NEW PROOF OF THE $T(1)$ THEOREM FOR TRIEBEL–LIZORKIN SPACES

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ABSTRACT. We give a new proof of the $T(1)$ theorem for the reflexive homogeneous Triebel–Lizorkin spaces $\dot{F}_p^{s,q}$, which uses neither maximal functions nor atomic decompositions. The substitute tool is an estimate on translations in the $L_p(\ell_q)$ spaces.

1. INTRODUCTION

After the publication of the remarkable $T(1)$ theorem of G. David and J.-L. Journé [1] concerning general conditions of L_2 boundedness of singular integral operators

$$Tf(x) = \int_{\mathbb{R}^N} K(x,y)f(y)dy,$$

several authors ([2, 3, 4, 5, 6]) have proved results of similar flavour on various different function spaces, such as the homogeneous Besov spaces $\dot{B}_p^{s,q}$ and the homogeneous Triebel–Lizorkin spaces $\dot{F}_p^{s,q}$ (which include the spaces $L_p = \dot{F}_p^{s,2}$ for $1 < p < \infty$). In a different direction of generalization, T. Figiel [7] has proved an analogue of the $T(1)$ theorem for X-valued functions $f \in L_p(X)$, where X is a Banach space with a certain additional property (UMD) but the kernel K is still scalar-valued. Quite recently, L. Weis and one of the present authors [8] gave a new proof of Figiel's $T(1)$ theorem and extended it to operator-valued kernels K. The two main analytic ingredients in [8] were the well-known Littlewood–Paley decomposition and a rather less-known, but quite powerful, square-function estimate, due to J. Bourgain [9], for the translation operators $\tau_h : f \mapsto f(\cdot - h)$. It was subsequently observed by the second-named author [10, 11] that the approach of [8] was adaptable to obtain proofs of $T(1)$ theorems also for the vector-valued homogeneous Besov $\dot{B}_p^{s,q}(X)$ and Bessel potential $\dot{H}_p^s(X)$ spaces. The latter ones, in the scalar-valued case, coincide with $\dot{F}_p^{s,2}$ for $1 < p < \infty$.

The purpose of the present note is to show that essentially the same approach carries over to the whole scale of the (reflexive) Triebel–Lizorkin spaces $\dot{F}_p^{s,q}$,

$1 < p, q < \infty$. Thus, our aim is to give a new proof of the $T(1)$ theorem for these spaces which makes no use whatsoever of either maximal functions or atomic decompositions, the methods employed in the earlier proofs of analogous results. The tool that replaces them in our proof is an appropriate modification to the present situation of the above mentioned square function estimate for translations (see Lemma 3.1).

We conclude the introduction with our statement of the $T(1)$ theorem for the Triebel–Lizorkin spaces. The definitions of these spaces, as well as of the various conditions appearing in the theorem, are given in Section 2. The crucial translation lemma is proved in Section 3, and the proof of Theorem 1.1 is then given in the final Section 4.

Theorem 1.1. *Let $n \in \mathbb{N}$ and $\nu \in (0,1)$. Suppose $T \in \mathrm{CZO}_{n+\nu}$ satisfies the weak boundedness property and the condition $T(u^\alpha) = 0$ for all $|\alpha| \leq n$. Then T extends to a bounded linear operator from $\dot{F}_p^{s,q}(\mathbb{R}^N)$ to $\dot{F}_p^{s,q}(\mathbb{R}^N)$ for each $s \in (0, n+\nu)$ and each $p, q \in (1, \infty)$.*

If in addition $T' \in \mathrm{CZO}_{n+\nu}$ and $T'(u^\alpha) = 0$ for all $|\alpha| \leq n$, then the assertion holds for all $|s| < n + \nu$.

2. Spaces and operators

We denote $\mathbb{N} := \{0, 1, 2, \dots\} \supsetneq \mathbb{Z}_+ := \{1, 2, \dots\}$. We fix a number $N \in \mathbb{Z}_+$, and all our functions and distributions will be defined on $\mathbb{R}^N$. By $\mathcal{D}(\mathbb{R}^N) \subset \mathcal{S}(\mathbb{R}^N) \subset \mathcal{S}'(\mathbb{R}^N)$ we denote the compactly supported smooth functions, the rapidly decreasing smooth functions and the corresponding tempered distributions, respectively. The pairing of $\mathcal{S}(\mathbb{R}^N)$ and $\mathcal{S}'(\mathbb{R}^N)$ is denoted by $\langle \cdot, \cdot \rangle$. $\mathcal{Z}(\mathbb{R}^N)$ is the space of all Schwartz functions $\varphi \in \mathcal{S}(\mathbb{R}^N)$ such that $D^\alpha \widehat{\varphi}(0) = 0$ for all multiindices $\alpha \in \mathbb{N}^N$, where $\widehat{\varphi}$ is the Fourier transform of φ. Then $\mathcal{Z}(\mathbb{R}^N)$ is a closed subspace of $\mathcal{S}(\mathbb{R}^N)$. If $\mathcal{Z}'(\mathbb{R}^N)$ denotes the space of all continuous linear functionals on $\mathcal{Z}(\mathbb{R}^N)$, then $\mathcal{S}'(\mathbb{R}^N)/\mathcal{P}(\mathbb{R}^N)$ and $\mathcal{Z}'(\mathbb{R}^N)$ are isomorphic, where $\mathcal{P}(\mathbb{R}^N)$ is the space of polynomials in N real variables (cf. [12, 5.1.2]).

Homogeneous Triebel-Lizorkin spaces. Let $\widehat{\phi} \in \mathcal{D}(\mathbb{R}^N)$ be radial, equal to 1 in $\overline{B}(0,1)$, and supported in $\overline{B}(0,2)$. Let $\widehat{\varphi} = \widehat{\phi} - \widehat{\phi}(2\cdot)$ and $\widehat{\varphi}_{2^j} = \widehat{\varphi}(2^j \cdot)$, $j \in \mathbb{Z}$.

Let $p, q \in [1, \infty)$ and $s \in \mathbb{R}$. The *homogeneous Triebel-Lizorkin space* $\dot{F}_p^{s,q}(\mathbb{R}^N)$ is the space consisting of all $f \in \mathcal{Z}'(\mathbb{R}^N)$ such that

$$\|f\|_{\dot{F}_p^{s,q}(\mathbb{R}^N)} := \left\| \left(\sum_{j \in \mathbb{Z}} \left(2^{-js} |f * \varphi_{2^j}| \right)^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^N)}$$

is finite. One can show that different choices of φ lead to equivalent norms and that $\dot{F}_p^{s,q}(\mathbb{R}^N)$, endowed with $\|\cdot\|_{\dot{F}_p^{s,q}(\mathbb{R}^N)}$, is a Banach space. Moreover $\mathcal{Z}(\mathbb{R}^N) \overset{d}{\hookrightarrow} \dot{F}_p^{s,q}(\mathbb{R}^N) \hookrightarrow \mathcal{Z}'(\mathbb{R}^N)$ (cf. [12, 5.1.5]).

Singular integral operators. Our main object of study is a continuous linear operator $T : \mathcal{S}(\mathbb{R}^N) \to \mathcal{S}'(\mathbb{R}^N)$. Its adjoint T' is a similar operator defined by $\langle \psi, T'\varphi \rangle := \langle \varphi, T\psi \rangle$.

Suppose that $K : \{(u,v) \in \mathbb{R}^N \times \mathbb{R}^N : u \neq v\} \to \mathbb{C}$ is a locally integrable function. We say that T is a *singular integral operator associated with K* if

$$\langle \varphi, T\phi \rangle = \int_{\mathbb{R}^N} \varphi(u) \int_{\mathbb{R}^N} K(u,v)\phi(v) dv \, du \tag{1}$$

holds for all $\varphi, \phi \in \mathcal{D}(\mathbb{R}^N)$ with $\mathrm{supp}\varphi \cap \mathrm{supp}\phi = \emptyset$. When this is the case, then T' is a singular integral operator with associated kernel K' given by $K'(u,v) = K(v,u)$ for $u \neq v$.

Next we introduce some more specific conditions on K and T:

The class $\mathrm{CZO}_{n+\nu}$. Let $n \in \mathbb{N}$ and $\nu \in (0,1)$. For a measurable kernel $K : \{(u,v) \in \mathbb{R}^N \times \mathbb{R}^N : u \neq v\} \to \mathbb{C}$, we consider the standard estimates

(SE$_n$) K is continuously differentiable up to order n with respect to the first variable and

$$\mathcal{C}_\alpha(K) = \sup\{|u-v|^{N+|\alpha|}|(\partial_u^\alpha K)(u,v)| : u \neq v\}$$

is finite for *all* multiindices $\alpha \in \mathbb{N}^N$ with $|\alpha| \leq n$;

(SE$_{n+\nu}$) K satisfies (SE$_n$) and

$$\mathcal{C}_{\alpha,\nu}(K) = \sup \left\{ |u-v|^{N+n+\nu} \frac{|(\partial_u^\alpha K)(u,v) - (\partial_u^\alpha K)(u_0,v)|}{|u-u_0|^\nu} : \atop |u-v| > 2|u-u_0| > 0 \right\}$$

is finite for *some* multiindex $\alpha \in \mathbb{N}^N$ with $|\alpha| = n$.

We say that $T \in \mathrm{CZO}_{n+\nu}$ if T is a singular integral operator associated with a kernel K satisfying (SE$_{n+\nu}$).

Note that $T \in \mathrm{CZO}_{n+\nu}$ does *not* imply that $T' \in \mathrm{CZO}_{n+\nu}$.

Definition of $T(u^\alpha)$. Let α be a multiindex with $|\alpha| \leq n$ and u^α the associated monomial. For $T \in \mathrm{CZO}_{n+\nu}$, it can be shown (see [10]) that $u^\alpha T'\varphi$ agrees with an integrable function in the exterior of any neighbourhood of $\mathrm{supp}\varphi$, provided that

$$\varphi \in \mathcal{D}^n(\mathbb{R}^N) := \{\varphi \in \mathcal{D}(\mathbb{R}^N) : \int_{\mathbb{R}^N} u^\beta \varphi(u) du = 0 \text{ for all } |\beta| \leq n\}.$$

Then we define

$$\langle \varphi, T(u^\alpha) \rangle := \langle \chi, u^\alpha T'\varphi \rangle + \langle 1 - \chi, u^\alpha T'\varphi \rangle,$$

where $\chi \in \mathcal{D}(\mathbb{R}^N)$ is any test function equal to unity in a neighbourhood of $\mathrm{supp}\varphi$. The first pairing above is the usual one between a test function and a distribution, while the second one may be evaluated as a convergent Lebesgue integral. The definition is independent of χ and produces a well-defined object $T(u^\alpha) \in (\mathcal{D}^n(\mathbb{R}^N))'$.

The weak boundedness property. We say that φ is a *normalized bump function* associated with the unit ball if $\varphi \in \mathcal{D}(\mathbb{R}^N)$ with $\mathrm{supp}\varphi \subseteq \overline{B}(0,1)$ and $\|D^\alpha\varphi\|_\infty \leq 1$ for all $|\alpha| \leq M$, where M is a large fixed number. ϕ is a normalized bump function associated with the ball $\overline{B}(u,r)$ if $\phi(\cdot) = r^{-N}\varphi(r^{-1}(\cdot - u))$, where φ is a normalized bump function associated with the unit ball. The operator T has the *weak boundedness property* if for every pair of normalized bump functions φ, ϕ associated with any ball $\overline{B}(u,r)$ we have $|\langle \phi, T\varphi \rangle| \leq Cr^{-N}$.

The various conditions defined above will be used through the following implied estimates, which are proved in [10] (see also [8]):

Lemma 2.1. *Let $k \in \mathbb{N}$, $a > 0$, $w \in \mathbb{R}^N$, and let $\varphi, \phi \in \mathcal{D}^0(\mathbb{R}^N)$ be normalized bump functions associated with $\overline{B}(0,a)$ and $\overline{B}(w, 2^k a)$ respectively. Suppose $T \in \mathrm{CZO}_{n+\nu}$ satisfies the weak boundedness property. Then:*

(a) there is a constant $C < \infty$ such that for all $v \in \mathbb{R}^N$

$$|\langle \phi(\cdot - v), T[\varphi(\cdot - v)] \rangle| \leq C \frac{1+k}{(a2^k)^N}\left(1 + \frac{|w|}{a2^k}\right)^{-N-\nu};$$

(b) if $T(u^\alpha) = 0$ for all $|\alpha| \leq n$, then there are constants $C < \infty$ and $\delta > 0$ such that for all $v \in \mathbb{R}^N$,

$$|\langle \phi(\cdot - v), T'[\varphi(\cdot - v)] \rangle| \leq C(a2^k)^{-N-n-\nu}\left(1 + \frac{|w|}{a2^k}\right)^{-N-\delta}.$$

3. The translation lemma

This section is devoted to the statement and proof of an estimate for translations in the reflexive $L_p(\ell_q)$ spaces. For $q = 2$, this is a square function estimate which was proved, apparently independently, by J. Bourgain [9] (with application to vector-valued singular integrals) and M. Yamazaki [13] (with application to pseudodifferential operators). In fact, Bourgain's result covers a much more general (and deeper) Banach space -valued situation, but this generality is to a different direction than our present needs. The proof given below is in the same spirit as Yamazaki's (i.e., vector-valued Calderón–Zygmund theory), but is not an immediate modification since we do not have the Hilbert space structure of ℓ_2 available here.

Lemma 3.1. *Let $1 < p, q < \infty$, and $h_i = 2^i k_i$, where $|k_i| \leq K$ for some $K > 2$ and for all $i \in \mathbb{Z}$. Let $\varphi : \mathbb{R}^N \to \mathbb{C}$ be a differentiable function with*

$$\|\varphi\|_{L_1} \leq c, \qquad |\nabla\varphi(x)| \leq c(1 + |x|)^{-N-1}, \qquad c > 0,$$

and denote $\varphi_{2^i} = 2^{-iN}\varphi(2^{-i}\cdot)$ for $i \in \mathbb{Z}$. Then there is a constant $C < \infty$ depending only on p, q, N and c, such that

$$\|(\tau_{h_i}\varphi_{2^i} * f_i)\|_{L_p(\ell_q)} \leq C \log K \|(f_i)\|_{L_p(\ell_q)}.$$

Here τ_{h_i} denotes the translation operator $f \mapsto f(\cdot - h_i)$.

In particular, if $\mathrm{supp}\widehat{f_i} \subseteq \{|\xi| \leq 2^{-i}\}$ for all $i \in \mathbb{Z}$, then

$$\|(\tau_{h_i}f_i)\|_{L_p(\ell_q)} \leq C \log K \|(f_i)\|_{L_p(\ell_q)}.$$

Proof. Note first that, once we have proved the first assertion, the second one immediately follows by choosing $\varphi \in \mathcal{S}(\mathbb{R}^N)$ such that $\widehat{\varphi}(\xi) = 1$ for $|\xi| \leq 1$; then $\widehat{\varphi_{2^i}}(\xi) = \widehat{\varphi}(2^i \xi) = 1$ on $\operatorname{supp}\widehat{f}_i$ so that $\varphi_{2^i} * f_i = f_i$. Let us thus consider the first assertion.

For $p = q$, we have $L_q(\ell_q) = \ell_q(L_q)$ and $\|\tau_{h_i}\varphi_{2^i} * f_i\|_{L_q} = \|\varphi_{2^i} * f_i\|_{L_q} \leq \|\varphi_{2^i}\|_{L_1}\|f_i\|_{L_q}$, and $\|\varphi_{2^i}\|_{L_1} = \|\varphi\|_{L_1} \leq c$, so the assertion holds with c in place of $C \log K$. The main part of the proof will consist of verifying that the diagonal-operator-valued kernel $\mathcal{K}(x) = \left(2^{-iN}\varphi(2^{-i}x - k_i)\right)_{i=-\infty}^{\infty}$ of the convolution operator $(f_i) \mapsto (\tau_{h_i}\varphi_{2^i} * f_i)$ satisfies the Hörmander integral condition

$$\int_{|x|>2|y|} \|\mathcal{K}(x - y) - \mathcal{K}(x)\|_{\mathcal{L}(\ell_q)}dx \leq C \log K. \tag{2}$$

This then yields the $L_p(\ell_q)$ boundedness of our operator for all $1 < p < \infty$, with the desired norm estimate, by the well-known vector-valued extension of the theory of singular integrals (see [14]).

For the proof of (2), note first that

$$
\begin{aligned}
&\|\mathcal{K}(x - y) - \mathcal{K}(x)\|_{\mathcal{L}(\ell^q)} \\
&= \sup_{i \in \mathbb{Z}} 2^{-iN}|\varphi(2^{-i}(x - y) - k_i) - \varphi(2^{-i}x - k_i)| \\
&= \sup_{i \in \mathbb{Z}} 2^{-i(N+1)}\left|\int_0^1 y \cdot \nabla\varphi(2^{-i}(x - \lambda y) - k_i)d\lambda\right| =: \sup_{i \in \mathbb{Z}} \mathcal{K}_i(x, y) \\
&\leq \sup_{2^i \leq |x|/4K} \mathcal{K}_i(x, y) + \sup_{|x|/4K < 2^i < |x|} \mathcal{K}_i(x, y) + \sup_{|x| \leq 2^i} \mathcal{K}_i(x, y).
\end{aligned}
\tag{3}
$$

For $|x| \geq \max(4K \cdot 2^i, 2|y|)$ we have

$$|2^{-i}(x - y) - k_i| \geq 2^{-i}(1 - 2^{-1} - 4^{-1})|x| = 2^{-i-2}|x|,$$

and hence

$$\mathcal{K}_i(x, y) \leq 2^{-i(N+1)}|y|C(2^{-i}|x|)^{-N-1} = C|y| \cdot |x|^{-N-1}.$$

For $|x| \leq 2^i$ we may estimate

$$\mathcal{K}_i(x, y) \leq 2^{-i(N+1)}|y|C \leq C|y| \cdot |x|^{-N-1}.$$

Since

$$\int_{|x|>2|y|} |x|^{-N-1}dx = C|y|^{-1},$$

we have handled the Hörmander estimate the first and third terms in (3), in fact with C in place of $C \log K$.

Concerning the second term in (3), we first consider the integral over $|x| > 4K|y|$ and estimate the supremum by the sum:

$$\int_{|x|>4K|y|} \sum_{|x|/4K<2^i<|x|} 2^{-i(N+1)}|y| \int_0^1 |\nabla\varphi(2^{-i}(x-\lambda y)+k_i)|d\lambda\,dx$$

$$= \sum_{2^i>|y|} 2^{-i(N+1)}|y| \int_0^1 \int_{2^i\vee 4K|y|<|x|\le 4K\cdot 2^i} |\nabla\varphi(2^{-i}(x-\lambda y)+k_i)|dx\,d\lambda$$

$$\le \sum_{2^i>|y|} 2^{-i}|y|\cdot\|\nabla\varphi\|_{L_1} \le C,$$

where in the first equality we could restrict the summation range of the i variable, since for $2^i \le |y|$ the x integration is over the empty set.

The remaining part in (3) yet to be estimated will give the principal contribution to the estimate, and is in particular responsible for the logarithmic factor $\log K$. As above, we estimate the supremum by the sum and interchange the order of summation and integration, arriving at

$$\sum_{|y|/2K<2^i<4K|y|} 2^{-iN} \int |\varphi(2^{-i}(x-y)-k_i) - \varphi(2^{-i}x-k_i)|dx$$

$$\le \sum_{|y|/2K<2^i<4K|y|} 2\|\varphi\|_{L_1} \le C\log K,$$

where the last estimate was simply counting the number of terms in the sum. This completes the proof of the Hörmander condition (2) and of the Lemma. $\square$

4. Proof of Theorem 1.1

The proof of Theorem 1.1 is done in three steps.

In the first step we make a decomposition of the operator T. For this purpose, we choose $\Phi \in \mathcal{D}^n(\mathbb{R}^N)$ and $\Psi \in \mathcal{S}(\mathbb{R}^N)$ such that

- Φ is real and radial,
- $\widehat{\Phi}, \widehat{\Psi} \ge 0$,
- $\widehat{\Phi}(u) \ge 1$ for $\frac{1}{2} \le |u| \le 2$,
- $\operatorname{supp}\widehat{\Psi} \subseteq \{\frac{1}{2} \le |u| \le 2\}$, and
- $\sum_{j\in\mathbb{Z}} \widehat{\Phi}(2^j u)\widehat{\Psi}(2^j u) = 1$ for all $u \in \mathbb{R}^N \setminus \{0\}$.

The existence of such Φ, Ψ is shown in [10] (see also [8]). For $j \in \mathbb{Z}$, we denote $\Phi_{2^j}(u) := 2^{-Nj}\Phi(2^{-j}u)$, $\Psi_{2^j}(u) := 2^{-Nj}\Psi(2^{-j}u)$, $P_j f := \Phi_{2^j} * f$, and $Q_j f = \Psi_{2^j} * f$. Now we can write

$$\langle g, Tf \rangle = \sum_{j,l\in\mathbb{Z}} \langle P_l Q_l g, T P_j Q_j f \rangle = \sum_{j,k\in\mathbb{Z}} \langle Q_{j+k}g, P_{j+k} T P_j Q_j f \rangle,$$

if $f, g \in \mathcal{S}(\mathbb{R}^N)$. The operator $T_{j+k,j} := P_{j+k} T P_j$ is associated with the kernel $K_{j+k,j}$ given by

$$K_{j+k,j}(u, v) = \langle \Phi_{2^{j+k}}(\cdot - u), T[\Phi_{2^j}(\cdot - v)] \rangle.$$

In the second step we fix $k \in \mathbb{Z}$ and estimate the sum over $j \in \mathbb{Z}$ in the above decomposition.

We first look at the case $k \in \mathbb{N}$. For $f, g \in \mathcal{Z}(\mathbb{R}^N)$,

$$\left| \sum_{j \in \mathbb{Z}} \langle Q_{j+k} g, T_{j+k,j} Q_j f \rangle \right| = \left| \sum_{j \in \mathbb{Z}} \langle 2^{(j+k)s} (T_{j+k,j})' Q_{j+k} g, 2^{-(j+k)s} Q_j f \rangle \right|$$

$$\leq \left\| \left(\sum_{j \in \mathbb{Z}} |2^{(j+k)s} (T_{j+k,j})' Q_{j+k} g|^{q'} \right)^{1/q'} \right\|_{L_{p'}} \left\| \left(\sum_{j \in \mathbb{Z}} |2^{-(j+k)s} Q_j f|^q \right)^{1/q} \right\|_{L_p},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. The second factor is bounded by $C 2^{-ks} \| f \|_{\dot{F}_p^{s,q}}$ (cf. [12, 5.1.5]). To estimate the first factor, we note that for $v \in \mathbb{R}^N$

$$[(T_{j+k,j})' \phi](v) = \int_{\mathbb{R}^N} K_{j+k,j}(u, v) \phi(u) du$$

$$= \int_{\mathbb{R}^N} 2^{jN} K_{j+k,j}(v + 2^j u, v) \phi(v + 2^j u) du,$$

and therefore

$$A := \left\| \left(\sum_{j \in \mathbb{Z}} |2^{(j+k)s} (T_{j+k,j})' Q_{j+k} g|^{q'} \right)^{1/q'} \right\|_{L_{p'}}$$

$$\leq \int_{\mathbb{R}^N} \left\| \left(\sum_{j \in \mathbb{Z}} |2^{(j+k)s} 2^{jN} K_{j+k,j}(\cdot + 2^j u, \cdot)(Q_{j+k} g)(\cdot + 2^j u)|^{q'} \right)^{1/q'} \right\|_{L_{p'}} du.$$

Using Lemma 2.1 (a), we obtain the estimate

$$\sup_{v \in \mathbb{R}^N} \sup_{j \in \mathbb{Z}} |2^{jN} K_{j+k,j}(v + 2^j u, v)| \leq C \frac{1+k}{(a 2^k)^N} \left(1 + \frac{|u|}{a 2^k} \right)^{-N-\nu},$$

where $a > 0$ is such that $\operatorname{supp} \Phi \subseteq \overline{B}(0, a)$. So with Lemma 3.1 we obtain

$$A \leq C \int_{\mathbb{R}^N} \frac{1+k}{(a 2^k)^N} \left(1 + \frac{|u|}{a 2^k} \right)^{-N-\nu} \log \left(2 + \frac{|u|}{2^k} \right) \| g \|_{\dot{F}_{p'}^{-s,q'}} du$$

$$\leq C(1+k) \| g \|_{\dot{F}_{p'}^{-s,q'}}.$$

Combining with the bound for the f factor, we have

$$\left| \sum_{j \in \mathbb{Z}} \langle Q_{j+k} g, T_{j+k,j} Q_j f \rangle \right| \leq C(1+k) 2^{-ks} \| f \|_{\dot{F}_p^{s,q}} \| g \|_{\dot{F}_{p'}^{-s,q'}}.$$

If k is negative, we proceed in a similar way. Here we use the integral representation for $T_{j+k,j}$ and Lemma 2.1 (b) to obtain

$$\left| \sum_{j \in \mathbb{Z}} \langle Q_{j+k} g, T_{j+k,j} Q_j f \rangle \right| \leq C 2^{k(n+\nu-s)} \| f \|_{\dot{F}_p^{s,q}} \| g \|_{\dot{F}_{p'}^{-s,q'}}.$$

As the third step, we carry out the summation over k. By our results from the previous step,

$$\left|\sum_{j,k\in\mathbb{Z}}\langle Q_{j+k}g, T_{j+k,j}Q_j f\rangle\right| \leq C\|f\|_{\dot{F}_p^{s,q}}\|g\|_{\dot{F}_{p'}^{-s,q'}}$$

if $s \in (0, n+\nu)$. Since $\mathcal{Z}(\mathbb{R}^N)$ is dense both in $\dot{F}_p^{s,q}(\mathbb{R}^N)$ and in $\dot{F}_{p'}^{-s,q'}$, the first part of the theorem is proved.

For the second part, we observe that our additional assumptions imply that we can use Lemma 2.1 (b) in Step 2 to show that

$$\left|\sum_{j\in\mathbb{Z}}\langle Q_{j+k}g, T_{j+k,j}Q_j f\rangle\right| \leq C2^{-|k|(n+\nu-|s|)}\|f\|_{\dot{F}_p^{s,q}}\|g\|_{\dot{F}_{p'}^{-s,q'}}.$$

This completes the proof of the theorem.

Acknowledgements

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