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Vibrations of Piezoceramic Rods

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Presently, most of the research on vibrations of monolithic piezoelectric rods at weak electric fields is restricted to longitudinal oscillations of such structural members where free and forced vibrations have been dealt with and in the case of resonance conditions not only linear but also nonlinear effects within the constitutive relations have been incorporated. On the other hand, bending and torsional vibrations of piezoceramic one-parametric rods have not been examined yet. The present contribution develops a linear vibration theory of rods with a focus to bending vibrations taking into account rotatory inertia and shear deformation. The governing boundary value problem for beams with both longitudinal and as well transversal polarization is derived, in particular free vibrations are analyzed. Also nonlinear extensions not only of physical nature but also geometrical ones are addressed. A possible technical application is given.

1 Physical model

Consider a uniform, slender, flexible rod with an axi-symmetric cross-section. Stressless, it is straight and has length ℓ (undeformed arc length z, cross-sectional coordinates x, y). The rod is assumed to be a monolithic piezoelectric solid with a preference direction by polarization so that there is in a good approximation a planar isotropy. As a first case, an axial polarization (i.e. in z-direction as usual) is assumed. Space- and time-dependent extensional, bending/shearing and torsional vibrations $w(z,t), u(z,t)$ (in x-), $v(z,t)$ (in y-direction), $\beta(z,t)$ (about the y-), $\alpha(z,t)$ (about the x-axis) and $\psi(z,t)$, respectively, coupled with the electric potential $\varphi(z,t)$ are discussed. A linear theory will be established, i.e., both linear strain-displacement relations and linear constitutive equations will be formulated as a first step developing a generalized nonlinear theory.

2 Formulation

The linear strain-displacement relations for such a Timoshenko beam are well-known (see [1], for example) and read

$$S_{zz} = w' + y\alpha' - x\beta', \quad 2S_{xz} = u' - \beta - y\psi', \quad 2S_{yz} = v' + \alpha + x\psi'. \tag{1}$$

where $(.)' = \partial(.)/\partial z$. Within a rod theory, usual constraint conditions with respect to the stresses are

$$T_{xx} = T_{yy} = T_{xy} = 0. \tag{2}$$

Similar as for the strains (see eq. (1)), also for the electric potential ϕ which in general depends from x, y, z and t, a linear dependence from the cross-sectional coordinates is assumed (a generalization to another more realistic dependence, see [3], for instance, is straightforward):

$$\phi(x,y,z,t) = (1 + ax + by)\varphi(z,t) \tag{3}$$

where the electric field coordinates are related to ϕ by $E_i = -\partial\phi/\partial i,\ i \in x, y, z$ and a, b are appropriate constants. Under the assumptions noticed, the well-known constitutive equations for linear piecoceramics (polarized in z-direction) can therefore be simplified to

$$\begin{aligned} T_{zz} &= E_{\mathrm{ax}}^{(0)} S_{zz} - \gamma_{\mathrm{ax}}^{(0)} E_z, \quad T_{xz} = G_{\mathrm{ax}}^{(0)} S_{zz} - e_{15} E_x, \quad T_{yz} = G_{\mathrm{ax}}^{(0)} S_{yz} - e_{15} E_y, \\ D_x &= e_{15} S_{xz} + \varepsilon_{11}^S E_x, \quad D_y = e_{15} S_{yz} + \varepsilon_{11}^S E_y, \quad D_z = \gamma_{\mathrm{ax}}^{(0)} S_{zz} + \nu_{\mathrm{ax}}^{(0)} E_z \end{aligned} \tag{4}$$

where $E_{\mathrm{ax}}^{(0)}$ and $G_{\mathrm{ax}}^{(0)}$ are, respectively elastic modulus and shear modulus (depending from the usual stiffness constants c_{ij}^E), $\gamma_{\mathrm{ax}}^{(0)}$ is a piezoelectric coupling factor (depending from c_{ij}^E and the piezoelectric constants e_{ij}) and $\nu_{\mathrm{ax}}^{(0)}$ is a dielectric property (depending from c_{ij}^E, e_{ij} and the dielectric constants ε_{ij}^S). The D_i are the dielectric displacements.

To derive the governing value problem by Hamilton's principle, kinetic energy density T and enthalpy density H of the rod and the virtual work W_{virt} of all potentialless actions in the system have to be determined. Using the governing kinematics and the constitutive equations (4), one obtains (if free vibrations neglecting any dissipative effects will be considered)

$$\begin{aligned} T &= \rho[\dot{u}^2 + \dot{v}^2 + (\dot{w} + y\dot{\alpha} - x\dot{\beta})^2 + (x^2+y^2)\dot{\psi}^2], \quad W_{\mathrm{virt}} = 0, \\ H &= \frac{E_{\mathrm{ax}}^{(0)}}{2} S_{zz}^2 - \gamma_{\mathrm{ax}}^{(0)} S_{zz} E_z - \frac{\nu_{\mathrm{ax}}^{(0)}}{2} E_z^2 + \frac{G_{\mathrm{ax}}^{(0)}}{2}(S_{xz}^2 + S_{yz}^2) - e_{15}(S_{xz}E_x + S_{yz}E_y) - \frac{\varepsilon_{11}^S}{2}(E_x^2 + E_y^2) \end{aligned} \tag{5}$$

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where $\dot{(.)} = \partial(.)/\partial t$. The resulting boundary value problem (defining in the usual way cross-sectional moments of inertia I_x, I_y, I_p and introducing both a corrected torsional stiffness $G_{\rm ax}^{(0)} I_T$ as well as shear stiffnesses $\kappa_{x,y} G_{\rm ax}^{(0)} A$) is then composed by a set of partial differential equations

$$\begin{aligned}
\rho\ddot{u} - \kappa_x G_{\rm ax}^{(0)}(u'' - \beta') - e_{15} a\varphi' = 0, \quad \rho\ddot{v} - \kappa_y G_{\rm ax}^{(0)}(v'' + \alpha') - e_{15} b\varphi' &= 0,\\
\rho I_y \ddot{\beta} - E_{\rm ax}^{(0)} I_y \beta'' + \kappa_x G_{\rm ax}^{(0)} A(u' - \beta) + \gamma_{\rm ax}^{(0)} a I_y \varphi'' - e_{15} A a\varphi &= 0,\\
\rho I_x \ddot{\alpha} - E_{\rm ax}^{(0)} I_x \alpha'' + \kappa_y G_{\rm ax}^{(0)} A(v' + \alpha) - \gamma_{\rm ax}^{(0)} b I_x \varphi'' + e_{15} A b\varphi &= 0,\\
\rho\ddot{w} - E_{\rm ax}^{(0)} w'' - \gamma_{\rm ax}^{(0)} \varphi'' &= 0,\\
\nu_{\rm ax}^{(0)}(A + a^2 I_y + b^2 I_x)\varphi'' - \varepsilon_{11} A(a^2 + b^2)\varphi - \gamma_{\rm ax}^{(0)}(A w'' - b I_x \alpha'' + a I_y \beta'') &= 0, \qquad (6)
\end{aligned}$$

$$\rho I_p \ddot{\psi} - G_{\rm ax}^{(0)} I_T \psi'' = 0 \qquad (7)$$

to be supplemented by "geometric" or/and dynamic boundary conditions at the ends $z = 0, \ell$ of the rod not specified here.

3 Qualitative Results

As expected for an axi-symmetric cross-section and a polarization in axial direction, the torsional oscillations $\psi(z,t)$ are purely elastic and decoupled from all other variables $u(z,t), v(z,t), \alpha(z,t), \beta(z,t), w(z,t)$ and $\varphi(z,t)$. It is obvious that for a electric potential depending from the cross-sectional coordinates ($a, b \neq 0$) a complicated coupling of the remaining variables appears. Calculations of the corresponding eigenfrequencies show that the coupling even for large a, b is weak and there is only a marginal correction compared with the case that the electric potential is constant in the transverse direction ($a, b = 0$). For that case, the coupling between elastic bending and piezoelectric axial vibrations vanishes. The remaining boundary value problem in $w(z,t)$ and $\varphi(z,t)$ describing piezoelectric longitudinal vibrations coincide with formulations given in the literature (see [5], for instance) and leads to a significant correction of the eigenvalues for elastic rods.

4 Supplements

The governing boundary value problem in the case of a transversally polarized rod can straightforwardly deduced from that of an axially polarized rod if the axial coordinate is renamed as y, for instance and the cross-sectional coordinates become x and (as direction of polarization) z. The axial oscillations are then denoted as $v(y,t)$, the bending vibrations are $u(y,t)$ and $\alpha(y,t)$ (in and respectively about the x-axis) as well $w(y,t)$ and $\beta(y,t)$ (in and respectively about the z-axis) and the torsional oscillation remains unchanged: $\psi(y,t)$. Performing all calculation steps as before (introducing corresponding geometrical and material rod parameters where $G_{\rm tr}^{(0)}$ is an averaged value of $G_{{\rm tr}\,x}^{(0)}$ and $G_{{\rm tr}\,z}^{(0)}$) leads to the final set of differential equations

$$\begin{aligned}
\rho\ddot{u} - \kappa_x G_{{\rm tr}\,x}^{(0)}(u'' + \beta') = 0, \quad \rho I_z \ddot{\beta} - E_{\rm tr}^{(0)} I_z \beta'' - \kappa_x G_{{\rm tr}\,x}^{(0)} A(u' + \beta) &= 0, \qquad (8)\\
\rho\ddot{w} - \kappa_z G_{{\rm tr}\,z}^{(0)}(w'' - \alpha') - e_{15}\varphi'' = 0, \quad \rho I_x \ddot{\alpha} - E_{\rm tr}^{(0)} I_x \alpha'' - \kappa_z G_{{\rm tr}\,z}^{(0)} A(w' - \alpha) - e_{15} A\varphi' &= 0,\\
\rho\ddot{v} - E_{\rm tr}^{(0)} v'' - \gamma_{\rm tr}^{(0)} a\varphi' = 0, \quad \rho I_p \ddot{\psi} - G_{\rm tr}^{(0)} I_T \psi'' + e_{15} I_z b\varphi'' &= 0,\\
\varepsilon_{11}^S (A + a^2 I_x + b^2 I_z)\varphi'' - (\nu_{{\rm tr}\,z}^{(0)} a^2 + \nu_{{\rm tr}\,x}^{(0)} b^2) A\varphi + \gamma_{\rm tr}^{(0)} A a v' - e_{15} A(w'' - \alpha') + e_{15} I_z b\psi' &= 0 \qquad (9)
\end{aligned}$$

to be closed by corresponding boundary conditions. As expected for a rod polarized in the direction of one of the principal axes of inertia, the elastic bending vibrations $u(y,t)$, $\beta(y,t)$ in the orthogonal transverse direction are decoupled from all the other variables $w(y,t), \alpha(y,t), v(y,t), \psi(y,t)$ and $\varphi(y,t)$ representing coupled piezoelectric motions. For a potential constant in transverse direction, there remain coupled piezoelectric bending vibrations $w(y,t), \alpha(y,t), \varphi(y,t)$ which decouple from the torsional vibrations $\psi(y,t)$ and also the axial oscillations $v(y,t)$ both purely elastic in that case.

To add viscous effects and to extend the considerations to both geometrical and physical nonlinearities is straightforward (see [1] and [5, 2], respectively) but involves great calculation expense.

A possible practical application where a geometrically and physically nonlinear theory is needed to analyze the function in a quantitative manner, might be a rod-shaped angular rate sensor (see [4], for instance) .

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