Characterization of quasimonotonicity by means of functional inequalities

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Dedicated to the Memory of Raymond Moos Redheffer,
April 17, 1921 - May 13, 2005.

Abstract. It is known that quasimonotonicity of a continuous function can be characterized by means of differential inequalities. Using this we give a characterization by means of functional inequalities.

1 Notations

Let \( \mathbb{R} \) denote the reals, let \( E \) be a real Hausdorff topological vector space, and let \( K \) be a wedge in \( E \), i.e. a non-void subset satisfying

\[
\lambda \geq 0, \ x \in K, \ y \in K \Rightarrow \lambda(x + y) \in K.
\]

We suppose \( K \) to be closed and such that

\[
\text{Int } K \neq \emptyset.
\]

For \( x, y \in E \) we write

\[
\begin{align*}
& x \leq y \iff y - x \in K, \\
& x \ll y \iff y - x \in \text{Int } K.
\end{align*}
\]

\( K^* \) denotes the dual wedge of \( K \), i.e. the set of all linear, continuous \( \varphi : E \to \mathbb{R} \) satisfying \( \varphi(x) \geq 0 \) for \( x \in K \).

A function

(1) \[
f(t, x) : D \to E
\]

(where \( D \subseteq \mathbb{R} \times E \)) is called \textit{quasimonotone increasing} with respect to \( x \), if

\[
(t, x), (t, y) \in D, \ x \leq y, \varphi \in K^*, \varphi(x) = \varphi(y) \Rightarrow \varphi(f(t, x)) \leq \varphi(f(t, y)).
\]

For functions \( u : [t_0, t_1] \to E \) and \( t_0 \leq t \leq t_1 \) we mean by \( u'(t) \) the strong derivative

\[
\begin{align*}
u'(t) &= \lim_{h \to 0} \frac{u(t + h) - u(t)}{h} \text{ (if it exists).}
\end{align*}
\]

2 Known results and a question

The here used quasimonotonicity stems from [7]; Herzog [4] gives a survey of results. For functions (1) being quasimonotone increasing with respect to \( x \) the following is known (cf. [7]):
(P) If \( v, w : [t_0, t_1] \to E \) are continuous functions fulfilling \( v(t_0) \ll w(t_0) \)
and \( v'(t) - f(t, v(t)) \ll w'(t) - f(t, w(t)) \) \((t_0 < t \leq t_1)\), then \( v(t) \ll w(t) \) \((t_0 \leq t \leq t_1)\).

According to Uhl [6] we have the following (converse) result (which for Banach spaces \( E \) is known from [5]):

**Theorem A** Let \( D \) be an open subset of \( R \times E \), and let \( f : D \to E \) be a continuous function, for which (P) holds. Then \( f(t, x) \) is quasimonotone increasing with respect to \( x \).

In [8] quasimonotonicity occurs in the context of functional equations

\[
(2) \quad u(F(t)) + f(t, u(t)) = 0 \quad (t_0 \leq t \leq t_1)
\]

(cf. the surveys [2] and [1] for such equations), where

\[
(3) \quad t_0 \leq F(t) \leq t.
\]

According to [8] (and inspired by a talk of Brydak [3]) the following holds for functions (1) being quasimonotone increasing with respect to \( x \):

(Q) If \( v, w : [t_0, t_1] \to E \) are continuous functions fulfilling \( v(t_0) \ll w(t_0) \)
and \( w(F(t)) + f(t, w(t)) \ll v(F(t)) + f(t, v(t)) \) (with \( F \) satisfying (3) for \( t_0 < t \leq t_1 \)), then \( v(t) \ll w(t) \) \((t_0 \leq t \leq t_1)\).

Looking at Theorem A now the question arises: Suppose function (1) to be continuous \((D \) being an open subset of \( R \times E \)). Can we use property (Q) to characterize the quasimonotonicity of \( f \)?

### 3 A negative result

In this paragraph we assume

\[
(4) \quad f(t, x) : R \times E \to E \text{ continuous.}
\]

Suppose \( v, w : [t_0, t_1] \to E \) and \( F : [t_0, t_1] \to [t_0, t_1] \) are such that the hypotheses of (Q) are fulfilled. Passing to the limit \( t \downarrow t_0 \) in the functional inequality leads to

\[
w(t_0) + f(t_0, w(t_0)) \leq v(t_0) + f(t_0, v(t_0)).
\]

With

\[
(5) \quad v(t_0) \ll w(t_0)
\]

we then get

\[
(6) \quad f(t_0, w(t_0)) \ll f(t_0, v(t_0)).
\]
Now, if for \( t \in \mathbb{R} \) and \( a, b \in E \) we always have

\[ a \ll b \Rightarrow "f(t, b) \ll f(t, a)" \text{ does not hold}, \]

then (5), (6) cannot occur simultaneously, so the hypotheses of (Q) cannot be satisfied, hence (Q) is (vacuously) true. If \( K \neq E \), then a special case of (7) is a (weakly) monotone increasing function, i.e.

\[ a \leq b \Rightarrow f(t, a) \leq f(t, b). \]

On the other hand, if \( K = E \), then the conclusion of (Q) is always vacuously true. Summarizing we can state:

**Remark 1** If function (4) is monotone increasing with respect to \( x \) (cf. (8)), then (Q) is vacuously true.

Despite of this, (Q) will be used in a certain sense for a characterization of quasimonotonicity (cf. the next paragraph). But let us first state:

**Remark 2** Theorem A does not remain true, when (P) is replaced by (Q).

Let us give an example: \( E = \mathbb{R}^2 \) with its usual topology, ordered by \( K = R^2_+ = \{(x_1, x_2) \mid x_1 \geq 0, x_2 \geq 0\} \), and function (4) defined by

\[ f(t, x) = f(t, x_1, x_2) = (-x_2, 0). \]

This linear function is not quasimonotone increasing. On the other hand, (7) holds, hence also (Q).

### 4 A positive result

The starting point is the observation that function (1) remains quasimonotone increasing with respect to \( x \) if it is changed into

\[ f_1(t, x) = \lambda(t)x + h(t)f(t, x) \quad ((t, x) \in D) \]

with arbitrary

\[ \lambda : \mathbb{R} \rightarrow \mathbb{R},\ h : \mathbb{R} \rightarrow [0, \infty[. \]

Then we have (Q) also with all the functions (9), and this leads to an analogue of Theorem A, viz.

**Theorem B** Let \( D \) be an open subset of \( \mathbb{R} \times E \), and let \( f : D \rightarrow E \) be continuous. Suppose (Q) always to be true if \( f \) is replaced by \( f_1 \) from (9), the \( \lambda, h \) being as in (10). Then \( f(t, x) \) is quasimonotone increasing with respect to \( x \).

**Proof.** If not, then (P) does not hold (according to Theorem A). So there
are continuous \( v, w : [t_0, t_1] \to E \) (on an appropriate interval \([t_0, t_1]; \ t_0 < t_1\))
satisfying
\[
(11) \quad v(t_0) \ll w(t_0),
\]
\[
(12) \quad v'(t) - f(t, v(t)) \ll w'(t) - f(t, w(t)) \ (t_0 < t \leq t_1),
\]
but such that
\[
(13) \quad v(t) \ll w(t) \ (t_0 \leq t \leq t_1) \text{ does not hold.}
\]

Suppose \( t_0 < t \leq t_1 \). In (12) we approximate the derivatives \( v'(t), w'(t) \)
by left-handed difference quotients in such a manner that the inequality \( \ll \)
remains true:
\[
(14) \quad \frac{v(t) - v(t - h(t))}{h(t)} - f(t, v(t)) \ll \frac{w(t) - w(t - h(t))}{h(t)} - f(t, w(t)),
\]
where \( t_0 \leq t - h(t) < t \), hence \( h(t) > 0 \ (t_0 < t \leq t_1) \). Now
\[ F(t) = t - h(t) \ (t_0 < t \leq t_1) \]
has property (3), and (14) can be written as
\[ w(F(t)) - w(t) + h(t)f(t, w(t)) \ll v(F(t)) - v(t) + h(t)f(t, v(t)) \]
for \( t_0 < t \leq t_1 \). Together with (11) we therefore have the hypotheses of (Q)
fulfilled with \( f \) replaced by the function
\[ f_1(t, x) = -x + h(t)f(t, x) \ ((t, x) \in D) \]
\( (h(t) \geq 0 \text{ being defined arbitrarily for } t \notin [t_0, t_1]) \). By the hypotheses of Theo-
rem B we get \( v(t) \ll w(t) \ (t_0 \leq t \leq t_1) \), which is a contradiction to (13).

**Remark 3** In Uhl’s proof for Theorem A (cf. [6]), (P) is only needed for
linear functions \( v(t) = a + tp, w(t) = b + tq \ (a, b, p, q \in E) \). Taking this into
account, other versions of Theorem B are possible. Our approach reflects
some kind of idea of a general comparison of the functional equation (2) and
the differential equation \( u'(t) = f(t, u(t)) \).

**Acknowledgement:**
The research of both authors was supported by the Mathematics Depart-
ment of the Silesian University at Katowice (program “Iterative Functional
Equations and Real Analysis”). The research of the first author also was
supported by the DFG (Deutsche Forschungsgemeinschaft).
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Typescript: Marion Ewald

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