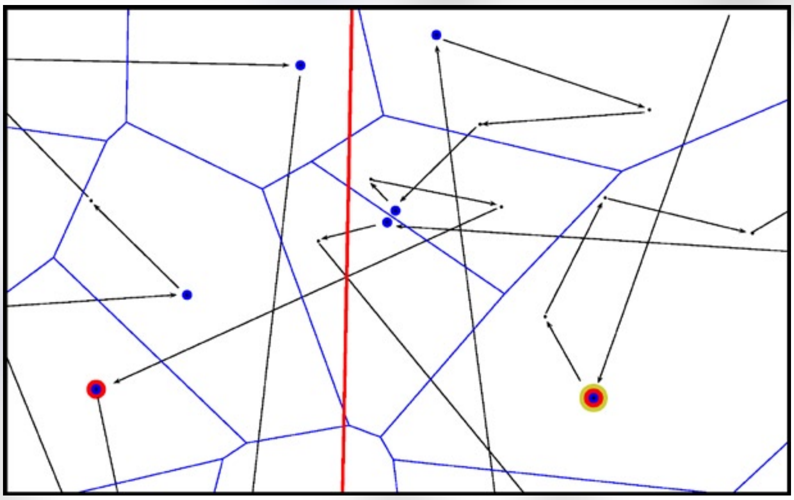
Matthias Heveling

Bijective point maps, point-stationarity and characterization of Palm measures



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Korreferent: Prof. Dr. Wolfgang Weil

Meinen Eltern in Dankbarkeit gewidmet.

Vorwort

Diese Dissertation entstand im Rahmen meiner Tätigkeit als wissenschaftlicher Mitarbeiter an der Universität Karlsruhe, zunächst im Mathematischen Institut II, dann im Institut für Mathematische Stochastik. Ich möchte all jenen danken, die zum Gelingen dieser Arbeit, und zum Gelingen dieser fast dreieinhalb Jahre beigetragen haben, die ich als lebendige, erfüllte und bereichernde Zeit empfinde.

Mein besonderer Dank gilt Herrn Prof. Dr. Günter Last, der mich an die hier behandelten Fragestellungen, wie auch an andere Themen seiner aktuellen Forschung, herangeführt hat. In diesen Dank möchte ich Herrn Prof. Dr. Wolfgang Weil einbeziehen, der mir als Korreferent der Arbeit ebenso hilfreich zur Seite stand. Beide haben meine Arbeit mit zahlreichen wertvollen Anregungen und Ratschlägen unterstützt und waren immer bereit, meine Fragen und Probleme zu diskutieren.

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Contents

Chapter 1

Introduction

The subject of the probabilistic part of this thesis are Palm measures and Palm distributions of stationary point processes and, more generally, of stationary random measures. The so-called Palm theory of stationary point processes named after the Swedish engineer Conny Palm is today an integral part of modern probability theory and its applications. While the distribution of a stationary point process describes the statistical properties as seen from a "randomly chosen" site in space, the Palm distribution is the conditional distribution given that there is a "typical" point of the process at the origin. Such a change of viewpoint from an absolute to an intrinsic frame of reference is not only useful for understanding point process properties but is also important for many applications.

The thesis of Palm (1943) himself includes both a general mathematical section with contributions to stochastic point processes, and a specialized section on teletraffic intensity variations. Today this field is intregrated into queueing theory, where Palm probabilities provide the natural framework for formulating and studying basic relationships between time and event averages. Another important field of applications is stochastic geometry. Already the definition of some of the basic characteristics, as the distribution of the volume of the typical cell of a random tessellation or of a rose of directions of a random surface process, require the usage of Palm distributions.

Recent years have seen some interesting and partly even surprising developments in the theory of stationary spatial point processes. One starting point was the observation made by Thorisson (1999), that a point process is "point stationary" under its Palm distribution, meaning that its statistical properties do not change, when the origin is shifted to another point of the process, provided this point is chosen in an unbiased ("bijective") way. In fact point stationarity is just a special case of some invariance properties found by Mecke (1974). Thorisson then asked whether point stationarity is characteristic for Palm distributions, a problem that has finally been solved in Heveling and Last (2005).

Another quite intriguing and stimulating question asked by Ferrari, Landim und Thorisson (2004) was for the existence of a linear order between the points of a stationary Poisson process that is almost surely preserved under translations. In low dimensions ($d = 2, 3$), they proved the existence of an equivariant random tree on the points of the Poisson process that can be transformed into a linear order. This approach inspired consecutive work by Holroyd and Peres (2003), who extended the results to higher dimensions, and by Timar (2004), who could even show that any stationary point process (with finite intensity) allows such a linear order.

The problem of linear orderings seems only on first glance to be unrelated to Palm measures. Actually, in the case of the real line, where a trivial linear order exists, a well-known result for point processes states that, under some mild side conditions, a point process is the Palm measure of some stationary point process if and only if it is invariant under the shift, which translates the succesor of 0 to the origin. In higher dimensions, it turns out that certain countable families of bijective point maps are sufficient to take the role of the universal point map in this characterization of Palm measures. The definition of such families is crucial for this appoach, which relies on Mecke's (1967) intrinsic characterization of Palm measures, and provides a new interpretation of it. Indeed, it turns out that the integral equation in this characterization theorem is equivalent with point stationarity.

The first part of this thesis, which comprises Chapters 2 and 3, is completely measure-free. We work on a locally compact, second countable Hausdorff (lcscH) group G, which is assumed to be Abelian and equipped with its Borel σ-field $\mathcal{G}$. A flow of translation or shift operators is introduced and various classes of shift equivariant and shift invariant functions are discussed, including (extended) index functions, selection functions, factor graphs, (extended) point maps and point shifts.

Point maps are defined without any reference to a point process or probability space, as measurable functions that map a locally finite subset φ to a point of φ, whenever 0 is a point of φ. We associate a (deterministic) point shift $\theta_\sigma : \mathbf{L}(G) \to \mathbf{L}(G)$ with the point map σ, which translates a locally finite set φ in such a way that the point $\sigma(\varphi)$ selected by the point map is moved to the origin. The extended point map $\tilde{\sigma}$ is defined as a mapping on the product space $\mathbf{L}(G) \times G$ and yields an interpretation of the point map σ as a function of a locally finite set φ and a point $x \in \varphi$. This interpretation motivates the non-standard definition of bijective point maps. We establish that the set of bijective point maps equipped with the composition is a group, which acts as a group operation on $\mathbf{L}(G)$.

In Chapter 3, various examples of families of point maps are defined. On the real line, there exists a single (universal) point map τ that generates a complete family of bijective point shifts, meaning that the iterates of τ and its inverse τ^{-1} exhaust the points of any locally finite subset φ of $\mathbb{R}$ that contains the origin. On general lcscH groups G, and even on $\mathbb{R}^d$, it turns out that it is much more difficult to define complete families of point maps, and the application of point maps to simple examples of periodic sets $\varphi \in \mathbf{L}(\mathbb{R}^d)$ shows that one cannot hope for a single point map to generate a complete family. However, a countable and quasi-complete family of self-inverse, bijective point maps (matchings) on $\mathbf{L}(G)$ is given (Theorem 3.2.6). In the special case $G = \mathbb{R}^d$, a bijective point map is defined, that is complete on large subclasses of $\mathbf{L}_0(\mathbb{R}^d)$, that turn out to be meaningful in a probabilistic framework (cf. Chapter 5). Moreover, we can show that two bijective point maps suffice to generate a complete family of point maps on $\mathbf{L}(\mathbb{R}^d)$ (Theorem 3.4.1).

The second part of this thesis, starting with Chapter 4, begins with a short review of the now classical Palm theory for stationary measures on $\mathbf{M}(G)$, the space of locally finite measures on a lcscH group G. We follow Mecke's (1967) exposition, which generalizes earlier work by Matthes (1963) for the case of the real line. In particular, we state the intrinsic characterization theorem for Palm measures (Theorem 4.2.2), where Palm measures are identified as the only σ-finite measures that satisfy a certain integral equation. We do not have anything to add to this particularly beautiful result per se, but the characterization results of Chapter 6 may be regarded as a new interpretation of the integral formula.

In Chapter 5, we take up the discussion from Chapter 3 in a probabilistic setting. A recent result by Timar (2004) defines a linear order on the points of any stationary point process, which is almost surely preserved under translations. We restate this result in terms of point maps and provide an alternative proof using Palm calculus (Theorem 5.2.3). In particular, the existence of a universal point map on general stationary point processes allows a generalisation to higher dimensions of one direction of the key stationarity theorem, that provides a one-to-one correspondance between stationary point processes on $\mathbb{R}$ and stationary sequences of non-negative random variables (Proposition 5.3.3).

In the final chapter, the discussions of point maps and Palm measures merge to the central problem of this thesis: The characterization of Palm measures by point stationarity. This concept was introduced by Thorisson (1999), and formalizes the intuitive idea of a point process for which the behaviour relative to a given point of the process is independant of the point selected as origin. Now, where large families of bijective point shifts are known (even in the general case of a lcscH group), point stationarity can be (and actually is) defined as invariance under bijective point shifts. Thorisson (1999) showed that the Palm version of a stationary point process is invariant under bijective point shifts. A more general, but implicit, proof of this fact was given by Mecke(1974). The converse result was established by Heveling and Last (2005), where, for the first time, a countable, complete family of bijective point maps was defined and used in the proof of the characterization result. Slightly adapting this technique of proof, we generalize the characterization result from $\mathbb{R}^d$ to a general lcscH group G, and from simple counting measures to discrete measures (Theorem 6.3.7). Both generalizations are needed in the last section, where an appoximation procedure for general random measures on $\mathbb{R}^d$ is defined that allows a further extension of the characterization results (Theorem 6.4.8).

Chapter 2

Flow equivariant and flow invariant analysis

2.1 Preliminaries

We will first introduce the general setting of this thesis. Let $(G, +)$ be an Abelian group and denote, as usual, the inverse element of $x \in G$ by $-x$ and the neutral element in G by 0. We assume that G is equipped with a topology $\mathcal{T}$, that the addition is a continuous mapping from $G \times G$ (equipped with the product topology) to G and that the mapping $G \to G, x \mapsto -x$ is also continuous.

Moreover, we assume that $(G, \mathcal{T})$ is a locally compact, second countable Hausdorff space (*lcscH space*). Throughout, we will fix a dense sequence (z_n) in G, and a countable base $\mathcal{B} = (B_n)$ of the topology of G, where $B_n, n \in \mathbb{N}$, are assumed to be open and relatively compact, i.e., with compact closure (cf. Theorem A.1.3). The Borel σ-field on G will be denoted by $\mathcal{G}$.

By Theorem A.1.1, there exists a metric d on G, which is compatible with the topology of G. Given a non-empty subset $A \subset G$, we will write $d(x, A) := \inf\{d(x, y) : y \in A\}$ for the distance of a point $x \in G$ from a set $A \subset G$. The open ball with centre x and radius r will be denoted by $B_d(x, r) := \{y \in G : d(x, y) < r\}$.

The locally finite subsets $\mathbf{L}(G)$ comprise all subsets φ of G such that $\varphi \cap C$ is a finite set for all relatively compact subsets C of G. On $\mathbf{L}(G)$ we introduce the *hit-and-miss σ-field* $\mathcal{L}(G)$, which is generated by the mappings $h_B : \mathbf{L}(G) \to \{0, 1\}, B \in \mathcal{G}$, defined by

$$h_B(\varphi) := \mathbf{1}\{\varphi \cap B \neq \emptyset\}.$$

We denote by $\mathbb{N} = \{1, 2, \ldots\}$ the natural numbers starting with one, and the power set of a set A by $\mathcal{P}(A)$.

We will begin with some general measurability results of elementary mappings on the spaces that we have introduced so far. Similar results and a by far more complete account on measurability are given in [1], [24] and [15].

Lemma 2.1.1. *The mapping $f : \mathbf{L}(G) \times G \to \{0, 1\}$ defined by $f(\varphi, x) := \mathbf{1}\{x \in \varphi\}$ is $(\mathcal{L}(G) \otimes \mathcal{G}, \mathcal{P}(\{0, 1\}))$-measurable.*

PROOF: Define $C := \{(\varphi, x) \in \mathbf{L}(G) \times G : x \in \varphi\}$. We have $f^{-1}(\{1\}) = C$, and it is sufficient to show that C is a $\mathcal{L}(G) \otimes \mathcal{G}$-measurable set. Recall that (z_n) is a dense sequence in G. We claim that

$$C = \bigcap_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \left(\{\psi \in \mathbf{L}(G) : \psi \cap B_d(z_n, 1/m) \neq \emptyset\} \times B_d(z_n, 1/m)\right).$$

Indeed, if $(\varphi, x) \in C$, then for all $m \in \mathbb{N}$ there exists an $n \in \mathbb{N}$ such that $d(x, z_n) < 1/m$, hence, $\varphi \cap B_d(z_n, 1/m) \neq \emptyset$ and $(\varphi, x) \in \{\psi \in \mathbf{L}(G) : \psi \cap B_d(z_n, 1/m) \neq \emptyset\} \times B_d(z_n, 1/m)$.

Conversely, assume that $(\varphi, x) \notin C$. Then there exists $\varepsilon > 0$ such that $\varphi \cap B_d(x, \varepsilon) = \emptyset$. For all $m > 2/\varepsilon$, we have $B_d(z_n, 1/m) \subset B_d(x, \varepsilon)$ whenever $d(x, z_n) < 1/m$, and hence, $\varphi \cap B_d(z_n, 1/m) = \emptyset$ whenever $x \in B_d(z_n, 1/m)$. □

Lemma 2.1.2. *The counting mappings $c_B : \mathbf{L}(G) \to [0, \infty], B \in \mathcal{G}$, defined by $c_B(\varphi) := \mathrm{card}(\varphi \cap B)$ are $(\mathcal{L}(G), \mathcal{B}([0, \infty]))$-measurable, where $\mathcal{B}([0, \infty])$ denotes the usual Borel σ-field on $[0, \infty]$.*

PROOF: Let P_n be the partition of G, that is generated by the first n elements of the countable base $\mathcal{B}$, i.e., $P_1 = \{B_1, B_1^c\}, P_2 = \{B_1 \cap B_2, B_1 \cap B_2^c, B_1^c \cap B_2, B_1^c \cap B_2^c\}, \ldots$. Then define $c_{n,B} : \mathbf{L}(G) \to [0, \infty]$ by

$$c_{n,B}(\varphi) := \sum_{C\in P_n} \mathbf{1}\{\varphi \cap (C \cap B) \neq \emptyset\}.$$

Then $c_{n,B}$ is an increasing sequence of $(\mathcal{L}(G), \mathcal{B}([0, \infty]))$-measurable functions on $\mathbf{L}(G)$. Since, for $x, y \in G$ and $x \neq y$, there exists some $n \in \mathbb{N}$ such that $x \in B_n$ and $y \notin B_n$, we have

$$\lim_{n\to\infty} c_{n,B}(\varphi) = c_B(\varphi),$$

so c_B is measurable as the pointwise limit function of a sequence of measurable functions. □

A *total order relation* $\leq$ on a set $\mathbb{X}$ satisfies the axioms of an order relation (reflexivity, anti-symmetry, transitivity) and for all distinct $x, y \in \mathbb{X}$ we have either $x \leq y$ or $y \leq x$.

Lemma 2.1.3. *There exists a total order relation $\prec$ on G, such that any locally finite subset φ of G equipped with the restriction of $\prec$ to $\varphi \times \varphi$ is a well-ordered set, i.e., any subset $\psi \subset \varphi$ has a minimal element with respect to $\prec$. Moreover, the sets $S_x := \{y \in G : y \prec x\}, x \in G$, are Borel subsets of G.*

PROOF: Define a function $f : G \to \{0, 1\}^{\mathbb{N}}$ by $f := (f_n)$ and $f_n(x) := \mathbf{1}\{x \in B_n\}$. As we have mentioned in the proof of Lemma 2.1.2, for $x, y \in G$ and $x \neq y$, there exists some $n \in \mathbb{N}$ such that $x \in B_n$ and $y \notin B_n$, hence, f is injective on G.

On $\{0, 1\}^{\mathbb{N}}$, we define the (reverse lexicographic) order relation $\leq$ by

$$(a_n) \leq (b_n) :\Leftrightarrow \begin{cases} (a_n) = (b_n) \text{ or} \\ \text{there exists } n \in \mathbb{N} \text{ such that } a_k = b_k \text{ for all } 1 \leq k < n \text{ and } a_n > b_n. \end{cases}$$

Using the function f and the order relation $\leq$ on $\{0,1\}^{\mathbb{N}}$ we introduce an order relation on G by

$$x \prec y :\Leftrightarrow f(x) \leq f(y). \tag{2.1}$$

Clearly, the order relation $\prec$ is total on G and we will now show that an arbitrary set $\varphi \in \mathbf{L}(G)$ equipped with the restriction of $\prec$ to $\varphi \times \varphi$ is a well-ordered set. Assume that $\varphi \neq \emptyset$. Then, for $\psi \subset \varphi$, there exists a number $n \in \mathbb{N}$ such that $\psi \cap B_k = \emptyset$ for all $k < n$ and $\psi \cap B_n \neq \emptyset$. Since B_n is relatively compact, $\psi \cap B_n = \{x_1, \ldots, x_m\}$ is a finite set. Also, by the definition of $\prec$, we have $x_i \prec y$ for all $1 \leq i \leq m$ and $y \in \psi \setminus \{x_1, \ldots, x_m\}$. Hence, the minimum w.r.t. $\prec$ of the finite set $\{x_1, \ldots, x_m\}$ is also the minimum of ψ and we conclude that φ is well-ordered by $\prec$.

Finally, for $x \in G$, the equation

$$S_x = \{x\} \cup \bigcup_{n \in \mathbb{N}} \Big(\{y \in G : f_n(y) > f_n(x)\} \cap \bigcap_{k<n} \{y \in G : f_k(y) = f_k(x)\} \Big)$$

yields that S_x is a measurable subset of G, so the lemma is proved. □

We will now define a sequence of mappings that enumerate the points of any locally finite set $\varphi \subset G$. Add an element Δ to the measurable space G, whose role is similar to the cemetery state which is often added to the state space of a stochastic process, and which allows us here to treat locally finite sets of finite and infinite cardinality simultaneously. A σ-field $\mathcal{G}_\Delta$ on $G \cup \{\Delta\}$ is defined by all subsets $A \subset G \cup \{\Delta\}$ such that $A \setminus \{\Delta\} \in \mathcal{G}$.

Proposition 2.1.4. *There exists a sequence (ξ_n) of $(\mathcal{L}(G), \mathcal{G}_\Delta)$-measurable mappings $\xi_n : \mathbf{L}(G) \to G \cup \{\Delta\}$ such that, for any $\varphi \in \mathbf{L}(G)$, we have $\varphi = \{\xi_n(\varphi) : n \in \mathbb{N}\} \setminus \{\Delta\}$, and, for $n \neq m$, $\xi_n(\varphi) \neq \xi_m(\varphi)$ or $\xi_n(\varphi) = \xi_m(\varphi) = \Delta$.*

Proof: Let $C := \{(\varphi, x) \in \mathbf{L}(G) \times G : x \in \varphi\}$. By Lemma 2.1.1, we have $C \in \mathcal{L}(G) \otimes \mathcal{G}$. Using Lemma 2.1.3, we first introduce the ranking function $f : C \to \mathbf{L}(G) \times \mathbb{N}$ by

$$f(\varphi, x) := (\varphi, \mathrm{card}(\varphi \cap S_x)), \tag{2.2}$$

that maps a point set $\varphi \in \mathbf{L}(G)$ and a point $x \in \varphi$ to the point set φ and the rank of $x \in \varphi$ with respect to $\prec$. Since there exists $n \in \mathbb{N}$ such that $x \in B_n$, we have

$$\mathrm{card}(\varphi \cap S_x) \leq \mathrm{card}(\cup_{k \leq n} (\varphi \cap B_k)) < \infty, \tag{2.3}$$

so f is well defined. Also, for two distinct elements $x, y \in \varphi$ such that $x \prec y$, we have $S_x \subset S_y \setminus \{y\}$, and deduce that $\mathrm{card}(\varphi \cap S_x) \leq \mathrm{card}(\varphi \cap S_y) - 1$. Hence, f is injective, and we denote by g the inverse function of f defined on $\mathrm{Im}(f) := \{f(\varphi, x) : (\varphi, x) \in C\}$, the image of C under f. We extend this definition to $\mathbf{L}(G) \times \mathbb{N}$ by letting $g(\varphi, n) := \Delta$ if $(\varphi, n) \notin \mathrm{Im}(f)$ and obtain a left-inverse function of f. Then define functions $\xi_n : \mathbf{L}(G) \to G \cup \{\Delta\}, n \in \mathbb{N}$, by

$$\xi_n(\varphi) := p_2(g(\varphi, n)), \quad \varphi \in \mathbf{L}(G), \tag{2.4}$$

where the projection on the second component of the product space is denoted by p_2.

Our proof that the functions $\xi_n, n \in \mathbb{N}$, are measurable, is along the lines of the proof of Lemma 3.1.7 in [24]. Indeed, for $B \in \mathcal{G}_\Delta$, we have

$$\begin{aligned}\xi_n^{-1}(B) &= \{\varphi \in \mathbf{L}(G) : \xi_n(\varphi) \in B\} \\ &= \bigcup_{j\in\mathbb{N}} \bigcup_{I\subset\{1,\dots,j\}} \Big\{ \varphi \in \mathbf{L}(G) : \operatorname{card}\Big(\varphi \cap B \cap \bigcap_{i\in I} B_i \cap \bigcap_{i\in\{1,\dots,j\}\setminus I} B_i^c\Big) = 1, \\ &\qquad \operatorname{card}\Big(\varphi \cap \bigcap_{i\in I} B_i \cap \bigcap_{i\in\{1,\dots,j\}\setminus I} B_i^c\Big) = 1, \\ &\qquad \operatorname{card}\Big(\varphi \cap \bigcup_{k=1}^{j} \bigcup_{\substack{I\cap\{1,\dots,k\}\subset J\subset\{1,\dots,k\} \\ I\cap\{1,\dots,k\}\neq J}} \Big(\bigcap_{i\in J} B_i\Big)\Big) = n-1 \Big\}.\end{aligned}$$

This formula can be interpreted in the following way. An element $\varphi \in \mathbf{L}(G)$ is in the set on the right hand side if and only if, for some $j \in \mathbb{N}$, there exist $I \subset \{1,\dots,j\}$ and a unique point x in $\varphi \cap \bigcap_{i\in I} B_i \cap \bigcap_{i\in\{1,\dots,j\}\setminus I} B_i^c$, this point satisfies $x \in B$, and there are exactly $n-1$ other points in φ that are strictly smaller than x w.r.t. $\prec$. We have shown that $\xi_n^{-1}(B)$ for any $B \in \mathcal{G}_\Delta$, and, hence, $\xi_n, n \in \mathbb{N}$, are $(\mathcal{L}(G), \mathcal{G}_\Delta)$-measurable mappings.

From the injectivity of f on C, we obtain that $\xi_n(\varphi) \neq \xi_m(\varphi)$, whenever $n \neq m$ and either $n \leq \operatorname{card}(\varphi)$ or $m \leq \operatorname{card}(\varphi)$. Also, using (2.3), we have $\operatorname{Im}(g) \subset \{(\varphi, \xi_n(\varphi)) : n \in \mathbb{N}\}$, where $\operatorname{Im}(g)$ is the image of g. The choice of g as left-inverse function of f yields $C = \operatorname{Im}(g) \setminus (\mathbf{L}(G) \times \{\Delta\})$, and we conclude that $\varphi = \{\xi_n(\varphi) : n \in \mathbb{N}\} \setminus \{\Delta\}$. □

Let us now introduce the space $\mathbf{M}(G)$ of locally finite measures on G, where local finiteness of $\mu \in \mathbf{M}(G)$ expresses that $\mu(B) < \infty$ for all relatively compact sets $B \in \mathcal{G}$, and the subspace $\mathbf{N}(G)$ of simple, locally finite counting measures. We equip $\mathbf{M}(G)$ with the *cylindrical σ-field* $\mathcal{M}(G)$ which is generated by the evaluation mappings $e_B : \mathbf{M}(G) \to [0,\infty], B \in \mathcal{G}$, given by $e_B : \mu \mapsto \mu(B)$, where $[0,\infty]$ is endowed with the usual (Borel) σ-field.

We denote by $\mathcal{N}(G)$ the restriction of $\mathcal{M}(G)$ to $\mathbf{N}(G)$, or equivalently, the σ-field which is generated by the restrictions of the mappings $e_B, B \in \mathcal{G}$, to $\mathbf{N}(G)$. Moreover, for $\mu \in \mathbf{M}(G)$, we denote by $\operatorname{supp}(\mu)$ the *support* of the measure μ.

Lemma 2.1.5. *The measurable spaces* $(\mathbf{L}(G), \mathcal{L}(G))$ *and* $(\mathbf{N}(G), \mathcal{N}(G))$ *are isomorphic.*

PROOF: We denote by δ_x the Dirac measure on G with a unit mass in x. Then define mappings f and g as follows. Let $f : \mathbf{L}(G) \to \mathbf{N}(G)$ be the mapping given by $f(\varphi)(\cdot) := \sum_{n\in\mathbb{N}} \delta_{\xi_n(\varphi)}(\cdot)$, and $g : \mathbf{N}(G) \to \mathbf{L}(G)$ by $g(\psi) := \operatorname{supp}(\psi)$. It is straightforward to show that g is measurable, and, using Proposition 2.1.4, the same is true for f. Also, we have $f \circ g = \mathrm{id}_{\mathbf{N}(G)}$ and $g \circ f = \mathrm{id}_{\mathbf{L}(G)}$, finishing the proof of the Lemma. □

2.2 The flow of translation mappings

Let us now define the class of translation operators on G, $\mathcal{G}$ and $\mathbf{M}(G)$. The group G acts on itself by addition of the inverse element, i.e., every element $x \in G$ induces a bijective mapping on G as follows.

Definition 2.2.1. For $x \in G$ we define an automorphism (of the measurable space $(G, \mathcal{G})$) $\theta_x : G \to G$ by

$$\theta_x y := y - x, \quad y \in G,$$

and call θ_x a *translation* or *shift operator* on G.

For two shift operators θ_x, θ_y we write $\theta_x \circ \theta_y$ for the composed mapping on G. It is easy to prove that $\theta_{x+y} = \theta_x \circ \theta_y$, and that $(\theta_x)^{-1} = \theta_{-x}$. Hence, for $\Theta_G := \{\theta_x : x \in G\}$, $(\Theta_G, \circ)$ is a group which is isomorphic to $(G, +)$.

The domain of the shift operators $\theta_x, x \in G$, extends naturally to the Borel sets of G and locally finite measures on G. Define

$$\theta_x B := B - x := \{y - x : y \in B\}, \quad B \in \mathcal{G}, \tag{2.5}$$
$$\theta_x \mu(\cdot) := \mu \circ \theta_x^{-1}(\cdot), \quad \mu \in \mathbf{M}(G). \tag{2.6}$$

One may easily verify that these definitions satisfy the axioms of a group operation (cf. [11], Chapter I 5) of Θ_G on the respective space. The operation of Θ_G on $\mathcal{G}$ can be restricted to $\mathbf{L}(G)$, and the operation of Θ_G on $\mathbf{M}(G)$ to $\mathbf{N}(G)$. More generally, if Θ_G operates on some space $\mathbb{X}$, then also on any Θ_G*-stable subspace* $\mathbb{Y} \subset \mathbb{X}$, i.e. a subspace that satisfies $\theta_x \mathbb{Y} \subset \mathbb{Y}$ for all $x \in G$. If Θ_G acts on spaces $\mathbb{X}_i, i \in I$, where I denotes an arbitrary index set, then we define the action of Θ_G on the product space $\prod_{i \in I} X_i$ by

$$\theta_x((x_i)_{i \in I}) := (\theta_x x_i)_{i \in I}. \tag{2.7}$$

Finally, if Θ_G acts on $\mathbb{X}$ and $\mathbf{F}(\mathbb{X})$ denotes the non-negative, real functions on $\mathbb{X}$, then Θ_G acts on $\mathbf{F}(\mathbb{X})$ through the definition

$$\theta_x(f) := f \circ \theta_x^{-1}, \quad f \in \mathbf{F}(\mathbb{X}). \tag{2.8}$$

The space $\mathbf{M}(G)$, topologized with the vague topology as described in Section A.2, is a Polish space. A complete metric ρ on $\mathbf{M}(G)$, that is compatible with the vague topology, is defined in (A.1). It is a remarkable fact that the associated Borel σ-field on $\mathbf{M}(G)$ coincides with the cylindrical σ-field $\mathcal{M}(G)$ (cf. Theorem A.2.1). Let us define $\theta : \mathbf{M}(G) \times G \to \mathbf{M}(G)$ by $\theta(\varphi, x) := \theta_x \varphi$. Then we have the following continuity result.

Proposition 2.2.2. *The mapping θ is continuous with respect to the product topology on $\mathbf{M}(G) \times G$. In particular, θ is $(\mathcal{M}(G) \otimes \mathcal{G}, \mathcal{M}(G))$-measurable.*

PROOF: The spaces $\mathbf{M}(G)$ and G are both metrizable (even Polish, cf. Theorem A.2.1 and Theorem A.1.2), hence, the same is true for the product space $\mathbf{M}(G) \times G$, and so it is sufficient to show sequential continuity of θ. Let (φ_n, x_n) be a converging sequence in $\mathbf{M}(G) \times G$ with limit (φ, x) and fix an arbitrary continuous function $f : G \to [0, \infty)$ with compact support. The claim of the propositon follows from

$$\int_G f(y) \theta_{x_n} \varphi_n(\mathrm{d}y) \to \int_G f(y) \theta_x \varphi(\mathrm{d}y) \quad \text{as } n \to \infty, \tag{2.9}$$

and we will use Theorem A.2.2 to show (2.9).

Define functions $f_n : G \to [0,\infty)$ by $f_n(y) := f(y - x_n)$ and $f_0 : G \to [0,\infty)$ by $f_0(y) := f(y - x)$. For $n \geq 0$, the support $\mathrm{supp}(f_n)$ of the function f_n is compact in G as a translate of the compact set $\mathrm{supp}(f)$. Let V be a relatively compact neighbourhood of 0. Then Tychonov's theorem yields that $\mathrm{supp}(f_0) \times \mathrm{cl}(V)$, where $\mathrm{cl}(V)$ denotes the closure of V, is compact in $G \times G$. The addition is a continuous mapping from $G \times G$ to G (and G has the Hausdorff property), hence, the Minkowski sum

$$\mathrm{supp}(f_0) \oplus \mathrm{cl}(V) = \{y + z : y \in \mathrm{supp}(f), z \in \mathrm{cl}(V)\}$$

is also a compact set in G. Moreover, there exists $n_0 \in \mathbb{N}$ such that $x - x_n \in V$ for all $n \geq n_0$, and we have $f_n(y) > 0$ if and only if $f_0(y + x - x_n) > 0$, hence,

$$\mathrm{supp}(f_n) \subset \mathrm{supp}(f_0) \oplus \mathrm{cl}(V) \quad \text{for all } n \geq n_0.$$

We deduce that the support of any of the functions $f_n, n \geq 0$, is contained in the compact set $K := \cup_{n<n_0} \mathrm{supp}(f_n) \cup (\mathrm{supp}(f_0) \oplus \mathrm{cl}(V))$. It is obvious from the definition of $f_n, n \geq 0$, that $\{f_n : n \geq 0\}$ is a uniformly bounded family of measurable functions, and that $f_n(y) \to f_0(y)$ as $n \to \infty$. Then Theorem A.2.2 yields that

$$\int_G f_n(y)\varphi_n(\mathrm{d}y) \to \int_G f_0(y)\varphi(\mathrm{d}y) \quad \text{as } n \to \infty,$$

which is equivalent to (2.9). □

The measurability of the mapping θ transfers to the restriction of θ to $\mathbf{N}(G) \times G$. We deduce that the mapping $\mathbf{L}(G) \times G \to \mathbf{L}(G), (\varphi, x) \mapsto \theta_x\varphi$ is also measurable, a fact that will be used several times in the sequel. Let us now define the notion of a flow (see also [7], p. 183).

Definition 2.2.3. Let H be a semigroup and $(\mathbb{X}, \mathcal{X})$ a measurable space. A family $\Theta_H := \{\theta_z : z \in H\}$ of measurable transformations $\theta_z : \mathbb{X} \to \mathbb{X}$ is called a *(generalized) flow on* $\mathbb{X}$ if $\theta_{y+z} = \theta_y \circ \theta_z$ for all $y, z \in H$, and $\theta_0 = \mathrm{id}_{\mathbb{X}}$. If H is equipped with a σ-field $\mathcal{H}$ and the mapping $(\varphi, y) \mapsto \theta_y\varphi$ is $(\mathcal{X} \otimes \mathcal{H}, \mathcal{X})$-measurable, then we call Θ_H a *measurable flow on* $\mathbb{X}$.

In particular, we have shown in Proposition 2.2.2 that the family Θ_G is a *measurable flow* on $\mathbf{M}(G)$. In the remainder of this chapter, and througout this thesis, we will be interested in mappings that commute with the operators of this flow, or are invariant under the composition with $\theta_x, x \in G$.

Definition 2.2.4. Assume that θ_H is a flow that acts on measurable spaces $\mathbb{X}$ and $\mathbb{Y}$. Then a function $f : \mathbb{X} \to \mathbb{Y}$ is called *equivariant* if $f \circ \theta_x = \theta_x \circ f$ for all $x \in H$. For an arbitrary third space $\mathbb{Z}$, a function $f : \mathbb{X} \to \mathbb{Z}$ is called *invariant* if $f \circ \theta_x = f$ for all $x \in H$.

2.3 Point maps on $\mathbf{L}(G)$

A simple description of point maps can be given as follows. A point map σ is a measurable mapping that assigns to $\varphi \in \mathbf{L}(G)$ a point $x \in \varphi$ if $0 \in \varphi$, and 0 otherwise. We write $\mathbf{L}_0(G) := \{\varphi \in \mathbf{L}(G) : 0 \in \varphi\}$ for all locally finite subsets of G that contain the origin. As we shall see, we can define an extended point map on $\mathbf{L}(G) \times G$, which is closely related to the original point map. We choose, however, to begin with separate definitions of point maps and extended point maps.

Definition 2.3.1. A *point map* is a measurable mapping $\sigma : \mathbf{L}(G) \to G$ such that $\sigma(\varphi) \in \varphi$ if $\varphi \in \mathbf{L}_0(G)$ and $\sigma(\varphi) = 0$ otherwise. A measurable, equivariant mapping $\tau : \mathbf{L}(G) \times G \to \mathbf{L}(G) \times G$ such that $\tau(\varphi, x) \in \{\varphi\} \times \varphi$ for all $\varphi \in \mathbf{L}(G)$ and $x \in \varphi$, and $\tau(\varphi, x) = (\varphi, x)$ for all $\varphi \in \mathbf{L}(G)$ and $x \notin \varphi$ is called an *extended point map.*

It is clear from the definition that a point map σ is determined by its restriction on $\mathbf{L}_0(G)$ and that it is measurable if and only if the restriction of σ to $\mathbf{L}_0(G)$ is measurable. Let us now show that there is a natural one-to-one correspondance between point maps and extended point maps.

Lemma 2.3.2. *For any point map σ, an extended point map $\tilde{\sigma}$ is defined by*

$$\tilde{\sigma}(\varphi, x) := (\varphi, \sigma(\theta_x \varphi) + x), \quad \varphi \in \mathbf{L}(G), x \in G.$$

Conversely, for an extended point map τ, a point map $\hat{\tau}$ is defined by

$$\hat{\tau}(\varphi) := p_2 \circ \tau(\varphi, 0), \quad \varphi \in \mathbf{L}(G),$$

where $p_2 : \mathbf{L}(G) \times G \to G$ denotes the projection on the second coordinate. Moreover, we have $\widehat{(\tilde{\sigma})} = \sigma$ for any point map σ and $\widetilde{(\hat{\tau})} = \tau$ for any extended point map τ.

Proof: The measurability of $\tilde{\sigma}$ follows from the measurability of σ and Proposition 2.2.2. The equivariance of $\tilde{\sigma}$ follows from

$$\begin{aligned} \tilde{\sigma} \circ \theta_z(\varphi, x) &= \tilde{\sigma}(\theta_z \varphi, x - z) \\ &= (\theta_z \varphi, \sigma(\theta_{x-z}(\theta_z \varphi) + x - z) \\ &= (\theta_z \varphi, \sigma(\theta_x \varphi) + x - z) = \theta_z \circ \tilde{\sigma}(\varphi, x), \quad z \in G. \end{aligned}$$

Finally, for $\varphi \in \mathbf{L}(G)$ and $x \in G$, we have

$$p_2(\tilde{\sigma}(\varphi, x)) \in \varphi \Leftrightarrow \sigma(\theta_x \varphi) + x \in \varphi \Leftrightarrow \sigma(\theta_x \varphi) \in \theta_x \varphi \Leftrightarrow \theta_x \varphi \in \mathbf{L}_0 \Leftrightarrow x \in \varphi. \qquad (2.10)$$

The measurability of $\hat{\tau}$ follows from the measurability of $\varphi \mapsto \tau(\varphi, 0)$ and the measurability of p_2. Moreover we have $\hat{\tau}(\varphi) \in \varphi$ if and only if $p_2(\tau(\varphi, 0)) \in \varphi$, which is the case if and only if $\varphi \in \mathbf{L}_0(G)$. The last claim is an immediate consequence of the definitions and easily verified. $\square$

Definition 2.3.3. We call a point map σ *bijective* if the extended point map $\tilde{\sigma}$ is bijective from $\mathbf{L}(G) \times G$ to $\mathbf{L}(G) \times G$. The set of bijective point maps will be denoted by Π.

In general, a bijective point map is not a one-to-one mapping. However, this terminology, which was introduced by Thorisson in [29] and also used in [4], gives the right intuition for the induced mapping $x \mapsto \sigma(\theta_x \varphi) + x$ on the points of some fixed $\varphi \in \mathbf{L}(G)$. In Proposition 2.4.4, we will give an additional argument for this choice of terminology.

The *composition* $\sigma \circ \tau$ of two point maps $\sigma : \mathbf{L}(G) \to G$ and $\tau : \mathbf{L}(G) \to G$ is defined via the associated extended point maps by

$$\sigma \circ \tau(\varphi) := \widehat{\tilde{\sigma} \circ \tilde{\tau}}(\varphi).$$

Lemma 2.3.2 yields that the composition of two point maps is again a point map, and that $\widetilde{\sigma \circ \tau} = \tilde{\sigma} \circ \tilde{\tau}$. Moreover, if σ and τ are bijective, so is $\sigma \circ \tau$, and we define the inverse point map of σ by $\sigma^{-1} := \widehat{\tilde{\sigma}^{-1}}$. Let us show that the composition $\circ$ defines a group operation on the set of bijective point maps.

Proposition 2.3.4. *Consider the set of bijective point maps* Π *equipped with the composition of mappings* $\circ$*. Then* $(\Pi, \circ)$ *is a group.*

PROOF: We have already mentioned that $\sigma \circ \tau \in \Pi$ for all $\sigma, \tau \in \Pi$. Let us now show that $\circ$ is an associative operation. The composition of extended point maps is trivially associative. For $\sigma_1, \sigma_2, \sigma_3 \in \Pi$, we then have

$$(\sigma_1 \circ \sigma_2) \circ \sigma_3 = ((\tilde{\sigma}_1 \widehat{\circ \tilde{\sigma}_2) \circ} \tilde{\sigma}_3) = (\tilde{\sigma}_1 \widehat{\circ (\tilde{\sigma}_2 \circ} \tilde{\sigma}_3)) = \sigma_1 \circ (\sigma_2 \circ \sigma_3).$$

Next, we have to show the existence of a neutral element and inverse elements. Clearly, the trivial point map $\nu : \mathbf{L}(G) \to G$ defined by $\nu(\varphi) := 0$ for all $\varphi \in \mathbf{L}(G)$ is a neutral element with respect to the composition.

Then fix $\sigma \in \Pi$. Since the extended point map $\tilde{\sigma} : \mathbf{L}(G) \times G \to \mathbf{L}(G) \times G$ is a one-to-one mapping, there exists an inverse mapping $\tilde{\sigma}^{-1}$, which is equivariant, because $\tilde{\sigma}$ is equivariant. Moreover, since $\mathbf{L}(G) \times G$ is a Borel space, the measurability of $\tilde{\sigma}^{-1}$ follows from a theorem by Kuratowski (cf. Theorem A 1.3 in [7]), which states that the inverse of a measurable bijection between two Borel spaces is again measurable. Using Lemma 2.3.2, we conclude that σ^{-1} is the inverse element of σ in $(\Pi(A), \circ)$. $\square$

To conclude this section, we will now complement our terminology for point maps and families of point maps.

Definition 2.3.5. Let σ be a bijective point map on $\mathbf{L}(G)$. If $\sigma = \sigma^{-1}$ then we call σ a *matching*. If, for some $n \in \mathbb{N}$, we have $\sigma^n(\varphi) = 0$ for all $\varphi \in \mathbf{L}(G)$, then we call σ *cyclic of order* n.

Definition 2.3.6. Let $A \subset \mathbf{L}_0(G)$. A family of point maps $\{\sigma_i : i \in I\}$ is called *complete on* A if $\varphi = \{\sigma_i(\varphi) : i \in I\}$ for all $\varphi \in A$. It is called *quasi-complete on* A if

$$\{x \in \varphi : \theta_x \varphi \neq \varphi\} \subset \{\sigma_i(\varphi) : i \in I\}$$

for all $\varphi \in A$.

A family of bijective point maps $\{\sigma_i, i \in I\}$ generates a – possibly strictly – larger family of bijective point maps in the following way.

Definition 2.3.7. Let $\{\sigma_i : i \in I\}$ be a family of bijective point maps. Then the family of compositions of finitely many of the point maps or their inverse point maps

$$\{\sigma_{i_1}^{k_1} \circ \ldots \circ \sigma_{i_n}^{k_n} : n \in \mathbb{N}, i_j \in I, k_j \in \mathbb{Z}, 1 \leq j \leq n\}$$

is called the *generated family* and denoted by $\langle\{\sigma_i : i \in I\}\rangle$. In particular, we denote by $\langle\{\sigma\}\rangle = \{\sigma^n : n \in \mathbb{Z}\}$ the family of point maps generated by σ.

Definition 2.3.8. Given a bijective point map σ on $\mathbf{L}(G)$ and $\varphi \in \mathbf{L}(G)$, we call the set $O_\sigma(\varphi) := \{\sigma^n(\varphi) : n \in \mathbb{Z}\}$ the *orbit of* 0 *in* φ *under* σ. For $A \subset \mathbf{L}_0(G)$, the point map σ is called *complete on* A or *universal point map on* A if $O_\sigma(\varphi) = \varphi$ for all $\varphi \in A$.

Clearly, σ is complete on A if and only if the family $\langle\{\sigma\}\rangle$ of point maps generated by σ is complete on A. The existence of such a universal point map on the locally finite subsets $\mathbf{L}(\mathbb{R})$ of $\mathbb{R}$ is a triviality (cf. Section 3.1.1). In higher dimensions, i.e., on $\mathbf{L}(\mathbb{R}^d)$, it has been the subject of recent papers such as [2], [5] and [30]. The surprising results will be summarized in Section 3.3.

2.4 Point shifts on $\mathbf{L}(G)$

Let us now introduce the second important family of mappings, the point shifts, which are closely related to point maps.

Definition 2.4.1. Let $\sigma : \mathbf{L}(G) \to G$ be a point map. The composed mapping $\theta_\sigma : \mathbf{L}(G) \to \mathbf{L}(G)$ defined by

$$\theta_\sigma(\varphi) := \theta_{\sigma(\varphi)}\varphi$$

is called the *point shift* associated with σ.

Lemma 2.4.2. *Let $\sigma : \mathbf{L}(G) \to G$ be a point map, then the associated point shift $\theta_\sigma : \mathbf{L}(G) \to \mathbf{L}(G)$ is $(\mathcal{L}(G), \mathcal{L}(G))$-measurable.*

PROOF: Clearly, θ_σ is a composition of measurable mappings. □

Simple examples show that point shifts are *not* (θ_x)-equivariant mappings. However, they fit well into the framework of this chapter (cf. Corollary 2.4.6). Let us begin our analysis of points shifts with some simple computation rules.

Proposition 2.4.3. *Let $\sigma, \tau : \mathbf{L}(G) \to G$ be point maps and $\pi : \mathbf{L}(G) \to G$ a matching. The following equalities involving shift operators, point maps and point shifts hold.*

(a) $\theta_\sigma \circ \theta_x(\varphi) = \theta_{\sigma(\theta_x\varphi)+x}(\varphi), \quad \varphi \in \mathbf{L}(G)$,

(b) $(\sigma \circ \tau)(\varphi) = \sigma \circ \theta_\tau(\varphi) + \tau(\varphi), \quad \varphi \in \mathbf{L}(G)$,

(c) $\pi \circ \theta_\pi(\varphi) = -\pi(\varphi), \quad \varphi \in \mathbf{L}(G)$,

(d) $\theta_{\sigma\circ\tau} = \theta_\sigma \circ \theta_\tau$.

PROOF: Throughout the proof we write $\varphi \in \mathbf{L}(G)$ for an arbitrary locally finite subset of G. Parts (a) and (b) follow directly from the definitions above and one line of computation, i.e.,

$$\theta_\sigma \circ \theta_x(\varphi) = \theta_{\sigma(\theta_x\varphi)}(\theta_x\varphi) = \theta_{\sigma(\theta_x\varphi)+x}(\varphi)$$

implies (a) and from

$$(\sigma \circ \tau)(\varphi) = \widehat{\tilde{\sigma} \circ \tilde{\tau}}(\varphi) = p_2(\tilde{\sigma}(\varphi, \tau(\varphi)) = \sigma(\theta_\tau(\varphi)) + \tau(\varphi) = \sigma \circ \theta_\tau(\varphi) + \tau(\varphi)$$

we obtain (b). Using (b) for the second equality, we have

$$\pi \circ \theta_\pi(\varphi) = \pi(\theta_\pi(\varphi)) + \pi(\varphi) - \pi(\varphi) = \pi \circ \pi(\varphi) - \pi(\varphi) = \pi \circ \pi^{-1}(\varphi) - \pi(\varphi) = -\pi(\varphi),$$

hence (c) is proved. Finally, using (b) one more time, we obtain

$$\theta_{\sigma\circ\tau}(\varphi) = \theta_{\sigma(\theta_{\tau(\varphi)}\varphi)+\tau(\varphi)}(\varphi) = \theta_{\sigma(\theta_{\tau(\varphi)}(\varphi))} \circ \theta_{\tau(\varphi)}(\varphi) = \theta_\sigma(\theta_{\tau(\varphi)}(\varphi)) = \theta_\sigma \circ \theta_\tau(\varphi),$$

so (d) holds and the proposition is proved. □

Proposition 2.4.4. *The point shift $\theta_\sigma : \mathbf{L}(G) \to \mathbf{L}(G)$ associated with the point map σ is bijective if and only if σ is a bijective point map. In this case, we have $\theta_{\sigma^{-1}} = (\theta_\sigma)^{-1}$.*

PROOF: Assume that σ is a bijective point map. There exists an inverse point map σ^{-1} and $\sigma \circ \sigma^{-1}(\varphi) = 0 = \sigma^{-1} \circ \sigma(\varphi)$ for all $\varphi \in \mathbf{L}(G)$. Proposition 2.4.3 (d) yields that

$$\theta_\sigma \circ \theta_{\sigma^{-1}} = \theta_{\sigma\circ\sigma^{-1}} = \theta_0 = \theta_{\sigma^{-1}\circ\sigma} = \theta_{\sigma^{-1}} \circ \theta_\sigma.$$

Hence, $\theta_{\sigma^{-1}}$ is the inverse element of θ_σ, and, in particular, θ_σ is bijective.

Let us now assume that θ_σ is bijective on $\mathbf{L}(G)$. We fix $\varphi \in \mathbf{L}(G)$ and show that $\tilde{\sigma}(\varphi, \cdot)$ is a bijective mapping from G to G. Let $x, y \in G$ such that $x \neq y$. If $\theta_x\varphi = \theta_y\varphi$ then trivially $\sigma(\theta_x\varphi) + x \neq \sigma(\theta_y\varphi) + y$. If $\theta_x\varphi \neq \theta_y\varphi$, then, by Proposition 2.4.3 (a),

$$\theta_{\sigma(\theta_x\varphi)+x}(\varphi) = \theta_\sigma \circ \theta_x(\varphi) = \theta_\sigma(\theta_x\varphi) \neq \theta_\sigma(\theta_y(\varphi)) = \theta_\sigma \circ \theta_y(\varphi) = \theta_{\sigma(\theta_y\varphi)+y}(\varphi),$$

hence, also $\sigma(\theta_x\varphi) + x \neq \sigma(\theta_y\varphi) + y$. We deduce that $\tilde{\sigma}$ is injective.

To show surjectivity, let $(\varphi, x) \in \mathbf{L}(G) \times G$. The mapping θ_σ is surjective, hence, there exists a set $\psi \in \mathbf{L}(G)$ such that $\theta_\sigma(\psi) = \theta_x\varphi$. Moreover, the set ψ is a translate of φ, so there exists $y \in G$ such that $\theta_\sigma(\theta_y\varphi) = \theta_x\varphi$. From Proposition 2.4.3 (a) we infer that $\theta_{\sigma(\theta_y\varphi)+y}(\varphi) = \theta_x\varphi$ and hence $\theta_y\varphi = \theta_{-\sigma(\theta_y\varphi)+x}(\varphi)$. Define $z := -\sigma(\theta_y\varphi) + x \in G$, then

$$\tilde{\sigma}(\varphi, z) = (\varphi, \sigma(\theta_z\varphi) + z) = (\varphi, \sigma(\theta_y\varphi) + z) = (\varphi, x),$$

finishing the proof of the proposition. □

Corollary 2.4.5. *The set of point shifts associated with bijective point maps* $\Theta_\Pi = \{\theta_\sigma : \sigma \in \Pi\}$*, equipped with the composition* $\circ$*, is a group. Moreover, the mapping* $\sigma \mapsto \theta_\sigma$ *is a group homomorphism and a point map* σ *is mapped to the neutral element* θ_0 *if and only if* $\varphi - \sigma(\varphi) = \varphi$ *for all* $\varphi \in \mathbf{L}(G)$.

To conclude this section, we will show how bijective point shifts fit into the framework of this chapter.

Corollary 2.4.6. *The group* $\{\theta_\sigma : \sigma \in \Pi\}$ *is a generalized flow on* $\mathbf{L}(G)$*, the flow of bijective point shifts.*

Proof: We have shown in Corollary 2.3.4 that $(\Pi, \circ)$ is a group, and in Proposition 2.4.3 (d) that $\theta_{\sigma\circ\pi} = \theta_\sigma \circ \theta_\pi$ for all $\sigma, \pi \in \Pi$. $\square$

2.5 Index functions

In the remaining sections of this chapter, we will provide tools for the definition and interpretation of (families of) point maps. We will begin with index functions, that were first introduced on point processes by Holroyd and Peres in [5]. They defined index functions in an isometry invariant way in order to generalize a tree construction for random graphs on $\mathbb{R}^d$. Here we will only postulate invariance under translation (shift) operators. In [30], Timar gave a more detailed account on index functions. We will adapt this concept to our deterministic setting.

Definition 2.5.1. An injective, measurable function $f : \mathbf{L}(G) \to [0, \infty)$ is called *index function*. The associated *extended index function* $\tilde{f} : \mathbf{L}(G) \times G \to [0, \infty)$ is defined as the composed mapping

$$\tilde{f}(\varphi, x) := f(\theta_x \varphi), \quad \varphi \in \mathbf{L}(G), x \in G.$$

The flow of shift operators Θ_G operates on the product $\mathbf{L}(G) \times G$ of measurable spaces as defined in (2.7), and it is easy to check that the extended index function is a shift invariant mapping.

Lemma 2.5.2. Let $\varphi \in \mathbf{L}(G)$ and $\tilde{f}$ an extended index function. The mapping

$$x \mapsto \tilde{f}(\varphi, x)$$

is an injective function from G to $[0, \infty)$, if and only if, for all $x, y \in G$, $x \neq y$, we have $\theta_x \varphi \neq \theta_y \varphi$.

Proof: The lemma follows directly from the definition of an index function $\square$

Note that in [5] and [30], almost sure injectivity in the sense of Lemma 2.5.2 is a defining property for the index function of a random point set.

The following example of an index function is constructed along the lines that Holroyd and Peres propose in [5]. Recall from Section 2.1 that $\mathcal{B} = (B_n)$ denotes a countable base of the topology of G, and $B_n, n \in \mathbb{N}$, are open sets with compact closure. We then define the *standard index function* $I : \mathbf{L}(G) \to [0, \infty)$ by

$$I(\varphi) := \sum_{n \in \mathbb{N}} 2^{-n} \, \mathbf{1}\{\varphi \cap B_n \neq \emptyset\}. \tag{2.11}$$

Lemma 2.5.3. *The function I is an index function. In particular, for $\varphi, \eta \in \mathbf{L}(G)$, we have $\varphi = \eta$ if and only if $I(\varphi) = I(\eta)$.*

Proof: As a pointwise limit function of measurable functions, I is also measurable. For $\varphi, \eta \in \mathbf{L}(G)$ such that $\varphi \not\subset \eta$, there exists $x \in \varphi \setminus \eta$. Since $G \setminus \eta$ is an open subset of G, there exists $n \in \mathbb{N}$ such that $x \in B_n \subset G \setminus \eta$. Hence, $\varphi \cap B_n \neq \emptyset$ and $\eta \cap B_n = \emptyset$, and we conclude that $I(\varphi) \neq I(\eta)$. □

We will now introduce the notion of *periodicity* of locally finite subsets of G.

Definition 2.5.4. We define the *periodicity lattice mapping* $L : \mathbf{L}(G) \to \mathbf{L}(G)$ by

$$L(\varphi) := \{x \in G : \theta_x \varphi = \varphi\} \quad \text{for all } \varphi \neq \emptyset,$$

and by $L(\emptyset) := \emptyset$. An element $\varphi \in \mathbf{L}(G)$ is called *aperiodic* if $L(\varphi) = \{0\}$, otherwise it is called *periodic*.

We may summarize the link between index function and periodicity lattice by the following equivalences, which are often used in the sequel,

$$\tilde{I}(\varphi, x) = \tilde{I}(\varphi, y) \Leftrightarrow \varphi - x = \varphi - y \Leftrightarrow x - y \in L(\varphi), \quad \varphi \in \mathbf{L}(G), x, y \in G.$$

Moreover, the periodicity lattice has the following properties.

Lemma 2.5.5. *The mapping $L : \mathbf{L}(G) \to \mathbf{L}(G)$ is measurable and shift invariant. For $\varphi \in \mathbf{L}(G) \setminus \{\emptyset\}$, the periodicity lattice $L(\varphi)$ is a discrete subgroup of G.*

Proof: Let $\varphi \in \mathbf{L}(G) \setminus \{\emptyset\}$ and $y \in \varphi$. Then $L(\varphi) \subset \theta_y \varphi$, and, in particular, $L(\varphi)$ is locally finite. Recall that we have already shown the measurability of the functions ξ_n (cf. Proposition 2.1.4), the mapping $(\varphi, x) \mapsto \theta_x \varphi$ (cf. Proposition 2.2.2), and of the index function I (cf. Lemma 2.5.3). Since $x \in L(\varphi)$ if and only if $x = y - z$ for some $z \in \varphi$ with $\varphi - y = \varphi - z$, the measurability of L follows from

$$\begin{aligned}
&\{\varphi \in \mathbf{L}(G) : L(\varphi) \cap B \neq \emptyset\} \\
&= \bigcup_{n \in \mathbb{N}} \{\varphi \in \mathbf{L}(G) : \operatorname{card}(\varphi) \geq n, \tilde{I}(\varphi, \xi_1(\varphi)) = \tilde{I}(\varphi, \xi_n(\varphi)), \xi_1(\varphi) - \xi_n(\varphi) \in B\}
\end{aligned}$$

for all $B \in \mathcal{G}$. Moreover, L is shift invariant, because $\varphi = \theta_x \varphi$ if and only if $\theta_z \varphi = \theta_x(\theta_z \varphi)$ for all $\varphi \in \mathbf{L}(G)$ and $x, z \in G$. Finally, the subgroup property follows from $\theta_0 \varphi = \varphi$ and $\theta_{x-y} \varphi = \varphi$, whenever $\theta_x \varphi = \varphi$ and $\theta_y \varphi = \varphi$. □

The extended index function $\tilde{I}$ may also be used to define an intrinsic order relation on φ as follows

$$x \ll_\varphi y :\Leftrightarrow \tilde{I}(\varphi, x) \leq \tilde{I}(\varphi, y), \quad x, y \in \varphi. \tag{2.12}$$

We will refer to $\ll_\varphi$ as the *standard order relation* on φ and, abusing notation, we will write $\ll$ for $\ll_\varphi$, whenever the set φ is uniquely determined from the context.

Lemma 2.5.6. *The standard order relation is equivariant in the sense that, for $x, y \in \varphi$ and $z \in G$, we have $x \ll_\varphi y$ if and only if $x - z \ll_{\theta_z \varphi} y - z$. It is a total order relation on φ if and only if φ is aperiodic.*

PROOF: Shift invariance of the extended index function yields the equivariance of the order relation. Also, $\varphi \in \mathbf{L}(G)$ is aperiodic if and only if the function $x \mapsto \tilde{I}(\varphi, x)$ is injective, hence, if and only if $\ll_\varphi$ is a total order relation on φ. □

2.6 Selection functions and thinning procedures

We will begin this section with the introduction of global and local selection functions.

Definition 2.6.1. A *selection function* is an equivariant, measurable mapping $\Phi : \mathbf{L}(G) \to \mathbf{L}(G)$ such that $\Phi(\varphi) \subset \varphi$. A *local selection function* is a measurable mapping $\Psi : \mathbf{L}(G) \to \mathbf{L}(G)$ such that $\Psi(\varphi) \subset \varphi$ and $\theta_x \circ \Psi(\varphi) = \Psi \circ \theta_x(\varphi)$ for all $\varphi \in \mathbf{L}(G)$ and $x \in \Psi(\varphi)$ (*local equivariance property*).

Lemma 2.6.2. *Any selection function Φ is also a local selection function. For selection functions Φ_1, Φ_2 and local selection functions Ψ_1, Ψ_2, the composed mapping $\Phi_1 \circ \Phi_2$ is a selection function, the composed mapping $\Psi_1 \circ \Psi_2$ a local selection function, and the mapping $\Psi_1 \cap \Psi_2 : \mathbf{L}(G) \to \mathbf{L}(G)$ defined by $(\Psi_1 \cap \Psi_2)(\varphi) := \Psi_1(\varphi) \cap \Psi_2(\varphi)$ is also a local selection mapping.*

PROOF: The first two claims follow almost immediately from the definitions, so that we will only prove that $\Psi_1 \circ \Psi_2$ is a local selection function. The mapping $\Psi_1 \circ \Psi_2$ is measurable, because it is the composition of measurable mappings. Also, for $\varphi \in \mathbf{L}(G)$ and $x \in \Psi_1 \circ \Psi_2(\varphi)$, we have $x \in \Psi_2(\varphi)$ and then

$$(\Psi_1 \circ \Psi_2) \circ \theta_x(\varphi) = \Psi_1 \circ (\Psi_2 \circ \theta_x)(\varphi) = \Psi_1 \circ (\theta_x \circ \Psi_2)(\varphi) = \theta_x \circ (\Psi_1 \circ \Psi_2)(\varphi),$$

where we have used $x \in \Psi_2(\varphi)$ for the second equation and $x \in \Psi_1(\Psi_2(\varphi))$ for the last equation. □

We have shown that the classes of (local) selection functions are stable with respect to compositions. More importantly, as we will show in the following proposition, the composition of a (bijective) point map with a local selection function that satisfies a side condition, is again a (bijective) point map. This fact will be used frequently in Chapter 3, where we will explicitly define various families of point maps.

Proposition 2.6.3. *Let σ be a point map resp. bijective point map and Ψ a local selection function that satisfies $\Psi(\varphi) \in \mathbf{L}_0(G)$ for all $\varphi \in \mathbf{L}_0(G)$. Then $\sigma \circ \Psi$ is a point map resp. bijective point map.*

PROOF: The mapping $\sigma \circ \Psi$ is measurable as a composition of measurable mappings. If $\varphi \in \mathbf{L}_0(G)$ we have $\Psi(\varphi) \in \mathbf{L}_0(G)$ and then

$$\sigma \circ \Psi(\varphi) = \sigma(\Psi(\varphi)) \in \Psi(\varphi) \subset \varphi.$$

Also, $\varphi \notin \mathbf{L}_0(G)$ yields $\Psi(\varphi) \notin \mathbf{L}_0(G)$ and hence $\sigma \circ \Psi(\varphi) = 0$. Hence, $\sigma \circ \Psi$ is a point map.

If σ is bijective then there exists an inverse point map σ^{-1}, and we claim that $\sigma^{-1} \circ \Psi$ is the inverse point map of $\sigma \circ \Psi$. Lemma 2.4.3(b) yields

$$0 = \sigma^{-1} \circ \sigma(\Psi(\varphi)) = \sigma^{-1} \circ \theta_\sigma(\Psi(\varphi)) + \sigma(\Psi(\varphi)) = \sigma^{-1} \circ \theta_{\sigma \circ \Psi(\varphi)} \circ \Psi(\varphi) + \sigma(\Psi(\varphi)) \quad (2.13)$$

for all $\varphi \in \mathbf{L}(G)$. Also, we have shown above that $(\sigma \circ \Psi)(\varphi) \in \Psi(\varphi)$ whenever $\varphi \in \mathbf{L}_0(G)$, hence, by the local equivariance property of Ψ

$$\Psi \circ \theta_{\sigma \circ \Psi}(\varphi) = \theta_{\sigma \circ \Psi} \circ \Psi(\varphi), \quad \varphi \in \mathbf{L}_0(G). \quad (2.14)$$

For $\varphi \in \mathbf{L}(G) \setminus \mathbf{L}_0(G)$, equation (2.14) is trivially satisfied, because $\Psi \circ \Psi(\varphi) \subset \Psi(\varphi) \subset \varphi$ and, hence, $\sigma \circ \Psi \circ \Psi(\varphi) = \sigma \circ \Psi(\varphi) = 0$. We deduce that

$$\begin{aligned}
&(\sigma^{-1} \circ \Psi) \circ (\sigma \circ \Psi)(\varphi) \\
&= \sigma^{-1} \circ \Psi \circ \theta_{\sigma \circ \Psi(\varphi)}(\varphi) + \sigma \circ \Psi(\varphi) \\
&= \sigma^{-1} \circ \theta_{\sigma \circ \Psi(\varphi)} \circ \Psi(\varphi) + \sigma \circ \Psi(\varphi) \\
&= 0
\end{aligned}$$

for all $\varphi \in \mathbf{L}(G)$, where we have used Lemma 2.4.3 (b) for the first, (2.14) for the second and (2.13) for the third equation. We have proved that the point map $\sigma^{-1} \circ \Psi$ is the left inverse point map of $\sigma \circ \Psi$, and, exchanging the roles of σ and σ^{-1}, we obtain that it is also the right inverse point map of $\sigma \circ \Psi$. Hence, $(\sigma \circ \Psi)^{-1} = \sigma^{-1} \circ \Psi$ and, in particular, $\sigma \circ \Psi$ is a bijective point map. □

Let us now define r-regular subsets of locally finite sets, and then introduce r-regularity of selection functions.

Definition 2.6.4. For $\varphi \in \mathbf{L}(G)$ and $r > 0$, we call ψ an *r-regular subset* of φ if $\psi \subset \varphi$, $d(x, \psi) \leq r$ for all $x \in \varphi$ and $d(y, z) > r$ for all distinct $y, z \in \psi$.

Definition 2.6.5. Let $r > 0$. A selection function Φ is called *r-regular on $A \subset \mathbf{L}(G)$* if $\Phi(\varphi)$ is an r-regular subset of φ for all $\varphi \in A$.

We will now define an example of a selection function Φ_r, which is r-regular on the aperiodic, locally finite subsets of G, i.e., on $A = \{\varphi \in \mathbf{L}(G) : L(\varphi) = \{0\}\}$, where $L(\varphi)$ denotes the periodicity lattice of φ (cf. Definition 2.5.4). It is inspired by similar constructions in [5] and [30].

Fix an enumeration (q_n) of the non-negative rational numbers contained in $(0,1)$ and let $r > 0$. Define the mappings $\Psi_{n,r} : \mathbf{L}(G) \to \mathbf{L}(G), n \in \mathbb{N}$, by

$$\Psi_{n,r}(\varphi) := \{x \in \varphi : |\tilde{I}(\varphi, x) - q_n| \leq |\tilde{I}(\varphi, y) - q_n| \text{ for all } y \in B_d(x,r)\},$$

where $\tilde{I}$ denotes the extended version of the standard index function defined in (2.11). A point $x \in \varphi$ is in $\Psi_{n,r}(\varphi)$ if and only if the extended index function $\tilde{I}$ applied to (φ, x) is closer to q_n than the extended index function applied to (φ, y) for all $y \in \varphi$ such that $d(x,y) \leq r$.

For $B \in \mathcal{G}$ and $r > 0$, we define the *parallel set* $B_{\oplus r}$ at distance r by

$$B_{\oplus r} := \{x \in G : \inf\{d(x,y) : y \in B\} \leq r\}.$$

Only in the case of a translation invariant metric d, i.e., $d(x,y) = d(x-z, y-z)$ for all $x, y, z \in G$, and for closed sets $F \subset G$, the equality $F_{\oplus r} = F \oplus \mathrm{cl}(B_d(0,r))$ holds, where $\mathrm{cl}(B_d(0,r))$ denotes the closed unit ball in G. It follows from the continuity of the addition in G that, for a measurable set $B \in \mathcal{G}$ and $r \geq 0$, the parallel set $B_{\oplus r}$ is also measurable. Define $\Phi_{n,r} : \mathbf{L}(G) \to \mathbf{L}(G), n \in \mathbb{N}$, inductively by $\Phi_{1,r}(\varphi) := \Psi_{1,r}(\varphi)$ and then

$$\Phi_{n,r}(\varphi) := \Phi_{n-1,r}(\varphi) \cup (\Psi_{n,r}(\varphi) \setminus (\Phi_{n-1,r}(\varphi))_{\oplus r}), \quad n \geq 2.$$

Finally, define $\Phi_r : \mathbf{L}(G) \to \mathbf{L}(G)$ by

$$\Phi_r(\varphi) := \bigcup_{n \in \mathbb{N}} \Phi_{n,r}(\varphi). \tag{2.15}$$

Let us show that Φ_r is a selection function.

Lemma 2.6.6. *The mapping Φ_r is equivariant and measurable.*

Proof: The equivariance of the mappings $\Psi_{n,r}, n \in \mathbb{N}$, can be seen from their definition. One deduces inductively that the mappings $\Phi_{n,r}, n \in \mathbb{N}$, are equivariant and, hence, the same is true for Φ_r.

The measurability of $\Psi_{n,r}, n \in \mathbb{N}$, follows from

$$\begin{aligned}&\big\{\varphi \in \mathbf{L}(G) : \Psi_{n,r}(\varphi) \cap B \neq \emptyset\big\} \\ &= \bigcup_{k \in \mathbb{N}} \bigcap_{m \in \mathbb{N}} \big\{\varphi \in \mathbf{L}(G) : \xi_k(\varphi) \in B, |\tilde{I}(\varphi, \xi_k(\varphi)) - q_n| \leq |\tilde{I}(\varphi, \xi_m(\varphi)) - q_n| \\ &\qquad \text{or } d(\xi_k(\varphi), \xi_m(\varphi)) > r\big\}, \quad B \in \mathcal{G},\end{aligned}$$

and the measurability of $\xi_k, k \in \mathbb{N}$, (cf. Propositon 2.1.4) and the extended standard index function $\tilde{I}$ (cf. Lemma 2.5.3). Define $\Phi_{0,r}(\varphi) := \emptyset$ for all $\varphi \in \mathbf{L}(G)$, then the measurability of the mappings $\Phi_{n,r}, n \in \mathbb{N}$, follows inductively from

$$\begin{aligned}&\{\varphi \in \mathbf{L}(G) : \Phi_{n,r}(\varphi) \cap B \neq \emptyset\} \\ &= \{\varphi \in \mathbf{L}(G) : \Phi_{n-1,r}(\varphi) \cap B \neq \emptyset\} \\ &\quad \cup \bigcup_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} \{\varphi \in \mathbf{L}(G) : \xi_k(\varphi) \in \Psi_{n,r}(\varphi) \cap B, \xi_k(\varphi) \notin \Phi_{n-1,r}(\varphi) \cap B_{\oplus r}\}, \quad B \in \mathcal{G}.\end{aligned}$$

Finally, we have

$$\{\varphi \in \mathbf{L}(G) : \Phi_r(\varphi) \cap B \neq \emptyset\} = \bigcup_{n\in\mathbb{N}} \{\varphi \in \mathbf{L}(G) : \Phi_{n,r}(\varphi) \cap B \neq \emptyset\}, \quad B \in \mathcal{G},$$

and, hence, Φ_r is measurable and equivariant, i.e., a selection function. □

Lemma 2.6.7. *Let $\varphi \in \mathbf{L}(G)$ be aperiodic. Then $\Phi_r(\varphi)$ is a r-regular subset of φ.*

PROOF: For $x \in \varphi$, there exists $k \geq 1$ such that $|\tilde{I}(\varphi, x) - q_k| < |\tilde{I}(\varphi, y) - q_k|$ for all $y \in \varphi \cap B_d(x, r) \setminus \{x\}$, because φ is locally finite, f injective on φ, and (q_n) dense in $[0, 1]$. We deduce that $x \in \Psi_{k,r}(\varphi)$. Hence, either $x \in (\Phi_{k-1,r}(\varphi))_{\oplus r} \subset (\Phi_r(\varphi))_{\oplus r}$ or $x \notin (\Phi_{k-1,r}(\varphi))_{\oplus r}$, and then $x \in \Phi_{k,r}(\varphi) \subset (\Phi_r(\varphi))_{\oplus r}$.

Moreover, it follows directly from the definition of Φ_r and the injectivity of $x \mapsto I(\varphi, \tilde{x})$ for all aperiodic $\varphi \in \mathbf{L}(G)$ that $d(x, y) > r$ for all distinct $x, y \in \Phi_r(\varphi)$. □

Let us now introduce families of selection mappings (T_n) that map a locally finite set φ to a decreasing sequence $(T_n(\varphi))$ of subsets of φ.

Definition 2.6.8. A sequence (φ_n) of subsets of $\varphi \in \mathbf{L}(G)$, such that $\varphi_{n+1} \subset \varphi_n$ for all $n \in \mathbb{N}$ is called *thinning of* φ. It is called *non-trivial* if $\varphi_n \neq \emptyset$ for all $n \in \mathbb{N}$, and *complete* if $\varphi_n \downarrow \emptyset$.

Definition 2.6.9. Let (r_n) be an increasing sequence of non-negative real numbers such that $r_n \uparrow \infty$. A thinning (φ_n) of $\varphi_0 \in \mathbf{L}(G)$ is called (r_n)*-regular* if φ_n is a r_n-regular subset of φ_{n-1} for all $n \in \mathbb{N}$.

Lemma 2.6.10. *If (φ_n) is a (r_n)-regular thinning of φ then*

$$\operatorname{card}\Big(\bigcap_{n\in\mathbb{N}} T_n(\varphi)\Big) \leq 1.$$

If $\varphi \in \mathbf{L}(G)$ satisfies $0 < \operatorname{card}(\varphi) < \infty$, then there exists $n_0 \in \mathbb{N}$ such that $\operatorname{card}(T_n(\varphi)) = 1$ for all $n \geq n_0$.

PROOF: Assume that there are two distinct points $x, y \in G$ in $\bigcap_{n\in\mathbb{N}} T_n(\varphi)$. There exists $n_0 \in \mathbb{N}$ such that $r_n > d(x, y)$, and, hence, $\{x, y\} \subset T_n(\varphi)$ implies that $T_n(\varphi)$ is not a r_n-regular subset, a contradiction. The second claim follows from the fact that a finite set is always bounded. □

Definition 2.6.11. A *thinning procedure* (T_n) is a sequence of selection functions $T_n, n \in \mathbb{N}$, such that $(T_n(\varphi))$ is a thinning for all $\varphi \in \mathbf{L}(G)$. It is called *non-trivial* resp. *complete* resp. (r_n)*-regular on* $A \subset \mathbf{L}(G) \setminus \emptyset$ if the thinning $(T_n\varphi)$ is non-trivial resp. complete resp. (r_n)-regular for all $\varphi \in A$.

Let (r_n) be a sequence of non-negative real numbers such that $r_n \uparrow \infty$. Using the selection function Φ_r from (2.15), we define an example of a thinning procedure (T_n), that is (r_n)-regular on $A \subset \mathbf{L}(G)$, the aperiodic, locally finite subsets of G. Indeed, let $T_0 := \mathrm{id}_{\mathbf{L}(G)}$ and then inductively

$$T_{n+1}(\varphi) := \Phi_{r_{n+1}}(T_n(\varphi)), \quad n \in \mathbb{N}. \tag{2.16}$$

In the special case, where $r_n = c^n$ for some $c > 1$, we will call this thinning procedure the *c-exponential thinning procedure.*

Proposition 2.6.12. *The thinning procedure* (T_n) *defined above is* (r_n)*-regular on the aperiodic, locally finite subsets* $A \subset \mathbf{L}(G)$.

PROOF: The claim of the proposition follows immediately from Lemma 2.6.7. □

2.7 Graphs on point sets

In the final section of this chapter, we will define factor graphs, i.e., equivariant mappings, that map a locally finite point set φ to a graph $\Gamma(\varphi)$ with vertex set $V(\varphi) \subset \varphi$. Many of the recent results on graphs defined on point processes (cf. [2], [4], [5] and [30], also the book [20]) can be stated in terms of equivariant mappings, allowing a strict separation of the deterministic construction of the graph, and the study of its random properties, which depend on the underlying point process.

In this thesis, the introduction of factor graphs has a double purpose. First, point maps can be most easily visualized (and interpreted) by means of the associated (equivariant) graph, and, secondly, recent findings in random graph theory will lead to examples of point maps.

Definition 2.7.1. A *factor graph* $\Gamma : \mathbf{L}(G) \to \mathbf{L}(G) \times \mathbf{L}(G \times G)$ is given by a pair of measurable, equivariant mappings $\Gamma = (V, E)$, where $V : \mathbf{L}(G) \to \mathbf{L}(G)$ satisfies $V(\varphi) \subset \varphi$ for all $\varphi \in \mathbf{L}(G)$, and $E : \mathbf{L}(G) \to \mathbf{L}(G \times G)$ is such that $E(\varphi) \subset V(\varphi) \times V(\varphi)$ for all $\varphi \in \mathbf{L}(G)$. We call V the *vertex mapping* and E the *edge mapping* of Γ.

Let us now shortly summarize some basic terminology from graph theory, which refers to a directed graph $\Gamma = (V, E)$. An edge $(x, y) \in E$ connects two vertices, the *starting point* $x \in V$ and the *endpoint* $y \in V$. If connected by an edge, we call x and y *adjacent* or *neighbours.* The *degree* of a vertex is the number of its neighbours. If every vertex has finite degree, we call the graph *locally finite.* A finite sequence of pairwise distinct vertices $(x_1, \dots, x_n)$, $n \geq 2$, and edges $(x_i, x_{i+1}), i \in \{1, \dots, n-1\}$ is called a *finite path.* If any two points are distinct except $x_1 = x_n$, then $(x_1, \dots, x_n)$ is a *circuit.* A sequence of pairwise distinct vertices $(x_i), i \in \mathbb{Z}$, such that $(x_i, x_j) \in E(\varphi)$ if and only if $j = i + 1$ for all $i, j \in \mathbb{Z}$ is called a *directed doubly infinite path.*

An edge of the form (x, x) is called a *loop.* A subset of vertices $S \subset V$ in a graph is *connected* if there is a path between any two distinct vertices in S. A *tree* is a connected graph without loops and circuits. A *connected component* of a graph is a maximal connected subset, and if every connected component in a graph is a tree, the graph is called a *forest.*

For a more complete account we refer the reader to [13]. Let us illustrate the definition of a factor graph with a well-known example of an undirected graph. For $r \geq 0$, we define the *geometric graph* $\Gamma_r = (V_r, E_r)$ by $V_r(\varphi) := \varphi$ and $E_r(\varphi) := \{(x, y) \in \varphi \times \varphi : d(x, y) \leq r\}$.

Lemma 2.7.2. *The geometric graph Γ_r for distance $r \geq 0$ is a factor graph.*

PROOF: The equivariance of Γ_r follows directly from the definition. We have to prove the measurability of E_r. For $B \in \mathcal{G} \otimes \mathcal{G}$, we have

$$\begin{aligned}&\big\{\varphi \in \mathbf{L}(G) : E_r(\varphi) \cap B \neq \emptyset\big\}\\ &= \bigcup_{n,m \in \mathbb{N}} \big\{\varphi \in \mathbf{L}(G) : (\xi_n(\varphi), \xi_m(\varphi)) \in B \text{ and } d(\xi_n(\varphi), \xi_m(\varphi)) \leq r\big\},\end{aligned}$$

so the measurability follows from Proposition 2.1.4. □

There exists an extensive literature on random geometric graphs, i.e. the composition of a geometric graph and a point process. For an introduction to the subject and many recent results, we refer the reader to the monograph [20]. Let us now establish the link between factor graphs and point maps.

Definition 2.7.3. Let σ be a point map. The *associated graph* $\Gamma_\sigma = (V_\sigma, E_\sigma)$ is defined by $V_\sigma(\varphi) := \varphi$ and $E_\sigma(\varphi) := \{(x, \sigma(\theta_x\varphi) + x) : x \in \varphi\}$.

Proposition 2.7.4. *The associated graph Γ_σ of a point map σ is a factor graph.*

PROOF: The measurability and equivariance of V_σ are obvious. The measurability of E_σ follows from

$$\{\varphi \in \mathbf{L}(G) : E_\sigma(\varphi) \cap B \neq \emptyset\} = \bigcup_{n \leq \mathrm{card}(\varphi)} \{\varphi \in \mathbf{L}(G) : (\xi_n(\varphi), \sigma(\theta_{\xi_n(\varphi)}\varphi) + \xi_n(\varphi)) \in B\}$$

for all $B \in \mathcal{G} \otimes \mathcal{G}$. Finally, the equivariance of the extended point map (cf. Lemma 2.3.2) yields $\tilde{\sigma}(\varphi, x) = (\varphi, y)$ if and only if $\tilde{\sigma}(\theta_z\varphi, x - z) = (\theta_z\varphi, y - z)$, and we deduce that $(x, y) \in \Gamma_\sigma(\varphi)$ if and only if $(x - z, y - z) \in E_\sigma(\theta_z\varphi)$. We conclude that $E_\sigma \circ \theta_z(\varphi) = \theta_z \circ E_\sigma(\varphi)$. □

Proposition 2.7.5. *A point map σ is bijective if and only if the associated graph $\Gamma_\sigma = (V_\sigma, E_\sigma)$ satisfies*

$$\mathrm{card}(\{(x, y) \in E_\sigma(\varphi) : x \in \varphi\}) = 1, \quad \varphi \in \mathbf{L}(G), y \in \varphi, \tag{2.17}$$

i.e., if and only if, for all $\varphi \in \mathbf{L}(G)$, any $y \in \varphi$ is the endpoint of exactly one directed edge (x, y) in $\Gamma_\sigma(\varphi)$.

PROOF: Let $\varphi \in \mathbf{L}(G)$ and $x \in \varphi$. Again, we use the link between the associated graph and the extended point map stated at the beginning of the proof of Proposition 2.7.4. The extended point map $\tilde{\sigma}$ is bijective if and only if, for all $\varphi \in \mathbf{L}(G)$, the mapping $x \to \sigma(\theta_x\varphi)+x$ is a bijection on φ. We claim that the latter is true, if and only if any point $y \in \varphi$ is the endpoint of exactly one directed edge ends in (x, y). Indeed, $\tilde{\sigma}$ is surjective, whenever every point $y \in \varphi$ is the endpoint of at least one directed edge (x, y), and $\tilde{\sigma}$ is injective, whenever every point $y \in \varphi$ is the endpoint of at most one directed edge (x, y) in $E_\sigma(\varphi), \varphi \in \mathbf{L}(G)$. $\square$

Proposition 2.7.6. *A point map σ is a universal point map on $A \subset \mathbf{L}_0(G)$, i.e., $\langle\{\sigma\}\rangle$ is a complete family of point maps on $A \subset \mathbf{L}_0(G)$, if and only if $\Gamma_\sigma(\varphi)$ is a doubly-infinite, directed path or a finite circuit for all $\varphi \in A$.*

Proof: If σ is a universal point map on A and $\varphi \in A$ then $\varphi = \{\sigma^n(\varphi) : n \in \mathbb{Z}\}$. We distinguish two cases. If the set $\{n \in \mathbb{N} : \sigma^n(\varphi) = 0\}$ is empty, then $\sigma^k(\varphi) \neq \sigma^m(\varphi)$ for all $k, m \in \mathbb{Z}, k < m$, because otherwise we have

$$0 = \sigma^{-k} \circ \sigma^k(\varphi) = \sigma^{-k} \circ \sigma^m(\varphi) = \sigma^{m-k}(\varphi),$$

a contradiction. Hence, $\Gamma_\sigma(\varphi)$ is a doubly-infinite, directed path on φ. Otherwise, define $n_0 := \min(\{n \in \mathbb{N} : \sigma^n(\varphi) = 0\})$, then $\Gamma_\sigma(\varphi)$ is a finite circuit of length n_0 on φ.

For the converse direction, we assume that, for all $\varphi \in A$, $\Gamma_\sigma(\varphi)$ is a doubly-infinite, directed path or a finite circuit. Then for all $\varphi \in \mathbf{L}(G)$, any $y \in \varphi$ is the endpoint of exactly one directed edge, hence, by Proposition 2.7.5, the point map σ is bijective. Moreover, the connectedness of $\Gamma_\sigma(\varphi)$ yields that σ is complete on φ. $\square$

Chapter 3

Point maps

3.1 The universal point map

We will now seek to define complete families of bijective point maps on $\mathbf{L}(G)$, i.e., families that exhaust the points of any given locally finite subset $\varphi \in \mathbf{L}_0(G)$. In [4], it was shown that there exists a countable families of bijective point maps that has this property on $\mathbf{L}_0(\mathbb{R}^d)$. In special cases, even a single bijective point map σ generates such a family. The existence of such a *universal point map* σ is trivial in the case of locally finite point sets on $\mathbb{R}$ (cf. Section 3.1.1), but it turns out to be a triggering – and not yet completely solved – problem in higher dimensions.

3.1.1 A fundamental point map on the line

In the early papers by Kaplan [10] and Ryll-Nardzewski [22], stationary, $\mathbb{Z}$-indexed sequences of positive valued random variables (X_n) were studied, where stationarity expresses distributional invariance under a shift of the indices, i.e.,

$$(X_n) \stackrel{d}{=} (X_{n+1}).$$

Denote by $\mathbf{L}_0^\infty(\mathbb{R})$ the locally finite subsets $\varphi \in \mathbf{L}_0(\mathbb{R})$ that are two-sided infinite, i.e., $\inf \varphi = -\infty$ and $\sup \varphi = \infty$. In this setting, the index shift corresponds to the fundamental example of a point shift on $\mathbf{L}(\mathbb{R})$, which shifts a point configuration $\varphi \in \mathbf{L}_0^\infty(\mathbb{R})$ in such a way that the first strictly positive point x of φ is translated to the origin. This particular point shift is associated with the point map $\sigma : \mathbf{L}_0(\mathbb{R}) \to \mathbb{R}$, which is defined by

$$\sigma(\varphi) := \begin{cases} \min\{x \in \varphi \ : x > 0\} & \text{if } \varphi \in \mathbf{L}_0^\infty, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

Proposition 3.1.1. *The point map σ is bijective. The family of point maps $\langle\{\sigma\}\rangle$ generated by σ is complete on $\mathbf{L}_0^\infty(\mathbb{R})$.*

We omit the obvious proof of the proposition and recall that, by definition, any point map π on $\mathbf{L}(\mathbb{R})$ is required to satisfy $\pi(\varphi) = 0$ for all $\varphi \in \mathbf{L}(\mathbb{R}) \setminus \mathbf{L}_0(\mathbb{R})$, so it is sufficient to give an explicit definition on $\mathbf{L}_0(\mathbb{R})$. The definition of σ can be modified in such a way that the new point map $\tau : \mathbf{L}(\mathbb{R}) \to \mathbb{R}$ is bijective and the family $\langle\{\tau\}\rangle$ is complete on $\mathbf{L}_0(\mathbb{R})$. For $\varphi \in \mathbf{L}_0(\mathbb{R})$, we define

$$\tau(\varphi) := \begin{cases} \min\{x \in \varphi \ : x > 0\} \\ \qquad \text{if } \varphi \in \mathbf{L}_0^\infty \text{ or if } \inf\varphi > -\infty \text{ and } 0 < \sup\varphi < \infty, \\ \min\{x \in \varphi\} \\ \qquad \text{if } \inf\varphi > -\infty \text{ and } \sup\varphi = 0, \\ \min\{x \in \varphi \ : \ \text{there exists } y \in \varphi \text{ such that } 0 < y < x\} \\ \qquad \text{if } \inf\varphi = -\infty \text{ and } \operatorname{card}(\varphi \cap (0,\infty)) \in 2\mathbb{N}, \\ \max\{x \in \varphi \ : \ \text{there exists } y \in \varphi \text{ such that } x < y < 0\} \\ \qquad \text{if } \inf\varphi = -\infty \text{ and } \operatorname{card}(\varphi \cap (0,\infty)) \in \mathbb{N} \setminus 2\mathbb{N}, \\ \max\{x \in \varphi \ : x < 0\} \\ \qquad \text{if } \inf\varphi = -\infty \text{ and } \sup\varphi = 0, \\ \max\{x \in \varphi \ : \ \text{there exists } y \in \varphi \text{ such that } x < y < 0\} \\ \qquad \text{if } \sup\varphi = \infty \text{ and } \operatorname{card}(\varphi \cap (-\infty,0)) \in 2\mathbb{N}, \\ \min\{x \in \varphi \ : \ \text{there exists } y \in \varphi \text{ such that } 0 < y < x\} \\ \qquad \text{if } \sup\varphi = \infty \text{ and } \operatorname{card}(\varphi \cap (-\infty,0)) \in \mathbb{N} \setminus 2\mathbb{N}, \\ \min\{x \in \varphi \ : x > 0\} \\ \qquad \text{if } \sup\varphi = \infty \text{ and } \inf\varphi = 0. \end{cases} \tag{3.2}$$

The technicality comes from the challenge to handle all cases by a single point map. However, from Figure 3.1 it is clear that the mapping itself is in no way complicated.

$\inf\varphi = -\infty, \sup\varphi = \infty$:

$\inf\varphi > -\infty, \sup\varphi < \infty$:

$\inf\varphi > -\infty, \sup\varphi = \infty$:

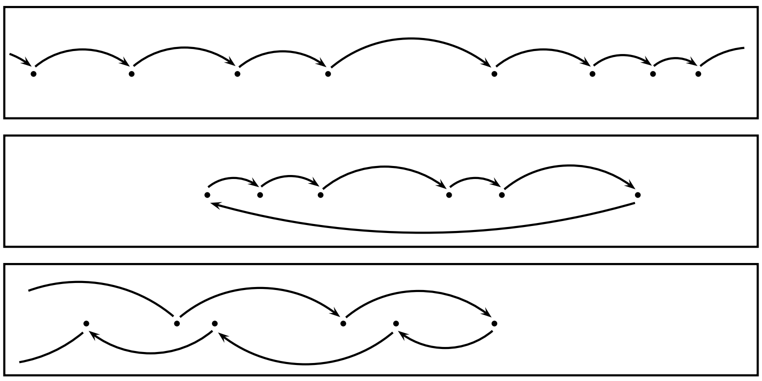

Figure 3.1: The factor graph Γ_τ associated with τ applied to subsets of the real line

3.1.2 Completeness in higher dimensions

Given the point map from Subsection 3.1.1, it is a natural question to ask whether there is also a bijective point map on $\mathbf{L}(\mathbb{R}^d)$ that is complete on $\mathbf{L}_0(\mathbb{R}^d)$. In the following proposition, we will show that this is not the case.

Proposition 3.1.2. *For $d > 1$, there is no (bijective) point map on $\mathbf{L}(\mathbb{R}^d)$, which is complete on $\mathbf{L}_0(\mathbb{R}^d)$.*

PROOF: Assume that σ is a bijective point map on $\mathbf{L}(\mathbb{R}^d)$ and complete on $\mathbf{L}_0(\mathbb{R}^d)$. Let $\varphi \in \mathbf{L}_0(\mathbb{R}^d)$ be a periodic set that is not contained in any half-space of $\mathbb{R}^d$. There exists $0 \neq x \in \varphi$ such that $\varphi - x = \varphi$ and $n \in \mathbb{Z}$ such that $\sigma^n(\varphi) = x$. Then it follows from the equivariance of σ that $\{\sigma^k(\varphi) : k \in \mathbb{Z}\} \subset \{\sigma^m(\varphi) + kx : 0 \leq m < n, k \in \mathbb{Z}\}$, which is a contradiction to the assumption that φ is not contained in any half-space of $\mathbb{R}^d$. $\square$

The proof of Proposition 3.1.2 relies on the equivariance of point maps and its implications for point maps on periodic sets, see Figure 3.2 for an illustration.

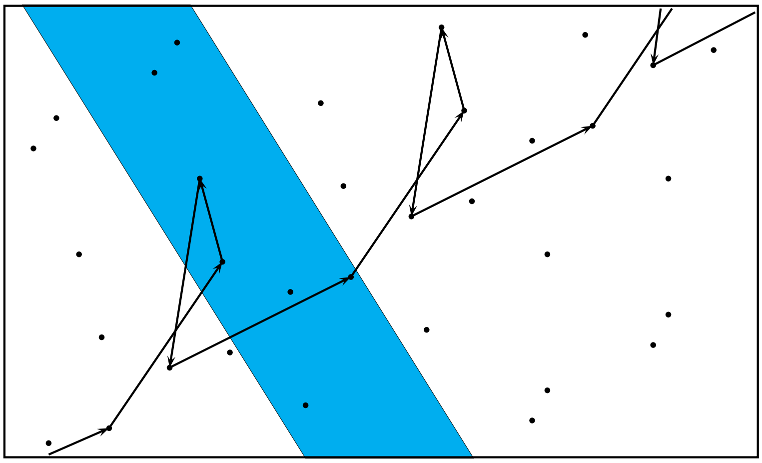

Figure 3.2: The graph Γ_σ associated with σ on a periodic set $\varphi \in \mathbb{R}^2$

Taking into account Proposition 3.1.2 we state the following fundamental problem which, for the moment, remains unsolved.

Open problem 1. Is there a bijective point map on $\mathbf{L}(\mathbb{R}^d)$, which is complete on the set of aperiodic, locally finite subsets $\mathbb{R}^d$, $d > 1$, that contain the origin?

We will have to settle with partial answers to the above problem. Indeed, in Section 3.2.4, we will show that there exists a countable family of matchings on $\mathbf{L}(G)$, that is quasi-complete on $\mathbf{L}_0(G)$. This result is crucial for the characterization results on Palm measures via bijective point shifts in Chapter 6.

In the special case $G = \mathbb{R}^d$, we will show in Section 3.3 that there exists a bijective point map on $\mathbf{L}(\mathbb{R})$ that is complete on large subclasses of $\mathbf{L}_0(\mathbb{R}^d)$, which in Chapter 5 will be shown to include almost all realizations of any stationary point process. Finally, in Section 3.4, we define two bijective point maps that generate a complete family on the aperiodic, locally finite subsets of $\mathbb{R}^d$ that contain the origin.

3.2 Cyclic point maps

In this section we will discuss various families of *cyclic point maps* in the general setting of a lcscH group G. Recall from Definition 2.3.5 that a point map σ on $\mathbf{L}(G)$ is called cyclic of order n if it satisfies $\sigma^n(\varphi) = 0$ for all $\varphi \in \mathbf{L}(G)$. We will begin our discussion of point maps in higher dimensions with matchings, i.e., self-inverse bijective point maps, which were also historically the first bijective point maps to be defined on $\mathbf{L}(\mathbb{R}^d)$.

3.2.1 First examples of matchings

The following example of a bijective point map on $\mathbf{L}(\mathbb{R}^d)$ was devised by Olle Häggström and is known as *mutual nearest neighbour matching*. It generalizes without any modification to the setting of a lcscH group G. Informally, a point map $\pi : \mathbf{L}(G) \to G$ is defined by $\pi(\varphi) := x$ if $0, x \in \varphi$ are mutual unique nearest neighbours in φ, and by $\pi(\varphi) := 0$ otherwise. The symmetry in the condition ensures that $\pi(\varphi) = x$ if and only if $\pi(\theta_x \varphi) + x = 0$. Hence π is a matching and, in particular, a bijective point map.

A moderate variation of mutual nearest neighbour matching, was defined in [4], Section 4, the matchings by *symmetric area search*. In this example, the symmetric nearest neighbour condition is replaced by a unique neighbour condition in a symmetric area as follows.

Fix a Borel set $B \in \mathcal{G}$. Then call $y \in \varphi$ a *B-neighbour* of $x \in \varphi$ if $y \in x + B$. Clearly, y is a B-neighbour of x if and only if x is a $(-B)$-neighbour of y, i.e., $x \in y + (-B)$, where the reflected set $-B$ is defined by $-B := \{-x : x \in B\}$. We say that y is the *unique B-neighbour* of x if $(x + B) \cap \varphi = \{y\}$. Note that y can be the unique B-neighbour of x and x a non-unique $(-B)$-neighbour of y. We then define, again informally, a point map $\pi_B : \mathbf{L}(G) \to G$ by

$$\pi_B(\varphi) := \begin{cases} x & \text{if } 0, x \in \varphi \text{ are mutual unique } (B \cup (-B))\text{-neighbours,} \\ 0 & \text{otherwise.} \end{cases}$$

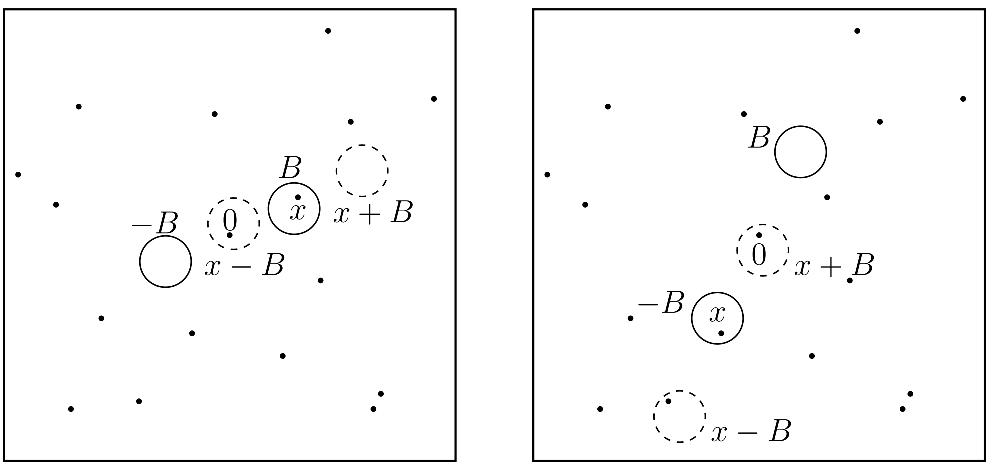

Figure 3.3: Matching by symmetric area search

In Figure 3.3, we illustrate on the left hand side the case where $\pi_B(\varphi) = x$. Not so on the right hand side, where we have $\pi_B(\varphi) = 0$, because there are two points in $x + (B \cup (-B))$.

Let us then check whether the family $\{\pi_B : B \in \mathcal{G}\}$ is quasi-complete on $\mathbf{L}_0(G)$. It is easy to show that, for $\varphi \in \mathbf{L}_0(G)$ we have

$$\{\pi_B(\varphi) : B \in \mathcal{G}\} = \{x \in \varphi : \{-x, 2x\} \cap \varphi = \emptyset\} \cup \{0\}. \tag{3.3}$$

In particular, the family of matchings $\{\pi_B : B \in \mathcal{G}\}$ is complete on the large subclass of $\mathbf{L}_0(\mathbb{R}^d)$ given by

$$D := \{\varphi \in \mathbf{L}_0(\mathbb{R}^d) : \text{ there are no distinct points } x, y, z \text{ in } \varphi \text{ such that } x - y = y - z\}. \tag{3.4}$$

However, if $0 \neq x$ and $-x$ (or $2x$) are points in $\varphi \in \mathbf{L}_0(G)$ then none of these matchings satisfies $\pi(\varphi) = x$. In the following three subsections, we will define a variation of the above symmetric area search in order to obtain a quasi-complete family of matchings on $\mathbf{L}_0(G)$. This is an optimal result as we can see from the following proposition.

Proposition 3.2.1. *Let $\varphi \in \mathbf{L}_0(G), 0 \neq x \in \varphi$ and assume that $\varphi = \varphi - x$. Then there is no matching π that satisfies $\pi(\varphi) = x$.*

PROOF: Let π be a bijective point map such that $\pi(\varphi) = x$. Then

$$\pi^2(\varphi) = \pi(\theta_x \varphi) + x = 2x,$$

hence, π is not a matching. □

3.2.2 Matchings by two area search

We will now introduce *matchings by two area search*, which, at first glance, may look more complicated, and hardly more powerful than the symmetric area search. However, the definition of these matchings is subject to further refinement (cf. Subsections 3.2.3 and 3.2.4). Also, the definition of the matchings introduced here can be adapted to define cyclic point maps of higher order (cf. Subsection 3.2.5).

Let (a_k) be a dense sequence of points in G, and U_m a sequence of symmetric, open neighbourhoods of 0 with compact closure and such that $U_{m+1} \subset U_m$ and $\cap_{m \in \mathbb{N}} U_m = \{0\}$. Fix $m \in \mathbb{N}$, distinct $k_1, k_2 \in \mathbb{N}$, and $i \in \{1, 2\}$, and define an auxiliary function $f_{m,i,k_1,k_2} : \mathbf{L}(G) \to \{0, 1\}$ by

$$f_{m,i,k_1,k_2}(\varphi) := \prod_{j=1}^{2} \mathbf{1}\{\mathrm{card}(\varphi \cap (U_m + a_{k_j} - a_{k_i})) = 1\} \quad \text{for all } \varphi \in \mathbf{L}_0(G),$$

and $f_{m,i,k_1,k_2}(\varphi) := 0$ for all $\varphi \in \mathbf{L}(G) \setminus \mathbf{L}_0(G)$. We have $f_{m,i,k_1,k_2}(\varphi) = 1$ if and only if there is a unique point in each of the two sets $\varphi \cap (U_m + a_{k_j} - a_{k_i}), j = 1, 2$, with the origin at

ith position. We need to have direct access to these points and, thus define the mappings $y_j^{m,i,k_1,k_2} : \mathbf{L}(G) \to G, j = 1, 2$, by

$$y_j^{m,i,k_1,k_2}(\varphi) := \begin{cases} x & \text{if } f_{m,i,k_1,k_2}(\varphi) = 1 \text{ and } (\varphi \cap U_m + a_{k_j} - a_{k_i}) = \{x\}, \\ 0 & \text{otherwise,} \end{cases}$$

In particular, we always have $y_i^{m,i,k_1,k_2} = 0$. For convenience, we will write y_j^i for y_j^{m,i,k_1,k_2} in the sequel, omitting the indices m, k_1, k_2, which can be retrieved from the context.

Let us comment on the above definition of y_j^i. If the origin is the unique point of φ in U_m and there exists a unique point y in $\varphi \cap U_m + a_{k_j} - a_{k_i}$, we write $y = y_j^i(\varphi)$ to express that y has jth position (lower index) in the unique pair of points detected by two area search and including the origin at ith position (upper index). Accordingly, we have $y_i^i(\varphi) = 0$. Moreover, assume that the origin is also the unique point in $\theta_y\varphi \cap U_m$ and that there is a unique point in $\theta_y\varphi \cap U_m + a_{k_i} - a_{k_j}$. Then this unique point is given by $y_i^j(\theta_y\varphi)$, i.e., the point at ith position in the unique pair of points detected by two area search and including the origin at jth position, and we also have $y_j^j(\theta_y\varphi) = 0$. In the following lemma, we will prove that, for $\ell \in \{1, 2\}$, the unique point in $\theta_y\varphi \cap U_m + a_{k_j} - a_{k_\ell}$ is the image of $y_\ell^i(\varphi)$ under θ_y, i.e., the point $y_\ell^i(\varphi) - y$.

Using the *Kronecker symbol* $\varepsilon_{j,\ell}$ defined by $\varepsilon_{j,\ell} = 1$ if $j = \ell$ and $\varepsilon_{j,\ell} = 0$ otherwise, the lemma is formally stated as follows.

Lemma 3.2.2. *Let $\varphi \in \mathbf{L}_0(G), i \in \{1, 2\}$, and assume that $f_{m,j,k_1,k_2}(\theta_{y_\ell^i(\varphi)}\varphi) = \varepsilon_{j,\ell}$ for all $1 \le j, \ell \le 2$. Then we have*

$$y_\ell^j(\theta_{y_j^i(\varphi)}\varphi) = \theta_{y_j^i(\varphi)}(y_\ell^i(\varphi)), \quad 1 \le j, \ell \le 2. \tag{3.5}$$

Proof: The special case $j = \ell = i$ yields $f_{m,i,k_1,k_2}(\varphi) = 1$ and, by the definition of y_j^i, we have

$$\varphi \cap (U_m + a_j - a_i) = \{y_j^i(\varphi)\}, \quad 1 \le j \le 2.$$

From the symmetry of U_m and $y_i^i(\varphi) = 0$ we deduce that

$$\theta_{y_j^i(\varphi)}(y_\ell^i(\varphi)) \in U_m + a_\ell - a_j, \quad 1 \le j, \ell \le 2.$$

Since we have assumed that $f_{m,j,k_1,k_2}(\theta_{y_j^i(\varphi)}(\varphi)) = 1$, we obtain

$$\theta_{y_j^i(\varphi)}\varphi \cap (U_m + a_\ell - a_j) = \{\theta_{y_j^i(\varphi)}(y_\ell^i(\varphi))\}, \quad 1 \le j, \ell \le 2,$$

and conclude that (3.5) holds. $\square$

A local selection function $\Xi_{m,k_1,k_2} : \mathbf{L}(G) \to \mathbf{L}(G)$ is defined by

$$\Xi_{m,k_1,k_2}(\varphi) := \begin{cases} \{y_1^1(\varphi), y_2^1(\varphi)\} & \text{if } f_{m,j,k_1,k_2}(\theta_{y_\ell^1(\varphi)}\varphi) = \varepsilon_{j,\ell} \text{ for } 1 \le j, \ell \le 2, \\ \{y_1^2(\varphi), y_2^2(\varphi)\} & \text{if } f_{m,j,k_1,k_2}(\theta_{y_\ell^2(\varphi)}\varphi) = \varepsilon_{j,\ell} \text{ for } 1 \le j, \ell \le 2, \\ \varphi & \text{if } \varphi \in \mathbf{L}(G) \setminus \mathbf{L}_0(G), \\ \{0\} & \text{otherwise.} \end{cases} \tag{3.6}$$

The local equivariance property follows from Lemma 3.2.2.

Define the very simple matching $\pi : \mathbf{L}(G) \to G$ by

$$\pi(\varphi) := \begin{cases} x & \text{if } \varphi = \{0, x\}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.7}$$

Proposition 3.2.3. *For all $\varphi \in \mathbf{L}_0(G)$, the family of matchings defined by a composition of one of the local selection functions $\Xi_{m,k_1,k_2}, m, k_1, k_2 \in \mathbb{N}$, and the matching π satisfies*

$$\{x \in \varphi : \{-x, 2x\} \cap \varphi = \emptyset\} \cup \{0\} \subset \{\pi \circ \Xi_{m,k_1,k_2}(\varphi) : m, k_1, k_2 \in \mathbb{N}\}. \tag{3.8}$$

PROOF: Assume that $\varphi \in \mathbf{L}_0(G)$ and that $x \in \varphi$ such that $\{-x, 2x\} \cap \varphi = \emptyset$. For $m_0 \in \mathbb{N}$ and $y \in G$, the set $\varphi \cap (U_{m_0} - y)$ contains only finitely many points, because U_{m_0} is assumed to be relatively compact. Moreover, we have $\cap_{m \in \mathbb{N}}(\varphi \cap U_m + y) \subset \{y\}$. We deduce that there exist $m, k_1, k_2 \in \mathbb{N}$ such that

(i) $\varphi \cap (U_m + a_{k_2} - a_{k_1}) = \{x\},\ \varphi \cap U_m = \{0\}$, hence $f_{m,1,k_1,k_2} = 1$,

(ii) $\varphi \cap (U_m + a_{k_1} - a_{k_2}) = \emptyset,\ \varphi \cap U_m = \{0\}$, hence $f_{m,2,k_1,k_2} = 0$,

(iii) $\theta_x\varphi \cap (U_m + a_{k_2} - a_{k_1}) = \emptyset,\ \theta_x\varphi \cap U_m = \{0\}$, hence $f_{m,1,k_1,k_2} = 0$,

(iv) $\theta_x\varphi \cap (U_m + a_{k_1} - a_{k_2}) = \{-x\},\ \theta_x\varphi \cap U_m = \{0\}$, hence $f_{m,2,k_1,k_2} = 1$.

We then have $\Xi_{m,k_1,k_2}(\varphi) = \{0, x\}$, and conclude that $\pi \circ \Xi_{m,k_1,k_2}(\varphi) = x$. □

Depending on $\varphi \in \mathbf{L}_0(G)$, there may be equality or strict inequality in (3.8), so the family of matchings is at least as rich as the family that we had obtained in the preceding subsection by symmetric area search.

3.2.3 The induced permutation

Our aim is now to refine the point maps from the last subsection. The major difference between the symmetric area search and the two area search is that an enumeration of the points that depends only on their relative positions, is intrinsic in the procedure. This particularity will be exploited now.

Let $\varphi \in \mathbf{L}_0(G), m, k_1, k_2 \in \mathbb{N}$ and assume that $f_{m,i,k_1,k_2}(\varphi) = 1$. Then there are two points in φ that we have called $y_j^i(\varphi) \in \varphi, 1 \le j \le 2$, that satisfy $U_m + a_j - a_i \cap \varphi = \{y_j^i(\varphi)\}, 1 \le j \le 2$. We assign to each point its position, i.e., its lower index j. It follows from Lemma 3.2.2, that this enumeration of the two points only depends on their relative positions, but not on the fact, which one of them is in the origin.

Let $\Delta : \{1, 2\} \to \{1, 2\}$ be the constant function $\Delta \equiv 1$. We introduce the induced permutation mapping $\varrho_{m,i,k_1,k_2} : \mathbf{L}(G) \to \mathcal{S}_2 \cup \{\Delta\}$, which maps each $\varphi \in \mathbf{L}(G)$ to an element of the symmetric group $\mathcal{S}_2$ of permutations of the set $\{1, 2\}$ or Δ. We define

$$\varrho_{m,i,k_1,k_2}(\varphi)(j) := \operatorname{card}\{\ell \in \{1, 2\} : y_\ell^i(\varphi) \ll_\varphi y_j^i(\varphi)\}$$

if $f_{m,i,k_1,k_2}(\varphi) = 1$ and $\tilde{I}(\varphi, y_1^i(\varphi)) \neq \tilde{I}(\varphi, y_2^i(\varphi))$, and $\varrho_{m,i,k_1,k_2}(\varphi) := \Delta$ otherwise.

This gives us the means to refine the definition of local selection functions from (3.6), and subsequently, extend the resulting family of matchings. Let $\gamma \in \mathcal{S}_2$ and $m, k_1, k_2 \in \mathbb{N}$, then we define

$$\Phi_{m,k_1,k_2,\gamma}(\varphi) := \begin{cases} \{y_1^1(\varphi), y_2^1(\varphi)\} & \text{if } f_{m,j,k_1,k_2}(\theta_{y_\ell^1(\varphi)}\varphi)\,\mathbf{1}\{\varrho_{m,j,k_1,k_2}(\theta_{y_\ell^1(\varphi)}\varphi) = \gamma\} = \varepsilon_{j,\ell} \\ & \text{for } 1 \leq j, \ell \leq 2, \\ \{y_1^2(\varphi), y_2^2(\varphi)\} & \text{if } f_{m,j,k_1,k_2}(\theta_{y_\ell^2(\varphi)}\varphi)\,\mathbf{1}\{\varrho_{m,j,k_1,k_2}(\theta_{y_\ell^2(\varphi)}\varphi) = \gamma\} = \varepsilon_{j,\ell} \\ & \text{for } 1 \leq j, \ell \leq 2, \\ \varphi & \text{if } \varphi \in \mathbf{L}(G) \setminus \mathbf{L}_0(G), \\ \{0\} & \text{otherwise.} \end{cases} \tag{3.9}$$

If we are in the first case, the condition $f_{m,j,k_1,k_2}(\theta_{y_\ell^1(\varphi)}\varphi)\,\mathbf{1}\{\varrho_{m,j,k_1,k_2}(\theta_{y_\ell^1(\varphi)}\varphi) = \gamma\} = \varepsilon_{j,\ell}$ yields $f_{m,1,k_1,k_2}(\varphi) = 1$ and $\varrho_{m,1,k_1,k_2}(\varphi) = \gamma$ when $j = \ell = 1$ and $f_{m,2,k_1,k_2}(\varphi)\,\mathbf{1}\{\varrho_{m,2,k_1,k_2}(\varphi) = \gamma\} = 0$ for $j = 2, \ell = 1$. In the second case we have $f_{m,2,k_1,k_2}(\varphi) = 1$ and $\varrho_{m,2,k_1,k_2}(\varphi) = \gamma$. So the first and second case are exclusive and the function is well defined. We can now state a refined version of Proposition 3.2.3.

Proposition 3.2.4. *The family of matchings defined by a composition of one of the local selection functions $\Phi_{m,k_1,k_2,\gamma}, m, k_1, k_2 \in \mathbb{N}, \gamma \in \mathcal{S}_2$, and the matching π satisfies*

$$\begin{aligned} &\{x \in \varphi : \{-x, 2x\} \cap \varphi = \emptyset \text{ or } \tilde{I}(\varphi, 0) \neq \tilde{I}(\varphi, x) = \tilde{I}(\varphi, -x)\} \\ &\subset \{\pi \circ \Phi_{m,k_1,k_2,\gamma}(\varphi) : m, k_1, k_2 \in \mathbb{N}, \gamma \in \mathcal{S}_2\}, \quad \varphi \in \mathbf{L}_0(G). \end{aligned} \tag{3.10}$$

Proof: Clearly, for any $m, k_1, k_2 \in \mathbb{N}$, we have

$$\{\pi \circ \Xi_{m,k_1,k_2}(\varphi)\} \subset \bigcup_{\gamma \in \mathcal{S}_2} \{\pi \circ \Phi_{m,k_1,k_2,\gamma}(\varphi)\},$$

so let us assume that $\varphi \in \mathbf{L}_0(G)$ and $x \in \varphi$ satisfies $\tilde{I}(\varphi, 0) \neq \tilde{I}(\varphi, x) = \tilde{I}(\varphi, -x)$. Then $\theta_x\varphi = \theta_{-x}\varphi$, hence, the fact that $0, x \in \varphi$ yields $\{-x, 0, x, 2x\} \subset \varphi$. We denote the two elements in $\mathcal{S}_2$ by $\mathrm{id}_{\{1,2\}}$ for the identity mapping on $\{1, 2\}$ and τ for the non-trivial transposition. Without restriction of generality, we assume here that $0 \ll_\varphi x$, the case $x \ll_\varphi 0$ is treated in the same way. There exist $m, k_1, k_2 \in \mathbb{N}$ such that

(i) $\varphi \cap (U_m + a_{k_2} - a_{k_1}) = \{x\}$, $\varphi \cap U_m = \{0\}$, hence $f_{m,1,k_1,k_2}(\varphi) = 1, y_1^1(\varphi) = 0, y_2^1(\varphi) = x$ and $\varrho_{m,1,k_1,k_2}(\varphi) = \mathrm{id}_{\{1,2\}}$,

(ii) $\varphi \cap (U_m + a_{k_1} - a_{k_2}) = \{-x\}$, $\varphi \cap U_m = \{0\}$, hence $f_{m,2,k_1,k_2}(\varphi) = 1, y_1^2(\varphi) = -x, y_2^2 = 0$ and $\varrho_{m,1,k_1,k_2}(\varphi) = \tau$,

(iii) $\theta_x\varphi \cap (U_m + a_{k_2} - a_{k_1}) = \{x\}$, $\theta_x\varphi \cap U_m = \{0\}$, hence $f_{m,1,k_1,k_2}(\theta_x\varphi) = 1, y_1^1(\varphi) = 0, y_2^1(\varphi) = x$ and $\varrho_{m,1,k_1,k_2}(\varphi) = \tau$.

(iv) $\theta_x\varphi \cap (U_m + a_{k_1} - a_{k_2}) = \{-x\}$, $\theta_x\varphi \cap U_m = \{0\}$, hence $f_{m,2,k_1,k_2}(\varphi) = 1, y_1^2(\varphi) = -x, y_2^2(\varphi) = 0$ and $\varrho_{m,1,k_1,k_2}(\varphi) = \mathrm{id}_{\{1,2\}}$.

We conclude that $\Phi_{m,k_1,k_2,\mathrm{id}_{\{1,2\}}}(\varphi) = \{0, x\}$, finishing the proof of the proposition. □

3.2.4 A quasi complete family of matchings

We have shown in Proposition 3.2.4 that there exists a countable family of matchings with the property that, for $\varphi \in \mathbf{L}_0(G)$ and $x \in \varphi$ such that $-x, 2x \notin \varphi$ or $\varphi \neq \varphi - x = \varphi + x$, (at least) one of the matchings σ satisfies $\sigma(\varphi) = x$. The challenge is now to define matchings for the case, where $\varphi - x \neq \varphi, \{-x, 2x\} \cap \varphi \neq \emptyset$, and $\tilde{I}(\varphi, -x) \neq \tilde{I}(\varphi, x)$. Our approach to this problem is quite simple. In the definition of $\Phi_{m,k_1,k_n,\gamma}$ we replace f by $f \circ \Psi$, where Ψ is a selection function that removes any point y such that $\tilde{I}(\varphi, y) \notin \{\tilde{I}(\varphi, 0), \tilde{I}(\varphi, x)\}$ in a suitable area around 0.

Let $(\bar{q}_n)$ be an enumeration of $(\mathbb{Q} \cap (0,1))^2$, i.e., the pairs $\bar{q}_n = (q_n^1, q_n^2)$ of rational numbers contained in the open interval $(0, 1)$. For $s, t \in \mathbb{N}$, define a selection function $\Psi_{s,t} : \mathbf{L}(G) \to \mathbf{L}(G)$ by

$$\Psi_{s,t}(\varphi) := \{x \in \varphi : \min_{1 \le i \le 2} |\tilde{I}(\varphi, x) - q_s^i| < t^{-1}\}. \tag{3.11}$$

These selection functions have the following property.

Lemma 3.2.5. *For $\varphi \in \mathbf{L}_0(G), x \in \varphi$ and $c > 0$ there exist $s_0, t_0 \in \mathbb{N}$ such that $0, x \in \Psi_{s_0,t_0}(\varphi)$ and $\tilde{I}(\varphi, y) \in \{\tilde{I}(\varphi, 0), \tilde{I}(\varphi, x)\}$ for all $y \in \Psi_{s_0,t_0}(\varphi) \cap B_d(0, c)$.*

PROOF: Define $\varepsilon = \varepsilon(\varphi, 0, x)$ as the minimal, strictly positive difference of the value of the index function applied to (φ, y), where $y \in \varphi \cap B_d(0, c)$, and the values of the standard index function applied to $(\varphi, 0)$ or (φ, x), i.e.,

$$\varepsilon := \inf\{|\tilde{I}(\varphi, y) - \tilde{I}(\varphi, z)| : y \in \varphi \cap B_d(0, c), z \in \{0, x\}, \tilde{I}(\varphi, y) \neq \tilde{I}(\varphi, z)\},$$

where, for convenience, we let $\inf \emptyset := \infty$. There are only finitely many points in the set $\varphi \cap B_d(0, c)$, so the infimum is attained or infinite and, hence, strictly positive. Choose $t_0 \in \mathbb{N}$ such that $t_0 > 2/\varepsilon$, and $s_0 \in \mathbb{N}$ such that $|\tilde{I}(\varphi, 0) - q_{s_0}^1| < t_0^{-1}$ and $|\tilde{I}(\varphi, x) - q_{s_0}^2| < t_0^{-1}$, then the claim of the Lemma follows by a straightforward application of the triangle inequality. □

Then define local selection functions $\Phi_{m,k_1,k_2,\gamma,s,t}$, $m, k_1, k_2, s, t \in \mathbb{N}$ and $\gamma \in \mathcal{S}_2$, by

$$\Phi_{m,k_1,k_2,\gamma,s,t}(\varphi) := \begin{cases} \{y_1^1(\varphi), y_2^1(\varphi)\} & \text{if } f_{m,j,k_1,k_2}(\Psi_{s,t}(\theta_{y_\ell^1(\varphi)}\varphi))\, \mathbf{1}\{\varrho_{m,j,k_1,k_2}(\theta_{y_\ell^1(\varphi)}\varphi) = \gamma\} = \varepsilon_{j,\ell} \\ & \text{for } 1 \le j, \ell \le 2, \\ \{y_1^2(\varphi), y_2^2(\varphi)\} & \text{if } f_{m,j,k_1,k_2}(\Psi_{s,t}(\theta_{y_\ell^2(\varphi)}\varphi))\, \mathbf{1}\{\varrho_{m,j,k_1,k_2}(\theta_{y_\ell^2(\varphi)}\varphi) = \gamma\} = \varepsilon_{j,\ell} \\ & \text{for } 1 \le j, \ell \le 2, \\ \varphi & \text{if } \varphi \in \mathbf{L}(G) \setminus \mathbf{L}_0(G), \\ \{0\} & \text{otherwise.} \end{cases} \tag{3.12}$$

Note that $\varrho_{m,j,k_1,k_2}(\theta_{y_\ell^i(\varphi)}\varphi) = \gamma$ yields $f_{m,j,k_1,k_2}(\theta_{y_\ell^i(\varphi)}\varphi) = 1$ for all $1 \le i, j, \ell \le n$. Hence,

$$f_{m,j,k_1,k_2}(\Psi_{s,t}(\theta_{y_\ell^i(\varphi)}\varphi))\, \mathbf{1}\{\varrho_{m,j,k_1,k_2}(\theta_{y_\ell^i(\varphi)}\varphi) = \gamma\} = 1$$

implies $f_{m,j,k_1,k_2}(\Psi_{s,t}(\theta_{y^i_\ell(\varphi)}\varphi)) = f_{m,j,k_1,k_2}(\theta_{y^i_\ell(\varphi)}\varphi) = 1$ and, hence, $y^j_1(\varphi) = y^j_1(\Psi_{s,t}(\varphi))$ and $y^j_1(\varphi) = y^j_1(\Psi_{s,t}(\varphi))$. We are now ready to state and prove the result announced in the introduction of this thesis on the existence of a countable family of matchings that is quasi-complete on $\mathbf{L}_0(G)$.

Theorem 3.2.6. *The family of matchings M defined by a composition of one of the local selection functions $\Phi_{m,k_1,k_2,\gamma,s,t}$, $m, k_1, k_2, s, t \in \mathbb{N}, \gamma \in \mathcal{S}_2$ and the matching π satisfies*

$$\{\pi \circ \Phi_{m,k_1,k_2,\gamma,s,t}(\varphi) : m, k_1, k_2, s, t \in \mathbb{N}, \gamma \in \mathcal{S}_2\} = \{x \in \varphi : \tilde{I}(\varphi, 0) \neq \tilde{I}(\varphi, x)\} \cup \{0\} \quad (3.13)$$

for all $\varphi \in \mathbf{L}_0(G)$. Hence, the countable family of matchings M is quasi-complete on $\mathbf{L}_0(G)$.

Proof: Let $\varphi \in \mathbf{L}_0(G)$ and $x \in \varphi$ and assume that $\tilde{I}(\varphi, 0) \neq \tilde{I}(\varphi, x)$. If $\{-x, 2x\} \cap \varphi = \emptyset$ or $\tilde{I}(\varphi, x) = \tilde{I}(\varphi, -x)$, then we may use that $\Psi_{1,1}$ is the identity mapping on $\mathbf{L}(G)$ and conclude by Proposition 3.2.4 that there exists a matching $\sigma \in M$ such that $\sigma(\varphi) = x$.

If $\{-x, 2x\} \cap \varphi \neq \emptyset$, $\tilde{I}(\varphi, 0) \neq \tilde{I}(\varphi, x)$ and $\tilde{I}(\varphi, x) \neq \tilde{I}(\varphi, -x)$, then we have $\varphi - x \neq \varphi$ and $\varphi - x \neq \varphi + x$, and, hence, also $\tilde{I}(\varphi, x) \neq \tilde{I}(\varphi, 2x)$ and $\tilde{I}(\varphi, 0) \neq \tilde{I}(\varphi, 2x)$. We may now apply Lemma 3.2.5 for some $c > \max\{d(0, 2x), d(0, -x)\}$, and obtain $s_0, t_0 \in \mathbb{N}$ such that $0, x \in \Psi_{s_0,t_0}(\varphi)$ and $\{-x, 2x\} \cap \Psi_{s_0,t_0}(\varphi) = \emptyset$. As before, in the proof of Proposition 3.2.4, we can then show that there exist $m, k_1, k_2 \in \mathbb{N}$ and $\gamma \in \mathcal{S}_2$ such that $\Phi_{m,k_1,k_2,\gamma,s_0,t_0}(\varphi) = \{0, x\}$, and conclude that there is a matching $\tau \in M$ such that $\tau(\varphi) = x$, finishing the proof of the theorem. □

3.2.5 Cyclic point maps of order n

Let us now generalize the definition of matchings to a definition of general *cyclic point maps of order n*, i.e., point maps σ that satisfy $\sigma^n(\varphi) = 0$. For this purpose, we will adopt a similar strategy as in Subsections 3.2.2-3.2.4, and adapt the steps made therein to obtain a family C of cyclic point maps of order n with the following property. For any $\varphi \in \mathbf{L}_0(G)$ and n points $x_1, \ldots, x_n \in \varphi$ such that $x_1 = 0$ and $\tilde{I}(\varphi, x_i) \neq \tilde{I}(\varphi, x_j), 1 \leq i < j \leq n$, there is a n-cyclic point map $\pi \in C$ such that $\{\pi^i(\varphi) : 1 \leq i \leq n\} = \{x_i : 1 \leq i \leq n\}$.

We begin with the definition of a point map on finite point configurations, which generalizes the definition of the matching π in (3.7). Compare also to the case of a finite φ in the definition of the universal point map τ in (3.2). Define $\sigma : \mathbf{L}(G) \to G$ by

$$\sigma(\varphi) := \begin{cases} \min\{x \in \varphi \setminus \{0\} : 0 \ll_\varphi x\} & \text{if } 1 \leq \operatorname{card}(\varphi) < \infty, 0 \in \varphi \text{ and } 0 \neq \max_{\ll_\varphi}(\varphi), \\ \min_{\ll_\varphi}(\varphi) & \text{if } 1 \leq \operatorname{card}(\varphi) < \infty \text{ and } 0 = \max_{\ll_\varphi}(\varphi), \\ 0 & \text{otherwise,} \end{cases} \quad (3.14)$$

where $\min_{\ll_\varphi}$ resp. $\max_{\ll_\varphi}$ denote the minimum resp. the maximum with respect to the standard order relation $\ll_\varphi$ on φ defined in (2.12). These minima and maxima exist, because a finite point set is always aperiodic, and so Lemma 2.5.6 yields that the standard order relation is total on any non-empty $\varphi \in \mathbf{L}(G)$ with finitely many elements. In particular, the mapping σ is cyclic of order n on

$$\{\varphi \in \mathbf{L}_0(G) : \operatorname{card}(\varphi) = n\}.$$

We will now define local selection functions, that extract the n points from a suitable $\varphi \in \mathbf{L}_0(G)$. We will do this in close analogy to the procedure from the last subsections.

As before, we denote by (a_k) a dense sequence of points in G. Then, let (U_m) be a sequence of symmetric, open neighbourhoods of 0 with compact closure and such that $\cap_{m\in\mathbb{N}} U_m = \{0\}$. Moreover, we require that $U_{m+1} \oplus U_{m+1} \subset U_m$. In general, the existence of such a sequence follows from the continuity of the addition on G, in the special case $G = \mathbb{R}^d$, we may choose open balls around the origin with radius 2^{-m}.

For fixed $m \in \mathbb{N}$, pairwise distinct $k_1, \ldots, k_n \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$, we define the auxiliary function $f_{m,i,k_1,\ldots,k_n} : \mathbf{L}(G) \to \{0,1\}$ by

$$f_{m,i,k_1,\ldots,k_n}(\varphi) := \prod_{j=1}^{n} \mathbf{1}\{\operatorname{card}(\varphi \cap (U_m + a_{k_j} - a_{k_i})) = 1\} \quad \text{for all } \varphi \in \mathbf{L}_0(G),$$

and $f_{m,i,k_1,\ldots,k_n}(\varphi) := 0$ for all $\varphi \in \mathbf{L}(G) \setminus \mathbf{L}_0(G)$. We have $f_{m,i,k_1,\ldots,k_n}(\varphi) = 1$ if and only if there is a unique point in each of the n sets $\varphi \cap (U_m + a_{k_j} - a_{k_i}), 1 \le j \le n$, with the origin at ith position. Again, we define mappings $y_j^i = y_j^{m,i,k_1,\ldots,k_n} : \mathbf{L}(G) \to G, j = 1, \ldots, n$, by

$$y_j^i(\varphi) := \begin{cases} x & \text{if } f_{m,i,k_1,\ldots,k_n}(\varphi) = 1 \text{ and } (\varphi \cap U_m + a_{k_j} - a_{k_i}) = \{x\}, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, we always have $y_i^{m,i,k_1,\ldots,k_n} = 0$. It is easy to verify that $f_{m,i,k_1,\ldots,k_n}(\varphi) = f_{m+1,i,k_1,\ldots,k_n}(\varphi)$ implies $y_j^{m,i,k_1,\ldots,k_n}(\varphi) = y_j^{m+1,i,k_1,\ldots,k_n}(\varphi)$ for all $1 \le j \le n$. This fact enables us to write y_j^i for $y_j^{m,i,k_1,\ldots,k_n}$ as before, even though m will not always be assumed to be fixed in the sequel.

The following lemma is an adaptation of Lemma 3.2.2. Here, we will use our additional assumption on the sequence (U_m), i.e., $U_{m+1} \oplus U_{m+1} \subset U_m$.

Lemma 3.2.7. *Let $\varphi \in \mathbf{L}(G), i \in \{1, \ldots, n\}$ and $m, k_1, \ldots, k_n \in \mathbb{N}, m \ge 2$, and assume that*

$$f_{m-1,i,k_1,\ldots,k_n}(\varphi) = f_{m+1,i,k_1,\ldots,k_n}(\varphi) = 1.$$

Then

$$f_{m,j,k_1,\ldots,k_n}(\theta_{y_j^i(\varphi)}\varphi) = 1, \quad 1 \le j \le n,$$

and we have

$$y_\ell^j(\theta_{y_j^i(\varphi)}\varphi) = y_\ell^i(\varphi) - y_j^i(\varphi), \quad 1 \le j, \ell \le n. \tag{3.15}$$

Proof: We have

$$y_j^i(\varphi) \in \varphi \cap (U_{m+1} + a_{k_j} - a_{k_i}), \quad 1 \le j \le n.$$

Hence, the assumption on the sequence (U_m) yields

$$y_\ell^i(\varphi) - y_j^i(\varphi) \in U_{m+1} \oplus U_{m+1} + a_{k_\ell} - a_{k_j} \subset U_m + a_{k_\ell} - a_{k_j}, \quad 1 \le j, \ell \le n.$$

For fixed j, ℓ such that $1 \le j, \ell \le n$, the fact that

$$y_\ell^i(\varphi) - y_j^i(\varphi) \in \theta_{y_j^i(\varphi)}\varphi$$

yields that $\mathrm{card}(\theta_{y^i_j(\varphi)}\varphi \cap (U_m + a_{k_\ell} - a_{k_j})) \geq 1$. Moreover, any $x \in \theta_{y^i_j(\varphi)}\varphi \cap (U_m + a_{k_\ell} - a_{k_j})$ satisfies

$$x + y^i_j(\varphi) \in \varphi \cap (U_m + a_{k_\ell} - a_{k_j} + y^i_j(\varphi)) \subset \varphi \cap (U_{m-1} + a_{k_\ell} - a_{k_i}).$$

Since, again by the assumption, $\varphi \cap (U_{m-1} + a_{k_\ell} - a_{k_i})$ contains exactly one element, we have $x + y^i_j(\varphi) = y^i_\ell(\varphi)$. We conclude that $\mathrm{card}(\theta_{y^i_j(\varphi)}\varphi \cap (U_m + a_{k_\ell} - a_{k_j})) = 1$ and that

$$\theta_{y^i_j(\varphi)}\varphi \cap (U_m + a_{k_\ell} - a_{k_j}) = \{y^i_\ell(\varphi) - y^i_j(\varphi)\},$$

proving both claims of the lemma. □

As before in (3.6), the lemma yields the possibility to introduce local selection functions. However, here we will directly proceed to introduce the induced permutation.

Let $\varphi \in \mathbf{L}_0(G), m, k_1, \ldots, k_n \in \mathbb{N}$ and assume that $f_{m,i,k_1,\ldots,k_n}(\varphi) = 1$. Then there are n points $y^i_j(\varphi) \in \varphi, 1 \leq j \leq n$, that satisfy $(U_m + a_j - a_i) \cap \varphi = \{y^i_j(\varphi)\}, 1 \leq j \leq n$. Lemma 3.2.7 yields that the position of each point, i.e., the lower index, only depends on the relative positions of the n points and not on the fact which point is the origin.

Let $\Delta : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be the constant function $\Delta \equiv 1$. The induced permutation mapping $\varrho_{m,i,k_1,\ldots,k_n} : \mathbf{L}(G) \to \mathcal{S}_n \cup \{\Delta\}$ is defined by

$$\varrho_{m,i,k_1,\ldots,k_n}(\varphi)(j) := \mathrm{card}\{\ell \in \{1, \ldots, n\} : y^i_\ell(\varphi) \ll_\varphi y^i_j(\varphi)\}$$

if $f_{m,i,k_1,\ldots,k_n}(\varphi) = 1$ and $\prod_{1 \leq j < \ell \leq n} \mathbf{1}\{\tilde{I}(\varphi, y^i_j(\varphi)) \neq \tilde{I}(\varphi, y^i_\ell)(\varphi))\} = 1$, and by

$$\varrho_{m,i,k_1,\ldots,k_n}(\varphi) := \Delta$$

otherwise. As a last preparation, we will now generalize the selection functions defined in (3.11). Let $(\bar{q}_k)$ be an enumeration of $(\mathbb{Q} \cap (0,1))^n$, i.e., the n-tuples $\bar{q}_k = (q^1_k, \ldots, q^n_k)$ of rational numbers contained in the open interval $(0, 1)$. For $s, t \in \mathbb{N}$, define a selection function $\Psi_{s,t,n} : \mathbf{L}(G) \to \mathbf{L}(G)$ by

$$\Psi_{s,t,n}(\varphi) := \{x \in \varphi : \min_{1 \leq i \leq n} |\tilde{I}(\varphi, x) - q^i_s| < t^{-1}\}.$$

These selection functions have the following properties.

Lemma 3.2.8. *For $\varphi \in \mathbf{L}_0(G), c > 0$ and $x_1, \ldots, x_n \in \varphi$ such that $x_1 = 0$ and $\tilde{I}(\varphi, x_i) \neq \tilde{I}(\varphi, x_j), 1 \leq i, j \leq n$, there exist $s, t \in \mathbb{N}$ such that $x_i \in \Psi_{s,t,n}(\varphi), 1 \leq i \leq n$, and $\tilde{I}(\varphi, y) \in \{\tilde{I}(\varphi, x_i) : 1 \leq i \leq n\}$ for all $y \in \Psi_{s,t,n}(\varphi) \cap B_d(0, c)$.*

We omit the proof, which is a straightforward adaptation of the proof of Lemma 3.2.5. Generalizing the definition of the local selection functions from (3.12), we define, for $\gamma \in \mathcal{S}_n$

and $m, k_1, \ldots, k_n, s, t \in \mathbb{N}$ the local selection function $\Phi_{m,k_1,\ldots,k_n,\gamma,s,t,n} : \mathbf{L}(G) \to \mathbf{L}(G)$ by

$$\Phi_{m,k_1,\ldots,k_n,\gamma,s,t,n}(\varphi) := \tag{3.16}$$
$$\begin{cases} \{y_1^1(\varphi),\ldots,y_n^1(\varphi)\} & \text{if } f_{m-1,j}(\theta_{y_\ell^1(\varphi)}\varphi) f_{m+1,j}(\Psi_{s,t,n}(\theta_{y_\ell^1(\varphi)}\varphi))\, \mathbf{1}\{\varrho_{m,j}(\theta_{y_\ell^1(\varphi)}\varphi) = \gamma\} = \varepsilon_{j,\ell} \\ & \text{for all } 1 \le j, \ell \le n, \\ \ldots & \ldots \\ \{y_1^n(\varphi),\ldots,y_n^n(\varphi)\} & \text{if } f_{m-1,j}(\theta_{y_\ell^n(\varphi)}\varphi) f_{m+1,j}(\Psi_{s,t,n}(\theta_{y_\ell^n(\varphi)}\varphi))\, \mathbf{1}\{\varrho_{m,j}(\theta_{y_\ell^n(\varphi)}\varphi) = \gamma\} = \varepsilon_{j,\ell} \\ & \text{for all } 1 \le j, \ell \le n, \\ \varphi & \text{if } \varphi \in \mathbf{L}(G) \setminus \mathbf{L}_0(G), \\ \{0\} & \text{otherwise,} \end{cases}$$

where we have exceptionally omitted the indices $k_1, \ldots, k_n$ in $f_{m-1,j,k_1,\ldots,k_n}$, $f_{m+1,j,k_1,\ldots,k_n}$ and $\varrho_{m,j,k_1,\ldots,k_n}$.

Note that $\varrho_{m,j,k_1,\ldots,k_n}(\theta_{y_\ell^i(\varphi)}\varphi) = \gamma$ yields $f_{m,j,k_1,\ldots,k_n}(\theta_{y_\ell^i(\varphi)}\varphi) = 1$ for all $1 \le i,j,\ell \le n$. Hence,

$$f_{m+1,j,k_1,\ldots,k_n}(\Psi_{s,t}(\theta_{y_\ell^i(\varphi)}\varphi))\, \mathbf{1}\{\varrho_{m,j,k_1,\ldots,k_n}(\theta_{y_\ell^i(\varphi)}\varphi) = \gamma\} = 1$$

implies $f_{m+1,j,k_1,\ldots,k_n}(\Psi_{s,t}(\theta_{y_\ell^i(\varphi)}\varphi)) = f_{m,j,k_1,\ldots,k_n}(\theta_{y_\ell^i(\varphi)}\varphi) = 1$ and, hence, $y_1^j(\varphi) = y_1^j(\Psi_{s,t}(\varphi))$ and $y_1^j(\varphi) = y_1^j(\Psi_{s,t}(\varphi))$.

The local selection functions $\Phi_{m,k_1,\ldots,k_n,\gamma,s,t,n}$ are well defined, because the conditions of the first n cases are exclusive. Indeed, if, for $i \in \{1,\ldots,n\}$, the condition

$$f_{m+1,j,k_1,\ldots,k_n}(\theta_{y_\ell^i(\varphi)}\varphi)\, \mathbf{1}\{\varrho_{m+1,i,k_1,k_2}(\theta_{y_\ell^i(\varphi)}\varphi) = \gamma\} = \varepsilon_{j,\ell} \tag{3.17}$$

holds, then the special case $i = \ell$ yields $f_{m+1,j,k_1,k_2}(\varphi)\, \mathbf{1}\{\varrho_{m+1,j,k_1,k_2}(\varphi) = \gamma\} = 1$ if and only if $j = i$. Hence, (3.17) does not hold for any $j \in \{1,\ldots,n\} \setminus \{i\}$. Moreover, Lemma 3.2.7 yields the local equivariance property.

Theorem 3.2.9. *The family C of cyclic point maps of order n defined by the composition of one of the local selection functions $\Phi_{m,k_1,\ldots,k_n,\gamma,s,t,n}, m, k_1, k_2, s, t \in \mathbb{N}, \gamma \in \mathcal{S}_n$ and the point map σ has the following property. For any $\varphi \in \mathbf{L}_0(G)$ and n points $x_1, \ldots, x_n \in \varphi$ such that $x_1 = 0$ and $\tilde{I}(\varphi, x_i) \neq \tilde{I}(\varphi, x_j)$ for all $1 \le i < j \le n$, there exist $m, k_1, \ldots, k_n, s, t \in \mathbb{N}$ and $\gamma \in \mathcal{S}_n$ such that*

$$\{(\sigma \circ \Phi_{m,k_1,\ldots,k_n,\gamma,s,t,n})^i(\varphi) : 1 \le i \le n\} = \{x_1, \ldots, x_n\}. \tag{3.18}$$

In particular, C is a countable family of cyclic point maps of order n that is quasi-complete on $\{\varphi \in \mathbf{L}_0(G) : \mathrm{card}\{\tilde{I}(\varphi, x) : x \in \varphi\} \ge n\}$.

Proof: Let $\varphi \in \mathbf{L}_0(G)$ and $x_1, \ldots, x_n \in \varphi$ such that $x_1 = 0$ and $\tilde{I}(\varphi, x_i) \neq \tilde{I}(\varphi, x_j)$ for all $1 \le i < j \le n$. Define

$$c := 2 \max_{1 \le i < j \le n} d(x_i, x_j) + 2.$$

By Lemma 3.2.8, there exist $s, t \in \mathbb{N}$ such that

$$x_i \in \Psi_{s,t,n}(\varphi), \quad 1 \le i \le n,$$

and

$$\tilde{I}(\varphi, y) \in \{\tilde{I}(\varphi, x_i) : 1 \le i \le n\} \tag{3.19}$$

for all $y \in \Psi_{s,t,n}(\varphi) \cap B_d(0,c)$. Moreover, there exist $m, k_1, \ldots, k_n \in \mathbb{N}, m \ge 2$, such that

$$f_{m-1,i,k_1,\ldots,k_n}(\theta_{x_i}\varphi) = f_{m+1,i,k_1,\ldots,k_n}(\Psi_{s,t,n}(\theta_{x_i}\varphi)) = 1, \quad 1 \le i \le n, \tag{3.20}$$

and $y^i_\ell(\theta_{x_i}\varphi) + x_i = y^i_\ell(\Psi_{s,t,n}(\theta_{x_i}\varphi)) + x_i = x_\ell, 1 \le i, \ell \le n$. Also, from (3.20) we obtain $f_{m+1,i,k_1,\ldots,k_n}(\theta_{x_i}\varphi) = 1, 1 \le i \le n$, and Lemma 3.2.7 yields

$$f_{m,\ell,k_1,\ldots,k_n}(\theta_{y^i_\ell(\theta_{x_i}\varphi)}\varphi) = 1, \quad 1 \le i, \ell \le n. \tag{3.21}$$

By assumption, we have $\tilde{I}(\varphi, x_i) \ne \tilde{I}(\varphi, x_j)$ for all $1 \le i < j \le n$, and by Lemma 3.2.7, the induced permutation $\gamma := \varrho_{m,1,k_1,\ldots,k_n}(\varphi)$ satisfies $\gamma \in \mathcal{S}_n$. We claim that

$$\mathbf{1}\{\varrho_{m,j,k_1,\ldots,k_n}(\theta_{x_\ell}\varphi) = \gamma\} = \varepsilon_{j,\ell} \tag{3.22}$$

for all $1 \le j, \ell \le n$. Indeed, for $j = \ell$, Lemma 3.2.7 and the fact that the extended standard index function $\tilde{I}$ is invariant under shifts applied to both arguments yield $\varrho_{m,i,k_1,\ldots,k_n}(\theta_{x_i}\varphi) = \gamma$. In the case $j \ne \ell$, let us assume that

$$f_{m,j,k_1,\ldots,k_n}(\theta_{x_\ell}\varphi) = 1$$

and that

$$\prod_{1 \le q,r \le n} \mathbf{1}\{\tilde{I}(\theta_{x_\ell}\varphi, y^\ell_q(\theta_{x_\ell}\varphi)) \ne \tilde{I}(\theta_{x_\ell}\varphi, y^\ell_r(\theta_{x_\ell}\varphi))\} = 1,$$

because otherwise we have $\varrho_{m,j,k_1,\ldots,k_n}(\theta_{x_\ell}\varphi) = \Delta \ne \gamma$. The last equation and (3.19) yield

$$\{\tilde{I}(\theta_{x_\ell}\varphi, y^\ell_r(\theta_{x_\ell}\varphi)) : 1 \le r \le n\} = \{\tilde{I}(\varphi, x_r) : 1 \le r \le n\}.$$

Hence,

$$\varrho_{m,j,k_1,\ldots,k_n}(\theta_{x_\ell}\varphi)(j) = \varrho_{m,1,k_1,\ldots,k_n}(\varphi)(\ell) = \gamma(i),$$

which implies $\varrho_{m,j,k_1,\ldots,k_n}(\theta_{x_\ell}\varphi) \ne \gamma$. We conclude that

$$\Phi_{m,k_1,\ldots,k_n,\gamma,s,t} = \{x_i : 1 \le i \le n\}$$

and, hence, $\sigma \circ \Phi_{m,k_1,\ldots,k_n,\gamma,s,t}$ is a cyclic point map of order n that satisfies (3.18). ☐

These point maps will be used in Section 3.4, where we will define two point maps that generate a family of point maps on $\mathbf{L}(\mathbb{R}^d)$ that is complete on the aperiodic sets in $\mathbf{L}_0(\mathbb{R}^d)$.

3.3 Directed doubly infinite paths

The study of directed, doubly infinite paths on point processes initiated with the paper [2] by Ferrari, Landim and Thorisson (see also Problems 1.1, 1.2, 1.3 in [2], which are closely related to the subjects of this thesis). Progress was made by Holroyd and Peres in [5] and Timar in [30]. Their probabilistic results will be briefly reviewed in Chapter 5, here we will summarize only some of the ideas from the graph construction in [30], which are essential for the point maps defined in this section. For details, and a careful description of the related problems in random graph theory, we refer the reader to the three papers cited above.

3.3.1 Clumpings, locally finite trees and doubly infinite directed paths

In [30], Timar defines isometry equivariant factor graphs on the points of a point process N on $\mathbb{R}^d$ with finite intensity and isometry invariant distribution. He assumes that a given (isometry invariant) index function is injective on the points of N almost surely, which is shown to be equivalent with the fact that almost surely there is no non-trivial isometry that leaves N invariant. To stick with our setting, we will describe the definition of the graph on locally finite subsets φ of $\mathbb{R}^d$ that satisfy $\operatorname{card}(\varphi) = \infty$ and $\varphi \neq S(\varphi)$ for all translations or rotations $S \neq \mathrm{id}_{\mathbb{R}^d}$ on $\mathbb{R}^d$. Both properties are almost surely satisfied by the point processes described above.

The index function is used to define a thinning procedure (T_n) such that the thinning $(T_n(\varphi))$ satisfies a hardcore property, i.e., the interpoint distances are bounded from below. Then a partition of $\mathbb{R}^d$ is defined by the superposition of the Voronoi mosaics associated with the sets $T_n(\varphi), n \in \mathbb{N}$. Every cell of the partition is bounded and, hence, contains at most finitely many points from φ. The index function is again used to declare one of the points in each cell the leader of the points in the cell. Then the Voronoi mosaic associated with $T_1(\varphi)$ is taken away from the superposition, which yields a coarser partition, where, among the leaders of the first generation, the leaders of the second generation are chosen in each cell. This procedure is iterated, the Voronoi mosaics associated with $T_2(\varphi), T_3(\varphi), \ldots$ are removed, giving way to a sequence of coarser and coarser partitions of $\mathbb{R}^d$, which is called a clumping.

A locally finite graph is then defined on the points of φ, by defining an edge from the leader of kth generation, to the leaders of $k-1$th generation, $k \in \mathbb{N}$, that are in the same cell of the partition generated by the superposition of Voronoi mosaics associated with the thinned sets $T_n(\varphi), n \geq k$. By a combination of geometric and probabilistic arguments it is finally shown that the so defined graph is a one-ended tree, and satisfies a side condition on each of its points. It is then a well known fact (cf. [2], [5]), which is also used for search algorithms in computer sciences, that a doubly infinite, directed path can be defined on the points of such a tree.

In the following subsection, we will take over many of these ideas, and give purely determistic results on the resulting graphs. Probabilistic arguments and results will then be given in Chapter 5.

3.3.2 The succession point map

In the remainder of the chapter, we specialize to the case where $G = \mathbb{R}^d$ equipped with the Euclidean metric $d(\cdot,\cdot)$ and the lexicographic order $\ll$ on $\mathbb{R}^d$. Clearly, $\mathbb{R}^d$ is a lcscH group, so the results from Section 2 apply. We define a point map σ on $\mathbf{L}(\mathbb{R}^d)$, such that, for any aperiodic point set $\varphi \in \mathbf{L}(\mathbb{R}^d)$, the associated graph $\Gamma_\sigma(\varphi)$ consists of only finitely many infinite directed paths (cf. Theorem 3.3.9). Moreover, the number of connected components (paths) of the graph can be bounded by a constant that depends only on the dimension d.

The definition is based on the constructions in [5] and [30] of a doubly infinite path on stationary point processes. As before, we denote by (ξ_k) a sequence of mappings that enumerates the points of an arbitrary locally finite subset $\varphi \in \mathbf{L}(\mathbb{R}^d)$. Starting with an aperiodic, locally finite set $\varphi \in \mathbf{L}(\mathbb{R}^d)$ we consider a sequence (r_n) of real numbers such that $r_n \uparrow \infty$ and we denote by T_n the (r_n)-thinning procedure defined in (2.16). For convenience, we define $T_0 := \mathrm{id}_{\mathbf{L}(\mathbb{R}^d)}$.

For $n \in \mathbb{N} \cup \{0\}$, we say that a point $x \in G$ is a *point of the nth generation in* φ, if and only if $x \in T_n(\varphi) \setminus T_{n+1}(\varphi)$. We introduce the mapping $g : \mathbf{L}(\mathbb{R}^d) \to \mathbb{N} \cup \{0, -\infty\}$ that maps $\varphi \in \mathbf{L}(\mathbb{R}^d)$ to the number of the generation of 0 with respect to φ. Formally, we define

$$g(\varphi) := \sup\{n \in \mathbb{N} \cup \{0\} : 0 \in T_n(\varphi)\}, \quad \varphi \in \mathbf{L}(\mathbb{R}^d),$$

where $\sup \emptyset = -\infty$. We define an extended function $\tilde{g} : \mathbf{L}(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{N} \cup \{0, -\infty\}$ by $\tilde{g}(\varphi, x) := g(\theta_x \varphi)$. Then $\tilde{g}$ is a shift invariant function and the number of the generation of an arbitrary point $x \in \mathbb{R}^d$ w.r.t. φ is given by $g(\varphi, x)$. Moreover, selection functions $\Psi_n : \mathbf{L}(\mathbb{R}^d) \to \mathbf{L}(\mathbb{R}^d), n \in \mathbb{N}$, are defined by

$$\Psi_n(\varphi) := T_n(\varphi) \setminus T_{n+1}(\varphi), \quad \varphi \in \mathbf{L}(\mathbb{R}^d).$$

The selection function Ψ_n maps a set $\varphi \in \mathbf{L}(\mathbb{R}^d)$ to the subset of φ consisting of the points of the nth geneneration.

Let us now introduce the (measurable) empty-space function $r : \mathbf{L}(\mathbb{R}^d) \setminus \{\emptyset\} \to [0, \infty)$ by

$$r(\varphi) := \inf\{s \geq 0 : \varphi \cap B_d(0, s) \neq \emptyset\},$$

and then define the *nearest neighbour point maps* $\nu_n : \mathbf{L}(\mathbb{R}^d) \mapsto \mathbb{R}^d, n \in \mathbb{N}$, associated with the thinning procedure (T_n) by

$$\nu_n(\varphi) := \begin{cases} x & \text{if } \varphi \in \mathbf{L}_0(\mathbb{R}^d) \text{ and } \min(T_n(\varphi) \cap B_d(0, r(T_n\varphi)) = x, \\ 0 & \text{otherwise.} \end{cases}$$

The mapping ν_n maps a point set $\varphi \in \mathbf{L}_0(\mathbb{R}^d)$ to the lexicographically smallest among the nearest neighbours of 0 in $T_n(\varphi)$. Then we define inductively the n-ancestors $\alpha_n, n \geq 0$, of 0 in φ by $\alpha_0(\varphi) := 0$ and

$$\alpha_n(\varphi) := \nu_n(\theta_{\alpha_{n-1}}(\varphi)) + \alpha_{n-1}(\varphi), \quad \varphi \in \mathbf{L}_0(\mathbb{R}^d), n \in \mathbb{N}.$$

The n-ancestor of an arbitrary point $x \in \varphi$ is given by $\alpha_n(\theta_x \varphi) + x$. In particular, unlike in genealogy, a point of the nth generation is considered to be its own k-ancestor for all $k \leq n$, and a k-ancestor is in general not a point of the kth generation. If $x \neq z := \alpha_n(\theta_x \varphi) + x$ for some $n \in \mathbb{N}$, we also say that x is a *descendant* of z, and we define the set of descendants of 0 in φ by

$$D(\varphi) := \bigcup_{n \in \mathbb{N}} \{y \in \varphi \setminus \{0\} : \alpha_n(\theta_y \varphi + y) = 0\}.$$

The classification of the points in φ in generations, with well defined relations of ancestors and descendants is called the (T_n)-*Voronoi hierarchy* on φ.

Figure 3.4 shows the Voronoi mosaics associated with the subsets $T_1(\varphi)$ and $T_2(\varphi)$ of a locally finite set $\varphi \in \mathbf{L}(\mathbb{R}^2)$. The point $y \in \varphi$ is in the Voronoi cell with center z of the

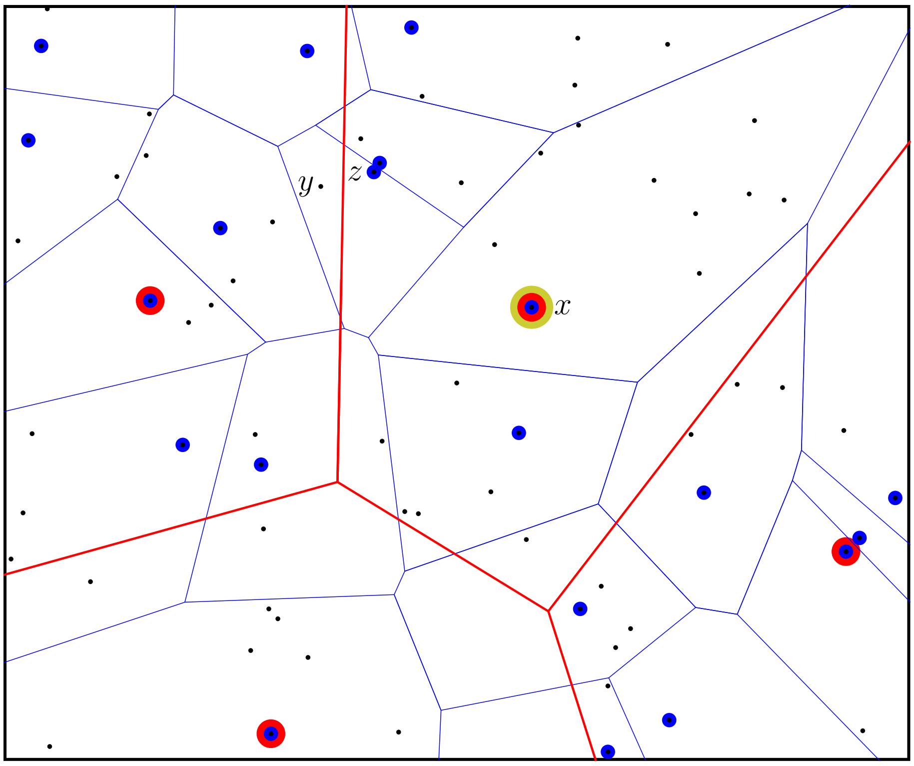

Figure 3.4: The Voronoi mosaics associated with $T_n(\varphi), 1 \leq n \leq 2$.

Voronoi mosaic associated with $T_1(\varphi)$. Hence, the nearest neighbour of y in $T_1(\varphi)$ is z, so z is the 1-ancestor of y in the (T_n)-Voronoi hierarchy. The point z is its own 1-ancestor, and its 2-ancestor is x, because x is the nearest neighbour of y in $T_2(\varphi)$. The point x is of third generation, so it is its own $1, 2$ and 3-ancestor, and also the 3-ancestor of y and z.

Lemma 3.3.1. *The mappings $\alpha_n, n \in \mathbb{N}$, are point maps.*

PROOF: We first proof that the mappings $\nu_n, n \in \mathbb{N}$, are measurable. Indeed, we have

$$
\begin{aligned}
&\{\varphi \in \mathbf{L}_0(\mathbb{R}^d) : \nu_n(\varphi \in B)\} \\
&\quad = \bigcup_{k\in\mathbb{N}} \bigcap_{j\in\mathbb{N}} \big(\{\varphi \in \mathbf{L}_0(\mathbb{R}^d) : \xi_k(\varphi) \in B \cap T_n(\varphi) \text{ and } \xi_j(\varphi) \notin T_n(\varphi)\} \\
&\qquad \cup \{\varphi \in \mathbf{L}_0(\mathbb{R}^d) : \xi_k(\varphi) \in B \cap T_n(\varphi) \text{ and } \|\xi_k(\varphi)\| < \|\xi_j(\varphi)\|\} \\
&\qquad \cup \{\varphi \in \mathbf{L}_0(\mathbb{R}^d) : \xi_k(\varphi) \in B \cap T_n(\varphi) \text{ and } \|\xi_k(\varphi)\| = \|\xi_j(\varphi)\| \text{ and } \xi_k(\varphi) \ll \xi_j(\varphi)\}\big),
\end{aligned}
$$

and, hence, the measurability of ν_n follows from the measurability of $\xi_k, k \in \mathbb{N}$, and T_n. By induction, we then obtain the measurability of $\alpha_n, n \in \mathbb{N}$. Moreover, we have $\alpha_0(\varphi) = 0 \in \varphi$ if and only if $0 \in \varphi$, and, again by induction, that $\alpha_n(\varphi) \in \varphi$ if and only if $0 \in \varphi$. □

We define two auxiliary point maps $\sigma_1, \sigma_2 : \mathbf{L}(\mathbb{R}^d) \to \mathbb{R}^d$ by

$$
\sigma_1(\varphi) := \begin{cases} \min(\varphi) & \text{if } \varphi \in \mathbf{L}_0(\mathbb{R}^d) \text{ and } 1 \leq \operatorname{card}(\varphi) < \infty, \\ 0 & \text{otherwise,} \end{cases} \tag{3.23}
$$

and

$$\sigma_2(\varphi) := \begin{cases} \min(\{y \in \varphi \setminus \{0\} : 0 \ll y\}) & \text{if } \varphi \in \mathbf{L}_0(\mathbb{R}^d) \\ & \text{and } 1 \leq \operatorname{card}(\{y \in \varphi \setminus \{0\} : 0 \ll y\}) < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Given $\varphi \in \mathbf{L}_0(\mathbb{R}^d)$ of finite cardinality, σ_1 maps φ to its lexicographically smallest element, and σ_2 maps φ to the lexicographically smallest among all elements that are bigger than 0. The composition $\sigma_i \circ f, i \in \{1, 2\}$, of σ_1 or σ_2 with any measurable mapping $f : \mathbf{L}(\mathbb{R}^d) \to \mathbf{L}(\mathbb{R}^d)$ such that $f(\varphi) \in \mathbf{L}_0(\mathbb{R}^d)$ if and only if $\varphi \in \mathbf{L}_0(\mathbb{R}^d)$ is again a point map. We will use this fact to define two more auxiliary point maps, $\pi_1, \pi_2 : \mathbf{L}(\mathbb{R}^d) \to \mathbb{R}^d$.

Let $\varphi \in \mathbf{L}_0(\mathbb{R}^d)$ and assume that $g(\varphi) = n \geq 1$. Then, for $m < n$ we have $\alpha_m(\varphi) = 0$, and the descendants of 0 in the mth generation are given by $D(\varphi) \cap \Psi_m(\varphi)$. With the point map $\pi_1 : \mathbf{L}(\mathbb{R}^d) \to \mathbb{R}^d$, we want to map φ to the lexicographically smallest of those elements that have 0 as an ancestor of minimal generation, i.e., with

$$h(\varphi) := \inf\{0 \leq k < g(\varphi) : \{x \in D(\varphi) : \alpha_{k+1}(\theta_x\varphi) + x = 0\} \neq \emptyset\},$$

where $\inf \emptyset := \infty$, we define

$$\pi_1(\varphi) := \begin{cases} \sigma_1(\{x \in D(\varphi) : \alpha_{h(\varphi)+1}(\theta_x\varphi) + x = 0\}) & 0 \leq h(\varphi) < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

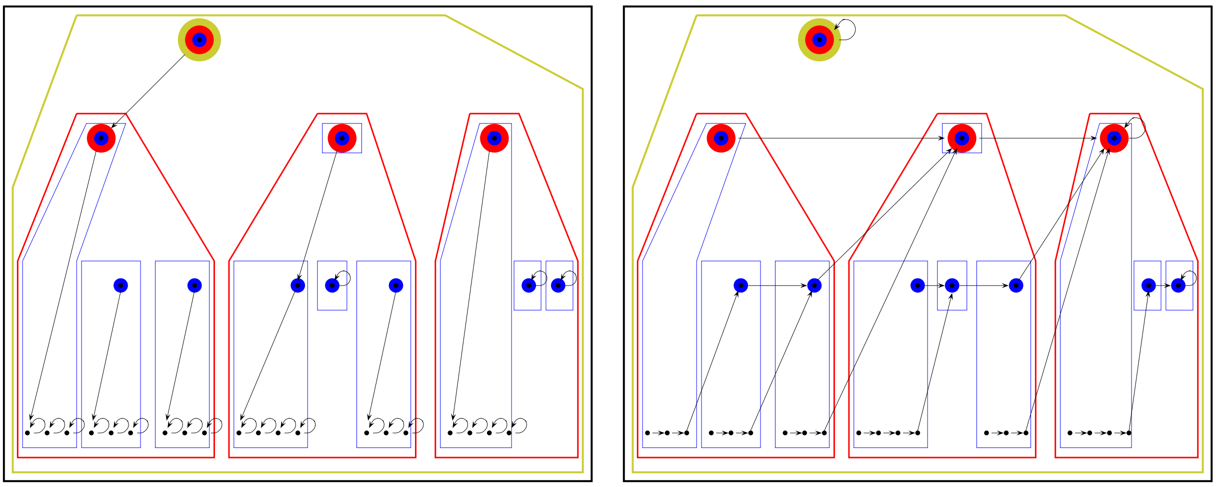

Figure 3.5: Γ_{π_1} and Γ_{π_2} on a finite set φ, rearranged in the (T_n)-Voronoi hierarchy

The second auxiliary point map π_2 is supposed to map $\varphi \in \mathbf{L}_0(\mathbb{R}^d)$ to the smallest among the lexicographically bigger sisters of the origin. If there is no such point, then φ is mapped to a point of higher generation as indicated on the right hand side of figure 3.5. Formally, the minimal generation k, that does not contain $\alpha_k(\varphi)$ but a different point x that has the same $k+1$-ancestor as 0, i.e., $\alpha_{k+1}(\theta_x\varphi) + x = \alpha_{k+1}(\varphi)$, is defined by

$$f_1(\varphi) := \inf\{k \geq g(\varphi) : g(\alpha_k(\varphi)) \neq k \text{ and } D(\alpha_{k+1}(\varphi)) \cap \Psi_k(\varphi) \neq \emptyset\},$$

where $\inf \emptyset := \infty$. Moreover, the minimal generation ℓ that actually contains the ℓ-ancestor of 0, i.e., $g(\alpha_\ell(\varphi)) = \ell$, and at least one more point that is bigger with respect to the lexicographic order than $\alpha_\ell(\varphi)$ is defined by

$$f_2(\varphi) := \inf\{\ell \geq g(\varphi) : g(\alpha_\ell(\varphi)) = \ell \text{ and } \max(D(\alpha_{\ell+1}(\varphi)) \cap \Psi_\ell(\varphi)) \neq \alpha_\ell(\varphi)\}.$$

Then define $\pi_2 : \mathbf{L}(\mathbb{R}^d) \to \mathbb{R}^d$ by

$$\pi_2(\varphi) := \begin{cases} \sigma_1(D(\alpha_{f_1(\varphi)+1}(\varphi)) \cap \Psi_{f_1(\varphi)}(\varphi)) & \text{if } f_1(\varphi) < f_2(\varphi), \\ \sigma_2(D(\alpha_{f_2(\varphi)+1}(\varphi) \cap \Psi_{f_2(\varphi)}(\varphi)) & \text{if } f_2(\varphi) < f_1(\varphi), \\ 0 & \text{otherwise.} \end{cases}$$

The *succession point map* $\eta : \mathbf{L}(\mathbb{R}^d) \to \mathbb{R}^d$ associated with the (T_n)-Voronoi hierarchy is then defined by

$$\eta(\varphi) := \begin{cases} \pi_1(\varphi) & \text{if } \pi_1(\varphi) \neq 0, \\ \pi_2(\varphi) & \text{otherwise.} \end{cases} \tag{3.24}$$

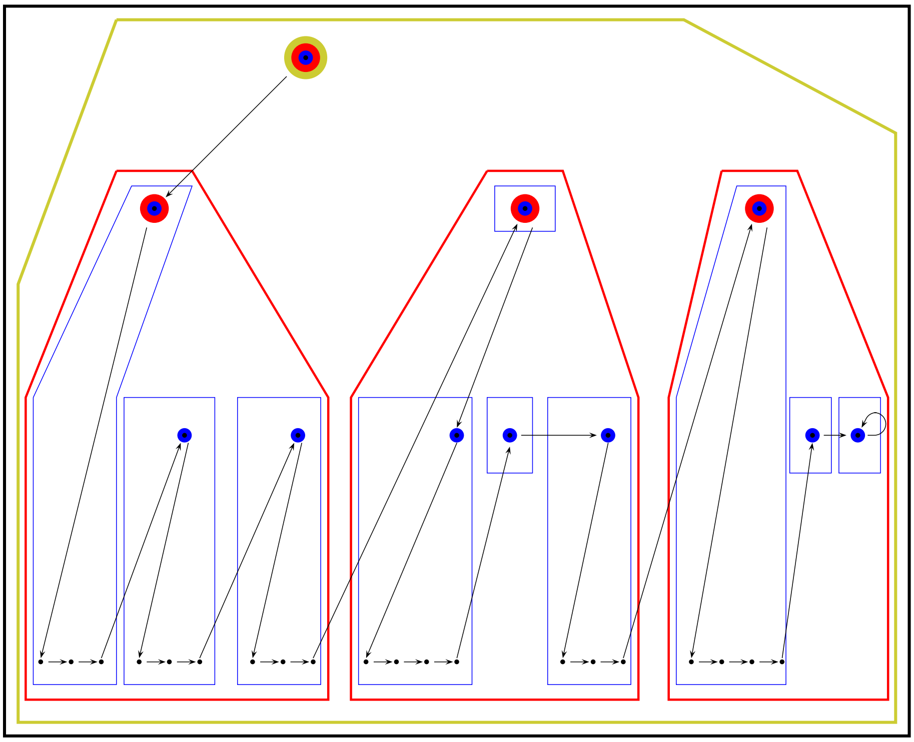

Figure 3.6: The graph Γ_η on the (T_n)-Voronoi hierarchy of a finite and rearranged set φ

The graph Γ_η associated with η has a loop in a point $x \in \varphi$ if and only if $\pi_1(\theta_x\varphi) = \pi_2(\theta_x\varphi) = 0$. Let us define the set of such points by

$$\Lambda_\eta(\varphi) := \{x \in \varphi : (x, x) \in E_\eta(\varphi)\}, \quad \varphi \in \mathbf{L}(\mathbb{R}^d).$$

Lemma 3.3.2. *The mapping Λ_η is a selection function on $\mathbf{L}(\mathbb{R}^d)$.*

PROOF: The fact that the extended point map $\tilde{\eta}$ is equivariant yields the equivariance of Λ_η. Also, the measurability of Λ_η follows from the measurability of $\tilde{\eta}$. $\square$

Proposition 3.3.3. *The mapping η is a point map with the following properties. For fixed aperiodic $\varphi \in \mathbf{L}_0(G)$ such that $\cap_{n\in\mathbb{N}} T_n(\varphi) = \emptyset$, there are only finitely many descendants of 0 in φ and, for $m := \mathrm{card}(D(\varphi))$, we have*

$$D(\varphi) = \{\eta^k(\varphi) : 1 \leq k \leq m\}, \tag{3.25}$$

i.e., the first m iterations of η map the set φ to all m descendants of 0 in φ. Conversely, there exists an increasing sequence (k_n) of numbers in $\mathbb{N} \cup \{0\}$, such that $\eta^{k_n}(\theta_{\alpha_n}(\varphi)) + \alpha_n(\varphi) = 0$ and, if $\mathrm{card}(\varphi) = \infty$, then $k_n \uparrow \infty$.

PROOF: We have already shown that π_1 and π_2 are point maps, so η is also a point map. For the remainder of the proof let us fix $\varphi \in \mathbf{L}_0(\mathbb{R}^d)$, and recall that the generation of the origin is denoted by $g(\varphi)$, and that $g(\varphi) < \infty$ because we have assumed that $\cap_{n\in\mathbb{N}} T_n(\varphi) = \emptyset$. Then, by (r_n)-regularity of the thinning $(T_n(\varphi))$, an arbitrary descendant x of 0 satisfies $d(x, 0) \leq r_1 + \ldots + r_{g(\varphi)}$. Hence, $D(\varphi) \subset \varphi \cap B_d(0, r_1 + \ldots + r_{g(\varphi)})$ is a set of finite cardinality.

Let us now prove (3.25) by an induction over m. If $m = 0$, then there are no descendants of 0 in φ and the claim is trivially satisfied. Now assume that $g(\varphi) \geq 1$, that $m > 0$ and that the claim is true for all $0 \leq k < m$. Let

$$\ell := \max\{0 \leq k < g(\varphi) : D(\varphi) \cap \Psi_k(\varphi) \geq 1\},$$

then the set $\Psi_\ell(\varphi) \cap \Phi_n(\varphi)$ consists of the j descendants $\{x_1, \ldots, x_j\}$ of 0 of maximal generation $\ell < g(\varphi)$. We have $1 \leq j \leq m$ and we may assume $x_1 \lll \ldots \lll x_j$.

The descendants of $x_i, 1 \leq i \leq j$, in φ are given by $D(\theta_{x_i}\varphi) + x_i$ and we define $m_0 := \mathrm{card}(\{x \in D(\varphi) : \alpha_\ell(\theta_x\varphi) + x = 0\})$ and $m_i := \mathrm{card}(D(\theta_{x_i}\varphi))$. By the induction hypothesis, for $1 \leq i \leq j$, we obtain a path on the $m_i + 1$ points in $D(\theta_{x_i}\varphi) + x_i \cup \{x_i\}$. Moreover, we have $\eta^{m_0+1}(\varphi) = x_1$ and $\eta^{m_i+1}(x_i) = x_{i+1}$ for all $1 \leq i < j$. We conclude that (3.25) holds.

From what we have proved above we deduce that for $n \in \mathbb{N}$ there exists a $k_n \in \mathbb{N} \cup \{0\}$ such that $k_n \leq \mathrm{card}(D(\theta_{\alpha_n}(\varphi)))$ and $\eta^{k_n}(\alpha_n(\varphi)) = 0$. Finally, the fact that the thinning is complete yields $g(\varphi, \alpha_n(\varphi)) < \infty$ for all $n \in \mathbb{N}$, and, hence, the sequence $(\alpha_n(\varphi))$ of ancestors of 0 in φ contains infinitely many distinct elements of φ. In particular, $k_n \uparrow \infty$. $\square$

Corollary 3.3.4. *Let $\varphi \in \mathbf{L}(\mathbb{R}^d)$ such that $\cap_{n\in\mathbb{N}} T_n(\varphi) = \emptyset$ and $x, y \in \varphi$. Then there exists $\ell \in \mathbb{N}$ such that $\eta^\ell(\theta_x\varphi) + x = y$ or $\eta^\ell(\theta_y\varphi) + y = x$ if and only if x and y have a joint ancestor in the T_n-Voronoi hierarchy.*

PROOF: If x and y have a joint k-ancestor $\alpha_k(\theta_x\varphi) + x = \alpha_k(\theta_y\varphi) + y$ then Proposition 3.3.3 applied to $\theta_{\alpha_k}(\varphi) \in \mathbf{L}_0(\mathbb{R}^d)$ yields that there are $i, j \in \mathbb{N} \cup \{0\}$ such that $\eta^i(\theta_{\alpha_k}(\varphi)) + \alpha_k(\varphi) = x$ and $\eta^j(\theta_{\alpha_k}(\varphi)) + \alpha_k(\varphi) = y$. Hence, $\ell := |i - j|$ satisfies the claim.

Conversely, let $x, y \in \varphi$ such that $\alpha_n(\theta_x\varphi) + x \neq \alpha_n(\theta_y\varphi) + y$ for all $n \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$, we must have $T_n(\varphi) \geq 2$, so Lemma 2.6.10 yields $\mathrm{card}(\varphi) = \infty$. Let us assume that $\eta^\ell(\theta_x\varphi)+x=y$. By Proposition 3.3.3 applied to $\theta_x\varphi$, there exists a sequence (k_n) such that $k_n \uparrow \infty$ and $\eta^{k_n}(\theta_{\alpha_n}\circ\theta_x(\varphi)+\alpha_n(\theta_x\varphi)+x)=x$. In particular, there exists $n_0\in\mathbb{N}$ such that $k_{n_0}\geq \ell$. We conclude, again using Proposition 3.3.3, that $\alpha_{n_0}(\theta_y\varphi)+y=\alpha_{n_0}(\theta_x\varphi)+x$, a contradiction. Hence, $\eta^\ell(\theta_x\varphi)+x\neq y$ for all $\ell\in\mathbb{N}\cup\{0\}$ and the same argument yields $\eta^\ell(\theta_y\varphi)+y\neq x$ for all $\ell\in\mathbb{N}\cup\{0\}$. □

Let us now assume that (r_n) is an increasing sequence of positive real numbers such that $r_n\geq\sum_{i=1}^{n-1}r_i$ for all $n\in\mathbb{N}$. Moreover, let (T_n) be the (r_n)-regular thinning procedure defined in (2.16). Then we have the following main result of this section.

Theorem 3.3.5. *For $\varphi\in\mathbf{L}(\mathbb{R}^d)$ such that $\cap_{n\in\mathbb{N}}T_n(\varphi)=\emptyset$, the number of connected components of the graph $\Gamma_\eta(\varphi)$ associated with the succession point map η on φ is bounded by a constant $b(d)\in\mathbb{N}$ that depends only on the dimension of the space. A connected component $C\subset\varphi$ in $\Gamma_\eta(\varphi)$ contains at most one point from $\Lambda(\varphi)$ and it is a doubly infinite path if and only if $C\cap\Lambda(\varphi)=\emptyset$.*

PROOF: Througout the whole proof, we fix $\varphi\in\mathbf{L}(\mathbb{R}^d)$. By Corollary 3.3.4, two points $x,y\in\varphi$ are in the same connected component of $\Gamma_\eta(\varphi)$ if and only if, for some $k\in\mathbb{N}$, they have a joint k-ancestor in the (T_n)-Voronoi hierarchy. Assume now that $F=\{x_1,\dots,x_m\}$ is a set of points in φ such that $\alpha_n(\theta_{x_i}\varphi+x_i)\neq\alpha_n(\theta_{x_j}\varphi+x_j)$ for all $n\in\mathbb{N}$ and $i\neq j$. Let $\varrho:=\max\{d(x_i,x_j):i\neq j\}$ be the maximal distance between two points in F. By the r_n-regularity of (T_n), we have

$$d(x_i,\alpha_n(x_i))\leq\sum_{\ell=1}^n r_\ell,\quad n\in\mathbb{N}.$$

Hence, the ancestors of x_i and x_j in the nth generation satisfy

$$r_n<d(\alpha_n(\theta_{x_i}\varphi)+x_i,\alpha_n(\theta_{x_j}\varphi)+x_j)\leq\varrho+2\sum_{\ell=1}^n r_\ell,\tag{3.26}$$

and, for $n\to\infty$, we conclude that there cannot be more than

$$b(d):=\max\{\mathrm{card}(A):1\leq d(x,y)\leq 3\text{ for all distinct }x,y\in A\}\tag{3.27}$$

points in F. Hence, there are at most $b(d)$ connected components in $\Gamma_\eta(\varphi)$.

Let us now fix one connected component $C\subset\varphi$ in $\Gamma_\eta(\varphi)$ and assume that there are two distinct points $x,y\in C\cap\Lambda_\eta(\varphi)$. Corollary 3.3.4 yields that there exists $\ell\in\mathbb{N}$ such that $\eta^\ell(\theta_x\varphi)+x=y$ or $\eta^\ell(\theta_y\varphi)+y=x$. In particular, there can not be a loop in x and a loop in y, a contradiction.

We will prove the last claim of the theorem by showing that, if $C\cap\Lambda_\eta(\varphi)=\emptyset$, every point $x\in C$ has a unique predecessor and a unique successor in $\Gamma_\eta(\varphi)$, and that C is not a cycle. By the equivariance of the graph, we may assume that $x=0$. The unique successor of $0\in C$ is the point $\eta(\varphi)$. It is distinct from 0, because $0\notin\Lambda_\eta(\varphi)$. Let $\ell:=g(\varphi)$ and $\alpha_{\ell+1}(\varphi)$ be the first ancestor of 0 that is not 0 itself. By Proposition 3.3.3 (applied to $\theta_{\alpha_{l+1}}(\varphi)$, there exists

$k := k_{\ell+1} \in \mathbb{N}$ such that $\eta^k(\theta_{\alpha_{\ell+1}}(\varphi)) + \alpha_{\ell+1}(\varphi) = 0$, and, hence, $\eta^{k-1}(\theta_{\alpha_{\ell+1}}(\varphi)) + \alpha_{\ell+1}(\varphi)$ is a predecessor of 0 in $\Gamma_\eta(\varphi)$. Moreover, since any predecessor of 0 satisfies this relation, the predecessor of 0 in $\Gamma_\eta(\varphi)$ is unique. Finally, assume that C is a cycle of points and $x \in C$. Then $\alpha_n(\theta_x\varphi) + x \in C$ for all $n \in \mathbb{N}$, hence, there is a point $y \in C \cap (\cap_{n\in\mathbb{N}} T_n(\varphi))$, a contradiction.

Conversely, if there exists a point $x \in C \cap \Lambda_\eta(\varphi)$ then C is not a doubly infinite, directed path because $\eta(\theta_x\varphi) + x = x$. □

Lemma 3.3.6. *For all $\varphi \in \mathbf{L}(G)$ we have $\cap_{n\in\mathbb{N}} T_n(\varphi) \subset \Lambda_\eta(\varphi)$, i.e., there is loop at every point x such that $g(\varphi, x) = \infty$.*

Proof: Assume that $x \in \cap_{n\in\mathbb{N}} T_n(\varphi)$. Then it follows straight from the definition of the point map η, where the non-trivial part refers only to points of finite generation in the (T_n)-Voronoi hierarchy, that $\eta(\theta_x\varphi) + x = x$, hence, there is a loop at the point x in $\Gamma_\eta(\varphi)$, and we have $x \in \Lambda_\eta(\varphi)$. □

Corollary 3.3.7. *There exists a point map $\zeta : \mathbf{L}(\mathbb{R}^d) \to \mathbb{R}^d$ such that $\zeta(\theta_x\varphi)+x = \zeta(\theta_y\varphi)+y$ for all $x, y \in \varphi$, whenever $\varphi \in \mathbf{L}(G)$ is aperiodic and satisfies $\Lambda_\eta(\varphi) \neq \emptyset$.*

Proof: Recall the definition of σ_1 in (3.23), then define the point map ζ by

$$\zeta(\varphi) := \begin{cases} \sigma_1(\cap_{n\in\mathbb{N}} T_n(\varphi)) & \text{if } \cap_{n\in\mathbb{N}} T_n(\varphi) \neq \emptyset, \\ \sigma_1 \circ \Lambda_\eta(\varphi) & \text{otherwise.} \end{cases}$$

The claim follows from Theorem 3.3.5 and Lemma 3.3.6. □

From all $\varphi \in \mathbf{L}(\mathbb{R}^d)$ in the Θ_G-stable set

$$\{\varphi \in \mathbf{L}(G) : \Lambda_\eta(\varphi) \neq \emptyset\}$$

we can thus pick a single point by ζ, which will serve as the anchor point in the following definition of a doubly infinite, directed path on φ. Let f be an arbitrary bijection from $\mathbb{N}$ to $\mathbb{Z}$. The function f is used to attribute a predecessor and a successor to a natural number, i.e., the natural numbers $f^{-1}(f(n) - 1)$ and $f^{-1}(f(n) + 1)$. Then, recall the definition of the order relation $\prec$ in (2.1), let $S_0 := \{x \in \varphi : x \prec 0\}$ and define $h : \mathbf{L}(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{N}$ by

$$h(\varphi, x) := \begin{cases} \operatorname{card}(\varphi \cap S_x) & \text{if } \varphi \in \mathbf{L}_0(\mathbb{R}^d), \\ 0 & \text{otherwise.} \end{cases}$$

We have already used and discussed this function in the proof of Proposition 2.1.4, it attributes to every $\varphi \in \mathbf{L}(\mathbb{R}^d)$ and $x \in \varphi$ the index n such that $\xi_n(\varphi) = x$. Then we define a mapping $\tau : \mathbf{L}(\mathbb{R}^d) \to \mathbb{R}^d$ by

$$\tau(\varphi) := \begin{cases} \xi_{f^{-1}(f \circ h(\theta_\zeta\varphi, -\zeta(\varphi))+1)} & \text{if } \varphi \in \mathbf{L}_0(\mathbb{R}^d) \text{ is aperiodic, } \operatorname{card}(\varphi) = \infty \text{ and } \Lambda_\eta(\varphi) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

where, in particular, $h(\theta_\zeta\varphi, -\zeta(\varphi))$ is the position of 0 seen from $\zeta(\varphi)$.

Proposition 3.3.8. *The mapping τ is a bijective point map. For all aperiodic $\varphi \in \mathbf{L}_0(\mathbb{R}^d)$ such that* $\mathrm{card}(\varphi) = \infty$ *and $\Lambda_\eta(\varphi) \neq \emptyset$, the graph $\Gamma_\tau(\varphi)$ is a doubly infinite directed path.*

PROOF: The mapping τ is measurable, because the functions f, h and ζ are measurable. Moreover, if $0 \in \varphi$ then $\tau(\varphi) \in \varphi$, hence, τ is a point map. If, in the definition of τ, we replace h by $g := -h$, which is also a bijection from $\mathbb{N}$ to $\mathbb{Z}$, we obtain a point map $\hat{\tau}$ which is the inverse point map of τ. In particular, τ is bijective.

Now assume that $\varphi \in \mathbf{L}_0(\mathbb{R}^d)$ is aperiodic, that $\mathrm{card}(\varphi) = \infty$ and that $\Lambda_\eta(\varphi) \neq \emptyset$. Then

$$\tau^n(\varphi) = \xi_{f^{-1}(f \circ h(\theta_\zeta \varphi, -\zeta(\varphi)) + n)}, \quad n \in \mathbb{Z},$$

hence, the point map τ is complete on $\{\varphi\}$, and, in particular, is not cyclic. Moreover, the unique predecessor of $x \in \varphi$ in $\Gamma_\tau(\varphi)$ is given by $\hat{\tau}(\theta_x \varphi) + x$ and the unique successor of $x \in \varphi$ in $\Gamma_\tau(\varphi)$ is given by $\tau(\theta_x \varphi) + x$. Hence, $\Gamma_\tau(\varphi)$ is a doubly infinite directed path on φ. □

Theorem 3.3.9. *There exists a point map $\sigma : \mathbf{L}(\mathbb{R}^d) \to \mathbb{R}^d$, such that $\Gamma_\sigma(\varphi)$ is a finite collection of doubly infinite paths, whenever φ is aperiodic and* $\mathrm{card}(\varphi) = \infty$, *and a cycle that contains all the points of φ, whenever* $\mathrm{card}(\varphi) < \infty$.

PROOF: It remains to put together the pieces that we already have. We define a point map $\sigma : \mathbf{L}(\mathbb{R}^d) \to \mathbb{R}^d$ by

$$\sigma(\varphi) := \begin{cases} \sigma_2(\varphi) & \text{if } \mathrm{card}(\varphi) < \infty \text{ and } 0 \neq \max(\varphi), \\ \sigma_1(\varphi) & \text{if } \mathrm{card}(\varphi) < \infty \text{ and } 0 = \max(\varphi), \\ \eta(\varphi) & \text{if } \mathrm{card}(\varphi) = \infty \text{ and } \Lambda_\eta(\varphi) = \emptyset, \\ \tau(\varphi) & \text{if } \mathrm{card}(\varphi) = \infty \text{ and } \Lambda_\eta(\varphi) \neq \emptyset. \end{cases}$$

Then the claim of the propositon follows from Theorem 3.3.5 and Proposition 3.3.8. □

3.4 A complete family generated by two point maps

We will now define a bijective point map π that, together with the point map σ from Theorem 3.3.9, generates a complete family on the aperiodic sets in $\mathbf{L}_0(\mathbb{R}^d)$. In particular, if the open problem 1 from Section 3.1 has a negative answer, i.e., if there is no universal point map on the aperiodic sets in $\mathbf{L}_0(\mathbb{R}^d)$, then $\{\sigma, \pi\}$ is a minimal generator of such a complete family.

We have seen in Theorem 3.3.9 that the point map σ partitions any $\varphi \in \mathbf{L}_0^\infty(\mathbb{R}^d)$ into finitely many doubly infinite paths, the $n(\varphi)$ connected components of the associated graph $\Gamma_\sigma(\varphi)$. We then want to define a mapping π that is $n(\varphi)$-cyclic and such that the orbit $O_\pi(\varphi, x)$ of a point x in φ under π is either trivial, or contains one point from each connected component of $\Gamma_\sigma(\varphi), \varphi \in \mathbf{L}_0^\infty(\mathbb{R}^d)$. From Theorem 3.2.9 we infer that there exists a countable family of cyclic point maps $\{\pi_m : m \in \mathbb{N}\}$ such that, for any aperiodic $\varphi \in \mathbf{L}_0^\infty(\mathbb{R}^d)$, $n \in \mathbb{N}$ and n distinct points $x_1, \ldots, x_n \in \varphi$, there exists an index m_0 such that π_{m_0} is n-cyclic and $\{x_1, \ldots, x_n\} = \{\sigma^i(\theta_{x_1}\varphi) + x_1 : 1 \leq i \leq n\}$.

We will now modify the point maps $\pi_m, m \in \mathbb{N}$, in such a way that we only keep cycles that involve exactly one point from every path in $\Gamma_\eta(\varphi)$. We define the point maps $\tau_m : \mathbf{L}(\mathbb{R}^d) \to \mathbb{R}^d$ by

$$\tau_m(\varphi) := \begin{cases} \pi_m(\varphi) & \text{if } \pi_m^{n(\varphi)}(\varphi) = 0 \text{ and} \\ & \sigma^k \circ \pi_m^i(\varphi) \neq \pi_m^j(\varphi) \text{ for all } k \in \mathbb{Z} \text{ and } 1 \leq i < j \leq n(\varphi), \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.2.9, as it is restated above, then yields that

$$m_0(\varphi) := \inf\{m \in \mathbb{N} : \tau_m \not\equiv 0\},$$

where $\inf \emptyset := \infty$, is finite for all aperiodic $\varphi \in \mathbf{L}_0(\mathbb{R}^d)$. The measurability of $m_0 : \mathbf{L}_0(G) \to \mathbb{N} \cup \{\infty\}$ is a consequence of the measurability of the mappings $\tau_m, m \in \mathbb{N}$, which are all bijective point maps. Hence, the definition of $\pi : \mathbf{L}(\mathbb{R}^d) \to \mathbb{R}^d$ by

$$\pi(\varphi) := \tau_{m_0(\varphi)}(\varphi), \quad \varphi \in \mathbf{L}(\mathbb{R}^d),$$

yields a bijective point map.

Theorem 3.4.1. *The family of bijective point maps generated by σ and π is complete on the aperiodic locally finite subsets of $\mathbb{R}^d$ that contain the origin, i.e., on*

$$A = \{\varphi \in \mathbf{L}_0(\mathbb{R}^d) : \varphi \textit{ is aperiodic}\}.$$

Proof: Let $\varphi \in A$. We define

$$\kappa_1(\varphi) := \inf\{k \geq 0 : \pi \circ \sigma^k(\varphi) \neq 0\}, \ \kappa_2(\varphi) := \inf\{k \in \mathbb{N} : \pi \circ \sigma^{-k}(\varphi) \neq 0\}$$

and, finally,

$$\kappa(\varphi) := \begin{cases} \kappa_1(\varphi) & \text{if } \kappa_1 < \infty, \\ -\kappa_2(\varphi) & \text{if } \kappa_1 = \infty. \end{cases}$$

The definition of π yields $\kappa(\varphi) \in \mathbb{Z}$, and we have

$$\{\sigma^k \circ \pi^j \circ \sigma^{\kappa(\varphi)} : k \in \mathbb{Z}, 0 \leq j < n(\varphi)\} = \varphi,$$

and we conclude that

$$\{\sigma^k \circ \pi^j \circ \sigma^i(\varphi) : i, k \in \mathbb{Z}, j \in \mathbb{N} \cup \{0\}\} = \varphi$$

for all φ in A. Hence, $\langle\{\sigma, \pi\}\rangle$ is a countable family of bijective point maps that is complete on A. $\square$

Chapter 4

Palm measure and characterization theorems

In this chapter, we give a brief summary of the classical theory of Palm measures associated with stationary measures on $\mathbf{M}(G)$, with emphasis on the characterization results that were established by Mecke in [16]. More complete accounts on Palm theory are given in Chapter 12 of [1], also in [7], [15] and [18].

In the first section, we introduce the Palm measure associated with a stationary measure $\mathbb{P}$ on $\mathbf{M}(G)$, the space of locally finite measures on a lcscH Abelian group G. The treatment of the subject follows the exposition in [16] and, in the second section, the intrinsic characterization theorem for Palm measures (cf. Theorem 4.2.2) is stated. We work in a canonical framework, and for some of the proofs, we refer the reader to the original work.

4.1 Stationarity and Palm measure

In Section 2.2, we have introduced the measurable flow $\Theta_G = \{\theta_x : x \in G\}$ of translation mappings that operates as a group operation on G, and also on the locally finite subsets $\mathbf{L}(G)$ of G, and on the space $\mathbf{M}(G)$ of locally finite measures on G. The operation on $\mathbf{M}(G)$ transfers to an operation on the space of σ-finite measures on $\mathbf{M}(G)$. For a σ-finite measure $\mathbb{P}$ on $\mathbf{M}(G)$ we define

$$\theta_x(\mathbb{P}) := \mathbb{P} \circ \theta_x^{-1}.$$

Definition 4.1.1. A σ-finite measure $\mathbb{P}$ on $(\mathbf{M}(G), \mathcal{M}(G))$ is called *stationary* if it is shift invariant, i.e., if $\theta_x(\mathbb{P}) = \mathbb{P}$ for all $x \in G$.

A standard monotone convergence argument yields that a σ-finite measure $\mathbb{P}$ on $\mathbf{M}(G)$ is stationary if and only if, for all $y \in G$ and $f \in \mathbf{F}(\mathbf{M}(G))$, we have

$$\int f \circ \theta_y(\varphi)\mathbb{P}(\mathrm{d}\varphi) = \int f(\varphi)\mathbb{P}(\mathrm{d}\varphi). \tag{4.1}$$

The *Campbell measure* $\mathbb{C}$ associated with a σ-finite measure $\mathbb{P}$ on $\mathbf{M}(G)$ is defined on the product space $(\mathbf{M}(G) \times G, \mathcal{M}(G) \otimes \mathcal{G})$ by

$$\mathbb{C}(A) := \int \mathbf{1}\{(\varphi, x) \in A\}\varphi(\mathrm{d}x)\mathbb{P}(\mathrm{d}\varphi).$$

The Campbell measure is again σ-finite. Now assume that $\mathbb{P}$ is stationary, and fix $B \in \mathcal{M}(G)$. Define a mapping $\theta : \mathbf{M}(G) \times G$ by $\theta(\varphi, x) := (\theta_x\varphi, x)$. Then, for $C \in \mathcal{G}$ and $y \in G$, we have

$$\begin{aligned}
\theta(\mathbb{C})(B \times C) &= \int \mathbf{1}\{\theta(\varphi, x) \in B \times C\}\varphi(\mathrm{d}x)\mathbb{P}(\mathrm{d}\varphi) \\
&= \int \mathbf{1}\{(\theta_x\varphi, x) \in B \times C\}\varphi(\mathrm{d}x)\theta_y(\mathbb{P})(\mathrm{d}\varphi) \\
&= \int \mathbf{1}\{(\theta_x \circ \theta_y\varphi, x) \in B \times C\}\theta_y\varphi(\mathrm{d}x)\mathbb{P}(\mathrm{d}\varphi) \\
&= \int \mathbf{1}\{(\theta_{x-y} \circ \theta_y(\varphi), x - y) \in B \times C\}\varphi(\mathrm{d}x)\mathbb{P}(\mathrm{d}\varphi) \\
&= \int \mathbf{1}\{(\theta_x\varphi, x) \in B \times (C + y)\}\varphi(\mathrm{d}x)\mathbb{P}(\mathrm{d}\varphi) \\
&= \theta(\mathbb{C})(B \times (C + y)).
\end{aligned}$$

We deduce that the set function $\theta(\mathbb{C})$ is invariant under shifts in G. Then Theorem A.3.2 yields the existence of a unique σ-finite measure $\mathbb{P}^0$ on $\mathbf{M}(G)$ such that $\theta(\mathbb{C}) = \mathbb{P}^0 \otimes \lambda$, where λ is the unique (up to normalization) invariant measure on G, called *Haar measure* on G (cf. Theorem A.3.1). By a standard monotone convergence argument we deduce that, for $f \in \mathbf{F}(\mathbf{M}(G) \times G)$, we have

$$\int_{\mathbf{M}(G)} \int_G f(\varphi, x)\lambda(\mathrm{d}x)\mathbb{P}^0(\mathrm{d}\varphi) = \int_{\mathbf{M}(G)} \int_G f(\theta_x\varphi, x)\varphi(\mathrm{d}x)\mathbb{P}(\mathrm{d}\varphi). \tag{4.2}$$

Equation 4.2 is known as the *refined Campbell formula*. In particular, given a non-negative, measurable function $g : G \to [0, \infty)$ such that $\int_G g(x)\lambda(\mathrm{d}x) = 1$, we obtain

$$\mathbb{P}^0(B) = \int_{\mathbf{M}(G)} \int_G g(x)\, \mathbf{1}\{\theta_x\varphi \in B\}\varphi(\mathrm{d}x)\mathbb{P}(\mathrm{d}\varphi), \quad B \in \mathcal{M}(G). \tag{4.3}$$

Definition 4.1.2. Let $\mathbb{P}$ be a σ-finite, stationary measure on $(\mathbf{M}(G), \mathcal{M}(G))$. Then the σ-finite measure $\mathbb{P}^0$ defined above is called the *Palm measure* associated with $\mathbb{P}$.

Recall from Section 2.1 that $\mathbf{N}(G)$ denotes the subspace of simple counting measures in $\mathbf{M}(G)$. We introduce a second measurable subspace of $(\mathbf{M}(G), \mathcal{M}(G))$,

$$\mathbf{M}_{\mathrm{d}}(G) := \{\varphi \in \mathbf{M}(G) : \mathrm{supp}(\varphi) \in \mathbf{L}(G)\}, \tag{4.4}$$

the set of locally finite measures on G with discrete support, equipped with the σ-field $\mathcal{M}_{\mathrm{d}}(G) := \{A \cap \mathbf{M}_{\mathrm{d}}(G) : A \in \mathcal{M}(G)\}$, the restriction of the cylindrical σ-field on $\mathbf{M}(G)$. Finally, we define the subspaces of $\mathbf{M}_{\mathrm{d}}(G)$ resp. $\mathbf{N}(G)$ that contain all measures φ with positive mass in 0 by

$$\mathbf{M}_{\mathrm{d},0}(G) := \{\varphi \in \mathbf{M}_{\mathrm{d}}(G) : \varphi(\{0\}) \neq 0\} \text{ resp. } \mathbf{N}_0(G) := \{\varphi \in \mathbf{N}(G) : \varphi(\{0\}) \neq 0\}.$$

Lemma 4.1.3. *If the stationary measure $\mathbb{P}$ on $(\mathbf{M}(G), \mathcal{M}(G))$ is concentrated on $\mathbf{M}_{\mathrm{d}}(G)$ resp. $\mathbf{N}(G)$, then the Palm measure $\mathbb{P}^0$ satisfies*

$$\mathbb{P}^0(\{\varphi \in \mathbf{M}(G) : \varphi(\{0\}) = 0\}) = 0, \tag{4.5}$$

and $\mathbb{P}^0$ is concentrated on $\mathbf{M}_{\mathrm{d},0}(G)$ resp. $\mathbf{N}_0(G)$.

Proof: Let g be a non-negative function on G such that $\int_G g(x)\lambda(\mathrm{d}x) = 1$. We apply the refined Campbell formula (4.2) to the measurable function

$$f(\varphi, x) := g(x)\, \mathbf{1}\{\varphi(\{0\}) = 0\}$$

and obtain

$$\begin{aligned}
\mathbb{P}^0(\{\varphi \in \mathbf{M}(G) : \varphi(\{0\}) = 0\}) &= \int_{\mathbf{M}(G)} \int_G f(\varphi, x)\lambda(\mathrm{d}x)\mathbb{P}^0(\mathrm{d}\varphi) \\
&= \int_{\mathbf{M}(G)} \int_G g(x)\, \mathbf{1}\{\theta_x\varphi(\{0\}) = 0\}\varphi(\mathrm{d}x)\mathbb{P}(\mathrm{d}\varphi) \\
&= \int_{\mathbf{M}(G)} \sum_{x \in \mathrm{supp}(\varphi)} g(x)\, \mathbf{1}\{\theta_x\varphi(\{0\}) = 0\}\varphi(\{x\})\mathbb{P}(\mathrm{d}\varphi) \\
&= 0.
\end{aligned}$$

Hence, (4.5) is satisfied. Moreover, $\mathbf{M}_{\mathrm{d}}(G)$ resp. $\mathbf{N}(G)$ are Θ_G-stable subspaces of $\mathbf{M}(G)$, hence, if $\mathbb{P}$ is concentrated on either one of the two spaces, then the same is true for the Palm measure $\mathbb{P}^0$. □

The Palm measure $\mathbb{P}^0$ has been defined with respect to a σ-finite, stationary measure $\mathbb{P}$ on $(\mathbf{M}(G), \mathcal{M}(G))$. Let us now ask the reverse question. Given a Palm measure $\mathbb{P}^0$, can the stationary measure $\mathbb{P}$ be retrieved? Denote the trivial zero-measure in $\mathbf{M}(G)$ by $\mathbf{0}$. It is a simple consequence of (4.2) applied to $f(\varphi, x) := \mathbf{1}\{\varphi = \mathbf{0}\}$ that the mass of $\mathbb{P}$ on $\{\mathbf{0}\}$ cannot be recovered from $\mathbb{P}^0$. However, we can retrieve the restriction of $\mathbb{P}$ to $\mathbf{M}(G) \setminus \{\mathbf{0}\}$.

For a non-negative, measurable function $g : G \to [0, \infty)$ such that $\int_G g(x)\lambda(\mathrm{d}x) = 1$ and $A \in \mathcal{M}(G)$, the refined Campbell formula (4.2) applied to $f(\varphi, x) := g(x)\, \mathbf{1}\{\varphi \in A\}$ yields the defining equation for the Palm measure because the inner integral over G is always 1 and disappears from the equation. In the same way, we obtain a defining equation for the restriction of the measure $\mathbb{P}$ to $\mathbf{M}(G) \setminus \{\mathbf{0}\}$, if there exists a measurable function $h : \mathbf{M}(G) \times G \to [0, \infty]$ such that

$$\int_G h(\varphi, x)\varphi(\mathrm{d}x) = 1 \tag{4.6}$$

for all $\varphi \in \mathbf{M}(G) \setminus \{\mathbf{0}\}$. We will now define such a function h.

A lcscH group G is a σ-compact space (cf. Theorem A.1.3). In particular, there exists a partition of G into countably many, pairwise disjoint sets $G_n \in \mathcal{G}$ with compact closure in G.

For all $\varphi \in \mathbf{M}(G)$ and $n \in \mathbb{N}$, we have $\varphi(G_n) < \infty$. We define a function $\bar{h} : \mathbf{M}(G) \times G \to [0, \infty]$ by

$$\bar{h}(\varphi, x) := \sum_{n \in \mathbb{N}} 2^{-n} \varphi(G_n)^{-1} \mathbf{1}\{x \in G_n\}, \tag{4.7}$$

where $\varphi(G_n)^{-1} := \infty$ if $\varphi(G_n) = 0$. Then, for all $\varphi \in \mathbf{M}(G) \setminus \{\mathbf{0}\}$, we have

$$0 < \int_G \bar{h}(\varphi, x)\varphi(\mathrm{d}x) \leq 1. \tag{4.8}$$

Normalizing the function $\bar{h}$, we define $h : \mathbf{M}(G) \times G \to [0, \infty)$ by

$$h(\varphi, x) := \begin{cases} \left(\int_G \bar{h}(\varphi, y)\varphi(\mathrm{d}y)\right)^{-1} \bar{h}(\varphi, x) & \text{if } \varphi \neq \mathbf{0}, \\ 0 & \text{if } \varphi = \mathbf{0}. \end{cases} \tag{4.9}$$

Hence, h is a measurable function on $\mathbf{M}(G) \times G$ that satisfies (4.6).

Theorem 4.1.4. *Let $\mathbb{P}$ be a σ-finite and stationary measure on $(\mathbf{M}(G), \mathcal{M}(G))$ and $\mathbb{P}^0$ the associated Palm measure. Then, for all $f \in \mathbf{F}(\mathbf{M}(G))$ such that $f(\mathbf{0}) = 0$, we have*

$$\int_{\mathbf{M}(G)} f(\varphi)\mathbb{P}(\mathrm{d}\varphi) = \int_{\mathbf{M}(G)} \int_G h(\theta_{-x}\varphi, x) f(\theta_{-x}\varphi)\mathrm{d}x\mathbb{P}^0(\mathrm{d}\mu). \tag{4.10}$$

Proof: Define a measurable function $u : \mathbf{M}(G) \times G \to [0, \infty)$ by

$$u(\varphi, x) := h(\theta_{-x}\varphi, x) f(\theta_{-x}\varphi).$$

The refined Campbell formula (4.2) applied to u yields (4.10). □

4.2 Integral characterization of Palm measures

In the preceding section, the Palm measure $\mathbb{P}^0$ was defined with respect to a stationary measure $\mathbb{P}$. We will now summarize results by Mecke from [16], that provide an intrinsic characterization of Palm measures.

Theorem 4.2.1. *For σ-finite measures $\mathbb{P}$ and $\mathbb{Q}$ on $(\mathbf{M}(G), \mathcal{M}(G))$ the following two assertions are equivalent.*

(a) *$\mathbb{P}$ is stationary and $\mathbb{Q} = \mathbb{P}^0$.*

(b) *For all $f \in \mathbf{F}(\mathbf{M}(G) \times G)$, we have*

$$\int_{\mathbf{M}(G)} \int_G f(\varphi, x)\mathrm{d}x\mathbb{Q}(\mathrm{d}\varphi) = \int_{\mathbf{M}(G)} \int_G f(\theta_x\varphi, x)\varphi(\mathrm{d}x)\mathbb{P}(\mathrm{d}\varphi). \tag{4.11}$$

PROOF: cf. [16], Satz 2.3.

In particular, it is shown in the proof that (4.11) yields the stationarity of $\mathbb{P}$. We will now state the main result of this section, in which Palm measures are characterized without any reference to an explicit stationary measure $\mathbb{P}$.

Theorem 4.2.2. *A measure $\mathbb{Q}$ on $(\mathbf{M}(G), \mathcal{M}(G))$ is the Palm measure of some σ-finite stationary measure $\mathbb{P}$ on $(\mathbf{M}(G), \mathcal{M}(G))$ if and only if the following three conditions are satisfied.*

(i) $\mathbb{Q}$ *is σ-finite,*

(ii) $\mathbb{Q}(\{\mathbf{0}\}) = 0$,

(iii) *for all $f \in \mathbf{F}(\mathbf{M}(G) \times G)$, we have*

$$\int_{\mathbf{M}(G)} \int_G f(\theta_x \varphi, -x)\varphi(\mathrm{d}x)\mathbb{Q}(\mathrm{d}\varphi) = \int_{\mathbf{M}(G)} \int_G f(\varphi, x)\varphi(\mathrm{d}x)\mathbb{Q}(\mathrm{d}\varphi).$$

PROOF: cf. [16], Satz 2.5.

Chapter 5

The universal point map on point processes

The central subject of this chapter is a probabilistic version of Open Problem 1 (cf. page 27) that has been discussed in various recent papers: For which classes of point processes is there a factor graph on the points of the process, that is a doubly infinite, directed path almost surely? The answer given in Theorem 5.2.3 yields a partial extension of a classical result on the relationship between time stationarity and cycle-stationarity from $\mathbb{R}$ to $\mathbb{R}^d$. Throughout the chapter, we will restrict ourselves to the case $G = \mathbb{R}^d$. We will begin with some basic probabilistic terminology, and then give a brief overview over the related results from [2], [5] and [30] (in chronological order).

A (simple) *point process* on $\mathbb{R}^d$ is a $\mathbf{N}(\mathbb{R}^d)$-valued random variable defined on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$. Sticking to the canonical framework that is used througout the whole thesis, we will equip the space $(\mathbf{N}(\mathbb{R}^d), \mathcal{N}(\mathbb{R}^d))$ with a probability measure $\mathbb{P}$. Then the identity mapping N on $\mathbf{N}(\mathbb{R}^d)$ is a point process. In view of the identification provided by Lemma 2.1.5, we can and will identify $\mathbf{N}(G)$ and $\mathbf{L}(G)$. We call N *stationary* resp. *aperiodic*, if $\mathbb{P}$ is stationary resp. aperiodic, i.e., if $\theta_x \mathbb{P} = \mathbb{P}$ for all $x \in \mathbb{R}^d$ resp. $\mathbb{P}(L(N) \neq \{0\}) = 0$, where L denotes the periodicity lattice mapping defined in Definition 2.5.4. Moreover, if N is stationary, we define the (possibly infinite) *intensity* of N under $\mathbb{P}$ by

$$\lambda_{\mathbb{P}} := \int_{\mathbb{R}^d} \varphi([0,1])^d \mathbb{P}(\mathrm{d}\varphi).$$

5.1 Paths on Poisson processes

In [2], the following example of a factor graph Γ is defined. Let $\varphi \in \mathbf{L}(\mathbb{R}^d)$ and for each point $x = (x_1, \dots, x_d) \in \varphi$ call the first $d-1$ coordinates $s(x) := (x_1, \dots, x_d) \in \mathbb{R}^{d-1}$ and the remaining coordinate $t(x) := x_d$. In this way $x = (s(x), t(x))$, and $s(x)$ is interpreted as the space coordinate and $t(x)$ as the time coordinate of x. For $s \in \mathbb{R}^{d-1}$, denote by $B(s) := \{s' \in \mathbb{R}^{d-1} : \|s - s'\| \leq 1\}$ the (closed) $(d-1)$-dimensional Euclidean unit ball centered at s. With every point $x = (s(x), t(x))$ we associate the *obstacle* $Z(x) := \{(s', t(x)) : s' \in B(s(x))\}$. A directed edge (x, x') is defined, if and only if the first obstacle hit by a half line starting in x in direction of the (positive) time axis is the one centered in x'. We then call x the mother of x', and x' a daughter of x and denote the resulting factor graph by $\Gamma(\varphi)$. When this construction is applied to the points of a stationary Poisson process N, the resulting graph $\Gamma(N)$ has no loops and in every point $x \in N$ a unique edge starts. Moreover, Ferrari, Landim and Thorisson obtained the following result.

Theorem 5.1.1. *Let N be a stationary Poisson process on $\mathbb{R}^d$. Then, almost surely, the random graph $\Gamma(N)$ associated with N has the following properties.*

(a) *$\Gamma(N)$ is a locally finite forest.*

(b) *Every vertex in $\Gamma(N)$ has a unique mother vertex.*

(c) *Using the order of the first coordinate in $\mathbb{R}^d$, each vertex has an ancestor with a lexicographically smaller sister.*

(d) *If $d = 2$ or $d = 3$ then $\Gamma(N)$ has a unique connected component; if $d \geq 4$, then $\Gamma(N)$ has infinitely many connected components.*

Proof: cf. [2], Theorem 3.1.

Also, it is observed in [2], that if Γ is a factor graph, and $\Gamma(\varphi)$ has the properties (a) (b) and (c) for all φ in some set $A \subset \mathbf{N}(\mathbb{R}^d)$, then there exists a factor graph Γ' such that $\Gamma'(\varphi)$ is a doubly-infinite, directed path for all $\varphi \in A$. Hence, by Theorem 5.1.1, for $d \in \{2, 3\}$, there exists a factors graph Γ_d defined on $\mathbf{N}(\mathbb{R}^d)$ such that, $\Gamma_d(N)$ is a directed, doubly infinite path on the points of the Poisson process N almost surely. In the same paper, Ferrari, Landim and Thorisson ask the question whether the same assertion is true for the Poisson process on $\mathbb{R}^d$ for $d \geq 4$. In [5], Holroyd and Peres answer this question by giving a deterministic construction of factor graphs, which are equivariant not only under translations but also under rotations, and have the properties stated in the following theorem.

Theorem 5.1.2. *Let N be a stationary Poisson process on $\mathbb{R}^d$.*

(a) *There is a factor graph Γ such that $\Gamma(N)$ is almost surely a locally finite one-ended tree.*

(b) *There is a factor graph Γ' such that $\Gamma'(N)$ is almost surely a directed, doubly infinite path.*

Proof: cf. [5], Theorem 1.

Using the same argument as in [2], it is shown that the first assertion implies the second one. Moreover, a similar result is given for each component of the graph $\Gamma(N)$, when N is a non-equidistant processes (a process such that the distribution is concentrated on the set D defined in (3.4)), which is ergodic and invariant under isometries (cf. [5], Theorem 2).

5.2 Paths on stationary point processes

In [30], Timar then generalized the results cited in the preceding subsection as follows.

Theorem 5.2.1. *Let N be an isometry invariant point process with finite intensity, such that the subgroup of isometries of $\mathbb{R}^d$ that leave N invariant is trivial almost surely. Then there exists a factor graph Γ, which is equivariant under all isometries, and such that $\Gamma(N)$ is a doubly infinite directed path almost surely.*

PROOF: cf. [30], Section 3.

The generality of this result is striking. The condition that the subgroup of isometries leaving N invariant is trivial almost surely yields, in particular, that N is aperiodic almost surely. The focus of Timar's paper is then to show that there are more general grid factor graphs as stated in the following theorem.

Theorem 5.2.2. *Let N be an isometry invariant point process with finite intensity, such that the subgroup of isometries of $\mathbb{R}^d$ that leave N invariant is trivial almost surely. Then there exists an isometry equivariant factor graph Γ such that $\Gamma(N)$ is isomorphic with the distance-one graph on $\mathbb{Z}^d$ almost surely.*

PROOF: cf. [30], Section 4.

We will now present an alternative proof of Theorem 5.2.1, with the following minor modification: We postulate that the point process N is stationary, not invariant under all isometries, and then show that there exists a factor graph Γ in the sense of Definition 2.7.1, i.e., not isometry equivariant but only shift equivariant, such that $\Gamma(N)$ is a doubly infinite path almost surely. Indeed, it is proved that the factor graph Γ_η associated with the succession point map η from Theorem 3.3.5 has this property. Using Palm calculus as it is introduced in the first section of Chapter 4, we show that the hypothesis of finite intensity can be omitted. Note, however, that one can also adapt the original proof to this more general case as follows. Let T be a c-regular selection function and define a path on $T(N)$, which is a stationary point process of finite intensity. Then include the points from $N \setminus T(N)$ into the path by some equivariant, deterministic rule.

Theorem 5.2.3. *Let N be a stationary, aperiodic point process. Then, almost surely, the factor graph $\Gamma_\eta(N)$ associated with the succession point map η defined in (3.24) is a doubly infinite, directed path on N.*

PROOF: Consider the sequence $r_n := 2^{n^2}, n \in \mathbb{N}$. We first show that the r_n-regular thinning procedure T_n yields a complete thinning on N almost surely. Indeed, the selection functions $T_n, n \in \mathbb{N}$, are equivariant and measurable, hence, $\cap_{n\in\mathbb{N}} T_n(N)$ is again a stationary point process, and by Lemma 2.6.10, we have $\text{card}(\cap_{n\in\mathbb{N}} T_n(N)) \in \{0, 1\}$. Since there is no stationary distribution for a $\mathbb{R}^d$-valued random variable, we deduce that $\cap_{n\in\mathbb{N}} T_n(N) = \emptyset$ almost surely, and denote by A_1 a subset of $\mathbf{N}(\mathbb{R}^d)$ that satisfies $\mathbb{P}(A_1) = 1$ and $\cap_{n\in\mathbb{N}} T_n(\varphi) = \emptyset$ for all $\varphi \in A_1$.

By Lemma 3.3.2, Λ_η is a selection function and, hence, $\Lambda_\eta(N)$ is also a stationary point process. For all $\varphi \in A_1$, Theorem 3.3.5 yields that $0 \leq \mathrm{card}(\Lambda_\eta(\varphi)) < \infty$, and whenever $\Lambda_\eta(\varphi)$ is not empty the choice of the lexicographic smallest point in $\Lambda_\eta(\varphi)$ yields again an equivariant selection of a single point from N. Hence, the same argument as above applies and there exists a set $A_2 \subset A_1$ such that $\mathbb{P}(A_2) = 1$ and $\Lambda_\eta(\varphi) = \emptyset$ for all $\varphi \in A_2$.

We may now apply Theorem 3.3.5 and obtain that, for all $\varphi \in A_2$, the graph $\Gamma_\eta(\varphi)$ consists of finitely many doubly infinite, directed paths on φ. It remains to show that, for almost all $\varphi \in A_2$, there is a unique path in $\Gamma_\eta(\varphi)$, i.e., that $\Gamma_\eta(\varphi)$ is connected. By Corollary 3.3.4, this is equivalent with the fact that for almost all $\varphi \in A_2$, any two points $x, y \in \varphi$ have a joint ancestor in the (T_n)-Voronoi hierarchy.

We will first argue in a determistic setting. Let $\varphi \in \mathbf{L}(G)$, $x \in \varphi$ and denote by $V(\varphi, x)$ the cell with center $x \in \varphi$ in the Voronoi mosaic associated with φ. Define the union of the boundaries of the cells by

$$Z(\varphi) := \bigcup_{x \in \varphi} \partial V(\varphi, x).$$

The set $Z(\varphi)$ consists of all points in $\mathbb{R}^d$ that do not have a unique nearest neighbour in φ, often this set is also called the *exoskeleton* of φ.

Now assume that there are two points $x, y \in \varphi \cap B_d(0, k)$ such that $\alpha_n(\theta_x\varphi) + x \neq \alpha_n(\theta_y\varphi) + y$. Then the points $\alpha_{n-1}(\theta_x\varphi) + x$ and $\alpha_{n-1}(\theta_y\varphi) + y$ are in two different cells of the Voronoi mosaic associated with the set $T_n(\varphi)$. Defining $s_n := k + \sum_{i=1}^{n-1} r_i$, we have $d(0, \alpha_{n-1}(\theta_x\varphi) + x) \leq s_n$ and $d(0, \alpha_{n-1}(\theta_y\varphi) + y) \leq s_n$, so both points are contained in the ball $B_d(0, s_n)$, and we deduce that this ball must have a non-empty intersection with $Z(T_n(\varphi))$. Hence, we have $d(0, Z(T_n(\varphi))) \leq s_n$, or equivalently, $0 \in Z(T_n(\varphi))_{\oplus s_n}$.

Finally, we get to the probabilistic part of the argument. Fix $n \in \mathbb{N}$ and write $\mathbb{P}_n$ for the stationary measure $T_n(\mathbb{P})$ and $\mathbb{P}_n^0$ for the associated Palm measure. We have

$$\begin{aligned}
&\mathbb{P}(\{\text{there exist } x, y \in N \cap B_d(0,k) \text{ with distinct } n\text{-ancestors}\}) \\
&\leq \mathbb{P}(\{0 \in Z(T_n(N))_{\oplus s_n}\}) \\
&= \int_{[0,1]^d} \int_{\mathbf{N}(\mathbb{R}^d)} \mathbf{1}\{x \in Z(\varphi)_{\oplus s_n}\} \mathbb{P}_n(\mathrm{d}\varphi) \lambda^d(\mathrm{d}x) \\
&= \int_{[0,1]^d} \int_{\mathbf{N}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \mathbf{1}\{x \in V(\varphi, y) \cap Z(\varphi)_{\oplus s_n}\} \varphi(\mathrm{d}y) \mathbb{P}_n(\mathrm{d}\varphi) \lambda^d(\mathrm{d}x) \qquad (5.1) \\
&= \int_{\mathbf{N}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}\{x \in [0,1]^d\} \, \mathbf{1}\{x - y \in V(\theta_y\varphi, 0) \cap Z(\theta_y\varphi)_{\oplus s_n}\} \lambda^d(\mathrm{d}x) \varphi(\mathrm{d}y) \mathbb{P}_n(\mathrm{d}\varphi) \\
&= \int_{\mathbf{N}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}\{x \in [0,1]^d\} \, \mathbf{1}\{x - y \in V(\varphi, 0) \cap Z(\varphi)_{\oplus s_n}\} \lambda^d(\mathrm{d}x) \lambda^d(\mathrm{d}y) \mathbb{P}_n^0(\mathrm{d}\varphi) \\
&= \int_{\mathbf{N}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \mathbf{1}\{x \in [0,1]^d\} \lambda^d(V(\varphi, 0) \cap Z(\varphi)_{\oplus s_n}) \lambda^d(\mathrm{d}x) \mathbb{P}_n^0(\mathrm{d}\varphi).
\end{aligned}$$

The set φ is $\mathbb{P}_n^0$-almost surely a r_n-regular set with $0 \in \varphi$, so the Voronoi cell $V(\varphi, 0)$ is a polytope, and its inradius is at least $r_n/2$. Denote by x the center of an open ball with radius $r_n/2$ that is contained in $V(\varphi, 0)$. Then the volume of each of the pyramides P, defined as the convex hull of a facet F of $V(\varphi, 0)$ and x, satisfies

$$\frac{1}{d}\,\frac{r_n}{2} S_{d-1}(F) \leq \lambda^d(P),$$

where $S_{d-1}(F)$ denotes the $d-1$-dimensional Hausdorff measure of F (or simply the $d-1$-dimensional volume of F in the $d-1$ dimensional affine subspace generated by F). Summing up over the finitely many facets of $V(\varphi, 0)$, we obtain

$$\frac{r_n}{2d} S_{d-1}(\partial V(\varphi, 0)) \leq \lambda^d(V(\varphi, 0)). \tag{5.2}$$

Using mixed volumes (see Chapter 5 in [23]) it is possible to derive this inequality for general convex bodies (i.e., convex, compact sets). Moreover, an elementary inequality for convex bodies is given by

$$\lambda^d(V(\varphi, 0) \cap (\partial V(\varphi, 0))_{\oplus s}) \leq s S_{d-1}(\partial V(\varphi, 0)). \tag{5.3}$$

Since $V(\varphi, 0) \cap Z(\varphi) = \partial V(\varphi, 0)$, we deduce from (5.1), (5.2) and (5.3) that

$$\mathbb{P}(\{0 \in Z(T_n(N))_{\oplus s_n}\}) \leq \frac{2ds_n}{r_n} \int_{\mathbf{N}(\mathbb{R}^d)} \lambda^d(V(\varphi, 0))\mathbb{P}_n^0(\mathrm{d}\varphi).$$

A similar computation as above (in the reverse direction) yields that

$$\begin{aligned}
&\int_{\mathbf{N}(\mathbb{R}^d)} \lambda^d(V(\varphi, 0))\mathbb{P}_n^0(\mathrm{d}\varphi) \\
&= \int_{\mathbf{N}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}\{x \in [0,1]^d\}\, \mathbf{1}\{x - y \in V(\varphi, 0)\}\lambda(\mathrm{d}x)\lambda(\mathrm{d}y)\mathbb{P}_n^0(\mathrm{d}\varphi) \\
&= \int_{\mathbf{N}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}\{x \in [0,1]^d\}\, \mathbf{1}\{x - y \in V(\theta_y\varphi, 0)\}\lambda(\mathrm{d}x)\varphi(\mathrm{d}y)\mathbb{P}_n(\mathrm{d}\varphi) \\
&= \int_{\mathbf{N}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}\{x \in [0,1]^d\}\, \mathbf{1}\{x \in V(\varphi, y)\}\lambda(\mathrm{d}x)\varphi(\mathrm{d}y)\mathbb{P}_n(\mathrm{d}\varphi) \\
&= \int_{\mathbf{N}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \mathbf{1}\{x \in [0,1]^d\}\lambda(\mathrm{d}x)\mathbb{P}_n(\mathrm{d}\varphi) \\
&= 1.
\end{aligned}$$

We have $s_n \leq 2^{(n-1)^2+1} + k$, and so $s_n/r_n \leq (1+k)2^{-n}$ holds for all $n \geq 2$. We conclude that

$$\mathbb{P}(\{\text{there exist } x, y \in N \cap B_d(0,k) \text{ with distinct } n\text{-ancestors}\}) \leq \frac{d(1+k)}{2^{n-1}}.$$

Since the series over n with the general term from the right of the last equation is summable, the Borel-Cantelli lemma yields that $\mathbb{P}$-almost surely any two points $x, y \in N \cap B_d(0,k)$ have only finitely many distinct n-ancestors, i.e., the event

$$D_k := \{\text{there exist } x, y \in N \cap B_d(0,k) \text{ with distinct } n\text{-ancestors for all } n \in \mathbb{N}\}$$

satisfies $\mathbb{P}(D_k) = 0$. Clearly, we then also have $\mathbb{P}(\cup_{k\in\mathbb{N}} D_k) = 0$, and conclude that the event $A_3 := A_2 \setminus (\cup_{k\in\mathbb{N}} D_k)$ satisfies $\mathbb{P}(A_3) = 1$ and $\Gamma_\eta(\varphi)$ is a doubly infinite, directed path on φ for all $\varphi \in A_3$. $\square$

5.3 On the key stationarity theorem

In this section, we will state and discuss a theorem, that (in a more general, marked version, cf. [29], Ch. 8) has been called the key stationarity theorem. We begin with the introduction of an invariance property for point processes on $\mathbb{R}$. Recall the definition of the universal point map τ defined in (3.2).

Definition 5.3.1. A σ-finite measure $\mathbb{Q}$ on $(\mathbf{N}(\mathbb{R}), \mathcal{N}(\mathbb{R}))$ is called *cycle-stationary* if

$$\mathbb{Q}(0 \notin N) = 0 \text{ and } \theta_\tau(\mathbb{Q}) = \mathbb{Q}.$$

If $\mathbb{Q}$ is a probability measure, we also call the point process N cycle-stationary (in this case $\theta_\tau(N) \stackrel{d}{=} N$).

If N is a cycle-stationary point process on $\mathbb{R}$, then the successive differences defined by

$$Z_n := \tau^{n+1}(N) - \tau^n(N), \quad n \in \mathbb{Z}, \tag{5.4}$$

form a stationary sequence of real random variables, i.e., $(Z_n)_{n\in\mathbb{Z}} \stackrel{d}{=} (Z_{n+1})_{n\in\mathbb{Z}}$, and a version of the cycle-stationary point process N can be recovered from the sequence (Z_n) via

$$N = \{0\} \cup \left\{ \sum_{i=1}^{n} Z_i : n \in \mathbb{N} \right\} \cup \left\{ \sum_{i=0}^{n} -Z_i : n \in \mathbb{N} \cup \{0\} \right\} \quad \mathbb{Q}\text{-almost surely.}$$

The following theorem states a fundamental relationship between stationarity and cycle-stationarity, and can easily be derived from Theorem 11.4 in [7].

Theorem 5.3.2. *There exists a one-to-one correspondence between the distributions $\mathbb{P}$ of stationary point processes on $\mathbb{R}$ such that $\lambda_{\mathbb{P}} < \infty$ and $\mathbb{P}(N = \mathbf{0}) = 0$, where $\mathbf{0}$ denotes the zero-measure on $\mathbb{R}$, and the distributions $\mathbb{Q}$ of cycle-stationary point processes such that $\mathbb{Q}(\{\tau(N) \le 0\}) = 0$ and $\int_{\mathbf{N}(G)} \tau(\varphi)\mathbb{Q}(\mathrm{d}\varphi) < \infty$.*

The one-to-one correspondence can even be made explicit, using normalized versions of Palm measures (4.3) in one direction, and the inversion formula (4.10) in the other (cf. [7], Theorem 11.4).

Let us now consider the analogous problem in d dimensions. Assume that $\mathbb{P}$ is the distribution of a stationary and aperiodic point process with finite intensity. Then the associated Palm measure $\mathbb{P}^0$ has finite total mass and normalizaton yields the associated Palm distribution $\hat{\mathbb{P}}^0$. Moreover, Theorem 5.2.3 yields that, $\mathbb{P}$-a.s., $\Gamma(N)$ is a doubly infinite path on the points of N. By the equivariance of the factor graph Γ_η and the refined Campbell formula (4.2), we obtain that, $\mathbb{P}^0$-a.s., and then also $\hat{\mathbb{P}}^0$-a.s., $\Gamma(N)$ is a doubly infinite path on the points of N. It will be shown in Proposition 6.2.2 that $\mathbb{P}^0$ is invariant under the point shift θ_η which is associated with the succession point map η from (3.24), a property that transfers to $\hat{\mathbb{P}}^0$. Hence, we can define a stationary sequence of $\mathbb{R}^d$-valued random variables just as in the one-dimensional case.

Proposition 5.3.3. *Let N be a stationary and aperiodic point process on $\mathbb{R}^d$, and let η be the point map from* (3.24)*. Then the sequence of random variables in $\mathbb{R}^d$ defined by*

$$Z_n := \eta^{n+1}(N) - \eta^n(N), \quad n \in \mathbb{Z},$$

is stationary under $\hat{\mathbb{P}}^0$ and

$$N = \{0\} \cup \left\{ \sum_{i=1}^{n} Z_i : n \in \mathbb{N} \right\} \cup \left\{ \sum_{i=0}^{n} -Z_i : n \in \mathbb{N} \cup \{0\} \right\} \quad \hat{\mathbb{P}}^0\text{-almost surely.} \tag{5.5}$$

In view of Theorem 5.3.2, the following problems arise.

Open problem 2. Let $(Z_n)_{n\in\mathbb{Z}}$ be a stationary random sequence in $\mathbb{R}^d$ with distribution $\mathbb{Q}$ (on $(\mathbb{R}^d)^{\mathbb{Z}}$) and define

$$M((Z_n)_{n\in\mathbb{Z}}) := \{0\} \cup \left\{ \sum_{i=1}^{n} Z_i : n \in \mathbb{N} \right\} \cup \left\{ \sum_{i=1}^{n} -Z_i : n \in \mathbb{N} \cup \{0\} \right\}.$$

What are suitable conditions on $\mathbb{Q}$ to ensure that $M((Z_n)_{n\in\mathbb{Z}})$ is the Palm version of a stationary point process?

Less ambitious, one could ask for examples of random sequences that yield a stationary point process.

Open problem 3. What examples of stationary probability distributions $\mathbb{Q}$ on $(\mathbb{R}^d)^{\mathbb{Z}}$ can be explicitly given, such that $M((Z_n)_{n\in\mathbb{Z}})$ is the Palm version of a stationary point process?

We are aware that, given a stationary point process N, the definition of η hardly gives any insights on the distribution of the sequence (Z_n) defined in Proposition 5.3.3. Still, there may be a chance to find examples that solve Open Problem 3. After all, Proposition 5.3.3 yields that the Palm distribution of any stationary, aperiodic point process with finite intensity can be obtained in this way.

At the end of this chapter, we return to the framework of σ-finite measures that will also be used in the remainder of this thesis. Then Theorem 5.3.2 has the following, smooth formulation.

Theorem 5.3.4. *There exists a one-to-one correspondence between σ-finite stationary measures $\mathbb{P}$ on $\mathbf{N}(\mathbb{R})$ such that $\mathbb{P}(\{\mathbf{0}\}) = 0$, and σ-finite cycle-stationary measures $\mathbb{Q}$.*

The following, concluding chapter will be concerned with the characterization of Palm measures on a lcscH group G, and Theorem 5.3.4 is a consequence of the characterization of Palm measures on $\mathbf{N}(\mathbb{R})$ derived in Theorem 6.2.2, Theorem 6.2.3 and the remark made after the latter theorem.

Chapter 6

Point shift characterization of Palm measures

In this final chapter we will bring together the three key words that appear in the title of this thesis: bijective point maps, point-stationarity and Palm measures. We will begin with the formal definition of point-stationarity, which involves bijective points shifts and will then explore the intimate connection of point-stationarity and Palm measures, beginning with measures on the space of simple counting measures $\mathbf{N}(G)$.

6.1 Point-stationarity

In this section, the notion of point-stationarity, which was first introduced and discussed by Thorisson (cf. [26], [28], [29]), will be defined. Informally, a point processs N is called point-stationary if it contains the origin almost surely and "looks statistically the same from any of its points", i.e., if by some unbiased rule we "pick a point x from $N \setminus \{0\}$" then the process $\theta_x N$ has the same distribution as N. Formally, we will first define point stationarity for σ-finite mesures $\mathbb{Q}$ on $\mathbf{N}(G)$, where, as before, we will identify a measure $\varphi \in \mathbf{N}(G)$ and its support $\operatorname{supp}(\varphi) \in \mathbf{L}(G)$ (cf. Lemma 2.1.5). The key to a formal approach to point-stationarity are bijective point shifts.

Definition 6.1.1. A σ-finite measure $\mathbb{Q}$ on $\mathbf{N}(G)$ is called *point-stationary* if

$$\mathbb{Q}(\mathbf{N}(G) \setminus \mathbf{N}_0(G)) = 0$$

holds and $\mathbb{Q}$ is invariant under bijective point shifts.

Indeed, a bijective point map σ provides the possibility to pick a point from $\varphi \in \mathbf{N}_0(G)$. This choice is made deterministically, however, the bijectivity of the point map assures that for every $x \in \varphi$ there exists a unique $y \in \varphi$ such that $\sigma(\theta_x\varphi) + x = y$, and the invariance of $\mathbb{Q}$ under θ_σ reflects the fact that $\theta_x\varphi$ has "the same weight" under $\mathbb{Q}$ as $\theta_y\varphi$. For a formalization of the property that the process "looks statistically the same from any of its points", it is then necessary to find a family of bijective point maps $\{\sigma_i : i \in I\}$, such that, for $\mathbb{Q}$-almost all $\varphi \in \mathbf{N}_0(G)$ and for any $x \in \varphi$ there exists $i \in I$ such that $\sigma_i(\varphi) = x$, or, equivalently, a complete family of point maps on some θ_G-stable set $A \subset \mathbf{N}_0(G)$ that satisfies $\mathbb{Q}(\mathbf{N}(G) \setminus A) = 0$.

A solution to this problem is given in Chapter 9.5 of [29], where randomized bijective point shifts are introduced. In [4], a countable family of deterministic bijective point shifts is defined that is complete on $\mathbf{N}_0(\mathbb{R}^d)$. In Section 3.2 of this thesis, we have defined a countable family of matchings that is quasi complete on $\mathbf{N}_0(G)$, where G is an arbitrary lcscH group.

6.2 Simple point processes

We will now discuss the connections of point-stationarity and Palm measures. For fixed $\varphi \in \mathbf{N}_0(G)$, (the second component of) an extended bijective point map $\tilde{\sigma}$ is a permutation of the points of φ. As we will show in the following lemma, this fact entails that the Campbell measure $\mathbb{C}(\mathrm{d}(\varphi, x)) = \varphi(\mathrm{d}x)\mathbb{P}(\mathrm{d}\varphi)$ on $\mathbf{N}(G) \times G$ associated with a measure $\mathbb{P}$ on $\mathbf{N}(G)$ is invariant under $\tilde{\sigma}$, a fact that is tacitly used in the proof of Theorem 3.1 in [4].

Lemma 6.2.1. *Let $\mathbb{P}$ be an arbitrary σ-finite measure on $\mathbf{N}(G)$, and $\mathbb{C}$ the associated Campbell measure on $\mathbf{N}(G) \times G$. If π is a bijective point map on $\mathbf{N}(G)$, then $\mathbb{C}$ is invariant under the extended point map $\tilde{\pi}$.*

Proof: For $f \in \mathbf{F}(\mathbf{N}(G) \times G)$, we have

$$\begin{aligned}
\int_{\mathbf{N}(G)\times G} f(\varphi, x)\mathbb{C}(\mathrm{d}(\varphi, x)) &= \int_{\mathbf{N}(G)} \sum_{x\in\varphi} f(\varphi, x)\mathbb{P}(\mathrm{d}\varphi) \\
&= \int_{\mathbf{N}(G)} \sum_{x\in\varphi} f(\varphi, \pi(\theta_x\varphi) + x)\mathbb{P}(\mathrm{d}\varphi) \\
&= \int_{\mathbf{N}(G)\times G} f(\varphi, x)\tilde{\pi}(\mathbb{C})(\mathrm{d}(\varphi, x)),
\end{aligned}$$

so the lemma is proved. □

We will now state and prove our first theorem establishing the link between point-stationarity and Palm measures. The theorem is a special case of Satz 4.3 in [17], see also Theorem 9.4.1 in [29] and Theorem 3.1 in [4].

Theorem 6.2.2. *Let $\mathbb{P}^0$ be the Palm measure of a stationary measure $\mathbb{P}$ on $\mathbf{N}(G)$ and π a bijective point map. Then $\mathbb{P}^0$ is invariant under the associated point shift θ_π.*

Proof: Let $g \in \mathbf{F}(G)$ such that $\int g(x)\mathrm{d}x = 1$, and denote by $\mathbb{C}$ the Campbell measure associated with $\mathbb{P}$. For any $f \in \mathbf{F}(\mathbf{N}(G))$, we have

$$\begin{aligned}
\int_{\mathbf{N}(G)} f(\varphi)\theta_\pi(\mathbb{P}^0)(\mathrm{d}\varphi) &= \int_{\mathbf{N}(G)} f \circ \theta_\pi(\varphi)\mathbb{P}^0(\mathrm{d}\varphi) \\
&= \int_{\mathbf{N}(G)} \int_G g(x) f \circ \theta_\pi(\theta_x\varphi)\varphi(\mathrm{d}x)\mathbb{P}(\mathrm{d}\varphi) \\
&= \int_{\mathbf{N}(G)} \int_G g(x) f(\theta_{\pi(\theta_x\varphi)+x}\varphi)\varphi(\mathrm{d}x)\mathbb{P}(\mathrm{d}\varphi) \\
&= \int_{\mathbf{N}(G)\times G} g(\pi^{-1}(\theta_x\varphi)+x) f(\theta_x\varphi)\tilde{\pi}(\mathbb{C})(\mathrm{d}(\varphi,x)) \\
&= \int_{\mathbf{N}(G)\times G} g(\pi^{-1}(\theta_x\varphi)+x) f(\theta_x\varphi)\mathbb{C}(\mathrm{d}(\varphi,x)) \\
&= \int_{\mathbf{N}(G)} \int_G g(\pi^{-1}(\theta_x\varphi)+x) f(\theta_x\varphi)\varphi(\mathrm{d}x)\mathbb{P}(\mathrm{d}\varphi) \\
&= \int_{\mathbf{N}(G)} \int_G g(\pi^{-1}(\varphi)+x) f(\varphi)\mathrm{d}x\mathbb{P}^0(\mathrm{d}\varphi) \\
&= \int_{\mathbf{N}(G)} f(\varphi)\mathbb{P}^0(\mathrm{d}\varphi),
\end{aligned}$$

where we have used Proposition 2.4.3, the refined Campbell formula (4.2) and Lemma 6.2.1. □

Hence, any σ-finite Palm measure is point-stationary. The main result in [4] establishes, that for $G = \mathbb{R}^d$ the converse is also true, i.e., that point-stationarity is a characteristic property of Palm measures. We will now generalize this result to the case of a lcscH group G.

Theorem 6.2.3. *A measure $\mathbb{Q}$ on $(\mathbf{N}(G), \mathcal{N}(G))$ is the Palm measure of some stationary σ-finite measure $\mathbb{P}$ if and only if $\mathbb{Q}$ is σ-finite and point-stationary.*

PROOF: Assume that $\mathbb{Q}$ is σ-finite and point-stationary, i.e., $\mathbb{Q}(\mathbf{N}(G) \setminus \mathbf{N}_0(G)) = 0$ and $\mathbb{Q}$ is invariant under bijective point shifts. By Mecke's characterization theorem (cf. Theorem 4.2.2), we have to show that, for any $f \in \mathbf{F}(\mathbf{N}(G) \times G)$, we have

$$\int_{\mathbf{N}(G)} \int_G f(\theta_x\varphi, -x)\varphi(\mathrm{d}x)\mathbb{Q}(\mathrm{d}\varphi) = \int_{\mathbf{N}(G)} \int_G f(\varphi, x)\varphi(\mathrm{d}x)\mathbb{Q}(\mathrm{d}\varphi). \tag{6.1}$$

Fix $f \in \mathbf{F}(\mathbf{N}(G) \times G)$ and let π be a matching on $\mathbf{N}(G)$. Invariance of $\mathbb{Q}$ under the point shift θ_π yields

$$\int_{\mathbf{N}(G)} f(\varphi, \pi(\varphi))\theta_\pi\mathbb{Q}(\mathrm{d}\varphi) = \int_{\mathbf{N}(G)} f(\theta_\pi\varphi, -\pi(\varphi))\mathbb{Q}(\mathrm{d}\varphi), \tag{6.2}$$

where we have used that, for a matching π, we have $\pi = \pi^{-1}$ and $\pi \circ \theta_\pi = -\pi$ (cf. Lemma 2.4.3 (c)). By Theorem 3.2.6, there exists a countable family of matchings $M = \{\pi_n : n \in \mathbb{N}\}$

on $\mathbf{N}(G)$ that is quasi-complete on $\mathbf{N}_0(G)$. Using the invariance of $\mathbb{Q}$ under bijective point shifts and (6.2) for the second equality, we obtain

$$\begin{aligned}
&\int_{\mathbf{N}(G)}\int_G \mathbf{1}\{\theta_x\varphi \neq \varphi\} f(\varphi,x)\varphi(\mathrm{d}x)\mathbb{Q}(\mathrm{d}\varphi)\\
&= \int_{\mathbf{N}(G)}\sum_{n\in\mathbb{N}} \mathbf{1}\{\pi_n(\varphi)\neq 0\}\,\mathbf{1}\{\pi_n(\varphi)\neq \pi_m(\varphi) : 1\le m<n\} f(\varphi,\pi_n(\varphi))\theta_{\pi_n}\mathbb{Q}(\mathrm{d}\varphi)\\
&= \int_{\mathbf{N}(G)}\sum_{n\in\mathbb{N}} \mathbf{1}\{-\pi_n(\varphi)\neq 0\}\,\mathbf{1}\{-\pi_n(\varphi)\neq \pi_m(\theta_{\pi_n}\varphi) : 1\le m<n\} f(\theta_{\pi_n}\varphi,-\pi_n(\varphi))\mathbb{Q}(\mathrm{d}\varphi)\\
&= \int_{\mathbf{N}(G)}\sum_{n\in\mathbb{N}} \mathbf{1}\{\pi_n(\varphi)\neq 0\}\,\mathbf{1}\{\pi_m(\varphi)\neq \pi_n(\varphi) : 1\le m<n\} f(\theta_{\pi_n}\varphi,-\pi_n(\varphi))\mathbb{Q}(\mathrm{d}\varphi)\\
&= \int_{\mathbf{N}(G)}\int_G \mathbf{1}\{\theta_x\varphi\neq\varphi\} f(\theta_x\varphi,-x)\varphi(\mathrm{d}x)\mathbb{Q}(\mathrm{d}\varphi),
\end{aligned}\tag{6.3}$$

where, in the penultimate step, we have used that

$$-\pi_n(\varphi)\neq\pi_m(\theta_{\pi_n}\varphi) \iff \pi_m(\theta_{\pi_n}\varphi)+\pi_n(\varphi)\neq 0 \iff \pi_m\circ\pi_n(\varphi)\neq 0 \iff \pi_m(\varphi)\neq\pi_n(\varphi).$$

Since, for $\varphi\in\mathbf{N}(G)$ and $x\in G$ such that $\theta_x\varphi=\varphi$, we have $\theta_{-x}\varphi=\varphi$ and, in particular, $x\in\varphi$ if and only if $-x\in\varphi$, we obtain

$$\begin{aligned}
&\int_{\mathbf{N}(G)}\int_G \mathbf{1}\{\theta_x\varphi = \varphi\} f(\varphi,x)\varphi(\mathrm{d}x)\mathbb{Q}(\mathrm{d}\varphi)\\
&= \int_{\mathbf{N}(G)}\int_G \mathbf{1}\{\theta_x\varphi = \varphi\} f(\theta_{-x}\varphi,x)\varphi(\mathrm{d}x)\mathbb{Q}(\mathrm{d}\varphi)\\
&= \int_{\mathbf{N}(G)}\int_G \mathbf{1}\{\theta_x\varphi = \varphi\} f(\theta_x\varphi,-x)\varphi(\mathrm{d}x)\mathbb{Q}(\mathrm{d}\varphi).
\end{aligned}$$

Hence, (6.1) holds and the proof of the theorem is concluded. □

In the proof of the theorem, we have only used the invariance of $\mathbb{Q}$ under the point shifts associated with a quasi-complete family of bijective point maps $\{\pi_n : n\in\mathbb{N}\}$ such that

$$\begin{aligned}
\{x\in\varphi : \theta_x\varphi\neq\varphi\} &= \{\pi_n(\varphi) : \pi_n(\varphi)\neq 0 \text{ and } \pi_n(\varphi)\neq\pi_m(\varphi) \text{ for all } 1\le m<n, n\in\mathbb{N}\}\\
&= \{\pi_n^{-1}(\varphi) : \pi_n(\varphi)\neq 0 \text{ and } \pi_n(\varphi)\neq\pi_m(\varphi) \text{ for all } 1\le m<n, n\in\mathbb{N}\}
\end{aligned}$$

for all $\varphi\in\mathbf{N}_0(G)$. Clearly, any family of matchings that is quasi complete on $\mathbf{N}_0(G)$ has this property. In the special case $G=\mathbb{R}$, the universal point map τ generates the complete family of bijective point maps $\{\tau^n : n\in\mathbb{Z}\}$. Moreover, for $\varphi\in\mathbf{N}_0(\mathbb{R})$ such that $\mathrm{card}(\varphi)=\infty$, we have

$$\varphi=\{\tau^n(\varphi) : n\in\mathbb{Z}\}=\{\tau^{-n}(\varphi) : n\in\mathbb{Z}\},$$

and, for $\varphi\in\mathbf{N}_0(\mathbb{R})$ such that $\mathrm{card}(\varphi)<\infty$, we have

$$\varphi=\{\tau^n(\varphi) : 1\le n\le\mathrm{card}(\varphi)\}=\{\tau^{-n}(\varphi) : 1\le n\le\mathrm{card}(\varphi)\},$$

where in each of the four cases $\tau_k(\varphi)\neq\tau_\ell(\varphi)$ whenever $k\neq\ell$. Hence, the above proof can easily be adapted to show that cycle-stationarity, i.e., invariance under the point shift θ_τ, is characteristic for Palm measures on $\mathbf{N}(\mathbb{R})$ (cf. Theorem 5.3.4).

6.3 Discrete random measures

In this section, we have the objective to generalize the results from the previous section to general discrete random measures. Recall the definition

$$\mathbf{M}_{\mathrm{d}}(G) := \{\varphi \in \mathbf{M}(G) : \mathrm{supp}(\varphi) \in \mathbf{L}(G)\}$$

from (4.4). The space $\mathbf{M}_{\mathrm{d}}(G)$ contains the locally finite measures on G with discrete support and is equipped with the σ-field $\mathcal{M}_{\mathrm{d}}(G)$, the restriction of the cylindrical σ-field on $\mathbf{M}(G)$. The mapping $\mathrm{supp} : \mathbf{M}_{\mathrm{d}}(G) \to \mathbf{L}(G), \varphi \mapsto \mathrm{supp}(\varphi)$, that maps a measure in $\mathbf{M}_{\mathrm{d}}(G)$ to its support, is measurable. Let us now extend the definitions of point maps and point shifts and the notion of point-stationarity.

Definition 6.3.1. Let σ be a point map and θ_π the associated point shift. We extend the domain of σ and define $\sigma : \mathbf{M}_{\mathrm{d}}(G) \to G$ by $\sigma(\varphi) := \sigma(\mathrm{supp}(\varphi))$. Accordingly, we extend the domain and the range of the associated point shift and define $\theta_\sigma : \mathbf{M}_{\mathrm{d}}(G) \to \mathbf{M}_{\mathrm{d}}(G)$ by $\theta_\sigma(\varphi) := \theta_{\sigma(\varphi)}(\varphi)$.

Definition 6.3.2. A σ-finite measure $\mathbb{Q}$ on $(\mathbf{M}_{\mathrm{d}}(G), \mathcal{M}_{\mathrm{d}}(G))$ is called *point-stationary* if $\mathbb{Q}(\mathbf{M}_{\mathrm{d}}(G) \setminus \mathbf{M}_{\mathrm{d},0}(G)) = 0$ and $\mathbb{Q}$ is invariant under bijective point shifts, i.e., $\theta_\sigma(\mathbb{Q}) = \mathbb{Q}$ for all bijective point maps π.

As we will see in the following, elementary example, the Palm measure $\mathbb{P}^0$ associated with a stationary measure $\mathbb{P}$ on $\mathbf{M}_{\mathrm{d}}(G)$ is (in general) not invariant under point shifts.

Example 6.3.3. Generalizing our notation for Dirac measures on G (see Lemma 2.1.5), we denote by δ_ψ the measure on $\mathbf{M}_{\mathrm{d}}(G)$ defined by $\delta_\psi(A) := \mathbf{1}\{\psi \in A\}$, $A \in \mathcal{M}_{\mathrm{d}}(G)$. Now assume that $G = \mathbb{R}$ and define a locally finite measure on $\mathbb{R}$ by $\varphi := \delta_0 + 2\delta_1$. A stationary measure $\mathbb{P}$ on $(\mathbf{M}_{\mathrm{d}}(\mathbb{R}), \mathcal{M}_{\mathrm{d}}(\mathbb{R}))$ is then given by

$$\mathbb{P}(A) := \int_{\mathbb{R}} \delta_{\theta_x \varphi}(A) \mathrm{d}x, \quad A \in \mathcal{M}_{\mathrm{d}}(\mathbb{R}).$$

We define the events $A_n := \{\varphi \in \mathbf{M}_{\mathrm{d}} : \varphi([-n, n]) \neq 0\}$. Then $\mathbb{P}(A_n) = 2n + 1$ and $\cup_{n \in \mathbb{N}} A_n \cup \{\mathbf{0}\} = \mathbf{M}_{\mathrm{d}}(\mathbb{R})$, hence, $\mathbb{P}$ is σ-finite. Using the function $g : \mathbb{R} \to [0, \infty)$ defined by $g(x) := \mathbf{1}\{x \in (0, 1]\}$, the definition of the Palm measure $\mathbb{P}^0$ associated with $\mathbb{P}$ yields

$$\begin{aligned}
\mathbb{P}^0(A) &= \int_{\mathbf{M}_{\mathrm{d}}(\mathbb{R})} \int_{\mathbb{R}} \mathbf{1}\{y \in (0,1]\} \, \mathbf{1}\{\theta_y \varphi \in A\} \varphi(\mathrm{d}y) \mathbb{P}(\mathrm{d}\varphi) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbf{1}\{y \in (0,1]\} \, \mathbf{1}\{\theta_y \theta_x \varphi \in A\} \theta_x \varphi(\mathrm{d}y) \mathrm{d}x \\
&= \delta_\varphi(A) + 2\delta_{\theta_1 \varphi}(A), \quad A \in \mathcal{M}_{\mathrm{d}}(\mathbb{R}),
\end{aligned}$$

where, in the last step, the integral with respect to x over $(-1, 0]$ corresponds to $\delta_\varphi(A)$, and the integral over $(0, 1]$ to $2\delta_{\theta_1 \varphi}(A)$. The integrand is zero if $y \notin \{-x, 1 - x\}$.

Now suppose that π is a bijective point map such that $\pi(\varphi) \neq 0$. Then $\pi(\{0, 1\}) = 1$ and, by bijectivity, $\pi(\{0, -1\}) = -1$. The associated point shift θ_π satisfies $\theta_\pi(\varphi) = \theta_1 \varphi$ and $\theta_\pi(\theta_1 \varphi) = \varphi$, hence,

$$\theta_\pi(\mathbb{P}^0) = 2\delta_\varphi + \delta_{\theta_1 \varphi}.$$

We conclude that $\mathbb{P}^0$ is not invariant under bijective point shifts.

Taking into account Example 6.3.3, we propose the following transformation for σ-finite measures on $(\mathbf{M}_{\mathrm{d}}(G), \mathcal{M}_{\mathrm{d}}(G))$.

Definition 6.3.4. For a σ-finite measure $\mathbb{Q}$ on $(\mathbf{M}_{\mathrm{d}}(G), \mathcal{M}_{\mathrm{d}}(G))$ we define the *zero compensated version* $\zeta(\mathbb{Q})$ by

$$\zeta(\mathbb{Q})(A) := \mathbb{Q}(A \setminus \mathbf{M}_{\mathrm{d},0}(G)) + \int_{A \cap \mathbf{M}_{\mathrm{d},0}(G)} \varphi(\{0\})^{-1} \mathbb{Q}(\mathrm{d}\varphi), \quad A \in \mathcal{M}_{\mathrm{d}}(G).$$

Lemma 6.3.5. *Let $\mathbb{Q}$ be a σ-finite measure on $(\mathbf{M}_{\mathrm{d}}(G), \mathcal{M}_{\mathrm{d}}(G))$. Then the zero compensated version $\zeta(\mathbb{Q})$ is also σ-finite. Moreover, if $\mathbb{Q}$ is concentrated on $\mathbf{M}_{\mathrm{d},0}(G)$ then $\zeta(\mathbb{Q})$ is also concentrated $\mathbf{M}_{\mathrm{d},0}(G)$.*

PROOF: Both claims follow from the fact that $\zeta(\mathbb{Q})$ has the positive density

$$f(\varphi) := \mathbf{1}\{\varphi \in \mathbf{M}_{\mathrm{d}}(G) \setminus \mathbf{M}_{\mathrm{d},0}(G)\} + \varphi(\{0\})^{-1} \mathbf{1}\{\varphi \in \mathbf{M}_{\mathrm{d},0}(G)\}$$

with respect to $\mathbb{Q}$. □

Proposition 6.3.6. *A measure $\mathbb{Q}$ on $(\mathbf{M}_{\mathrm{d}}(G), \mathcal{M}_{\mathrm{d}}(G))$ is invariant under zero compensation if and only if*

$$\mathbb{Q}(\{\varphi \in \mathbf{M}_{\mathrm{d}}(G) : \varphi(\{0\}) \notin \{0, 1\}\}) = 0.$$

PROOF: If $\mathbb{Q}$ is concentrated on $\{\varphi \in \mathbf{M}^d : \varphi(\{0\}) \in \{0,1\}\}$, then it is clearly invariant under ζ. Otherwise, define $A_+ := \{\varphi \in \mathbf{M}^d : \varphi(\{0\}) > 1\}$ and $A_- := \{\varphi \in \mathbf{M}^d : 0 < \varphi(\{0\}) < 1\}$. If $\mathbb{Q}(A_+) > 0$ then $\zeta(\mathbb{Q})(A_+) < \mathbb{Q}(A_+)$, and, if $\mathbb{Q}(A_-) > 0$ then $\zeta(\mathbb{Q})(A_-) > \mathbb{Q}(A_-)$. We conclude that $\mathbb{Q}$ is not invariant under ζ, whenever $\mathbb{Q}(A_+ \cup A_-) \neq 0$. □

Let us now briefly return to Example 6.3.3. The zero-compensated version of the Palm measure $\mathbb{P}^0$ is given by

$$\zeta(\mathbb{P}^0) = \delta_\varphi + \delta_{\theta_1 \varphi}.$$

It is invariant under the bijective point shift θ_π. Indeed, generalizing Theorem 6.2.2, we will now show that the zero-compensated version of a Palm measure is always invariant under bijective point shifts. This fact can also be derived from Satz 4.3 in [17], where invariance properties of Palm measures are studied. Moreover, we also generalize Theorem 6.2.3 and show that invariance of the zero-compensated version under all bijective point shifts is a characterizing property of Palm measures.

Theorem 6.3.7. *A measure $\mathbb{Q}$ on $(\mathbf{M}_{\mathrm{d}}(G), \mathcal{M}_{\mathrm{d}}(G))$ is the Palm measure of some stationary σ-finite measure $\mathbb{P}$ if and only if $\mathbb{Q}$ is σ-finite and its zero-compensated version $\zeta(\mathbb{Q})$ is point-stationary.*

PROOF: First assume that $\mathbb{Q}$ is the Palm measure of some stationary, σ-finite measure $\mathbb{P}$. Then $\mathbb{Q}$ is also σ-finite. From Lemma 4.1.3 and Lemma 6.3.5, we obtain that $\zeta(\mathbb{Q})(\mathbf{M}_{\mathrm{d}} \setminus$

$\mathbf{M}_{\mathrm{d},0}) = 0$. Let π be a bijective point map, $f \in \mathbf{F}(\mathbf{M}_{\mathrm{d}}(G))$ and $g \in \mathbf{F}(G)$ such that $\int g(x)\mathrm{d}x = 1$. An adaptation of the proof of Theorem 6.2.2 then yields

$$\begin{aligned}
&\int_{\mathbf{M}_{\mathrm{d}}(G)} \int_G f(\varphi)\theta_\pi \circ \zeta(\mathbb{Q})(\mathrm{d}\varphi) \\
&= \int_{\mathbf{M}_{\mathrm{d}}(G)} \int_G f(\theta_\pi(\varphi))\zeta(\mathbb{Q})(\mathrm{d}\varphi) \\
&= \int_{\mathbf{M}_{\mathrm{d}}(G)} \int_G g(\pi(\varphi)+x) f(\theta_{\pi(\varphi)}(\varphi)\varphi(\{0\})^{-1}\mathrm{d}x\mathbb{Q}(\mathrm{d}\varphi) \\
&= \int_{\mathbf{M}_{\mathrm{d}}(G)} \int_G g(\pi(\theta_x\varphi)+x) f(\theta_{\pi(\theta_x\varphi)}(\theta_x\varphi)\theta_x\varphi(\{0\})^{-1}\varphi(\mathrm{d}x)\mathbb{P}(\mathrm{d}\varphi) \\
&= \int_{\mathbf{M}_{\mathrm{d}}(G)} \sum_{x\in\mathrm{supp}(\varphi)} g(\pi(\theta_x\varphi)+x) f(\theta_{\pi(\theta_x\varphi)+x}\varphi)\mathbb{P}(\mathrm{d}\varphi) \\
&= \int_{\mathbf{M}_{\mathrm{d}}(G)} \sum_{x\in\mathrm{supp}(\varphi)} g(x) f(\theta_x\varphi)\mathbb{P}(\mathrm{d}\varphi),
\end{aligned}$$

and from a similar computation in the reverse direction we obtain

$$\begin{aligned}
&\int_{\mathbf{M}_{\mathrm{d}}(G)} \sum_{x\in\mathrm{supp}(\varphi)} g(x) f(\theta_x\varphi)\mathbb{P}(\mathrm{d}\varphi) \\
&= \int_{\mathbf{M}_{\mathrm{d}}(G)} \int_G g(x) f(\theta_x\varphi)\varphi(\{x\})^{-1}\varphi(\mathrm{d}x)\mathbb{P}(\mathrm{d}\varphi) \\
&= \int_{\mathbf{M}_{\mathrm{d}}(G)} f(\varphi)\varphi(\{0\})^{-1}\mathbb{Q}(\mathrm{d}\varphi) \\
&= \int_{\mathbf{M}_{\mathrm{d}}(G)} f(\varphi)\zeta(\mathbb{Q})(\mathrm{d}\varphi).
\end{aligned}$$

We have shown that $\zeta(\mathbb{Q})$ is point-stationary. Now we suppose that $\mathbb{Q}$ is σ-finite and $\zeta(\mathbb{Q})$ point-stationary. Then

$$\mathbb{Q}(\mathbf{M}_{\mathrm{d}}(G) \setminus \mathbf{M}_{\mathrm{d},0}(G)) = \zeta(\mathbb{Q})(\mathbf{M}_{\mathrm{d}}(G) \setminus \mathbf{M}_{\mathrm{d},0}(G)) = 0,$$

and $\zeta(\mathbb{Q})$ is invariant under bijective point shifts. By Mecke's characterization theorem (cf. Theorem 4.2.2), we have to show that, for any measurable $f \in \mathbf{F}(\mathbf{M}_{\mathrm{d}}(G) \times G)$, we have

$$\int_{\mathbf{M}_{\mathrm{d}}(G)} \int_G f(\theta_x\varphi, -x)\varphi(\mathrm{d}x)\mathbb{Q}(\mathrm{d}\varphi) = \int_{\mathbf{M}_{\mathrm{d}}(G)} \int_G f(\varphi, x)\varphi(\mathrm{d}x)\mathbb{Q}(\mathrm{d}\varphi). \tag{6.4}$$

Fix $f \in \mathbf{F}(\mathbf{M}_{\mathrm{d}}(G) \times G)$, then invariance of $\zeta(\mathbb{Q})$ under the bijective point shift θ_π associated with a matching π yields

$$\begin{aligned}
&\int_{\mathbf{M}_{\mathrm{d}}(G)} \mathbf{1}\{\theta_\pi(\varphi) \neq \varphi\} f(\varphi, \pi(\varphi))\varphi(\{\pi(\varphi)\})\mathbb{Q}(\mathrm{d}\varphi) \\
&= \int_{\mathbf{M}_{\mathrm{d}}(G)} \mathbf{1}\{\theta_\pi(\varphi) \neq \varphi\} f(\varphi, \pi(\varphi))\varphi(\{\pi(\varphi)\})\varphi(\{0\})\zeta(\mathbb{Q})(\mathrm{d}\varphi) \\
&= \int_{\mathbf{M}_{\mathrm{d}}(G)} \mathbf{1}\{\varphi \neq \theta_\pi(\varphi)\} f(\theta_\pi(\varphi), -\pi(\varphi))\theta_\pi(\varphi)(\{-\pi(\varphi)\})\theta_\pi(\varphi)(\{0\})\zeta(\mathbb{Q})(\mathrm{d}\varphi) \\
&= \int_{\mathbf{M}_{\mathrm{d}}(G)} \mathbf{1}\{\varphi \neq \theta_\pi(\varphi)\} f(\theta_\pi(\varphi), -\pi(\varphi))\varphi(\{0\})\varphi(\{\pi(\varphi)\})\zeta(\mathbb{Q})(\mathrm{d}\varphi) \\
&= \int_{\mathbf{M}_{\mathrm{d}}(G)} \mathbf{1}\{\varphi \neq \theta_\pi(\varphi)\} f(\theta_\pi(\varphi), -\pi(\varphi))\varphi(\{\pi(\varphi)\})\mathbb{Q}(\mathrm{d}\varphi).
\end{aligned}$$

By Theorem 3.2.6, there exists a countable family of matchings $\{\pi_n : n \in \mathbb{N}\}$ on $\mathbf{L}(G)$ that is quasi-complete $\mathbf{L}_0(G)$. The above equation applied to the matchings of the family yields

$$\begin{aligned}
&\int_{\mathbf{M}_{\mathrm{d}}(G)} \int_G \mathbf{1}\{\theta_x\varphi \neq \varphi\} f(\varphi, x)\varphi(\mathrm{d}x)\mathbb{Q}(\mathrm{d}\varphi) \\
&= \int_{\mathbf{M}_{\mathrm{d}}(G)} \sum_{n\in\mathbb{N}} \mathbf{1}\{\pi_n(\varphi) \neq 0\}\, \mathbf{1}\{\pi_n(\varphi) \neq \pi_m(\varphi) : 1 \leq m < n\} f(\varphi, \pi_n(\varphi))\varphi(\pi_n(\varphi))\mathbb{Q}(\mathrm{d}\varphi) \\
&= \int_{\mathbf{M}_{\mathrm{d}}(G)} \sum_{n\in\mathbb{N}} \mathbf{1}\{\pi_n(\varphi) \neq 0\}\, \mathbf{1}\{\pi_n(\varphi) \neq \pi_m(\varphi) : 1 \leq m < n\} f(\theta_{\pi_n}\varphi, -\pi_n(\varphi))\theta_{\pi_n}\varphi(0)\mathbb{Q}(\mathrm{d}\varphi) \\
&= \int_{\mathbf{M}_{\mathrm{d}}(G)} \int_G \mathbf{1}\{\theta_x\varphi \neq \varphi\} f(\theta_x\varphi, -x)\varphi(\mathrm{d}x)\mathbb{Q}(\mathrm{d}\varphi),
\end{aligned}$$

where the same reasoning as after (6.3) justifies that $\pi_n(\varphi) \neq \pi_m(\varphi)$ if and only if $\pi_n(\theta_{\pi_n}\varphi) \neq \pi_m(\theta_{\pi_n}\varphi)$. Since, for $\varphi \in \mathbf{M}_{\mathrm{d}}(G)$ and $x \in G$ such that $\theta_x\varphi = \varphi$, we have $\theta_{-x}\varphi = \varphi$ and, in particular, $\varphi(\{-x\} = \varphi(\{x\})$, we obtain

$$\begin{aligned}
&\int_{\mathbf{M}_{\mathrm{d}}(G)} \int_G \mathbf{1}\{\theta_x\varphi = \varphi\} f(\varphi, x)\varphi(\mathrm{d}x)\mathbb{Q}(\mathrm{d}\varphi) \\
&= \int_{\mathbf{M}_{\mathrm{d}}(G)} \int_G \mathbf{1}\{\theta_x\varphi = \varphi\} f(\theta_{-x}\varphi, x)\varphi(\mathrm{d}x)\mathbb{Q}(\mathrm{d}\varphi) \\
&= \int_{\mathbf{M}_{\mathrm{d}}(G)} \int_G \mathbf{1}\{\theta_x\varphi = \varphi\} f(\theta_x\varphi, -x)\varphi(\mathrm{d}x)\mathbb{Q}(\mathrm{d}\varphi).
\end{aligned}$$

This implies (6.4), concluding the proof of the theorem. □

6.4 Discretisation of general random measures

The generalization of the point shift characterization to Palm measures of general random measures appears to be a difficult problem. In particular, only little is known about bijective point maps that are defined on (subclasses different from $\mathbf{L}(G)$ of) the closed subsets $\mathbf{C}(G)$ of G.

Open problem 4. What families of bijective point maps can be defined on the closed subsets of G? Are there complete (in a sense which is to be defined!) families of point maps on $\mathbf{C}_0(G)$, i.e. the closed subsets of G that contain 0?

To circumvent these difficulties for random measures on $\mathbb{R}^d$, we will propose here a discretization procedure. Denote by $\mathbb{D}_n$ the subgroup of $(\mathbb{R}^d, +)$ of the dyadic numbers of order $n \in \mathbb{N}$, i.e.,

$$\mathbb{D}_n = \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_i = k_i/2^n, (k_1, \dots, k_d) \in \mathbb{Z}^d\}.$$

Equipped with the discrete topology, $(\mathbb{D}_n, +)$ is a lcscH group. We denote by $C_n := [-2^{-n-1}, 2^{-n-1})^d$ the product of d half-open intervals $[-2^{-n-1}, 2^{-n-1})$ and by $d_n : \mathbb{R}^d \to \mathbb{D}_n$ the mapping that sends $x \in \mathbb{R}^d$ to the (unique) point in $\mathbb{D}_n$ such that $x \in C_n + d_n(x)$. Conversely, we denote by $i_n : \mathbb{D}_n \to \mathbb{R}^d$ the embedding of $\mathbb{D}_n$ into $\mathbb{R}^d$.

There exists a translation invariant measure λ_n on $\mathbb{D}_n$, which is unique up to a constant. We normalize λ_n in such a way, that $\lambda_n(\mathbb{D}_n \cap [0,1)^d) = 1$. With this normalization, $\lambda(C_n)^{-1}\lambda_n$ is the counting measure on $\mathbb{D}_n$. Then, for $n \in \mathbb{N}$, define a discretisation operator D_n, and an embedding operator I_n as follows.

Definition 6.4.1. The mapping $D_n : \mathbf{M}(\mathbb{R}^d) \to \mathbf{M}(\mathbb{D}_n)$ defined by

$$D_n(\mu) := d_n(\mu) = \sum_{x \in \mathbb{D}_n} \mu(C_n + x)\delta_x,$$

is called a *dyadic lattice discretisation operator* of order n. Conversely, an embedding $I_n : \mathbf{M}(\mathbb{D}_n) \to \mathbf{M}(\mathbb{R}^d)$ is defined by

$$I_n(\mu) := i_n(\mu) = \sum_{x \in \mathbb{D}_n} \mu(\{x\})\delta_{i_n(x)}.$$

We have $D_n(\lambda^d) = \lambda_n$ and $\lambda_n \xrightarrow{v} \lambda^d$ for $n \to \infty$. More generally, the operators D_n and I_n have the following properties.

Lemma 6.4.2. *The composition $D_n \circ I_n$ is the identity mapping on $\mathbf{M}(\mathbb{D}_n)$. Conversely, for $\varphi \in \mathbf{M}(\mathbb{R}^d)$, the sequence of measures $(I_n \circ D_n(\varphi))$ converges vaguely towards φ, i.e., $I_n \circ D_n(\varphi) \xrightarrow{v} \varphi$ for $n \to \infty$.*

Proof: The first claim follows immediately from the definitions of I_n and D_n. The second is a consequence of the fact that continuous functions with compact support on $\mathbb{R}^d$ are Riemann integrable. □

Proposition 6.4.3. *Let $\mathbb{P}$ be a σ-finite measure on $(\mathbf{M}(\mathbb{R}^d), \mathcal{M}(\mathbb{R}^d))$.*

(a) *If $\mathbb{P}$ is stationary, then $D_n(\mathbb{P})$ is also stationary, i.e., $\theta_x \circ D_n(\mathbb{P}) = D_n(\mathbb{P})$ for all $x \in \mathbb{D}_n$.*

(b) *If $\mathbb{P}(\mathbf{M}(\mathbb{R}^d)) < \infty$, then the sequence $(I_n \circ D_n(\mathbb{P}))_{n \in \mathbb{N}}$ converges weakly towards $\mathbb{P}$.*

PROOF: To show (a), we fix $x \in \mathbb{D}_n$. Then $\theta_x \circ D_n = D_n \circ \theta_{i_n(x)}$, and, hence, $\theta_x \circ D_n(\mathbb{P}) = D_n(\mathbb{P})$. For (b), we fix a bounded and continuous function $f : \mathbf{M}(\mathbb{R}^d) \to \mathbb{R}$. Then

$$\int_{\mathbf{M}(\mathbb{R}^d)} f(\mu) I_n \circ D_n(\mathbb{P})(\mathrm{d}\mu) = \int_{\mathbf{M}(\mathbb{R}^d)} f(I_n \circ D_n(\mu))\mathbb{P}(\mathrm{d}\mu).$$

By Lemma 6.4.2, we have vague convergence $I_n \circ D_n(\mu) \xrightarrow{v} \mu$ and the continuity of f yields $f(I_n \circ D_n(\mu)) \xrightarrow{n\to\infty} f(\mu)$. Using the dominated convergence theorem we conclude that

$$\int_{\mathbf{M}(\mathbb{R}^d)} f(\mu) I_n \circ D_n(\mathbb{P})(\mathrm{d}\mu) \to \int_{\mathbf{M}(\mathbb{R}^d)} f(\mu)\mathbb{P}(\mathrm{d}\mu),$$

finishing the proof of the proposition. □

Lemma 6.4.4. *Let (x_n) be a converging sequence in $\mathbb{R}^d$ with limit $x \in \mathbb{R}^d$.*

(a) *For $\varphi \in \mathbf{M}(\mathbb{R}^d)$, the sequence of measures $\theta_{x_n}\varphi$ converges vaguely towards φ, i.e. $\theta_{x_n}(\varphi) \xrightarrow{v} \varphi$ for $n \to \infty$.*

(b) *For a finite measure $\mathbb{P}$ on $\mathbf{M}(\mathbb{R}^d)$, the sequence $(\theta_{x_n}(\mathbb{P}))$ converges weakly towards $\mathbb{P}$.*

PROOF: Part (a) is a corollary of the continuity result proved in Proposition 2.2.2. The proof of (b) is identical to the proof of part (b) in Proposition 6.4.3. □

Let $\mathbb{P}$ be a stationary measure on $\mathbf{M}(\mathbb{R}^d)$. Then, by Proposition 6.4.3 (b), the measure $D_n(\mathbb{P})$ is a stationary measure on $\mathbf{M}(\mathbb{D}_n)$, so it is natural to ask for the existence of an associated Palm measure and its connections with the Palm measure associated with $\mathbb{P}$. However, as we have seen in Section 4, σ-finiteness of the stationary measure is a necessary condition for the definition of the associated Palm measure. We will see in the following example that σ-finiteness is a property that is, in general, not preserved by the discretisation operators.

Example 6.4.5. For $c \in [0,1]$ we define a density function $f_c : \mathbb{R} \to [0,\infty)$ by $f_c(x) = c + (1-c)2(2x - \lfloor 2x \rfloor)$, where $\lfloor \cdot \rfloor$ denotes the Gaussian brackets, that map a real number x to the biggest integer that is smaller than x. Then we define locally finite measures μ_c on $\mathbb{R}$ by $\mu_c := f_c \cdot \lambda^1$, where λ^1 denotes the Lebesgue measure on $\mathbb{R}$. A σ-finite measure $\mathbb{P}$ on $(\mathbf{M}(\mathbb{R}), \mathcal{M}(\mathbb{R}))$ is then defined by

$$\mathbb{P}(A) := \int_{[0,1]} \mathbf{1}\{\mu_c \in A\} c^{-1} \lambda^1(\mathrm{d}c), \quad A \in \mathcal{M}(\mathbb{R}).$$

However, since $D_1(\mu_c) = \lambda_1$ for all $c \in [0,1]$, the measure $D_1(\mathbb{P})$ is not σ-finite, but the degenerate measure $D_1(\mathbb{P})(\cdot) = \infty \cdot \delta_{\lambda_1}(\cdot)$.

Proposition 6.4.6. *Let $\mathbb{P}$ be a σ-finite, stationary measure on $(\mathbf{M}(\mathbb{R}^d), \mathcal{M}(\mathbb{R}^d))$ and $\mathbb{P}^0$ the corresponding Palm measure. Define a measure $\mathbb{Q}_n$ by*

$$\mathbb{Q}_n(A) := \frac{1}{\lambda^d(C_n)} \int_{C_n} D_n \circ \theta_{-z}(\mathbb{P}^0)(A)\lambda^d(\mathrm{d}z), \quad A \in \mathcal{M}(\mathbb{D}_n).$$

Then we have

$$\int_{\mathbf{M}(\mathbb{D}_n)} \int_{\mathbb{D}_n} f(\theta_x\mu, x)\mu(\mathrm{d}x)D_n(\mathbb{P})(\mathrm{d}\mu) = \int_{\mathbf{M}(\mathbb{D}_n)} \int_{\mathbb{D}_n} f(\mu, y)\lambda_n(\mathrm{d}y)\mathbb{Q}_n(\mathrm{d}\mu) \tag{6.5}$$

for all $f \in \mathbf{F}(\mathbf{M}(\mathbb{D}_n) \times \mathbb{D}_n)$ *and if* $D_n(\mathbb{P})$ *is* σ*-finite then* $\mathbb{Q}_n$ *is the associated Palm measure on* $(\mathbf{M}(\mathbb{D}_n), \mathcal{M}(\mathbb{D}_n))$*.*

Proof: Let us first show that (6.5) is satisfied. For $f \in \mathbf{F}(\mathbf{M}(\mathbb{D}_n) \times \mathbb{D}_n)$, we have

$$\begin{aligned}
&\int_{\mathbf{M}(\mathbb{D}_n)} \int_{\mathbb{D}_n} f(\theta_x\mu, x)\mu(\mathrm{d}x)D_n(\mathbb{P})(\mathrm{d}\mu) \\
&= \int_{\mathbf{M}(\mathbb{R}^d)} \int_{\mathbb{D}_n} f(\theta_x \circ D_n(\mu), x)D_n(\mu)(\mathrm{d}x)\mathbb{P}(\mathrm{d}\mu) \\
&= \int_{\mathbf{M}(\mathbb{R}^d)} \int_{\mathbb{R}^d} f(\theta_{d_n(x)} \circ D_n(\mu), d_n(x))\mu(\mathrm{d}x)\mathbb{P}(\mathrm{d}\mu) \\
&= \int_{\mathbf{M}(\mathbb{R}^d)} \int_{\mathbb{R}^d} f(D_n \circ \theta_{i_n\circ d_n(x)}(\mu), d_n(x))\mu(\mathrm{d}x)\mathbb{P}(\mathrm{d}\mu) \\
&= \int_{\mathbf{M}(\mathbb{R}^d)} \int_{\mathbb{R}^d} f(D_n \circ \theta_{i_n\circ d_n(x)-x}(\mu), d_n(x))\lambda^d(\mathrm{d}x)\mathbb{P}^0(\mathrm{d}\mu) \\
&= \int_{\mathbf{M}(\mathbb{R}^d)} \sum_{y\in\mathbb{D}_n} \left(\int_{C_n} f(D_n \circ \theta_{-z}(\mu), y)\lambda^d(\mathrm{d}z)\right) \mathbb{P}^0(\mathrm{d}\mu) \\
&= \int_{C_n} \int_{\mathbf{M}(\mathbb{R}^d)} \sum_{y\in\mathbb{D}_n} f(D_n \circ \theta_{-z}(\mu), y)\mathbb{P}^0(\mathrm{d}\mu)\lambda^d(\mathrm{d}z) \\
&= \frac{1}{\lambda^d(C_n)} \int_{C_n} \int_{\mathbf{M}(\mathbb{D}_n)} \int_{\mathbb{D}_n} f(\mu, y)\lambda_n(\mathrm{d}y)D_n \circ \theta_{-z}(\mathbb{P}^0)(\mathrm{d}\mu)\lambda^d(\mathrm{d}z) \\
&= \int_{\mathbf{M}(\mathbb{D}_n)} \int_{\mathbb{D}_n} f(\mu, y)\lambda_n(\mathrm{d}y)\mathbb{Q}_n(\mathrm{d}\mu),
\end{aligned}$$

where we have used the refined Campbell formula (4.2) for the forth equation. If $D_n(\mathbb{P})$ is σ-finite, then Theorem 4.2.1 yields that $D_n(\mathbb{P})$ is a stationary measure with Palm measure $\mathbb{Q}_n$. This concludes the proof of the theorem. □

In Theorem 4.1.4, we have seen how (the restriction of) the stationary measure $\mathbb{P}$ (to $\mathbf{M}(G) \setminus \{\mathbf{0}\}$) can be retrieved from the Palm measure $\mathbb{P}^0$. We will now make the choice of the function h on $\mathbb{R}^d \times \mathbf{M}(\mathbb{R}^d)$ more explicit by fixing an enumeration (u_n) of the elements of $\mathbb{Z}^d$ and defining $G_n := u_n + C_1$. Clearly, (G_n) is a partition of $\mathbb{R}^d$ into relatively compact sets. As before in (4.7), (4.8) and (4.9), we define a function $h : \mathbf{M}(\mathbb{R}^d) \times \mathbb{R}^d \to [0, \infty]$.

Lemma 6.4.7. *The function h satisfies*

$$h(\mu, x) = h(\mu, i_n \circ d_n(x)) = h(I_n \circ D_n(\mu), i_n \circ d_n(x)) \tag{6.6}$$

for all $\mu \in \mathbf{M}(\mathbb{R}^d)$ *and* $x \in \mathbb{R}^d$. *Moreover, for* $n \in \mathbb{N}$, *the function* $h_n : \mathbf{M}(\mathbb{D}_n) \times \mathbb{D}_n \to [0, \infty]$ *defined by* $h_n(\mu, x) := h(I_n(\mu), i_n(x))$ *satisfies*

$$\int_{\mathbb{D}_n} h_n(\mu, x)\mu(\mathrm{d}x) = 1$$

for all $\mu \in \mathbf{M}(\mathbb{D}_n)$.

Proof: First we prove (6.6). Indeed, for $\bar{h}$ defined in (4.7) and $\mu \neq 0$, we have

$$\begin{aligned}
\bar{h}(\mu, x) &= \sum_{n\in\mathbb{N}} 2^{-n}(\mu(G_n))^{-1}\,\mathbf{1}\{x \in G_n\} \\
&= \sum_{n\in\mathbb{N}} 2^{-n}(\mu(G_n))^{-1}\,\mathbf{1}\{i_n \circ d_n(x) \in G_n\} = \bar{h}(\mu, i_n \circ d_n(x)) \\
&= \sum_{n\in\mathbb{N}} 2^{-n}(I_n \circ D_n(\mu)(G_n))^{-1}\,\mathbf{1}\{i_n \circ b_n(x) \in G_n\} = \bar{h}(I_n \circ D_n(\mu), i_n \circ d_n(x)).
\end{aligned}$$

The above equality yields

$$\int_{\mathbb{R}^d} \bar{h}(I_n \circ D_n(\mu), y) I_n \circ D_n(\mu)(\mathrm{d}y) = \int_{\mathbb{R}^d} \bar{h}(I_n \circ D_n(\mu), i_n \circ d_n(y))(\mathrm{d}y) = \int_{\mathbb{R}^d} \bar{h}(\mu, y)\mu(\mathrm{d}y),$$

so, using the definition of h from (4.9), the first claim is proved. For $\mu \in \mathbf{M}(\mathbb{D}_n)$, we have

$$\int_{\mathbb{D}_n} h_n(\mu, x)\mu(\mathrm{d}x) = \int_{\mathbb{D}_n} h(I_n(\mu), i_n(x))\mu(\mathrm{d}x) = \int_{\mathbb{R}^d} h(I_n(\mu), x) I_n(\mu)(\mathrm{d}x) = 1,$$

so the lemma is proved. □

We are now prepared to state the main theorem of this section.

Theorem 6.4.8. *Let* $\mathbb{Q}$ *be a measure on* $(\mathbf{M}(\mathbb{R}^d), \mathcal{M}(\mathbb{R}^d))$ *and assume that*

$$\int_{\mathbf{M}(\mathbb{R}^d)} \int_{\mathbb{R}^d} h(\theta_{-x}\mu, x)\mathrm{d}x\mathbb{Q}(\mathrm{d}\mu) < \infty. \tag{6.7}$$

Then $\mathbb{Q}$ *is a Palm measure on* $(\mathbf{M}(\mathbb{R}^d), \mathcal{M}(\mathbb{R}^d))$ *if and only if, for all* $n \in \mathbb{N}$, *the zero-compensated version of the measure* $\mathbb{Q}_n$ *defined by*

$$\mathbb{Q}_n(A) := \frac{1}{\lambda^d(C_n)} \int_{C_n} D_n \circ \theta_{-z}(\mathbb{Q})(A)\lambda^d(\mathrm{d}z), \quad A \in \mathcal{M}(\mathbb{D}_n)$$

is point-stationary.

Proof: It follows from Proposition 6.4.6, that the condition of the theorem is necessary. Let us now show that it is sufficient.

Let us first fix $n \in \mathbb{N}$. Using the characterization of Palm measures on lcscH groups given in Theorem 6.3.7, we will show that $\mathbb{Q}_n$ is a Palm measure on $(\mathbf{M}(\mathbb{D}_n), \mathcal{M}(\mathbb{D}_n))$. We have to show that $\mathbb{Q}_n$ is σ-finite. Indeed, the function $u_n : \mathbf{M}(\mathbb{D}_n) \to [0, \infty)$ defined by

$$u_n(\mu) := \int_{\mathbb{D}_n} h_n(\theta_{-x}\mu, x)\lambda_n(\mathrm{d}x)$$

is strictly positive on $\mathbf{M}(\mathbb{D}_n) \setminus \{\mathbf{0}\}$, so from (6.7) we obtain

$$\begin{aligned}
&\int_{\mathbf{M}(\mathbb{D}_n)} u_n(\mu)\mathbb{Q}_n(\mathrm{d}\mu) \\
&= \frac{1}{\lambda^d(C_n)} \int_{C_n} \int_{\mathbf{M}(\mathbb{D}_n)} \int_{\mathbb{D}_n} h_n(\theta_{-x}\mu, x)\lambda_n(\mathrm{d}x) D_n \circ \theta_{-z}(\mathbb{Q})(\mathrm{d}\mu)\lambda^d(\mathrm{d}z) \\
&= \frac{1}{\lambda^d(C_n)} \int_{C_n} \int_{\mathbf{M}(\mathbb{D}_n)} \int_{\mathbb{D}_n} h(\theta_{-i_n(x)}(I_n(\mu)), i_n(x))\lambda_n(\mathrm{d}x) D_n \circ \theta_{-z}(\mathbb{Q})(\mathrm{d}\mu)\lambda^d(\mathrm{d}z) \\
&= \frac{1}{\lambda^d(C_n)} \int_{C_n} \int_{\mathbf{M}(\mathbb{R}^d)} \int_{\mathbb{R}^d} h(\theta_{-x} \circ I_n \circ D_n \circ \theta_{-z}(\mu), x) I_n(\lambda_n)(\mathrm{d}x)\mathbb{Q}(\mathrm{d}\mu)\lambda^d(\mathrm{d}z) \\
&= \frac{1}{\lambda^d(C_n)} \int_{\mathbf{M}(\mathbb{R}^d)} \int_{C_n} \int_{\mathbb{R}^d} h(I_n \circ D_n \circ \theta_{-x-z}(\mu), x) I_n(\lambda_n)(\mathrm{d}x)\lambda^d(\mathrm{d}z)\mathbb{Q}(\mathrm{d}\mu) \\
&= \int_{\mathbf{M}(\mathbb{R}^d)} \int_{C_n} \sum_{x \in \mathbb{D}_n} h(I_n \circ D_n \circ \theta_{-i_n(x)-z}(\mu), i_n(x) + z)\lambda^d(\mathrm{d}z)\mathbb{Q}(\mathrm{d}\mu) \\
&= \int_{\mathbf{M}(\mathbb{R}^d)} \int_{\mathbb{R}^d} h(\theta_{-y}(\mu), y)\lambda^d(\mathrm{d}y)\mathbb{Q}(\mathrm{d}\mu) < \infty,
\end{aligned}$$

where we have used (6.6) for the penultimate equality. Hence, there exists a σ-finite, stationary measure $\mathbb{P}_n$ on $(\mathbf{M}(\mathbb{D}_n), \mathcal{M}(\mathbb{D}_n))$ such that $\mathbb{Q}_n$ is the Palm measure associated with $\mathbb{P}_n$. In particular, the inversion formula (cf. Theorem 4.1.4) yields

$$\int_{\mathbf{M}(\mathbb{D}_n)} f(\mu)\mathbb{P}_n(\mathrm{d}\mu) = \int_{\mathbf{M}(\mathbb{D}_n)} \int_{\mathbb{D}_n} h_n(\theta_{-x}\mu, x) f(\theta_{-x}\mu)\lambda_n(\mathrm{d}x)\mathbb{Q}_n(\mathrm{d}\mu), \tag{6.8}$$

and from the special case $f \equiv 1$ we deduce that all $\mathbb{P}_n, n \in \mathbb{N}$, are finite measures. Let us now show that $(I_n(\mathbb{P}_n))$ is a weakly convergent sequence of finite measures, and that the limit measure is given by

$$\mathbb{P}(A) := \int \int h(\theta_{-x}\mu, x) \, \mathbf{1}\{\theta_{-x}\mu \in A\} \mathrm{d}x \mathbb{Q}(\mathrm{d}\mu), \quad A \in \mathcal{M}(\mathbb{R}^d). \tag{6.9}$$

For an arbitrary measurable and bounded function $f : \mathbf{M}(\mathbb{R}^d) \to \mathbb{R}$, we have

$$\begin{aligned}
&\int_{\mathbf{M}(\mathbb{R}^d)} f(\mu) I_n(\mathbb{P}_n)(\mathrm{d}\mu) \\
&= \int_{\mathbf{M}(\mathbb{D}_n)} f(I_n(\mu))\mathbb{P}_n(\mathrm{d}\mu) \\
&= \int_{\mathbf{M}(\mathbb{D}_n)} \int_{\mathbb{D}_n} h_n(\theta_{-x}\mu, x) f(I_n \circ \theta_{-x}(\mu))\lambda_n(\mathrm{d}x)\mathbb{Q}_n(\mathrm{d}\mu) \\
&= \frac{1}{\lambda^d(C_n)} \int_{C_n} \int_{\mathbf{M}(\mathbb{D}_n)} \int_{\mathbb{D}_n} h_n(\theta_{-x}\mu, x) f(I_n \circ \theta_{-x}(\mu))\lambda_n(\mathrm{d}x) D_n \circ \theta_{-z}(\mathbb{Q})(\mathrm{d}\mu)\lambda^d(\mathrm{d}z) \\
&= \frac{1}{\lambda^d(C_n)} \int_{C_n} \int_{\mathbf{M}(\mathbb{R}^d)} \int_{\mathbb{D}_n} h(I_n \circ \theta_{-x} \circ D_n \circ \theta_{-z}(\mu), i_n(x)) \\
&\qquad\qquad f(I_n \circ \theta_{-x} \circ D_n \circ \theta_{-z}(\mu))\lambda_n(\mathrm{d}x)\mathbb{Q}(\mathrm{d}\mu)\lambda^d(\mathrm{d}z) \\
&= \frac{1}{\lambda^d(C_n)} \int_{C_n} \int_{\mathbf{M}(\mathbb{R}^d)} \int_{\mathbb{D}_n} h(I_n \circ D_n \circ \theta_{i_n(-x)-z}(\mu), i_n(x)) \\
&\qquad\qquad f(I_n \circ D_n \circ \theta_{i_n(-x)-z}(\mu))\lambda_n(\mathrm{d}x)\mathbb{Q}(\mathrm{d}\mu)\lambda^d(\mathrm{d}z) \\
&= \int_{\mathbf{M}(\mathbb{R}^d)} \int_{C_n} \sum_{x \in \mathbb{D}_n} h(I_n \circ D_n \circ \theta_{-i_n(x)-z}(\mu), i_n(x) + z) \\
&\qquad\qquad f(I_n \circ D_n \circ \theta_{-i_n(x)-z}(\mu))\lambda^d(\mathrm{d}z)\mathbb{Q}(\mathrm{d}\mu) \\
&= \int_{\mathbf{M}(\mathbb{R}^d)} \int_{\mathbb{R}^d} h(\theta_{-y}(\mu), y) f(I_n \circ D_n \circ \theta_{-y}(\mu))\lambda^d(\mathrm{d}y)\mathbb{Q}(\mathrm{d}\mu).
\end{aligned}$$

Assume now that f is a continuous, bounded function on $\mathbf{M}(\mathbb{R}^d)$. Then, by Proposition 6.4.3, the sequence $f(I_n \circ D_n \circ \theta_{-y}(\mu))$ tends to $f(\theta_{-y}(\mu))$ for $n \to \infty$ and from equation (6.7) we know that the function

$$\mu \mapsto \int h(\theta_{-x}\mu, x)\mathrm{d}x \sup\{|f(\nu)| : \nu \in \mathbf{M}(\mathbb{R})\}$$

is integrable with respect to $\mathbb{Q}$. Hence,

$$\int_{\mathbf{M}(\mathbb{R}^d)} f(\mu) I_n(\mathbb{P}_n)(\mathrm{d}\mu) \overset{n\to\infty}{\longrightarrow} \int_{\mathbf{M}(\mathbb{R}^d)} f(\mu)\mathbb{P}(\mathrm{d}\mu),$$

where $\mathbb{P}$ is defined in (6.9). Finally, for $n \in \mathbb{N}$, the measures $I_m(\mathbb{P}_m), m \geq n$, are invariant under θ_x for all $x \in i_n(\mathbb{D}_n) \subset \mathbb{R}^d$, and we infer that the limit measure $\mathbb{P}$ is also invariant under θ_x for all $x \in i_n(\mathbb{D}_n)$. Clearly, $\bigcup_{n\in\mathbb{N}} i_n(\mathbb{D}_n)$ is dense in $\mathbb{R}^d$. Hence, for an arbitrary $y \in \mathbb{R}^d$, there exists a converging sequence $y_n \in i_n(\mathbb{D}_n), n \in \mathbb{N}$, with limit y, and $\theta_{y_n}(\mathbb{P}) = \mathbb{P}$ and Lemma 6.4.4(b) yield $\theta_y(\mathbb{P}) = \mathbb{P}$. Hence, $\mathbb{P}$ is a stationary measure on $(\mathbf{M}(\mathbb{R}^d), \mathcal{M}(\mathbb{R}^d))$ and $\mathbb{Q}$ the associated Palm measure. $\square$

Appendix A

Appendix

A.1 General topological complements

As a general reference on basic topology we refer the reader to [25], [21] or [3]. Here, we will only review some results which are needed in this thesis. A topological space $(\mathbb{X}, \mathcal{T})$ is called a *T_1-space* if each of its points is a closed set. A T_1-space is called *normal*, if, for any two disjoint closed sets F_1, F_2, there exist disjoint open sets G_1, G_2 such that $F_1 \subset G_1$ and $F_2 \subset G_2$. Every normal space is, in particular, a *Hausdorff space*. The space $\mathbb{X}$ is called *second countable*, if its topology has a countable base. Any metric space is normal and the following theorem is on a converse statement.

Theorem A.1.1. (Urysohn embedding theorem). *If $\mathbb{X}$ is a second countable, normal space, then there exists a homeomorphism f of $\mathbb{X}$ onto a subspace of $\ell^2(\mathbb{R})$, and $\mathbb{X}$ is therefore metrizable.*

PROOF: cf. [25] Section 29, Theorem A.

A topological space $\mathbb{X}$ is called Polish if it is separable with a complete metrization. It is *locally compact* if every point $x \in \mathbb{X}$ has a neighbourhood with compact closure. A locally compact Hausdorff space is normal and we have the following complement to the above theorem.

Theorem A.1.2. *If $\mathbb{X}$ is locally compact, metrizable and second countable then it is Polish.*

PROOF: cf. [21], Satz 13.17.

A topological space $\mathbb{X}$ is called σ-compact if it is a countable union of compact sets. Let us now summarize some properties of locally compact, second countable Hausdorff (*lcscH*) spaces.

Theorem A.1.3. *Let $(\mathbb{X}, \mathcal{T})$ be lcscH space. There exists a countable base of the topology of $\mathbb{X}$ that consists of open sets with compact closure. Moreover, there exist compact sets $K_n, n \in \mathbb{N}$, such that $K_n \subset \operatorname{int}(K_{n+1})$, where $\operatorname{int}(K)$ denotes the interior of K, and $\mathbb{X} = \cup_{n\in\mathbb{N}} K_n$. In particular, $\mathbb{X}$ is σ-compact.*

Proof: Fix an arbitrary countable base (B_n) of the topology $\mathcal{T}$ and show that the subfamily of sets in (B_n) with compact closure is again a base of the topology. Let G be an arbitrary open subset of $\mathbb{X}$ and $x \in G$. Since $\mathbb{X}$ is locally compact, there exists an open neighbourhood N_x of x with compact closure, hence, $G \cap N_x$ is also an open set and $G \cap N_x = \cup_{i \in I_x} B_i$ for some $I_x \subset \mathbb{N}$. All B_i have compact closure, because $B_i \subset N_x, i \in I_x$. Then we can write $G = \cup_{x \in G} \cup_{i \in I_x} B_i$ as the union of sets from (B_n), all of which have compact closure.

Let us now write (B_k) for a base of the topology such that B_k has compact closure for all $k \in \mathbb{N}$. Inductively, the sequence of compact sets $K_n, n \in \mathbb{N}$, is defined as follows. Let $K_1 := \mathrm{cl}(B_1)$. Then assume that the first $n-1$ sets in the sequence are defined. The compact set $K_{n-1} \cup \mathrm{cl}(B_n)$ is included in the union of finitely many of the B_k and we let K_n be the closure of the union of these sets. □

A.2 Vague and weak topology on measure spaces

Following Section A2 in [7] and Section A 2.3 in [1], we will now introduce the notions of vague and weak topology, and discuss some of their properties. Let $(\mathbb{X}, \mathcal{T})$ be a lcscH space with Borel σ-field $\mathcal{X}$, and let $\hat{\mathcal{X}}$ denote the relatively compact sets in $\mathbb{X}$. By Theorem A.1.1 and Theorem A.1.2, $\mathbb{X}$ is Polish. The family $C_K^+(\mathbb{X})$ of continuous functions $f : \mathbb{X} \to [0, \infty)$ with compact support is separable in the uniform metric.

We denote by $\mathbf{M}(\mathbb{X})$ the space of locally finite measures on $\mathbb{X}$, and, for $B \in \mathcal{X}$ and $f \in \mathbf{F}(\mathbb{X})$ (a measurable, non-negative function on $\mathbb{X}$), write π_B and π_f for the mappings $\mu \mapsto \mu(B)$ and $\mu \mapsto \int_G f(x)\mu(\mathrm{d}x)$, respectively, on $\mathbf{M}(\mathbb{X})$. The vague topology is generated by the maps $\pi_f, f \in C_K^+(\mathbb{X})$, and we write the *vague convergence* of μ_n to μ as $\mu_n \xrightarrow{v} \mu$. For any $\mu \in \mathbf{M}(\mathbb{X})$, define $\hat{\mathcal{X}}_\mu := \{B \in \hat{\mathcal{X}} : \mu(\partial B) = 0\}$. We then have the following properties of the vague topology.

Theorem A.2.1. *For any lcscH space* $\mathbb{X}$ *we have*

(a) $\mathbf{M}(\mathbb{X})$ *is Polish in the vague topology;*

(b) *a set* $A \subset \mathbf{M}(\mathbb{X})$ *is vaguely relatively compact if and only if* $\sup\{\mu f : \mu \in A\} < \infty$ *for all* $f \in C_K^+(\mathbb{X})$*;*

(c) *if* $\mu_n \xrightarrow{v} \mu$ *and* $B \in \hat{\mathcal{X}}_\mu$ *then* $\mu_n(B) \to \mu(B)$*;*

(d) *The Borel* σ*-field* $\mathcal{B}(\mathbf{M}(\mathbb{X}))$ *generated by the open sets of the vague topology is generated by the maps* $\pi_f, f \in C_K^+(\mathbb{X})$*, and also for any* $\mu \in \mathbf{M}(\mathbb{X})$ *by the maps* $\pi_B, B \in \hat{\mathcal{S}}_\mu$*.*

Proof: cf. [7], Theorem A 2.3.

Let (f_n) be a dense sequence in $C_K^+(\mathbb{X})$. Then a complete metric ρ on $\mathbf{M}(\mathbb{X})$ is defined by

$$\rho(\mu, \varphi) := \sum_{n \in \mathbb{N}} 2^{-n}(|\pi_{f_n}(\mu) - \pi_{f_n}(\varphi)| \wedge 1), \quad \mu, \nu \in \mathbf{M}(\mathbb{X}), \tag{A.1}$$

and the topology generated by ρ coincides with the vague topology (cf. [7], Theorem A 2.3).

Theorem A.2.2. *Let $\mu_n \in \mathbf{M}(G), n \geq 0$, with $\mu_n \xrightarrow{v} \mu_0$, and let $B_n \in \mathcal{G}, n \geq 0$, such that $\mu_n(G \setminus B_n) = 0, n \geq 0$. Further suppose that $f_n : G \to [0, \infty), n \geq 0$, are uniformly bounded measurable functions with uniformly bounded supports, such that $f_n(y_n) \to f_0(y_0)$ whenever $y_n \in B_n, n \geq 0$, with $y_n \to y_0$. Then $\pi_{f_n}(\mu_n) \to \pi_{f_0}(\mu_0)$.*

Proof: cf. [8], Theorem A 7.3.

Let $\mathbb{Y}$ be a metric space, with Borel σ-field $\mathcal{Y}$, and let $C_b^+(\mathbb{Y})$ denote the family of bounded, continuous functions $f : \mathbb{Y} \to [0, \infty)$. Write $\mathbf{M}_{\mathrm{f}}(\mathbb{Y})$ for the space of finite measures on $\mathbb{Y}$. The weak topology is generated by the maps $\pi_f, f \in C_b^+(\mathbb{Y})$, and we write *weak convergence* of μ_n to μ as $\mu_n \xrightarrow{w} \mu$.

Theorem A.2.3. *If $\mathbb{Y}$ is Polish and $\mathbf{M}_{\mathrm{f}}(\mathbb{Y})$ is equipped with the topology of weak convergence, then $\mathbf{M}_{\mathrm{f}}(\mathbb{Y})$ is also Polish.*

Proof: The theorem is an immediate consequence of Theorem A 2.5.III in [1].

A.3 Measure theoretical complements

Assume now, that $(G, \mathcal{T})$ is a topological group, i.e., a topological space equipped with a group operation $+$ such that the mapping $(x, y) \mapsto x + (-y)$, where $-y$ denotes the inverse element of $y \in G$, is continuous with respect to the product topology. A *Radon* measure is a measure μ on the Borel σ-field $\mathcal{G} := \sigma(\mathcal{T})$ of G which satisfies $\mu(K) < \infty$ for all compact sets $K \subset G$. If $(G, \mathcal{T})$ is a locally compact, second countable Hausdorff space, we have the following fundamental result on the existence and uniqueness of an invariant Radon measure.

Theorem A.3.1. (Haar measure) *On G there exists, uniquely up to normalization, a left-invariant Radon measure $\lambda \neq 0$. If G is compact, then λ is also right-invariant.*

Proof: cf. [7], Theorem 2.27.

Finally, we state the following factorization theorem.

Theorem A.3.2. *Let $(S, \mathcal{S})$ be a measurable space, and μ a σ-finite measure on the product space $G \times S$, such that μ is left-invariant under shifts in G. Then $\mu = \lambda \otimes \nu$ for a unique, σ-finite measure ν on S.*

Proof: cf. [9], Lemma 4.2.

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