Veech Groups of Origamis

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Preface

An *origami* can be described as follows: Take finitely many copies of the unit square in $\mathbb{C}$ and glue them together such that each left edge is glued with a right edge and each upper edge with a lower one. This defines a compact surface. In fact the surface is naturally punctured by removing the vertices of the squares. Lifting the euclidean structure of $\mathbb{C} \cong \mathbb{R}^2$ via the squares defines a natural translation structure on the surface.

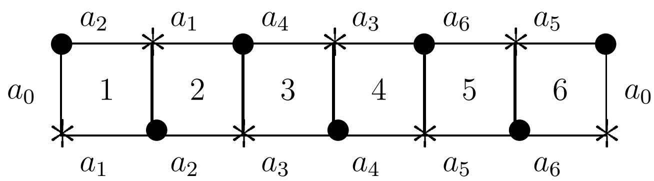

Figure 1: An origami

The central object of this thesis is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ associated to the origami, called the *Veech group*. One obtains it in the following way:
Due to the translation structure we may consider affine diffeomorphisms on the surface. Locally they are of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto A \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}, \quad \text{with } A \in \mathrm{GL}_2(\mathbb{R}) \text{ and } t = (t_1, t_2)^t \in \mathbb{R}^2.$$

The translation vector t varies with the local charts, whereas the matrix A is globally the same. All matrices obtained this way from affine diffeomorphisms form a subgroup of $\mathrm{GL}_2(\mathbb{R})$: the Veech group of the origami.

One motivation for studying Veech groups is given as follows: Each translation surface defines a *Teichmüller disk* in an appropriate Teichmüller space $T_{g,n}$ of Riemann surfaces of genus g with n marked points. Sometimes this Teichmüller disk projects onto an affine curve in the corresponding moduli space. This curve is then called a *Teichmüller curve.* The Veech group of the translation surface "knows" whether this happens or not. It knows as well of which type the Teichmüller curve is.

The aim of this thesis is to study the Veech groups of origamis. It turns out that they are in fact subgroups of $\mathrm{SL}_2(\mathbb{Z})$ of finite index. It is natural to ask which subgroups of $\mathrm{SL}_2(\mathbb{Z})$ occur. This is not known in general, but for a large number of subgroups of $\mathrm{SL}_2(\mathbb{Z})$ we show that they are the Veech group of some origami.

With Theorem 1 we provide a *characterization of the Veech groups of origamis* in terms of subgroups of F_2, the free group on two generators.
We start from the fact that F_2 is the fundamental group of the punctured torus.

Each origami defines an unramified covering of it, see Chapter 1. It is associated to a finite index subgroup of F_2. As second important ingredient we will use that $\mathrm{GL}_2(\mathbb{Z})$ is isomorphic to the outer automorphism group of F_2. We consider stabilizing groups in $\mathrm{Aut}^+(F_2)$, consisting of all automorphisms that stabilize a given finite index subgroup of F_2. We prove in Theorem 1, that

$\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is the Veech group of an origami	$\Leftrightarrow$	it is the image of such a stabilizing group in $\mathrm{Aut}^+(F_2)$.

This characterization is the basis of an *algorithm that determines the Veech group of an origami*, see Chapter 4:
More precisely, the algorithm obtains a set of generators and a system of coset representatives in $\mathrm{SL}_2(\mathbb{Z})$. It calculates the genus and the number of the cusps of the corresponding Teichmüller curve. Furthermore, one obtains a suitable fundamental domain of the action of the Veech group on the upper half plane.

The free group F_2 in the characterization can also be considered as the group of deck transformations of the universal covering of the punctured torus. In this sense it is embedded into $\mathrm{PSL}_2(\mathbb{R})$, the automorphism group of the upper half plane $\mathbb{H}$. This embedding depends on the complex structure that one has on the punctured torus. In Chapter 2, we study the *Teichmüller space* $T_{1,1}$ of once punctured tori and obtain coordinates for it which are especially appropriate in this context. In particular we obtain "nice" generators and "nice" fundamental domains for this family of Fuchsian groups isomorphic to F_2.

The genus of the *Teichmüller curves associated to origamis* is not bounded. In Chapter 5 we explicitly present an infinite sequence X_k of origamis whose associated Teichmüller curves have increasing genus.
On the other hand, in each moduli space M_g there are Teichmüller curves of genus 0. We show this in Theorem 2 by finding an explicit origami for each $g \in \mathbb{N}$, such that the normalization of the associated Teichmüller curve in the moduli space M_g is a projective line without two points and similarly for the projective line without three points.
The origamis presented in this chapter all have Veech groups that are well known congruence groups.

In fact, many *congruence groups* occur as Veech groups of origamis.
In Theorem 4 we show that actually all congruence groups of prime level, with possibly five exceptions, are the Veech group of some origami.
We study congruence groups of level n by their action on $(\mathbb{Z}/n\mathbb{Z})^2$ and consider the orbit spaces they have. In Theorem 3 we prove that all congruence groups which are maximal with respect to their orbit space are the Veech group of some origami.
Theorem 4 then shows that almost any congruence group of prime level has this

property. The theorem can be generalized for arbitrary level n, which is done in Theorem 5.

It is then natural to ask, whether maybe all Veech groups are congruence groups. But this is not true at all. In fact it seems that there exist many examples which are not. In Chapter 7 we give two origamis whose Veech groups are *non congruence groups* and develop a method to obtain infinitely many other examples out of them.

In Chapter 1 we introduce the basic objects that we study: origamis, translation surfaces and Veech groups. Furthermore, we explain the construction that leads to Teichmüller curves in the moduli space in order to state the context in which origamis are studied. In 1.5 we give a glance on what is known in general about Veech groups and Teichmüller curves.

The results of Chapter 3 and 4 are published in [Sh 04]. The content of chapter 5 is submitted for publication as [Sh 05]. I thank Professor Kirsch, the dean of the Department of Mathematics, for allowing me to do so.

At this place I would like to thank all persons who have contributed to this thesis, in particular:

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Furthermore I thank the *Studienstiftung des deutschen Volkes* for having supported my thesis and providing a grant.

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Contents

Chapter 1

Origamis and Teichmüller curves

1.1 Origamis

In this section we define origamis and describe some different ways how they can be presented. I have learnt this theory from [Lo 03].

Definition 1.1. *An* origami *consists of a finite set of copies of the euclidian unit square that are glued observing the following rules:*

- *Each left edge of a square is identified by a translation with a right edge.*
- *Each upper edge is identified by a translation with a lower one.*
- *The closed (topological) surface X, that one obtains, is connected.*

This definition by giving simple rules that define a combinatorial object motivated the name *origami* introduced in [Lo 03]. In fact, the term *origami* is used there for somewhat more general objects. We here restrict to what one might call *oriented origamis* in the terminology there.

Example 1.2. As a first example we consider an origami consisting of four squares, see Figure 3.

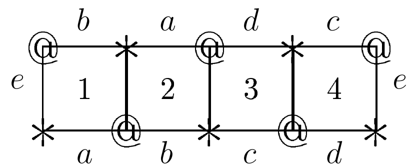


Figure 2: A first example of an origami.

After identifying edges labeled by the same letter one obtains a closed surface X of genus 2: it is divided into 4 squares with 8 edges (after identification) and the two vertices $*$ and @. Hence the Euler characteristic is -2 and the genus is 2.

If one labels the squares of an origami by the numbers $1, \ldots, d$, then the identification of the edges is given by two permutations σ_h and σ_v in S_d, where σ_h indicates how the horizontal edges and σ_v how the vertical edges are glued. In Example 1.2 we have $\sigma_h = (1\ 2\ 3\ 4)$ and $\sigma_v = (1\ 2)(3\ 4)$.

One may present an origami also as a *finite oriented graph* $\mathcal{G}$ as follows: $\mathcal{G}$ has d vertices $v_1, \ldots, v_d$ that correspond to the squares of the origami. The edges of $\mathcal{G}$ are labeled by x or y. Two vertices v_i and v_j are connected by an edge with label x if and only if the right edge of the square that corresponds to v_i is glued with the left edge of the square that corresponds to v_j. They are connected by an edge with label y if and only if the upper edge of the square corresponding to v_i is glued with the lower edge of the one that corresponds to v_j. We will come back to this presentation of an origami in Section 4.2.

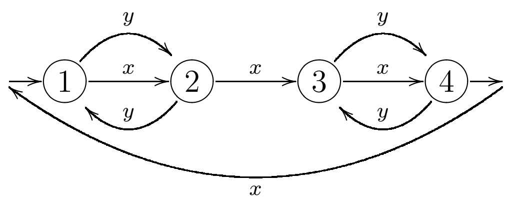
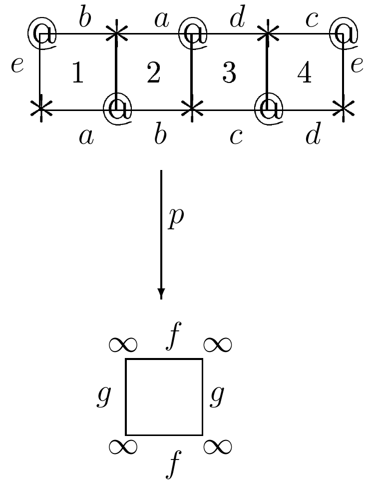


Figure 3: The graph $\mathcal{G}$ for the origami in Example 1.2.

1.2 Origamis as coverings of the once punctured torus

The "smallest" example for an origami consists of only one square. There is only one possible way to glue the edges. One obtains a closed surface of genus 1, i.e. a torus E. The four vertices of the square are glued to one point on E that we denote as ∞.

Let now O be an arbitrary origami and X the closed surface that it defines. The images of the squares on X define a (topological) covering p from X onto E. This covering is ramified at most over the point ∞. The degree of the covering is the number of squares and the preimages of ∞ on X are the vertices of the square tiling.
In Example 1.2 we have a covering of degree 4 ramified in the two points $*$ and @.



Let $E^* := E - \{\infty\}$ and $X^* := X - p^{-1}(\infty)$. Then $p : X^* \to E^*$ is a finite unramified covering of the punctured torus.

Conversely, each finite unramified covering $p : X^* \to E^*$ defines an origami O as follows: Let d be the degree of the covering and let $\rho : \pi_1(E^*) \to S_d$ be the monodromy of p. The fundamental group $\pi_1(E^*)$ of the once punctured torus E^* is isomorphic to the free group on two generators x and y. We identify x with the horizontal simple closed path on E^* and y with the vertical simple closed path as indicated in the figure.

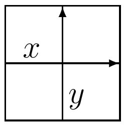


Figure 4: Generators of $\pi_1(E^*)$.

Set $\pi_h := \rho(x)$ and $\pi_v := \rho(y)$. Then we obtain O as the origami that consists of d squares that are glued according to the permutations π_h and π_v, i.e. the left edge of the ith square is glued to the right edge of the $\pi_h(i)$th square and the upper edge of the ith square is glued to the lower edge of the $\pi_v(i)$th square.

These two constructions are inverse to each other. Therefore we may use equivalently the following definition for origamis.

Definition 1.3. *An* origami *O (of genus $g \geq 1$) is a (topological) unramified covering $p : X^* \to E^*$, where X^* is obtained by removing finitely many points from a compact surface X of genus g. We call $X - X^*$ the* set of vertices *of O.*

1.3 Veech groups of origamis

In this section we introduce the main object that we study, the *Veech group of an origami.*

We start with the definition of general translation structures on surfaces.

Definition 1.4. *An atlas on a surface X such that all transition maps are translations is called* translation atlas. *It defines a* translation structure *μ on X. $X_\mu := (X, \mu)$ is called* translation surface.

We shall identify throughout the whole thesis $\mathbb{C}$ with $\mathbb{R}^2$ by sending $\{1, i\}$ to the standard basis of $\mathbb{R}^2$. With this identification, a translation is a holomorphic map, thus a translation atlas defines in particular a complex structure on X and (X, μ) is in this sense a Riemann surface.

For a translation surface $X_\mu = (X, \mu)$ one calls

$$\text{Aff}^+(X_\mu) := \{f : X \to X \mid f \text{ orientation preserving affine diffeomorphism}\}$$

the *affine group*, where the diffeomorphisms have to be affine with respect to the translation structure μ on X. Here *affine* means that the diffeomorphism is locally defined as a real-affine map

$$z \mapsto A \cdot z + t \quad \text{with some } A \in \text{GL}_2(\mathbb{R}),\ t \in \mathbb{C}. \tag{1.1}$$

Since all transition maps in the atlas of X are translations, the matrix A in (1.1) is independent of the chart. Furthermore, composition of diffeomorphisms leads to multiplication of the matrices. Thus we obtain a homomorphism

$$\text{der} : \text{Aff}^+(X_\mu) \;\to\; \text{GL}_2(\mathbb{R}),\ \ f \mapsto A \quad (\text{ with } A \text{ as above}).$$

We call the matrix $\text{der}(f) = A$ the *derivative* of the diffeomorphism f. The derivative is in $\text{GL}_2^+(\mathbb{R})$ iff f preserves the orientation. If X is of finite volume, then A is in fact in $\text{SL}_2(\mathbb{R})$.

Definition 1.5. *The image* $\Gamma(X_\mu) := \text{der}(\text{Aff}^+(X_\mu))$ *of the affine group in* $\text{SL}_2(\mathbb{R})$ *is called the* Veech group *of* X_μ.

Let us consider as a first example the torus: Let B be a matrix in $\text{SL}_2(\mathbb{R})$ and Λ_B the lattice in $\mathbb{C}$ with

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}) \;\Rightarrow\; \Lambda_B := \;<\vec{v}_1 := \begin{pmatrix} a \\ c \end{pmatrix}, \vec{v}_2 := \begin{pmatrix} b \\ d \end{pmatrix} > .$$

Then the torus $\mathbb{C}/\Lambda_B$ has the natural translation structure ν_B descending from $\mathbb{C}$. We denote the translation surface $(\mathbb{C}/\Lambda_B, \nu_B)$ by E_B.

The affine diffeomorphisms on E_B can all be lifted to the universal covering $\mathbb{C}$. Conversely, an affine diffeomorphism $z \mapsto A \cdot z + b$ on $\mathbb{C}$ descends to E_B if and only if it respects the lattice Λ_B. But this is the case if and only if $BAB^{-1} \in \text{SL}_2(\mathbb{Z})$. Thus we obtain the Veech group

$$\Gamma(E_B) = B^{-1}\text{SL}_2(\mathbb{Z})B, \quad \text{for all } B \in \text{SL}_2(\mathbb{R}).$$

In particular one has

$$\Gamma(E_I) = \text{SL}_2(\mathbb{Z}), \quad \text{where } I \text{ is the identity matrix in } \text{SL}_2(\mathbb{R}).$$

In general the group $\text{SL}_2(\mathbb{R})$ acts on the translation structures of an arbitrary surface X from the right, in the following sense: For $B \in \text{SL}_2(\mathbb{R})$ let $\varphi_B : \mathbb{C} \to \mathbb{C}$ be the $\mathbb{R}$-linear map $z \mapsto B \cdot z$. For a given translation structure μ on X, B defines a translation structure $\mu \cdot B$ by composing each chart with φ_B.

The following remark states how the Veech groups are related to each other.

Remark 1.6. *Let μ and $\mu \cdot B$ be defined as above. Then*

$$\mathrm{Aff}^+(X_\mu) \cong \mathrm{Aff}^+(X_{\mu \cdot B}) \quad \text{and} \quad \Gamma(X_{\mu \cdot B}) = B\Gamma(X_\mu)B^{-1}.$$

Proof. The map $\psi : X_\mu \to X_{\mu \cdot B}$ that is topologically the identity on X is an affine diffeomorphism with derivative B and induces the group isomorphism

$$\mathrm{Aff}^+(X_\mu) \to \mathrm{Aff}^+(X_{\mu \cdot B}), \quad f \mapsto \psi \circ f \circ \psi^{-1}.$$

Since the derivative of ψ is B, we have $\mathrm{der}(\psi f \psi^{-1}) = B \cdot \mathrm{der}(f) \cdot B^{-1}$. □

Let us return to origamis now. Let $O = (p : X^* \to E^*)$ be an origami, using Definition 1.3. Similarly as for the torus we obtain a bunch of natural translation structures on X^*. More precisely, one defines for each $B \in \mathrm{SL}_2(\mathbb{R})$ a translation structure μ_B on X^* as follows: Let us fix some homeomorphism $h : E \to E_I = \mathbb{H}/\Lambda_I$. Denote $E_I^* := E_I - h(\infty)$. Then h defines a translation structure μ_I on X^* by lifting $\nu_I|_{E_I^*}$ via $h \circ p$ to X^*. Now, we may define μ_B as $\mu_B := \mu_I \cdot B$.

Note, that one obtains the same translation structure μ_B from E_B, if one proceeds the following way: The affine map $\varphi_B : z \mapsto B \cdot z$ descends to a map $\bar{\varphi}_B : E_I \to E_B$. Then μ_B is the lift of the translation structure $\nu_B|_{E_B^*}$ via the map $\bar{\varphi}_B \circ h \circ p$. Here, we denote similarly as above $E_B^* := E_B - \bar{\varphi}_B(h(\infty))$

$$\begin{array}{ccccc} X^* & & & & \\ \downarrow p & & & & \\ E^* & \xrightarrow{h} & E_I^* & \xrightarrow{\bar{\varphi}_B} & E_B^* \end{array}$$

Definition 1.7. *Let $O = (p : X^* \to E^*)$ be an origami. For each B let μ_B be the translation structure on X^* as explained above. We denote by X_B^* the translation surface (X^*, μ_B).*

By Remark 1.6 the Veech groups of all these translation surfaces are conjugated in $\mathrm{SL}_2(\mathbb{R})$. Therefore we may for the study of Veech groups restrict to $B = I$. This motivates the following definition.

Definition 1.8. *Let $O = (p : X^* \to E^*)$ be an origami. We call $\Gamma(X_I^*)$ the* Veech group of the origami O *and denote*

$$\Gamma(O) := \Gamma(X^*) := \Gamma(X_I^*)$$

We shall see later in Chapter 3 that $\Gamma(O)$ is always a subgroup of $\Gamma(E_I) = \mathrm{SL}_2(\mathbb{Z})$ of finite index.

1.4 Relation to Teichmüller curves

An important motivation to study Veech groups is their relation to Teichmüller disks in the Teichmüller space and to Teichmüller curves in the moduli space. In this section we briefly introduce this concept. For more details one might look e.g. in [EG 97], [Lo 03], [McM 03] and references therein.

Let (X, μ) be a translation surface. We will suppose for the rest of this section that the Riemann surface X defined by the structure μ is of finite type. Let g be its genus and n the number of punctures. Let $T_{g,n}$ be the Teichmüller space of Riemann surfaces of genus g with n punctures and $M_{g,n}$ the corresponding moduli space.

We have as explained in Section 1.3 for each $B \in \mathrm{SL}_2(\mathbb{R})$ a translation structure $\mu \cdot B$ on X. Let us denote in the following $X_B := (X, \mu \cdot B)$.

In order to obtain points in the Teichmüller space $T_{g,n}$ we may choose the topological surface X and the identity map $\mathrm{id} : X \to X_B$ as marking for the Riemann surface X_B. This defines a point $[X_B]$ in $T_{g,n}$. Thus we have a map:

$$\mathrm{SL}_2(\mathbb{R}) \to T_{g,n}, \quad B \mapsto [X_B]. \tag{1.2}$$

For two matrices B and B' with $B' \cdot B^{-1} \in \mathrm{SO}_2(\mathbb{R})$, the identity map on X considered as diffeomorphism $\mathrm{id} : X_B \to X_{B'}$ is holomorphic. Thus $[X_B]$ and $[X_{B'}]$ are the same point in $T_{g,n}$. Therefore the map in (1.2) factors through $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \cong \mathbb{H}$ and one has a map:

$$i : \mathbb{H} \hookrightarrow T_{g,n}, \quad B \cdot \mathrm{SO}_2(\mathbb{R}) \mapsto [X_B]. \tag{1.3}$$

Since one has also vice versa that the identity map is (isotopic) to a holomorphic map only if $B' \cdot B^{-1}$ is in $\mathrm{SO}_2(\mathbb{R})$, one obtains that the map in (1.3) is injective.

The image Δ of the map i in $T_{g,n}$ is called a *Teichmüller disk*. In fact, $i : \mathbb{H} \to \Delta$ is an isometry with respect to the Poincaré metric on $\mathbb{H}$ and the Teichmüller metric on $T_{g,n}$, therefore Δ is also called *geodesic disk*. A proof for this result can be found for e.g. in [Ga 87].

The affine group $\mathrm{Aff}^+(X_I)$ of X_I acts on Δ by

$$\mathrm{Aff}^+(X_I) \to \mathrm{Aut}(\Delta), \quad f \cdot [X_B] = [X_B] \circ f = [X_{A \cdot B}] \quad \text{with } A := \mathrm{der}(f).$$

Here one defines $[X_B] \circ f$ by the composition of the diffeomorphism f with the marking id : $X \rightarrow X_B$. If one considers now the following commutative diagram,

$$\begin{array}{ccccc} X_I & \xrightarrow{f} & X_I & \xrightarrow{\text{id}} & X_B \\ & & \searrow^{\text{id}} & & \uparrow \\ & & & & X_{A\cdot B} \end{array}$$

then the map $X_{A\cdot B} \rightarrow X_B$ is as a composition of affine maps itself affine and has derivative I. Thus it is biholomorphic and $[X_B] \circ f = [X_{A\cdot B}]$.

This action fits together with the action of the Veech group $\Gamma(X)$ on $\mathbb{H}$ by Möbius transformations, see e.g. [McM 03, Prop. 3.2]. The action of $\Gamma(X)$ is discrete on $\mathbb{H}$, as was proven in [Ve 89], i.e. $\Gamma(X)$ is a Fuchsian group.

Now one considers the image of the Teichmüller disk Δ in the moduli space $M_{g,n}$ under the natural projection $T_{g,n} \rightarrow M_{g,n}$. This is sometimes an algebraic curve. In fact, this happens if and only if the Veech group $\Gamma(X)$ is a lattice in $\mathrm{SL}_2(\mathbb{R})$, i.e. $\mathbb{H}/\Gamma(X)$ has finite hyperbolic volume. If this happens the image is called a *Teichmüller curve*. Its normalization is then the affine nonsingular curve $\mathbb{H}/\Gamma(X)$ (or rather its mirror image), see e.g. [McM 03].

Thus the Veech group $\Gamma((X, \mu))$ determines whether (X, μ) defines a Teichmüller curve in the moduli space. In the case it does, the Veech group determines furthermore the Teichmüller curve up to a birational map.

1.5 A view beyond this work

Teichmüller curves and Veech groups can already be found in the works of Thurston and Veech, see [Th 88], [Ve 89]. We have summarized in this section some of the results that are known so far.

As mentioned in the last section, the Veech group of a translation surface is always a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$. One might ask now whether, vice versa, all discrete subgroups occur as Veech groups. However this is not the case, see [HuSc 01].

Since the construction for the Teichmüller geodesic disk leads to a Teichmüller curve if and only if the Veech group is a lattice in $\mathrm{SL}_2(\mathbb{R})$, there is a particular interest in such translation surfaces; sometimes they are called *Veech surfaces*.

First examples were given by Veech himself, e.g. the surfaces obtained by gluing parallel sides of two regular n-gons (see [Ve 89]). Their Veech groups are the hyperbolic triangle groups $\Delta(2, n, \infty)$ if n is odd and $\Delta(m, \infty, \infty)$ if $n = 2m$, $n \geq 5$.

Here $\Delta(r, s, t)$ denotes the Fuchsian triangle group with signature r, s, t. One gets these translation surfaces also using the construction in [KaZe 75] starting from billiard tables with the shape of an isosceles triangle with base angles π/n.

Other examples were found by the same construction starting from rational triangles with angles (q_1, q_2, q_3): The Veech groups associated to isosceles triangles with base angles $q_1 = q_2 = \frac{2k-1}{4k}\pi$ and $q_1 = q_2 = \frac{k}{2k+1}\pi$ $(k \geq 2)$ are the triangle groups $\Delta(2k, \infty, \infty)$ and $\Delta(2k+1, \infty, \infty)$, respectively ([EG 97], [HuSc 01]); those associated to the triangles defined by $q_1 = \frac{\pi}{2n}$, $q_2 = \frac{\pi}{n}$, $q_3 = \frac{2n-3}{2n}\pi$ $(n \geq 4)$ are the triangle groups $\Delta(3, n, \infty)$ ([Wa 98]). The three triangles where (q_1, q_2, q_3) equals $(\frac{\pi}{4}, \frac{\pi}{3}, \frac{5}{12}\pi)$, $(\frac{\pi}{5}, \frac{\pi}{3}, \frac{7}{15}\pi)$ and $(\frac{2}{9}\pi, \frac{\pi}{3}, \frac{4}{9}\pi)$ (in [Vo 96] and [KS 00]) also have Veech groups that are lattices, namely $\Delta(6, \infty, \infty)$, $\Delta(15, \infty, \infty)$ and $\Delta(9, \infty, \infty)$, respectively ([HuSc 01]).[1]

Not all Veech groups are commensurable to a triangle group. Starting with L-shaped billiard tables instead of triangles, McMullen finds in [McM 03] an infinite sequence of Veech surfaces of genus 2, among them surfaces whose Veech groups are not commensurable to a triangle group. Their associated Teichmüller curves belong to the infinite family of curves in M_2 mentioned below.

Nevertheless, being a lattice should be considered to be an exception for a Veech group. For example, the last three triangle-shaped billiard tables given above are the only acute non-isosceles triangles whose associated Veech group is a lattice, see [KS 00], [Pu 01].

The Veech group of an origami, however, is always a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of finite index and thus a lattice. In fact, Gutkin and Judge obtain the following equivalence in [GJ 00]: A translation surface has a Veech group commensurable to $\mathrm{SL}_2(\mathbb{Z})$ if and only if it covers a flat torus with at most one branch point. In other words, origamis can be characterized as those Veech surfaces whose Veech groups are arithmetic.

Hence, the quotient $\mathbb{H}/\Gamma$ is always an affine algebraic curve and all origamis define thus Teichmüller curves in the moduli space. We also call them *origami curves.* It is not difficult to see that they are defined over $\overline{\mathbb{Q}}$, see 4.4.
One has even more: Also the **embedded** curve C in $M_{g,n}$ is an irreducible component of a Hurwitz space and thus also defined over $\overline{\mathbb{Q}}$, see [Mö 03]. In [Lo 03], Pierre Lochak suggests to study them in the context of the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on combinatorial objects, in some sense as generalization of the study of dessins d'enfants, see also [LS 05]. The group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the set of *origami curves* in $M_{g,n}$, and this action is faithful as was shown in [Mö 03]. More recently, it was

[1] For a more detailed overview see e.g. [Le 02].

shown in [Mö 04], using different methods, that arbitrary Teichmüller curves are also defined over $\overline{\mathbb{Q}}$.

It would be interesting to know "all" Teichmüller curves.
For the case of genus 2 many things are already known. For example, [McM 03] and [Ca 05] classify Teichmüller curves in the moduli space M_2 using different methods (Jacobians with real multiplication; Kenyon-Smillie invariants defined in [KS 00]). [McM 03] obtains an infinite family of *primitive Teichmüller curves*, where "primitive" means that the structure defining the Teichmüller curve is not the pullback from a surface of lower genus.
Explicit examples of general Teichmüller curves are studied e.g. in [EG 97], [Lo 03], [McM 03], [Ve 89] to list some.

Although origamis are in some aspects more accessible then general translation surfaces, it is still not that much known about the geometry of origami curves in the moduli spaces. Equations for particular examples are given e.g. in [Mö 03], [HeSh 05] and in [He 05].
In [HeSh 05], a particular origami curve in genus 3 is studied that is intersected by infinitely many other origami curves. Origami curves in genus 2 with one singularity are explored in [HL1 04]. In particular, it is shown that for each n there are only two different origami curves in $M_{2,n}$ in this stratum.
An especially "large" example, a Teichmüller curve in genus 37 of an origami with 108(!) squares, is studied in [Ba 05]. In [Ma 05] it is detected which strata of the boundary of the moduli spaces are met by origami curves.

Chapter 2

The Teichmüller space $T_{1,1}$

We study the Teichmüller space $T_{1,1}$ of once punctured complex Riemann surfaces of genus 1 and give particular coordinates for it by suitably chosen generators of the fundamental groups.

Recall that $T_{1,1}$ is isomorphic to the upper half plane $\mathbb{H}$. Thus for $g = 1$ and $n = 1$, the isometric embedding $\mathbb{H} \hookrightarrow T_{1,1}$ which was described in Section 1.4 is surjective. This means that the Teichmüller disk defined by the trivial origami which consists of only one square is the full $T_{1,1}$.

For a general origami $O = (p : X^* \to E^*)$ the Teichmüller disk is parametrized by the translation structures that are lifted from the once punctured torus. Therefore we may also consider the associated Teichmüller disk as parametrized by $T_{1,1}$.

In Chapter 3 we will study origamis via subgroups of the free group F_2 on two generators, which is embedded into $\mathrm{PSL}_2(\mathbb{R})$. The translation structure is then given by this embedding. In the current chapter we choose particular generators of the embedded free groups F_2 and find appropriate fundamental domains for them.

Let us start with a definition of $T_{1,1}$. There are many ways to define the Teichmüller space, see e.g. [Ab 80] or [IT 92]. Here we will use the following one.

Definition 2.1.

$$T_{1,1} := \{\, \chi : F_2 := F_2(\alpha,\beta) \hookrightarrow \mathrm{PSL}_2(\mathbb{R}) \ \textit{with}: \ X = \mathbb{H}/\mathrm{Im}(\chi) \textit{ is a punctured torus and } \chi(\alpha), \chi(\beta) \textit{ cover geodesics on } X \textit{ with oriented intersection number } 1 \,\}/\sim$$

Here, $F_2(\alpha, \beta)$ denotes the free group in the two generators α and β and

$$\chi_1 \sim \chi_2 :\Leftrightarrow \exists\, C \in \mathrm{PSL}_2(\mathbb{R}) \text{ with } \chi_1 = C \cdot \chi_2 \cdot C^{-1}.$$

Furthermore, a geodesic on X covered by some hyperbolic $D \in \mathrm{PSL}_2(\mathbb{R})$ has a natural orientation since the axis of D has the following orientation: from the repelling fixed point to the attracting fixed point.

The topology of $T_{1,1}$ is inherited from the usual topology on $\mathrm{PSL}_2(\mathbb{R})$ that one obtains by the embedding of $\mathrm{SL}_2(\mathbb{R})$ into $\mathbb{R}^4$ that sends a matrix to the vector that contains the entries of the matrix. Since embeddings of the free group F_2 are given by the images of the two generators, one may represent them by points in $\mathrm{PSL}_2(\mathbb{R})^2$. The equivalence relation embeds $T_{1,1}$ into a quotient of $\mathrm{PSL}_2(\mathbb{R})^2$ which will be endowed with the quotient topology.

We may equivalently consider the Poincaré disk model $\mathbb{D}$. We will use the isomorphism

$$\theta: \ \mathbb{D} \to \mathbb{H}, \quad z \mapsto \frac{z+i}{iz+1} \tag{2.1}$$

in order to switch between the disk model and the upper half plane. We denote by G' the image of a Fuchsian group G in the automorphism group $\mathrm{Aut}(\mathbb{D})$ of $\mathbb{D}$. Recall that one has:

$$\mathrm{Aut}(\mathbb{D}) \ = \ \{\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{C})\} \ = \ \mathrm{PSU}(1,1).$$

With respect to the Poincaré disk model, we obtain the following definition of $T_{1,1}$ equivalent to that in Definition 2.1.

Definition 2.2.

$$\begin{aligned} T_{1,1} \ = \ \{ \ \chi' : \ & F_2 := F_2(\alpha,\beta) \hookrightarrow \mathrm{Aut}(\mathbb{D}) \ \ \textit{with}: \\ & X = \mathbb{D}/\mathrm{Im}(\chi') \textit{ is a punctured torus and } \chi'(\alpha), \chi'(\beta) \textit{ cover} \\ & \textit{geodesics on } X \textit{ with oriented intersection number } 1 \qquad \} / \sim \end{aligned}$$

We now want to find "nice" generators and "nice" fundamental domains of the Fuchsian groups G that define the points in $T_{1,1}$.

For this purpose we define the following *normalization condition* for pairs (A, B) of Möbius transformations. We will prove in Proposition 2.4 that in each equivalence class defining a point in $T_{1,1}$, there is a unique representative χ that sends α, β to such a normalized pair.

Condition 2.1. *Normalization for a pair (A, B) in $\mathrm{PSL}_2(\mathbb{R})$:*

- *A and B are hyperbolic and their commutator $[A, B]$ is parabolic.*
- *B has 0 as repelling fixed point and ∞ as attracting fixed point.*

- *The axes of A and B intersect in i and their oriented intersection number is* 1.

We define the equivalent condition in the Poincaré disk model for A', B' in $\mathrm{Aut}(\mathbb{D})$ in the same way.

Condition 2.2. *Normalization for a pair (A', B') in Aut($\mathbb{D}$):*

- *A' and B' are hyperbolic (i.e. they have each two fixed points on the boundary of $\mathbb{D}$) and their commutator $[A', B']$ is parabolic (i.e. it has one fixed point on the boundary of $\mathbb{D}$).*
- *B' has $-i$ as repelling fixed point and i as attracting fixed point.*
- *The axes of A' and B' intersect in* 0 *and their oriented intersection number is* 1.

Furthermore we consider the following *normalization conditions* for quadrangles in $\mathbb{H}$, respectively in $\mathbb{D}$, which have their vertices on the boundary.

Condition 2.3. *Normalization for a quadrangle Q in $\mathbb{H}$ with vertices at infinity:*

- *Q is a hyperbolic quadrangle $Q(z_0, z_1, z_2, z_3)$ with its vertices z_0, z_1, z_2 and z_3 on the boundary of $\mathbb{H}$ but not at* 0 *or* ∞ .
- *The diagonals of Q intersect in i.*

Respectively, in the Poincaré model:

Condition 2.4. *Normalization for a quadrangle Q' in $\mathbb{D}$ with vertices at infinity:*

- *Q' is a hyperbolic quadrangle with its vertices on the boundary of $\mathbb{D}$ but not at i or $-i$.*
- *The diagonals of Q' intersect in* 0.

A quadrangle that fulfils the normalization conditions looks like that in Figure 5, if it is in $\mathbb{H}$, or that in Figure 6, if it is in $\mathbb{D}$.
We will prove that for each such normalized quadrangle Q in $\mathbb{H}$ there is a unique pair (A, B) of matrices fulfilling the normalization condition 2.1 such that A and B identify the opposite edges of Q, respectively the analogous statement for a normalized quadrangle Q' in $\mathbb{D}$.

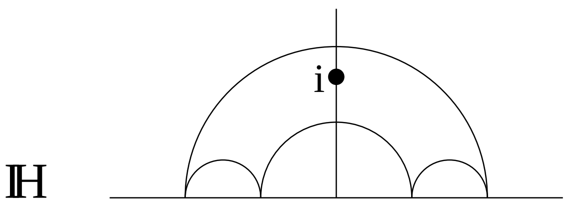


Figure 5: *A normalized quadrangle in* $\mathbb{H}$

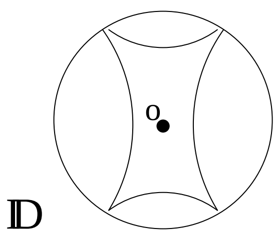


Figure 6: *A normalized quadrangle in* $\mathbb{D}$

We restrict first to the model of the upper half plane. The transfer to the analogous statements in the language of the disk model $\mathbb{D}$ will be done afterwards.

The goal will be to obtain that $T_{1,1}$ can be parametrized by the set of triples (G, A, B) fulfilling Condition 2.1 or equivalently by the set of quadrangles fulfilling Condition 2.3, see Proposition 2.5.

For the beginning we show that for each point $[(G, A, B)]$ in $T_{1,1}$ there is a unique triple (G, A, B) fulfilling Condition 2.1, see Proposition 2.4. We will use the following remark.

Remark 2.3. *If $\chi : F_2 \hookrightarrow \mathrm{PSL}_2(\mathbb{R})$ defines the point $\bar{\chi}$ in $T_{1,1}$, then $A := \chi(\alpha)$ and $B := \chi(\beta)$ are hyperbolic, their commutator $[A, B]$ is parabolic and their axes intersect with oriented intersection number* 1.

Proof. A and B are hyperbolic since they cover geodesics on X. Let γ_A and γ_B be elements of the fundamental group of X covered by A and B. Let γ_C and γ_D be standard generators of the fundamental group. Thus, $[\gamma_C, \gamma_D]$ is a simple loop around the puncture.
Since γ_A and γ_B also generate $\pi_1(X)$, the map $\gamma_C \mapsto \gamma_A, \gamma_D \mapsto \gamma_B$ defines an automorphism of $\pi_1(X)$. By the theorem of Dehn and Nielsen there is a homeomorphism $f : X \to X$ such that $[\gamma_A] = [f \circ \gamma_C]$ and $[\gamma_B] = [f \circ \gamma_D]$. Since $[\gamma_C, \gamma_D]$ is freely homotopic to a simple loop around the puncture, this is also true for $[\gamma_A, \gamma_B]$. Hence $[A, B]$ is parabolic.
The axes of A and B intersect with oriented intersection number 1, since γ_A and γ_B do so by definition. □

We now may use the remark and obtain by conjugation generators that have the desired properties.

Proposition 2.4. *Let $\bar{\chi}$ be in $T_{1,1}$. There is a unique triple (G, A, B) consisting of a subgroup G of* $\mathrm{Aut}(\mathbb{H})$ *generated by two elements $A, B \in \mathrm{Aut}(\mathbb{H})$ such that*

- *(A, B) fulfils Condition 2.1 and*
- *the map $\chi : F_2 \hookrightarrow \mathrm{Aut}(\mathbb{H}), \quad \alpha \mapsto A, \ \beta \mapsto B$ is an element of the equivalence class $\bar{\chi}$.*

Proof. Let $\tilde{\tilde{\chi}} : F_2 \hookrightarrow \mathrm{PSL}_2(\mathbb{R})$ be a representative of $\bar{\chi}$ and $\tilde{\tilde{A}} := \tilde{\tilde{\chi}}(\alpha)$ and $\tilde{\tilde{B}} := \tilde{\tilde{\chi}}(\beta)$. Let z_1 and z_2 be the repelling and attracting fixed point of $\tilde{\tilde{B}}$, respectively. Choose $C_1 \in \mathrm{PSL}_2(\mathbb{R})$ with $C_1(z_1) = 0$ and $C_1(z_2) = \infty$. Set

$$\tilde{A} := C_1 \tilde{\tilde{A}} C_1^{-1} \text{ and } \tilde{B} := C_1 \tilde{\tilde{B}} C_1^{-1}.$$

Hence, $\tilde{B}$ has 0 as repelling fixed point and ∞ as attracting fixed point.
By Remark 2.3 the axes of $\tilde{A}$ and $\tilde{B}$ intersect with intersection number 1, let us say in $r \cdot i$ ($r \in \mathbb{R}$, $r > 0$). Set

$$C_2 := \begin{pmatrix} \frac{1}{\sqrt{r}} & 0 \\ 0 & \sqrt{r} \end{pmatrix} \quad \text{and} \quad A := C_2 \tilde{A} C_2^{-1}, \quad B := C_2 \tilde{B} C_2^{-1}.$$

Since $C_2(0) = 0$, $C_2(\infty) = \infty$ and $C_2(r \cdot i) = i$, the Möbius transformation B still has the fixed points 0 and ∞. Furthermore the axes of A and B intersect in i still with oriented intersection number 1. Therefore, (A, B) fulfils Condition 2.1. Thus if G is the subgroup of $\mathrm{PSL}_2(\mathbb{R})$ generated by A and B, then the triple (G, A, B) has the desired properties.

Furthermore, (A, B) is unique, since the conjugation matrix $C := C_1 C_2$ is uniquely determined by Condition 2.1. □

As indicated above, we may now improve the statement in Proposition 2.4: We will see in Proposition 2.5 that in fact for each (A, B) that fulfils Condition 2.1

- the map $\chi : F_2 \to \mathrm{PSL}_2(\mathbb{R}), \; \alpha \mapsto A, \beta \mapsto B$ defines a point $\bar{\chi}$ in $T_{1,1}$ and
- we find a unique quadrangle Q fulfilling Condition 2.3 such that Q is a fundamental domain for G and A and B identify the opposite edges of Q.

Proposition 2.5. *The points in $T_{1,1}$ correspond to the normalized generator pairs (i.e. pairs (A, B) that fulfil Condition 2.1) and to the normalized quadrangles (i.e. quadrangles Q that fulfil Condition 2.3). More precisely: Define the following sets*

$$\begin{aligned} \mathcal{P} &:= \{(A,B) \text{ with } A, B \in \mathrm{PSL}_2(\mathbb{R}) \mid (A,B) \text{ fulfils Condition } 2.1\} \\ \mathcal{F} &:= \{\text{quadrangle } Q \mid Q \text{ fulfils Condition } 2.3\} \end{aligned}$$

One has the natural homeomorphisms Ψ_1 and Ψ_2:

$$\begin{aligned} \Psi_1 &: T_{1,1} \to \mathcal{P}, \quad \bar{\chi} \mapsto (A,B) \text{ as in Prop. 2.4} \\ \Psi_2 &: \mathcal{P} \to \mathcal{F}, \quad (A,B) \mapsto \text{the quadrangle } Q \text{ with the vertices} \\ &\qquad z_0 := \text{ the fixed point of } [A,B],\; z_1 := B^{-1}(z_0), \\ &\qquad z_2 := A^{-1}(z_1),\; z_3 := B(z_2). \end{aligned}$$

Proof. We will show in Lemma 2.8 that Ψ_2 is well defined, i.e. that the diagonal geodesics $z_2 z_0$ and $z_1 z_3$ intersect in i. Furthermore we will show in the proof of Lemma 2.10 that Q is a fundamental domain of the group G generated by A and B. By the definition of Q one has:

$$A(z_2 z_3) = z_1 z_0 \quad \text{and} \quad B(z_2 z_1) = z_3 z_0.$$

Thus A maps the "left vertical" edge of Q to the "right vertical" edge and the "bottom horizontal edge" to the "top" one.

Finally, we show in Lemma 2.9 and 2.10 that Ψ_1 and Ψ_2 are bijective. All maps that we use are continuous. Therefore this will finish the proof of Proposition 2.5. □

It turns out to be useful for the proof of Proposition 2.5 to determine the matrices that fulfil Condition 2.1 explicitely. As a preparation we state the following geometrical fact.

Lemma 2.6. *Let x and y be in $\mathbb{R} \cup \{\infty\}$. Then the hyperbolic line l through x and y contains i iff $x \cdot y = -1$ or $\{x, y\} = \{0, \infty\}$.*

Proof. If x or y is in $\{0, \infty\}$, then the statement is true. Thus let us suppose that $x, y \in \mathbb{R}$ and $x < y$. If l contains i, we must have $x < 0 < y$. Thus we are in the situation of Figure 7.

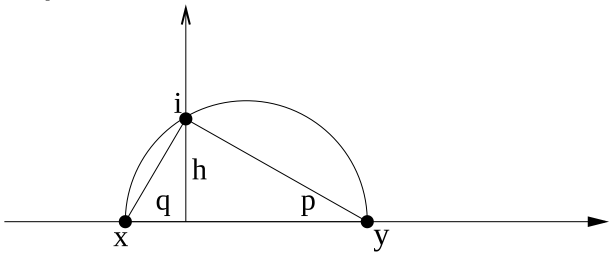


Figure 7: Geodesic containing i.

Consider now the triangle Δ with the vertices x, y and i. We have that l contains i iff Δ has a right interior angle at i, by Thales' Theorem. This is, by the Pythagorean theorem of height, the case iff

$$1 = h^2 = p \cdot q = x \cdot (-y), \quad \text{compare Figure 7.}$$

This is true iff $x \cdot y = -1$.

□

Lemma 2.7. *Let $\mathcal{P}$ be defined as in Proposition 2.5. Then*

$$\mathcal{P} = \{(\begin{pmatrix} a & b \\ b & \frac{1+b^2}{a} \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}) \mid b = \frac{2\lambda}{\lambda^2 - 1},\ a > 0,\ \lambda > 1\} \tag{2.2}$$

Proof. We proceed in three steps.

First step:
Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq \pm I$ be in $\mathrm{SL}_2(\mathbb{R})$. Then:

$$A \text{ is hyperbolic and its axis contains } i \Leftrightarrow b = c \Leftrightarrow A = \begin{pmatrix} a & b \\ b & \frac{1+b^2}{a} \end{pmatrix} \text{ and } a \neq 0.$$

To prove this, suppose first that $c \neq 0$. Then the fixed points of A are

$$z_{1/2} = -\frac{d-a}{2c} \pm \frac{1}{2c}\sqrt{(d-a)^2 + 4bc}$$

A is hyperbolic and its axis contains $i \Leftrightarrow z_1 \cdot z_2 = -1$ (by Lemma 2.6). This is equivalent to

$$(\frac{d-a}{2c})^2 - (\frac{1}{2c})^2 \cdot ((d-a)^2 + 4bc) = -1 \Leftrightarrow -\frac{4bc}{4c^2} = -1 \Leftrightarrow b = c.$$

Using $\det(A) = 1$ one obtains:

$$A = \begin{pmatrix} a & b \\ b & \frac{1+b^2}{a} \end{pmatrix} \qquad (a \neq 0 \text{ since } -b^2 = 1 \text{ has no solution}). \tag{2.3}$$

If $c = 0$, then ∞ is a fixed point of A. If the axis of A contains i, then the other fixed point is 0. Hence, $b = c = 0$ and $a \neq 0$ since $\det(A) \neq 0$.
Conversely, if

$$A = \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \qquad \text{with } a \in \mathbb{R} - \{0, 1, -1\},$$

then A is hyperbolic with fixed points 0 and ∞, i.e. i lies on the axis of A. This finishes the first step.

If (A,B) is in $\mathcal{P}$, then by the normalization condition

$$B = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \qquad (|\lambda| > 1) \tag{2.4}$$

and by the first step, A is of the form (2.3) with $b \neq 0$, since it does not have the fixed points 0 and ∞. In the second step, we obtain for $[A, B]$ a condition to be parabolic.

Second step:
Let A be as in (2.3) and B be as in (2.4) (with $a, b \neq 0$, $|\lambda| > 1$), then:

$$[A, B] \text{ is parabolic } \Leftrightarrow b = \pm\frac{2\lambda}{\lambda^2 - 1}$$

In order to prove this, let us calculate the commutator $[A, B]$:

$$A \cdot B = \begin{pmatrix} \lambda a & \frac{b}{\lambda} \\ \lambda b & \frac{1+b^2}{\lambda a} \end{pmatrix}, \qquad A^{-1} \cdot B^{-1} = \begin{pmatrix} \frac{1+b^2}{\lambda a} & -\lambda b \\ -\frac{b}{\lambda} & \lambda a \end{pmatrix}$$

$$\Rightarrow [A, B] = \begin{pmatrix} 1 + b^2 - \frac{b^2}{\lambda^2} & ab(-\lambda^2 + 1) \\ \frac{b(1+b^2)}{a}(1 - \frac{1}{\lambda^2}) & 1 + b^2 - \lambda^2 b^2 \end{pmatrix} \tag{2.5}$$

$$\begin{aligned}[A,B] \text{ is parabolic} \Leftrightarrow\ & \text{the trace of } [A,B] \text{ is } \pm 2\\ \Leftrightarrow\ & 2(1+b^2) - \frac{b^2}{\lambda^2} - \lambda^2 b^2 \ = \ \pm 2.\end{aligned}$$

The left side of the last equation is equal to $2 + b^2(2 - \frac{1}{\lambda^2} - \lambda^2)$. But this is smaller than 2, since $\lambda^2 + \frac{1}{\lambda^2} > 2$.

Thus we have:

$$\begin{aligned}2(1+b^2) - \frac{b^2}{\lambda^2} - \lambda^2 b^2 &= -2\\ \Leftrightarrow \quad -b^2(\lambda - \tfrac{1}{\lambda})^2 &= -4\\ \Leftrightarrow \quad b(\lambda - \tfrac{1}{\lambda}) = \pm 2 \quad &\Leftrightarrow \quad b = \tfrac{\pm 2\lambda}{\lambda^2 - 1}.\end{aligned}$$

This finishes the second step.

If we consider elements of $\mathrm{PSL}_2(\mathbb{R})$ we can choose $\lambda > 1$ and $b = \frac{+2\lambda}{\lambda^2-1}$.
As last step we consider the condition that the oriented intersection number of the axes of A and B is 1.

Third step:
Let A be as in (2.3) and B as in (2.4) with $a \neq 0$, $b = \frac{2\lambda}{\lambda^2-1}$, $\lambda > 1$. Then:

The axes of A and of B have oriented intersection number $1 \Leftrightarrow a > 0$.

Since B has the oriented axis (0∞), the intersection number is 1, if and only if the repelling fixed point of A is < 0 and the attracting fixed point is > 0. Since the two axes intersect in i this is equivalent to $\mathrm{Re}(A(i)) > 0$. We have:

$$\begin{aligned}A(i) \quad &= \quad \frac{ai+b}{bi + \frac{1+b^2}{a}} = \frac{a^2 i + ab}{abi + 1 + b^2}\\ &= \quad \frac{(a^2 i + ab)(1 + b^2 - abi)}{(1+b^2)^2 + a^2 b^2}.\end{aligned}$$

Since the denominator is real and positive it is sufficient to consider the real part of the numerator:

$$ab(1+b^2) + a^3 b = ab(1 + b^2 + a^2) > 0 \Leftrightarrow a > 0 \quad (\text{since } b > 0).$$

From these three steps the Equation (2.2) follows.

□

The following Lemma shows that the map Ψ_2 introduced in Proposition 2.5 is well defined.

Lemma 2.8. *Let (A, B) be in $\mathcal{P}$ and Q the quadrangle with the vertices z_0, z_1, z_2 and z_3, where z_0 is the fixed point of $[A, B]$, $z_1 = B^{-1}(z_0)$, $z_2 = A^{-1}(z_1)$ and $z_3 = B(z_2)$. Then Q fulfils Condition 2.3.*

Proof. By Lemma 2.7 we have that for some a and some λ in $\mathbb{R}$:

$$A = \begin{pmatrix} a & b \\ b & \frac{1+b^2}{a} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \quad \text{with } a > 0,\ b = \frac{2\lambda}{\lambda^2 - 1},\ \ \lambda > 1.$$

In the following, we show that

$$z_0 = a \cdot \lambda \cdot \frac{\lambda^2 - 1}{\lambda^2 + 1}, \quad z_1 = \frac{a}{\lambda} \cdot \frac{\lambda^2 - 1}{\lambda^2 + 1}, \quad z_2 = -\frac{1}{z_0}, \quad z_3 = -\frac{1}{z_1}. \tag{2.6}$$

Then we have in particular that $z_0 \cdot z_2 = -1$ and $z_1 \cdot z_3 = -1$. By Lemma 2.6 it follows that the hyperbolic geodesic through z_0 and z_2 and that through z_1 and z_3 intersect in i. Furthermore we obtain from (2.6) that z_0, z_1, z_2, z_3 are not 0 or ∞. Thus we have shown that Q fulfils Condition 2.3.

Proof of the equalities (2.6):

First, we calculate z_0:

Let $\begin{pmatrix} r & s \\ t & u \end{pmatrix}$ be an arbitrary matrix in $\mathrm{SL}_2(\mathbb{R})$. If $t \neq 0$ the fixed points are

$$z_{1,2} := -\frac{u - r}{2t} \pm \sqrt{\frac{(u - r)^2}{4t^2} + \frac{s}{t}}.$$

If the matrix is parabolic, one has the unique fixed point $-\frac{u-r}{2t}$.
Thus - using (2.5) - the fixed point of $[A, B]$ is:

$$z_0 = -\frac{\frac{b^2}{\lambda^2} - \lambda^2 b^2}{2\frac{b(1+b^2)}{a}(1 - \frac{1}{\lambda^2})} = \frac{-ab^2 \frac{(1-\lambda^4)}{\lambda^2}}{2b(1 + b^2)\frac{(\lambda^2-1)}{\lambda^2}} = \frac{ab}{2(1 + b^2)} \cdot (1 + \lambda^2)$$

Using $b = \frac{2\lambda}{\lambda^2-1}$ and hence $b^2 + 1 = \frac{(\lambda^2+1)^2}{(\lambda^2-1)^2}$ one obtains:

$$z_0 = \frac{a}{2} \cdot \frac{2\lambda}{\lambda^2 - 1} \cdot \frac{(\lambda^2 - 1)^2}{(\lambda^2 + 1)^2} \cdot (\lambda^2 + 1) = a \cdot \lambda \cdot \frac{\lambda^2 - 1}{\lambda^2 + 1}. \tag{2.7}$$

Now, we obtain z_1, z_2 and z_3:

$$z_1 = B^{-1}(z_0) = \frac{a}{\lambda} \cdot \frac{\lambda^2-1}{\lambda^2+1} \tag{2.8}$$

$$\begin{aligned} z_2 &= A^{-1}(z_1) = \frac{\frac{1}{a} \cdot (\frac{\lambda^2+1}{\lambda^2-1})^2 \cdot \frac{a}{\lambda} \cdot \frac{\lambda^2-1}{\lambda^2+1} - \frac{2\lambda}{\lambda^2-1}}{-\frac{2\lambda}{\lambda^2-1} \cdot \frac{a}{\lambda} \cdot \frac{\lambda^2-1}{\lambda^2+1} + a} \\ &= \frac{(\lambda^2+1-2\lambda^2)(\lambda^2+1)}{\lambda(\lambda^2-1) \cdot a(-2+\lambda^2+1)} = \frac{-(\lambda^2+1)}{\lambda \cdot a(\lambda^2-1)} \end{aligned} \tag{2.9}$$

$$z_3 = B(z_2) = -\frac{\lambda}{a} \cdot \frac{\lambda^2+1}{\lambda^2-1}. \tag{2.10}$$

□

Finally, we show in the following two lemmas that the maps Ψ_1 and Ψ_2 in Proposition 2.5 are bijective.

Lemma 2.9. *The map Ψ_2 is bijective.*

Proof. Let us first go back to the set $\mathcal{F}$:
Let Q be a quadrangle in $\mathcal{F}$ with the vertices z_0, z_1, z_2 and z_3. We may assume without loss of generality that we have $z_0 > z_1 > z_2 > z_3$. Then the two diagonals are the geodesic through z_0 and z_2 and that through z_1 and z_3, respectively. By Lemma 2.6, it follows that $z_2 = -\frac{1}{z_0}$ and $z_3 = -\frac{1}{z_1}$. Furthermore we have $z_0 > z_1 > 0$.
Conversely, if z_0, z_1 are in $\mathbb{R}$ with $z_0 > z_1 > 0$, then the quadrangle with the vertices z_0, z_1, $z_2 := -\frac{1}{z_0}$ and $z_3 = -\frac{1}{z_1}$ is in $\mathcal{F}$.
Thus we may identify

$$\mathcal{F} \leftrightarrow \{(z_0, z_1) \in \mathbb{R}^2 | \, 0 < z_1 < z_2\}.$$

Similarly, using the description of $\mathcal{P}$ that we gave in Lemma 2.7, we may identify

$$\mathcal{P} \leftrightarrow \{(\lambda, a) \in \mathbb{R}^2 | \, a > 0, \lambda > 1\}.$$

Thus we may rewrite the map Ψ_2 as follows:

$$\begin{aligned} \{(\lambda, a) \in \mathbb{R}^2 | \, \lambda > 1, a > 0\} &\rightarrow \{(z_0, z_1) \in \mathbb{R}^2 | \, 0 < z_1 < z_0\} \\ (\lambda, a) &\mapsto (z_0 := a \cdot \lambda \cdot \frac{\lambda^2-1}{\lambda^2+1}, \ z_1 := \frac{a}{\lambda} \cdot \frac{\lambda^2-1}{\lambda^2+1}). \end{aligned}$$

This map has the inverse map:

$$\Psi_3 : \mathcal{F} \rightarrow \mathcal{P} \text{ defined by } (z_0, z_1) \mapsto (\lambda := \sqrt{\frac{z_0}{z_1}}, a := \sqrt{z_0 z_1} \cdot \frac{z_0 + z_1}{z_0 - z_1}).$$

That Ψ_3 is inverse to Ψ_2 follows from the following calculations:

$$\text{for } \Psi_3 \circ \Psi_2 = \text{ id}: \quad z_0 + z_1 = a \cdot \frac{\lambda^2 - 1}{\lambda^2 + 1} \cdot (\lambda + \frac{1}{\lambda}) = \frac{a}{\lambda} \cdot (\lambda^2 - 1)$$

$$z_0 - z_1 = \frac{a}{\lambda} \cdot \frac{(\lambda^2 - 1)^2}{\lambda^2 + 1}$$

$$\text{for } \Psi_2 \circ \Psi_3 = \text{ id}: \quad \frac{\lambda^2 - 1}{\lambda^2 + 1} = \frac{\frac{z_0}{z_1} - 1}{\frac{z_0}{z_1} + 1} = \frac{z_0 - z_1}{z_0 + z_1}$$

□

Lemma 2.10. *The map Ψ_1 defined in Proposition 2.5 is bijective.*

Proof. Ψ_1 is injective since we have shown in Proposition 2.4 that the triple (G, A, B) is uniquely determined by the required properties.
It remains to show surjectivity: Let (A, B) be in $\mathcal{P}$ and $Q := \Psi_2((A, B))$ be the quadrangle with the vertices z_0, z_1, z_2 and z_3. By definition of Ψ_2 we had that z_0 is the fixed point of $[A, B]$, $z_1 = B^{-1}(z_0)$, $z_2 = A^{-1}(z_1)$ and $z_3 = B(z_2)$. Thus the edge z_3z_2 of the polygon Q is mapped by A to the edge z_0z_1 and the edge z_1z_2 is mapped by B to the edge z_0z_3. This gives for the group $G := < A, B >$ a pairing of the sides of the polygon Q, and the vertices z_0, z_1, z_2, z_3 are fixed by parabolic elements in G. It follows by the theorem of Poincaré (see e.g. [Be 83, Thm. 9.8.4, Exc. 2 (p. 251) and the construction on p. 266]) that G is Fuchsian and Q is a fundamental polygon. The pairing shows that $\mathbb{H}/G$ is a punctured torus, hence the homomorphism $\chi : F_2(\alpha, \beta) \hookrightarrow \mathrm{PSL}_2(\mathbb{R})$, $\alpha \mapsto A$, $\beta \mapsto B$ is injective and $\bar{\chi}$ is in $T_{1,1}$. Thus we have: $(A, B) = \Psi_1(\bar{\chi})$.

□

With Lemma 2.10 we have provided the last piece needed for the proof of Proposition 2.5. We may now turn to the Poincaré disk model $\mathbb{D}$.

The normalization conditions concerning the upper half plane model and those concerning the Poincaré disk model are equivalent via the isomorphism θ in (2.1). More precisely, we have:

A pair (A, B) in $\mathrm{PSL}_2(\mathbb{R})$ fulfils Condition 2.1
$\Leftrightarrow$ The pair (A', B') in $\mathrm{Aut}(\mathbb{D})$ with $A' := \theta^{-1}A\theta$, $B' := \theta^{-1}B\theta$ fulfils Condition 2.2 and

A quadrangle Q with vertices z_0, z_1, z_2, z_3 on the boundary of $\mathbb{H}$ fulfils Condition 2.3
$\Leftrightarrow$ The quadrangle Q' with the vertices $\theta^{-1}(z_0), \theta^{-1}(z_1), \theta^{-1}(z_2), \theta^{-1}(z_3)$ on the boundary of $\mathbb{D}$ fulfils Condition 2.4.

Thus the statements given in terms of Fuchsian subgroups of $\mathrm{PSL}_2(\mathbb{R})$ can be transferred to subgroups of $\mathrm{Aut}(\mathbb{D})$.

One obtains as analogon to Proposition 2.4 the following proposition. We use here the description of $T_{1,1}$ given in Definition 2.2.

Proposition 2.11. *Let $\bar{\chi}'$ be in $T_{1,1}$. Then there is a unique triple (G', A', B') consisting of a subgroup G' of $\mathrm{Aut}(\mathbb{D})$ generated by two elements $A', B' \in \mathrm{Aut}(\mathbb{D})$ such that*

- *(A', B') fulfils Condition 2.2 and*
- *the map $\chi' : F_2 \hookrightarrow \mathrm{Aut}(\mathbb{D}),\ \alpha \mapsto A', \beta \mapsto B'$ is an element of the equivalence class $\bar{\chi}'$.*

The analogon to Proposition 2.5 in terms of subgroups of $\mathrm{Aut}(\mathbb{D})$ is given as follows.

Proposition 2.12. *The points in $T_{1,1}$ correspond to the pairs (A', B') in $\mathrm{Aut}(\mathbb{D})$ that fulfil Condition 2.2. Furthermore, they correspond to the quadrangles with vertices on the boundary of $\mathbb{D}$ fulfilling Condition 2.4.*

More precisely: Define the following sets

$$\begin{aligned}\mathcal{P}' &:= \{(A', B') \text{ with } A', B' \in \mathrm{Aut}(\mathbb{D}) \mid (A', B') \text{ fulfils Condition } 2.2\}\\ \mathcal{F}' &:= \{\text{ quadrangle } Q' \mid Q' \text{ fulfils Condition } 2.4\}\end{aligned}$$

One has the natural homeomorphisms Ψ'_1 and Ψ'_2:

$$\begin{aligned}\Psi'_1 &: T_{1,1} \to \mathcal{P}', \quad \bar{\chi}' \mapsto (A', B') \text{ as in Prop. 2.11}\\ \Psi'_2 &: \mathcal{P}' \to \mathcal{F}', \quad (A', B') \mapsto \text{ the quadrangle } Q' \text{ with the vertices}\\ &\qquad z_0 := \text{ the fixed point of } [A', B'],\ z_1 := B'^{-1}(z_0),\\ &\qquad z_2 := A'^{-1}(z_1),\ z_3 := B'(z_2).\end{aligned}$$

Finally, we may also describe the matrices in $\mathcal{P}'$ explicitely.

Proposition 2.13.

$$\begin{aligned}\mathcal{P}' := \Big\{(A', B') \mid A' &= \frac{1}{2} \cdot \begin{pmatrix} a + \frac{1+b^2}{a} & 2b + (a - \frac{1+b^2}{a})i \\ 2b - (a - \frac{1+b^2}{a})i & a + \frac{1+b^2}{a} \end{pmatrix},\\ B' &= \begin{pmatrix} \lambda + \frac{1}{\lambda} & (\lambda - \frac{1}{\lambda})i \\ -(\lambda - \frac{1}{\lambda})i & \lambda + \frac{1}{\lambda} \end{pmatrix} \text{ with } b = \frac{2\lambda}{\lambda^2 - 1},\ a > 0,\ \lambda > 1\Big\}\end{aligned}$$

Proof. (of Proposition 2.13)
The claim follows from Proposition 2.5 by conjugation with the isomorphism given in (2.1) and the following calculations:

$$\begin{aligned}
\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} a & b \\ b & \frac{1+b^2}{a} \end{pmatrix}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} &= \begin{pmatrix} a-ib & b-\frac{1+b^2}{a}i \\ -ai+b & -ib+\frac{1+b^2}{a} \end{pmatrix}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \\
&= \begin{pmatrix} a+\frac{1+b^2}{a} & 2b+(a-\frac{1+b^2}{a})i \\ 2b-(a-\frac{1+b^2}{a})i & a+\frac{1+b^2}{a} \end{pmatrix} \text{ and} \\
\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} &= \begin{pmatrix} \lambda & -\frac{1}{\lambda}i \\ -i\lambda & \frac{1}{\lambda} \end{pmatrix}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \\
&= \begin{pmatrix} \lambda+\frac{1}{\lambda} & (\lambda-\frac{1}{\lambda})i \\ -(\lambda-\frac{1}{\lambda})i & \lambda+\frac{1}{\lambda} \end{pmatrix}
\end{aligned}$$

□

Chapter 3

A characterization for Veech groups of origamis

In this chapter we present in Theorem 1 a characterization of the Veech groups of origamis in terms of *stabilizing subgroups* of $\mathrm{Aut}^+(F_2)$. This result will be the initial point for our investigations in the subsequent chapters.

Let F_2 be the free group in two generators and $\mathrm{Aut}(F_2)$ its group of automorphisms. We will use the fact that $\mathrm{GL}_2(\mathbb{Z})$ is isomorphic to $\mathrm{Out}(F_2)$, the group of outer automorphisms of F_2, and denote by

$$\hat{\beta}: \ \mathrm{Aut}(F_2) \ \to \ \mathrm{Out}(F_2) \cong \mathrm{GL}_2(\mathbb{Z})$$

the canonical projection, see Proposition 3.5. In fact, in the following we will restrict to the preimage of $\mathrm{SL}_2(\mathbb{Z})$, which we denote as $\mathrm{Aut}^+(F_2)$. We call $\mathrm{Aut}^+(F_2)$ the group of *orientation preserving automorphisms* of F_2.

We will associate to an origami O a subgroup H of F_2, see Section 3.1.

Theorem 1. *Let O be an origami associated with $H \subseteq F_2$. Let*

$$\mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(H) \ := \ \{\gamma \in \mathrm{Aut}^+(F_2) \mid \gamma(H) = H\}.$$

Then the Veech group $\Gamma(O)$ of the origami O is given by

$$\Gamma(O) \ = \ \hat{\beta}(\mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(H)).$$

Proof. The statement will follow from Corollary 3.4 and Proposition 3.5. □

We call a subgroup $\mathcal{G}$ of $\mathrm{Aut}^+(F_2)$ *stabilizing group*, if there is some subgroup H of F_2 of finite index such that

$$\mathcal{G} \ = \ \mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(H)\,.$$

With this notation we immediately obtain the following characterization for origami Veech groups from Theorem 1.

Corollary 3.1. *Γ is a Veech group of some origami if and only if it is the image in $\mathrm{SL}_2(\mathbb{Z})$ of some stabilizing group.*

By Theorem 1 studying Veech groups for origamis becomes the purely group theoretical problem of detecting stabilizing groups in $\mathrm{Aut}^+(F_2)$. In particular, we may omit the restriction to finite index subgroups of F_2. This suggests the following definition.

Definition 3.2. *Let U be a subgroup of F_2. Then we call*

$$\Gamma(U) := \hat{\beta}(\mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(U))$$

the Veech group *of U.*

3.1 Origamis and subgroups of F_2

In this section, we point out that we may think of an origami as a subgroup of finite index of F_2.

We fix a (topological) unramified universal covering of the punctured torus E^*:

$$v : \tilde{E}^* \rightarrow E^*$$

Let $\mathrm{Gal}(\tilde{E}^*/E^*)$ be the Galois group of v, i.e. the group of deck transformations. It is by the theorem of the universal covering naturally isomorphic to the fundamental group $\pi_1(E^*)$ of the punctured torus E^*, which in turn is isomorphic to F_2. We fix an isomorphism

$$\alpha : F_2 \rightarrow \pi_1(E^*) \overset{\text{nat}}{\cong} \mathrm{Gal}(\tilde{E}^*/E^*). \tag{3.1}$$

Let now $O = (p : X^* \rightarrow E^*)$ be an origami. It defines a finite index subgroup of F_2 as described in the following:

By the theorem of the universal covering applied to v there exists an unramified covering

$$u : \tilde{E}^* \rightarrow X^* \quad \text{with} \quad v = p \circ u.$$

Furthermore, the Galois group $H := \mathrm{Gal}(\tilde{E}^*/X^*)$ of the covering u is a subgroup of $\mathrm{Gal}(\tilde{E}^*/E^*)$.

We may now consider H by (3.1) as a finite index subgroup of F_2:

$$H := \mathrm{Gal}(\tilde{E}^*/X^*) \subseteq \mathrm{Gal}(\tilde{E}^*/E^*) \cong F_2. \tag{3.2}$$

Conversely, each finite index subgroup H of F_2 defines an origami O as follows:

$$O \ := \ (p: \ \tilde{E}^*/H \ \to \ \tilde{E}^*/F_2 \cong E^*).$$

Here again the isomorphism $\tilde{E}^*/F_2 \ \cong \ E^*$ is given by the identification of F_2 with $\mathrm{Gal}(\tilde{E}^*/E^*)$ fixed in (3.1).

Thus we obtain the relation between origamis and finite index subgroups of F_2 described as follows.

We say that two origamis $O_1 = (p_1 : X^* \to E^*)$ and $O_2 = (p_2 : Y^* \to E^*)$ are *equivalent* or the *same*, if there exists a homeomorphism $\varphi : X^* \to Y^*$ such that $p_1 = p_2 \circ \varphi$. Then by the theorem of the universal covering, one obtains that two origamis O_1 and O_2 are equivalent, if and only if the subgroups H_1 and H_2 of F_2 associated to them as above are conjugated in F_2. Thus we have a *one-to-one-correspondence* between origamis (up to equivalence) and finite index subgroups of F_2 (up to conjugation).

3.2 First part of the proof

We start with some definitions and facts for translation surfaces and more general G – structures, which we have mainly learnt from [GJ 00] and [Th 97].

Let $X_\mu := (X, \mu)$ be a translation surface and $u : \tilde{X} \to X$ a (topological) universal covering. We may lift the structure μ on X via u to a translation structure η on $\tilde{X}$. We denote the translation surface $(\tilde{X}, \eta)$ also by $\tilde{X}_\eta$.

A fixed chart (U, η_U) of $\tilde{X}_\eta$ defines a translation map $\mathrm{dev} : \tilde{X}_\eta \to \mathbb{C}$, called the *developing map*, such that

$$\eta_U = \mathrm{dev}|_U \quad \text{ and } \quad \eta_{U'} = t \circ \mathrm{dev}|_{U'} \text{ for a translation } t := t(U', \eta_{U'})$$

for any other chart $(U', \eta_{U'})$ of $\tilde{X}_\eta$. Here, we consider $\mathbb{C}$ as a translation surface with the global chart id.

Since $\tilde{X}_\eta$ is simply connected, for any affine diffeomorphism $\hat{f}$ of $\tilde{X}_\eta$ there is a unique affine diffeomorphism $\mathrm{aff}(\hat{f})$ of $\mathbb{C}$ such that $\mathrm{dev} \circ \hat{f} = \mathrm{aff}(\hat{f}) \circ \mathrm{dev}$, and one obtains a group homomorphism

$$\mathrm{aff}: \ \mathrm{Aff}^+(\tilde{X}_\eta) \ \to \ \mathrm{Aff}^+(\mathbb{C}), \quad \hat{f} \ \mapsto \ \mathrm{aff}(\hat{f}).$$

The group $\mathrm{Gal}(\tilde{X}_\eta/X_\mu)$ of deck transformations is a subgroup of $\mathrm{Aff}^+(\tilde{X}_\eta)$, since a deck transformation is locally the identity. The *holonomy map*

$$\mathrm{hol}: \ \pi_1(X_\mu) \cong \mathrm{Gal}(\tilde{X}_\eta/X_\mu) \ \to \ \mathrm{Aff}^+(\mathbb{C})$$

is the restriction of aff to $\mathrm{Gal}(\tilde{X}_\eta/X_\eta)$. Furthermore let

$$\mathrm{proj}: \mathrm{Aff}^+(\mathbb{C}) \rightarrow \mathrm{GL}_2(\mathbb{R})$$

be the natural projection. Then the group homomorphism

$$\begin{array}{rlll} \mathrm{der}: & \mathrm{Aff}^+(X_\mu) & \rightarrow & \mathrm{GL}_2(\mathbb{R}), \\ & f & \mapsto & \mathrm{proj}(\mathrm{aff}(\hat{f})) \quad \text{where } \hat{f} \text{ is some lift of } f \text{ to } \tilde{X}_\eta \end{array}$$

is well defined and called *derived map*. This definition is equivalent to the one we gave in Chapter 1 in the paragraph before Definition 1.5. Recall that the *Veech group* was defined by $\Gamma(X_\mu) := \mathrm{der}(\mathrm{Aff}^+(X_\mu))$.

Now, let us return to the particular situation of an origami $O = (p: X^* \rightarrow E^*)$. In Section 1.3, we have defined for each $B \in \mathrm{SL}_2(\mathbb{R})$ a natural translation structure ν_B on E^* and μ_B on X^*. Since μ_B was defined as lift of ν_B via p, the covering p becomes a translation covering between the translation surfaces $X_B^* := (X^*, \mu_B)$ and $E_B^* := (E^*, \nu_B)$, i.e. it is locally given by translations.

Recall that all Veech groups $\Gamma(X_B^*)$ for different B are conjugated in $\mathrm{GL}_2(\mathbb{R})$, see Remark 1.6, and that we defined the Veech group of $\Gamma(O)$ to be the group $\Gamma(X_I^*)$.

Therefore, in the following we will always consider X^* and E^* as endowed with the translation structure ν_I and μ_I, respectively, and denote $\tilde{X}^* = \tilde{X}_I^*$, the universal covering. Furthermore we write $X^* = X_I^*$, $E = E_I$, $\Lambda = \Lambda_I$, $E^* := E_I^*$ and $\mu := \mu_I$.

By the uniformization theorem there exists a biholomorphic map $\delta: \mathbb{H} \rightarrow \tilde{X}^* = \tilde{X}_I^*$, where $\mathbb{H}$ is the complex upper half plane. In the following we will identify $\mathbb{H}$ with $\tilde{X^*} = \tilde{X^*}_I$ and thus consider it also as a translation surface.

Furthermore, we may choose free generators x and y of F_2 in such a way that under the isomorphism α in (3.1) $\alpha(x)$ is the horizontal and $\alpha(y)$ is the vertical simple closed path, compare Figure 4 in Section 1.2. Then it follows in particular for x and y considered as elements in $\mathrm{Gal}(\mathbb{H}/E^*)$ that:

$$\mathrm{aff}(x) = (z \mapsto z + \begin{pmatrix} 1 \\ 0 \end{pmatrix}) \text{ and } \mathrm{aff}(y) = (z \mapsto z + \begin{pmatrix} 0 \\ 1 \end{pmatrix}). \tag{3.3}$$

We will use this in the proof of Proposition 3.5.

We may now show the following proposition as first step in order to obtain the characterization.

Proposition 3.3. *Let $O = (p: X^* \rightarrow E^*)$ be an origami and $\mathbb{H}$ be the upper half plane, endowed with the translation structure as above. Then we have:*

1. $\Gamma(O)$ *is a subgroup of* $\Gamma(\mathbb{H})$.

2. $\Gamma(E^*) = \Gamma(\mathbb{H}) = \mathrm{SL}_2(\mathbb{Z})$.

3. *Let* f *be in* $\mathrm{Aff}^+(X^*)$. *Then* f *descends via* p *to some* $\bar{f} \in \mathrm{Aff}^+(E^*)$ *and Diagram 8 becomes commutative with* $A := \mathrm{der}(f)$, *with* $\hat{f}$ *some lift of* f *to* $\mathbb{H}$ *and with some* $b \in \mathbb{Z}^2$.

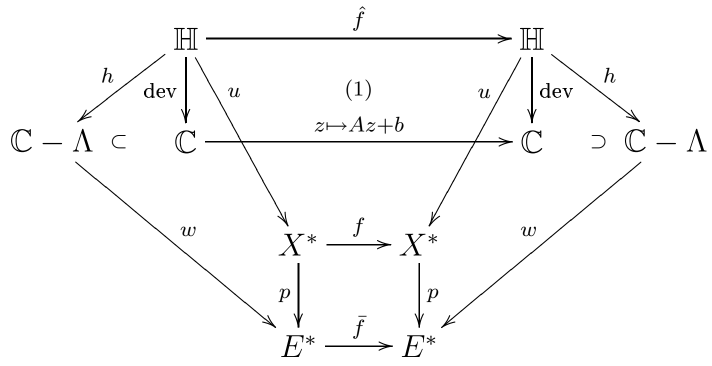


Diagram 8

Proof.
1.: Let f be in $\mathrm{Aff}^+(X^*)$ and $\hat{f}$ be some lift of f via u. Since the translation structure on $\mathbb{H}$ is lifted via u, $\hat{f}$ is also affine and $\mathrm{der}(\hat{f}) = \mathrm{der}(f)$. Hence, $\Gamma(O) \subseteq \Gamma(\mathbb{H})$.

2.: Let $\mathbb{C} \to E$ be the universal covering and $w : \mathbb{C} - \Lambda \to E^*$ its restriction to $\mathbb{C} - \Lambda$. Since $v = p \circ u$ is the universal covering of E^*, there is an unramified covering $h : \mathbb{H} \to \mathbb{C} - \Lambda$, such that $w \circ h = v = p \circ u$. But the structure on $\mathbb{H}$ was obtained by lifting the translation structure on E^* via v. The map h is therefore locally a chart of $\mathbb{H} = \tilde{X}_I^*$. Thus, the map h is a developing map and the image of this developing map dev is $\mathbb{C} - \Lambda$.
Now, let A be in $\Gamma(\mathbb{H})$, then $A = \mathrm{der}(\hat{f})$ for some $\hat{f} \in \mathrm{Aff}^+(\mathbb{H})$. By the definition of the maps der and dev, Part (1) of Diagram 8 is commutative for some $b \in \mathbb{Z}^2$, i. e.

$$(z \mapsto Az + b) \circ \mathrm{dev} = \mathrm{dev} \circ \hat{f}.$$

Since the image of dev is $\mathbb{C} - \Lambda$, the map $z \mapsto Az + b$ respects $\Lambda = \mathbb{Z}^2$. Thus, A is in $\mathrm{SL}_2(\mathbb{Z})$. Hence, we have: $\Gamma(\mathbb{H}) \subset \mathrm{SL}_2(\mathbb{Z})$.
Conversely, taking a matrix $A \in \mathrm{SL}_2(\mathbb{Z})$, the map $z \mapsto Az$ descends to an affine diffeomorphism $\bar{f} \in \mathrm{Aff}^+(E^*)$. This can be lifted to some $\hat{f} \in \mathrm{Aff}^+(\mathbb{H})$ with $\mathrm{der}(\hat{f}) = A$. Thus, we have: $\mathrm{SL}_2(\mathbb{Z}) \subset \Gamma(\mathbb{H})$.
Using the same arguments it follows that also $\Gamma(E^*) = \mathrm{SL}_2(\mathbb{Z})$.

3.: Let $\hat{f} \in \mathrm{Aff}^+(\mathbb{H})$ be some lift of f to $\mathbb{H}$. If follows by the proof of 2. that $\hat{f}$ descends via $w \circ h = v$ to some $\bar{f} \in \mathrm{Aff}^+(E^*)$, and that Diagram 8 is commutative. □

An immediate consequence of Proposition 3.3 is

Corollary 3.4.

$$\Gamma(O) = \{A \in \mathrm{SL}_2(\mathbb{Z}) \mid A = \mathrm{der}(\hat{f}) \text{ for some } \hat{f} \in \mathrm{Aff}^+(\mathbb{H}) \text{ that descends to } X^* \text{ via } u\}.$$

To prove Theorem 1 from Corollary 3.4 we have to state a condition for $\hat{f}$ in $\mathrm{Aff}^+(\mathbb{H})$ to descend via u to some $f \in \mathrm{Aff}^+(X^*)$.

3.3 Second part of the proof

We will now use the relation of origamis to subgroups of F_2 described in Section 3.1 in order to state a condition for an affine diffeomorphism $\hat{f} \in \mathrm{Aff}^+(\mathbb{H})$ to descend to X^*.

Recall that we have:

$$H := \mathrm{Gal}(\mathbb{H}/X^*) \subseteq F_2 = \mathrm{Gal}(\mathbb{H}/E^*) \subseteq \mathrm{PSL}_2(\mathbb{R}), \quad \text{see} \quad (3.2).$$

We may use that

$$F_2 = \mathrm{Gal}(\mathbb{H}/E^*) = \{\hat{f} \in \mathrm{Aff}^+(\mathbb{H}) \mid \mathrm{der}(\hat{f}) = I\}. \tag{3.4}$$

Thus we obtain the following group homomorphism:

$$\begin{array}{rccl} \star : & \mathrm{Aff}^+(\mathbb{H}) & \to & \mathrm{Aut}^+(F_2) \\ & \hat{f} & \mapsto & (\hat{f}_\star : \sigma \mapsto \hat{f} \circ \sigma \circ \hat{f}^{-1}) \end{array}$$

The map $\star$ is well defined, since $\hat{f} \circ \sigma \circ \hat{f}^{-1}$ is again affine with the derivative $\mathrm{der}(\hat{f}) \cdot I \cdot \mathrm{der}(\hat{f})^{-1} = I$ and thus in $F_2 = \mathrm{Gal}(\mathbb{H}/E^*)$.

Proposition 3.5. *We have the following properties of $\star$:*

1. *The following two sequences are exact and the diagram is commutative:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F_2 & \longrightarrow & \mathrm{Aff}^+(\mathbb{H}) & \xrightarrow{\mathrm{der}} & \mathrm{SL}_2(\mathbb{Z}) & \longrightarrow & 1 \\ & & \cong\big\downarrow\alpha & (A) & \cong\big\downarrow\star & (B) & \cong\big\uparrow\beta & & \\ 1 & \longrightarrow & \mathrm{Inn}(F_2) & \longrightarrow & \mathrm{Aut}^+(F_2) & \longrightarrow & \mathrm{Out}^+(F_2) & \longrightarrow & 1 \end{array}$$

Diagram 9

Here, $\mathrm{Inn}(F_2)$ *is the group of inner automorphisms of* F_2, α *is the natural isomorphism* $F_2 \to \mathrm{Inn}(F_2), x \mapsto (y \mapsto xyx^{-1})$, $\beta : \mathrm{Out}^+(F_2) \to \mathrm{SL}_2(\mathbb{Z})$ *is the group isomorphism induced by the natural homomorphism:*

$$\hat{\beta} : \mathrm{Aut}^+(F_2) \to \mathrm{SL}_2(\mathbb{Z}),\ \varphi \mapsto A := \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where a *is the number of* x *appearing in* $\varphi(x)$, b *the number of* x *appearing in* $\varphi(y)$, c *the number of* y *in* $\varphi(x)$ *and* d *the number of* y *in* $\varphi(y)$ *(see [LS 77, I 4.5, p.25]). Recall that for the canonical projection* $\mathrm{proj} : F_2 \to \mathbb{Z}^2$ *sending* x *to* $(1,0)^t$ *and* y *to* $(0,1)^t$ *one has:*

$$\forall \varphi \in \mathrm{Aut}^+(F_2) : \quad A := \hat{\beta}(\varphi) \quad \Rightarrow \quad \mathrm{proj} \circ \varphi = (z \mapsto A \cdot z) \circ \mathrm{proj}. \quad (3.5)$$

2. *An element* $\hat{f} \in \mathrm{Aff}^+(\mathbb{H})$ *descends to* X^* *via* p *if and only if* $\hat{f}_\star(H) = H$.

Proof.
1.: The exactness of the first sequence follows by (3.4) and by the second part of Proposition 3.3. The exactness of the second sequence is true by the definition of $\mathrm{Out}^+(F_2)$.
The commutativity of Part (A) of the Diagram is true by the definition of $\star$. We now prove the commutativity of Part (B):
Recall from (3.3), that:

$$\mathrm{aff}(x) = (z \mapsto z + \begin{pmatrix} 1 \\ 0 \end{pmatrix}) \quad \text{and} \quad \mathrm{aff}(y) = (z \mapsto z + \begin{pmatrix} 0 \\ 1 \end{pmatrix}).$$

Thus, $\mathrm{aff}|_{F_2}(= \mathrm{hol})$ is the natural projection $\mathrm{proj} : F_2 \to \mathbb{Z}^2$. Here we identify the group of translations of $\mathbb{C}$ along some vector in $\mathbb{Z}^2$ canonically with $\mathbb{Z}^2$.
We show that the following diagram is commutative with $A := \mathrm{der}(\hat{f})$.

$$\begin{array}{ccc} F_2 & \xrightarrow{\hat{f}_\star} & F_2 \\ \downarrow{\scriptstyle \mathrm{proj}} & & \downarrow{\scriptstyle \mathrm{proj}} \\ \mathbb{Z}^2 & \xrightarrow{z \mapsto A \cdot z} & \mathbb{Z}^2 \end{array}$$

Figure 10

Let σ be in $F_2 = \mathrm{Gal}(\mathbb{H}/E^*)$. We have to show that $\mathrm{proj}(\hat{f}_\star(\sigma)) = A \cdot \mathrm{proj}(\sigma)$.
We have $\mathrm{aff}(\sigma) = (z \mapsto z + c)$ and $\mathrm{aff}(\hat{f}) = (z \mapsto Az + b)$ for some $b, c \in \mathbb{Z}^2$. Thus we get:

$$\mathrm{proj}(\hat{f}_\star(\sigma)) = \mathrm{aff}(\hat{f}_\star(\sigma)) = \mathrm{aff}(\hat{f})\mathrm{aff}(\sigma)\mathrm{aff}(\hat{f}^{-1}) = (z \mapsto z + Ac).$$

Thus the diagram in Figure 10 is commutative with $A = \mathrm{der}(\hat{f})$.

To conclude we use that the same diagram is also commutative with $A = \hat{\beta}(\hat{f}_\star)$, see (3.5). Thus, $\mathrm{der}(\hat{f}) = \hat{\beta}(\hat{f}_\star)$, and (B) is commutative.
Finally, α and β are both isomorphisms, thus $\star$ is also an isomorphism.

5.4.: $\hat{f} \in \mathrm{Aff}^+(\mathbb{H})$ descends to X^* via $p \Leftrightarrow$ for all $z \in \mathbb{H}, \sigma \in H = \mathrm{Gal}(\mathbb{H}/X)$ there is some $\tilde{\sigma}_{z,\sigma} \in H$ such that $\tilde{\sigma}_{z,\sigma}(\hat{f}(z)) = \hat{f}(\sigma(z))$.
For $\tilde{\sigma} := \hat{f}_\star(\sigma)$, by definition of $\hat{f}_\star$ we have $\tilde{\sigma}(\hat{f}(z)) = \hat{f}(\sigma(z))$ for all $z \in \mathbb{H}$. Since F_2 operates without fixed points on $\mathbb{H}$ it follows from the last equation that $\tilde{\sigma}_{z,\sigma}$ has to be equal to $\tilde{\sigma} = \hat{f}_\star(\sigma)$. On the other hand, $\tilde{\sigma}_{z,\sigma}$ has to be in H.
This proves 2. .

□

With Corollary 3.4 and Proposition 3.5 we have finished the proof of Theorem 1.

3.4 Some applications

In this section we state some immediate conclusions of Theorem 1.

Corollary 3.6 (to Theorem 1).
$\Gamma(O)$ is a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

As mentioned in section 1.5, this result follows e. g. by [GJ 00, Thm. 5.5]. Their proof uses completely different methods.

Proof. Let H be defined as above and $d := [F_2 : H]$.
We have a natural action of $\mathrm{Aut}^+(F_2)$ on the subgroups of F_2 of index d, and

$$\mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(H) \;=\; \{\gamma \in \mathrm{Aut}^+(F_2) \mid \gamma(H) = H\}$$

is the stabilizer of H under this action. Since there are only finitely many subgroups of index d in F_2, the orbit of H under $\mathrm{Aut}^+(F_2)$ is finite and therefore we have

$$[\mathrm{Aut}^+(F_2) : \mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(H)] < \infty.$$

From Theorem 1 it then follows that $\Gamma(O) = \hat{\beta}(\mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(H))$ has finite index in $\mathrm{SL}_2(\mathbb{Z}) = \hat{\beta}(\mathrm{Aut}^+(F_2))$, too. □

As an other application of the Theorem we get: In order to check whether an element A of $\mathrm{SL}_2(\mathbb{Z})$ is in $\Gamma(O)$, we have to check if there exists some lift γ_A in $\mathrm{Aut}^+(F_2)$ of A (i.e. a preimage of A under $\hat{\beta}$) that fixes H. The following corollary translates this into a finite problem that can be left to a computer.

Corollary 3.7 (to Theorem 1).
Let as above

$$\begin{array}{rcl} O &=& (p: X^* \to E^*) \quad \textit{be an origami of degree } d, \\ F_2 &=& \mathrm{Gal}(\mathbb{H}/E^*) \quad \textit{and} \quad H = \mathrm{Gal}(\mathbb{H}/X^*). \end{array}$$

Let furthermore $h_1, \ldots, h_k$ be generators of H and $\sigma_1, \ldots, \sigma_d$ a system of right coset representatives of $H \backslash F_2$ (denote the right coset $H \cdot \sigma_i$ by $\bar{\sigma}_i$).
Further let $\gamma_A^0 \in \mathrm{Aut}^+(F_2)$ be some fixed lift of $A \in \mathrm{SL}_2(\mathbb{Z})$. Then

$$A \in \Gamma(O) \;\Leftrightarrow\; \exists\, i \in \{1, \ldots, d\} \textit{ such that } \bar{\sigma}_i \cdot \gamma_A^0(h_j) = \bar{\sigma}_i \textit{ for all } j \in \{1, \ldots, k\}.$$

Proof. Let γ_A be another lift of A. Thus $\gamma_A^0 = \sigma^{-1} \cdot \gamma_A \cdot \sigma$ for some $\sigma \in F_2$ and for all h in H we have:

$$\gamma_A(h) \in H \;\Leftrightarrow\; \sigma \cdot \gamma_A^0(h) \cdot \sigma^{-1} \in H \;\Leftrightarrow\; H \cdot \sigma \cdot \gamma_A^0(h) = H \cdot \sigma \;\Leftrightarrow\; \bar{\sigma} \cdot \gamma_A^0(h) = \bar{\sigma}$$

Hence the claim follows from Theorem 1. □

The last statement can be slightly generalized as follows.

Corollary 3.8 (to Theorem 1). *In the situation of Corollary 3.7, let U be the normalizer of H in F_2 and $\rho_1, \ldots, \rho_r$ a system of right coset representatives of $U \backslash F_2$. Then*

$$\begin{array}{rcl} A \in \Gamma(O) &\Leftrightarrow& \exists\, i \in \{1, \ldots, r\} \textit{ such that} \\ && \quad (H \cdot \rho_i) \cdot \gamma_A^0(h_j) = H \cdot \rho_i \textit{ for all } j \in \{1, \ldots, k\}. \end{array}$$

Proof. Let σ and σ' be in F_2 in the same right coset $\bar{\sigma} = \bar{\sigma}'$ of U in F_2, i.e. $\sigma' = u\sigma$ for some $u \in U = \mathrm{Norm}_{F_2}(H)$. Then

$$\begin{array}{rcl} H \cdot \sigma' \cdot \gamma_A^0(h_j) = H \cdot \sigma' &\Leftrightarrow& \sigma' \gamma_A^0(h_j) \sigma'^{-1} \in H \;\Leftrightarrow\; u\sigma\gamma_A^0(h_j)\sigma^{-1}u^{-1} \in H \\ &\overset{u \in U}{\Leftrightarrow}& \sigma\gamma_A^0(h_j)\sigma^{-1} \in H \;\Leftrightarrow\; (H \cdot \sigma) \cdot \gamma_A^0(h_j) = H \cdot \sigma \end{array}$$

Thus the claim follows by Corollary 3.7. □

By Theorem 1 it particularly follows that each characteristic subgroup of F_2 of finite index defines an origami with Veech group $\mathrm{SL}_2(\mathbb{Z})$. One may call such origamis *characteristic*. In fact there are many characteristic origamis.

Corollary 3.9. *There are infinitely many non trivial origamis having Veech group* $\mathrm{SL}_2(\mathbb{Z})$.

Proof. For an arbitrary origami O one obtains an origami O', which covers O and whose Veech group is the full group $\mathrm{SL}_2(\mathbb{Z})$, as follows:
Let H be the finite index subgroup of F_2 corresponding to the origami O. Then

$$N \;:=\; \bigcap_{\gamma \in \mathrm{Aut}^+(F_2)} \gamma(H)$$

is a characteristic subgroup of H. The corresponding origami O' covers O and has Veech group $\mathrm{SL}_2(\mathbb{Z})$ by Theorem 1. □

Several (in fact infinitely many) explicit examples for characteristic origamis can be found in [He 05]. There exist also origamis with Veech group $\mathrm{SL}_2(\mathbb{Z})$, such that the corresponding subgroup of F_2 is not characteristic, see e.g. [Sm 05].

In the following lemma we consider particular parabolic elements that can be found easily in Veech groups:
Consider a decomposition of an origami O into r cylinders. Without loss of generality we may assume that the cylinders are horizontal and their height is 1. For any other direction one may conjugate by a suitable matrix in $\mathrm{SL}_2(\mathbb{Z})$. Let $l_1, \ldots, l_r$ be the lengths of the cylinders and $l := \mathrm{lcm}(l_1, \ldots, l_r)$. One has the following well known fact.

Lemma 3.10. *The Veech group of O contains the parabolic element*

$$A_l := \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}.$$

This can be translated into our language of finite index subgroups of F_2 as follows:

Let O be an origami and H the corresponding finite index subgroup of F_2.
The d squares of the origami correspond to the right cosets $H\sigma_1, \ldots, H\sigma_d$ of H in F_2. The group F_2 acts on the set of right cosets by multiplication from the right. The horizontal cylinders correspond to the orbits of $< x >$. The length of a cylinder is the length of the corresponding orbit. If $l_1, \ldots, l_r$ are these lengths, we have:

$$l = \mathrm{lcm}(l_1, \ldots, l_r) \quad \Leftrightarrow \quad \begin{array}{l} l \text{ is the smallest number in } \mathbb{N} \text{ such that} \\ \forall\, w \in F_2 : \; wx^l w^{-1} \in H. \end{array}$$

In this language we obtain Lemma 3.10 by Theorem 1 as follows.

Proof. Let $\gamma \in \mathrm{Aut}^+(F_2)$ be defined by

$$\gamma(x) = x, \qquad \gamma(y) = x^l y.$$

Then γ is a preimage of A_l under the projection $\hat{\beta} : \mathrm{Aut}^+(F_2) \to \mathrm{SL}_2(\mathbb{Z})$.

Since $ax^l a^{-1} \in H$ for all $a \in F_2$, we have:

$$\forall\, a \in F_2 : \; Hax^l = Ha. \tag{3.6}$$

Let h be an arbitrary element of H. It can be considered as a word in the letters x, y, x^{-1}, y^{-1}: $h = w(x, y, x^{-1}, y^{-1})$. Then $\gamma(h) = w(x, x^l y, x^{-1}, y^{-1} x^{-k})$. It follows that

$$H\gamma(h) \;=\; Hw(x, x^k y, x^{-1}, y^{-1}x^{-k}) \stackrel{(3.6)}{=} Hw(x, y, x^{-1}, y^{-1}) \;=\; Hh \;=\; H$$

Thus $\gamma(H) = H$ and $A_l \in \Gamma(H) = \Gamma(O)$ by Theorem 1. $\square$

Chapter 4

An algorithm for finding the Veech group of an origami

In this section we present an algorithm that determines the Veech group $\Gamma(O)$ of an origami O. It is based on the characterization given in Theorem 1.

We have subdivided the description into four parts: In 4.1 we describe how to find some lift $\gamma_A \in \mathrm{Aut}^+(F_2)$ for any matrix A in $\mathrm{SL}_2(\mathbb{Z}) \cong \mathrm{Out}^+(F_2)$, in 4.2 we show how to decide whether a given matrix $A \in \mathrm{SL}_2(\mathbb{Z})$ is in $\Gamma(O)$, in 4.3 we give an algorithm that determines generators and a system of coset representatives of $\Gamma(O)$ in $\mathrm{SL}_2(\mathbb{Z})$, and finally in 4.4 we state how to calculate the genus and the number of points at infinity of $\mathbb{H}/\Gamma(O)$, the normalization of the corresponding Teichmüller curve.
In order to illustrate the algorithm we will use the example $O = L(2,3)$.

Example 4.1 (The origami $O = L(2,3)$).

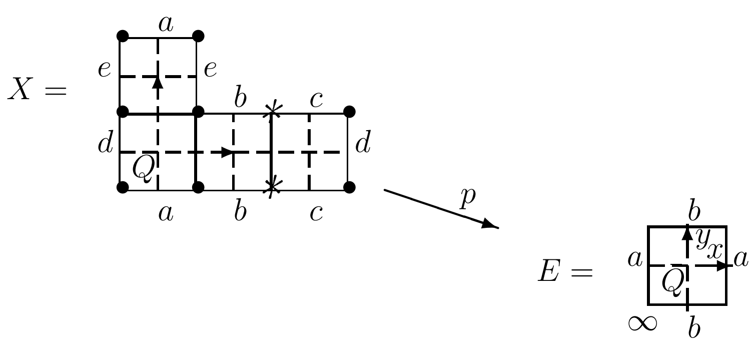


Diagram 11

In Example 4.1 edges labeled with the same letters shall be glued. This way X becomes a surface of genus 2. The squares describe the covering map to E. The point ∞ on E has 2 preimages on the surface X, namely the points $\bullet$ and $*$. Thus $X^* = X - \{\bullet, *\}$. The degree d of the covering p is 4.

We identify $F_2 = \mathrm{Gal}(\mathbb{H}/E^*)$ with the fundamental group of E^* with the base point $\bar{Q}$ and $H = \mathrm{Gal}(\mathbb{H}/X)$ with the fundamental group of X^* with the base point Q. The projection of the closed paths on X^* to E^* defines the embedding of H into F_2, x and y are the fixed generators of $F_2 = \pi_1(E^*)$. Since the $L(2,3)$-shape is simply connected, the generators of H are obtained by the identifications of the edges. Thus, $H = < x^3, x^2yx^{-2}, xyx^{-1}, yxy^{-1}, y^2 >$. The index $[F_2 : H]$ is equal to $d = 4$.

4.1 Lifts from $\mathbf{SL_2(\mathbb{Z})}$ to $\mathbf{Aut^+(}\boldsymbol{F_2}\mathbf{)}$

Let S and T be the following matrices in $\mathrm{SL}_2(\mathbb{Z})$:

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We will use the fact that $\mathrm{SL}_2(\mathbb{Z})$ is generated by S and T. Furthermore, one has $S^{-1} = -S$ and $T^{-1} = -STSTS$. Thus, every $A \in \mathrm{SL}_2(\mathbb{Z})$ can be written as $A = W(S,T)$ or $A = -W(S,T)$, where W is a word in the letters S and T.
The homomorphisms

$$\begin{aligned} \gamma_S &: F_2 \to F_2 \text{ defined by } \gamma_S(x) = y \text{ and } \gamma_S(y) = x^{-1}, \\ \gamma_T &: F_2 \to F_2 \text{ defined by } \gamma_T(x) = x \text{ and } \gamma_T(y) = xy \quad \text{ and} \\ \gamma_{-I} &: F_2 \to F_2 \text{ defined by } \gamma_{-I}(x) = x^{-1} \text{ and } \gamma_{-I}(y) = y^{-1} \end{aligned}$$

are in $\mathrm{Aut}^+(F_2)$ with $\hat{\beta}(\gamma_S) = S$, $\hat{\beta}(\gamma_T) = T$ and $\hat{\beta}(\gamma_{-I}) = -I$, where the morphism $\hat{\beta} : \mathrm{Aut}^+(F_2) \to \mathrm{SL}_2(\mathbb{Z})$ is the projection defined in Proposition 3.5.
Hence for $A = \pm W(S,T)$ the automorphism $\gamma_A := \pm W(\gamma_S, \gamma_T) \in \mathrm{Aut}^+(F_2)$ is a lift of A. Here we denote $-W(\gamma_S, \gamma_T) := \gamma_{-I} \circ W(\gamma_S, \gamma_T)$.

In order to find a word W such that $A = W(S,T)$ or $A = -W(S,T)$ we will define a sequence $A_1 := A$, A_2, …, A_N such that for $1 \leq n < N$:

$$A_{n+1} = A_n \cdot T^{-k_n} \cdot S \quad \text{with } k_n \in \mathbb{Z} \quad \text{and} \quad A_N = \pm T^{\pm b_N} \quad \text{with } b_N \in \mathbb{Z}.$$

From this we get that $A = \pm T^{\pm b_n} \cdot (-S) \cdot T^{k_{n-1}} \cdot \ldots \cdot (-S) \cdot T^{k_1}$. We will conclude using that $T^{-1} = -STSTS$.
These considerations give rise to the following algorithm, in which we denote

$$A_n =: \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \text{ with } a_n, b_n, c_n, d_n \in \mathbb{Z}.$$

Algorithm for finding a lift in Aut$^+(F_2)$:

Given: $A \in \mathrm{SL}_2(\mathbb{Z})$.

$n := 1; A_1 := A$.

1. If $c_n \neq 0$ find $k_n \in \mathbb{Z}$, such that

$$A_{n+1} := A_n T^{-k_n} S \text{ fulfills } |c_{n+1}| < |c_n|.$$

   $k_n := d_n \operatorname{div} c_n$ does this job: $d_n = k_n c_n + r_n$ with $r_n \in \{0, 1, \ldots, |c_n| - 1\}$

$$\Rightarrow A_{n+1} = \begin{pmatrix} -a_n k_n + b_n & -a_n \\ r_n & -c_n \end{pmatrix}.$$

   Increase n by 1.

2. Iterate Step (1) until $c_n = 0$. Thus

$$\begin{aligned} A_n &= \begin{pmatrix} \pm 1 & b_n \\ 0 & \pm 1 \end{pmatrix} = \pm T^{\pm b_n} \text{ and} \\ A &= \pm T^{\pm b_n} \cdot (-S) \cdot T^{k_{n-1}} \cdot \ldots \cdot (-S) \cdot T^{k_1} =: \pm \tilde{W}(S, T, T^{-1}). \end{aligned}$$

3. Replace in $\tilde{W}$ each T^{-1} by $-STSTS$
   $\Rightarrow$ Word W in S and T with $A = W(S,T)$ or $A = -W(S,T)$.

4. Compute $\gamma_A := W(\gamma_S, \gamma_T)$ or $\gamma_A := -W(\gamma_S, \gamma_T)$.

Result: $\gamma_A \in \mathrm{Aut}(F_2)$ with $\hat{\beta}(\gamma_A) = A$.

Example 4.2.

$$\begin{pmatrix} -3 & 5 \\ -2 & 3 \end{pmatrix} = -T^2 S T^3 S T S \;\Rightarrow\; \gamma_A^0 = \gamma_{-I} \gamma_T^2 \gamma_S \gamma_T^3 \gamma_S \gamma_T \gamma_S$$

$$\Rightarrow \gamma_A^0 : \begin{array}{l} x \mapsto x^{-2} y^{-1} x^{-2} y^{-1} x^{-2} y^{-1} x y x^2, \\ y \mapsto x^{-1} y x^2 y x^2 y x^2 \end{array}$$

4.2 Decide whether A is in the Veech group $\Gamma(O)$

Let A be in $\mathrm{SL}_2(\mathbb{Z})$. We want to decide whether A is in $\Gamma(O)$ or not. As in Corollary 3.7 let $h_1, \ldots, h_k$ be generators of $H = \mathrm{Gal}(\mathbb{H}/X) \subseteq F_2 = \mathrm{Gal}(\mathbb{H}/E^*)$, $\sigma_1, \ldots, \sigma_d$ a system of right coset representatives of H in F_2 ($\bar{\sigma}_i := H \cdot \sigma_i$) and γ_A^0 some fixed lift of A in $\mathrm{Aut}^+(F_2)$.
Corollary 3.7 suggests how to build the algorithm:

$$A \in \Gamma(O) \Leftrightarrow \exists i \in \{1, \ldots, d\} \text{ such that } \forall j \in \{1, \ldots, k\} : \; \bar{\sigma}_i \cdot \gamma_A^0(h_j) = \bar{\sigma}_i.$$

Hence, the *main step* will be to decide for some $\tau \in F_2$ whether

$$\bar{\sigma}_i \cdot \tau = \bar{\sigma}_i.$$

In order to do this we present the origami O as directed graph G with edges labeled by x and y (see Figure 12). The cosets $\bar{\sigma}_1, \ldots, \bar{\sigma}_d$ are the vertices of G. Each vertex $\bar{\sigma}_i$ is start point of one x-edge and one y-edge. The endpoint is $\overline{\sigma_i \cdot x}$ and $\overline{\sigma_i \cdot y}$, respectively.

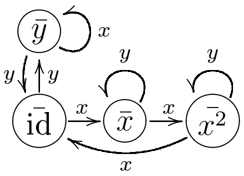


Figure 12: Graph for $O = L(2,3)$.

Writing $\tau \in F_2$ as word in x,y,x^{-1} and y^{-1} defines a not necessarily oriented path in G starting at the vertex $\bar{\sigma}_i$ with end point $\bar{\sigma}_i \cdot \tau$. We have:

$$\bar{\sigma}_i \cdot \tau = \bar{\sigma}_i \Leftrightarrow \text{ this path is closed.}$$

Thus we get the following algorithm.

Algorithm for deciding whether A is in $\Gamma(O)$:

Given: $A \in \mathrm{SL}_2(\mathbb{Z})$.

Calculate some lift $\gamma_A^0 \in \mathrm{Aut}^+(F_2)$ of A (see 4.1).
For $j = 1$ to k do: $\tilde{h}_j := \gamma_A^0(h_j)$.
result := false.
for $i = 1$ to d do
    help := true.
    for $j = 1$ to k do:
        if $\bar{\sigma}_i \cdot \tilde{h}_j \neq \bar{\sigma}_i$ (main step, see above) then help := false.
    if help = true then result := true.

Result: If the variable 'result' is true, then $A \in \Gamma(O)$, else $A \notin \Gamma(O)$.

Example 4.3 (for $O = L(2,3)$).
Let $A := \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Take the lift:

$$\gamma_A^0 : \; x \mapsto xyxyx^{-1} =: u \quad y \mapsto xyxyx^{-1}y^{-1}x^{-1} =: v$$

Generators of H (see Ex. 4.1) are:

$$h_1 := x^3, h_2 := xyx^{-1}, h_3 := x^2yx^{-2}, h_4 := yxy^{-1}, h_5 := y^2.$$

For example $\bar{\mathrm{id}} \cdot \gamma_A^0(h_2) = \bar{\mathrm{id}} \cdot uvu^{-1} = \bar{x}vu^{-1} = \bar{x}^2u^{-1} = \bar{x}^2 \;\Rightarrow\; \gamma_A^0(H) \neq H$.
But one has: $\bar{x} \cdot \gamma_A^0(h_i) = \bar{x} \; \forall i \in \{1, \ldots, 5\}$.
$\Rightarrow \gamma_A(H) = H$ for $\gamma_A = x \cdot \gamma_A^0 \cdot x^{-1}$ and $A \in \Gamma(O)$.

4.3 Generators and coset representatives of $\Gamma(O)$

Let $\bar{\Gamma}(O)$ be the *projective Veech group*, i.e. the image of $\Gamma(O)$ under the projection of $\mathrm{SL}_2(\mathbb{Z})$ to $\mathrm{PSL}_2(\mathbb{Z})$. We first give an algorithm that calculates a list **Gen** of generators and a list **Rep** of right coset representatives of $\bar{\Gamma}(O)$ in $\mathrm{PSL}_2(\mathbb{Z})$, then we determine $\Gamma(O)$. The way how we proceed is based on the Reidemeister-Schreier method ([LS 77], II.4).
We denote by $\bar{A}$ the image of an element $A \in \mathrm{SL}_2(\mathbb{Z})$ under the projection to $\mathrm{PSL}_2(\mathbb{Z})$ and, conversely, denote for $\bar{A}$ in $\mathrm{PSL}_2(\mathbb{Z})$ by A some lift of $\bar{A}$. Moreover, we write $A \sim B$ (respectively $\bar{A} \sim \bar{B}$) if they are in the same coset, i.e. $\Gamma(O) \cdot A = \Gamma(O) \cdot B$ (respectively $\bar{\Gamma}(O) \cdot \bar{A} = \bar{\Gamma}(O) \cdot \bar{B}$).
Each element of $\mathrm{PSL}_2(\mathbb{Z})$ can be presented as word in $\bar{S}$ and $\bar{T}$. We use the directed infinite tree shown in Figure 13: The vertices $v_0, v_1, v_2, \ldots$ of the tree are labelled by elements of $\mathrm{PSL}_2(\mathbb{Z})$. The root v_0 is labelled by $\bar{I}$, the image of the identity matrix. Each vertex is starting point of two edges, one labelled by $\bar{S}$, one labelled by $\bar{T}$.
Each element of $\mathrm{PSL}_2(\mathbb{Z})$ occurs as label of at least one vertex. Starting with v_0 we will visit each vertex v (with label $\bar{B}$) and check if it is not yet represented by the list **Rep**. In this case we will add it to **Rep**. Otherwise for each $\bar{D}$ in **Rep** that is in the same coset as $\bar{B}$, we add $\bar{B} \cdot \bar{D}^{-1}$ to the list **Gen** of generators.

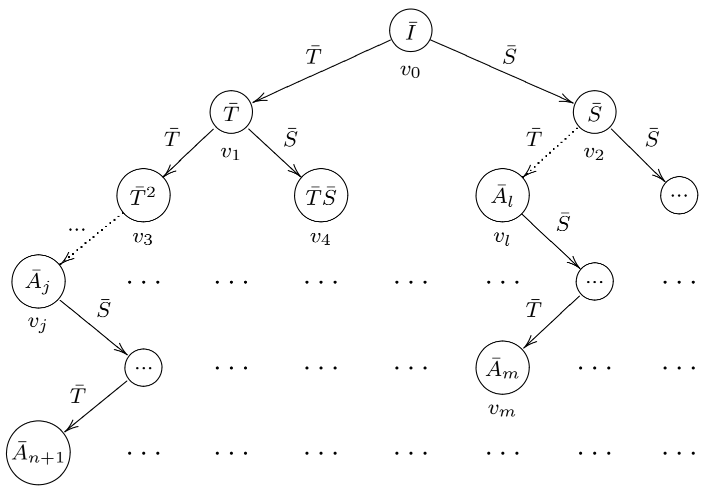


Figure 13: Tree labelled by the elements of $PSL_2(\mathbb{Z})$

We will first give the algorithm and then proof that the lists **Gen** and **Rep** that are calculated are what they should be.

Algorithm for Calculating $\bar{\Gamma}(O)$:

Given: Origami O.

Let **Rep** and **Gen** be empty lists.
Add $\bar{I}$ to **Rep**. $\bar{A} := \bar{I}$.
Loop:
$B := A \cdot T$, $C := A \cdot S$
Check whether $\bar{B}$ is already represented by **Rep**:
For each $\bar{D}$ in **Rep**, check whether $B \cdot D^{-1}$ is in $\Gamma(O)$ or $-B \cdot D^{-1}$ is in $\Gamma(O)$.
If so, add $\bar{B} \cdot \bar{D}^{-1}$ to **Gen**.
If none is found, add $\bar{B}$ to **Rep**.
Do the same for C instead of B.
If there exists a successor of $\bar{A}$ in **Rep**, let $\bar{A}$ be now this successor and go to the beginning of the loop. If not, finish the loop.

Result: **Gen**: list of generators of $\bar{\Gamma}(O)$, **Rep**: list of coset representatives in $\mathrm{PSL}_2(\mathbb{Z})$.

Remark 4.4.

1. *Any two elements of* **Rep** *belong to different cosets.*
2. *The algorithm stops after finitely many steps.*
3. *In the end each coset is represented by a member of* **Rep**.
4. *In the end* $\bar{\Gamma}(O)$ *is generated by the elements of* **Gen**.

Proof.
1.: The statement follows by induction. It is true in the beginning, since **Rep** contains only $\bar{I}$. After passing through the loop it is still true, since $\bar{B}$ (respectively $\bar{C}$) is only added if $\bar{B} \cdot \bar{D}^{-1}$ (resp. $\bar{C} \cdot \bar{D}^{-1}$) is not in $\bar{\Gamma}(O)$ for all $\bar{D}$ in **Rep**.

2.: Follows from 1, since $\bar{\Gamma}(O)$ has finite index in $\mathrm{PSL}_2(\mathbb{Z})$ (Corollary 3.6).

3.: Let $\bar{A}$ be an arbitrary element of $\mathrm{PSL}_2(\mathbb{Z})$. There is at least one vertex in the tree that is labelled by $\bar{A}$. Denote the vertices by v_0, v_1, v_2, ... as in Figure 13 and their labels by $\bar{A}_0$, $\bar{A}_1$, $\bar{A}_2$, ..., respectively.
We do induction by the numeration n of the vertices:
$\bar{A}_0 = \bar{I}$ is in **Rep**. Suppose for a certain $n \in \mathbb{N}$ all $\bar{A}_k$ with $k \leq n$ are represented by **Rep**.
If A_{n+1} is not itself in **Rep** then consider the path ω from v_0 to v_{n+1} and let v_j be the first vertex on ω that is not in **Rep**. Hence, its predecessor is in **Rep** and $\bar{A}_j$ was checked but not added. Thus, there is some $\bar{A}_l$ ($l < j$) in **Rep** such that $\bar{A}_j \cdot \bar{A}_l^{-1}$ is in $\bar{\Gamma}(O)$, i.e. $\bar{A}_j \sim \bar{A}_l$.
Let $\hat{\omega}$ be the path from v_j to v_{n+1} and $\bar{D}$ the product of the labels of the edges

on $\hat{\omega}$. Then $\bar{A}_{n+1} = \bar{A}_j \cdot \bar{D}$.
Walking 'the same path' as $\hat{\omega}$ starting at v_l (i.e. a path described by the same sequence of $\bar{S}$ and $\bar{T}$) leads to some vertex v_m with $m < n+1$ and label $\bar{A}_m = \bar{A}_l \cdot \bar{D}$. We have $\bar{A}_{n+1} = \bar{A}_j \cdot \bar{D} \sim \bar{A}_l \cdot \bar{D} = \bar{A}_m$ and by the assumption $\bar{A}_m$ is represented by **Rep**, hence also $\bar{A}_{n+1}$ is.

4.: Let G be the group generated by the elements of **Gen**. We have by construction of the list **Gen** that $G \subseteq \bar{\Gamma}(O)$.
We show again by induction that each label $\bar{A}_n$ in the tree that is in $\bar{\Gamma}(O)$ is also in G. This is true for $n = 0$. Suppose it is true for all $k \leq n$ with a certain $n \in \mathbb{N}$. If $\bar{A}_{n+1}$ is in $\bar{\Gamma}(O)$, we proceed as in (3) and find some $\bar{A}_j$, $\bar{A}_l$, $\bar{A}_m$ and $\bar{D}$ ($j, l, m < n+1$) such that $\bar{A}_j$ and $\bar{A}_l$ are in the same coset, $\bar{A}_j \cdot \bar{A}_l^{-1}$ is in the list **Gen** (hence, $\bar{A}_j \cdot \bar{A}_l^{-1} \in G$), $\bar{A}_{n+1} = \bar{A}_j \cdot \bar{D}$ and $\bar{A}_m = \bar{A}_l \cdot \bar{D}$. $\bar{A}_m$ is in the same coset as $\bar{A}_{n+1}$, thus it is an element of $\bar{\Gamma}(O)$. By the assumption $\bar{A}_m$ is then also in G. Hence, we have:

$$\bar{A}_{n+1} = \bar{A}_j \cdot \bar{A}_l^{-1} \cdot \bar{A}_l \cdot \bar{D} = (\bar{A}_j \cdot \bar{A}_l^{-1}) \cdot \bar{A}_m \in G.$$

□

Now — knowing $\bar{\Gamma}(O)$ —, it is easy to determine $\Gamma(O)$. We just have to distinguish the two cases, whether $-I$ is in $\Gamma(O)$ or not.

Algorithm for Calculation of $\boldsymbol{\Gamma(O)}$:

Given: Origami O.

Calculate **Gen** and **Rep**.
Let **Gen**$'$ and **Rep**$'$ be empty lists.
Check, whether $-I \in \Gamma(O)$.
If yes: For each $\bar{A} \in$ **Gen** add A to **Gen**$'$. Add $-I$ to **Gen**$'$.
For each $\bar{A} \in$ **Rep** add A to **Rep**$'$.
If no: For each $\bar{A} \in$ **Gen**, check whether $A \in \Gamma(O)$.
If it is, add A to **Gen**$'$; if it is not, add $-A$ to **Gen**$'$.
For each $\bar{A} \in$ **Rep** add A and $-A$ to **Rep**$'$.

Result: **Gen**$'$: list of generators of $\Gamma(O)$,
Rep$'$: list of right coset representatives of $\Gamma(O)$ in $\mathrm{SL}_2(\mathbb{Z})$.

Example 4.5 (for $O = L(2,3)$).

1) Result of calculating $\bar{\Gamma}(O)$:

Gen:

$$\begin{pmatrix}1 & 3\\ 0 & 1\end{pmatrix} = \bar{T}^3, \quad \begin{pmatrix}-1 & 3\\ -2 & 5\end{pmatrix} = \bar{T}\bar{S}\bar{T}^2\bar{S}\bar{T}^{-1}\bar{T}^{-1}, \quad \begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix} = \bar{T}\bar{S}\bar{T}\bar{S}\bar{T}^{-1}\bar{S},$$

$$\begin{pmatrix}3 & -5\\ 2 & -3\end{pmatrix} = \bar{T}^2\bar{S}\bar{T}\bar{S}\bar{T}^{-1}\bar{S}^{-1}\bar{T}^{-2}$$

is a list of generators of $\bar{\Gamma}(O)$. In fact, the algorithm produces more generators, compare Example 4.7. We eliminated here redundant ones.

Rep :

$$\bar{I},\ \bar{T},\ \bar{S},\ \bar{T}^2,\ \bar{T}\bar{S},\ \bar{S}\bar{T},\ \bar{T}^2\bar{S},\ \bar{T}\bar{S}\bar{T},\ \bar{T}^2\bar{S}\bar{T}$$

is a system of coset representatives of $\bar{\Gamma}(O)$ in $\mathrm{SL}_2(\mathbb{Z})$.

2) Result of calculating $\Gamma(O)$: $(-I \in \Gamma(O))$

$$\begin{aligned} \mathbf{Gen}' &= \mathbf{Gen} \cup \{-I\}. \\ \mathbf{Rep}' &= I,\ T,\ S,\ T^2,\ TS,\ ST,\ T^2S,\ TST,\ T^2ST \end{aligned}$$

Hence, $\Gamma(O)$ is a subgroup of index 9 in $\mathrm{SL}_2(\mathbb{Z})$.

4.4 The Geometrical type of the quotient $\mathbb{H}/\bar{\Gamma}(O)$

The group $\bar{\Gamma}(O)$ is a subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ and of finite index, see Corollary 3.6. It acts as Fuchsian group (via Möbius transformations) on $\mathbb{H}$ and $V := \mathbb{H}/\bar{\Gamma}(O)$ is an affine algebraic curve.

V is defined over $\bar{\mathbb{Q}}$ by the Theorem of Belyi: We have a covering from $\mathbb{H}/\bar{\Gamma}(O)$ to $\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z}) \cong \mathbb{A}^1(\mathbb{C}) = \mathbb{P}^1(\mathbb{C}) - \{\infty\}$ ramified at most over the images of i and $\rho = \frac{1}{2} + (\frac{1}{2}\sqrt{3})i$. Thus, by Belyi's theorem the projective curve $\overline{\mathbb{H}/\bar{\Gamma}(O)}$ and hence also the associated Teichmüller curve C introduced in Section 1.4 is defined over $\overline{\mathbb{Q}}$. In the following we want to determine the genus and the number of points at infinity of $V = \mathbb{H}/\bar{\Gamma}(O)$.

Let $\Delta := \Delta(P_0, P_1, P_\infty)$ be the standard fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$, i.e. the hyperbolic pseudo-triangle with vertices $P_0 := -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $P_1 := \frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $P_\infty := i\infty$.

We denote by $\bar{A}$ also the Möbius transformation defined by the matrix A. Then $\bar{T}$ and $\bar{S}$ (as Möbius transformations) send P_0P_∞ to P_1P_∞, respectively P_0P_1 to itself (fixing i).
Let **Rep** $= \{\bar{A}_1, \ldots, \bar{A}_k\}$ be the system of right coset representatives we calculated in Section 4.3. Then

$$F := \bigcup_{i=1}^{k} \bar{A}_i(\Delta)$$

is a simply connected fundamental domain of $\bar{\Gamma}(O)$.
The list **Gen** of generators prescribes how to glue the edges of F to obtain $\mathbb{H}/\bar{\Gamma}(O)$. This way, we get a triangulation of $\mathbb{H}/\bar{\Gamma}(O)$ (compare Figure 14). We calculate the numbers t, e, v of the triangles, the edges and the vertices of this triangulation as described in the following algorithm. Furthermore, the vertices defined by translates of P_∞ are exactly the cusps of $\mathbb{H}/\bar{\Gamma}(O)$. We denote their number by $\hat{v}$. Thus (using the formula of Euler for calculating the genus) we get the following result.

Remark 4.6. *Let t, e, v and $\hat{v}$ be the numbers of triangles, edges, vertices and marked vertices as calculated in the following algorithm. Then $\mathbb{H}/\bar{\Gamma}(O)$ is an affine curve of genus $g = \frac{2-(v-e+t)}{2}$ with $\hat{v}$ cusps.*

Algorithm determining the geometrical type of $\mathbb{H}/\bar{\Gamma}(O)$:

Generate a list of triangles $L := \{\bar{A}_1(\Delta), \ldots, \bar{A}_k(\Delta)\}$.
In the triangle $\bar{A}_i(\Delta)$ we call $\bar{A}_i(P_0)\bar{A}_i(P_1)$ (the image of the edge P_0P_1) 'the S-edge'. Similarly, we call $\bar{A}_i(P_1)\bar{A}_i(P_\infty)$ 'the T-edge' and $\bar{A}_i(P_0)\bar{A}_i(P_\infty)$ 'the T^{-1}-edge'.
For each $i, j \in \{1, \ldots, k\}$ identify

- the T-edge of $\bar{A}_j(\Delta)$ with the T^{-1}-edge of $\bar{A}_i(\Delta)$, if $\bar{A}_i \sim \bar{A}_j \cdot \bar{T}$, i.e. if $(\bar{A}_j\bar{T})\bar{A}_i^{-1} \in \bar{\Gamma}(O)$,
- the T^{-1}-edge of $\bar{A}_j(\Delta)$ with the T-edge of $\bar{A}_i(\Delta)$, if $\bar{A}_i \sim \bar{A}_j \cdot \bar{T}^{-1}$ and
- the S-edge of $\bar{A}_j(\Delta)$ with the S-edge of $\bar{A}_i(\Delta)$, if $\bar{A}_i \sim \bar{A}_j \cdot \bar{S}$.
  If an S-edge of some triangle $\bar{A}_j(\Delta)$ is identified with itself (i.e. $i = j$) create an additional triangle: Add a vertex in the middle of this S-edge and add an edge from this new vertex to the opposite vertex in the triangle $\bar{A}_j(\Delta)$. (Compare triangle $\bar{T}^2\bar{S}\bar{T}$ in Figure 14). This is done to get in the end a triangulation of the surface.

$t :=$ number of triangles. $e :=$ number of edges.
$v :=$ number of vertices, $\hat{v} :=$ number of vertices that are endpoints of T-edges.
$g := \frac{2-(v-e+t)}{2}$.
Result: g : genus of $\mathbb{H}/\bar{\Gamma}(O)$ $\hat{v}$: number of vertices at infinity of $\mathbb{H}/\bar{\Gamma}(O)$.

Example 4.7 (for $O = L(2,3)$).

Rep: $\bar{I}, \bar{T}, \bar{T}^2, \bar{T}^2\bar{S}, \bar{T}^2\bar{S}\bar{T}, \bar{T}\bar{S}, \bar{T}\bar{S}\bar{T}, \bar{S}, \bar{S}\bar{T}$.
Gen: $a := \bar{T}^3, b := \bar{S}\bar{T}\bar{S}\bar{T}^{-1}\bar{S}\bar{T}^{-1}, c := \bar{S}\bar{T}^2\bar{S}, d := \bar{T}\bar{S}\bar{T}^2\bar{S}\bar{T}^{-2}$,
$e := \bar{T}\bar{S}\bar{T}^{-2}\bar{S}\bar{T}^{-2}$, $f := \bar{T}^2\bar{S}\bar{T}\bar{S}\bar{T}^{-1}\bar{S}\bar{T}^{-2}$

We obtain the following fundamental domain for $\bar{\Gamma}(L(2,3))$.

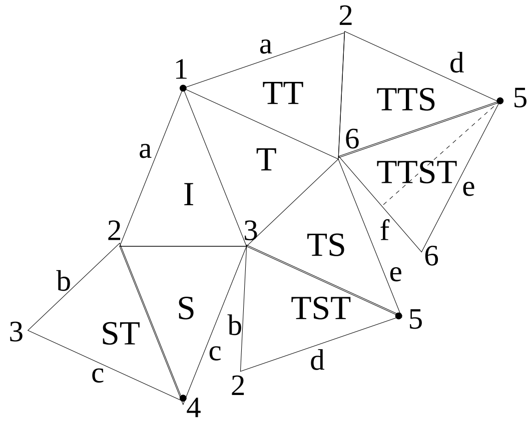


Figure 14: Fundamental domain of $\bar{\Gamma}(L(2,3))$.

In Figure 14 edges with the same letters are glued. In the triangle labeled with $\bar{T}\bar{T}\bar{S}\bar{T}$ there were an edge and a vertex added, since the 'S-edge' is glued to itself. Vertices with same numbers are identified. Vertices at infinity are marked by a filled circle. One can verify using the picture that the number of triangles is $t = 9 + 1$, the number of edges (after identification of those with same labels) is $e = 14 + 1$ and the number of vertices (again after identification) is $v = 6 + 1$. Furthermore the vertices with the labels 1,4 and 5 are vertices at ∞, thus $\hat{v} = 3$.

Result: $g = 0, \hat{v} = 3$. Hence,

$$\mathbb{H}/\bar{\Gamma}(L(2,3)) \cong \mathbb{P}^1 - \{0, 1, \infty\}.$$

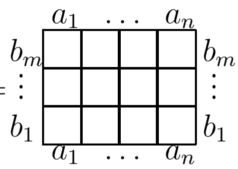
We may summarize this to the following statement about the corresponding Teichmüller curve.

Proposition 4.8. *The origami $L(2,3)$ defines a Teichmüller curve whose normalization is the projective line without three points.*

4.5 Some examples

In this section we give the results of the algorithm for some explicit examples.

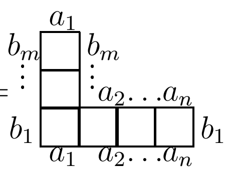
1. "Trivial Origamis":

$O =$ 

$$\Gamma(O) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) | b \equiv 0 \bmod n', c \equiv 0 \bmod m'\}$$

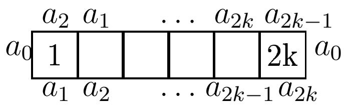
where $t := \gcd(m,n), n' := n/t, m' := m/t$

2. "*L*–Sequence":

$L(n,m) =$ 

Origami	Index	Genus	♯ Cusps
$L(2,2)$	3	0	2
$L(2,3)$	9	0	3
$L(2,4)$	18	0	5
$L(2,5)$	36	0	8
$L(2,6)$	54	0	10
$L(2,7)$	108	1	17
$L(3,3)$	9	0	3
$L(4,4)$	54	0	10

3. "*X*–Sequence":

$O_k =$ 

Origami	Index	Genus	♯ Cusps
O_1	3	0	2
O_2	6	0	3
O_3	12	0	4
O_4	24	0	6
O_5	36	0	8
O_6	48	0	10
O_7	72	1	12
O_8	96	2	14

Remarks on the examples:

As in Example 4.1 edges labeled with same letters are glued. The tables itemize for an origami O the index of the projective Veech group $\bar{\Gamma}(O)$ in $\mathrm{PSL}_2(\mathbb{Z})$ and the genus and number of cusps of $\mathbb{H}/\bar{\Gamma}(O)$.

The first example consists of origamis that are themselves elliptic curves. The Veech group $\Gamma(O)$ can be determined using Theorem 1.

The sequence in the second example was introduced to me by Pierre Lochak. The Veech group e.g. of $L(2,2)$ is given also in [Mö 03]. This sequence is studied in detail in [HL1 04], where e.g. estimates for the growth of the genus and the number of cusps are obtained. The Veech groups are not in general congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$, see Proposition 7.8 and the comment following it.

The X-sequence in the third example will be studied in Section 5.2. All Veech groups in this sequence are congruence groups. They can explicitely be determined for each k, see Section 5.2.

Chapter 5

Examples for Veech groups of origamis

In this chapter we calculate the Veech groups for some infinite sequences of origamis. In particular we show that the X - origamis we introduced in Section 4.5 have as Veech groups the well known congruence groups

$$\pm\Gamma_1(2k) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) | a \equiv \pm 1, b \equiv 0, d \equiv \pm 1 \bmod 2k \right\}.$$

Furthermore, we construct for each $g \geq 2$ origamis of genus g with Veech group $\Gamma(2)$. In particular we obtain the following result.

Theorem 2. *In each moduli space M_g ($g \geq 2$) there is a Teichmüller curve defined by an origami whose normalization is the projective line without three points.*

With a variant of this construction one obtains a projective line without two points in M_g.

In Section 5.1 we give some helpful properties for calculating Veech groups. Finally, in Sections 5.2 and 5.3 we obtain the results listed above.

5.1 A few properties of the stabilizer group

We list some properties of the stabilizer group that we will use in the next sections. By $N \trianglelefteq H$ we denote that N is a normal subgroup of H.

Let U be a subgroup of F_2. Then U defines three subgroups of F_2 as follows:

$$\begin{array}{rcll} \mathrm{Norm}(U) & := & \{w \in F_2 | wUw^{-1} = U\}, & \text{the normalizer of } U \text{ in } F_2 \\ << U >>_{F_2} & := & < wuw^{-1} | \, w \in F_2, u \in U >, & \text{the normal closure of } U \text{ in } F_2 \\ \mathrm{NT}(U) & := & \bigcap_{w \in F_2} wUw^{-1}, & \text{the biggest subgroup } N \text{ of } U \\ & & & \text{that is normal in } F_2 \end{array}$$

The properties listed in the following remark are easily verified.

Remark 5.1. *Let U be a subgroup of F_2, U_i with $i \in I$ a family of subgroups of F_2 and γ an automorphism in $Aut^+(F_2)$. One has the following properties:*

1. $\mathrm{Stab}(U) \subseteq \mathrm{Stab}(Norm(U))$,
2. $\mathrm{Stab}(U) \subseteq \mathrm{Stab}(<< U >>_{F_2})$
3. $\bigcap_{i \in I} \mathrm{Stab}(U_i) \subseteq \mathrm{Stab}(\bigcap_{i \in I} U_i)$
4. $\mathrm{Stab}(\gamma(U)) = \gamma \circ \mathrm{Stab}(U) \circ \gamma^{-1}$
5. $\mathrm{Stab}(U) \subseteq \mathrm{Stab}(NT(U))$

Let now O be an origami, $p : X^* \to E^*$ the unramified covering and U the finite index subgroup of F_2 corresponding to O. The groups $\mathrm{Norm}(U)$, $<< U >>_{F_2}$ and $\mathrm{NT}(U)$ are also finite index subgroups of F_2 and define origamis O_1, O_2 and O_3.
Again let $p_1 : X_1^* \to E^*$, $p_2 : X_2^* \to E^*$ and $p_3 : X_3^* \to E^*$ be the unramified coverings defined by these three origamis.
Then p_1 is the unramified covering of E^* of minimal degree such that it is covered normally by X^*, i.e. there exists a normal unramified covering $q_1 : X^* \to X_1^*$ with $p_1 \circ q_1 = p$.
Similarly, p_2 is the unramified covering of E^* of maximal degree that is normal and covered by X^*.
Finally, p_3 is the minimal unramified covering of E^* that factors through p by a normal q_3, i.e. there is a normal covering q_3 of X^* such that $p \circ q_3 = p_3$.

The properties of stabilizer groups listed above imply the following corollary 5.2 to Theorem 1.

Corollary 5.2. *The Veech group $\Gamma(O)$ of the origami O is contained in the Veech groups $\Gamma(O_1)$, $\Gamma(O_2)$ and $\Gamma(O_3)$ of the origamis O_1, O_2 and O_3.*

5.2 X-origamis

In this section we study in more detail the sequence O_k of origamis mentioned in Section 4.5; we call them *X-origamis* because of their shape, see Figure 15. We detect their Veech groups $\Gamma(O_k)$ as congruence groups of level $2k$.

Definition 5.3. *Let O_k be the origami with $2k$ squares given in Figure 15, i.e. the origami defined by the permutations*

$$\sigma_a := (1\ 2\ \dots\ 2k) \in S_{2k} \text{ and } \sigma_b := ((1\ 2)(3\ 4)\dots(2k-1\ \ 2k)) \in S_{2k}.$$

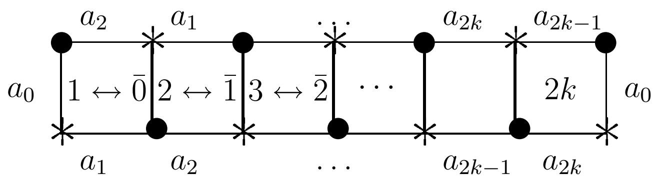


Figure 15

Edges with same labels are identified. We obtain a closed surface X_k. It is divided into $2k$ squares with $4k$ edges and two vertices $\bullet$ and $*$. The genus of X_k is k. Recall that O_k defines an unramified covering $p_k : X_k^* \to E^*$ of degree $2k$. The fundamental group $U_k = \pi_1(X_k^*) \subseteq \pi_1(E^*) = F_2$ is – if we choose the base point in the first square:

$$U_k = < x^{2k}, xy, yx^{-1}, x^2yx^{-3}, x^3yx^{-2}, \dots, x^{2k-2}yx^{-(2k-1)}, x^{2k-1}yx^{-(2k-2)} >$$

Proposition 5.4. *The Veech group of O_k is*

$$\Gamma(O_k) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) | 2b \equiv 0,\ a+b \equiv \pm 1 \bmod 2k,\ a+c \equiv b+d \equiv 1 \bmod 2\}.$$

In particular we have

- k *odd* $\Rightarrow$ $\Gamma(O_k)$ *is conjugated to* $\pm\Gamma_1(2k)$,
  where $\pm\Gamma_1(2k)$ *is defined as in the introduction to this chapter.*
- k *even* $\Rightarrow \Gamma(O_k)$ *has the same index as* $\pm\Gamma_1(2k)$ *but is not conjugated.*

Proof. The proof is divided into the following steps:

1. One obtains for the baby origami O_1 with two squares:

$$\Gamma(O_1) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) | a+c, b+d \text{ odd } \}$$

   and all Veech groups are contained in the first one, i.e. $\Gamma(O_k) \subseteq \Gamma(O_1)$.

2. The group U_k can be described alternatively as

$$U_k = \{w \in F_2 | \sharp_x(w) + \Delta_y \text{ is divisible by } 2k\}.$$

(precise definitions see below)

3. We solve the problem first in the principal congruence group $\Gamma(2)$:

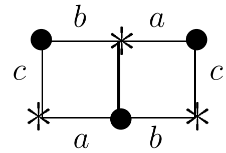
$$\Gamma(O_k) \cap \Gamma(2) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2) | 2b \equiv 0,\ a+b \equiv \pm 1 \mod 2k\}$$

4. Using 3. we show that $\Gamma(O_k)$ is the group claimed in the proposition.

5. Using 4. we show that $\Gamma(O_k)$ has the same index in $\mathrm{SL}_2(\mathbb{Z})$ as $\pm\Gamma_1(2k)$ and they are conjugated iff k is odd.

<u>1.:</u>

We consider the first element of the sequence O_1:



The corresponding subgroup of F_2 is

$$U_1 =< x^2, xy, yx^{-1} >=< x^2, xy, y^2 >= \{w \in F_2 | \text{ le}(w) \text{ is even}\},$$

where $\text{le}(w)$ denotes the length of w as word in x and y.
Hence for an automorphism $\gamma \in \text{Aut}^+(F_2)$ we have:

$$\begin{aligned} \gamma(U_1) = U_1 \quad &\Leftrightarrow \quad \gamma \text{ preserves the parity of the length of words} \\ &\Leftrightarrow \quad \text{le}(\gamma(x)) \text{ and } \text{le}(\gamma(y)) \text{ are odd} \\ &\Leftrightarrow \quad \sharp_x(\gamma(x)) + \sharp_y(\gamma(x)) \text{ odd and } \sharp_x(\gamma(y)) + \sharp_y(\gamma(y)) \text{ odd} \\ &\Leftrightarrow \quad a+c \text{ and } b+d \text{ are odd for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \hat{\beta}(\gamma), \end{aligned}$$

By Theorem 1 the Veech group $\Gamma(O_1) = \hat{\beta}(\text{Stab}(U_1))$. This proves the first part of 1.

Furthermore, U_1 contains U_k for all k and it is the normal closure of U_k in F_2, i.e. $U_1 =<< U_k >>_{F_2}$. This can be seen by checking the three generators of U_1: y^2 and xy are already elements of U_k and $x^2 = x(xy)x^{-1}(xy^{-1})$ with xy and xy^{-1}

in U_k is also in $<< U_k >>_{F_2}$.
Using Corollary 5.2 we obtain the second part of 1.

2.:

We identify the $2k$ squares of the origami O_k with the elements of

$$\mathbb{Z}/2k\mathbb{Z} = \{\bar{0}, \bar{1}, \ldots, \overline{2k-1}\}.$$

The group F_2 acts from the right on the set of the squares as follows: For $w \in F_2 = \pi_1(E^*)$ and $\bar{a}$ one of the squares, lift the path w on E^* via the covering p_k to a path on X_k^* with its starting point in the square $\bar{a}$. Let $\bar{b}$ be the square in which the ending point of the lifted path lies. Then define

$$\bar{a} \cdot w := \bar{b}.$$

Since we had chosen the base point for $U_k = \pi_1(X_k^*)$ in the square $\bar{0}$ one has for v, w in F_2 by definition:

$$w \in U_k \Leftrightarrow \bar{0} \cdot w = \bar{0} \qquad \text{and} \qquad vwv^{-1} \in U_k \Leftrightarrow \bar{b} \cdot w = \bar{b} \text{ with } \bar{b} := \bar{0} \cdot v \quad (5.1)$$

Let $\bar{a}$ be in $\mathbb{Z}/2k\mathbb{Z}$, then x, y, x^{-1}, y^{-1} act on $\bar{a}$ in the following way:

$$\begin{array}{rclrcl} \bar{a} \cdot x & = & \overline{a+1} & \bar{a} \cdot x^{-1} & = & \overline{a-1} \\ \bar{a} \cdot y & = & \left\{ \begin{array}{l} \overline{a+1}\ , \text{ if } \bar{a} \text{ even} \\ \overline{a-1}\ , \text{ if } \bar{a} \text{ odd}\ . \end{array} \right. & \bar{a} \cdot y^{-1} & = & \left\{ \begin{array}{l} \overline{a+1}\ , \text{ if } \bar{a} \text{ even} \\ \overline{a-1}\ , \text{ if } \bar{a} \text{ odd}\ . \end{array} \right. \end{array} \quad (5.2)$$

Here we use that x^{2k} is in U_k.

Now, we obtain the action of any w in F_2 on $\mathbb{Z}/k\mathbb{Z}$: Each $x^{\pm 1}$ contributes ± 1, each $y^{\pm 1}$ contributes 1 or -1 depending on the parity of the position of $y^{\pm 1}$ in w.

Definition 5.5. *For $w \in F_2$ let $\sharp_{|y|}(w|$ odd$)$ be the total number of occurrences of y and y^{-1} in w at an odd position (y^{-1} counted positive!). Similarly, denote by $\sharp_{|y|}(w|$ even$)$ the number of occurrences of y and y^{-1} in w at an even position. Furthermore define*

$$\Delta_y(w) := \sharp_{|y|}(w|\ \textit{odd}) - \sharp_{|y|}(w|\ \textit{even}).$$

E.g. for $w := xyxy^{-1}x^2y^{-1}$ one has $\sharp_{|y|}(w|\ \text{odd}) = 1$, $\sharp_{|y|}(w|\ \text{even}) = 2$ and $\Delta_y(w) = -1$.

Using (5.2) we obtain

$$\bar{a} \cdot w = \left\{ \begin{array}{l} \overline{a + \sharp_x(w) + \Delta_y(w)}\ , \text{ if } \bar{a} \text{ even} \\ \overline{a + \sharp_x(w) - \Delta_y(w)}\ , \text{ if } \bar{a} \text{ odd}\ . \end{array} \right. \quad (5.3)$$

Since $U_k = \{w \in F_2 | \, \bar{0} \cdot w = \bar{0}\}$, 2. follows from (5.3).

<u>3.:</u>

Before restricting to $\Gamma(2)$ we stay in the general setting and observe that it is sufficient to consider the two generators xy and y^2. More precisely: For $\gamma \in \mathrm{Aut}^+(F_2)$

$$\gamma \in \mathrm{Stab}(U_k) \Leftrightarrow \gamma \in \mathrm{Stab}(U_1) \text{ and } \gamma(y^2), \gamma(xy) \in U_k \tag{5.4}$$

$\Rightarrow$ follows by 1. and the definition of $\mathrm{Stab}(U_k)$.
$\Leftarrow$ is true since U_k is the subgroup of U_1 consisting of those words in the three generators $w_1 := x^2$, $w_2 := xy$ and $w_3 := y^2$ of U_1 for which the number of occurrences of w_1 is divisible by k (w_1^{-1} counted negative), i.e.

$$U_k = \{w = w(w_1, w_2, w_3) \in U_1 | \, \sharp_{w_1}(w) \text{ is divisible by } k\}.$$

U_k is generated as normal subgroup of U_1 by w_1^k, w_2 and w_3. Thus it is sufficient to check x^{2k}, xy and y^2 in order to find out, if a given $\gamma \in \mathrm{Stab}(U_1)$ fixes U_k. But $\gamma(x^{2k}) = \gamma(w_1)^k$ and the number of occurrences of each generator in it is divisible by k. Hence it follows (5.4).

As next step observe that in order to check whether A is in $\Gamma(O_k)$ it is sufficient to consider one preimage of A under $\hat{\beta}$: Let A be in $\mathrm{SL}_2(\mathbb{Z})$ and $\gamma_0 \in \mathrm{Aut}^+(F_2)$ such that $\hat{\beta}(\gamma_0) = A$. Since $\hat{\beta}$ is the quotient map $\mathrm{Aut}^+(F_2) \to \mathrm{Out}^+(F_2) \cong \mathrm{SL}_2(\mathbb{Z})$, an automorphism γ is mapped to A by $\hat{\beta}$ iff it is conjugated to γ_0. Thus we have:

$$\begin{aligned} A \in \Gamma(O_k) &\overset{\text{Thm.1}}{\Leftrightarrow} \exists \gamma \in \mathrm{Stab}(U_k) : \hat{\beta}(\gamma) = A \Leftrightarrow \exists w \in F_2 : w\gamma_0 w^{-1} \in \mathrm{Stab}(U_k) \\ &\overset{(5.1)}{\Leftrightarrow} \exists \bar{b} \in \mathbb{Z}/(2k\mathbb{Z}) : \bar{b} \cdot \gamma_0(u) = \bar{b} \text{ for all } u \in U_k. \end{aligned} \tag{5.5}$$

Observe that $\mathrm{Norm}(U_k) = U_1$ (by checking that it contains the three generators of U_1). Thus $\mathrm{Norm}(U_k)$ has index 2 in F_2 and $1, x$ are coset representatives. Thus in (5.5) it is sufficient to consider $\bar{b} \in \{\bar{0}, \bar{1}\}$. Together with (5.4) it follows that

$$\begin{aligned} A \in \Gamma(O_k) \Leftrightarrow \quad & (\bar{0} \cdot \gamma_0(y^2) = \bar{0} \text{ and } \bar{0} \cdot \gamma_0(xy) = \bar{0}\,) \text{ or} \\ & (\bar{1} \cdot \gamma_0(y^2) = \bar{1} \text{ and } \bar{1} \cdot \gamma_0(xy) = \bar{1}\,). \end{aligned} \tag{5.6}$$

Now, suppose that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in $\Gamma(2)$.

One has: $\Gamma(2) = < A_1 := \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, A_2 := \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, A_3 := \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} > .$ (5.7)

We define the three automorphisms

$$\gamma_1 : x \mapsto x, y \mapsto x^2y \qquad \gamma_2 : x \mapsto xy^2, y \mapsto y \qquad \gamma_{-I} : x \mapsto x^{-1}, y \mapsto y^{-1}$$

and set

$$G(2) :=< \gamma_1, \gamma_2, \gamma_{-I} > .$$

Thus $\hat{\beta}(G(2)) = \Gamma(2)$. Let γ_0 be in $G(2)$ with $\hat{\beta}(\gamma_0) = A$.
We will use in the following the fact proven below in Lemma 5.6 that $G(2)$ respects Δ_y, i.e.:

$$\forall \gamma \in G(2), w \in F_2 : \Delta_y(\gamma(w)) = \Delta_y(w).$$

Then we have:

$$\begin{array}{rcl} \bar{0} \cdot \gamma(xy) & \stackrel{(5.3)}{=} & \overline{0 + \sharp_x(\gamma(xy)) + \Delta_y(\gamma(xy))} \\ & \stackrel{\text{Lem. 5.6}}{=} & \overline{\sharp_x(\gamma(x)) + \sharp_x(\gamma(y)) + \Delta_y(xy)} \\ & = & \overline{a + b - 1}. \end{array}$$

Similarly one obtains

$$\begin{array}{rcl} \bar{1} \cdot \gamma(xy) & = & \bar{1} + \overline{a + b + 1} \\ \bar{0} \cdot \gamma(y^2) & = & \overline{2b} \quad \text{and} \quad \bar{1} \cdot \gamma(y^2) = \bar{1} + \overline{2b} \end{array}$$

Thus by (5.6) $A \in \Gamma(O_k)$ iff $2b \equiv 0$ and $a + b \equiv \pm 1$ modulo $2k$. This proves 3.

4.:

Recall that by 1. the Veech group $\Gamma(O_k)$ is a subgroup of $\Gamma(O_1)$. Suppose that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in $\Gamma(O_1) \backslash \Gamma(2)$. The index of $\Gamma(2)$ in $\Gamma(O_1)$ is 2, since by 1. any element of $\Gamma(O_1)$ maps to either $\bar{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\bar{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in $\mathrm{SL}_2(\mathbb{Z})/\Gamma(2) = \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$. Therefore A has a decomposition $A = B \cdot S$ for some matrix B in $\Gamma(2)$. We define the automorphism

$$\gamma_s : x \mapsto y, y \mapsto x^{-1},$$

then γ_s is a preimage of S under $\hat{\beta}$. Furthermore, we take a preimage γ_B of B in $G(2)$, then $\gamma_A := \gamma_B \circ \gamma_S$ is a preimage of A. One obtains:

$$\bar{0} \cdot \gamma_A(xy) \stackrel{(5.3)}{=} \overline{0 + \sharp_x(\gamma_A(xy)) + \Delta_y(\gamma_B(yx^{-1}))} \stackrel{Lem5.6}{=} \overline{a + b + 1}$$

Similarly, one calculates $\bar{0} \cdot \gamma_A(y^2)$, $\bar{1} \cdot \gamma_A(xy)$ and $\bar{1} \cdot \gamma_A(y^2)$ and obtains altogether:

$$\begin{array}{ll} \bar{0} \cdot \gamma_A(xy) = \overline{a + b + 1} & \bar{0} \cdot \gamma_A(y^2) = \overline{2b} \\ \bar{1} \cdot \gamma_A(xy) = \bar{1} + \overline{a + b - 1} & \bar{1} \cdot \gamma_A(y^2) = \bar{1} + \overline{2b} \end{array}$$

Thus it follows that $A \in \Gamma(O_k)$ iff $2b \equiv 0 \bmod 2k$ and $a + b \equiv \mp 1 \bmod 2k$. Together with 1. and 3. this finishes the proof of 4.

5.:

In order to obtain that $\Gamma(O_k)$ and $\pm\Gamma_1(2k)$ have the same index in $\mathrm{SL}_2(\mathbb{Z})$, we use the fact that $\Gamma(2k)$ is contained in $\Gamma(O_k)$ as well as in $\pm\Gamma_1(2k)$. Therefore it is sufficient to show that their images in $\mathrm{SL}_2(\mathbb{Z}/2k\mathbb{Z}) \cong \mathrm{SL}_2(\mathbb{Z})/\Gamma(2k)$ have the same number of elements.
Using 4. we obtain that the image of $\Gamma(O_k)$ in $\mathrm{SL}_2(\mathbb{Z})$ is:

$$\{\begin{pmatrix} \pm1 & 0 \\ e & \pm1 \end{pmatrix}, \begin{pmatrix} \pm1+k & k \\ o & \pm1+k \end{pmatrix} | e, o \in \mathbb{Z}/2k\mathbb{Z},\ e \text{ even}, o \text{ odd}\},\ \text{if } k \text{ odd} \quad (5.8)$$

$$\{\begin{pmatrix} \pm1 & 0 \\ e & \pm1 \end{pmatrix}, \begin{pmatrix} \pm1+k & k \\ e' & \pm1+k \end{pmatrix} | e, e' \in \mathbb{Z}/2k\mathbb{Z},\ e, e' \text{ even}\},\ \text{if } k \text{ even} \quad (5.9)$$

Thus the image has in both cases $4k$ elements. The image of $\pm\Gamma_1(2k)$ consists of $4k$ elements as well.

Observe by (5.9) that $\Gamma(O_k)$ is contained in $\pm\Gamma(2)$ if k is even. But $\Gamma(2)$ is normal and does not contain $\pm\Gamma_1(2k)$. Therefore $\Gamma(O_2)$ is not conjugated to $\pm\Gamma_1(2k)$ if k is even.
For k odd, one can check by a calculation in $\mathrm{SL}_2(\mathbb{Z}/2k\mathbb{Z})$ that

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \Gamma_1(2k) \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix} = \Gamma(O_k). \quad (5.10)$$

□

Lemma 5.6. *The number $\Delta_y(w) = \sharp_{|y|}(w|\ odd) - \sharp_{|y|}(w|\ even)$ is invariant under $G(2) = <\gamma_1, \gamma_2, \gamma_{-I}>$, i.e. if γ is in $G(2)$, then*

$$\forall w \in F_2 : \Delta_y(\gamma(w)) = \Delta_y(w).$$

Proof. It is sufficient to check the claim for the generators of $G(2)$:

$$\gamma_1 : x \mapsto x, y \mapsto x^2y, \quad \gamma_2 : x \mapsto xy^2, y \mapsto y \quad \text{and } \gamma_{-I} : x \mapsto x^{-1}, y \mapsto y^{-1}$$

Consider $\gamma := \gamma_1$: Let w be an arbitrary element in F_2, thus w is a reduced word in the four letters x, y, x^{-1}, y^{-1}:

$$w = w(x, y, x^{-1}, y^{-1}) \text{ and } \gamma(w) = w(x, x^2y, x^{-1}, y^{-1}x^{-2}).$$

Observe that for the words of replacement x, x^2y, x^{-1}, $y^{-1}x^{-2}$ the value of Δ_y is the same as for the original words x, y, x^{-1} and y^{-1}, their length is odd and that reduction also does not change the value of Δ_y. Hence $\Delta_y(\gamma_1(w)) = \Delta_y(w)$.
With the same arguments this is true for γ_2 and γ_{-I}. Thus the claim holds. □

Using this sequence of origamis one can construct origamis having Veech group $\pm\Gamma_1(2k)$ (for k odd). In the following corollary, we use the automorphism $\gamma : x \mapsto x, y \mapsto x^{-k}y$.

Corollary 5.7. *Let k be odd. Define $V_k := \gamma(U_k)$ with the group U_k defined in Proposition 5.4. Call P_k the origami that is defined by the finite index subgroup V_k of F_2. Then $\Gamma(P_k) = \pm\Gamma_1(2k)$.*

Proof. By Remark 5.1 we have $\mathrm{Stab}(V_k) = \gamma \circ \mathrm{Stab}(U_k) \circ \gamma^{-1}$. By Theorem 1 it follows that

$$\Gamma(P_k) = \hat{\beta}(\gamma)\Gamma(O_k)\hat{\beta}(\gamma^{-1}) = \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix} \Gamma(O_k) \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \overset{(5.10)}{=} \pm\Gamma_1(2k)$$

□

5.3 Stair-origamis

In this section we consider two infinite sequences G_k and St_k of origamis of genus k. We show for both that all origamis in the sequence have the same Veech group. Because of their shape (see Figures 16 and 17) they are called *stair origamis.*

The smallest example of the two sequences, the stairs with 3 and 4 squares, appear e.g. in [Mö 03], where the equations for the Teichmüller curves defined by these two origamis are calculated. The stair with three squares is because of its shape also called *L-origami* and is generalized in another sequence with origamis all in genus 2 (see e.g [HL2 04]).
The stairs with an odd number of squares occur in [He 05], where they are used to construct origamis that cover it having Veech group $\mathrm{SL}_2(\mathbb{Z})$.

Definition 5.8. *Let G_k be the origami with $2k$ squares ($k \geq 2$) in Figure 16 given by the permutations*

$$\sigma_a := (1\ 2)\ \ldots\ (2k-1\ 2k) \text{ and } \sigma_b := (2\ 3)\ \ldots\ (2k-2\ 2k-1) \in S_{2k}$$

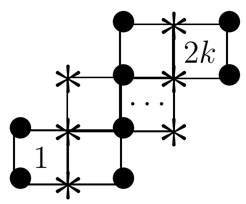


Figure 16

Here opposite edges are identified. One obtains a closed surface with the two marked points ● and ✳. Its genus is k. The fundamental group is

$$U_k =< y, (xy)^{k-1}xyx^{-1}(xy)^{-(k-1)}, (xy)^j x^2 (xy)^{-j},$$
$$(xy)^i xy^2 x^{-1}(xy)^{-i} | j \in \{0, \ldots, k-1\}, i \in \{0, \ldots, k-2\} >$$

Proposition 5.9. *The Veech group* $\Gamma(G_k)$ *is for all* $k \in \mathbb{N}$ *the principal congruence group* $\Gamma(2)$.

Proof. The proof is divided into two parts: In the first part we show $\Gamma(2)$ is a subgroup of $\Gamma(G_k)$; in the second part we show that it is not bigger.

$\Gamma(G_k)$ *is a subgroup of* $\Gamma(2)$*:*

Recall that the group $\Gamma(2)$ is generated by the three matrices A_1, A_2, A_3 given in (5.7). Take again the three preimages under $\hat{\beta}$:

$$\gamma_1 : \begin{cases} x & \mapsto & x \\ y & \mapsto & x^2y \end{cases}, \gamma_2 : \begin{cases} x & \mapsto & xy^2 \\ y & \mapsto & y \end{cases} \text{ and } \gamma_3 : \begin{cases} x & \mapsto & x^{-1} \\ y & \mapsto & y^{-1} \end{cases}$$

We show that $\gamma_i(U_k) = U_k$.

Observe that U_k contains $N := << x^2, y^2 >>_{F_2}$. More precisely, U_k is generated by N and the two elements y and cyc^{-1} with $c := (xy)^{k-1}x$.
Observe furthermore, that $\gamma_i(N) = N$ for $i = 1, 2, 3$:
E.g. $\gamma_1(x^2) = x^2 \in N$ and $\gamma_1(y^2) = x^2yx^2y = y((y^{-1}x^2y)x^2y^2)y^{-1} \in N$. This works similarly for $i = 2$ and $i = 3$. Thus we have $\gamma(N) = N$ for all $\gamma \in G(2)$.

Since $N \trianglelefteq F_2$ and $N \subseteq U_k$, it follows that

$$\forall n \in N, w, v \in F_2 : \ wnv = uwv \text{ with some } u \in U_k. \tag{5.11}$$

One obtains e.g.:
$\gamma_1(y) = x^2y \in U_k$ and
$\gamma_1(cyc^{-1}) = (\gamma_1(xy))^{k-1}xx^2yx^{-1}(\gamma_1(xy))^{-(k-1)} = (x^3y)^{k-1}xx^2yx^{-1}(x^3y)^{-(k-1)} \overset{(5.11)}{=} u(xy)^{k-1}xyx^{-1}(xy)^{-(k-1)} = ucyc^{-1}$ for some $u \in U$. Thus $\gamma_1(U_k) = U_k$.
This works similarly for $i = 2$, $i = 3$, which finishes the proof that $\Gamma(2) \subseteq \Gamma(G_k)$.

$\Gamma(2)$ *is the whole group* $\Gamma(G_k)$*:*

The matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, B_3 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix},$$
$$B_4 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, B_5 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

form a system of coset representatives of $\Gamma(2)$ in $\mathrm{SL}_2(\mathbb{Z})$. Thus it remains to show, that B_1, B_2, B_3, B_4 and B_5 are not in $\Gamma(G_k)$.
Observe that all generators and thus all elements of U_k contain an even number of occurrences of x. Since y is in U_k, the number $\sharp_x(\gamma(y))$ has to be even for an

automorphism γ in $\mathrm{Stab}(U_k)$. This implies that the top right entry of an element of $\Gamma(G_k)$ has to be even. From this argument it follows that B_1, B_2, B_3 and B_4 are not in $\Gamma(G_k)$.
It remains to check B_5. We take the preimage $\gamma_0 : x \mapsto xy, y \mapsto y$ in $\mathrm{Aut}^+(F_2)$ of B_5 under $\hat{\beta}$.
Then we have for each other preimage $\gamma := w \cdot \gamma_0 \cdot w^{-1}$ $(w \in F_2)$:
$\gamma(xy^{-1}xy^{-1}) = w\gamma_0(xy^{-1}xy^{-1})w^{-1} = wx^2w^{-1} \in N \subseteq U_k$.
But $xy^{-1}xy^{-1}$ is not in U_k, thus $\gamma \notin \mathrm{Stab}(U_k)$. From this it follows that $B_5 \notin \Gamma(G_k)$.

□

Definition 5.10. *Let St_k be the origami with $2k-1$ $(k \geq 2)$ squares in Figure 17 given by the permutations*

$$\sigma_a := (1\ 2)\ \ldots\ (2k-3\ 2k-2) \text{ and } \sigma_b := (2\ 3)\ \ldots\ (2k-2\ 2k-1) \in S_{2k-1}$$

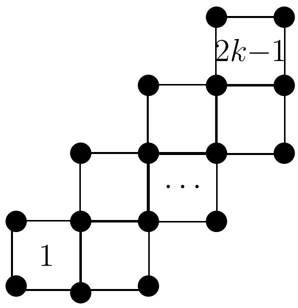


Figure 17

Again opposite edges are identified. One obtains a closed surface with one marked point: •. Its genus is k. The fundamental group is

$$U_k =< y, (xy)^{k-1}x(xy)^{-(k-1)}, (xy)^j x^2 (xy)^{-j}, \\ (xy)^j xy^2 x^{-1}(xy)^{-j} |\ j \in \{0, \ldots, k-2\} >$$

Proposition 5.11. *The Veech group $\Gamma(St_k)$ is for all $k \in \mathbb{N}$ the congruence group*

$$\Gamma := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) | a + c \text{ and } b + d \text{ odd } \right\}.$$

Proof. We have

$$A \in \Gamma \Leftrightarrow A \text{ is sent to the image of } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } B_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

in $\mathrm{SL}_2(\mathbb{Z})/\Gamma(2) = \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$ under the natural projection. Thus Γ is generated as normal subgroup of $\mathrm{SL}_2(\mathbb{Z})$ by $\Gamma(2)$ and the matrix B_2.
Take the automorphisms γ_1, γ_2 and γ_3 defined as in the proof of Proposition 5.9

and take the automorphism $\gamma_4 : x \mapsto y, y \mapsto x^{-1}$ as preimage of B_2 under $\hat{\beta}$. Observe that U_k again contains $N =<< x^2, y^2 >>_{F_2}$ and is generated by N and the two elements y and cxc^{-1} with $c := (xy)^{k-1}$.
We have already seen in the last proof that $\gamma_i(N) = N$ for $i \in \{1, 2, 3\}$ and it is easily seen that $\gamma_4(N) = N$. Furthermore, one can check similarly as in the last proof that $\gamma_i(y)$ and $\gamma_i(cyc^{-1})$ is in U_k. Hence Γ is contained in the Veech group of St_k.
Finally we show that $B_1 \notin \Gamma(St_k)$: Take one fixed preimage of B_1 under $\hat{\beta}$: $\gamma_5 : x \mapsto x, y \mapsto xy$. Then for each conjugated automorphism $\gamma := w\gamma_5 w^{-1}$ ($w \in F_2$) one has $\gamma(x^{-1}yx^{-1}y) = wy^2w^{-1} \in St_k$, but $x^{-1}yx^{-1}y \notin St_k$. Thus $\Gamma(St_k) \neq \mathrm{SL}_2(\mathbb{Z})$. It contains Γ which has index 3. Thus it is equal to Γ. □

One has

$$\mathbb{H}/\Gamma(2) \;\cong\; \mathbb{P}^1\backslash\{0, 1, \infty\} \;\text{ and }\; \mathbb{H}/\Gamma \;\cong\; \mathbb{P}^1\backslash\{0, \infty\},$$

where Γ is from Proposition 5.11. Thus it follows for all $k \in \mathbb{N}$ with $k \geq 2$, that

$$\mathbb{H}/\Gamma(G_k) \;\cong\; \mathbb{P}^1\backslash\{0, 1, \infty\} \quad\text{and}\quad \mathbb{H}/\Gamma(St_k) \cong \mathbb{P}^1\backslash\{0, \infty\}$$

Since the genus of G_k and that of St_k is k, we now have in particular proved Theorem 2.

Chapter 6

Congruence groups

In this chapter we construct origamis whose Veech groups are congruence groups. We show in Theorem 3 that all congruence groups of level n whose image in $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ is a stabilizing group in the sense of Definition 6.1 occur as Veech groups of origamis. In particular we obtain with this tool all principal and other common congruence groups, shown in Section 6.4 explicitely.
But in fact, we obtain much more: Provided with Theorem 3 we can show that "most" congruence groups occur as Veech groups of origamis, see Theorem 5. If we restrict to congruence groups of prime level, we can even prove that all of them are Veech groups of origamis, with possibly five exceptions for small primes, see Theorem 4.

The basic idea is to consider the group of affine diffeomorphisms on the trivial $n \times n$ - origami $\mathrm{Tr}(n, n)$ (see Figure 18). Its image in $\mathrm{SL}_2(\mathbb{R})$ is the full group $\mathrm{SL}_2(\mathbb{Z})$.

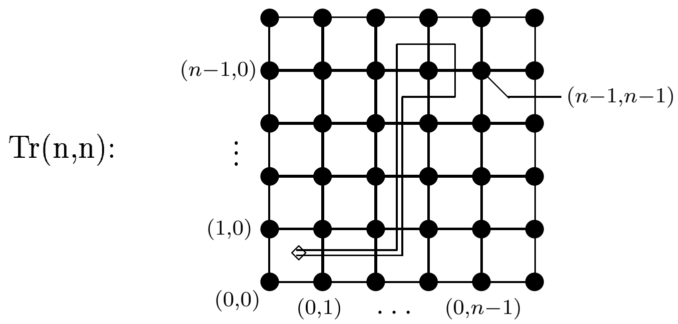


Figure 18: Trivial $n \times n$ - origami. Opposite sides are glued.

By definition affine diffeomorphisms respect the set of cusps. We will choose a suitable partition of the set of cusps. If one restricts to affine diffeomorphisms that respect this partition, i.e. that map each cusp to a cusp in the same class, then their image in $\mathrm{SL}_2(\mathbb{R})$ will be a congruence group of level n. We will consider coverings of $\mathrm{Tr}(n,n)$ that are ramified over the cusps in a way that fits to the partition. The Veech group of such a covering origami will be in general a subgroup of our congruence group. We will find a construction that leads to a covering whose Veech group is the full congruence group.

We start by defining the type of congruence groups that we obtain this way. Let us study congruence groups of level n by their natural action on $(\mathbb{Z}/n\mathbb{Z})^2$.

Definition 6.1. *Let* $B := \{b_1, \ldots, b_k\}$ *be a partition of* $(\mathbb{Z}/n\mathbb{Z})^2$*, i.e.*

$$\bigcup_{i\in\{1,\ldots,k\}} b_i = (\mathbb{Z}/n\mathbb{Z})^2, \quad b_i \cap b_j = \emptyset \ \ \forall\, i \neq j \quad \textit{and} \quad b_i \neq \emptyset \ \ \forall i \in \{1,\ldots,k\}.$$

B defines an equivalence relation on $(\mathbb{Z}/n\mathbb{Z})^2$ *by:*

$$x \sim_B y :\Leftrightarrow x, y \in b_i \textit{ for the same } i \in \{1,\ldots,k\}.$$

We define the stabilizing group *of B in* $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ *resp. the corresponding congruence group of level n as*

$$\begin{aligned} \bar{\Gamma}_B &:= \{\bar{A} \in \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z}) |\ \bar{A}\cdot x \sim_B x \ \ \forall\, x \in (\mathbb{Z}/n\mathbb{Z})^2\} \\ \Gamma_B &:= \textit{the preimage of } \bar{\Gamma}_B \textit{ in } \mathrm{SL}_2(\mathbb{Z}). \end{aligned}$$

Observe that $\bar{\Gamma}_B$ is a subgroup of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ and thus Γ_B is a congruence group of level n. Since for each $\bar{A}$ in $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ the point $(0,0)$ is a fixed point, it is appropriate to consider partitions B with $b_1 = \{(0,0)\}$.

We will prove in Theorem 3 that each Γ_B occurs as Veech group of an origami. The first part of this chapter is dedicated to the proof: We start in Section 6.1 by finding a subgroup N_B of F_2 having infinite index with $\Gamma(N_B) = \Gamma_B$. We use this to state in Corollary 6.9 a condition for subgroups of F_2 to have a Veech group that is contained in Γ_B. In Section 6.2 we show that there exist finite index subgroups fulfilling this condition. Thus they define origamis whose Veech group is a subgroup of Γ_B. Finally, in 6.3, we construct a finite index subgroup W of F_2 such that its Veech group $\Gamma(W)$ is equal to Γ_B. This corresponds by Theorem 1 to an origami O with Veech group $\Gamma(O) = \Gamma_B$.

In the second part of this chapter we investigate which congruence groups we can obtain with this method. In Section 6.4 we show by giving explicitely suitable partitions that the principal congruence groups and other common congruence groups are among them. In 6.5 we point out the relation between stabilizing groups and orbit spaces. We use this in 6.6 in order to characterize for p prime, which subgroups of $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ are stabilizing groups. In particular, if $p > 11$ all subgroups have this property. Thus each congruence group of level p with p prime and greater than 11 is Veech group of some origami, see Theorem 4. In 6.7 we generalize these results to congruence groups of level p^e. If $p > 11$ it is still true for $n = p^e$ $(e \in \mathbb{N})$ that all congruence of level n occur as Veech groups, see Proposition 6.40. Finally, we generalize this statement in 6.8 again and show that it is true for an arbitrary natural number n that is not divisible by 2, 3, 5, 7 or 11, see Theorem 5. More precisely, each congruence group of arbitrary odd level n is a Veech group of some origami, if its image in $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ does not have the same orbit space on $(\mathbb{Z}/n\mathbb{Z})^2$ as the full group $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$, see Theorem 5.

6.1 Infinite index origamis with Veech group Γ_B

Proposition 6.2. *Let $B := \{b_1, \ldots, b_k\}$ be a partition of $(\mathbb{Z}/n\mathbb{Z})^2$ such that $b_1 = \{(0,0)\}$. There is a subgroup N_B of F_2 having infinite index with*

$$\Gamma(N_B) = \Gamma_B.$$

The group N_B will be introduced in Definition 6.4. The proof of Proposition 6.2 is carried out in Lemma 6.7, where we calculate the stabilizer of N_B in $\mathrm{Aut}^+(F_2)$.

But first we need some notations.

Definition 6.3. *We call an element h in F_2* primitive, *if it is not a proper power of an other element, i.e. if there is no $h' \in F_2$ such that $h = h'^k$ with $k \in \mathbb{N}, k \geq 2$. Furthermore, we denote by $\mu(h)$ for an arbitrary element h the biggest natural number such that $h = h'^{\mu(h)}$ for some $h' \in F_2$.*

Thus, if h is parabolic it can be written as $h = v[x,y]^{\mu(h)}v^{-1}$ or $h = v[x,y]^{-\mu(h)}v^{-1}$ for some $v \in F_2$ and the number $\mu(h) \in \mathbb{N}$ is unique with this property.
We define furthermore the subsets C_0 and P_0 and the subgroup N_0 of F_2 as follows:

$$\begin{aligned} C_0 &:= \{w[x,y]w^{-1} \in F_2 | \ w \in F_2\} \\ P_0 &:= \{v[x,y]^k v^{-1} \in F_2 | \ v \in F_2, k \in \mathbb{Z}\} \\ N_0 &:= < C_0 > = < P_0 > \end{aligned}$$

Observe that P_0 is the set of all parabolic elements of F_2. C_0 is the set of all primitive oriented parabolic elements and corresponds to the set of cusps of F_2 on $\mathbb{H}$. N_0 is the kernel of the natural projection $p_{\mathbb{Z}^2} : F_2 \to \mathbb{Z}^2$, i.e. N_0 is the commutator $[F_2, F_2]$ of F_2. We will define the group N_B as a subgroup of N_0 generated by a certain subset of P_0.

We identify the cusps of $\mathrm{Tr}(n,n)$ with the elements in $(\mathbb{Z}/n\mathbb{Z})^2$ as indicated in Figure 18. The partition $B = \{b_1, \ldots, b_k\}$ defines a partition of the cusps. We choose k different natural numbers $r_1, \ldots, r_k$ greater than 1 and associate to each element in the class b_j the number r_j, i.e. we define a map

$$\bar{\alpha} : (\mathbb{Z}/n\mathbb{Z})^2 \to \mathbb{Z} \quad \text{with } s \mapsto r_j \;:\Leftrightarrow\; s \in b_j.$$

Furthermore, we denote

$$\alpha := \bar{\alpha} \circ p_{n,n} : \; F_2 \overset{p_{n,n}}{\to} (\mathbb{Z}/n\mathbb{Z})^2 \overset{\bar{\alpha}}{\to} \mathbb{Z} \quad \text{and} \quad \alpha_w := \alpha(w),$$

where $p_{n,n} : F_2 \to (\mathbb{Z}/n\mathbb{Z})^2$ is the composition of $p_{\mathbb{Z}^2}$ with the natural projection $\mathbb{Z}^2 \to (\mathbb{Z}/n\mathbb{Z})^2$.

Notice that α is not a group homomorphism! It induces a well defined map

$$l_B : C_0 \to \mathbb{Z} \quad \text{by} \quad w[x,y]w^{-1} \mapsto \alpha(w), \tag{6.1}$$

since:

$$\begin{aligned} v[x,y]v^{-1} = w[x,y]w^{-1} &\Rightarrow w^{-1}v \in \,<[x,y]> \\ &\Rightarrow p_{n,n}(v) = p_{n,n}(w) \Rightarrow \alpha(v) = \alpha(w). \end{aligned} \tag{6.2}$$

Note, that l_B is not the restriction of α to C_0! It can be understood as a labeling of the cusps in C_0.

With these notations we define the group N_B as follows.

Definition 6.4. *For α as above let*

$$\begin{aligned} C_\alpha &:= \{w[x,y]^{\alpha_w}w^{-1} \in F_2 | w \in F_2\} \;\subseteq\; P_0 \\ N_B &:= N_\alpha := \,< C_\alpha > \;\subseteq\; N_0 \end{aligned}$$

This definition is motivated as follows: As said above we identify $(\mathbb{Z}/n\mathbb{Z})^2$ with the cusps of $\mathrm{Tr}^*(n,n)$. They are divided into k classes by the partition B. Cusps in the same class are labelled by the same number. The (oriented) parabolic elements $v[x,y]v^{-1}$ in F_2 define loops around these cusps. For each cusp s we take the l-th powers of all loops around s, where l is the label of this cusp. N_B is the group generated by these parabolic elements.

Let $H_{n,n}$ be the Galois group $\mathrm{Gal}(\mathbb{H}/\mathrm{Tr}^*(n,n))$, i.e. $H_{n,n}$ is the kernel of the projection $p_{n,n} : F_2 \to (\mathbb{Z}/n\mathbb{Z})^2$. The set C_α is preserved by conjugation with elements in $H_{n,n}$: If $h \in H_{n,n}$, then $p_{n,n}(h) = (0,0)$. Thus $\alpha(hw) = \bar{\alpha}(p_{n,n}(hw)) = \bar{\alpha}(p_{n,n}(w)) = \alpha(w)$ and we have:

$$hw[x,y]^{\alpha_w}w^{-1}h^{-1} = hw[x,y]^{\alpha_{hw}}(hw)^{-1} \in C_\alpha.$$

Hence N_B is a normal subgroup of $H_{n,n}$.
The parabolic elements in N_B arise as l-th powers of a (maybe multiple) loop around a cusp, where l is the label of the cusp. This is stated in the following lemma.

Lemma 6.5. *Let $h = w[x,y]^k w^{-1}$ be a parabolic element in F_2.*

$$h \text{ is in } N_B \iff k \text{ is divisible by } \alpha_w$$

Proof. "$\Leftarrow$" follows immediately from the definition of N_B.

"$\Rightarrow$": Consider the unramified covering $\mathbb{C}\backslash\mathbb{Z}^2 \to E^*$. It defines an embedding of $\pi_1(\mathbb{C}\backslash\mathbb{Z}^2)$ into $\pi_1(E^*) = F_2$. The Galois group $\mathrm{Gal}(\mathbb{C}\backslash\mathbb{Z}^2/E^*)$ is $\mathbb{Z}^2$, the Galois group $\mathrm{Gal}(\mathbb{H}/(\mathbb{C}\backslash\mathbb{Z}^2))$ is the commutator $[F_2, F_2] = N_0$. Thus $\pi_1(\mathbb{C}\backslash\mathbb{Z}^2)$ is identified via this embedding with $[F_2, F_2] = N_0$. Using this we obtain that N_0 is generated by the loops around the cusps of $\mathbb{C}\backslash\mathbb{Z}^2$, i.e. around the points in the lattice $\mathbb{Z}^2$.
We choose for each cusp $i \in \mathbb{Z}^2$ a fixed loop $v_i[x,y]v_i^{-1}$. Two loops $v[x,y]v^{-1}$ and $v'[x,y]v'^{-1}$ belong to the same cusp iff $vv'^{-1} \in N_0$. Thus fixing v_i for each cusp corresponds to fixing a representative in each coset modulo $N_0 = [F_2, F_2]$. We may assume that $p_{\mathbb{Z}^2}(v_i) = i$ by choosing the base point of $\pi_1(\mathbb{C}\backslash\mathbb{Z}^2)$ suitably.
Then, $N_0 = \pi_1(\mathbb{C}\backslash\mathbb{Z}^2)$ is freely generated by $\{v_i[x,y]v_i^{-1} \in F_2 | \; i \in \mathbb{Z}^2\}$.

For each $j \in \mathbb{Z}^2$ we consider the group homomorphism

$$\varphi_j : N_0 \to (\mathbb{Z}, +), \quad w \mapsto \sharp_{v_j[x,y]v_j^{-1}}(w),$$

that counts the number of occurrences of $v_j[x,y]v_j^{-1}$ in w.
Let us consider an arbitrary parabolic element $v[x,y]^k v^{-1}$. Let $i \in \mathbb{Z}^2$ be the cusp to which $v[x,y]v^{-1}$ belongs, i.e. $i = p_{\mathbb{Z}^2}(v)$ and $vv_i^{-1} \in N_0$. Therefore we have:

$$\begin{aligned} \varphi_j(v[x,y]^k v^{-1}) &= \varphi_j((\underbrace{vv_i^{-1}}_{\in N_0})(v_i[x,y]^k v_i^{-1})\underbrace{(vv_i^{-1})^{-1}}_{\in N_0}) \\ &= \varphi_j(v_i[x,y]^k v_i^{-1}) = \begin{cases} 0, & \text{if } i \neq j \\ k, & \text{if } i = j \end{cases} \end{aligned} \tag{6.3}$$

We now return to the group N_B. For each generator $v[x,y]^{\alpha_v} v^{-1}$ using (6.3) we obtain:

$$\varphi_j(v[x,y]^{\alpha_v} v^{-1}) = \begin{cases} 0, & \text{if } p_{\mathbb{Z}^2}(v) \neq j \\ \alpha_v = \alpha_{v_j} =: \alpha_j, & \text{if } p_{\mathbb{Z}^2}(v) = j \end{cases}$$

Thus, the image $\varphi_j(N_B)$ in $\mathbb{Z}$ is the cyclic group $< \alpha_j >$.

Now, let $h = w[x,y]^k w^{-1}$ be in N_B and $j := p_{\mathbb{Z}^2}(w) \in \mathbb{Z}^2$. Then $\varphi_j(h) \overset{(6.3)}{=} k$ is in $< \alpha_j >$. Thus k is divisible by $\alpha_j = \alpha_{w_j} = \alpha_w$. □

In the next lemma we determine the stabilizer of N_B in $\mathrm{Aut}^+(F_2)$ and prove that its Veech group $\Gamma(N_B)$ is equal to Γ_B.

But first we describe how the partition B on $(\mathbb{Z}/n\mathbb{Z})^2$ defines a partition $\tilde{B}$ of the set of cusps C_0: We have a natural projection

$$C_0 \to (\mathbb{Z}/n\mathbb{Z})^2, \quad w[x,y]w^{-1} \mapsto p_{n,n}(w).$$

This map is well defined by the same argument that we used in (6.2).
We define $\tilde{B} := \{\tilde{b}_1, \dots, \tilde{b}_k\}$ by

$$\tilde{b}_i \text{ is the preimage of } b_i \text{ under the projection } C_0 \to (\mathbb{Z}/n\mathbb{Z})^2.$$

This is equivalent to

$$v[x,y]v^{-1} \text{ and } w[x,y]w^{-1} \text{ are in the same class } \Leftrightarrow \alpha(v) = \alpha(w).$$

Recall that $\mathrm{Aut}^+(F_2)$ acts on C_0. We will consider the group $\mathcal{G}_{\tilde{B}}$ of automorphisms that respect the partition $\tilde{B}$ of C_0 and show in Lemma 6.7 that this is the stabilizing group of N_B. Furthermore we show that its image in $\mathrm{SL}_2(\mathbb{Z})$ is Γ_B.

Definition 6.6. *Let $\mathcal{G}_{\tilde{B}}$ be the subgroup of $\mathrm{Aut}^+(F_2)$ defined as follows:*

$$\begin{aligned}\mathcal{G}_{\tilde{B}} \; &:= \; \{\gamma \in \mathrm{Aut}^+(F_2) \,|\, \forall w \in F_2 : \; \gamma(w[x,y]w^{-1}) \textit{ is in the same class as } \\ &\qquad\qquad\qquad\qquad\qquad\qquad w[x,y]w^{-1} \textit{ in } \tilde{B}\} \\ &= \; \{\gamma \in \mathrm{Aut}^+(F_2) \,|\, \forall w \in F_2 : \; \gamma(w[x,y]w^{-1}) = v[x,y]v^{-1} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \textit{with some } v \in F_2 \textit{ such that } \alpha_w = \alpha_v\}\end{aligned}$$

Lemma 6.7. *The stabilizer of N_B in $\mathrm{Aut}^+(F_2)$ is $\mathcal{G}_{\tilde{B}}$.*
The Veech group $\Gamma(N_B)$ is equal to Γ_B.

Proof. Recall that by definition $N_B = N_\alpha$. We proceed in four steps. We show:

1. $\mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(N_\alpha) = \mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(C_\alpha)$
2. $\mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(C_\alpha) = \mathcal{G}_{\tilde{B}}$.
3. $\gamma \in \mathcal{G}_{\tilde{B}} \Leftrightarrow \; \gamma([x,y]) = v_0[x,y]v_0^{-1}$ with some $v_0 \in H_{n,n}$ and
   $\forall w \in F_2 : \; \alpha(\gamma(w)) = \alpha(w)$.
4. $\hat{\beta}(\mathcal{G}_{\tilde{B}}) = \Gamma_B$.

Step 1: The group N_α is generated by C_α, thus we have:

$$\mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(C_\alpha) \subseteq \mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(N_\alpha).$$

Now, let γ be in $\mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(N_\alpha)$ and $h \in C_\alpha$, i.e. $h = w[x,y]^{\alpha_w}w^{-1}$ for some w in F_2. Since γ preserves parabolic elements and their order μ in F_2, we have $\gamma(h) = v[x,y]^{\alpha_w}v^{-1} \in N_\alpha$ for some v in F_2. By Lemma 6.5 it follows that $\alpha_v|\alpha_w$. Consider $h' := v[x,y]^{\alpha_v}v^{-1} \in C_\alpha$. Then $\gamma(h)$ is a power of h'. Thus h is a power of $\gamma^{-1}(h')$ and $\gamma^{-1}(h') = w[x,y]^{\alpha_v}w^{-1}$ is in N_α. Again by Lemma 6.5 we have $\alpha_w|\alpha_v$. Hence, $\alpha_v = \alpha_w$ and $\gamma(h) = v[x,y]^{\alpha_v}v^{-1}$ is in C_α. Thus γ stabilizes C_α.

Step 2: The claim follows by the following chain of equivalences:

$$\begin{aligned}
& \gamma \in \mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(C_\alpha) \\
\Leftrightarrow\quad & \forall w \in F_2 : \gamma(w[x,y]^{\alpha_w}w^{-1}) = v[x,y]^{\alpha_v}v^{-1} \text{ for some } v \in F_2 \\
\Leftrightarrow\quad & \text{(since } \gamma \text{ preserves (oriented)parabolics and their order } \mu) \\
& \forall w \in F_2 : \gamma(w[x,y]w^{-1}) = v[x,y]v^{-1} \text{ for some } v \text{ with } \alpha_w = \alpha_v \\
\Leftrightarrow\quad & \gamma \in \mathcal{G}_{\tilde{B}}
\end{aligned}$$

Step 3:
Let γ be in $\mathrm{Aut}^+(F_2)$. Choose $v_0 \in F_2$ such that $\gamma([x,y]) = v_0[x,y]v_0^{-1}$. Then we have:

$$\forall w \in F_2 : \gamma(w[x,y]w^{-1}) = \gamma(w)v_0[x,y]v_0^{-1}\gamma(w)^{-1} \tag{6.4}$$

"$\Leftarrow$":
Suppose, γ fulfills the condition on the right hand side of the equivalence. Then

$$\forall w \in F_2 : \alpha(\gamma(w)v_0) = \bar{\alpha}(p_{n,n}(\gamma(w)v_0)) \overset{v_0 \in H_{n,n}}{=} \bar{\alpha}(p_{n,n}(\gamma(w)) = \alpha(\gamma(w)) = \alpha(w).$$

Setting $v := \gamma(w)v_0$ we thus have $\alpha_v = \alpha_w$ and obtain by (6.4):

$$\gamma(w[x,y]w^{-1}) = v[x,y]v^{-1} \text{ with } \alpha_w = \alpha_v$$

Hence, $\gamma \in \mathcal{G}_{\tilde{B}}$.

"$\Rightarrow$": Let γ be in $\mathcal{G}_{\tilde{B}}$.
We have chosen v_0 above such that $\gamma([x,y]) = v_0[x,y]v_0^{-1}$. From the definition of $\mathcal{G}_{\tilde{B}}$ it follows that $\alpha(v_0) = \alpha(\mathrm{id})$. (Observe that if $\alpha(v_0') = \alpha(\mathrm{id})$ for some v_0' fulfilling $\gamma([x,y]) = v_0'[x,y]v_0'^{-1}$, than this is by (6.2) true for all v_0' fulfilling the same property.)
This means by the definition of α that $p_{n,n}(v_0)$ and $p_{n,n}(\mathrm{id})$ are in the same class b_1 of the partition B. But we required for B that $b_1 = \{(0,0)\}$, thus v_0 is in the kernel of $p_{n,n}$, i.e. $v_0 \in H_{n,n}$.
Now, let w be in F_2. Then by the definition of $\mathcal{G}_{\tilde{B}}$ there is some $v \in F_2$ with $\gamma(w[x,y]w^{-1}) = v[x,y]v^{-1}$ and $\alpha_v = \alpha_w$.

$$\begin{aligned}
&\overset{(6.4)}{\Rightarrow}\quad v[x,y]v^{-1} = \gamma(w)v_0[x,y](\gamma(w)v_0)^{-1} \;\Rightarrow\; p_{n,n}(v) = p_{n,n}(\gamma(w)\cdot v_0) \\
&\Rightarrow\quad \alpha(v) = \alpha(\gamma(w)\cdot v_0) \;\Rightarrow\; \alpha(\gamma(w)) \overset{v_0 \in H_{n,n}}{=} \alpha(\gamma(w)\cdot v_0) = \alpha(v) = \alpha(w)
\end{aligned}$$

Thus γ fulfills the required condition.

Step 4: Recall that $\hat{\beta} : \mathrm{Aut}^+(F_2) \to \mathrm{SL}_2(\mathbb{Z})$ is the natural projection. We define $\hat{\beta}_{n,n} : \mathrm{Aut}^+(F_2) \to \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ by composing $\hat{\beta}$ with the natural projection $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$.
Recall that $\hat{\beta}$ is compatible with $p_{\mathbb{Z}^2}$ (see Proposition 3.5). Thus we have for $\gamma \in F_2$ and $\bar{A} := \hat{\beta}_{n,n}(\gamma)$:

$$\forall w \in F_2 : \; p_{n,n}(\gamma(w)) = \bar{A} \cdot p_{n,n}(w) \tag{6.5}$$

We obtain the following chain of equivalences:

$$\begin{array}{rcl}
\gamma \in \hat{\beta}^{-1}(\Gamma_B) & \Leftrightarrow & \bar{A} \in \bar{\Gamma}_B \\
& \Leftrightarrow & \bar{A}\bar{w} \text{ is in the same class of } B \text{ as } \bar{w} \;\; \forall \bar{w} \in (\mathbb{Z}/n\mathbb{Z})^2 \\
& \Leftrightarrow & \bar{\alpha}(\bar{A}\bar{w}) = \bar{\alpha}(\bar{w}) \;\; \forall \bar{w} \in (\mathbb{Z}/n\mathbb{Z})^2 \\
& \Leftrightarrow & \bar{\alpha}(\bar{A}p_{n,n}(w)) = \bar{\alpha}(p_{n,n}(w)) \;\; \forall w \in F_2 \\
& \overset{(6.5)}{\Leftrightarrow} & \bar{\alpha}(p_{n,n}(\gamma(w))) = \bar{\alpha}(p_{n,n}(w)) \;\; \forall w \in F_2 \\
& \Leftrightarrow & \alpha(\gamma(w)) = \alpha(w) \;\; \forall w \in F_2
\end{array}$$

Thus the preimage of Γ_B in $\mathrm{Aut}^+(F_2)$ is

$$\hat{\beta}^{-1}(\Gamma_B) = \{\gamma \in \mathrm{Aut}^+(F_2) |\, \alpha(\gamma(w)) = \alpha(w) \text{ for all } w \in F_2\}.$$

By Step 3 we have therefore:

$$\mathcal{G}_{\tilde{B}} = \{\gamma \in \hat{\beta}^{-1}(\Gamma_B) |\, \gamma([x,y]) = v_0[x,y]v_0^{-1} \text{ with some } v_0 \in H_{n,n}\}.$$

From this it is immediate that $\hat{\beta}(\mathcal{G}_{\tilde{B}}) \subseteq \Gamma_B$.
Furthermore, each element γ' in $\hat{\beta}^{-1}(\Gamma_B)$ is conjugated to an element γ of $\mathcal{G}_{\tilde{B}}$: If $\gamma'([x,y]) = v_0'[x,y]v_0'^{-1}$ then set $\gamma := v_0'^{-1} \cdot \gamma \cdot v_0'$.
Hence, $\hat{\beta}(\mathcal{G}_{\tilde{B}}) = \Gamma_B$.

In Step 1 and Step 2 we have shown that

$$\mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(N_B) = \mathcal{G}_{\tilde{B}}\,.$$

By definition we have $\Gamma(N_B) = \hat{\beta}(\mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(N_B))$, thus by Step 3 and Step 4 it follows that $\Gamma(N_B) = \Gamma_B$. □

Corollary 6.8. *Let U be a subgroup of F_2 with $U \cap N_0 = N_B$. Then*

$$\mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(U) \subseteq \mathcal{G}_{\tilde{B}}.$$

Proof. Since N_0 is characteristic, i.e. its stabilizer is the whole group $\mathrm{Aut}^+(F_2)$, we have:

$$\begin{aligned}\mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(U) &= \mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(U) \cap \mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(N_0) \\ &\overset{\text{Lemma 10}}{\subseteq} \mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(U \cap N_0) \;=\; \mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(N_B) \\ &= \mathcal{G}_{\tilde{B}}.\end{aligned}$$

□

Let P_α be the set of parabolic elements in N_B, i.e.:

$$P_\alpha = \{v[x,y]^k v^{-1} \in F_2 |\, k \in \mathbb{Z}, v \in F_2 \text{ and } k \text{ is divisible by } \alpha_v\}.$$

Corollary 6.9. *If U is a subgroup of F_2 with $U \cap P_0 = P_\alpha$. Then*

$$\mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(U) \subseteq \mathcal{G}_{\tilde{B}} \quad \text{and in particular: } \Gamma(U) \subseteq \Gamma_B.$$

Proof. Let γ be in the stabilizer of U in $\mathrm{Aut}^+(F_2)$. Since γ stabilizes (oriented) parabolic elements, it stabilizes $U \cap P_0 = P_\alpha$. Since γ respects the order μ of parabolic elements, it stabilizes C_α by exactly the same argument as we used in Lemma 6.7 in Step 1. But the stabilizer of C_α is $\mathcal{G}_{\tilde{B}}$. □

6.2 An origami whose Veech group is a subgroup of the congruence group Γ_B

Proposition 6.10. *Let $B := \{b_1, \ldots, b_k\}$ be a partition of $(\mathbb{Z}/n\mathbb{Z})^2$ such that $b_1 = \{(0,0)\}$. There exist origamis whose Veech group is a subgroup of Γ_B.*

The proposition will follow from Lemma 6.11 and Corollary 6.12: We prove that for an origami O, that is obtained as ramified covering of $\mathrm{Tr}(n,n)$ with suitable ramification behaviour, the Veech group $\Gamma(O)$ is a subgroup of Γ_B. We show afterwards that such origamis exist and construct an example.

Lemma 6.11. *Let $O = p : (X^* \to E^*)$ be an origami such that p splits as $p : X^* \xrightarrow{q} Tr^*(n,n) \to E^*$ and the extension $q : X \to Tr(n,n)$ is ramified over the cusps of $Tr(n,n)$ in the following way: For each $x \in b_i$ the preimage $q^{-1}(x)$ consists of $\frac{N}{r_i}$ points with ramification index r_i, where N is the degree of q and the r_i's are defined as in Section 6.1. Then for $U := \mathrm{Gal}(\mathbb{H}/X^*) \subseteq F_2$ we have:*

$$U \cap P_0 = P_\alpha,$$

where α, N_0 and N_α are defined as in Section 6.1.

Using Corollary 6.9 and Theorem 1 one obtains immediately the following corollary.

Corollary 6.12. *In the situation of Lemma 6.11 we have*

$$\Gamma(O) \subseteq \Gamma_B.$$

Proof of Lemma 6.11.
Since $q : X^* \to \mathrm{Tr}^*(n,n)$ is an unramified covering we have:

$$\pi_1(X^*) \cong \mathrm{Gal}(\mathbb{H}/X^*) = U \subseteq H_{n,n} = \mathrm{Gal}(\mathbb{H}/\mathrm{Tr}^*(n,n)) \cong \pi_1(\mathrm{Tr}^*(n,n)).$$

As in Section 6.1 we label the cusps of $\mathrm{Tr}(n,n)$ by the elements of $(\mathbb{Z}/n\mathbb{Z})^2$. Each parabolic element of the form $v[x,y]v^{-1}$ is in $H_{n,n}$. It is a simple loop around one of the cusps. The base point of $\pi_1(\mathrm{Tr}^*(n,n))$ was chosen suitably such that $v[x,y]v^{-1}$ is a simple loop around the cusp i with $i := p_{n,n}(v)$. Lifting this loop to X^* one obtains a path that is not closed. Since all preimages of the cusp i have ramification index r_i, the smallest natural number k such that the loop $v[x,y]^k v^{-1}$ can be lifted to a closed loop on X^* is $k = r_i$. Thus

$$\forall v \in F_2 :\ r_i = \min\{k \in \mathbb{N}_{>0} \mid v[x,y]^k v^{-1} \in U\} \ \text{ with } i := p_{n,n}(v)$$
$$\text{and } v[x,y]^k v^{-1} \in U \Leftrightarrow k \text{ is divisible by } r_i. \tag{6.6}$$

This is equivalent to

$$U \cap P_0 = P_\alpha.$$

□

Origamis as required in Lemma 6.11 do exist. By the Riemann-Hurwitz formula one obtains the following restriction on $r_1, \ldots, r_k$, $|b_1|, \ldots, |b_k|$ and the degree N of the covering $q : X \to \mathrm{Tr}(n,n)$:

$$2g_X - 2 = N(2-2) + S \quad \text{with } S := \sum_{j=1}^{k} |b_j| \cdot \frac{N}{r_j}(r_j - 1) \ \text{ for some } g_X \in \mathbb{N}.$$

The genus of X is then $g_X := \frac{S+2}{2}$.

We give in the following an explicit example.

Example 6.13. If $k > 2$, choose $r_2, \ldots, r_k$ as different prime numbers with $\gcd(|b_j|, r_j) = 1$ for $j \in \{2, \ldots, k\}$. Set $N := r_2 \cdot \ldots \cdot r_k$ and $r_1 := N$.
If $k = 2$, choose two prime numbers p and q with $\gcd(q, n^2-1) = \gcd(p, n^2-1) = 1$ and set $r_1 := q$, $r_2 := q(n^2-1)$ and $N := pq(n^2-1)$.

The surface $\mathrm{Tr}^*(n,n)$ is an elliptic curve with n^2 cusps. Its fundamental group $\pi_1(\mathrm{Tr}^*(n,n))$ is generated by x^n, y^n and the simple loops $l_1, \ldots, l_{n^2}$ around the cusps with the relation

$$[x^n, y^n] l_1 \cdot \ldots \cdot l_{n^2} = \mathrm{id}.$$

We denote in the following for $i \in \{1, \ldots, n^2\}$ $\alpha_i := \bar{\alpha}(i)$, where i is understood as element of $(\mathbb{Z}/n\mathbb{Z})^2$, i.e. α_i is the label of the cusp i. Let $\sigma \in S_N$ be the N-cycle $(1 \ldots N)$. We now define a homomorphism φ to the symmetric group S_N as follows:

$$\begin{aligned} \varphi : \quad & \pi_1(\mathrm{Tr}^*(n,n)) \to S_N\,, \\ & x^n \mapsto \mathrm{id},\ y^n \mapsto \mathrm{id}, \\ & l_i \mapsto \sigma^{\frac{N}{\alpha_i}} \ \text{ for } i \in \{2, \ldots, n^2\}, \quad l_1 \mapsto \sigma^{-(\frac{N}{\alpha_2} + \ldots + \frac{N}{\alpha_k})} \end{aligned}$$

φ is well defined, since $\varphi([x^n, y^n] l_1 \cdot l_2 \cdot \ldots \cdot l_{n^2}) = \mathrm{id}$.
Furthermore, we have $\mathrm{ord}(\varphi(l_i)) = \alpha_i$ for $i \in \{1, \ldots, n^2\}$.
For $i > 1$, this is true by definition.
If $k > 2$, this is true for $i = 1$, since $-(\frac{N}{\alpha_2} + \ldots + \frac{N}{\alpha_{n^2}}) = -(|b_2| \cdot \frac{N}{r_2} + \ldots + |b_k| \cdot \frac{N}{r_k})$ is not divisible by any of the primes $r_2, \ldots, r_k$ and thus has order N.
If $k = 2$, then $\frac{N}{\alpha_2} = \ldots = \frac{N}{\alpha_{n^2}} = \frac{N}{r_2} = p$. Hence $\varphi(l_1) = \sigma^{-p(n^2-1)}$ and has order q.

Let $U \subseteq F_2$ be the kernel of φ and let $O = (p : X^* \to E^*)$ be the origami that corresponds to U. The embedding $U \subseteq \pi_1(\mathrm{Tr}^*(n,n))$ defines a covering $q : X^* \to \mathrm{Tr}^*(n,n)$. Its extension $q : X \to \mathrm{Tr}(n,n)$ has the required ramification behaviour by the definition of φ.

Remark 6.14. The result in Corollary 6.12 could – using terms of ramified coverings – also be described as follows:
Consider an affine diffeomorphism f in $\mathrm{Aff}^+(X^*)$. Since it is affine it descends via p to E^* (see Theorem 1). The covering $\mathrm{Tr}^*(n,n) \to E^*$ is characteristic, thus f descends also via q to some $\bar{f}$ on $\mathrm{Tr}^*(n,n)$. Denote the extension of f to X, resp. of $\bar{f}$ to $\mathrm{Tr}(n,n)$, also by f, resp. by $\bar{f}$. The diffeomorphism $\bar{f}$ acts on the set of cusps of $\mathrm{Tr}(n,n)$, which we identify as before with $(\mathbb{Z}/n\mathbb{Z})^2$. Since $\bar{f}$ is induced by f, it respects the ramification order, i.e. it preserves the partition B on $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. Hence it acts as an element $\bar{A}$ of $\bar{\Gamma}_B$.
The lift of $\bar{f}$ to the universal covering $\mathbb{C}$ of $\mathrm{Tr}(n,n)$ is an affine map of $\mathbb{C}$ which respects the partition of $\mathbb{Z}^2$ that one obtains as preimage of B. One can choose the lift such that 0 is a fixed point.
The derivative $A = \mathrm{der}(f)$ is the derivative of this lift as linear map on $\mathbb{C}$. Hence we have A is in Γ_B.

6.3 An origami with Veech group Γ_B

Theorem 3. *Let* $B := \{b_1, \ldots, b_k\}$ *be a partition of* $(\mathbb{Z}/n\mathbb{Z})^2$ *such that* $b_1 = \{(0,0)\}$. *Then* Γ_B *occurs as Veech group of an origami.*

Proof. We will first define a subgroup W of F_2 of finite index and then show that $\Gamma(W) = \Gamma_B$.

Let $U \subseteq F_2$ be a subgroup of finite index such that $U \cap P_0 = P_\alpha$. The existence of such groups was assured at the end of section 6.2.
Define W in the following way:

$$W := \bigcap_{\gamma \in \mathcal{G}_{\tilde{B}}} \gamma(U) \subseteq F_2.$$

All images $\gamma(U)$ have the same index in F_2. There are only finitely many subgroups of F_2 with this index. Thus W is a subgroup of finite index of F_2.

We now prove that $\mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(W) = \mathcal{G}_{\tilde{B}}$.

(i) "$\subseteq$": Since $W \subseteq U$ we have $P_0 \cap W \subseteq P_0 \cap U = P_\alpha$.
On the other hand $\mathcal{G}_{\tilde{B}}$ is the stabilizer of C_α, thus $\gamma(P_\alpha) = P_\alpha$ for all $\gamma \in \mathcal{G}_{\tilde{B}}$.
$P_\alpha \subseteq U$, therefore $P_\alpha \subseteq \gamma(U)$ for all $\gamma \in \mathcal{G}_{\tilde{B}}$ and we have $P_\alpha \subseteq W$.
Since anyway $P_\alpha \subseteq P_0$, we get $P_\alpha \subseteq W \cap P_0$.
It follows that $P_0 \cap W = P_\alpha$. By Corollary 6.9 we obtain the claim.

(ii) "$\supseteq$": Let γ be in $\mathcal{G}_{\tilde{B}}$ and $w \in W$, i.e. $w \in \gamma'(U)$ for all γ' in $\mathcal{G}_{\tilde{B}}$. Thus $\gamma(w)$ is in $\gamma(\gamma'(U))$ for all $\gamma' \in \mathcal{G}_{\tilde{B}}$ and hence also in W. It follows that γ stabilizes W.

So far, we have found a finite index subgroup W of F_2 such that

$$\mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(W) = \mathcal{G}_{\tilde{B}} \quad \text{and therefore} \quad \Gamma(W) = \hat{\beta}(\mathcal{G}_{\tilde{B}}) \overset{\text{Lemma 6.7}}{=} \Gamma_B.$$

Let $O = (X^* \to E^*)$ be the origami corresponding to W, i.e. $X^* = \mathbb{H}/W$. Then by Theorem 1 one has:

$$\Gamma(O) = \Gamma_B.$$

□

6.4 The principal congruence groups and other examples

In this section we show for some of the most common congruence groups that they occur as Veech groups of origamis. By Theorem 3 we just have to find for

each of them suitable partitions of $(\mathbb{Z}/n\mathbb{Z})^2$, where n is the respective level.
At the end of this section we study the case $n = 2$ and give an example for a subgroup of $\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$ that is not stabilizing group for any partition of $(\mathbb{Z}/2\mathbb{Z})^2$.

We will suppose from now on, without mentioning it explicitly, for all partitions $B = \{b_1, \ldots, b_k\}$ of $(\mathbb{Z}/n\mathbb{Z})^2$ which we consider, that the first class b_1 is equal to $\{(0,0)\}$. Furthermore, we sometimes will denote $\bar{\Gamma}_B$ also as $\bar{\Gamma}(B)$.

Corollary 6.15 (to Theorem 3).
For each $n \in \mathbb{N}$, the principal congruence group $\Gamma(n)$ occurs as Veech group of an origami.

Proof. Define the partition B of $(\mathbb{Z}/n\mathbb{Z})^2$ as follows:

$$B := \{\{i\} \,|\, i \in (\mathbb{Z}/n\mathbb{Z})^2\},$$

i.e. B consists of n^2 classes each containing one element. Then

$$\bar{\Gamma}_B = \{\bar{A} \in \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z}) |\, \bar{A} \cdot x = x \;\; \forall x \in (\mathbb{Z}/n\mathbb{Z})^2\} = \{\,\mathrm{id}\,\}.$$

Thus $\Gamma_B = \Gamma(n)$ and the claim follows immediately from Theorem 3. □

Remark 6.16. In the proof of Corollary 6.15, one could choose as well the partition $B := \{b_1, b_2, b_3, b_4\}$ with:

$$\begin{aligned} b_1 &:= \{(0,0)\},\; b_2 := \{(1,0)\},\; b_3 := \{(0,1)\}, \\ b_4 &:= \{i \in (\mathbb{Z}/n\mathbb{Z})^2 |\, i \notin \{(0,0),(1,0),(0,1)\}\} \end{aligned}$$

We then have

$$\bar{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \bar{\Gamma}_B \Leftrightarrow \bar{A} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \bar{A} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Leftrightarrow \bar{A} = \mathrm{id}.$$

This partition with only four classes might be more efficient for concrete computations. We will return in Section 6.5 to the aspect that the partition B is not uniquely determined.

Similarly as for the principal congruence groups we proceed for the following list of congruence groups of level n:

$$\begin{aligned}
\Gamma_1(n) &:= \{A \in \mathrm{SL}_2(\mathbb{Z}) |\ A \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mod n,\ \ b \in \mathbb{Z}\} \\
\Gamma_0(n) &:= \{A \in \mathrm{SL}_2(\mathbb{Z}) |\ A \equiv \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mod n,\ \ a, b, d \in \mathbb{Z}\} \\
\Gamma_1^{(m)}(n) &:= \{A \in \mathrm{SL}_2(\mathbb{Z}) |\ A \equiv \begin{pmatrix} 1 & b \cdot m \\ 0 & 1 \end{pmatrix} \mod n,\ \ b \in \mathbb{Z}\} \ \ (m \in \mathbb{N}, \gcd(m, n) \neq 1), \\
\Gamma_0^{(m)}(n) &:= \{A \in \mathrm{SL}_2(\mathbb{Z}) |\ A \equiv \begin{pmatrix} a & b \cdot m \\ 0 & d \end{pmatrix} \mod n,\ \ a, b, d \in \mathbb{Z}\} \ \ (m \in \mathbb{N}, \gcd(m, n) \neq 1), \\
\Gamma^S(n) &:= \{A \in \mathrm{SL}_2(\mathbb{Z}) |\ A \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^e \mod n,\ e \in \{1, 2, 3, 4\}\} \\
\Gamma^{-I}(n) &:= \{A \in \mathrm{SL}_2(\mathbb{Z}) |\ A \equiv \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^e \mod n,\ e \in \{1, 2\}\} \\
\Gamma^d(n) &:= \{A \in \mathrm{SL}_2(\mathbb{Z}) |\ A \equiv \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mod n,\ a, d \in \mathbb{Z}\}
\end{aligned}$$

Corollary 6.17 (to Theorem 3).
The congruence groups $\Gamma_1(n)$, $\Gamma_0(n)$, $\Gamma_1^{(m)}(n)$, $\Gamma_0^{(m)}(n)$, $\Gamma^S(n)$, $\Gamma^{-I}(n)$ and $\Gamma^d(n)$ and their conjugates in $\mathrm{SL}_2(\mathbb{Z})$ occur as Veech groups of origamis.

Proof. The case $n = 2$ is shown separately in Example 6.18, see Remark 6.20. Thus, we suppose that $n > 2$.
For any group Γ in the list above, let $\bar{\Gamma}$ denote its image in $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$. Note that Γ is the full preimage of $\bar{\Gamma}$. Therefore, by Theorem 3, we just have to find for each $\bar{\Gamma}$ a partition B of $(\mathbb{Z}/n\mathbb{Z})^2$ such that $\bar{\Gamma}$ is the stabilizer of B.

In the following, we list partitions $B_1, \ldots, B_7$, such that $\bar{\Gamma}_1(n)$, $\bar{\Gamma}_0(n)$, $\bar{\Gamma}_1^{(m)}(n)$, $\bar{\Gamma}_0^{(m)}(n)$, $\bar{\Gamma}^S(n)$, $\bar{\Gamma}^{-I}(n)$, $\bar{\Gamma}^d(n)$ is the stabilizer of $B_1, \ldots, B_7$, respectively.

$$\begin{aligned}
B_1 &:= \{b_1, b_2, b_3\} \text{ with } b_1 := \{(0,0)\},\ b_2 := \{(1,0)\}, \\
&\qquad b_3 := \{i \in (\mathbb{Z}/n\mathbb{Z})^2 |\ i \notin b_1 \cup b_2\} \\
B_2 &:= \{b_1, b_2, b_3\} \text{ with } b_1 := \{(0,0)\}, \\
&\qquad b_2 := \{(x, 0) \in (\mathbb{Z}/n\mathbb{Z})^2 |\ x \neq 0\}, \\
&\qquad b_3 := \{i \in (\mathbb{Z}/n\mathbb{Z})^2 | i \notin b_1 \cup b_2\} \\
B_3 &:= \{b_1, b_2, b_3, b_4\} \text{ with } b_1 := \{(0,0)\},\ b_2 := \{(1,0)\}, \\
&\qquad b_3 := \{(x, 1) \in (\mathbb{Z}/n\mathbb{Z})^2 |\ m \text{ divides } x \text{ in } \mathbb{Z}/n\mathbb{Z}\} \\
&\qquad b_4 := \{i \in (\mathbb{Z}/n\mathbb{Z})^2 | i \notin b_1 \cup b_2 \cup b_3\}
\end{aligned}$$

$$
\begin{aligned}
B_4 \;&:=\; \{b_1,b_2,b_3,b_4\} \text{ with } && b_1 := \{(0,0)\}\ ,\ b_2 := \{(x,0)\in(\mathbb{Z}/n\mathbb{Z})^2 |\, x\neq 0\},\\
&&& b_3 := \{(x,y)\in(\mathbb{Z}/n\mathbb{Z})^2 |\, y\neq 0,\ m|x\ \}\\
&&& b_4 := \{i\in(\mathbb{Z}/n\mathbb{Z})^2 | i\notin b_1\cup b_2\cup b_3\}\\
B_5 \;&:=\; \{b_1,b_2,b_3\} \text{ with } && b_1 := \{(0,0)\}\\
&&& b_2 := \{(1,0)\,,\,(0,1)\,,\,(-1,0)\,,\,(0,-1)\},\\
&&& b_3 := \{i\in(\mathbb{Z}/n\mathbb{Z})^2 | i\notin b_1\cup b_2\}\\
B_6 \;&:=\; \{b_1,b_2,b_3,b_4\} \text{ with } && b_1 := \{(0,0)\},\\
&&& b_2 := \{(1,0)\,,\,(-1,0)\}\ ,\ b_3 := \{(0,1)\,,\,(0,-1)\},\\
&&& b_4 := \{i\in(\mathbb{Z}/n\mathbb{Z})^2 | i\notin b_1\cup b_2\cup b_3\}\\
B_7 \;&:=\; \{b_1,b_2,b_3,b_4\} \text{ with } && b_1 := \{(0,0)\}\ ,\\
&&& b_2 := \{(w,0)\in(\mathbb{Z}/n\mathbb{Z})^2 |\, w \text{ is invertible in } \mathbb{Z}/n\mathbb{Z}\},\\
&&& b_3 := \{(0,w)\in(\mathbb{Z}/n\mathbb{Z})^2 |\, w \text{ is invertible in } \mathbb{Z}/n\mathbb{Z}\},\\
&&& b_4 := \{i\in(\mathbb{Z}/n\mathbb{Z})^2 | i\notin b_1\cup b_2\cup b_3\}
\end{aligned}
$$

We show exemplarily that $\bar{\Gamma}^S(n) = \bar{\Gamma}_{B_5}$ and $\bar{\Gamma}^d(n) = \bar{\Gamma}_{B_7}$.

$\underline{\bar{\Gamma}^S(n) = \bar{\Gamma}_{B_5}}$:

$\bar{\Gamma}^S(n)$ has four elements, since $n > 2$:

$$\bar{\Gamma}^S(n) = \{\ \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\ \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix},\ \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\ \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\}$$

Each element in $\bar{\Gamma}^S(n)$ preserves the classes b_1, b_2 and b_3 of B_5. Thus $\bar{\Gamma}^S(n) \subseteq \bar{\Gamma}_{B_5}$. Now, let $\bar{A}$ be in $\bar{\Gamma}_{B_5}$, i.e. it preserves B_5. Thus we have

$$\bar{A} = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \quad\Rightarrow\quad \begin{pmatrix}a\\ c\end{pmatrix} = \bar{A}\cdot\begin{pmatrix}1\\ 0\end{pmatrix} \in b_2,\ \begin{pmatrix}b\\ d\end{pmatrix} = \bar{A}\cdot\begin{pmatrix}0\\ 1\end{pmatrix} \in b_2.$$

Hence (a,c) and (b,d) are in $\{(1,0),(0,1),(-1,0),(0,-1)\}$. Since the determinant of $\bar{A}$ is equal to 1, the only possible combinations are

$$\bar{A} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\ \bar{A} = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix},\ \bar{A} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\ \bar{A} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

$\underline{\bar{\Gamma}^d(n) = \bar{\Gamma}_{B_7}}$:

Again, one can check that the elements in $\bar{\Gamma}^d(n)$ preserve the classes b_1, b_2, b_3 and b_4 of B_7.
Now, if $\bar{A} = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \bar{\Gamma}^d(n)$, it preserves B_7 and we have:

$$\begin{pmatrix}a\\ c\end{pmatrix} = \bar{A}\cdot\begin{pmatrix}1\\ 0\end{pmatrix} \in b_2,\ \begin{pmatrix}b\\ d\end{pmatrix} = \bar{A}\cdot\begin{pmatrix}0\\ 1\end{pmatrix} \in b_3.$$

Thus $b = c = 0$ and $\bar{A}$ is in $\bar{\Gamma}^d(n)$.

The proof for the other groups in our list is done similarly.
With Theorem 3 we obtain that all groups Γ in the list occur as Veech groups of origamis. By Remark 1.6, this is also true for their conjugates in $\mathrm{SL}_2(\mathbb{Z})$.

□

In the following example we study the case $n = 2$. In particular, we will obtain an example for a subgroup of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ that is not stabilizing group of any partition B of $(\mathbb{Z}/n\mathbb{Z})^2$.

Example 6.18. *($n = 2$)*

$$\begin{aligned}\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z}) = \{\; & \bar{A}_1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; \bar{A}_2 := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; \bar{A}_3 := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \\ & \bar{A}_4 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \bar{A}_5 := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \; \bar{A}_6 := \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad \}\end{aligned}$$

contains 6 elements and is isomorphic to S_3. It has six subgroups, namely the trivial one, three groups of order two, one of order 3 and S_3 itself:

$$\begin{array}{lll l} \bar{\Gamma}_1 & := & \{\mathrm{id}\} & \text{order } 1 \\ \bar{\Gamma}_2 & := & < \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} >, \; \bar{\Gamma}_3 := < \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} >, \; \bar{\Gamma}_4 := < \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} > & \text{order } 2 \\ \bar{\Gamma}_5 & := & \{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\} & \text{order } 3 \\ \bar{\Gamma}_6 & := & \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z}) \;\cong\; S_3 & \text{order } 6 \end{array}$$

The subgroups of order 2, namely $\bar{\Gamma}_2$, $\bar{\Gamma}_3$ and $\bar{\Gamma}_4$, are conjugated: $\bar{\Gamma}_3 = \bar{A}_4 \cdot \bar{\Gamma}_2 \cdot \bar{A}_4^{-1}$ and $\bar{\Gamma}_4 = \bar{A}_3 \cdot \bar{\Gamma}_2 \cdot \bar{A}_3^{-1}$.

Lemma 6.19. *Let $\Gamma_1, \ldots, \Gamma_6$ be the preimages of $\bar{\Gamma}_1, \ldots, \bar{\Gamma}_6$ in $\mathrm{SL}_2(\mathbb{Z})$, i.e. they are the congruence groups of level 2.*

a) Γ_1, Γ_2, Γ_3, Γ_4 and Γ_6 occur as Veech groups of origamis.

b) $\bar{\Gamma}_5$ is not stabilizing group for any partition B of $(\mathbb{Z}/2\mathbb{Z})^2$.

Proof.

a) We use again Theorem 3 and list suitable partitions of $(\mathbb{Z}/2\mathbb{Z})^2$:

$$\begin{array}{rcl} B_1 & := & \{b_1, b_2, b_3, b_4\} \text{ with } b_1 := \{(0,0)\},\ b_2 := \{(0,1)\}, \\ & & \qquad\qquad\qquad\quad b_3 := \{(1,0)\},\ b_4 := \{(1,1)\} \\ B_2 & := & \{b_1, b_2, b_3\} \text{ with } b_1 := \{(0,0)\},\ b_2 := \{(1,0)\},\ b_3 := \{(0,1)\,,\,(1,1)\} \\ B_3 & := & \{b_1, b_2, b_3\} \text{ with } b_1 := \{(0,0)\},\ b_2 := \{(0,1)\},\ b_3 := \{(1,0)\,,\,(1,1)\} \\ B_4 & := & \{b_1, b_2, b_3\} \text{ with } b_1 := \{(0,0)\},\ b_2 := \{(1,1)\},\ b_3 := \{(0,1)\,,\,(1,0)\} \\ B_6 & := & \{b_1, b_2\} \text{ with } b_1 := \{(0,0)\},\ \ b_2 := (\mathbb{Z}/2\mathbb{Z})^2 \backslash \{(0,0)\} \end{array}$$

Then we have for each $i \in \{1,2,3,4,6\}$ that $\bar{\Gamma}_i = \bar{\Gamma}_{B_i}$. By Theorem 3 we have thus that Γ_i is Veech group of some origami.
For Γ_3 and Γ_4 this follows by Remark 1.6 also from the fact that Γ_2 is a Veech group, since Γ_3 and Γ_4 are conjugated to Γ_2. The orbits B_3 and B_4 are the images of B_2 under the matrices $\bar{A}_4$ and $\bar{A}_3$.

b) With B_1, B_2, B_3, B_4 and B_6 we have all partitions of $(\mathbb{Z}/2\mathbb{Z})^2$, whose first class b_1 is $\{(0,0)\}$. $\bar{\Gamma}_5$ is not stabilizer of one of them. This proves the claim.

Observe that the orbit space of $\bar{\Gamma}_5$, i.e. the partition of $(\mathbb{Z}/2\mathbb{Z})^2$ defined by the set of its orbits, is

$$B_{\bar{\Gamma}_5} := \{b_1, b_2\} \text{ with } b_1 := \{(0,0)\},\ b_2 := \{(1,0),\ (0,1),\ (1,1)\},$$

i.e. $B_{\bar{\Gamma}_5} = B_6$ and $\bar{\Gamma}_5$ has the same orbit space as $\bar{\Gamma}_6$. In the next section we will see, that the claim already follows from the fact that $\bar{\Gamma}_5 \neq \bar{\Gamma}_{B_6}$.

□

Remark 6.20. Observe that $\Gamma_1 = \Gamma(2) = \Gamma^{-I}(2) = \Gamma^d(2)$, $\Gamma_2 = \Gamma_1(2) = \Gamma_0(2)$ and $\Gamma_4 = \Gamma^S(2)$. Furthermore, $\Gamma_1^{(m)}(2)$ and $\Gamma_0^{(m)}(2)$ are equal to $\Gamma_0(2) = \Gamma_1(2)$ or to the principal congruence group $\Gamma(2)$, which was already obtained in Corollary 6.15. Thus Lemma 6.19 proves the case $n = 2$ of Corollary 6.17.

Lemma 6.19 implies that Γ_5 cannot be detected as Veech group of an origami with the method that is provided by Theorem 3. Furthermore, the proof of the second part gives an example of two different subgroups of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ (here with $n = 2$), where the orbit spaces are equal. We will see later, in Section 6.6, that for n prime this happens only rarely.

6.5 Relation to Orbit spaces

Because of Theorem 3, we are interested in subgroups $\bar{\Gamma}$ of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$, such that $\bar{\Gamma}$ is stabilizing group for some partition B of $(\mathbb{Z}/n\mathbb{Z})^2$. The partition B is not unique (see e.g. Remark 6.16). But each subgroup $\bar{\Gamma}$ of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ defines in a natural way a partition of $(\mathbb{Z}/n\mathbb{Z})^2$, namely its orbit space $B(\bar{\Gamma})$. $\bar{\Gamma}$ is not always the stabilizing group of its orbit space (compare Lemma 6.19.b), but the orbit space is always the finest of all partitions that are stabilized by $\bar{\Gamma}$ (as will be shown in Corollary 6.24).

In this section we study these relations between subgroups of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ and partitions of $(\mathbb{Z}/n\mathbb{Z})^2$. We show that the stabilizing groups correspond one-to-one to the orbit spaces. This is first described in a more general context.

Definition 6.21. *Let G be a group acting on a set X.*

Let $\mathcal{H}$ be the set of all subgroups of G and
$\mathcal{B}$ the set of all partitions of X.

Between $\mathcal{H}$ and $\mathcal{B}$ we have the maps B and $\bar{\Gamma}$ as follows:

$$\begin{array}{lllllll} B & : & \mathcal{H} \to \mathcal{B}, & \bar{\Gamma} & \mapsto & B(\bar{\Gamma}) & := \{\bar{\Gamma}\cdot x \mid x \in X\} \\ &&&&&& = \text{the } \textit{orbit space} \text{ of } \bar{\Gamma} \\ \bar{\Gamma} & : & \mathcal{B} \to \mathcal{H}, & B = \{b_i : i \in I\} & \mapsto & \bar{\Gamma}(B) & := \{\gamma \in G \mid \gamma b_i = b_i \forall i \in I\} \\ &&&&&& = \text{the } \textit{stabilizing group} \text{ of } B \end{array} \tag{6.7}$$

$\mathcal{H}$ becomes an ordered set by the relation "$\subseteq$". On $\mathcal{B}$ we consider the relation "$\preceq$" defined as follows.

Definition 6.22. *Let B_1, B_2 be partitions of X. We say B_1 is finer than B_2 and denote*

$$\begin{array}{lll} B_1 \preceq B_2 & :\Leftrightarrow & B_1 \textit{ is a subpartition of } B_2, \\ & i.e & \forall x, y \in X : x \sim_{B_1} y \Rightarrow x \sim_{B_2} y. \\ & \Leftrightarrow & \forall\, b_i \in B_1\ \exists\, b'_j \in B_2 \textit{ with } b_i \subseteq b'_j. \end{array}$$

The maps B and $\bar{\Gamma}$ preserve these relations and $\bar{\Gamma} \circ B$ is increasing, whereas $B \circ \bar{\Gamma}$ is decreasing. This is summarized in the following lemma.

Lemma 6.23. *The maps B and $\bar{\Gamma}$ have the following properties:*

(1) $\bar{\Gamma}_1 \subseteq \bar{\Gamma}_2 \Rightarrow B(\bar{\Gamma}_1) \preceq B(\bar{\Gamma}_2) \quad \forall\, \bar{\Gamma}_1, \bar{\Gamma}_2 \in \mathcal{H}$

(2) $B_1 \preceq B_2 \Rightarrow \bar{\Gamma}(B_1) \subseteq \bar{\Gamma}(B_2) \quad \forall\, B_1, B_2 \in \mathcal{B}$

(3) $\bar{\Gamma} \;\subseteq\; \bar{\Gamma}(B(\bar{\Gamma})) \quad \forall \bar{\Gamma} \in \mathcal{H}$

(4) $B(\bar{\Gamma}(B)) \;\preceq\; B \quad \forall B \in \mathcal{B}$

Corollary 6.24. *As consequence of Lemma 6.23 one has:*

(5) $B(\bar{\Gamma})$ *is the finest partition, that is stabilized by* $\bar{\Gamma}$.

(6) $\bar{\Gamma}(B(\bar{\Gamma}(B))) \;=\; \bar{\Gamma}(B) \quad \forall B \in \mathcal{B}$

(7) $B(\bar{\Gamma}(B(\bar{\Gamma}))) \;=\; B(\bar{\Gamma}) \quad \forall \bar{\Gamma} \in \mathcal{H}$

Proof of Corollary 6.24.

(5): Let B be stabilized by $\bar{\Gamma} \overset{\text{Def.}}{\Rightarrow} \bar{\Gamma} \subseteq \bar{\Gamma}(B) \overset{(1)}{\Rightarrow} B(\bar{\Gamma}) \preceq B(\bar{\Gamma}(B)) \overset{(4)}{\preceq} B$.
(6): "$\supseteq$" follows by (3) and "$\subseteq$" follows by (4) and (2).
(7): "$\preceq$" follows by (4) and "$\succeq$" follows by (3) and (1).

□

Proof of Lemma 6.23.

(1): $\bar{\Gamma}_1 \subseteq \bar{\Gamma}_2 \;\Rightarrow\; \forall x \in X: \;\bar{\Gamma}_1 \cdot x \;\subseteq \bar{\Gamma}_2 \cdot x \;\Rightarrow\; B(\bar{\Gamma}_1) \preceq B(\bar{\Gamma}_2)$.

(2): Let $B_1 \preceq B_2$ and let γ be in $\bar{\Gamma}(B_1)$.
We have to show that γ stabilizes B_2, i.e.: $\forall b'_j \in B_2: \;\gamma b'_j = b'_j$.
Let x be in b'_j and b_i the class in B_1 containing x. By the assumption $B_1 \preceq B_2$, we have that $b_i \subseteq b'_j$. Since γ stabilizes B_1, it follows that $\gamma(x) \in b_i \subseteq b'_j$.
Hence, $\gamma b'_j = b'_j$.

(3): Let γ be in $\bar{\Gamma}$ and $b \in B(\bar{\Gamma})$, i.e. $b = \bar{\Gamma} \cdot x$ for some $x \in X$.
Then $\gamma \cdot b = \gamma \cdot (\bar{\Gamma} \cdot x) = (\gamma\bar{\Gamma}) \cdot x = \bar{\Gamma} \cdot x = b \Rightarrow \gamma \in \bar{\Gamma}(B(\bar{\Gamma}))$.

(4): Let b be in $B(\bar{\Gamma}(B))$, i.e. $b = \bar{\Gamma}(B) \cdot x$ for some $x \in X$.
One has to show that $b = \bar{\Gamma}(B) \cdot x \subseteq b_i$ for some $b_i \in B$.
Let b_i be the class in B that contains x. By the definition of $\bar{\Gamma}(B)$, one has $\bar{\Gamma}(B) \cdot x \subseteq b_i$.

□

Now, let us return to the original situation with $X = (\mathbb{Z}/n\mathbb{Z})^2$ and $G = \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$. The statements in Lemma 6.23 and Corollary 6.24 remain true, if we replace $\mathcal{B}$ by $\mathcal{B}_0$, the set of all partitions whose first class b_1 is $\{(0,0)\}$, since $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ acts on $\mathcal{B}_0$.
Furthermore, it is an immediate consequence of Corollary 6.24 that the stabilizing groups in $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ correspond one-to-one to the orbit spaces on $(\mathbb{Z}/n\mathbb{Z})^2$.

Corollary 6.25. *Let*

$$\begin{aligned}\mathcal{B}^{orb} &:= \{B \in \mathcal{B} |\, B \textit{ is orbit space of some subgroup of } \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})\} = B(\mathcal{H}),\\ \mathcal{H}^{stab} &:= \{\bar{\Gamma} \subseteq \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z}) |\, \bar{\Gamma} \textit{ is stabilizing group of a partition } B \textit{ of } (\mathbb{Z}/n\mathbb{Z})^2\}\\ &= \bar{\Gamma}(\mathcal{B}^{orb}),\end{aligned}$$

then the maps $\Gamma : \mathcal{B}^{orb} \to \mathcal{H}^{stab}$ *and* $B : \mathcal{H}^{stab} \to \mathcal{B}^{orb}$ *are bijections and inverse to each other.*

In particular, we obtain the following Corollary.

Corollary 6.26. *Let* $\bar{\Gamma}$ *be a subgroup of* $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$. *Then we have:*

$$\bar{\Gamma} \in \mathcal{H}^{stab} \Leftrightarrow \begin{array}{l}\textit{there is no } \bar{\Gamma}' \neq \bar{\Gamma} \textit{ with the same orbit space as } \bar{\Gamma},\\ \textit{such that } \bar{\Gamma}' \supset \bar{\Gamma}\end{array}$$

In this case, the preimage Γ *of* $\bar{\Gamma}$ *in* $\mathrm{SL}_2(\mathbb{Z})$ *is the Veech group of some origami.*

Proof.

"$\Rightarrow$": If $\bar{\Gamma}'$ has also the orbit space $B(\bar{\Gamma})$, then

$$\bar{\Gamma}' \overset{(3)}{\subseteq} \bar{\Gamma}(B(\bar{\Gamma}')) = \bar{\Gamma}(B(\bar{\Gamma})) \overset{Cor.6.25}{=} \bar{\Gamma}.$$

"$\Leftarrow$": By (3) we have $\bar{\Gamma} \subseteq \bar{\Gamma}(B(\bar{\Gamma}))$.
Furthermore, $\bar{\Gamma}(B(\bar{\Gamma}))$ has by (7) the same orbit space as $\bar{\Gamma}$. Thus, by the assumption, we have that $\bar{\Gamma} = \bar{\Gamma}(B(\bar{\Gamma})) \in \mathcal{H}^{\mathrm{stab}}$.

The last statement follows by Theorem 3. □

Finally, we study subgroups H of G that are not stabilizing groups.

Lemma 6.27. *Suppose that* H *is a subgroup of* G, *that is not a stabilizing group, i.e.* $H \in \mathcal{H} - \mathcal{H}^{stab}$. *Then* $\bar{\Gamma}(B(H))$ *has the following property:*

$$\forall\, x \in X : \quad \exists\, \gamma \neq \mathrm{id} \;\; \textit{in} \;\; \bar{\Gamma}(B(H)) \quad \textit{with} \;\; \gamma(x) \;=\; x.$$

Proof. Recall from Lemma 6.23 that we have: $H \subseteq \bar{\Gamma}(B(H))$. Let now γ_0 be an element of $\bar{\Gamma}(B(H)) - H$. By Corollary 6.24 we have that $B(H) = B(\bar{\Gamma}(B(H)))$. Thus for all $x \in X$ one has that $\gamma_0(x)$ and x are in the same orbit space of H an thus:

$$\exists\, \gamma_x \in H : \;\; \gamma_0(x) \;=\; \gamma_x(x).$$

From this it follows, that

$$\gamma_x^{-1}\gamma_0(x) \;=\; x \quad \text{with} \;\; \gamma_x^{-1}\gamma_0 \;\in\; \bar{\Gamma}(B(H)) \;\; \text{and} \;\; \gamma_x^{-1}\gamma_0 \;\neq\; \mathrm{id}.$$

□

6.6 Congruence groups of prime level

In the following $n =: p$ will be prime. Thus we have $\mathbb{Z}/n\mathbb{Z} = \mathbb{F}_p$ is a field and $\mathrm{SL}_2(\mathbb{F}_p)$ has order $(p-1) \cdot p \cdot (p+1)$.

We show in this section that almost all congruence subgroups of level p occur as Veech groups of origamis, see Theorem 4. To obtain this we will apply Theorem 3. Therefore we study which subgroups of $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ are stabilizing groups of some partition B. The answer to this question is given by Proposition 6.28: Almost all subgroups of $\mathrm{SL}_2(\mathbb{F}_p)$ are stabilizing groups.

As tool we will utilize the properties of orbit spaces and stabilizing groups introduced in Section 6.5, here for the group $G = \mathrm{SL}_2(\mathbb{F}_p)$ acting on the set $X = \mathbb{F}_p^2$. Throughout the rest of the chapter we will use the notations from Section 6.5, in particular the maps B and $\bar{\Gamma}$, see (6.7).

Proposition 6.28. *Let p be prime. The subgroups of $\mathrm{SL}_2(\mathbb{F}_p)$ that are the stabilizing groups of their orbit spaces are characterized as follows.*

a) If p is a prime number with $p > 11$, then all subgroups of $\mathrm{SL}_2(\mathbb{F}_p)$ are stabilizing group of their orbit space, i.e. $\forall\, \bar{\Gamma} \subseteq \mathrm{SL}_2(\mathbb{F}_p):\ \bar{\Gamma} = \bar{\Gamma}(B(\bar{\Gamma}))$.

b) If p is one of the five primes 2, 3, 5, 7 or 11, then a subgroup $\bar{\Gamma}$ of $\mathrm{SL}_2(\mathbb{F}_p)$ is a stabilizing group if its index in $\mathrm{SL}_2(\mathbb{F}_p)$ is different from p.

Proof. The claim is true for $\mathrm{SL}_2(\mathbb{F}_p)$ itself.
For a proper subgroup H of $\mathrm{SL}_2(\mathbb{F}_p)$ with $B(H) \neq B(\mathrm{SL}_2(\mathbb{F}_p))$ we prove the claim in Corollary 6.31.
In Lemma 6.32, we show that if $B(H) = B(\mathrm{SL}_2(\mathbb{F}_p))$, then H has index p in $\mathrm{SL}_2(\mathbb{F}_p)$. Finally, in Proposition 6.33 we obtain that $\mathrm{SL}_2(\mathbb{F}_p)$ has a subgroup of index p iff $p \in \{2, 3, 5, 7, 11\}$. □

As a consequence one obtains the following Corollary.

Corollary 6.29 (to Proposition 6.28).
If $p > 11$, then different subgroups of $\mathrm{SL}_2(\mathbb{F}_p)$ have different orbit spaces on $\mathbb{F}_p^2$.

Proof. Suppose $B(\bar{\Gamma}_1) = B(\bar{\Gamma}_2)$ for two subgroups $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ of $\mathrm{SL}_2(\mathbb{F}_p)$, then

$$\bar{\Gamma}_1 \overset{\text{Prop. 6.28}}{=} \bar{\Gamma}(B(\bar{\Gamma}_1)) = \bar{\Gamma}(B(\bar{\Gamma}_2)) \overset{\text{Prop. 6.28}}{=} \bar{\Gamma}_2.$$

□

We now prove the statements used in the proof of Proposition 6.28.

Lemma 6.30. *Let H be a subgroup of* $\mathrm{SL}_2(\mathbb{F}_p)$ *with the following property:*

$$\forall x \in \mathbb{F}_p^{\,2} : \ \exists A \neq \mathrm{id} \ \text{ in } \ H \ \text{such that } A \cdot x \ = \ x. \tag{6.8}$$

Then H is the full group $\mathrm{SL}_2(\mathbb{F}_p)$.

Proof. By (6.8) we have elements $A_0 \neq \mathrm{id}$ and $A_\infty \neq \mathrm{id}$ in H with

$$A_0 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{and} \qquad A_\infty \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Thus we have

$$A_0 \ = \ \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \in H \quad \text{and} \quad A_\infty \ = \ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in H \quad \text{for certain } b, c \in \mathbb{F}_p^\times.$$

In particular,

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ are in } H.$$

It follows that: $H = \mathrm{SL}_2(\mathbb{F}_p)$ (see e.g. [DSV 03, Lemma 3.2.1]). □

From this we obtain the following Corollary.

Corollary 6.31. *Let H be a proper subgroup of* $\mathrm{SL}_2(\mathbb{F}_p)$ *that does not have the same orbit space as* $\mathrm{SL}_2(\mathbb{F}_p)$, *i.e.*

$$B(H) \neq B_0 := \{b_1, b_2\} \ \text{with} \qquad b_1 := \{(0,0)\}\,,\ b_2 := \mathbb{F}_p^{\,2} \backslash \{(0,0)\}.$$

Then H is the stabilizing group of its orbit space.

Proof. Let $\bar{\Gamma} := \bar{\Gamma}(B(H))$ be the stabilizing group of the orbit space of H and suppose $\bar{\Gamma} \neq H$. By Lemma 6.27 if follows that $\bar{\Gamma}$ fulfills (6.8). Thus we have by Lemma 6.30 that $\bar{\Gamma} = \mathrm{SL}_2(\mathbb{F}_p)$. It follows that

$$B(H) \overset{\text{Cor. 6.24}}{=} B(\bar{\Gamma}(B(H))) = B(\bar{\Gamma}) = B(\mathrm{SL}_2(\mathbb{F}_p)) = B_0.$$

□

Lemma 6.32. *Let H be a proper subgroup of* $\mathrm{SL}_2(\mathbb{F}_p)$ *with the same orbit space as* $\mathrm{SL}_2(\mathbb{F}_p)$. *Then H has $p^2 - 1$ elements and is a subgroup of index p.*

Proof. By the assumption, the orbit space of H consists only of two orbits: $b_1 := \{(0,0)\}$ and $b_2 := \mathbb{F}_p^{\,2} \backslash \{(0,0)\}$. Thus, H acts transitively on b_2 and its order is divisible by $p^2 - 1 = |b_2|$. But H is a proper subgroup of $\mathrm{SL}_2(\mathbb{F}_p)$ which has order $p(p^2 - 1)$. Hence the order of H is $p^2 - 1$ and its index in $\mathrm{SL}_2(\mathbb{F}_p)$ is p.

□

Proposition 6.33. $\mathrm{SL}_2(\mathbb{F}_p)$ *has a subgroup of index* p *iff* $p \in \{2,3,5,7,11\}$.

Proof. The case $p = 2$ was already treated in Example 6.18. We obtained the subgroup $\bar{\Gamma}_5$ of index 2 in $\mathrm{SL}_2(\mathbb{F}_2)$.

Let now p be greater than 2. Suppose, $\mathrm{SL}_2(\mathbb{F}_p)$ has a subgroup H of index p. We use the theorem of Dickson, that classifies the subgroups of $\mathrm{PSL}_2(\mathbb{F}_p)$ (see e.g. [Hu 67, 8.27, p.213]).
Let $\overline{H}$ be the image of H in $\mathrm{PSL}_2(\mathbb{F}_p)$. Since p is odd, the index of $\overline{H}$ in $\mathrm{PSL}_2(\mathbb{F}_p)$ is again p. Thus, the order of $\overline{H}$ is $\frac{1}{2}(p-1)(p+1)$. In particular, this excludes all subgroups of $\mathrm{PSL}_2(\mathbb{F}_p)$ whose order is divisible by p and those whose order is smaller or equal to $\frac{1}{2}(p+1)$, since $p > 2$. In the list of Dickson that gives all subgroups of $\mathrm{PSL}_2(\mathbb{F}_p)$ there remain the following possibilities for $\overline{H}$:

1. Dihedral groups of order $2k$ with k a divisor of $\frac{p\pm1}{2}$.
2. The alternating group A_4.
3. The symmetric group S_4 for $p^2 \equiv 1 \mod 16$.
4. The alternating group A_5 for $p = 5$ or $p^2 \equiv 1 \mod 5$.

Let h be the order of $\overline{H}$, i.e. $h = \frac{1}{2}(p^2-1)$.

In the first case $\frac{h}{2}$ is a divisor of $\frac{p-1}{2}$ or $\frac{p+1}{2}$. This is fulfilled for $p = 3$.
For $p > 3$ one has:

$$\frac{h}{2} = \frac{(p-1)(p+1)}{4} \geq p+1 > \frac{p+1}{2} > \frac{p-1}{2}.$$

Hence, $\frac{h}{2}$ cannot be a divisor of $\frac{p\pm1}{2}$.

Since A_4 has order 12, $\overline{H} \cong A_4$ is possible only if $\frac{1}{2}(p^2-1) = 12$, i.e. $p = 5$.
Similarly, we obtain in the third case $p = 7$ and in the fourth case $p = 11$.

Conversely, if $p \in \{2,3,5,7,11\}$ one has a subgroup of index p: The theorem of Dickson assures that all groups in the list occur as subgroups of $\mathrm{PSL}_2(\mathbb{F}_p)$. Thus we obtain $\overline{H}$ of index p and define H as its preimage in $\mathrm{SL}_2(\mathbb{F}_p)$. □

By Proposition 6.28 and Theorem 3, we obtain immediately the following result.

Theorem 4. *Let* p *be prime.*

a) If p *is not in* $\{2,3,5,7,11\}$, *then all congruence groups of level* p *occur as Veech groups of origamis.*

b) If p *is in* $\{2,3,5,7,11\}$, *then all congruence groups of level* p *that have index different from* p *in* $\mathrm{SL}_2(\mathbb{Z})$ *occur as Veech groups of origamis.*

6.7 Congruence groups of level p^e

In this section we generalize the results of the last section to powers of odd primes and obtain that for almost all primes p all congruence groups of level p^e $(e \in \mathbb{N})$ occur as Veech groups of origamis, see Proposition 6.40.

We proceed similarly as in the last section: We study first which subgroups of $\mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})$ are stabilizing groups and obtain Proposition 6.34 as generalization of Proposition 6.28. The proof of the proposition is given by the lemmas and corollaries afterwards. Finally, we use Theorem 3 in order to obtain the statement on Veech groups in Proposition 6.40.

Proposition 6.34. *Let p be an odd prime and e be a natural number.*

a) If $p > 11$, then all subgroups of $\mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})$ are stabilizing group of their orbit space.

b) If $p \leq 11$, then all subgroups of $\mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})$ that do not have the same orbit space as $\mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})$ are stabilizing groups.

Proof. The claim is true for $\mathrm{SL}_2(\mathbb{F}_p)$ itself.
b) follows immediately from Corollary 6.38 below.
a) follows from Lemma 6.39 below together with again Corollary 6.38. □

As a consequence one obtains the following corollary.

Corollary 6.35 (to Proposition 6.34).
If $p > 11$, then different subgroups of $\mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})$ have different orbit spaces on $(\mathbb{Z}/p^e\mathbb{Z})^2$.

Proof. The proof works the same way as that of Corollary 6.29. □

We now provide the tools we need in order to prove the statements we have used in the proof of Proposition 6.34.

Let throughout the section p be an odd prime and e be a natural number. If $e \geq 2$, one has the natural projection

$$\beta := \beta^{p^e}_{p^{e-1}} : \ \mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z}) \ \rightarrow \ \mathrm{SL}_2(\mathbb{Z}/p^{e-1}\mathbb{Z}). \tag{6.9}$$

Let K be the kernel of β.

The morphism β is compatible with the actions of $\mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})$ and $\mathrm{SL}_2(\mathbb{Z}/p^{e-1}\mathbb{Z})$ on $(\mathbb{Z}/p^e\mathbb{Z})^2$ and $(\mathbb{Z}/p^{e-1}\mathbb{Z})^2$, respectively, i.e.:

$\forall\, A \in \mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})$: The diagram

$$\begin{array}{ccc} (\mathbb{Z}/p^e\mathbb{Z})^2 & \xrightarrow{x \mapsto A\cdot x} & (\mathbb{Z}/p^e\mathbb{Z})^2 \\ \downarrow & & \downarrow \\ (\mathbb{Z}/p^{e-1}\mathbb{Z})^2 & \xrightarrow{\bar{x} \mapsto \beta(A)\cdot \bar{x}} & (\mathbb{Z}/p^{e-1}\mathbb{Z})^2 \end{array} \quad \text{is commutative.} \tag{6.10}$$

The following lemma provides a tool in order to "lift" the results on subgroups of $\mathrm{SL}_2(\mathbb{F}_p)$ to $\mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})$. Its proof was explained to me by Stefan Kühnlein.

Lemma 6.36. *Let $e \geq 2$ and let H be a subgroup of $\mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})$ such that:*

(1) $\beta(H) = \mathrm{SL}_2(\mathbb{Z}/p^{e-1}\mathbb{Z})$, with β as in (6.9) and

(2) $H \cap K \neq \{\mathrm{id}\}$, where K is the kernel of β as above.

Then H is actually the whole group $\mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})$.

Proof. By the assumption (1) it remains to show, that the kernel K of β is contained in H.
The kernel K is given as:

$$K = \{ \begin{pmatrix} 1 + a'p^{e-1} & b'p^{e-1} \\ c'p^{e-1} & d \end{pmatrix} \mid a', b', c' \in \{1, \ldots, p\},\ d = 1 - a'p^{e-1} \}.$$

Hence the order of K is p^3 and we have an isomorphism of groups:

$$(\mathbb{F}_p^3, +) \;\to\; K \subseteq \mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z}), \qquad \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} \;\mapsto\; \begin{pmatrix} 1 + a'p^{e-1} & b'p^{e-1} \\ c'p^{e-1} & 1 - a'p^{e-1} \end{pmatrix} \tag{6.11}$$

It is seen immediately that this map is bijective and one can check by a short computation that the map is in fact a homomorphism of groups. In the following we identify these two groups via this isomorphism and write $\mathbb{F}_p^3 = K$.

Since K is normal in $\mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})$, the group $\mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})$ acts on $K = \mathbb{F}_p^3$ by conjugation:

$$\rho:\ \mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z}) \;\to\; \mathrm{Aut}(K) = \mathrm{GL}_3(\mathbb{F}_p), \quad A \;\mapsto\; (B \mapsto ABA^{-1}).$$

Since $K = \mathbb{F}_p^3$ is abelian, K acts as subgroup of $\mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})$ trivially on itself, thus ρ induces an action $\bar{\rho}$ of $\mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})/K \cong \mathrm{SL}_2(\mathbb{Z}/p^{e-1}\mathbb{Z})$ on K:

$$\bar{\rho}:\ \mathrm{SL}_2(\mathbb{Z}/p^{e-1}\mathbb{Z}) \;\to\; \mathrm{Aut}(K) = \mathrm{GL}_3(\mathbb{F}_p).$$

Let us consider the restriction $\rho|_H : H \to \mathrm{Aut}(K)$ of ρ to H. The subgroup $H \cap K$ of K is invariant under $\rho|_H$.
Again since K is abelian, $H \cap K \subseteq H$ acts trivially on K and we obtain the induced action of $H/H \cap K$ on K that still leaves $H \cap K$ invariant. But by assumption (1), $H/H \cap K$ is isomorphic to the full group $\mathrm{SL}_2(\mathbb{Z}/p^{e-1}\mathbb{Z})$. Thus the induced action is actually equal to $\bar{\rho}$ and $H \cap K$ is invariant under the action $\bar{\rho}$. We may consider $H \cap K$ as invariant subspace of the vector space $K = \mathbb{F}_p^3$.

We will show next, that the action $\bar{\rho}$ is irreducible, i.e. there are no nontrivial proper subspaces of $\mathbb{F}_p^3$ that are invariant. Since $H \cap K$ is nontrivial by assumption (2), it follows then, that $H \cap K = K$ and we obtain the claim by the assumption (1).

Suppose that we have a nontrivial subspace U of $\mathbb{F}_p^3 = K$ that is invariant under the action of $\bar{\rho}$. It is then in particular invariant under the action of any element, e.g. the element $\bar{T} := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}/p^{e-1}\mathbb{Z})$. Let us study the action of $\bar{T}$ on $K = \mathbb{F}_p^3$.

The standard basis of $\mathbb{F}_p^3$ is sent by the isomorphism (6.11) to

$$b_1 := \begin{pmatrix} 1+p^{e-1} & 0 \\ 0 & 1-p^{e-1} \end{pmatrix}, \quad b_2 := \begin{pmatrix} 1 & p^{e-1} \\ 0 & 1 \end{pmatrix}, \quad b_3 := \begin{pmatrix} 1 & 0 \\ p^{e-1} & 1 \end{pmatrix} \in K.$$

The action of $\bar{T}$ on this basis is given as follows:

$$\begin{aligned} \rho(\bar{T}) \cdot b_1 &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1+p^{e-1} & 0 \\ 0 & 1-p^{e-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1+p^{e-1} & -2p^{e-1} \\ 0 & 1-p^{e-1} \end{pmatrix} = (1,-2,0)^t \quad \text{as element in } \mathbb{F}_p^3. \end{aligned}$$

In the same way one obtains:

$$\rho(\bar{T}) \cdot b_2 = (0,1,0)^t \in \mathbb{F}_p^3 \quad \text{and} \quad \rho(\bar{T}) \cdot b_3 = (1,-1,1)^t \in \mathbb{F}_p^3.$$

Thus we have:

$$\rho(\bar{T}) = \rho(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) = \begin{pmatrix} 1 & 0 & 1 \\ -2 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} =: \mathcal{A} \quad \text{as element of } \mathrm{GL}_3(\mathbb{F}_p).$$

Observe that the two subspaces

$$U_1 := \mathbb{F}_p \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad U_2 := \mathbb{F}_p \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \oplus \mathbb{F}_p \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

are invariant under the action of $\mathcal{A}$. Since the Jordan canonical form of $\mathcal{A}$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

there are only two proper nontrivial subspaces of $\mathbb{F}_p^3$ which are invariant under $\mathcal{A}$. Hence, with U_1 and U_2 we have found them all. Thus $U = U_1$, $U = U_2$ or $U = \mathbb{F}_p^3$. In all three cases the vector $(0,1,0)^t$ is contained in U.

Now, let us additionally consider the action of the element $\bar{R} := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Proceeding as above for $\bar{T}$, one obtains:

$$\rho(\bar{R}) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix} =: \mathcal{B}.$$

Since U is supposed to be invariant under the whole action of $\mathrm{SL}_2(\mathbb{Z}/p^{e-1}\mathbb{Z})$ and $(0,1,0)^t$ is in U, we have:

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathcal{B} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathcal{B}^2 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix} \quad \text{are in } U.$$

But the span of these three vectors is the whole $\mathbb{F}_p^3$. Thus we obtain $U = \mathbb{F}_p^3$. Hence, we have shown that $\bar{\rho}$ is irreducible. ☐

Lemma 6.37. *Let H be a subgroup of $\mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})$ with the following property:*

$$\forall\, x \in (\mathbb{Z}/p^e\mathbb{Z})^2 : \ \exists\, A \neq \mathrm{id} \in H \text{ such that } A \cdot x = x. \tag{6.12}$$

Then H is the full group $\mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})$.

Proof. We make induction on e: By the proof of Lemma 6.31 the claim is true for $e = 1$. Suppose now, that $e \geq 2$ and that the claim is true for $e - 1$.

Let $\overline{H} := \beta(H)$. By (6.10) the property (6.12) "descends" to $\overline{H}$, i.e.:

$$\forall \bar{x} \in (\mathbb{Z}/p^{e-1}\mathbb{Z})^2 : \ \exists\, \bar{A} \in \overline{H} \ \text{ with } \ \bar{A} \cdot \bar{x} = \bar{x}.$$

By the assumption of the induction it follows that $\beta(H) = \overline{H} = \mathrm{SL}_2(\mathbb{Z}/p^{e-1}\mathbb{Z})$.

Next, we show that the intersection of the kernel K of β and H contains a nontrivial element:
By (6.12) we have some element $A \neq \mathrm{id}$ in H, that fixes the point $(1,0)$, i.e.

$$A = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad \text{with some } b \neq 0 \text{ in } \mathbb{Z}/p^e\mathbb{Z}.$$

We write $b = b' \cdot p^k$ with b' and p coprime and $k \leq e-1$. Then there exists some $z \in \mathbb{Z}/p^e\mathbb{Z}$ such that $b' \cdot z = 1$ and thus

$$A_1 := A^{p^{e-1-k}\cdot z} = \begin{pmatrix} 1 & b' \cdot p^k \cdot p^{e-1-k} \cdot z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & p^{e-1} \\ 0 & 1 \end{pmatrix} \in K \cap H.$$

Hence we have $K \cap H$ is nontrivial and $\beta(H) = \mathrm{SL}_2(\mathbb{Z}/p^{e-1}\mathbb{Z})$. It follows by Lemma 6.36, that $H = \mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})$. □

Corollary 6.38. *Let H be a subgroup of $\mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})$ and $e \geq 1$. Then H is either the stabilizing group of its orbit space, or it has the same orbit space as $\mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})$.*

Proof. The proof works the same way as that of Corollary 6.31. □

Lemma 6.39. *Let p be bigger than 11 and $e \geq 1$. If H is a proper subgroup of $\mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})$, then it does not have the same orbit space as $\mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})$.*

Proof. We proceed similarly as in Lemma 6.37, making induction on e again. The case $e = 1$ was shown in Lemma 6.32 and Proposition 6.33. Thus let us suppose now that $e \geq 2$ and the claim is true for $e-1$.

Suppose furthermore that H has the same orbit space on $(\mathbb{Z}/p^e\mathbb{Z})^2$ as $\mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})$, i.e.

$$B(H) = B(\mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})). \tag{6.13}$$

This property "descends" by (6.10) to $(\mathbb{Z}/p^{e-1}\mathbb{Z})^2$, i.e. we have:

$$B(\overline{H}) = B(\mathrm{SL}_2(\mathbb{Z}/p^{e-1}\mathbb{Z})), \qquad \text{with } \overline{H} = \beta(H).$$

Thus by the assumption of the induction it follows that $\overline{H} = \mathrm{SL}_2(\mathbb{Z}/p^{e-1}\mathbb{Z})$.

Next, we show that $H \cap K$ is nontrivial:

Suppose that this is not true.
By (6.17) there is some element A in H with

$$A \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 + p^{e-1} \\ 0 \end{pmatrix}.$$

Thus we have

$$A = \begin{pmatrix} 1 + p^{e-1} & b \\ 0 & 1 - p^{e-1} \end{pmatrix} \in H \quad \text{with some } b \in \mathbb{Z}/p^e\mathbb{Z}.$$

By the assumption that $K \cap H$ is trivial, we have that b is not divisible by p^{e-1}. Thus we may write $b = b' \cdot p^r$ with b' and p coprime and $r \leq e-2$.

Claim: For all $n \in \mathbb{N}$ we have:

$$A^n = \begin{pmatrix} 1+np^{e-1} & nb'p^r \\ 0 & 1-np^{e-1} \end{pmatrix}.$$

This follows by induction on n. The statement is true for $n = 0$. If $n \geq 1$ and the claim it is true for $n-1$, we obtain:

$$\begin{aligned} A^n &= \begin{pmatrix} 1+(n-1)p^{e-1} & (n-1)b'p^r \\ 0 & 1-(n-1)p^{e-1} \end{pmatrix} \cdot \begin{pmatrix} 1+p^{e-1} & b'p^r \\ 0 & 1-p^{e-1} \end{pmatrix} \\ &\overset{e\geq 1}{=} \begin{pmatrix} 1+np^{e-1} & b'p^r + (n-1)b'p^{e-1}p^r + (n-1)b'p^r - (n-1)b'p^r p^{e-1} \\ 0 & 1-np^{e-1} \end{pmatrix} \\ &= \begin{pmatrix} 1+np^{e-1} & nb'p^r \\ 0 & 1-np^{e-1} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z}). \end{aligned}$$

Set $n := p^{e-1-r}$. Then n is a natural number divisible by p, since $r \leq e-2$, and we have:

$$A^n = \begin{pmatrix} 1 & b'p^{e-1} \\ 0 & 1 \end{pmatrix}$$

Thus A^n is a nontrivial element in K. Recall that A was in H, hence we have $H \cap K \neq \{\mathrm{id}\}$.

Now we may again use Lemma 6.36 to finish the proof. □

Let us finally turn to congruence groups of level p^e that occur as Veech groups. Using Theorem 3 we obtain from Proposition 6.34 the following proposition.

Proposition 6.40. *Let p be an odd prime and $e \in \mathbb{N}$.*

a) Each congruence group Γ of level p^e, whose image $\bar{\Gamma}$ in $\mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})$ does not have the same orbit space as $\mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})$, is the Veech group of some origami.

b) If $p > 11$, then each congruence group of level p^e is the Veech group of some origami.

6.8 Congruence groups of level n

In this section we finally turn to congruence groups of arbitrary level n. In fact it will be necessary to restrict to odd numbers n. We then may generalize the results of the last section and obtain in Theorem 5 that "most" of the congruence groups of level n are Veech groups. If n is not divisible by our five exceptional primes, then we actually obtain all congruence groups of level n.

We start again, similarly as in the last two sections, by stating in Proposition 6.41 which stabilizing groups in $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ are obtained from generalizing Proposition 6.34 and give proofs for the needed statements afterwards. Finally, we deduce Theorem 5 from the proposition.

Proposition 6.41. *Let n be an odd natural number.*

a) If n is not divisible by 3, 5, 7 or 11 then each subgroup of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ is stabilizing group of its orbit space.

b) In general all subgroups of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ that do not have the same orbit space as $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ are stabilizing groups.

Proof. The claim is true for $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ itself.
b) follows immediately from Corollary 6.45 below.
a) follows from Lemma 6.46 below together with again Corollary 6.45. □

As a consequence one obtains, similarly as in the two previous sections, the following corollary.

Corollary 6.42 (to Proposition 6.41).
Suppose n is a natural number that is not divisible by 2, 3, 5, 7 or 11. Then different subgroups of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ have different orbit spaces on $(\mathbb{Z}/n\mathbb{Z})^2$.

Proof. One may proceed in the same way as for the proof of Corollary 6.29. □

We provide now the statements that we used in the poof of Proposition 6.41. Thus let n be an odd natural number and consider its decomposition into prime numbers $n = p_1^{e_1} \cdot \ldots \cdot p_r^{e_r}$ (p_i prime, $e_i \in \mathbb{N}$, $i \in \{1, \ldots, r\}$) with $p_1 < \ldots < p_r$.

For each $i \in \{1, \ldots, r\}$, let

$$\beta_i : \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/p_i^{e_i}\mathbb{Z})$$

be the natural projection.

The product morphism $\beta := \beta_1 \times \ldots \times \beta_r$ is an isomorphism:

$$\beta : \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/{p_1}^{e_1}) \times \ldots \times \mathrm{SL}_2(\mathbb{Z}/{p_r}^{e_r}). \qquad (6.14)$$

This follows, since we have by the Chinese remainder theorem the isomorphism: $\psi : \mathbb{Z}/p_1^{e_1}\mathbb{Z} \times \ldots \times \mathbb{Z}/p_r^{e_r}\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. We obtain the inverse map of β as

$$\beta^{-1} : (\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \ldots, \begin{pmatrix} a_r & b_r \\ c_r & d_r \end{pmatrix}) \mapsto \begin{pmatrix} \psi(a_1, \ldots, a_r) & \psi(b_1, \ldots, b_r) \\ \psi(c_1, \ldots, c_r) & \psi(d_1, \ldots, d_r) \end{pmatrix}$$

The determinant of the image is 1, again by the Chinese remainder theorem.

In the following we identify these two groups and write

$$\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z}/p_1^{e_1}) \times \ldots \times \mathrm{SL}_2(\mathbb{Z}/p_r^{e_r}).$$

For every $i \in \{1, \ldots, r\}$ the projection β_i is compatible with the actions of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ and $\mathrm{SL}_2(\mathbb{Z}/p_i^{e_i}\mathbb{Z})$ on $(\mathbb{Z}/n\mathbb{Z})^2$ and $(\mathbb{Z}/p_i^{e_i}\mathbb{Z})^2$, respectively, i.e.:

$\forall A \in \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$: The diagram

$$\begin{array}{ccc} (\mathbb{Z}/n\mathbb{Z})^2 & \xrightarrow{x \mapsto A\cdot x} & (\mathbb{Z}/n\mathbb{Z})^2 \\ \downarrow & & \downarrow \\ (\mathbb{Z}/p_i^{e_i}\mathbb{Z})^2 & \xrightarrow{\bar{x} \mapsto \beta_i(A)\cdot\bar{x}} & (\mathbb{Z}/p_i^{e_i}\mathbb{Z})^2 \end{array} \quad \text{is commutative.} \tag{6.15}$$

In the following we show that only the full group $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ projects surjectively onto all factors $\mathrm{SL}_2(\mathbb{Z}/p_i^{e_i}\mathbb{Z})$.

Lemma 6.43. *Let H be a subgroup of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ with:*

$$\beta_i(H) = \mathrm{SL}_2(\mathbb{Z}/p_i^{e_i}\mathbb{Z}) \ \text{ for all } i \in \{1, \ldots, r\}.$$

Then H is the full group $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$.

Proof. We make induction on r. For $r = 1$ the claim is true. Let us suppose that $r \geq 2$ and the claim is true for $r - 1$.

Let $m := p_1^{e_1} \cdot \ldots \cdot p_{r-1}^{e_{r-1}}$ and

$$\beta_a : \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z}) \ \text{ with } \beta_a := \beta_1 \times \ldots \times \beta_{r-1}.$$

Consider the image $\overline{H} := \beta_a(H)$. By the assumption of the lemma, the image of $\overline{H}$ under the natural projection $\mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/p_i^{e_i}\mathbb{Z})$ is the whole group $\mathrm{SL}_2(\mathbb{Z}/p_i^{e_i}\mathbb{Z})$ for all $i \in \{1, \ldots, r-1\}$.

Thus by the assumption of the induction it follows that $\beta_a(H) = \overline{H} = \mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z})$ and we have the following exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \{I\} \times \mathrm{SL}_2(\mathbb{Z}/p_r^{e_r}) & \longrightarrow & \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z}) & \xrightarrow{\beta_a} & \mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z}) & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \| & & \\ 1 & \longrightarrow & H \cap (\{I\} \times \mathrm{SL}_2(\mathbb{Z}/p_r^{e_r})) & \longrightarrow & H & \longrightarrow & \mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z}) & \longrightarrow & 1 \end{array}$$

Diagram 19

We will show that $\{I\} \times \mathrm{SL}_2(\mathbb{Z}/p_r^{e_r})$ actually is contained in H and therefore

$H = \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$.

The map $\beta_r : H \to \mathrm{SL}_2(\mathbb{Z}/p_r^{e_r}\mathbb{Z})$ is also surjective. Hence, there exist elements A and B in H with

$$\beta_r(A) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \beta_r(B) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \text{ in } \mathrm{SL}_2(\mathbb{Z}/p_r^{e_r}\mathbb{Z}).$$

Thus we may denote

$$A = (A_1, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) \text{ and } B = (B_1, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}) \text{ in } \mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}/p_r^{e_r}\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z}),$$

with A_1, B_1 in $\mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z})$.

Let k be the order of A_1. Then k is a divisor of

$$|\mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z})| = \Pi_{i=1}^{r-1}|\mathrm{SL}_2(\mathbb{Z}/p_i^{e_i}\mathbb{Z})| = \Pi_{i=1}^{r-1} p_i^{3e_i-2}(p_i - 1)(p_i + 1).$$

Recall that we have $2 < p_i < p_r$ for all $i \in \{1, \ldots, r-1\}$. It follows that p_r does not divide $p_i - 1$, p_i and $p_i + 1$. Thus,

$$\gcd(p_r, p_i^{3e_i-2}(p_i - 1)(p_i + 1)) = 1.$$

Therefore k and $p_r^{e_r}$ are coprime. Thus,

$$A^k = (I, \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}) \in \mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}/p_r^{e_r}\mathbb{Z}) \text{ with } \gcd(k, p_r^{e_r}) = 1$$

is in H. Similarly one obtains:

$$B^l = (I, \begin{pmatrix} 1 & 0 \\ l & 1 \end{pmatrix}) \in \mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}/p_r^{e_r}\mathbb{Z}) \text{ with some } l \in \mathbb{N} : \gcd(l, p_r^{e_r}) = 1$$

is an element of H.

But A^k and B^l generate $\{I\} \times \mathrm{SL}_2(\mathbb{Z}/p_r^{e_r})$. Thus, $\{I\} \times \mathrm{SL}_2(\mathbb{Z}/p_r^{e_r}\mathbb{Z})$ is a subgroup of H. From the exact sequences in Diagram 19 it follows now that $H = \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$. $\square$

Now, provided with Lemma 6.43, we proceed similarly as in Section 6.7 and "lift" the results on subgroups of $\mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})$ to $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$.

Lemma 6.44. *Let H be a subgroup of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ with the following property:*

$$\forall\, x \in (\mathbb{Z}/n\mathbb{Z})^2 : \ \exists\, A \in H,\ A \neq \mathrm{id} \text{ such that } A \cdot x = x. \tag{6.16}$$

Then H is the full group $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$.

Proof. By (6.15) the property (6.16) "descends" for all $i \in \{1, \ldots, r\}$ to $\beta_i(H)$, i.e.:

$$\forall \bar{x} \in (\mathbb{Z}/p_i^{e_i}\mathbb{Z})^2 : \exists \bar{A} \in \beta_i(H),\ \bar{A} \neq \mathrm{id}, \text{ such that } \bar{A} \cdot \bar{x} = \bar{x}.$$

By Lemma 6.37, it follows that $\beta_i(H) = \mathrm{SL}_2(\mathbb{Z}/p_i^{e_i}\mathbb{Z})$. Thus we have by Lemma 6.43 that $H = \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$. □

We obtain from the last lemma the following corollary by exactly the same proof as in Corollary 6.38, only using Lemma 6.44 instead of Lemma 6.37.

Corollary 6.45. *Let H be a subgroup of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$. Then H is either the stabilizing group of its orbit space, or it has the same orbit space as $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$.*

In order to generalize Lemma 6.39 using Lemma 6.43 we have to demand that all p_i are different from 2, 3, 5, 7 and 11.

Lemma 6.46. *Suppose that n is not divisible by 2, 3, 5, 7 or 11. If H is a proper subgroup of $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$, then H does not have the same orbit space as $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$.*

Proof. Suppose that H has the same orbit space on $(\mathbb{Z}/n\mathbb{Z})^2$ as $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$, i.e.

$$B(H) = B(\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})). \tag{6.17}$$

This property "descends" again by (6.15) for all $i \in \{1, \ldots, r\}$ to $(\mathbb{Z}/p_i^{e_i}\mathbb{Z})^2$, i.e. we have:

$$B(\beta_i(H)) = B(\mathrm{SL}_2(\mathbb{Z}/p_i^{e_i}\mathbb{Z})).$$

It follows by Lemma 6.39 that $\beta_i(H) = \mathrm{SL}_2(\mathbb{Z}/p_i^{e_i}\mathbb{Z})$. Again we use Lemma 6.43 and obtain that $H = \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$.

□

Now we may use again Theorem 3 and obtain the statement indicated above about congruence groups of level n that occur as Veech groups of origamis.

Theorem 5. *Let n be an odd natural number.*

a) Each congruence group of level n, whose image in $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ does not have the same orbit space as $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$, is the Veech group of some origami.

b) If n is not divisible by 2, 3, 5, 7 or 11, then each congruence group of level n is the Veech group of some origami.

Final remark: It is probably not difficult to improve the results in Section 6.7 and 6.8 on congruence groups of level p^e and n, by studying the five exceptional primes 2, 3, 5, 7 and 11 individually.

Chapter 7

Remarks on non congruence groups

In the last chapter we have seen that almost all of the congruence groups in $\mathrm{SL}_2(\mathbb{Z})$ of prime level occur as Veech groups of origamis. Now, one might ask, if there are also Veech groups that are non congruence groups. In Section 7.3 we give a positive answer to this question.
In fact it seems that there are plenty of Veech groups which are non congruence groups. Hubert and Lelièvre e.g. have shown that there are infinitely many origamis of genus 2 with this property, see [HuSc 01]. We will construct in this chapter for suitable origamis O whose Veech group $\Gamma(O)$ is a non congruence group an infinite sequence of origamis O_n, such that $\Gamma(O_n)$ is contained in $\Gamma(O)$. Thus all Veech groups $\Gamma(O_n)$ in this sequence are non congruence groups.

In particular we obtain the following result.

Theorem 6. *Each M_g $(g \geq 2)$ contains an origami curve whose Veech group is a non congruence group.*

The statement of this theorem will be proved in Section 7.4.

We start in Section 7.1 by introducing the general setting that we will use. In 7.2 we present a construction that defines for a basic origami O a sequence of origamis O_n, such that the Veech groups $\Gamma(O_n)$ are "controlled" by $\Gamma(O)$. Sections 7.1 and 7.2 are not aiming at non congruence groups exclusively. They might be used in general to determine Veech groups of origamis.
In 7.3 we present two origamis whose Veech groups are non congruence groups. In 7.4, finally, we take them as basic origamis and construct the infinite origami sequences for them by using the tool that we provided in Section 7.2.

7.1 Lifting stabilizing groups of quotients

In this section we pursue the following concept: Our aim is to study for a group G (in our case $G = F_2$) the stabilizing groups of subgroups H of G in its automorphism group. This is in general difficult. But we might find for a subgroup U of G a quotient group $\overline{U}$, that is easier accessible. In particular it should be easier to obtain for subgroup $\overline{H}$ of $\overline{U}$ their stabilizing groups in the automorphism group of $\overline{U}$. We formulate a setting in which they determine the stabilizing groups of the preimages H of $\overline{H}$ in U.

General setting

We formulate our purpose first in an a bit more general setting:

Suppose we have:

(1) two groups U and G such that U is a subgroup of G.

(2) a surjective group homomorphism $\mathrm{pr}: U \to \bar{U}$ with kernel N.

(3) a subgroup $\mathcal{G}$ of $\mathrm{Aut}(G)$ such that (7.1)

$$\forall \gamma \in \mathcal{G}: \quad \gamma(U) = U \text{ and } \gamma(N) = N, \qquad \text{i.e.}$$
$$\mathcal{G} \subseteq \mathrm{Stab}_{\mathrm{Aut}(G)}(U) \cap \mathrm{Stab}_{\mathrm{Aut}(G)}(N).$$

Remark 7.1. By (3), $\mathcal{G}$ acts via the restriction map $\rho: \mathcal{G} \to \mathrm{Aut}(U)$, $\gamma \mapsto \gamma|_U$ on U. Furthermore, this action is compatible with pr, i.e. we have a homomorphism

$$\beta : \mathcal{G} \to \mathrm{Aut}(\overline{U}) \quad \text{such that}$$

for all $\gamma \in \mathcal{G}$ the following diagram is commutative:

$$\begin{array}{ccc} U & \xrightarrow{\gamma} & U \\ {\scriptstyle \mathrm{pr}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}} \\ \overline{U} & \xrightarrow{\bar{\gamma}=\beta(\gamma)} & \overline{U} \end{array} \tag{7.2}$$

We define

$$\overline{\mathcal{G}} := \beta(\mathcal{G}) \subseteq \mathrm{Aut}(\overline{U}).$$

Let $\overline{H}$ be a subgroup of $\overline{U}$ and $H := \mathrm{pr}^{-1}(\overline{H})$.

Lemma 7.2. *One has the following relations between* $\mathrm{Stab}_{\mathcal{G}}(H)$ *and* $\mathrm{Stab}_{\overline{\mathcal{G}}}(\overline{H})$*:*

a)

$$\mathrm{Stab}_{\mathcal{G}}(H) = \beta^{-1}(\mathrm{Stab}_{\overline{\mathcal{G}}}(\overline{H})). \tag{7.3}$$

b)

$$[\mathcal{G} : \mathrm{Stab}_{\mathcal{G}}(H)] = [\overline{\mathcal{G}} : \mathrm{Stab}_{\overline{\mathcal{G}}}(\overline{H})]. \tag{7.4}$$

Proof.

a) Let γ be in $\mathcal{G}$. It follows:

$\gamma \in \mathrm{Stab}_{\mathcal{G}}(H) \;\Leftrightarrow\; \gamma(H) = H \;\Leftrightarrow\; \beta(\gamma)(\overline{H}) = \overline{H}$, by (7.2) and since $H = \mathrm{pr}^{-1}(\overline{H})$.

b) By a), one has the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & K & \longrightarrow & \mathcal{G} & \xrightarrow{\ \beta\ } & \overline{\mathcal{G}} & \longrightarrow & 1\,, \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & K & \longrightarrow & \mathrm{Stab}_{\mathcal{G}}(H) & \longrightarrow & \mathrm{Stab}_{\overline{\mathcal{G}}}(\overline{H}) & \longrightarrow & 1
\end{array}$$

where K is the kernel of β. Thus in particular, $[\mathcal{G} : \mathrm{Stab}_{\mathcal{G}}(H)] = [\overline{\mathcal{G}} : \mathrm{Stab}_{\overline{\mathcal{G}}}(\overline{H})]$.

□

We will apply this now to the situation of origamis or more precisely to subgroups of F_2 and their stabilizing groups in $\mathrm{Aut}^+(F_2)$.
For this purpose we set in (7.1)

$$G := F_2 \quad \text{and} \quad \mathcal{G} := \mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(U) \tag{7.5}$$

Condition (3) becomes this way a condition on pr:

(4) $N = \mathrm{kernel}(\mathrm{pr}) \quad \Rightarrow \quad \mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(N) \supseteq \mathcal{G} = \mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(U)$

If we require in addition for the subgroup H in Lemma 7.2 the following property:

(5) $\mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(H) \subseteq \mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(U) = \mathcal{G}$,

then we have:

$$\mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(H) = \mathrm{Stab}_{\mathcal{G}}(H). \tag{7.6}$$

Thus under the assumption of (4) and (5), Lemma 7.2 provides a description of $\mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(H)$.

In the construction that is described in the next section we will obtain these two requirements for free since we will be in the situation given in the following corollary.

Corollary 7.3. *Suppose we are in the setting of* (7.1) *with*

$$G := F_2 \quad \textit{and} \quad \mathcal{G} := \mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(U) \qquad \textit{as in } (7.5).$$

Suppose that $N = \mathrm{kernel}(\mathrm{pr})$ *is characteristic in* U.

Let $\overline{H}$ *be a subgroup of* $\overline{U}$ *and* $H := \mathrm{pr}^{-1}(\overline{H})$. *If* U *is*

- *the normalizer of* H *in* F_2*, i.e.* $U = \mathrm{Norm}_{F_2}(H)$*, or*
- *the normal closure of* H *in* F_2*, i.e.* $U = << H >>_{F_2}$*, or*
- *a characteristic subgroup of* F_2*,*

then it follows that:

$$\begin{aligned} \mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(H) &= \beta^{-1}(\mathrm{Stab}_{\overline{\mathcal{G}}}(\overline{H})) \quad \textit{and} \\ [\mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(U) : \mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(H)] &= [\overline{\mathcal{G}} : \mathrm{Stab}_{\overline{\mathcal{G}}}(\overline{H})] \end{aligned}$$

Proof. Since N is characteristic, we have $\mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(N) = \mathrm{Aut}^+(F_2)$. Thus (4) is fulfilled. By Remark 5.1, we have that $\mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(H) \subseteq \mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(U)$ in the first two cases. In the third case this inclusion is obvious. Therefore condition (5) holds, too. The claim follows now from Lemma 7.2 and (7.6). □

7.2 Multiple origamis

In this section we construct for an arbitrary finite index subgroup U of F_2 a system of normal subgroups $H_n \subseteq U$ $(n \in \mathbb{N} \cup \{\infty\})$, such that

$$\forall\, n, m \in \mathbb{N} : \;\; n | m \;\Rightarrow H_m \subseteq H_n \quad \text{and} \quad \bigcap_{n \in \mathbb{N}} H_n = H_\infty. \tag{7.7}$$

Each $H_n \subseteq U$ will fit into a setting as in (7.1), such that we may apply Corollary 7.3 in order to study their stabilizing groups.

Recall that all subgroups of F_2 are free groups (see e.g. [LS 77, Prop. 3.8.]). Let U be a subgroup of F_2 of rank k and $\{g_1, \ldots, g_k\}$ a set of free generators, i.e.:

$$U \;= < g_1, \ldots, g_k > \cong\; F(g_1, \ldots, g_k) \;=:\; F_k$$

Here $F(g_1, \ldots, g_k)$ denotes the free group in the generators $g_1, \ldots, g_k$. In order to emphasize that U is the free group in k generators we denote U also by $F_k = U$. The set of generators is considered fixed as $\{g_1, \ldots, g_k\}$.

Observe that in this situation the restriction map $\rho : \mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(U) \to \mathrm{Aut}(U)$ is injective, since F_2 is a free group and U is of finite index in F_2:
Suppose that γ is in the kernel of ρ, i.e. $\gamma|_U = \mathrm{id}$. U has finite index, therefore we have: $\forall w \in F_2\ \exists\, n \in \mathbb{N}$ with $w^n \in U$. Thus, $\gamma(w)^n = \gamma(w^n) \overset{w^n \in U}{=} w^n$. From this it follows that $\gamma(w) = w$, since F_2 is a free group.
Thus $\gamma = \mathrm{id}$ and we may denote: $\mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(U) \subseteq \mathrm{Aut}(U)$.

We now establish for U the setting (7.1) as follows:

$$
\begin{aligned}
&(1)\ \ G = F_2, \quad U = F_k \subseteq F_2 \\
&(2)\ \ \overline{U} := \overline{U}_\infty := \mathbb{Z}^k, \qquad \mathrm{pr}_\infty : F_k \to \mathbb{Z}^k \text{ the abelianization.} \\
&\qquad\qquad\qquad\qquad\quad\ \ N_\infty := \mathrm{kernel}(\mathrm{pr}_\infty) = [F_k, F_k] \\
&(3)\ \ \mathcal{G} := \mathrm{Stab}_{\mathrm{Aut}^+(F_2)}(F_k) \subseteq \mathrm{Aut}(F_k) \\
&(4)\ \ \beta_\infty : \mathcal{G} \to \mathrm{Aut}(\mathbb{Z}^k) \text{ is the restriction to } \mathcal{G} \text{ of the map} \\
&\qquad\qquad \mathrm{Aut}(F_k) \to \mathrm{GL}_k(\mathbb{Z}) \quad \text{induced by the abelianization.} \\
&(5)\ \ \overline{H}_\infty := \{(0, x_2, \ldots, x_k) \in \mathbb{Z}^k\} \subseteq \overline{U}_\infty \\
&\qquad H_\infty := \mathrm{pr}_\infty^{-1}(\overline{H}_\infty) = \mathrm{kernel}(\alpha_\infty) \subseteq U, \\
&\qquad\qquad \text{with } \alpha_\infty : F_k \to \mathbb{Z}, \ u \mapsto \sharp_{g_1}(u), \\
&\qquad\qquad \text{where } \sharp_{g_1}(u) \text{ is the number of occurrences of } g_1 \text{ in } u.
\end{aligned}
\tag{7.8}
$$

Since N_∞ is the commutator $[F_k, F_k]$, it is characteristic in $U = F_k$. Furthermore, the group H_∞ is the kernel of α_∞. Therefore we have $U \subseteq \mathrm{Norm}_{F_2}(H_\infty)$.
Thus we only need the condition $\mathrm{Norm}_{F_2}(H_\infty) \subseteq U$ in order to fulfill the requirements of Corollary 7.3. In Proposition 7.5 we shall formulate a condition on U (see (7.9)) which implies this inclusion.

Using the natural projection $\mathrm{p}_n^k : \mathbb{Z}^k \to (\mathbb{Z}/n\mathbb{Z})^k \quad (n \in \mathbb{N})$, we now obtain the setting (7.1) for each $n \in \mathbb{N}$ from (7.8), as follows:

(1) and (3) as in (7.8).

(2) $\overline{U}_n := (\mathbb{Z}/n\mathbb{Z})^k, \quad \mathrm{pr}_n : F_k \to (\mathbb{Z}/n\mathbb{Z})^k$ with $\mathrm{pr}_n := \mathrm{p}_n^k \circ \mathrm{pr}_\infty$.

(4) $\beta_n : \mathcal{G} \to \mathrm{Aut}((\mathbb{Z}/n\mathbb{Z})^k)$ is the restriction to $\mathcal{G}$ of the map

$$\mathrm{Aut}(F_k) \to \mathrm{GL}_k(\mathbb{Z}/n\mathbb{Z}) \quad \text{induced by } \mathrm{pr}_n.$$

(5) $\overline{H}_n := \{(0, x_2, \ldots, x_k) \in (\mathbb{Z}/n\mathbb{Z})^k\} \subseteq \overline{U}_n$
$H_n := \mathrm{pr}_n^{-1}(\overline{H}_n) = \mathrm{kernel}(\alpha_n) \subseteq U$,
with $\alpha_n : F_k \to \mathbb{Z}, \ u \mapsto \sharp_{g_1}(u) \mod n$.

As above we have that $N_n := \mathrm{kernel}(\mathrm{pr}_n)$ is characteristic in $U = F_k$, and H_n is normal in U. Thus we need again only that $\mathrm{Norm}_{F_2}(H_n) \subseteq U$, which will follow from the same condition (7.9) on U as mentioned above.

But first, let us describe the groups H_n by giving their generators.

Lemma 7.4. *The subgroups H_n of F_2 $(n \in \mathbb{N} \cup \{\infty\})$ are given as follows:*

a) If $n = \infty$:

$$H_\infty \;=\; << g_2, \ldots, g_k >>_U$$

b) If $n \in \mathbb{N}$:

$$H_n \;= < g_1^n,\;\; g_1^i\, g_j\, g_1^{-i} \;\in F_k \,|\;\; i \in \{0, \ldots, n-1\} \text{ and } j \in \{2, \ldots, k\} >$$

Proof.
a) Recall that $H_\infty = \text{kernel}(\alpha_\infty)$. The normal subgroup

$$H'_\infty \;:=\; << g_2, \ldots, g_k >>_U$$

is contained in H_∞.
Consider the quotient group U/H'_∞. We have that $g_2 H'_\infty = \ldots = g_k H'_\infty = 0$. Thus for u_1, u_2 in U one has: $\alpha_\infty(u_1) = \alpha_\infty(u_2) \;\Rightarrow\; u_1 H'_\infty = u_2 H'_\infty$. Furthermore, " $\Leftarrow$ " holds, since $H'_\infty \subseteq H_\infty$. Thus it follows equality.

b) We may proceed similarly as in a):
$H'_n := < g_1^n,\;\; g_1^i\, g_j\, g_1^{-i} \;\in F_k \,|\;\; i \in \{0, \ldots, n-1\},\; j \in \{2, \ldots, k\} >$ is a normal subgroup of $U = F_k \;= < g_1, \ldots, g_k >$.
H_n is the kernel of α_n, thus H'_n is a subgroup of H_n. In particular we have $|F_k/H'_n| \geq |F_k/H_n| = |\mathbb{Z}/n\mathbb{Z}| = n$.
Furthermore we have $g_2 H'_n = \ldots = g_k H'_n = 0$. Thus F_k/H'_n is generated by $g_1 H'_n$. But the order of $g_1 H'_n$ is smaller or equal to n, since g_1^n is in H'_n. It follows that $|F_k/H'_n| \leq n$. Thus we have $H_n = H'_n$. □

Observe that the groups H_n fulfill the conditions in (7.7), since for all $n, m \in \mathbb{N}$ with $n|m$, the morphism α_n factors as

$$F_k \overset{\alpha_\infty}{\to} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \quad \text{and} \quad F_k \overset{\alpha_m}{\to} \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z},$$

with the natural projections $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$.

We will now apply Corollary 7.3 to this setting. For simpler notations we denote in the following $\mathbb{Z}/\infty\mathbb{Z} \;:=\; \mathbb{Z}$.

Proposition 7.5. *Let $U \;= < g_1, \ldots, g_k >$ $\;(k \geq 2)$ be a subgroup of F_2 of rank k and $\{w_i\}_{i \in I}$ a system of coset representatives with $w_1 = \text{id}$. Suppose that U has the following property:*

$$\forall\, j \in I - \{1\} : \;\; w_j \;<< g_2, \ldots, g_k >>_U\; w_j^{-1} \not\subseteq U. \tag{7.9}$$

Then we have for all $n \in \mathbb{N} \cup \{\infty\}$:

$$\text{Stab}_{\text{Aut}^+(F_2)}(H_n) =$$
$$\beta^{-1}(\{A = (a_{i,j})_{1 \leq i,j \leq k} \in \text{SL}_k(\mathbb{Z}/n\mathbb{Z}) |\; a_{1,2} = \ldots = a_{1,k} = 0\}) \,\cap\, \mathcal{G}.$$

Proof. We first show, that (7.9) implies $\text{Norm}_{F_2}(H_n) = U$ for all n in $\mathbb{N} \cup \{\infty\}$: We already have that H_n is normal in U, i.e. $U \subseteq \text{Norm}_{F_2}(H_n)$. Let now w be an element of $F_2 \backslash U$. Hence, $w = w_j \cdot u$ for some $j \in I - \{1\}$, $u \in U$. By (7.9), there exists some $h_\infty \in << g_2, \ldots, g_k >>_U \stackrel{Lem.\ 7.4}{=} H_\infty$, such that $w_j h_\infty w_j^{-1} \notin U$. Therefore we have $w(u^{-1}h_\infty u)w^{-1} \notin U$. But $u^{-1}h_\infty u \in H_\infty \subseteq H_n$, since H_∞ is normal in U. This shows that $w \notin \text{Norm}_{F_2}(H_n)$.

Now, we may use Corollary 7.3 and obtain:

$$\text{Stab}_{\text{Aut}^+(F_2)}(H_n) \ = \ \beta^{-1}(\text{Stab}_{\overline{\mathcal{G}}_n}(\overline{H}_n))$$

The claim follows by the following calculation:

$$\begin{aligned}\text{Stab}_{\overline{\mathcal{G}}_n}(\overline{H}_n) \ &= \ \{A = (a_{i,j})_{1 \le i,j \le k} \in \overline{\mathcal{G}}_n | \quad (y_1, \ldots, y_k) = A \cdot (0, x_2, \ldots, x_k) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \Rightarrow \ y_1 = 0 \qquad \} \\ &= \ \{A = (a_{i,j}) \in \overline{\mathcal{G}}_n | \ a_{1,2} \ = \ \ldots \ = \ a_{1,k} \ = \ 0\}.\end{aligned}$$

□

We now use Proposition 7.5 in order to show that the properties of the groups H_n in (7.7) are passed down to their stabilizing groups and to their Veech groups.

Corollary 7.6. *Suppose that U has the property in (7.9). Then we have for all $n \in \mathbb{N}$:*

a) $\text{Stab}_{\text{Aut}^+(F_2)}(H_\infty) \subseteq \text{Stab}_{\text{Aut}^+(F_2)}(H_n)$ *and* $\Gamma(H_\infty) \subseteq \Gamma(H_n)$.

b) If $m \in \mathbb{N}$ with $n|m$, then:

$\text{Stab}_{\text{Aut}^+(F_2)}(H_m) \subseteq \text{Stab}_{\text{Aut}^+(F_2)}(H_n)$ *and* $\Gamma(H_m) \subseteq \Gamma(H_n)$.

c)

$$\text{Stab}_{\text{Aut}^+(F_2)}(H_\infty) = \bigcap_{n \in \mathbb{N}} \text{Stab}_{\text{Aut}^+(F_2)}(H_n) \quad \textit{and} \quad \Gamma(H_\infty) = \bigcap_{n \in \mathbb{N}} \Gamma(H_n)$$

Proof. a) and b)
Let $\gamma \in \text{Aut}^+(F_2)$. By Proposition 7.5 we have that

$$\begin{aligned}\forall n \in \mathbb{N}: \ \gamma \in \text{Stab}_{\text{Aut}^+(F_2)}(H_n) \quad &\Leftrightarrow \quad \beta_\infty(\gamma) = A = (a_{i,j}) \\ &\qquad \text{with } a_{1,2} \equiv \ldots \equiv a_{1,k} \equiv 0 \mod n \\ \text{and} \quad \gamma \in \text{Stab}_{\text{Aut}^+(F_2)}(H_\infty) \quad &\Leftrightarrow \quad \beta_\infty(\gamma) = A = (a_{i,j}) \\ &\qquad \text{with } a_{1,2} = \ldots = a_{1,k} = 0.\end{aligned}$$

Thus we have for all $n \in \mathbb{N}$ and for all $m \in \mathbb{N}$ with $n|m$, that

$$\text{Stab}_{\text{Aut}^+(F_2)}(H_\infty) \subseteq \text{Stab}_{\text{Aut}^+(F_2)}(H_m) \subseteq \text{Stab}_{\text{Aut}^+(F_2)}(H_n).$$

We have in particular by the definition of the Veech group of a subgroup of F_2:

$$\Gamma(H_\infty) \subseteq \Gamma(H_m) \subseteq \Gamma(H_n).$$

c)
$\subseteq$: Follows from a).
$\supseteq$: Follows from Remark 5.1.

□

We now return to the language of origamis: Let O be an origami, U the corresponding subgroup of F_2. Define for U the subgroups $H_n \quad (n \in \mathbb{N})$ as above and let O_n be the origamis corresponding to the groups H_n.
By Corollary 7.6 and Theorem 1 we obtain immediately the following result.

Proposition 7.7. *If U has the property (7.9), then*

$$\forall n \in \mathbb{N}: \ \Gamma(O_n) \subseteq \Gamma(O) \quad \textit{and} \quad \forall n, m \in \mathbb{N}: \ n|m \ \Rightarrow \ \Gamma(O_n) \subseteq \Gamma(O_m).$$

In particular, if $\Gamma(O)$ is a non congruence group, each $\Gamma(O_n)$ is a non congruence group. Thus we then obtain infinitely many origamis whose Veech group is a non congruence group.

7.3 Two non congruence origamis

In this section we present two origamis whose Veech groups are non congruence groups: the origami $L(2,3)$ and the origami D.

7.3.1 The origami $L(2,3)$

Let $O = (p : X^* \to E^*)$ be the origami $L(2,3)$ of Example 4.1, see Figure 20.

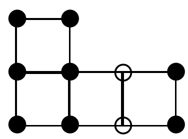


Figure 20: The origami $L(2,3)$: Opposite sides are glued.

In Example 4.5 we have shown that the Veech group is given as follows:

$$\Gamma(L(2,3)) = < \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 3 \\ -2 & 5 \end{pmatrix}, \begin{pmatrix} 3 & -5 \\ 2 & -3 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} > .$$

Furthermore we obtained in Example 4.7 that $\mathbb{H}/\bar{\Gamma}(L(2,3))$ has three cusps represented in Figure 14 by the vertices 1, 4 and 5. T^3, ST^2S^{-1} and $(TS)T^4(TS)^{-1}$

are parabolic elements that correspond to them, respectively, i.e. they define loops around these cusps. Their amplitudes are 3, 2 and 4. Here, we define the *amplitude* of a parabolic element A in $\mathrm{SL}_2(\mathbb{Z})$ to be the natural number n such that A is conjugated to

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix}.$$

The following proof, which shows that the described subgroup of $\mathrm{SL}_2(\mathbb{Z})$ is in fact a non congruence group, I have learned from Stefan Kühnlein.

Proposition 7.8. *$\Gamma(L(2,3))$ is a non congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$.*

Proof. One may use the following result of Wohlfahrt: Define the *general level* of a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ to be the least common multiple of the amplitudes of the cusps. Then Theorem 2 in [Wo 64] states that if Γ is a congruence group of general level m then $\Gamma(m)$ is contained in Γ.

Since the amplitudes of the three cusps of $\Gamma(L(2,3))$ are 3, 2 and 4, the general level m is $\mathrm{lcm}(3,2,4) = 12$.

Suppose that $\Gamma(L(2,3))$ is a congruence subgroup. By Wohlfahrt's theorem we would have:

$$\Gamma(12) \subseteq \Gamma(L(2,3)). \tag{7.10}$$

Let $p : \mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/3\mathbb{Z})$ be the natural projection. Then we have

$$p(\bar{\Gamma}(L(2,3))) = < \begin{pmatrix} \bar{1} & \bar{0} \\ \bar{2} & \bar{1} \end{pmatrix}, \begin{pmatrix} \bar{0} & \bar{1} \\ \bar{2} & \bar{0} \end{pmatrix} > = \mathrm{PSL}_2(\mathbb{Z}/3\mathbb{Z}).$$

Hence Diagram 21 is commutative with $N := \bar{\Gamma}(L(2,3)) \cap \bar{\Gamma}(3)$.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \bar{\Gamma}(3) & \longrightarrow & \mathrm{PSL}_2(\mathbb{Z}) & \longrightarrow & \mathrm{PSL}_2(\mathbb{Z}/3\mathbb{Z}) & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \| & & \\
1 & \longrightarrow & N & \longrightarrow & \bar{\Gamma}(L(2,3)) & \longrightarrow & \mathrm{PSL}_2(\mathbb{Z}/3\mathbb{Z}) & \longrightarrow & 1
\end{array}$$

Diagram 21

Since the index $[\mathrm{PSL}_2(\mathbb{Z}) : \bar{\Gamma}(L(2,3))]$ of $\bar{\Gamma}(L(2,3))$ in $\mathrm{PSL}_2(\mathbb{Z})$ is 9, it follows from Diagram 21 that $[\bar{\Gamma}(3) : N] = 9$.
By (7.10) we have: $\bar{\Gamma}(12) \subseteq N \subseteq \bar{\Gamma}(3)$. But $[\bar{\Gamma}(3) : \bar{\Gamma}(12)] = 2^4 \cdot 3$ (using [Sh 71], (1.6.2)). Thus $[\bar{\Gamma}(3) : N] = 9$ would have to be a factor of $2^4 \cdot 3$. Contradiction! □

Hubert and Lelièvre have generalized this result for the origamis $L(n,m)$ where $(n,m) \neq (2,2)$ as defined in Section 4.5, see [HL2 04].

7.3.2 The origami D

Let $D = (p : X^* \to E^*)$ be the origami of degree 5 given in Figure 22, with

$$\sigma_h = (123), \qquad \sigma_v = (145)(23)$$

The genus of X^* is 2 and it has the 3 marked points $\bullet$, $\star$, $\circ$ of order 2, 2 and 1.

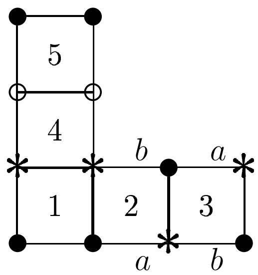


Figure 22: *The origami D: edges with the same label and unlabeled edges that are opposite are glued.*

The corresponding subgroup of F_2 is

$$H \;= < x^3 \,,\, y^3 \,,\, xyx^{-2} \,,\, x^2yx^{-1} \,,\, yxy^{-1} \,,\, y^2xy^{-2} >$$

The Veech group $\Gamma(H)$ has index 24 and the following generators:[1]

$$A_0 := \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I, \qquad A_1 := \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = T^3,$$

$$A_2 := \begin{pmatrix} 1 & 0 \\ -6 & 1 \end{pmatrix} = ST^6S^{-1}, \qquad A_3 := \begin{pmatrix} -7 & 16 \\ -4 & 9 \end{pmatrix} = (T^2S)T^4(T^2S)^{-1}.$$

$$A_4 := \begin{pmatrix} -3 & 4 \\ -4 & 5 \end{pmatrix} = (TS)T^4(TS)^{-1}, \qquad A_5 := \begin{pmatrix} -9 & 5 \\ -20 & 11 \end{pmatrix} = (TST^2S)T^5(TST^2S)^{-1},$$

$$A_6 := \begin{pmatrix} 7 & 2 \\ -18 & -5 \end{pmatrix} = (ST^3S)T^2(ST^3S)^{-1},$$

with

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

[1] The data of the group $\Gamma(H)$ we list here were calculated with a computer program that implements the algorithm presented in Section 4.

The following is a system of cosets representatives:

$$I\,,\ T\,,\ S\,,\ T^2\,,\ TS\,,\ ST\,,\ T^2S\,,\ TST\,,\ ST^2\,,\ STS\,,T^2ST\,,\ TST^2\,,$$
$$ST^5\,,\ ST^3\,,\ T^2S\,,\ TST^3\,,\ TST^2S\,,\ ST^4\,,\ ST^3S\,,\ TST^2ST^{-1}\,,$$
$$TST^2ST^{-2}\,,\ TST^2ST^{-3}\,;\ TST^2ST^{-4}\,,\ ST^3ST$$

The corresponding origami curve $C(D)$ has genus 0 and 6 cusps. It is shown with its natural triangulation (compare Section 4.4) in Figure 23.

More precisely we have the cusps C_1, C_2, C_3, C_4, C_5 and C_6 of amplitude 3,6,4,4,5 and 2. Hence, the general level of $\Gamma(D)$ is 60.

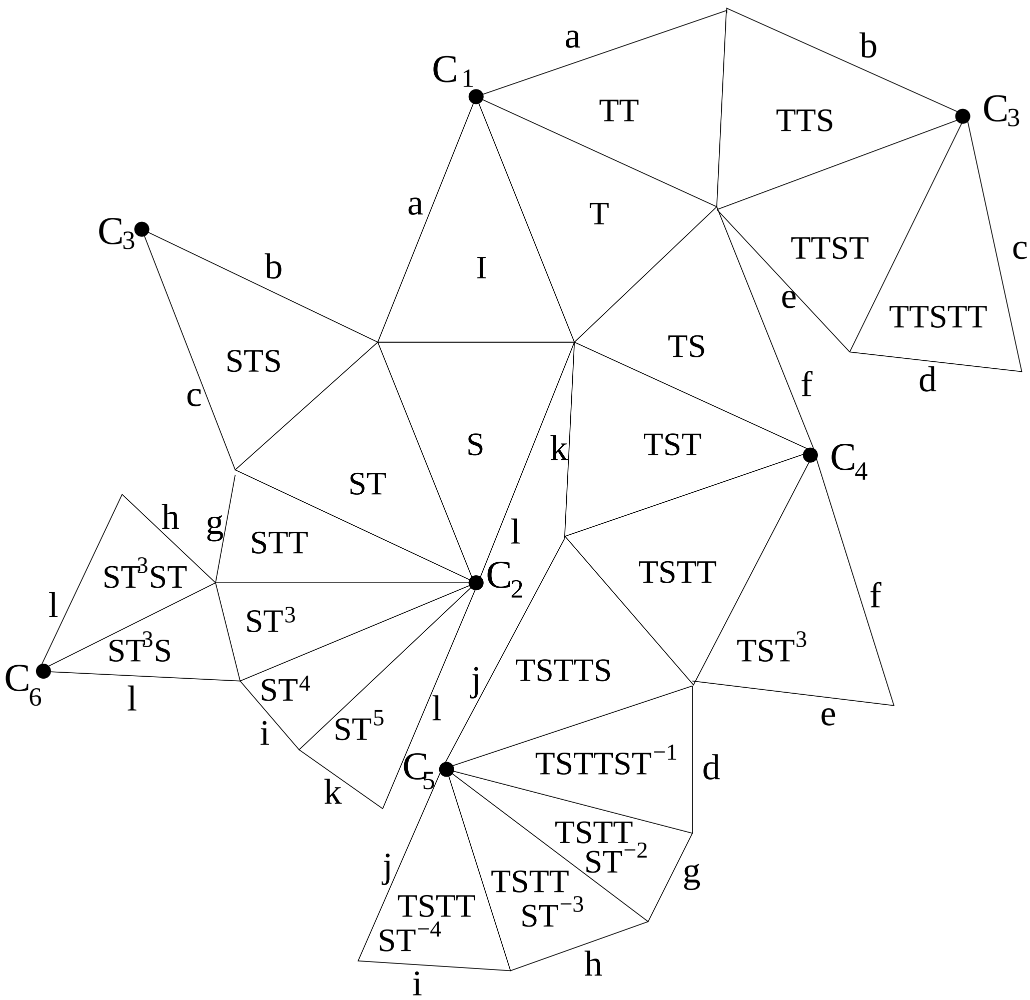


Figure 23: The origami curve to D.

Proposition 7.9. $\Gamma(D)$ *is a non congruence group.*

Proof. Suppose that $\Gamma := \Gamma(D)$ is a congruence group. Since the general level of Γ is 60, we have again by Theorem 2 in [Wo 64], that $\Gamma(60)$ is a subgroup of Γ.

We will use the following data:

$$A_1 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \in \Gamma, \quad A_6 = \begin{pmatrix} 7 & 2 \\ -18 & -5 \end{pmatrix} \in \Gamma \quad \text{and } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \notin \Gamma$$

and find an element in Γ whose projection to $\mathrm{SL}_2(\mathbb{Z}/60\mathbb{Z})$ is equal to that of T. Recall that

$$\mathrm{SL}_2(\mathbb{Z}/60\mathbb{Z}) \cong \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}/5\mathbb{Z}).$$

We identify in the following these two groups. Furthermore we denote by p_4, p_3, p_5 and p_{60} the projection from $\mathrm{SL}_2(\mathbb{Z})$ to $\mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$, $\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$, $\mathrm{SL}_2(\mathbb{Z}/5\mathbb{Z})$ and $\mathrm{SL}_2(\mathbb{Z}/60\mathbb{Z})$. Then $p_{60} = p_4 \times p_3 \times p_5$.
We have

$$\begin{aligned} p_{60}(A_1) &= (\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}) \qquad \text{and} \\ p_{60}(A_6) &= (\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}) \end{aligned}$$

The order of $p_4(A_1)$ in $\mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$ is 4, the order of $p_3(A_1)$ in $\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$ is 1 and the order of $p_5(A_1)$ in $\mathrm{SL}_2(\mathbb{Z}/5\mathbb{Z})$ is 5. We also say: *The order of* $p_{60}(A_1)$ *is* $(4,1,5)$. Since $7 \equiv 3 \mod 4$ and $7 \equiv 2 \mod 5$ we have

$$p_{60}(A_1)^7 = (\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) \tag{7.11}$$

Furthermore:

$$p_{60}(A_6)^2 = (\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 4 \\ 4 & 4 \end{pmatrix})$$

and with the same notation as above $p_{60}(A_6)$ *has the order* $(1,3,5)$. Thus

$$p_{60}(A_6)^{20} = (\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}). \tag{7.12}$$

From (7.11) and (7.12) it follows that

$$p_{60}(A_6^{20} \cdot A_1^7) = (\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) = p_{60}(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) = p_{60}(T)$$

But $A_6^{20} \cdot A_1^7 \in \Gamma$ and $T \notin \Gamma$, thus $\Gamma(60) = \ker(p_{60})$ cannot be contained in Γ. Therefore, Γ cannot be a congruence group of level 60. Contradiction! □

7.4 Multiple non congruence origamis

In this section we apply the multiple origami construction to the two origamis $L(2,3)$ and D introduced in Section 7.3. As consequence we obtain two infinite sequences D_n and L_n of origamis such that the Veech groups $\Gamma(L_n)$ and $\Gamma(D_n)$ are contained in $\Gamma(L(2,3))$, respectively $\Gamma(D)$. We use the first sequence to prove Theorem 6, see Corollary 7.10.

7.4.1 The multiple origamis to $L(2,3)$

Let $(p : X^* \to E^*)$ be the origami $L(2,3)$ as in Section 7.3.1.

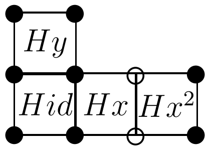


Figure 24: The origami $L(2,3)$: Opposite sides are glued.

Recall that the corresponding subgroup U of F_2 is

$$U \; = < g_1 := x^3, \; g_2 := y^2, \; g_3 := xyx^{-1}, g_4 := x^2yx^{-2}, \; g_5 := yxy^{-1} > .$$

Furthermore, the genus of X^* is $g = 2$ and we have $n = 2$ cusps $\{\circ, \bullet\}$. The rank r of the fundamental group is $r = 2g + n - 1 = 5$. Thus it is freely generated by $g_1, \ldots, g_5$.

We now carry out the construction of multiple origamis introduced in Section 7.2 for U and the generators $g_1, \ldots, g_5$ and obtain the origamis $L_n \quad (n \in \mathbb{N})$. By Lemma 7.4 they correspond to the groups

$$H_n \; = < g_1^n, \;\; g_1^i \, g_j \, g_1^{-i} \; \in F_2 \, | \;\; i \in \{0, \ldots, n-1\} \text{ and } j \in \{2, \ldots, 5\} > .$$

Claim: The corresponding origami L_n is the origami given in Figure 25 with

$$\sigma_h = (1\ 3\ 4\ 5\ 7\ 8\ 9\ 11\ 12\ \ldots\ 4n{-}3\ 4n{-}1\ 4n), \quad \sigma_v = (1\ 2)(5\ 6)\ldots(4n{-}3\ 4n{-}2).$$

The genus of L_n is $n+1$ and it has $2n$ cusps: n of order 3 (all n marked by $\bullet$ in Figure 25), n of order 1 (all n marked by $\circ$ in Figure 25).

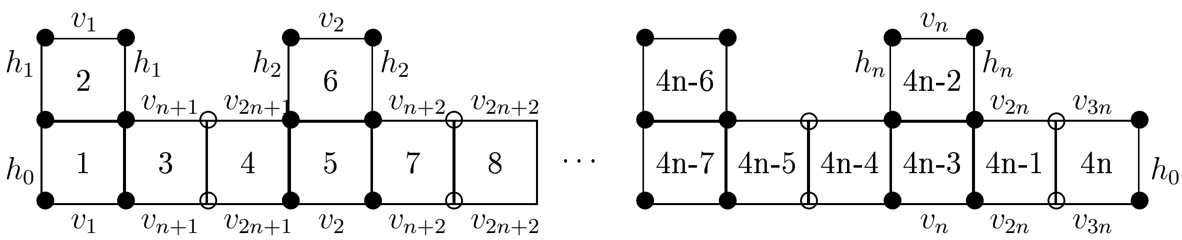


Figure 25: The origami L_n: Opposite sides are glued.

Proof of the claim:
The domain given in the figure is simply connected. Thus the fundamental group is generated by the paths corresponding to the edges that are glued. We choose the base point in square 1 and obtain the generators as follows:

edge h_0	:	$x^{3n} = g_1^n$
edge h_i	:	$x^{3(i-1)}yxy^{-1}x^{-3(i-1)} = g_1^{i-1}g_5g_1^{-(i-1)} \quad (i \in \{1,\dots,n\})$
edge v_i	:	$x^{3(i-1)}y^2x^{-3(i-1)} = g_1^{i-1}g_2g_1^{-(i-1)} \quad (i \in \{1,\dots,n\})$
edge v_{i+n}	:	$x^{3(i-1)}xyx^{-1}x^{-3(i-1)} = g_1^{i-1}g_3g_1^{-(i-1)} \quad (i \in \{1,\dots,n\})$
edge v_{i+2n}	:	$x^{3(i-1)}x^2yx^{-2}x^{-3(i-1)} = g_1^{i-1}g_4g_1^{-(i-1)} \quad (i \in \{1,\dots,n\})$

Thus the fundamental group is H_n.
The statement about the genus follows from the Euler formula: We have $4n$ squares, $8n$ edges and $2n$ vertices. Thus the Euler characteristic is $-2n$ and the genus is $n+1$.

The following result is obtained as a consequence using the fact provided in Proposition 7.8 that $\Gamma(L_1) = \Gamma(L(2,3))$ is a non congruence group.

Corollary 7.10 (to Proposition 7.7). *The Veech group of L_n is a subgroup of $\Gamma := \Gamma(L(2,3))$ for all $n \in \mathbb{N}$. If n divides m, then $\Gamma(L_m) \subseteq \Gamma(L_n)$.*
In particular all $\Gamma(L_n)$ are non congruence groups.

Proof. By Corollary 7.5, we have to show, that Condition (7.9) is fulfilled. The elements id, x, x^2 and y corresponding to the squares of $L(2,3)$ are a system of coset representatives for U in F_2. We verify (7.9) by finding for each representative $w \neq$ id an element g in $<< g_2, g_3, g_4, g_5 >>_U = H_\infty$ such that $wgw^{-1} \notin U$. It can be seen using Figure 24 that the elements we obtain are not in U.

$$\text{for } x: \quad x \cdot g_4 \cdot x^{-1} = x \cdot (x^2yx^{-2}) \cdot x^{-1} = x^3yx^{-3} \notin U$$
$$\text{for } x^2: \quad x^2 \cdot g_3 \cdot x^{-2} = x^2 \cdot (xyx^{-1}) \cdot x^{-2} = x^3yx^{-3} \notin U$$
$$\text{for } y: \quad y \cdot g_5 \cdot y^{-1} = y \cdot (yxy^{-1}) \cdot y^{-1} = y^2xy^{-2} \notin U$$

□

From Corollary 7.10 we obtain in particular Theorem 6 since the genus of L_n is $n+1$.

Proposition 7.11. *For all natural numbers n and m with $n \neq m$, we have:*

$$\Gamma(L_n) \neq \Gamma(L_m)$$

Proof. We prove the claim by showing that for all $n \in \mathbb{N}$:

$$\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \in \Gamma(L_n) \;\Leftrightarrow\; 3n \text{ divides } s.$$

$\Leftarrow$:
This is true by Lemma 3.10, since $3n$ is the smallest common multiple of the horizontal cylinder lengths.

$\Rightarrow$:
In the proof of Corollary 7.10 we saw that condition (7.9) is fulfilled. Thus, we have: $\mathrm{Norm}_{F_2}(H_n) = U$, as shown in the proof of Proposition 7.5. Furthermore, $\{1, x, x^2, y\}$ is a system of coset representatives of U. Furthermore let γ_s and A_s be defined as:

$$\gamma_s : \ x \mapsto x, \ y \mapsto x^s y, \qquad A_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

Thus γ_s is a preimage of A_s and we have by Corollary 3.8

$$A_s \in \Gamma(L_n) \ \Leftrightarrow \ \exists\, w \in \{1, x, x^2, y\} \text{ with } \forall\, h \in H_n : \ (H_n \cdot w)\gamma_s(h) = H_n \cdot w.$$

Take $h := xyx^{-1} \in H_n$. Then $\gamma_s(h) = x^{s+1}yx^{-1}$.

In the following we denote $\bar{1} := H_n$, $\bar{2} := H_n \cdot y$, $\bar{3} := H_n \cdot x$ and $\bar{4} := H_n \cdot x^2$. We have (see Figure 25):

$$\begin{array}{rcl} \bar{1} \cdot \gamma_s(h) & = & \bar{1} \cdot x^{s+1}yx^{-1} = \bar{1} \ \Leftrightarrow \ \bar{1} \cdot x^s = \bar{1} \cdot xy^{-1}x^{-1} = \bar{1} \ \Leftrightarrow \ 3n \,|\, s. \\ \bar{3} \cdot \gamma_s(h) & = & \bar{3} \cdot x^{s+1}yx^{-1} = \bar{3} \ \Leftrightarrow \ \bar{3} \cdot x^s = \bar{3} \cdot xy^{-1}x^{-1} = \bar{3} \ \Leftrightarrow \ 3n \,|\, s. \\ \bar{2} \cdot \gamma_s(h) & = & \bar{2} \cdot x^{s+1}yx^{-1} = \overline{4n} \neq \bar{2} \ \text{ for all } s. \\ \bar{4} \cdot \gamma_s(h) & = & \bar{4} \cdot x^{s+1}yx^{-1} = \bar{4} \ \Leftrightarrow \ \bar{4} = \bar{4} \cdot xy^{-1}x^{-(s+1)} = \bar{6} \neq \bar{2} \ \text{ for all } s. \end{array}$$

From this it follows that if A_s is in $\Gamma(L_n)$, then $3n$ divides s. □

As a consequence of Proposition 7.11 we obtain the following Corollary.

Corollary 7.12. *The Veech group* $\Gamma(H_\infty)$ *is a nontrivial subgroup of infinite index in* $\mathrm{SL}_2(\mathbb{Z})$.

Proof. From the proof of Proposition 7.11 it follows that for all $s \in \mathbb{N}$

$$A_s \ = \ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \notin \bigcap_{n \in \mathbb{N}} \Gamma(H_n) \overset{Cor.\ 7.6}{=} \Gamma(H_\infty).$$

Thus $\Gamma(H_\infty)$ has infinite index in $\mathrm{SL}_2(\mathbb{Z})$.

Observe furthermore that for all $s \in \mathbb{N}$ the least common multiple of the lengths of the vertical cylinders is 2. Thus by Lemma 3.10 we have

$$B_2 \ := \ \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \in \Gamma(H_n) \text{ for all } n \in \mathbb{N} \quad \text{and hence } \ B_2 \in \bigcap_{n \in \mathbb{N}} \Gamma(H_n) = \Gamma(H_\infty).$$

Therefore, $\Gamma(H_\infty)$ is nontrivial. □

7.4.2 The multiple origamis to D

We consider now the origami D from Section 7.3.2.

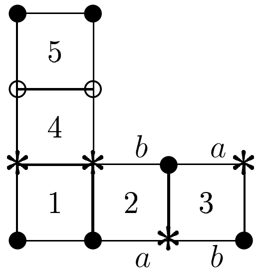


Figure 26: *The origami D: edges with the same label and unlabeled edges that are opposite are glued.*

Recall that the corresponding subgroup of F_2 is

$$U = < g_1 := x^3, g_2 := y^3, g_3 := xyx^{-2}, g_4 := x^2yx^{-1}, g_5 := yxy^{-1}, g_6 := y^2xy^{-2} >$$

If we carry out the multiple construction for U and $g_1 = x^3$, then we obtain similarly as in 7.4.1 the origami shown in Figure 27, with:

$$\sigma_h = (1\,2\,3\ \ 6\,7\,8\ \dots\ 5n-4\ 5n-3\ 5n-2),$$
$$\sigma_v = (1\,4\,5)(6\,9\,10)\dots(5n-4\ 5n-1\ 5n)(2\,3)(7\,8)\dots(5n-3\ 5n-2)$$

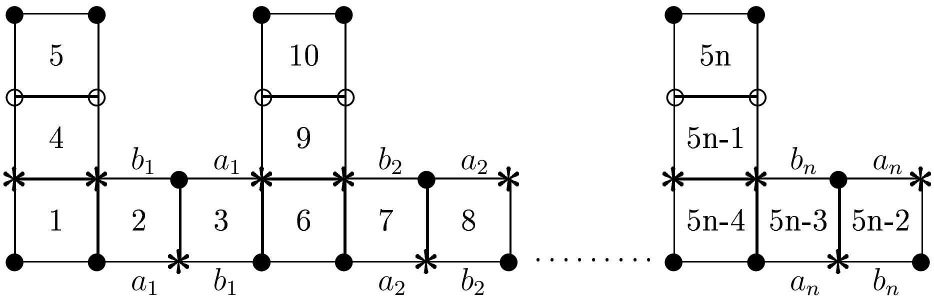


Figure 27: *The origami D_n: edges with the same label and unlabeled edges that are opposite are glued.*

The genus of D_n is $2n$ and it has $n+2$ marked points: 2 of order $2n$ (marked as $\bullet$ and $\star$) and n of order 1 (all n marked by $\circ$).
We proceed in the following as in Subsection 7.4.1.

Corollary 7.13 (to Proposition 7.7). *The Veech group of D_n is a subgroup of $\Gamma := \Gamma(L(2,3))$ for all $n \in \mathbb{N}$. If n divides m, then $\Gamma(D_m) \subseteq \Gamma(D_n)$. In particular all $\Gamma(D_n)$ are non congruence groups.*

Proof. Similarly as in the proof of Corollary 7.10 we have to verify (7.9) for U, $H_\infty =<< g_2, \ldots, g_6 >>_U$ and $w_2 := x$, $w_3 := x^2$, $w_4 := y$, $w_5 := y^2$. But we have for all w_i, $i \in \{2, \ldots, 5\}$:

$$w_i g_3 w_i^{-1} \;=\; w_i x y x^{-2} w_i^{-1} \notin U.$$

This can be seen with the help of Figure 26. □

Finally, we can show also for the origamis D_n, that their Veech groups are all different.

Proposition 7.14. *For all natural numbers n and m with $n \neq m$, we have:*

$$\Gamma(D_n) \neq \Gamma(D_m)$$

Proof. Again we proceed as in Subsection 7.4.1. We show that for all $n \in \mathbb{N}$:

$$\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \in \Gamma(D_n) \;\Leftrightarrow\; 3n \text{ divides } s.$$

$\Leftarrow$: as in the proof of Proposition 7.11.

$\Rightarrow$:
Again we use that the normalizer $\mathrm{Norm}_{F_2}(H_n) = U$. $\{1, x, x^2, y, y^2\}$ is a system of coset representatives of U corresponding to the squares 1, 2, 3, 4 and 5 in Figure 26. Furthermore we define γ_s and A_s again as:

$$\gamma_s : \; x \mapsto x, \; y \mapsto x^s y, \qquad A_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

We show, that if $3n$ does not divide s then

$$\forall w \in \{1, x, x^2, y, y^2\} \; \exists\, h \in H_n : \; (H_n \cdot w)\gamma_s(h) \neq H_n \cdot w.$$

The claim follows from this again with Corollary 3.8.

Take $h_1 := xyx^{-2}$, $h_2 := x^2yx^{-1} \in H_n$. Then $\gamma_s(h_1) = x^{s+1}yx^{-2}$ and $\gamma_s(h_2) = x^{s+2}yx^{-1}$.

We denote $\bar{1} := H_n$, $\bar{2} := H_n \cdot x$, $\bar{3} := H_n \cdot x^2$, $\bar{4} := H_n \cdot y$ and $\bar{5} := H_n \cdot y^2$. We see from Figure 27:

$$\begin{array}{rcl}
\bar{1} \cdot \gamma_s(h_1) & = & \bar{1} \cdot x^{s+1} y x^{-2} = \bar{1} \;\Leftrightarrow\; \bar{1} \cdot x^s = \bar{1} \cdot x^2 y^{-1} x^{-1} = \bar{1} \;\Leftrightarrow\; 3n \mid s. \\
\bar{2} \cdot \gamma_s(h_1) & = & \bar{2} \cdot x^{s+1} y x^{-2} = \bar{2} \;\Leftrightarrow\; \bar{2} \cdot x^s = \bar{2} \cdot x^2 y^{-1} x^{-1} = \overline{10} \neq \bar{2} \cdot x^s \quad \text{for all } s. \\
\bar{3} \cdot \gamma_s(h_2) & = & \bar{3} \cdot x^{s+2} y x^{-1} = \bar{3} \;\Leftrightarrow\; \bar{3} = \bar{3} \cdot x y^{-1} x^{-(s+2)} = \overline{10} \neq \bar{3} \quad \text{for all } s. \\
\bar{4} \cdot \gamma_s(h_1) & = & \bar{4} \cdot x^{s+1} y x^{-2} = \bar{5} \neq \bar{4} \quad \text{for all } s. \\
\bar{5} \cdot \gamma_s(h_1) & = & \bar{5} \cdot x^{s+1} y x^{-2} = \overline{5n-3} \neq \bar{5} \text{ for all } s.
\end{array}$$

From this it follows that if A_s is in $\Gamma(D_n)$, then $3n$ divides s. □

Also for this example we obtain that $\Gamma(H_\infty)$ is a nontrivial subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of infinite index.

Corollary 7.15. *The Veech group $\Gamma(H_\infty)$ is a nontrivial subgroup of infinite index in $\mathrm{SL}_2(\mathbb{Z})$.*

Proof. We use that :

$$\begin{array}{rrcll}
\forall s \in \mathbb{N}\, \exists n \in \mathbb{N}: & A_s & = & \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \notin \Gamma(H_n), & \text{see Proof of Prop. 7.14 and} \\
\forall n \in \mathbb{N}: & B_s & = & \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \in \Gamma(H_n), & \text{by Lemma 3.10.}
\end{array}$$

Again, the first fact shows that $\Gamma(H_\infty)$ has infinite index, and the second fact, that it contains nontrivial elements. □

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