Proceedings of ISROMAC 11
2006 International Symposium on Transport Phenomena and Dynamics of Rotating Machinery
Honolulu, Hawaii, USA, February 26-March 2 , 2006

STATIONARY SOLUTIONS AND TRANSITION CONDITIONS FOR THE NON-SMOOTH MOTION OF ROTOR SYSTEMS WITH UNILATERAL CONSTRAINTS

Bernd Waltersberger*
Institut für Technische Mechanik
Universität Karlsruhe
76128 Karlsruhe
Email: waltersberger@itm.uni-karlsruhe.de

Jörg Wauer
Institut für Technische Mechanik
Universität Karlsruhe
76128 Karlsruhe

ABSTRACT

This paper presents a method for providing stationary solutions for rotor systems with a considerable number of unilateral constraints, such as normal contact due to cracks or delaminations in rotating shafts. In more general manner, systems with moving continua or internal flow, for example a pipe resting on unilateral supports, rank among the same class of problems. It is shown that, by virtue of centrifugal effects, the existence of stationary solutions is not guaranteed without restrictions. Sufficient existence conditions are given for an important class of rotor systems and are based on findings from the theory of Complementarity Problems. Furthermore, the presented method comprises an approach to assess the stability of the solution and a modification is given to determine the velocity jumps when a contact configuration change occurs during the system's non-smooth evolution.

1 Introduction

An extensive literature on the dynamics of systems with unilateral constraints has been published in the last decades. Much of the work in this field has been motivated by problems in the dynamics of systems of rigid bodies, see for instance [1–4]. Unilateral constraints imposed on continuous structures are nowadays mainly investigated in the context of general FEM contact problems, cp. [5, 6]. In conjunction with impact and vibro-impact problems research often focusses on specific structures like impacting rods, beams or lumped masses impacting on flexible structures, [7–9], with fundamental contributions already in the 18th and 19th century, for instance by D. BERNOULLI and F. NEUMANN , cp. [10, 11].

Many structural imperfections such as cracks or delaminations of plies in composites can be typically modelled as unilateral constraints, in particular if *breathing* of the gaps due to these defects is involved. Such imperfections are also of interest in rotordynamics as numerous publications give evidence, cp. references in surveys and books, respectively, [12, 13]. Other types of rotordynamical contact problems arise from rotor-stator contacts or loose parts, cp. [14, 15]. Static problems of continua with stationary internal flow and unilateral supports can be tackled with the same methods as will be presented for rotorsystems.

Any investigation of contact problems is based on more or less drastically simplifying assumptions on the contact physics. So the modelling is often done in a way that keeps the number of potential contact points small or even fixed to one. In doing so, one avoids providing a huge number of different sets of equations of motion, one set for each possible contact configuration, which makes a total of 2^k sets, k being the number of potential contact points. This problem is often circumvented by replacing rigid constraints by compliant members such as unilateral springs, which can only transmit compressive forces. It is true that this approach keeps the degree of freedom constant but it introduces considerably stiff nonlinear elements – a fact that could make simulation more costly.

In rotordynamics described in a rotating reference frame and dealing with the above mentioned imperfections, constant centrifugal effects due to these defects often prevail compared to harmonic gravitational excitation. It might therefore be worthwhile to investigate oscillations about stationary solutions – os-

*Address all correspondence to this author.

cillations that can be harmonic provided the excitation is sufficiently small so that inactive constraints are not touched.

An investigation procedure of such a kind corresponds to linearization about stationary solutions in the case when differentiation is feasible. With unilateral constraints, differentiability is impaired. In this paper a procedure is given that circumvents this obstacle.

First of all, however, the stationary solutions and thus the set of active constraints have to be found. With unilateral constraints, the straightforward procedure would be checking each possible contact configuration and assessing afterwards wether the obtained solutions are physically feasible. This procedure involves assessing 2^k configurations, again k being the number of potential contact points – for $k \gg 1$ a costly technique. From the mathematical perspective, as will be seen, the mechanical problem corresponds, with some restrictions, to a well known algebraic problem. Hence, some useful findings known from the related algebraic theory can be successfully applied to the mechanical problem.

1.1 Introductory Example

The most simple model of a LAVAL-rotor with unilateral constraints is depicted in Fig. 1. The unilateral constraint be $y(\tau) \geq 0$. The equations of motion in the rotating frame read (nondimensionalized constants: rotational speed η, y-x-stiffness ratio κ, external damping δ, unbalance described by $\varepsilon, \alpha_\varepsilon$, gravity γ; nondimensionalized time τ)

$$\begin{pmatrix}\ddot{x}\\ \ddot{y}\end{pmatrix} + \begin{pmatrix}\delta & -2\eta\\ 2\eta & \delta\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} + \begin{pmatrix}1-\eta^2 & -\eta\delta\\ \eta\delta & \kappa-\eta^2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = -\gamma\begin{pmatrix}\cos\eta\tau\\ \sin\eta\tau\end{pmatrix} + \varepsilon\eta^2\begin{pmatrix}\cos\alpha_\varepsilon\\ \sin\alpha_\varepsilon\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix}\lambda, \quad (1)$$

$$\begin{pmatrix}0\\ 1\end{pmatrix}^{\mathsf{T}}\begin{pmatrix}x\\ y\end{pmatrix} = 0, \quad (2)$$

wherein Eqn. (2) expresses the auxiliary condition $y = 0$ for the bilaterally constrained case and λ is a Lagrangian-multiplier, which can be interpreted here as the constraint force in y-direction. As long as the constraint is inactive ($y > 0$), the equations of motion are obtained by discarding the auxiliary condition Eqn. (2) and setting $\lambda = 0$ in Eqn. (1). An active constraint ($y = 0$) changes the ordinary differential equation (ODE) into a *Differential-Algebraic*-equation (DAE). The transition from an inactive to an active constraint is not discussed, yet.

The stationary solutions are obtained by ignoring time dependant excitations and time derivatives. Following the mentioned straightforward strategy of assessing the physical plausibility of every solution obtained by cycling through every potential constraint configuration is not really costly for this example. Because there is only one constraint and hence are only two

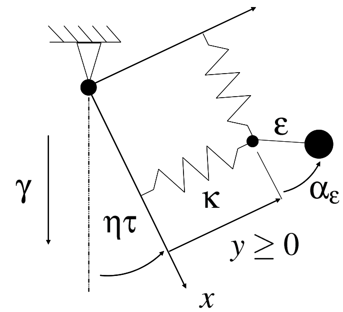

Figure 1. Most simple unilaterally constrained LAVAL-rotor model.

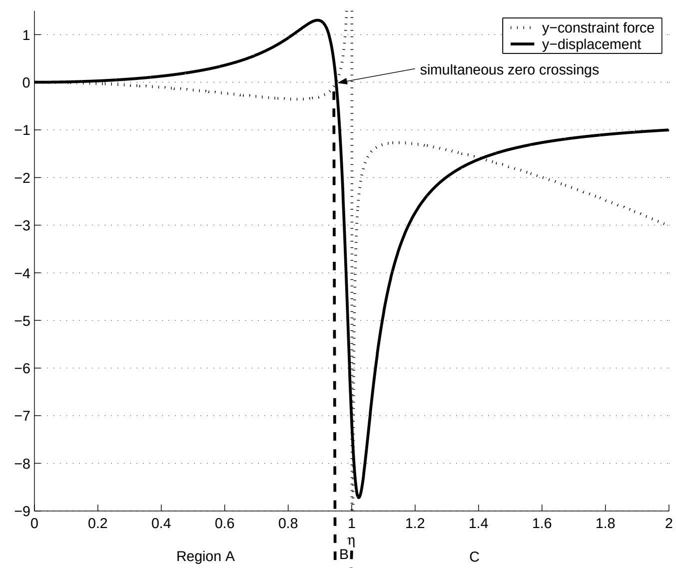

Figure 2. Stationary solutions and constraint forces in y-direction for unconstrained and constrained rotor, respectively, vs. rotational speed.

possibilities: *constraint active/ inactive*. Keeping all parameters fixed apart from the rotational speed η, we check the plausibility of the stationary solutions by monitoring y vs. η for the case without constraint ($\lambda = 0$) and λ vs. η for the case with constraint ($y = 0$) as plotted in Fig. 2. In this example the parameters are set as follows: $\kappa = 1, \delta = 0.1, \varepsilon = 1, \alpha_\varepsilon = \frac{\pi}{2}$.

A stationary solution y for the case *inactive constraint* is also admissible for the case of a unilateral constraint if $y \geq 0$, which physically plausibly means: *no penetration*. This is seen in region A in Fig. 2.

Correspondingly, a stationary solution λ for the case *active constraint* is also admissible for the case of a unilateral constraint if $\lambda \geq 0$, which physically plausibly means: *no tensile constraint force*. This behavior is depicted in region B.

A solution for the case of a unilateral constraint does **not** exist if $y < 0$ and $\lambda < 0$ for the cases *constraint inactive* and *constraint active*, respectively. In the plots this happens in region C where the rotor spins at overcritical speed $\eta > 1$.

One could imagine that non-unique solutions would have been found if there had been regions where $y \geq 0$ and $\lambda \geq 0$ simultaneously for the cases *constraint inactive* or *constraint active*, respectively. Regions with such a property are not found in the present example since the stationary solution plots for displacement and corresponding constraint force show simultaneous zero crossings, see Fig. 2, at least for $\eta < 1$. As will be explained later, this does not happen accidently.

1.2 Outline

The main problems with a class of unilateral constraints in rotor systems have all been touched in the introductory example and will be revisited in a more general manner in this paper. It is organized as follows:

Discrete equations of motion Upon applying HAMILTON's Principle for continuous structures in a formulation that accounts for auxiliary constraints and upon discretizing, the equations of motion are derived under the presumptions

- linear elasticity,
- formulation in rotating frame with constant rotational speed,
- constraint formulation is not explicitly time-dependant, i.e. cracks, stops etc. rotate with the frame,
- friction, i.e. tangential contact interaction, is negligible,
- little *breathing* of constraints during operation, i.e. unilateral auxiliary conditions can be reasonably assumed linearizeable.

Stationary Solutions Under certain assumptions on the continuous constraint force distribution the problem of finding stationary solutions is shown to be equivalent to a *Linear Complementarity Problem* (LCP). The related theory provides some useful results for the mechanical problem.

Minimal coordinates and stability Having found the stationary solutions the set of active constraints is determined and thus the actual degree of freedom. Since the underlying structure is assumed linear elastic and the constraint inequalities are assumed linear, too, it is possible to formulate the linear equations of motion in minimal coordinates, i.e. eliminating the constraint forces. At this stage it is possible to apply the usual matrix stability theory.

Transition conditions During the motion of the system, new constraints come into play what implies that the material points where new constraints become active, experience velocity jumps. Moreover, since reduced discretized models are used, velocity jumps at one point can immediately affect distant points. It is shown, how the transition from one contact configuration to an other one can be cast in a physically plausible formulation.

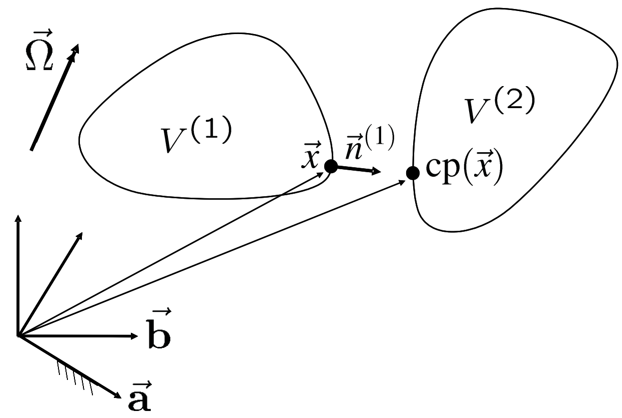

Figure 3. Sketch of potential contact.

2 Equations of motion

2.1 Hamilton's Principle for unilaterally constrained systems

As depicted in Fig. 3, it is given an inertial system $\vec{\mathbf{a}} = (\vec{a}_1\,\vec{a}_2\,\vec{a}_3)^\intercal$ and a rotating (constant angular velocity $\vec{\Omega}$) frame $\vec{\mathbf{b}} = (\vec{b}_1\,\vec{b}_2\,\vec{b}_3)^\intercal$, each with orthonormal basisvectors $\vec{a}_k, \vec{b}_k$, so that $\vec{\mathbf{a}}\vec{\mathbf{a}}^\intercal = \vec{\mathbf{b}}\vec{\mathbf{b}}^\intercal = \mathbf{I}$ equals the 3×3 identity matrix. It is thereby assumed that the scalar products between the vectors are evaluated according to the rules of matrix multiplication.

Given two elastic bodies $V^{(k)}, k = 1, 2$, or two separate parts of one body, whose boundaries $\partial V^{(k)}$ are subdivided into disjoint sets $\Gamma_\sigma^{(k)}$, $\Gamma_u^{(k)}$,$\Gamma_c^{(k)}$, where stress-, displacement- or impenetrability boundary conditions due to contact are applied, respectively. The potential contact point is formally found by means of the *closest-point-operator* $\mathrm{cp}(\cdot)$, the impenetrability condition can be expressed by a gap function g as

$$g(\vec{x}, \vec{u}(\vec{x})) \geq 0 \text{ on } \Gamma_c^{(1)} \tag{3}$$

with the related variation [5]:

$$\mathrm{cp}(\vec{x}) = \arg \min_{\vec{y} \in V^{(2)}} \|\vec{x} - \vec{y}\|, \tag{4}$$

$$g\left(\vec{x}, \vec{u}\left(\vec{x}\right)\right) = -\left(\vec{x} + \vec{u}^{(1)}\left(\vec{x}\right) - \mathrm{cp}(\vec{x}) - \vec{u}^{(2)}\left(\mathrm{cp}(\vec{x})\right)\right) \cdot \vec{n}^{(1)}, \tag{5}$$

$$\delta g = n^{(1)} \cdot \left(-\delta \vec{u}^{(1)} + \delta \vec{u}^{(2)}\right), \tag{6}$$

wherein $\vec{u}^{(k)}, \vec{n}^{(1)}$ are displacement vectors and normal unit vectors, respectively. If here and in what follows superscripting is omitted it is meant to be read: $\vec{u}(\vec{x}) = \vec{u}^{(k)}$ if $\vec{x} \in V^{(k)}$.

HAMILTON's Principle reads

$$\int_{t_0}^{t_1} \left(\delta(T-U) + W_{\text{virt}}\right) dt = 0, \tag{7}$$

with kinetic energy T and elastic potential energy U, the latter one as usual expandable to comprise other forces that have a potential and that are not taken into account in the virtual work W_{virt}.

Before dealing with the virtual work in more detail, we notice that upon partial integration in the time domain and exploiting vanishing variations at the time boundaries t_0, t_1, the variation of the kinetic energy can also be written as

$$\delta T = -\int_V \rho \delta\vec{r} \cdot \ddot{\vec{r}}\, dV \tag{8}$$

with density ρ and $V = V^{(1)} \cup V^{(2)}, \vec{r} = \vec{x} + \vec{u}$, Lagrangian reference point $\vec{x}$. Upon expressing the displacement with respect to the rotating frame by

$$\vec{u} = \vec{\mathbf{b}}^{\mathsf{T}}\mathbf{u};\ \mathbf{u} = (u_1\, u_2\, u_3)^{\mathsf{T}}, \tag{9}$$

the acceleration and the variation read

$$\ddot{\vec{r}} = \vec{\mathbf{b}}^{\mathsf{T}}\ddot{\mathbf{u}} + 2\vec{\Omega} \times \vec{\mathbf{b}}^{\mathsf{T}}\dot{\mathbf{u}} + \vec{\Omega} \times (\vec{\Omega} \times (\vec{x} + \vec{\mathbf{b}}^{\mathsf{T}}\mathbf{u})) \tag{10}$$

$$\delta\vec{r} = \delta\mathbf{u}^{\mathsf{T}}\vec{\mathbf{b}}. \tag{11}$$

The virtual work is composed of the usual part W_0, which does not take the contact into account, and the part W_c that includes traction on the contact boundary:

$$W_{\text{virt}} = W_0 + W_c, \tag{12}$$

where

$$W_0 = \sum_{k=1,2} \left(\int_V \vec{f} \cdot \delta\vec{u}\, dV + \int_{\Gamma_\sigma} \vec{t}^{\,*} \cdot \delta\vec{u}\, d\Gamma \right)^{(k)}, \tag{13}$$

$$W_c = \sum_{i=1,2} \int_{\Gamma_c^{(i)}} \vec{t}^{\,(i)} \cdot \delta\vec{u}^{(i)}\, d\Gamma \tag{14}$$

$$= \int_{\Gamma_c^{(1)}} \vec{t}^{\,(2)} \cdot (-\delta\vec{u}^{(1)} + \delta\vec{u}^{(2)})\, d\Gamma \tag{15}$$

$$= \int_{\Gamma_c^{(1)}} (\lambda_n \vec{n}^{(1)} + \underbrace{\vec{t}_T}_{\vec{0}}) \cdot (-\delta\vec{u}^{(1)} + \delta\vec{u}^{(2)})\, d\Gamma \tag{16}$$

$$= \int_{\Gamma_c^{(1)}} \lambda_n \delta g\, d\Gamma. \tag{17}$$

Herein is $\vec{f}$ a given volume force in V, is $\vec{t}^{\,*}$ the prescribed stress vector on Γ_σ and is $\vec{t}$ the unknown stress vector in the still unknown contact zone Γ_c. Equation (15) follows from $\vec{t}^{\,(2)} = -\vec{t}^{\,(1)}$, i.e. *Actio=Reactio*, Eqn. (16) from the decomposition of the stress vector into normal and tangential part, the latter one being zero since friction is neglected. Eventually, Eqn. (17) is obtained by plugging in Eqn. (6).

The difficulty remains that the contact zone itself is dependant on the displacement and thus not constant.

By simultaneously demanding

$$\lambda_n \geq 0,\ g \geq 0,\ \lambda_n g = 0, \tag{18}$$

the contact zone can be expanded to the constant boundary portion $\Gamma_c^{(1)} = \partial V^{(1)} \setminus \Gamma_\sigma^{(1)} \setminus \Gamma_u^{(1)}$. Because now with the conditions (18) integration in Eqn. (17) over the now constant domain $\Gamma_c^{(1)}$ gives no contribution to the integral value when there is no contact, i.e. gap function $g > 0$, since in this case the third condition in (18) ensures $\lambda_n = 0$. The conditions (18) are called the SIGNORINI complementarity conditions for normal contact.

2.2 Discretization

The discretization of the displacements is done in the rotating frame by substituting the location dependant displacement coordinates $\mathbf{u} \in \mathbb{R}^3$ in Eqn. (9) and their variations by

$$\mathbf{u} = \mathbf{\Phi}(\vec{x})^{\mathsf{T}}\mathbf{q}(t), \tag{19}$$

$$\delta\mathbf{u}^{\mathsf{T}} = \delta\mathbf{q}^{\mathsf{T}}\mathbf{\Phi}(\vec{x}), \tag{20}$$

with appropriately chosen admissible functions $\mathbf{\Phi}_k(\vec{x}) \in \mathbb{R}^3$, $k = 1,\dots,N$, concatenated in $\mathbf{\Phi}(\vec{x}) = (\mathbf{\Phi}_1(\vec{x}),\dots,\mathbf{\Phi}_N(\vec{x}))^{\mathsf{T}} \in \mathbb{R}^{3\times N}$ and N generalized coordinates $\mathbf{q} = (q_1,\dots,q_N)^{\mathsf{T}}$.

Due to the assumption on the constraints in section 1.2, the gap function in Eqn. (5) can be approximated by a discretized and linearized formulation. It can be assumed to have the following form, likewise its variation:

$$g = \mathbf{g}(\vec{x})^{\mathsf{T}}\mathbf{q} + g_0(\vec{x}), \tag{21}$$

$$\delta g = \delta\mathbf{q}^{\mathsf{T}}\mathbf{g}(\vec{x}). \tag{22}$$

Moreover, the distributed normal constraint force $\lambda_n(\vec{x},t) \in \mathbb{R}$ in Eqn. (17) has to be discretized:

$$\lambda_n(\vec{x},t) = \mathbf{\Lambda}(\vec{x})^{\mathsf{T}}\boldsymbol{\lambda}(t), \tag{23}$$

where $\mathbf{\Lambda}(\vec{x}) = (\Lambda_1(\vec{x}),\dots,\Lambda_M(\vec{x}))^{\mathsf{T}} \in \mathbb{R}^M$ contains appropriate shape functions for the force distribution whose scalar weighting factors are comprised in the time dependant column matrix $\boldsymbol{\lambda} \in \mathbb{R}^M$.

The SIGNORINI inequality conditions (18) are approximated by discretization through weighted averaging in $\Gamma_c^{(1)}$ to ensure that the inequalities are at least satisfied in an averaging sense. In order to preserve meaningful inequalities, the weighting functions must be non-negative in $\Gamma_c^{(1)}$. The equality constraint in the conditions (18) is sufficiently discretized by averaging with one constant weight function since, provided the factors are positive, the integral over the product vanishes if, and only if, the product itself is zero. Eventually, we get the discretized SIGNORINI conditions

$$\underbrace{\int_{\Gamma_c^{(1)}} \mathbf{\Lambda}\mathbf{g}^\mathsf{T}\, d\Gamma}_{\mathbf{J}^\mathsf{T}}\ \mathbf{q} + \underbrace{\int_{\Gamma_c^{(1)}} \mathbf{\Lambda} g_0\, d\Gamma}_{\mathbf{j}} \geq \mathbf{0}, \tag{24}$$

$$\underbrace{\int_{\Gamma_c^{(1)}} \mathbf{\Lambda}\mathbf{\Lambda}^\mathsf{T}\, d\Gamma}_{\mathbf{Y}}\ \boldsymbol{\lambda} \geq \mathbf{0}, \tag{25}$$

$$\boldsymbol{\lambda}^\mathsf{T}\left(\int_{\Gamma_c^{(1)}} \mathbf{\Lambda}\mathbf{g}^\mathsf{T}\, d\Gamma\, \mathbf{q} + \int_{\Gamma_c^{(1)}} \mathbf{\Lambda} g_0\, d\Gamma\right) = 0, \tag{26}$$

whereby the force shape functions $\mathbf{\Lambda}$ are simultaneously used as weight functions and hence have to be non-negative. The inequality sign between column matrices is meant to apply between the referring entries.

Upon discretizing the Hamilton functional Eqn. (7) by means of the displacement and variation approximations in Eqn. (19,20) and applying the reformulation of the kinetic energy variation in Eqn. (8-11) one eventually ends up with

$$\mathbf{M}\ddot{\mathbf{q}} + (\mathbf{G}+\mathbf{D})\dot{\mathbf{q}} + (\mathbf{K}+\mathbf{P})\mathbf{q} = \mathbf{f} + \mathbf{J}\boldsymbol{\lambda}, \tag{27}$$

$$\mathbf{J}^\mathsf{T}\mathbf{q} + \mathbf{j} \geq \mathbf{0}, \tag{28}$$

$$\mathbf{Y}\boldsymbol{\lambda} \geq \mathbf{0}, \tag{29}$$

$$\boldsymbol{\lambda}^\mathsf{T}\left(\mathbf{J}^\mathsf{T}\mathbf{q} + \mathbf{j}\right) = 0, \tag{30}$$

together with auxiliary conditions (24-26). Hereby is

$$\mathbf{M} = \int_V \rho \mathbf{\Phi}\mathbf{\Phi}^\mathsf{T}\, dV \tag{31}$$

the symmetric and positive-definite mass matrix,

$$\mathbf{G} = \int_V 2\rho \mathbf{\Phi}\tilde{\mathbf{\Omega}}\mathbf{\Phi}^\mathsf{T}\, dV \tag{32}$$

the skew-symmetric gyroscopic matrix. The skew-symmetry is due to the skew-symmetry of the cross-product tensor's coordinate matrix with respect to the rotating frame

$$\tilde{\mathbf{\Omega}} = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}, \tag{33}$$

with $\vec{\Omega} = \sum_{k=1}^{3} \Omega_k \vec{b}_k$. Furthermore is $\mathbf{D} = \mathbf{D}_{\text{int}} + \mathbf{D}_{\text{ext}}$ the symmetric damping matrix, comprising internal damping effects, for instance modelled as *Rayleigh*-Damping and external damping effects

$$\mathbf{D}_{\text{ext}} = \int_{\partial V} d_{\text{ext}} \mathbf{\Phi}\mathbf{\Phi}^\mathsf{T}\, d\Gamma \tag{34}$$

so that together with the skew symmetric damping matrix

$$\mathbf{P} = \int_{\partial V} d_{\text{ext}} \mathbf{\Phi}\tilde{\mathbf{\Omega}}\mathbf{\Phi}^\mathsf{T}\, d\Gamma, \tag{35}$$

the virtual work of the external damping forces can be expressed with a given damping constant d_{ext} as

$$-\int_{\partial V} d_{\text{ext}} \delta\vec{r} \cdot \dot{\vec{r}}\, dV = -\delta\mathbf{q}^\mathsf{T}\left(\mathbf{D}_{\text{ext}}\dot{\mathbf{q}} + \mathbf{P}\mathbf{q}\right). \tag{36}$$

The overall stiffness matrix $\mathbf{K} = \mathbf{K}_0 + \mathbf{K}_c$ comprises elastic and centrifugal effects, respectively, the latter ones by

$$\mathbf{K}_c = \int_V \rho \mathbf{\Phi}\tilde{\mathbf{\Omega}}\tilde{\mathbf{\Omega}}\mathbf{\Phi}^\mathsf{T}\, dV. \tag{37}$$

The in general rectangular gradient matrix $\mathbf{J}$ is obtained by evaluating the contact integral in Eqn. (17) together with the discretized versions of the normal constraint force and variation from Eqn. (23, 22) and comparing with the abbreviation in Eqn. (24).

3 Stationary solutions

3.1 Formulation as Linear Complementarity Problem

Upon cancelling the explicitly time dependant forces in time derivatives in Eqn. (27) the problem of the stationary solution reads

$$\overbrace{(\mathbf{K}+\mathbf{P})}^{\mathbf{Q}}\mathbf{q} = \mathbf{f} + \mathbf{J}\boldsymbol{\lambda}, \tag{38}$$

$$\mathbf{J}^\mathsf{T}\mathbf{q} + \mathbf{j} \geq \mathbf{0}, \tag{39}$$

$$\mathbf{Y}\boldsymbol{\lambda} \geq \mathbf{0}, \tag{40}$$

$$\boldsymbol{\lambda}^\mathsf{T}\left(\mathbf{J}^\mathsf{T}\mathbf{q} + \mathbf{j}\right) = 0, \tag{41}$$

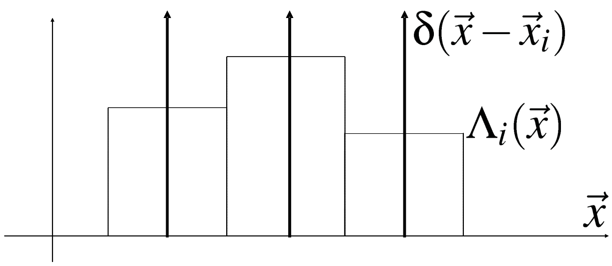

Figure 4. Ansatz functions for constraint force.

wherein **f** is constant, i.e. it comprises forces constant in the rotating frame, for instance centrifugal unbalance effects. Eliminating the generalized displacements **q** and thereby assuming **Q** to be invertible yields

$$\mathbf{A}\boldsymbol{\lambda}+\mathbf{a} \geq \mathbf{0}, \tag{42}$$
$$\mathbf{Y}\boldsymbol{\lambda} \geq \mathbf{0}, \tag{43}$$
$$\boldsymbol{\lambda}^{\mathsf{T}}(\mathbf{A}\boldsymbol{\lambda}+\mathbf{a}) = 0, \tag{44}$$

with

$$\mathbf{A} = \mathbf{J}^{\mathsf{T}}\mathbf{Q}^{-1}\mathbf{J}, \tag{45}$$
$$\mathbf{a} = \mathbf{J}^{\mathsf{T}}\mathbf{Q}^{-1}\mathbf{f}+\mathbf{j}. \tag{46}$$

In comparison with the inequality system

$$\mathbf{A}\boldsymbol{\lambda}+\mathbf{a} \geq \mathbf{0}, \tag{47}$$
$$\boldsymbol{\lambda} \geq \mathbf{0}, \tag{48}$$
$$\boldsymbol{\lambda}^{\mathsf{T}}(\mathbf{A}\boldsymbol{\lambda}+\mathbf{a}) = 0 \tag{49}$$

with given square and column matrices **A** and **a**, respectively, which is the general form of a much studied algebraic problem known as *Linear Complementarity Problem* – usually abbreviated by LCP(**A**,**a**) – for the unknown column matrix $\boldsymbol{\lambda}$, cp. [16], Eqn. (42-44) differ only in the matrix **Y**.

It is sufficient for the equivalence of Eqn. (48) and (43) that **Y** is diagonal, since it is guaranteed that the diagonal entries are non-negative as the ansatz functions in $\boldsymbol{\Lambda}$, cp. Eqn. (23), are chosen to be non-negative. The matrix **Y** is diagonal if, and only if, the ansatz functions have disjoint supports, for example generalized rectangle-functions or Dirac δ-functions, see Fig. 4. Such kinds of discretization can be interpreted as an averaged sectionwise constraint enforcement or pointwise constraint enforcement (collocation), respectively

3.2 Solution of the LCP

According to the LCP theory, cp. [16], the LCP in Eqn. (47-49) is guaranteed to have a unique solution if the LCP matrix **A** is *positive-definite* (*pd*), i.e. $\mathbf{q}^{\mathsf{T}}\mathbf{A}\mathbf{q} > 0$ for any fitting column matrix **q**.

If **A** is not *pd*, the LCP(**A**,**a**) can have from 0 up to an infinite number of solutions. To the authors' knowledge there is no known generally applicable method to predict the non-existence of solutions.

The LCP matrix **A** does not have to be symmetric. However, it solely depends on its symmetric part, wether **A** is *pd*.

For rotor systems described in the rotating frame, the LCP matrix $\mathbf{A} = \mathbf{J}^{\mathsf{T}}\mathbf{Q}^{-1}\mathbf{J}$ in Eqn. (45) typically is obtained from

$$\mathbf{Q} = \underbrace{\mathbf{K}_0 + \overbrace{\left(-\Omega^2 \int_V \rho \boldsymbol{\Phi}\mathbf{E}_k\boldsymbol{\Phi}^{\mathsf{T}}\, dV\right)}^{=\mathbf{K}_c,\ \text{negative definite}}}_{\text{symmetric part}} + \mathbf{P}, \tag{50}$$

wherein $\mathbf{K}_0$ is the symmetric and in this paper presumedly positive definite portion that is derived from the elastic potential. According to Eq. (37), centrifugal effects are taken into account by the matrix $\mathbf{K}_c$ where moreover is assumed that the angular velocity vector with magnitude Ω is parallel to the k-th coordinate axis in the rotating frame $\vec{\mathbf{b}}$. Due to Eq. (33) this implies $\tilde{\boldsymbol{\Omega}}^2 = -\Omega^2\mathbf{E}_k$ with $\mathbf{E}_k = \mathbf{diag}(1-\delta_{ik}; i = 1,2,3)$.

In order to obtain a *pd* LCP matrix **A**, $\mathbf{Q}^{-1}$ has to be *pd* and **J** of full rank. If the latter condition were not satisfied, for instance for linearly dependant unilateral constraints, **A** would in general be only *positive-semi-definite* (*psd*), a fact that can cost the uniqueness of the solution. It can be shown, that for any invertible matrix **Q** applies, that **Q** being *pd* or *psd* is equivalent to $\mathbf{Q}^{-1}$ being *pd* or *psd*, respectively.

For the given rotor system two conclusions thus can be drawn:

1. A unique stationary solution exists if the overall stiffness matrix $\mathbf{K} = \mathbf{K}_0 - \Omega^2 \int_V \rho\boldsymbol{\Phi}\mathbf{E}_k\boldsymbol{\Phi}^{\mathsf{T}}\, dV$, including centrifugal effects, is *pd* and the constraints **J** have full rank. This means, at subcritical angular velocity, the unilaterally constraint rotor has a uniquely determined stationary solution. The existence condition is independent from the skew-symmetric external damping matrix **P** and the applied constant forces.
2. Presently, no theory exists that can a priori give existence conditions in case of supercritical speed.

Coming back to the introductory example in section 1.1, it can now be understood that at subcritical speed $\eta < 1$ a solution has to uniquely exist in regions *A* and *B* in Fig. 2. This also explains that the zero crossings of the constraint force and the

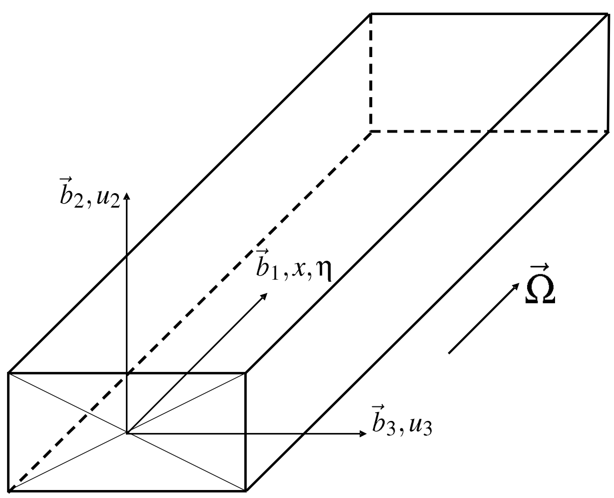

Figure 5. Rotating EULER-BERNOULLI-Beam.

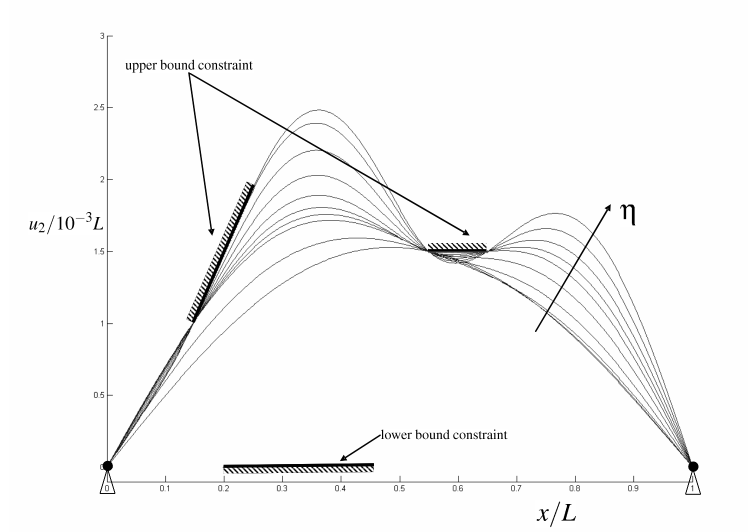

Figure 6. Stationary solutions u_2 of unilaterally constrained rotating EULER-BERNOULLI-Beam.

displacement plots in Fig. 2 have to happen simultaneously at subcritical speed, since otherwise no solution would exist. The non-existence of a solution at supercritical speed cannot be predicted.

3.2.1 Numerical solution of LCP

As to the actual solution technique, LEMKE's algorithm can be used to solve an LCP(**A**,**a**), cp. [16, 17], a pivoting scheme that is proved to find a solution in a finite number of steps if **A** is *pd*. If **A** is not *pd* there again are no known general conditions for the algorithm to find a solution. However, experience shows, that in a slightly modified version, cp. [17], where an initial guess can be provided to the algorithm, it often reliably works if a good guess value is applied. In view of Eqn. (50) this observation suggests a homotopy method, using a homotopy parameter $0 \leq h \leq \Omega^2$. Starting with $h = 0$ yields a *pd* LCP-matrix and hence a starting guess value, since the centrifugal portion tending to negative-definiteness is faded out.

3.3 Example: Rotating Euler-Bernoulli Beam

Upon upgrading the introductory example in section 1.1 to a simple continuous model one ends up with the rotating EULER-BERNOULLI-Beam, cp. Fig. 5. The original constraint in Eq. (2) is replaced by a sectionwise continuous constraint as lower bound $g_{lb}(x)$ according to the deflections plotted in Fig. 6. In this example, all constraints act in $\vec{b}_2$-direction. In addition, an upper bound constraint $g_{ub}(x)$ is introduced in two seperate domains. Upon discretization of the displacements according to Eq. (19), the constraints

$$u_2 = \vec{b}_2 \cdot \vec{u} \leq g_{ub}(x), \tag{51}$$

$$\vec{b}_2 \cdot \vec{u} \geq g_{lb}(x) \tag{52}$$

transform to

$$\begin{bmatrix} -\vec{b}_2 \cdot \vec{\mathbf{b}}^\intercal \mathbf{\Phi}^\intercal \\ \vec{b}_2 \cdot \vec{\mathbf{b}}^\intercal \mathbf{\Phi}^\intercal \end{bmatrix} \mathbf{q} + \begin{pmatrix} g_{ub}(x) \\ -g_{lb}(x) \end{pmatrix} \geq \mathbf{0}. \tag{53}$$

Each row of this equation corresponds to a discretized gap function according to Eq. (21). Thereby, eigenfunctions of a simply supported beam are employed, with a $\vec{b}_3$-$\vec{b}_2$-stiffness ratio $\kappa_3 = 1.1$.

In this example, the beam is deflected only due to a constant unbalance (nondimensionalized quantity) $\varepsilon_2(x) = 0.001$ in $\vec{b}_2$-direction and external damping (nondimensionalized) $\delta = 0.1$. The plots in Fig. 6 show the sationary solutions in $\vec{b}_2$-direction for ascending (nondimensionalized) rotational speeds $0 \leq \eta \leq 7\pi^2$, i.e. supercritical speed for $\eta > \pi^2$. Without upper bound constraints, no stationary solution can be computed for supercritical speed, which in this case seams reasonable, since at least for a one mode repesantation in either lateral direction, unbalance tended to be compensated due to self-balancing at supercritical speed, were the unilateral constraint not present. This, of course, cannot happen with the active constraint.

4 Transformation to Minimal Coordinates

4.1 Updating the actual constraints

Solving the LCP in Eqn. (42-44) yields a solution for the constraint force ansatz multipliers $\boldsymbol{\lambda}$. The entries in $\boldsymbol{\lambda}$ being zero indicate that the referring constraint are inactive, the actual constraint gradient matrix $\mathbf{J}_A$ of active constraints emerges from $\mathbf{J}$ by cancelling the referring columns of the inactive constraints. That means, for a finite time interval the system appears like a bilaterally constrained one, as long as the constraint configuration does not change. The equations of motion (27-30) in this time interval therefore read

$$\mathbf{M}\ddot{\mathbf{q}} + (\mathbf{G}+\mathbf{D})\dot{\mathbf{q}} + (\mathbf{K}+\mathbf{P})\mathbf{q} = \mathbf{f} + \mathbf{J}_A\boldsymbol{\lambda}, \tag{54}$$

$$\mathbf{J}_A{}^{\mathsf{T}}\mathbf{q} + \mathbf{j}_A = \mathbf{0}. \tag{55}$$

Be the columns of the overall constraint matrix $\mathbf{J}$ denoted according to

$$\mathbf{J} = [\mathbf{n}_1, \ldots, \mathbf{n}_m]. \tag{56}$$

For the sake of brevity, $\mathbf{J}$ is assumed to have full rank. Furthermore be

$$\mathbf{J}'_0 = [\mathbf{n}'_1, \ldots, \mathbf{n}'_m] \text{ so that } \mathbf{J}^{\mathsf{T}}\mathbf{J}'_0 = \mathbf{I}_{m\times m} \tag{57}$$

a pseudoinverse and the columns of

$$\mathbf{B}_0 = [\mathbf{b}_1, \ldots, \mathbf{b}_{n-m}] \text{ so that } \mathbf{J}^{\mathsf{T}}\mathbf{B}_0 = \mathbf{O} \tag{58}$$

a basis for the kernel $\ker(\mathbf{J})$, i.e. nullspace, of $\mathbf{J}$. Without limitations one can assume the constraint with gradient $\mathbf{n}_1$ to become inactive – enforced by reordering, in case of several inactive constraints by repetitive application of the following scheme – and hence cancelling the referring row gives

$$\mathbf{J}_A = [\mathbf{n}_2, \ldots, \mathbf{n}_m]. \tag{59}$$

The active constraints reduce the space of possible motions, which is now spanned by

$$\dot{\mathbf{q}} = \underbrace{[\mathbf{B}_0, \mathbf{n}'_1]}_{\mathbf{B}} \underbrace{\begin{pmatrix} \dot{\mathbf{z}}_1 \\ \dot{z}_2 \end{pmatrix}}_{\dot{\mathbf{z}}}. \tag{60}$$

This implies, in view of Eqn. (57,58), that $\mathbf{B}^{\mathsf{T}}\mathbf{J}_A = \mathbf{0}$ and thus the constraint forces can be eliminated by left-multiplication of $\mathbf{B}^{\mathsf{T}}$. Upon integration of Eqn. (60) with the stationary solution as initial condition and differentiation, respectively, substitution in Eqn. (54,55) gives the equations of motion

$$\mathbf{B}^{\mathsf{T}}\mathbf{M}\mathbf{B}\ddot{\mathbf{z}} + \mathbf{B}^{\mathsf{T}}(\mathbf{G}+\mathbf{D})\mathbf{B}\dot{\mathbf{z}} + \mathbf{B}^{\mathsf{T}}(\mathbf{K}+\mathbf{P})\mathbf{B}\mathbf{z} = \mathbf{B}^{\mathsf{T}}\mathbf{f} \tag{61}$$

in minimal-coordinates. This equation corresponds to the linearized equations of *small* – so the meaning of *small* will have been explained below – vibrations about stationary solutions with differentiable non-linearities. For practical applications it is the starting point for the usual matrix-stability theory in order to decide wether the stationary solution is stable. In practice, the kernel basis $\mathbf{B}_0$ is computed by means of a Singular-Value-Decomposition [18]. It has to be noted that the costly kernel computation has to be done only once with all ever possible constraints.

4.2 Small forced vibrations

In the context of discrete unilateral constraints, the term *small* with vibrations is meant to indicate that during the forced motion the constraint configuration does not change and hence the generalized constraint forces remain positive and inactive constraints are never touched. Then forced vibrations with excitation $\mathbf{f} = \hat{\mathbf{f}}e^{i\omega t}$ and response $\mathbf{z} = \hat{\mathbf{z}}e^{i\omega t}$ are described by

$$\left(-\omega^2\mathbf{B}^{\mathsf{T}}\mathbf{M}\mathbf{B} + i\omega\mathbf{B}^{\mathsf{T}}(\mathbf{G}+\mathbf{D})\mathbf{B} + \mathbf{B}^{\mathsf{T}}(\mathbf{K}+\mathbf{P})\mathbf{B}\right)\hat{\mathbf{z}} = \mathbf{B}^{\mathsf{T}}\hat{\mathbf{f}}, \tag{62}$$

$$\hat{\boldsymbol{\lambda}} = \mathbf{J}'_A{}^{\mathsf{T}}\left(\left(-\omega^2\mathbf{M}\mathbf{B} + i\omega(\mathbf{G}+\mathbf{D})\mathbf{B} + (\mathbf{K}+\mathbf{P})\mathbf{B}\right)\hat{\mathbf{z}} - \hat{\mathbf{f}}\right), \tag{63}$$

$$\mathbf{J}'_A = [\mathbf{n}'_2, \ldots, \mathbf{n}'_m], \tag{64}$$

wherein $i = \sqrt{-1}$ and $\mathbf{J}'_A$ is the pseudinverse of the active constraints, which is obtained by cancelling column $\mathbf{n}'_1$ in the overall pseudoinverse in Eqn. (57) with assumedly inactive constraint with gradient $\mathbf{n}_1$ as above.The column matrix $\hat{\boldsymbol{\lambda}}$ contains the complex constraint force amplitudes of the active constraints. Its magnitudes must not be greater than the non-zero entries in the stationary solution $\boldsymbol{\lambda}$ from Eqn. (42-44) since otherwise negative constraint forces occur.

5 Transition conditions for percussive normal contacts

In the previous sections a change of the constraint configuration was tacitely excluded. The motion actually was a smooth one. Non-smoothness in form of velocity-jumps comes into play if an inactive unilateral constraint is touched. It can be shown, cp. [1], that the motion remains smooth if an active constraint becomes inactive, so that only the transition inactive to active constraint has to be considered.

In this case the Eqn. of motion (27-30) are valid only *almost everywhere*, since the generalized acceleration $\ddot{\mathbf{q}}$ does not exist if a jump in the velocities occurs. The generalized velocity jumps from $\dot{\mathbf{q}} = \dot{\mathbf{q}}^-$ during the infinitesimally small time interval $I = [t^-, t^+]$ immediately before the impact at t^- to $\dot{\mathbf{q}} = \dot{\mathbf{q}}^+$ immediately after the impact at t^+. The jumps are calculated under the usual assumptions in impact mechanics of rigid bodies, cp. [3]: All forces that are no constraint forces, which in turn have to be compressive, are neglected during I as well as the position remains unchanged, i.e. ${}^+\mathbf{q} = {}^-\mathbf{q}$.

Immediately before impact, the set of active constraints be expanded by the constraints that just have been touched and reduced by those constraints that are about to become inactive, i.e. whose referring entries in $\mathbf{J}_A^\mathsf{T}\dot{\mathbf{q}}^-$ are positive. The problem now is to decide, which of those constraints remain active immediately after the impact at t^+.

Upon integrating Eqn. (27) under the above assumptions and additionally assuming constraint force ansatz functions so that $\mathbf{Y}$ in Eqn. (43) becomes diagonal, for the same reasons as discussed in section 3.1, one gets

$$\mathbf{M}(\dot{\mathbf{q}}^+ - \dot{\mathbf{q}}^-) = \mathbf{J}_A \bar{\boldsymbol{\lambda}}, \tag{65}$$

$$\mathbf{J}_A^\mathsf{T}\dot{\mathbf{q}}^+ \geq -\varepsilon \mathbf{J}_A^\mathsf{T}\dot{\mathbf{q}}^-, \tag{66}$$

$$\bar{\boldsymbol{\lambda}} \geq \mathbf{0}, \tag{67}$$

$$\bar{\boldsymbol{\lambda}}^\mathsf{T}(\mathbf{J}_A^\mathsf{T}\dot{\mathbf{q}}^+ + \varepsilon \mathbf{J}_A^\mathsf{T}\dot{\mathbf{q}}^-) = 0, \tag{68}$$

wherein inequality (66) is a Newtonian restitution law with positive scalar restitution coefficient ε that demands that for each constraint the depart velocity after impact be at least $-\varepsilon$ times the approach velocity before impact, inequality (67) demands that the impact impulses be non-tensile and eventually Eqn. (68) ensures that impulses be only transmitted if inequality (66) is satisfied as equality. The latter condition ensures, that active constraints that were already active before impact can become inactive after impact.

Upon elimination of the unknown $\dot{\mathbf{q}}^+$, which is feasible since $\mathbf{M}$ is *pd*, Eqn. (65-68) can again be transferred to a LCP($\mathbf{A}$,$\mathbf{a}$) for $\bar{\boldsymbol{\lambda}}$ with

$$\mathbf{A} = \mathbf{J}_A^\mathsf{T}\mathbf{M}^{-1}\mathbf{J}_A, \tag{69}$$

$$\mathbf{a} = (1+\varepsilon)\mathbf{J}_A^\mathsf{T}\dot{\mathbf{q}}^-. \tag{70}$$

This LCP always has a solution, which is unique if $\mathbf{A}$ is *pd*. The latter property is given if $\mathbf{J}_A$ has full rank.

The upcoming time interval before the next contact configuration change occurs is now governed by a new set of active constraints. These are found to be those whose referring entries in the column matrix $\mathbf{J}_A^\mathsf{T}\dot{\mathbf{q}}^+$ are zero, meaning the constraint remains active. Applying the very same scheme as in section 4.1 gives the updated equations of motion for the new constraint configuration.

6 Conclusion

A variational formulation upon HAMILTON's Principle was given for systems with tangentially frictionless unilateral constraints, which are assumed to be explicitly time-independent in the rotating frame. The discrete equations of motion with respect to the rotating frame were derived together with discretized SIGNORINI contact conditions. The problem of finding stationary solutions was transferred to a *Linear Complementarity Problem*. Finally, transition conditions on the basis of a Newtonian restitution law for the non-smooth motion with changing contact configurations were given adopting a formulation of rigid-body mechanics.

It was shown that a stationary solution always exists at subcritical rotational speed, even in the case of external damping uniquely for linearly independent constraints. An efficient method for the transformation on minimal coordinates was presented, which comes with a practical method to assess stability of the stationary solution.

REFERENCES

[1] Moreau, J., 1988. "Unilateral contact and dry friction in finite freedom dynamics". In Nonsmooth Mechanics and Application, J. Moreau and P. Panagiotopoulos, eds., no. 302 in CISM Courses and Lectures, Springer.

[2] Brogliato, B., 1999. *Nonsmooth Mechanics*, 2 ed. Springer, London.

[3] Pfeiffer, F., and Glocker, C., 1996. *Multibody dynamics with unilateral contacts*. Wiley, New York.

[4] Stewart, D., 2000. "Rigid-body dynamics with friction and impact". *SIAM Review,* **42**(1).

[5] Laursen, T., 2002. *Computational Contact and Impact Mechanics*. Springer, Berlin.

[6] Eberhard, P., 2000. *Kontaktuntersuchungen durch hybride Mehrkörpersystem/ Finite Element Simulationen*. Shaker, Aachen.

[7] Cox, H., 1849. "On impacts on elastic beams". *Cambridge Philosophical Transactions*.

[8] Babickij, V., 1998. *Theory of vibro-impact systems and applications*. Springer, Berlin.

[9] Stronge, W., 2000. *Impact Mechanics*. Cambridge University Press.

[10] Szabó, I., 1979. *Geschichte der mechanischen Prinzipien*, 2 ed. Birkhäuser, Basel.

[11] Neumann, F., 1885. *Vorlesungen über die Theorie der Elasticität der festen Körper und des Lichtäthers*. Teubner, Leipzig.

[12] Wauer, J., 1990. "On the dynamics of cracked rotors: A literature survey". *Applied Mechanics Reviews*, **43**(1).

[13] Muszyńska, A., 2005. *Rotordynamics*. Taylor & Francis, Boca Raton.

[14] Fumagalli, M., Varadi, P., and Schweitzer, G., 1994. "Impact dynamics of high speed rotors in retained bearings and measurement concept". In Proceedings of Fourth International Symposium on Magnetic Bearings.

[15] Adams, M., and Afshari, F., 2000. "An experiment to measure the restitution coefficient for rotor-stator impacts". In Proceedings of IMechE, no. C576/080.

[16] Cottle, R., Pang, J., and Stone, R., 1992. *The Linear Complementarity Problem*. Academic Press, Boston.

[17] Miranda, M., and Fackler, P., 2002. *Applied Computational Economics and Finance*. MIT Press, Cambridge, Mass.

[18] Bebendorf, M., 2004. *Numerische Lineare Algebra*. No. 23 in Lecture Notes. Max-Planck-Institut für Mathematik in den Naturwissenschaften, Leipzig.