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Characterization of quasimonotonicity by means of functional inequalities

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Dedicated to the Memory of Raymond Moos Redheffer,
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Abstract. It is known that quasimonotonicity of a continuous function can be characterized by means of differential inequalities. Using this we give a characterization by means of functional inequalities.

1 Notations

Let R denote the reals, let E be a real Hausdorff topological vector space, and let K be a wedge in E, i.e. a non-void subset satisfying

$$\lambda \geq 0,\ x \in K,\ y \in K \Rightarrow \lambda(x+y) \in K.$$

We suppose K to be closed and such that

$$\text{Int } K \neq \emptyset.$$

For $x, y \in E$ we write

$$x \leq y \Leftrightarrow y - x \in K,$$
$$x \ll y \Leftrightarrow y - x \in \text{Int } K.$$

K^* denotes the dual wedge of K, i.e. the set of all linear, continuous $\varphi : E \to R$ satisfying $\varphi(x) \geq 0$ for $x \in K$.

A function

$$f(t,x) : D \to E \tag{1}$$

(where $D \subseteq R \times E$) is called *quasimonotone increasing* with respect to x, if

$$(t,x), (t,y) \in D, x \leq y, \varphi \in K^*, \varphi(x) = \varphi(y) \Rightarrow \varphi(f(t,x)) \leq \varphi(f(t,y)).$$

For functions $u : [t_0, t_1] \to E$ and $t_0 \leq t \leq t_1$ we mean by $u'(t)$ the strong derivative

$$u'(t) = \lim_{h \to 0} \frac{u(t+h) - u(t)}{h}$$

(if it exists).

2 Known results and a question

The here used quasimonotonicity stems from [7]; Herzog [4] gives a survey of results. For functions (1) being quasimonotone increasing with respect to x the following is known (cf. [7]):

(P) If $v, w : [t_0, t_1] \to E$ are continuous functions fulfilling $v(t_0) \ll w(t_0)$ and $v'(t) - f(t, v(t)) \ll w'(t) - f(t, w(t))$ $(t_0 < t \le t_1)$, then $v(t) \ll w(t)$ $(t_0 \le t \le t_1)$.

According to Uhl [6] we have the following (converse) result (which for Banach spaces E is known from [5]):

Theorem A Let D be an open subset of $R \times E$, and let $f : D \to E$ be a continuous function, for which (P) holds. Then $f(t, x)$ is quasimonotone increasing with respect to x.

In [8] quasimonotonicity occurs in the context of functional equations

$$u(F(t)) + f(t, u(t)) = 0 \ (t_0 \le t \le t_1) \tag{2}$$

(cf. the surveys [2] and [1] for such equations), where

$$t_0 \le F(t) \le t. \tag{3}$$

According to [8] (and inspired by a talk of Brydak [3]) the following holds for functions (1) being quasimonotone increasing with respect to x:

(Q) If $v, w : [t_0, t_1] \to E$ are continuous functions fulfilling $v(t_0) \ll w(t_0)$ and $w(F(t)) + f(t, w(t)) \ll v(F(t)) + f(t, v(t))$ (with F satisfying (3) for $t_0 < t \le t_1$), then $v(t) \ll w(t)$ $(t_0 \le t \le t_1)$.

Looking at Theorem A now the question arises: Suppose function (1) to be continuous (D being an open subset of $R \times E$). Can we use property (Q) to characterize the quasimonotonicity of f?

3 A negative result

In this paragraph we assume

$$f(t, x) : R \times E \to E \text{ continuous.} \tag{4}$$

Suppose $v, w : [t_0, t_1] \to E$ and $F :]t_0, t_1] \to [t_0, t_1]$ are such that the hypotheses of (Q) are fulfilled. Passing to the limit $t \downarrow t_0$ in the functional inequality leads to

$$w(t_0) + f(t_0, w(t_0)) \le v(t_0) + f(t_0, v(t_0)).$$

With

$$v(t_0) \ll w(t_0) \tag{5}$$

we then get

$$f(t_0, w(t_0)) \ll f(t_0, v(t_0)). \tag{6}$$

Now, if for $t \in R$ and $a, b \in E$ we always have

$$a \ll b \Rightarrow \text{“}f(t,b) \ll f(t,a) \text{ does not hold”}, \tag{7}$$

then (5), (6) cannot occur simultaneously, so the hypotheses of (Q) cannot be satisfied, hence (Q) is (vacuously) true. If $K \neq E$, then a special case of (7) is a (weakly) monotone increasing function, i.e.

$$a \leq b \Rightarrow f(t,a) \leq f(t,b). \tag{8}$$

On the other hand, if $K = E$, then the conclusion of (Q) is always vacuously true. Summarizing we can state:

Remark 1 If function (4) is monotone increasing with respect to x (cf. (8)), then (Q) is vacuously true.

Despite of this, (Q) will be used in a certain sense for a characterization of quasimonotonicity (cf. the next paragraph). But let us first state:

Remark 2 Theorem A does not remain true, when (P) is replaced by (Q).

Let us give an example: $E = R^2$ with its usual topology, ordered by $K = R^2_+ = \{(x_1, x_2) \mid x_1 \geq 0, x_2 \geq 0\}$, and function (4) defined by

$$f(t,x) = f(t, x_1, x_2) = (-x_2, 0).$$

This linear function is not quasimonotone increasing. On the other hand, (7) holds, hence also (Q).

4 A positive result

The starting point is the observation that function (1) remains quasimonotone increasing with respect to x if it is changed into

$$f_1(t,x) = \lambda(t)x + h(t)f(t,x) \quad ((t,x) \in D) \tag{9}$$

with arbitrary

$$\lambda : R \to R,\ h : R \to [0, \infty[. \tag{10}$$

Then we have (Q) also with all the functions (9), and this leads to an analogue of Theorem A, viz.

Theorem B Let D be an open subset of $R \times E$, and let $f : D \to E$ be continuous. Suppose (Q) always to be true if f is replaced by f_1 from (9), the λ, h being as in (10). Then $f(t,x)$ is quasimonotone increasing with respect to x.
P r o o f. If not, then (P) does not hold (according to Theorem A). So there

are continuous $v, w : [t_0, t_1] \to E$ (on an appropriate interval $[t_0, t_1];\ t_0 < t_1$) satisfying

$$v(t_0) \ll w(t_0), \tag{11}$$

$$v'(t) - f(t, v(t)) \ll w'(t) - f(t, w(t)) \ (t_0 < t \leq t_1), \tag{12}$$

but such that

$$v(t) \ll w(t) \ (t_0 \leq t \leq t_1) \text{ does not hold.} \tag{13}$$

Suppose $t_0 < t \leq t_1$. In (12) we approximate the derivatives $v'(t), w'(t)$ by left-handed difference quotients in such a manner that the inequality $\ll$ remains true:

$$\frac{v(t) - v(t - h(t))}{h(t)} - f(t, v(t)) \ll \frac{w(t) - w(t - h(t))}{h(t)} - f(t, w(t)), \tag{14}$$

where $t_0 \leq t - h(t) < t$, hence $h(t) > 0$ $(t_0 < t \leq t_1)$. Now

$$F(t) = t - h(t) \ (t_0 < t \leq t_1)$$

has property (3), and (14) can be written as

$$w(F(t)) - w(t) + h(t)f(t, w(t)) \ll v(F(t)) - v(t) + h(t)f(t, v(t))$$

for $t_0 < t \leq t_1$. Together with (11) we therefore have the hypotheses of (Q) fulfilled with f replaced by the function

$$f_1(t, x) = -x + h(t)f(t, x) \ ((t, x) \in D)$$

($h(t) \geq 0$ being defined arbitrarily for $t \notin]t_0, t_1]$). By the hypotheses of Theorem B we get $v(t) \ll w(t)$ $(t_0 \leq t \leq t_1)$, which is a contradiction to (13).

Remark 3 In Uhl's proof for Theorem A (cf. [6]), (P) is only needed for linear functions $v(t) = a + tp, w(t) = b + tq$ $(a, b, p, q \in E)$. Taking this into account, other versions of Theorem B are possible. Our approach reflects some kind of idea of a general comparison of the functional equation (2) and the differential equation $u'(t) = f(t, u(t))$.

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References

1. Karol Baron, Recent results in the theory of functional equations in a single variable, http://www.mathematik.uni-karlsruhe.de/~semlv, Sem. LV, No. 15, 16 pp. (2003).

2. – and Witold Jarczyk, Recent results on functional equations in a single variable, perspectives and open problems, Aequationes Math. **61**, 1-48 (2001).

3. Dobiesław Brydak, On a linear functional inequality, Talk at the 8th International Conference on Functional Equations and Inequalities, Złockie, September 10-15, 2001, Abstract in Ann. Acad. Paedagog. Cracov. **13** (Studia Mathematica II), 66-67 (2002).

4. Gerd Herzog, Quasimonotonicity, Nonlinear Analysis **47**, 2213-2224 (2001).

5. Alice Simon and Peter Volkmann, Remark on quasimonotonicity, World Scientific Singapore, World Scientific Series in Applicable Analysis 3, 543-548 (1994).

6. Roland Uhl, Ordinary differential inequalities and quasimonotonicity in ordered topological vector spaces, Proc. Amer. Math. Soc. **126**, 1999-2003 (1998).

7. Peter Volkmann, Gewöhnliche Differentialungleichungen mit quasimonoton wachsenden Funktionen in topologischen Vektorräumen, Math. Z. **127**, 157-164 (1972).

8. –, Funktionalungleichungen in topologischen Vektorräumen, http://www.mathematik.uni-karlsruhe.de/~semlv, Sem. LV, No. 12, 4 pp. (2002).

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