations in the whole circuit, described by FDT. As shown below, the two types of fluctuations can be treated separately.

In the regime of incoherent Cooper-pair tunneling, the I-V characteristics of the detector can be described by a perturbation theory in the Josephson coupling energy

E_J. This yields for the current through JJ [12, 13]

$$I_m(V_m) = \frac{\pi e E_J^2}{\hbar} \left[P_{\phi(t)}(2eV_m) - P_{-\phi(t)}(-2eV_m) \right]. \quad (1)$$

Here $P_{\phi(t)}(E) \equiv \int dt e^{iEt/\hbar} \langle e^{i\phi(t)} e^{-i\phi(0)} \rangle /(2\pi\hbar)$ describes the Cooper-pair tunneling through the JJ due to the fluctuations $\phi(t)$ of the phase difference across the junction. This result makes no assumptions about the form of the phase fluctuations - only that they do not modify E_J itself. As these fluctuations are connected to the fluctuations $\delta V_B(t)$ of the voltage over the JJ via the Josephson relation, $\phi(t) = \phi(0) + 2e \int_0^t dt' \delta V_B(t')/\hbar$, $I_m(V_m)$ provides information on these fluctuations. Note that, although the coupling capacitance acts as a high-pass filter for voltage noise, the conversion to phase noise allows to transmit noise down to low frequencies. This way the device can be operated non-hysteretically [10] and in Coulomb blockade.

One can identify $P_\phi(E)$ as the Fourier transform of the moment-generating function $\kappa_\phi(\chi) = \langle e^{\chi\phi(t)} e^{-\chi\phi(0)} \rangle$ of $\phi(t) - \phi(0)$, evaluated at $\chi = i$. This can be expanded in the cumulants $C_n^\phi(t)$, $\kappa_\phi(\chi) = \exp[\chi^2/(2!)C_2^\phi(t) + \chi^3/(3!)C_3^\phi(t) + \cdots]$. These cumulants are defined such that $\phi(t)$ is ordered before $\phi(0)$ in the expectation value. The expansion defines a function $J_\phi(t) = \ln(\kappa_\phi(i)) = J_2(t) + J_3(t) + \cdots$, where for stationary fluctuations $J_2(t) = \langle [\phi(t) - \phi(0)]\phi(0) \rangle$ and $J_3(t) = \frac{i}{2}\langle \phi(t)[\phi(t) - \phi(0)]\phi(0) \rangle$, etc. In the Gaussian limit, $J_\phi(t) = J_{-\phi}(t) = J_2(t)$ coincides with that applied in Ref. [12].

The odd cumulants of the fluctuations in the source break the symmetry between the positive and the negative V_m. To separate the non-Gaussian effects, we consider the even and odd parts of the I-V, $I_{S/A}(V_m) \equiv (I_m(V_m) \pm I_m(-V_m))/2$. The odd part $I_A(V_m)$ describes the general behavior mostly due to the even cumulants (c.f., Eq. (6) below), and the even part $I_S(V_m)$ (Eq. (7)) responds to the odd cumulants, vanishing if $C_{2n+1} = 0$. We show below that tuning the voltage V_m and measuring $I_{A/S}(V_m)$ gives access to the frequency dependence of the lowest cumulants at large bandwidths.

We have to take the additional Gaussian fluctuations $J_\phi^D(t)$ from the total impedance of the setup into account, as they will inevitably influence the measurement. Thus, we split the fluctuations as parts $J_\phi(t) = J_\phi^D(t) + J_\phi^S(t)$ with the excess fluctuations due to the driven source denoted by $J_\phi^S(t)$. The latter includes the non-Gaussian effects for which $J_{-\phi}^S \neq J_\phi^S$. We also define $P_\phi^{D/S}(E) \equiv \int dt e^{iEt/\hbar} \langle e^{J_\phi^{D/S}(t)} \rangle /(2\pi\hbar)$. Using detailed balance [12], we obtain for the current $I_D(V_m) \equiv I_m(V_m; J_\phi^S = 0) = \pi e E_J^2 [P_\phi^D(2eV_m)(1 - \exp(-\beta 2eV_m))]/\hbar$. In the presence of the excess noise, the current is given by

$$I_m = \left[\frac{I_D\left(\frac{E}{2e}\right)}{1 - e^{-\beta E}} * \left(P_\phi^S(E) - e^{\beta(E - 2eV_m)} P_{-\phi}^S(-E) \right) \right], \quad (2)$$

evaluated at $E = 2eV_m$. Here $\beta^{-1} = k_B T$ and $*$ denotes the convolution over the energy. Thus, the detector characteristics, described by $P_\phi^D(E)$, do not need to be known exactly, but they can be calibrated by measuring the current $I_D(V_m)$ in the absence of the additional noise.

For connecting the fluctuations at the JJ and the source, we relate the mutually uncorrelated intrinsic fluctuations δI_a, δI_b and δI_m through the resistors R_a, R_b and R to the voltage fluctuations ΔV at point B, $\Delta V(\omega) = R[-\delta I_m(\omega) - i\omega C_m R_S(\delta I_b - \delta I_a - \delta I_m)]/G(\omega)$ through circuit analysis. Here $G(\omega) = 1 - i\omega(RC + R_S C_m) - \omega^2 RC_J R_S C_m$, $C = C_m + C_J$ and $R_S = (R_a^{-1} + R_b^{-1})^{-1}$. The equilibrium fluctuations in the source are present even for $V_N = 0$, and they can be included in the calibration through the FDT. The excess fluctuations due to driving produce excess phase fluctuations, characterized by the n'th order correlators $\langle \phi(\omega_1)\cdots\phi(\omega_n) \rangle \equiv 2\pi\delta(\omega_1 + \cdots + \omega_n)S_{n\phi}(\vec{\omega})$ [14],

$$S_{n\phi}(\vec{\omega}) = \lambda^n \frac{\tau^n}{e^n} \frac{S_{nI}^b(\vec{\omega}) + (-1)^n S_{nI}^a(\vec{\omega})}{G_n(\vec{\omega})}, \quad (3)$$

where $\vec{\omega} = (\omega_1, \ldots, \omega_n)$, $\lambda = \pi R_S C_m/(R_Q C)$, $R_Q = h/(4e^2)$, $\tau = RC$, $G_n(\vec{\omega}) = G(\omega_1)\cdots G(\omega_n)$, and $\langle \delta I(\omega_1)\cdots\delta I(\omega_n) \rangle = 2\pi\delta(\omega_1 + \ldots \omega_n)S_{nI}(\vec{\omega})$. Thus, we find that the excess phase noise and its cumulants are governed by powers of λ, the current $I_N \tau/e$ and the bandwidth of the current correlators, times τ. In the absence of the source ($C_m \to 0$), the detector phase fluctuations $J_{RC}(t)$ have been calculated in [16] (see also Fig. 11 of [12]). The equilibrium fluctuations in the source slightly modify this behavior [17]. If $R_S C_m \ll RC_J$, the resulting $J^D(t)$ is close to $J_{RC}(t)$, with the capacitance given by $C_m + C_J$ and resistance by R, hence the JJ is in the insulating state for $R > R_Q$. Below, we concentrate on this insulating parameter regime.

Now assume the source is driven weakly, such that $J^S(t) \ll 1$. In this case, one may expand the exponential $e^{J^S(t)} \approx 1 + J^S(t)$ [15]. Then,

$$P_\phi^S(E) = \delta(E) + \frac{S_{2\phi}(E/\hbar)}{2\pi\hbar} - \tilde{S}_{2\phi}(0)\delta(E) + \frac{K_\phi(E/\hbar)}{4\pi\hbar} + \cdots. \quad (4)$$

Here $S_{2\phi}(\omega)$ is the driven phase noise spectrum induced by the source, $\tilde{S}_{2\phi}(t)$ its inverse Fourier transform, and

$$K_\phi(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega' \text{Im} S_{3\phi}(\omega, \omega', -\omega - \omega') \quad (5)$$

describes the third cumulant of phase fluctuations. In the derivation of this form for K_ϕ, we used the hermiticity of $\phi(t)$ and the stationarity of the fluctuations. In what follows, we cut the expansion in the third cumulant [15].

Let us first concentrate on the antisymmetric part of the detector current. In the first order in $J^S(t)$, it only probes the even cumulants. In this case, it can be expressed as $I_A(V_m) \equiv I_D(V_m) + \delta I_A(V_m)$, where for sym-

metric $S_{2I}(\omega) = S_{2I}(-\omega)$ [18], using Eq. (3),

$$\delta I_A(V_m) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \mathcal{D}_A(\omega; V_m)(S_{2I}^b(\omega) + S_{2I}^a(\omega)). \quad (6)$$

Here $\mathcal{D}_A(\omega; V_m) \equiv \lambda^2\tau^2[I_D(V_m + \hbar\omega/(2e)) + I_D(V_m - \hbar\omega/(2e)) - 2I_D(V_m)]/(2e^2 G_2(\omega))$ characterizes the frequency band for the detection of $S_{2I}(\omega)$. This band can be tuned by tuning the bias V_m (see Fig. 2).
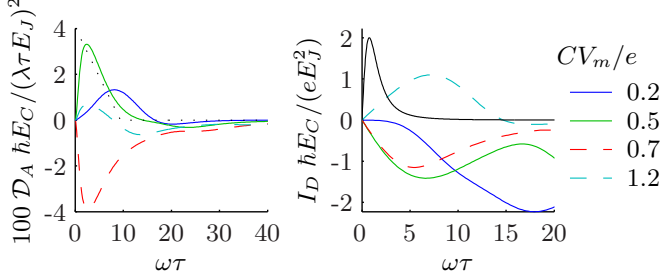

FIG. 2: (Color online) Left: characteristic function $\mathcal{D}_A(\omega; V_m)$ of the excess noise measurement, (c.f., Eq. (6)), probing the spectrum $S_{2I}(\omega)$ of current fluctuations. Dotted line shows the frequency dependence in $S_{2I}(\omega)$ (arbitrary units), for $T = 0$ and $V = e/C$. Right: function $[I_D(V_m + \hbar\omega/(2e)) - I_D(V_m - \hbar\omega/(2e))]/2$, characterizing the measurement of the third cumulant. The frequency dependence in $K_\phi(\omega)$ (arbitrary units) is shown in black. In the limit $eV_N \gg \hbar/\tau$, the width of $K_\phi(\omega)$ is given by $1/\tau$. Here, $E_C \equiv 2e^2/C$, $R = 6R_Q$, $R_a = R_b = 0.1R_Q$, and $C_m = 10C_J$. Changes in C_m do not essentially affect the figure.

The even part of the detector current can then be related to the odd cumulants of phase fluctuations. For a weakly driven source, assuming $K_\phi(-\omega) = -K_\phi(\omega)$ (see below), it is given by [18]

$$I_S(V_m) = \int_{-\infty}^{\infty} \frac{d\omega}{4\pi} I_D\left(V_m - \frac{\hbar\omega}{2e}\right) K_\phi(\omega). \quad (7)$$

In this way, $I_S(V_m)$ probes the frequency dependence of the third cumulant of source fluctuations, (c.f., Fig. 2).

Let us now analyze $\delta I_A(V_m)$ for the noise source shown in Fig. 1. The frequency dependence of the nonsymmetrized noise $S_I(\omega)$ is derived, e.g., in Ref. [8]. This derivation holds provided that the Thouless energy E_T for the resistors greatly exceeds eV_N. Subtracting the fluctuations at $V_N = 0$ we get the excess noise $S_{2I}^i(\omega) = F_2^i \mathcal{S}(V_i)/R_i$, where $i = \{a, b\}$, $V_i = R_i V_N/(R_a + R_b)$ and

$$\mathcal{S}(V) = \frac{eV \sinh(\frac{eV}{kT}) - 2\hbar\omega \coth(\frac{\hbar\omega}{2kT}) \sinh^2(\frac{eV}{2kT})}{\cosh(\frac{eV}{kT}) - \cosh(\frac{\hbar\omega}{kT})}. \quad (8)$$

An example of the frequency dependence of S_{2I} is plotted in Fig. 2. Note that it is symmetric with respect to the sign reversal of ω. This property can be traced to the fact that in our example the source impedance, characterizing quantum fluctuations, stays constant. Now, S_{2I} can be
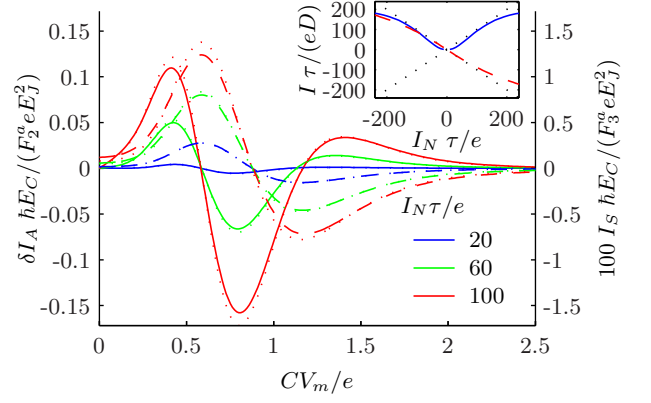

FIG. 3: (Color online) Antisymmetric $\delta I_A(V_m)$ (solid, left axis) and symmetric part $I_S(V_m)$ (dashed, right axis) of the detector current variation. Dotted lines are expansion results (Eqs. (6), (7)). Parameters are as in Fig. 2. The source is assumed to consist of a mesoscopic ($F_n^a \neq 0$) and a macroscopic resistor ($F_n^b = 0$) for which $S_{nI}(\omega = 0) = e^{n-1}F_n I_N$. Inset: $\delta I_A(V_m = e/2C)$ (solid) and $I_S(V_m = e/2C)$ (dashed) as functions of I_N. Scaling parameters are $D = F_2^a D_A \equiv eF_2^a \int d\omega \mathcal{D}_A(\omega; V_m)/(2\pi)$ for the antisymmetric part, and $D = F_3^a D_S \equiv F_3^a \frac{\tau^3}{2\pi e}\lambda^3 \int d\omega I_D(V_m - \hbar\omega/2e)\omega[4 + 5(\omega\tau)^2 + (\omega\tau)^4]^{-1}$ for the symmetric part.

substituted to Eq. (6) to find the effect of the driven noise on the detector current. Figure 3 shows an example of $\delta I_A(V_m)$, i.e., the probed shot noise, for a few bias voltages V_N. For $V_N \gg \hbar/\tau$, shot noise is essentially white over the detector bandwidth (Fig. 2) and thus the signal is linear in V_N. In this case, $I_A(V_m)/I_N$ (Fig. 3 inset) depends only on the factor $D_A(V_m)$, which can be related to the calibration measurement, and the Fano factors $F_2^{a,b}$, which can thus be measured from this curve.

Next, consider the detection of the third cumulant of current fluctuations. Its frequency dependence has been described in various limits for different systems in Ref. [19]. At $T = 0$, the third cumulant at zero frequency is of the form $S_{3I} = F_3 e^2 I_N$, and its dispersion occurs on the scale $\omega_c = \min(eV/\hbar, E_T/\hbar, I/e)$. For $R_s C_m \ll RC_J$ and a frequency independent S_{3I} within the detection bandwidth, the resulting $K_\phi(\omega)$ at $T = 0$ is given by

$$K_\phi(\omega) = 2\tau\lambda^3(F_3^b - F_3^a)\frac{I_N}{e/\tau}\frac{\omega\tau}{4 + 5(\omega\tau)^2 + (\omega\tau)^4}. \quad (9)$$

This is an antisymmetric function of ω, due to the simple form assumed for S_{3I}, and its frequency scale is given by $1/\tau$. With the knowledge of the detector calibration current $I_D(V_m)$, plugging $K_\phi(\omega)$ into Eq. (7) allows us to calculate the response of the symmetric detector current $I_S(V_m)$ to the third cumulant in principle for any type of resistors R_a and R_b. A few examples of $I_S(V_m)$ are shown in Fig. 3. As for the second cumulant, following $I_S(V_m)/D_S(V_m)$ allows to measure the Fano factors.

The general measurement scheme is valid at a finite temperature as long as the Coulomb-blockade condition
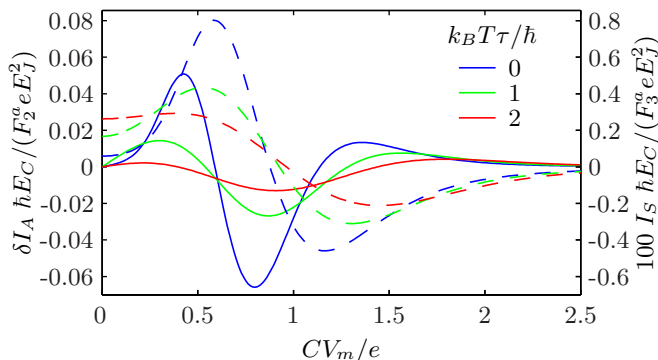
FIG. 4: (Color online) Antisymmetric (solid) and symmetric (dashed) detection currents $\delta I_A(V_m)$ and $I_S(V_m)$ for the same parameters as in Fig. 3 but with a few different temperatures at fixed $V_N = 2e/C$, i.e., $I_N = 60e/\tau$. For $I_S(V_m)$, we neglect the temperature dependence of the intrinsic fluctuations.

$T \lesssim E_C/k_B$ is satisfied. The effects of $T > 0$ are illustrated in Fig. 4. Compared to the previous example, two corrections arise: the function $I_D(V_m)$ characterizing the detector becomes smoother and its amplitude decreases, and the form of the source excess fluctuations changes. The two are characterized by different temperature scales, E_C/k_B and eV_N/k_B, respectively.

Measuring the calibration current $I_D(V_m)$ along with the antisymmetric and symmetric currents, $\delta I_A(V_m)$ and $I_S(V_m)$, allows to find the second and third cumulants of excess current fluctuations in the source within the bandwidth described in Fig. 2 (for typical parameters [10], in the range of 100 GHz) to an accuracy limited mostly only by the resolution of the current measurement (resolution of 0.1 pA yields ~ 100 (fA)^{2}s for the second and ~ 0.01 (fA)^{3}s^2 for the third cumulant [7, 10]). In the limit $C_m \gg C_J$, $R_S C_m \ll R C_J$, the only information required about the setup are the resistances R_a, R_b and R, and the sum capacitance $C_m + C_J$, all of which can be measured separately. Thus, the scheme allows an accurate determination of the second and third cumulants over a large bandwidth and for the third cumulant, overcomes the bandwidth problems encountered in Ref. [4].

In summary, we show in this Letter how incoherent Cooper-pair tunneling in small JJs with a high-impedance environment can be used for accurate and wide-band detection of voltage fluctuations. Via the symmetry of the detector output current, one can identify the contributions from the second and third cumulants separately. While the presented example is on the measurement of fluctuations in samples exhibiting no Coulomb blockade, this is not a limitation of the scheme itself.

Note added in proof: During the refereeing process, we became aware of Ref. [20], which points out that the Fano factors F_3 for the third cumulant depend on the definition of the measured observable. Given that F_3 are related to the nonsymmetrized observable, our results remain valid.

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