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# A Hopf bifurcation theorem for the vorticity formulation of the Navier-Stokes equations in three space dimensions

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**Abstract**

We prove a Hopf bifurcation theorem for the vorticity formulation of the Navier-Stokes equations in $\mathbb{R}^3$ in case of spatially localized external forcing. The difficulties are due to essential spectrum up to the imaginary axis for all values of the bifurcation parameter which a priori no longer allows to reduce the problem to a finite dimensional one.
Moreover, we discuss the nonlinear stability of the trivial solution and the exchange of spectral stability of the bifurcating time-periodic solutions with respect to spatially localized perturbations.

**To my parents**

# Contents

# 1 Introduction

The flow around some obstacle is a paradigm for bifurcation theory. For increasing velocities of the fluid the laminar flow becomes unstable and bifurcates into a time-periodic flow. The next bifurcation gives the von Karman vortex street, and finally turbulent flow can be observed.

Although a number of analytic results are known for the steady flow very few is known analytically about the bifurcation theory. The reason for this is the continuous spectrum up to the imaginary axis for all values of the bifurcation parameter. Hence classical methods as the center manifold theorem or the Lyapunov-Schmidt method fail a priori to reduce the bifurcation problem to a finite dimensional one. However, in [20] under the assumption that a family of steady solutions with certain spectral properties exist, the occurrence of a Hopf bifurcation has been shown. The proof is based on a Lyapunov-Schmidt reduction and on the invertibility of the Oseen operator from $L^p$ to $L^q$ with $p < q$ suitably chosen.

Motivated by the paper [21] in which the spatial structure of the bifurcating time-periodic solutions in reaction-diffusion convection problems with similar properties has been analyzed, it has been the purpose of this work to analyze the question if the nontrivial time-periodic part decays with some exponential rate in space. Due to the non smoothness of the symbol of the projection operator on the divergence-free vector fields this cannot be expected to be true for the velocity. Therefore, our aim is to prove a Hopf bifurcation theorem for the vorticity formulation of the Navier-Stokes equations. However, it turned out that even for the vorticity formulation the spatial localization cannot be proved with method of [21]. Instead of the obstacle problem we consider the Navier-Stokes equations in $\mathbb{R}^3$ with localized external, time-independent forcing.

In the rest of this introduction, we introduce the obstacle problem, explain the principle of a Hopf bifurcation, explain the underlying scheme of our Hopf bifurcation theorem and present our other results, as stability of the trivial solution and exchange of spectral stability.

## 1.1 The obstacle problem

The following figures 1.11.2[28] [1] show a von Karman vortex street.


Figure 1.1: Von Karman vortex street over the Aleutian Islands

[1]The first picture was shot by the satellite Landsat 7 on 4th July 2002 over the pacific in the area of the Aleutian Islands, the second on 15th September 1999 on the Selkirk Island also in the pacific by Landsat 7.

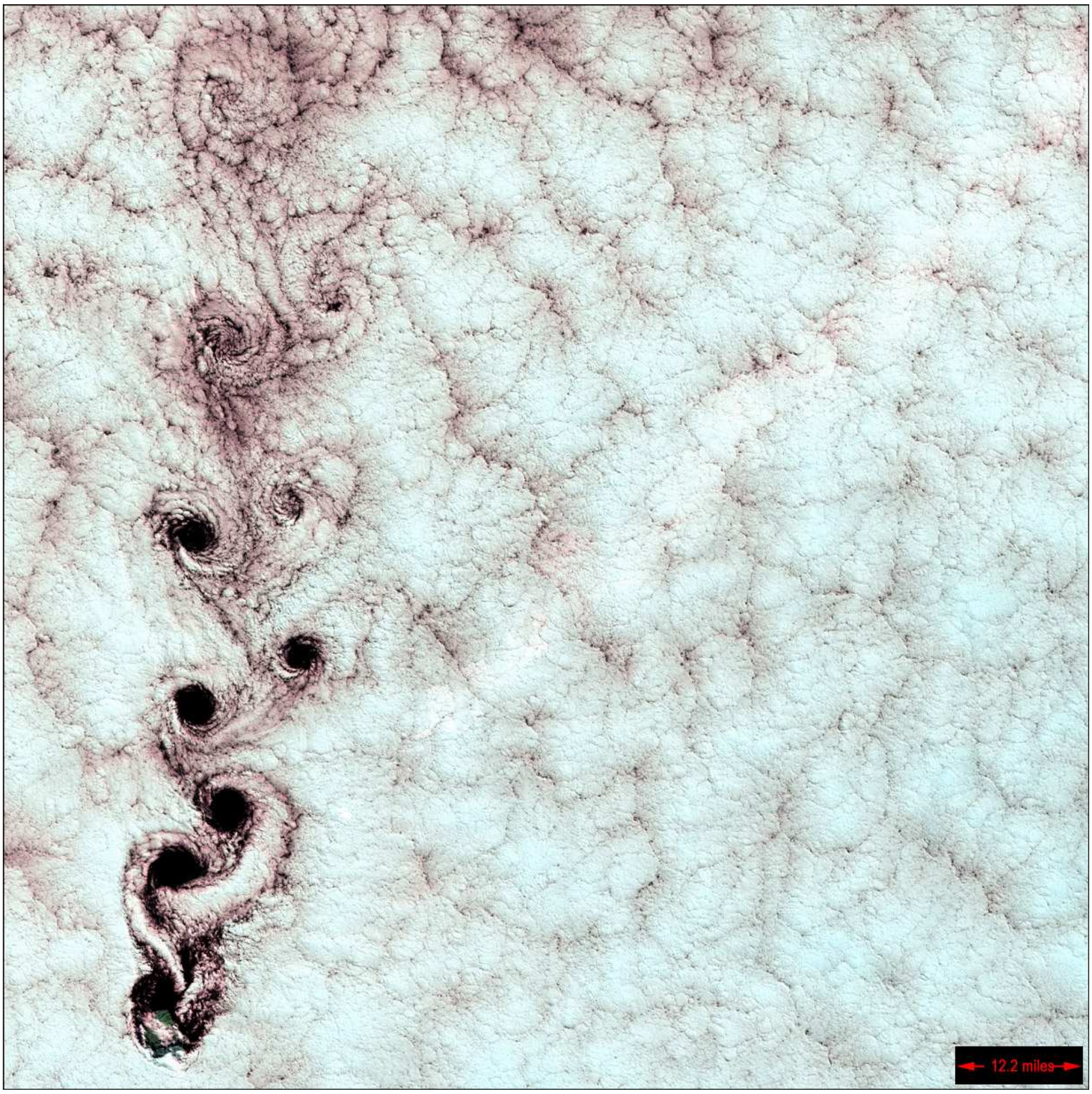



Figure 1.2: Von Karman vortex street over the Selkirk Island

The mathematical model for the flow around an obstacle (for instance see [2]) are the Navier-Stokes equations, which describe the flow of a viscous, incompressible fluid

$$\begin{aligned}
\rho\left(\frac{\partial}{\partial t}\tilde{u}(x,t) + (\tilde{u}(x,t)\cdot\nabla)\tilde{u}(x,t)\right) &= \mu\Delta\tilde{u}(x,t) - \nabla p(x,t) + f(x,t) \\
\nabla\cdot\tilde{u}(x,t) &= 0, \qquad (1.1)
\end{aligned}$$

with the velocity field $\tilde{u}(x,t) \in \mathbb{R}^d$, and the pressure field $p(x,t) \in \mathbb{R}$ at time $t$ in the point $x \in \mathbb{R}^d$. $\rho$ and $\mu$ denote the constant density and viscosity of the fluid and $f$ is the external force, which is zero for this problem. The scalar product of two vectors in $\mathbb{R}^d$ is denoted by $\cdot$. The $\nabla$-operator is the vector $\nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d})^T$ and $\Delta = \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$ denotes the Laplacian. Furthermore we have $(\tilde{u}(x,t) \cdot \nabla)\tilde{u}(x,t) = \sum_{j=1}^d \tilde{u}_j(x,t)\frac{\partial}{\partial x_j}\tilde{u}(x,t)$ and $d = 2,3$ is the space dimension.

Further some initial and boundary values are needed to describe the problem completely

$$
\begin{aligned}
\tilde{u}(x,t) &= 0, \text{ for all } x \in \Sigma \text{ and all } t \in [0,T] \\
\lim_{|x| \to \infty} \tilde{u}(x,t) &= \tilde{u}_\infty, \text{ for all } t \in [0,T] \\
\tilde{u}(x,0) &= \tilde{u}^0(x), \text{ for all } x \in \mathcal{E},
\end{aligned} \tag{1.2}
$$

where $\Sigma = \partial\Omega$ denotes the surface of the body $\Omega$ and $\tilde{u}^0$ is the initial velocity at time $t = 0$. $\mathcal{E} := \mathbb{R}^d \backslash (\Omega \cup \Sigma)$, $d = 2,3$ is the area taken by the fluid and $\tilde{u}_\infty$ denotes the velocity of the fluid at infinity. For a detailed mathematical theory we refer to the papers of Finn [5, 6], Shibata [22, 23, 24, 25], Bemelmans [2] or the book of Galdi [7, 8] and the work of Meister [16].

Due to the reasons explained above (1.1) is considered as an equation for the vorticity $u = \nabla \times \tilde{u}$, namely

$$
\begin{aligned}
\rho\Big(\frac{\partial}{\partial t}u(x,t) &+ (\tilde{u}(x,t) \cdot \nabla)u(x,t) - (u(x,t) \cdot \nabla)\tilde{u}(x,t)\Big) \\
&= \mu\Delta u(x,t) + \nabla \times f(x,t) \\
\nabla \cdot u(x,t) &= 0,
\end{aligned} \tag{1.3}
$$

where $\tilde{u}(x,t)$ is given by the Biot-Savart law

$$
\tilde{u}(x,t) = -\frac{1}{4\pi} \int\limits_{\mathbb{R}^3} \frac{(x-y) \times u(y,t)}{|x-y|^3} dy. \tag{1.4}
$$

Note that the Biot-Savart law represents an integral relation with a singular kernel. Since the search of suitable boundary conditions for (1.3) is still a subject of the scientific discussion we circumvent this problem, by considerung (1.3) in $\mathbb{R}^3$

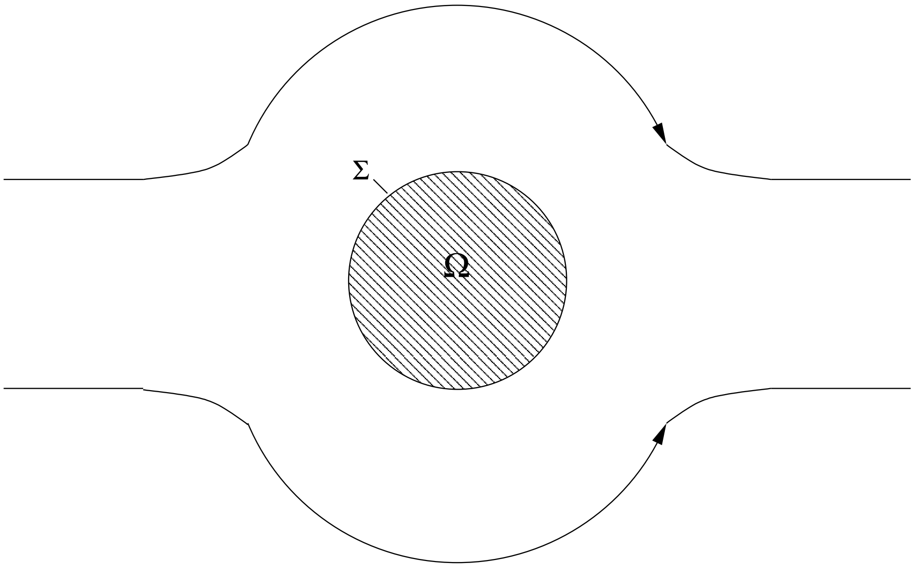



Figure 1.3: Flow around a body

but now with localized external forcing $f$ representing the obstacle. We assume that the external force $f$ is chosen such a way there exists a stationary flow $u_c$ with certain properties, see the sections 5 and 6. The deviation from $u_c$ satisfies

$$\begin{aligned} \frac{\partial}{\partial t} u &= \Delta u - c\frac{\partial}{\partial x_1} u + 2\nabla \cdot Q(u_c, u) + \nabla \cdot Q(u,u), \\ u(x,0) &= u^0(x), \end{aligned} \tag{1.5}$$

where $u(x,t) \in \mathbb{R}^3$ is the vorticity and $2Q(u,v) = v\tilde{u} + u\tilde{v} - \tilde{v}u - \tilde{u}v$ the nonlinearity.

In order to analyze the linear operator into an optimal way we transform (1.5) into Fourier space.

$$\begin{aligned} \frac{\partial}{\partial t} \widehat{u} &= -(|\xi|^2 + ic\xi_1)\widehat{u} + 2i\xi \cdot \widehat{Q}(\widehat{u}_c, \widehat{u}) + i\xi \cdot \widehat{Q}(\widehat{u}, \widehat{u}), \\ \widehat{u}(\xi, 0) &= \widehat{u}^0(\xi). \end{aligned} \tag{1.6}$$

with $\widehat{u}(\xi,t) \in \mathbb{R}^3$ and $2\widehat{Q}(\widehat{u},\widehat{v}) = \widehat{v} * \widehat{\tilde{u}} + \widehat{u} * \widehat{\tilde{v}} - \widehat{\tilde{v}} * \widehat{u} - \widehat{\tilde{u}} * \widehat{v}$.

## 1.2 Finite-dimensional Hopf bifurcation theory

Bifurcation theory is a widely used tool in the theory of dynamical systems, to describe the behavior of solutions associated to a dynamical system, which

depends on some parameter. Several types of bifurcations can occur. In the sequel we are interested in the theory of a Hopf bifurcation.
In general a bifurcation occurs if a stationary solution of a dynamical system loses its stability for some parameter value. This happens, when one or more eigenvalues of the linearization cross the imaginary axis.
Roughly speaking, a Hopf bifurcation occurs when a pair of complex conjugate eigenvalues of the linearization around a stationary solution cross the imaginary axis. Then the new bifurcating solutions are time-periodic.
We will now explain the principles of a Hopf bifurcation in a system of two ordinary differential equations

$$\begin{aligned} \dot{x} &= \mu x + y - x(x^2+y^2) \\ \dot{y} &= -x + \mu y - y(x^2+y^2). \end{aligned} \tag{1.7}$$

The linearized system around the trivial solution $(x, y) = (0, 0)$ has the form

$$\begin{aligned} \dot{\xi} &= \mu\xi + \eta \\ \dot{\eta} &= -\xi + \mu\eta, \end{aligned} \tag{1.8}$$

with the complex conjugate eigenvalues $\lambda_{1,2} = \mu \pm i$. For $\mu < 0$, the trivial solution is asymptotically stable and for $\mu > 0$, the trivial solution becomes unstable.
If we introduce polar coordinates $x = r\cos(\phi),\ y = r\sin(\phi)$ with $r \geq 0, \phi \in \mathbb{R}\backslash(2\pi\mathbb{Z})$ then equation (1.7) becomes

$$\begin{aligned} \dot{r} &= \mu r - r^3 \\ \dot{\phi} &= 1, \end{aligned} \tag{1.9}$$

possessing the solution

$$\begin{aligned} x(t) &= \sqrt{\mu}\sin(t+\phi_0) \\ y(t) &= \sqrt{\mu}\cos(t+\phi_0). \end{aligned} \tag{1.10}$$

We see that for $\mu = 0$ a family of periodic solutions bifurcates from the trivial solution. This behavior is called a supercritical Hopf bifurcation. This two dimensional example can be generalized to the $\mathbb{R}^d$, and the general result has been proven by Hopf [13] in the year 1942.

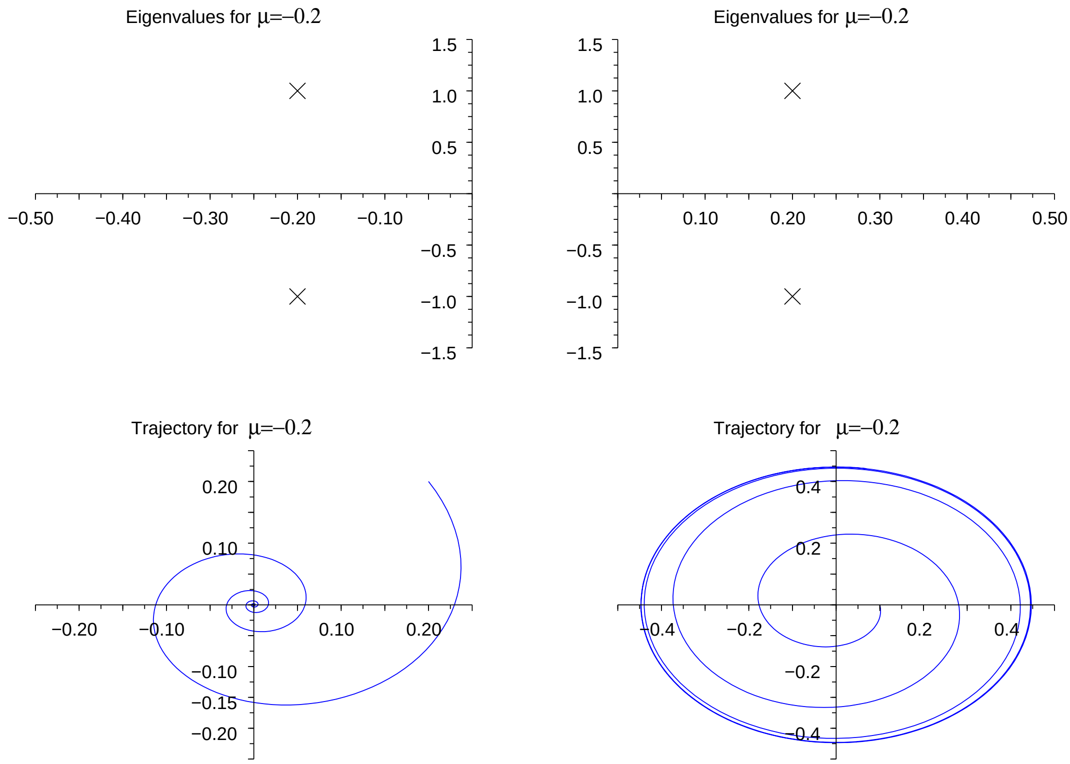



Figure 1.4: Scenario of a Hopf bifurcation

**Theorem 1.1** *Consider the system*

$$\dot{x} = f(x, \mu), \; x \in \mathbb{R}^d, \; \mu \in \mathbb{R}. \tag{1.11}$$

*Assume that this system has a stationary solution in* $(x_0, \mu_0)$ *i.e.* $f(x_0, \mu_0) = 0$ *and the Jacobian* $f_x(x_0, \mu_0)$ *has a single pair of purely imaginary eigenvalues* $\lambda(\mu_0) = \pm i\beta$*, no other eigenvalues with vanishing real part which are resonant, i.e.* $\pm in\beta$ *is not an eigenvalue of* $f_x(x_0, \lambda_0)$ *for* $n \in \mathbb{Z}\backslash\{-1, 1\}$ *and* $\frac{dRe(\lambda(\mu))}{d\mu} \neq 0$*. Then at* $(x_0, \mu_0)$ *periodic solutions will bifurcate.*

For detailed theory of this subject we refer the books of Guckenheimer, Holmes [9] and Wiggins [27].

An abstract setup for Hopf bifurcations in the infinite-dimensional case with applications to PDEs can be found in the book of Kielhöfer [15].

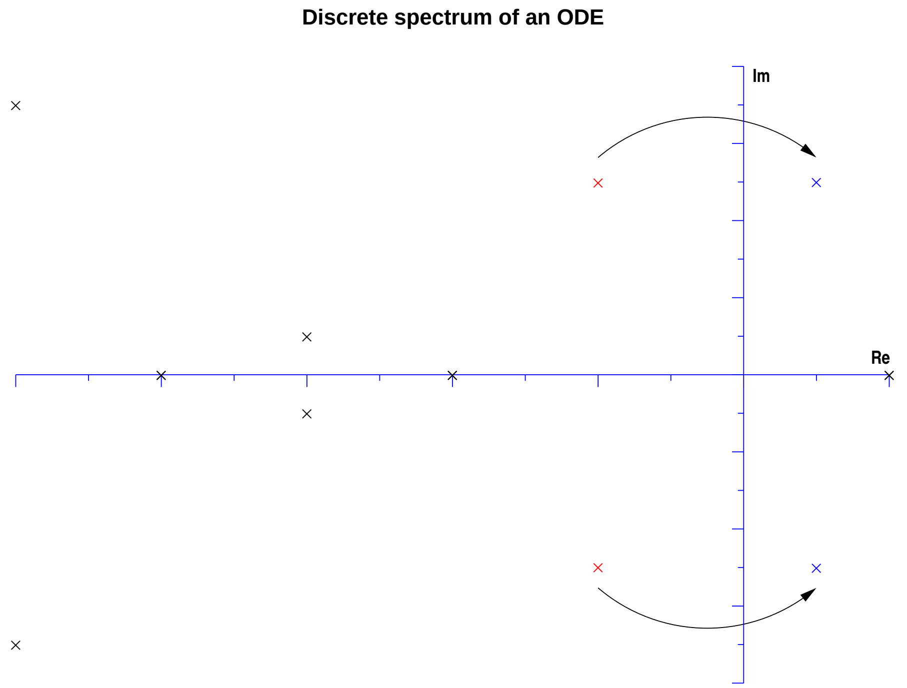



Figure 1.5: Spectrum in the ODE-Case: A pair of conjugate complex eigenvalues cross the imaginary axis

The proof is based on a Lyapunov-Schmidt reduction or a reduction to the center manifold under the assumption that the spectrum possesses a non-vanishing gap to the imaginary axis.

The application of this theory to the problem of the flow around an obstacle leads to some problems due to the fact that the associated spectrum has no gap to the imaginary axis, and so the traditional theory is no longer applicable.

Therefore due to the additional reason from page 9 our intention is to prove a Hopf bifurcation result for the vorticity formulation of the Navier-Stokes equations. Such a result has been shown for the velocity formulation in [20]. However our results covers a larger class of solutions.

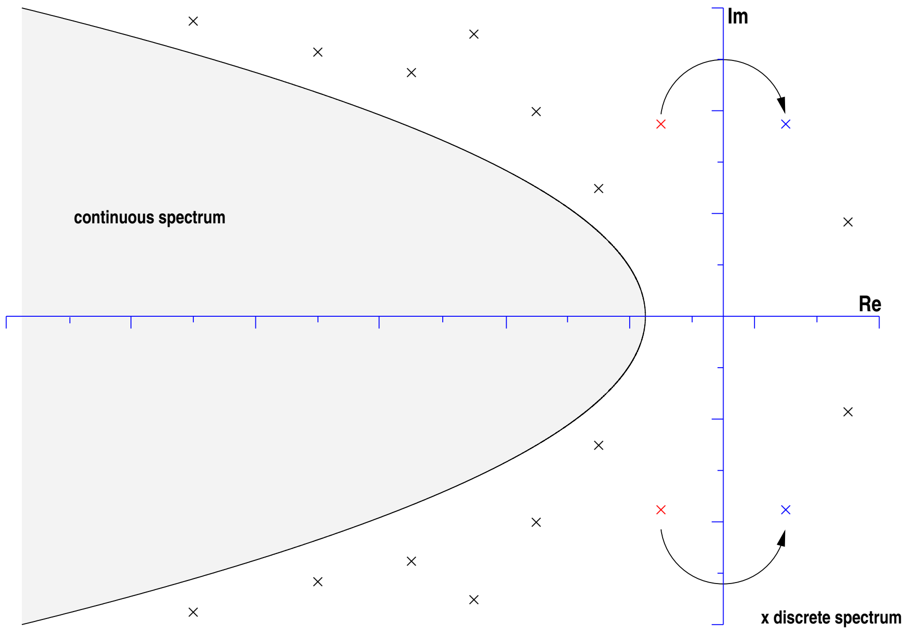



Figure 1.6: Spectrum in the PDE-case: Conjugate complex eigenvalues cross the imaginary axis with spectral gap

## 1.3 The Hopf bifurcation theorem

Let $\widehat{u}_c = \widehat{u}_c(\xi, \alpha)$ be a family of stationary solutions of the vorticity formulation depending on some parameter $\alpha$ in Fourier space. The vorticity formulation of the Navier-Stokes equations with $\widehat{u}_c$ as origin is then given by

$$\frac{\partial}{\partial t}\widehat{u} = \widehat{L}\widehat{u} + \widehat{N}(\widehat{u}), \tag{1.12}$$

where

$$\begin{aligned}
\widehat{L}\widehat{u} &= \widehat{B}\widehat{u} - 2i\xi \cdot \widehat{Q}(\widehat{u}_c, \widehat{u}), \\
\widehat{B}\widehat{u} &= -|\xi|^2\widehat{u} - ic\xi_1\widehat{u}, \\
2\widehat{Q}(\widehat{u}_c, \widehat{u}) &= \widehat{u}_c * \widehat{\widetilde{u}} + \widehat{u} * \widehat{\widetilde{u}}_c - \widehat{\widetilde{u}}_c * \widehat{u} - \widehat{\widetilde{u}} * \widehat{u}_c, \\
\widehat{N}(\widehat{u}) &= -i\xi \cdot (\widehat{\widetilde{u}} * \widehat{u} - \widehat{u} * \widehat{\widetilde{u}})
\end{aligned} \tag{1.13}$$

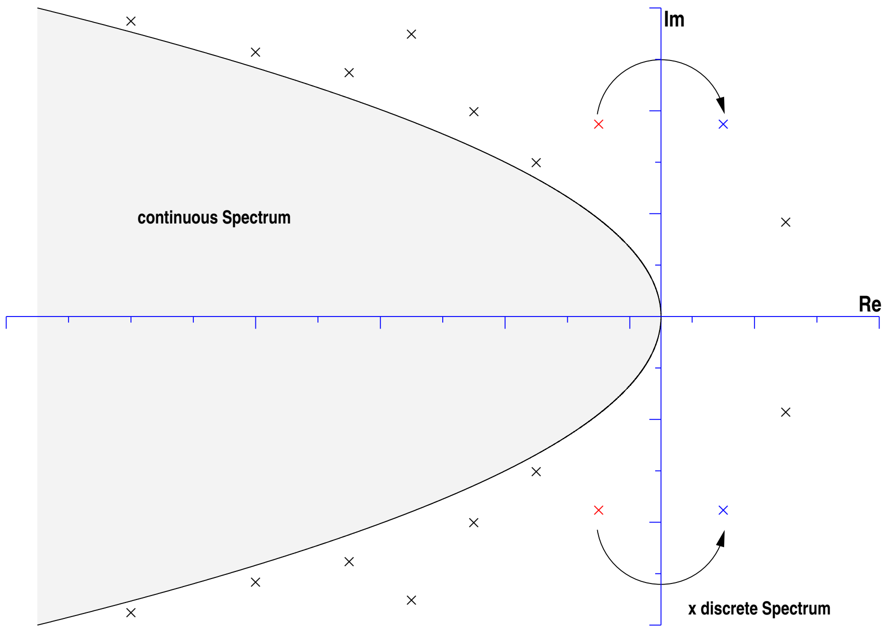



Figure 1.7: Spectrum in the PDE-Case: Conjugate complex eigenvalues cross the imaginary axis without spectral gap.

and where $\widehat{\widehat{u}}$ is the velocity field reconstructed from the vorticity field $\widehat{u}$ via the Biot-Savart Law in Fourier space. We note that $\widehat{u}*\widehat{u}$ is an abbreviation for $\widehat{u}*\widehat{u}^T$ [2].

At the bifurcation point $\alpha = \alpha_c$ the spectrum of $\widehat{L}$ consists of continuous spectrum up to the imaginary axis, two eigenvalues $\pm i\omega_0$ and a number of other discrete eigenvalues lying strictly in the left half plane. There exist projections $P_{c,\pm 1}$ onto the eigenspaces associated to the eigenvalues $\pm i\omega_0$. We define $P_s = I - P_{c,1} - P_{c,-1}$ and make the ansatz

$$\widehat{u}(\xi, t) = \sum_{n\in\mathbb{Z}} \widehat{u}_n(\xi) \exp(in\omega t) \tag{1.14}$$

for the bifurcating time-periodic solutions. With $P_n$ defined by

[2] For details we refer to section 2

$$P_n\widehat{u} = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \exp(-in\omega t)\widehat{u}(\xi,t)dt \tag{1.15}$$

we obtain the system

$$in\omega\widehat{u}_n = \widehat{L}\widehat{u}_n + P_n\widehat{N}(\widehat{u}) \tag{1.16}$$

for $n \in \mathbb{Z}$. The idea of the Lyapunov-Schmidt reduction method is to invert the linear operator $(in\omega I - \widehat{L})_{n\in\mathbb{Z}}$ in the biggest possible subspace in order to reduce the bifurcation problem to a finite dimensional one on the kernel of this operator. Due to the spectral assumptions on $\widehat{L}$, we have the invertibility of $in\omega I - \widehat{L}$ for $n = \pm 2, \pm 3, \dots$ and of $(in\omega I - \widehat{L})P_s$ for $n = \pm 1$. Moreover, we have the invertibility of $\widehat{L}$, i.e. for $n = 0$, as operator, if $\widehat{L}^{-1}$ is applied to $\xi$. In Fourier space $\widehat{L}^{-1}\xi_j$ is a bounded operator from $L^p \cap \widehat{L}^\infty$ to $L^p$ if

$$p \in [1,4), \tag{1.17}$$

cf. theorem 4.1.

By Young's inequality we have $\widehat{\widehat{u}} * \widehat{u} \in L^r$ for $\widehat{u} \in L^p$ and $\widehat{\widehat{u}} \in L^q$ if

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \tag{1.18}$$

especially $\widehat{\widehat{u}} * \widehat{u} \in L^\infty$, if $1 = \frac{1}{p} + \frac{1}{q}$.

The Biot-Savart law in Fourier space allows to construct the velocity $\widehat{\widehat{u}} \in L^q$ from the vorticity $\widehat{u} \in L^p$ if

$$\frac{1}{q} = \frac{1}{p} + \frac{1}{r^*} \tag{1.19}$$

for $1 \le q < p \le \infty$ and $r^* \in [1,3)$. Thus by choosing $p = 4 - \delta$ we find from (1.19) $q = 4/3 + \mathcal{O}(\delta)$ which gives $r^* = 2$ in (1.19), which is allowed. In fact we can chose $p \in (3,4)$.
As a consequence the first three equations of

$$\widehat{u}_n = (in\omega I - \widehat{L})^{-1}P_n\widehat{N}(\widehat{u}), \; n = \pm 2, \pm 3 \dots,$$

$$
\begin{aligned}
\widehat{u}_{n,s} &= (in\omega I - \widehat{L})^{-1} P_n P_s \widehat{N}(\widehat{u}), \ n = \pm 1, \\
\widehat{u}_0 &= \widehat{L}^{-1} P_0 \widehat{N}(\widehat{u}), \\
\widehat{u}_{n,c} &= (in\omega I - \widehat{L})^{-1} P_n P_c \widehat{N}(\widehat{u}), \ n = \pm 1
\end{aligned} \tag{1.20}
$$

can be solved for $\widehat{u}_n$, $\widehat{u}_{n,s} = P_s \widehat{u}_n$, $\widehat{u}_0$ in terms of $\widehat{u}_{1,c} = P_{c,1}\widehat{u}_1$ and $\widehat{u}_{-1,c} = P_{c,-1}\widehat{u}_{-1}$ in the space

$$
\widehat{\mathcal{X}}_s^p = \{\widehat{u} = (\widehat{u}_n)_{n\in\mathbb{Z}} : \|\widehat{u}\|_{\widehat{\mathcal{X}}_s^p} = \sum_{n\in\mathbb{Z}} \|\widehat{u}_n\|_{L_s^p} < \infty\} \tag{1.21}
$$

for every $p \in (3,4)$ and with $s > 1$ which is necessary also to have $\|\widehat{\widehat{u}} * \widehat{u}\|_{L_s^p} \leq C\|\widehat{u}\|_{L_s^p}^2$ in the case $n \neq 0$.

Thus, the bifurcation problem can be reduced to a problem for $\widehat{u}_{1,c}$ and $\widehat{u}_{-1,c}$ alone. Introducing coordinates $A_j$, where $u_j = A_j \widehat{\varphi}_j$ with $\widehat{\varphi}_j \in L_s^p$ the eigenfunctions associated to the eigenvalues $\pm i\omega_0$, we find the reduced problem

$$
\begin{aligned}
g_1(\alpha - \alpha_c, \omega - \omega_0, A_1, A_{-1}) &= 0, \\
g_{-1}(\alpha - \alpha_c, \omega - \omega_0, A_1, A_{-1}) &= 0
\end{aligned} \tag{1.22}
$$

where $g_j : \mathbb{R}^2 \times \mathbb{C}^2 \mapsto \mathbb{R}$ for $j = \pm 1$.

Since we have an autonomous problem, the reduced problem has to be invariant under $A_1 \mapsto A_1 \exp(i\phi)$ and $A_{-1} \mapsto A_{-1}\exp(i\phi)$ and so we have that $g_1$ and $g_{-1}$ are of the form

$$
\begin{aligned}
A_1 \tilde{g}_1(\alpha - \alpha_c, \omega - \omega_0, |A_1|^2) &= 0, \\
A_{-1} \tilde{g}_{-1}(\alpha - \alpha_c, \omega - \omega_0, |A_1|^2) &= 0.
\end{aligned} \tag{1.23}
$$

Introducing polar coordinates $A_1 = r\exp(i\phi)$ yields

$$
\begin{aligned}
(\alpha - \alpha_c) + \gamma r^2 + \mathcal{O}(|\alpha - \alpha_c|^2 + |\omega - \omega_0|^2 + r^4) &= 0, \\
\omega - \omega_0 + \mathcal{O}(|r|^2 + |\alpha - \alpha_c|^2 + |\omega - \omega_0|^2) &= 0,
\end{aligned} \tag{1.24}
$$

i.e. the well known reduced system in case of a Hopf bifurcation. This system possesses nontrivial solutions with $r = \mathcal{O}(|\alpha - \alpha_c|^{\frac{1}{2}})$ and $\omega = \mathcal{O}(|\alpha - \alpha_c|)$ which correspond to time-periodic solutions of the original system.

Thus we will prove

**Theorem 1.2** *If* $\gamma$ *in* (1.24) *is non-vanishing and the assumptions (A0)-(A4) stated in section 4.1 are valid, then equation* (1.12) *has a one-dimensional family of small time-periodic solutions* $\widehat{u}_{per}$*, i.e. there exists an* $\epsilon_0 > 0$ *such that for* $\epsilon \in (0, \epsilon_0)$

$$\widehat{u}_{per}(\xi, t) = \widehat{u}_{per}\left(\xi, t + \frac{2\pi}{\omega}\right) \tag{1.25}$$

*solves* (1.6) *for* $\alpha = \alpha_c - sgn(\gamma)\epsilon$. *Furthermore we have* $\omega = \omega_0 + \mathcal{O}(\epsilon)$ *and* $\|u_{per}\|_{C_b^0\left(\mathbb{R}^3 \times \left[0, \frac{2\pi}{\omega}\right]\right)} = \mathcal{O}(\sqrt{\epsilon})$.

We note that if the vorticity is time-periodic then the velocity is time-periodic too.

## 1.4 Other results

In section 6 we consider the stability of the trivial solution $\widehat{u} = 0$ of

$$\frac{\partial}{\partial t}\widehat{u} = \widehat{L}\widehat{u} + i\xi \cdot \widehat{Q}(\widehat{u}, \widehat{u}) \tag{1.26}$$

for parameter values $\alpha < \alpha_c$ with $\widehat{L} = \widehat{B} + 2i\xi \cdot \widehat{Q}(\widehat{u}_c, \cdot)$. We make extensive use of semigroup theory since $\widehat{L}$ generates an analytic semigroup [18]. Our main tool is the Duhamel formula

$$\widehat{u}(t) = \exp(\widehat{L}t)\widehat{u}^0 + \int_0^t \exp(t-\tau) i\xi \cdot \widehat{Q}(\widehat{u}, \widehat{u})(\tau) d\tau, \tag{1.27}$$

since any mild solution of (1.26) can be expressed with this formula. With this tool and the assumptions (ASS1) stated in section 6 we can show the nonlinear stability of the trivial solution $\widehat{u} = 0$ .

**Theorem 1.3** *Let* $p, q \in [1, \infty]$, $q \le p$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, $r < d = 3$, $s > 3\frac{p-1}{p}$ *and assume that (ASS1) holds. For all* $C_2$ *there exists a* $C_1 > 0$ *such that for* $\widehat{u}(0) \in L_s^p \cap L_s^q$ *with* $\|\widehat{u}(0)\|_{L_s^p \cap L_s^q} \le C_1$ *we have solutions* $\widehat{u}(t)$ *satisfying*

$$\|\widehat{u}(t)\|_{L_s^p} + (1+t)^{\frac{d}{2r}} \|\widehat{u}(t)\|_{L_s^q} \le C_2 \tag{1.28}$$

*for all* $t > 0$.

We prove the validity of (ASS1) for the semigroup generated by $\widehat{L}$, if the stationary solution $\widehat{u}_c(\xi)$ is small. To do this, initially we show the estimates of assumption (ASS1) for the operator $\widehat{B} = -|\xi|^2 - ic\xi_1$. The result for $\widehat{L}$ follows by some perturbation argument for small $\widehat{u}_c$. Finally, we state resolvent estimates for $\widehat{L}$ which also imply (ASS1).

In section 7 we show the exchange of spectral stability via the principle of reduced stability. Therefore we linearize about periodic solutions $\widehat{u}_p$ and consider the eigenvalue problem

$$\frac{\partial}{\partial t}\widehat{w} = (\widehat{L} - \lambda I)\widehat{w} + 2i\xi \cdot \widehat{Q}(\widehat{u}_p, \widehat{w}). \tag{1.29}$$

Via a Fourier series ansatz and the Lyapunov-Schmidt procedure from above we reduce this problem to a two-dimensional one, for which we can deduce that we have a zero eigenvalue and a negative eigenvalue. So we can show

**Theorem 1.4** *For* $\alpha - \alpha_c > 0$ *sufficiently small, the bifurcating time-periodic solutions of* (1.6) *are spectrally stable.*

At the end we discuss the validity of our results for the vorticity equation in $\mathbb{R}^2$. In two space dimensions the vorticity is a scalar in contrast to the three dimensional case where the vorticity is a vector.

# 2 The vorticity formulation

We consider the Navier-Stokes equations

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{u}(x,t)+(\tilde{u}(x,t)\cdot\nabla)\tilde{u}(x,t) &= \Delta\tilde{u}(x,t)-\nabla p(x,t)+f(x)\\ \nabla\cdot\tilde{u}(x,t) &= 0,\end{aligned}\tag{2.1}$$

with spatial variable $x\in\mathbb{R}^3$, time $t\in\mathbb{R}$, velocity field $\tilde{u}(x,t)\in\mathbb{R}^3$, pressure field $p(x,t)\in\mathbb{R}$, and external force $f(x)\in\mathbb{R}^3$. We assume that the external force $f(x)$ is chosen in such a way that there exists a stationary solution $(\tilde{u}_c,p_c)=(\tilde{u}_c,p_c)(x)$, i.e.

$$\begin{aligned}(\tilde{u}_c(x)\cdot\nabla)\tilde{u}_c(x) &= \Delta\tilde{u}_c(x)-\nabla p_c(x)+f(x)\\ \nabla\cdot\tilde{u}_c(x) &= 0.\end{aligned}\tag{2.2}$$

Further, we assume that $\tilde{u}_c(x)=v^c+v_c(x)$ with $v^c=c(1,0,0)^T$, $\lim_{|x|\to\infty}v_c(x)=0$. The deviations $v(x,t)=\tilde{u}(x,t)-\tilde{u}_c(x)$ and $q(x,t)=p(x,t)-p_c(x)$ from the stationary solution $(u_c,\ p_c)$ satisfy

$$\begin{aligned}\frac{\partial}{\partial t}v(x,t) &= \Delta v(x,t)-c\frac{\partial}{\partial x_1}v(x,t)-(v_c(x)\cdot\nabla)v(x,t)-(v(x,t)\cdot\nabla)v_c(x)\\ &\quad -\nabla q(x,t)-(v(x,t)\cdot\nabla)v(x,t)\\ \nabla\cdot v(x,t) &= 0.\end{aligned}\tag{2.3}$$

These equations can be rewritten as

$$\begin{aligned}\frac{\partial}{\partial t}v(x,t) &= \Delta v(x,t)-c\frac{\partial}{\partial x_1}v(x,t)\\ &\quad -\nabla\cdot(v_c(x)v(x,t))-\nabla\cdot(v(x,t)v_c(x))\\ &\quad -\nabla q(x,t)-\nabla\cdot(v(x,t)v(x,t))\\ \nabla\cdot v(x,t) &= 0.\end{aligned}\tag{2.4}$$

This is justified as follows (see [26]). For $v_c,v\in\mathbb{R}^3$ we define

$$v_cv \ := \ (v_{c,j}v_l)_{j,l=1}^d=\begin{pmatrix}v_{c,1}\\ \vdots\\ v_{c,d}\end{pmatrix}\cdot(v_1\dots v_d)=\begin{pmatrix}v_{c,1}v_1 & \dots & v_{c,d}v_1\\ \vdots & \ddots & \vdots\\ v_{c,1}v_d & \dots & v_{c,d}v_d\end{pmatrix}.$$

Hence

$$v_c v = (v_{c,1}v \ \dots v_{c,d}v).$$

$\nabla \cdot v_c = 0$ and the standard rules for the nabla operator yield

$$\begin{aligned}\nabla \cdot (v_c v) &= \nabla \cdot (v_{c,1}v \ \dots v_{c,d}v) = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_d} \end{pmatrix} \cdot \begin{pmatrix} v_{c,1}v \\ \vdots \\ v_{c,d}v \end{pmatrix} = \sum_{j=1}^{d} \frac{\partial}{\partial x_j} (v_{c,j}v) \\ &= \underbrace{\left( \sum_{j=1}^{d} \frac{\partial}{\partial x_j} v_{c,j} \right)}_{=\nabla \cdot v_c = 0} v + \sum_{j=1}^{d} v_{c,j} \frac{\partial}{\partial x_j} v = (v_c \cdot \nabla) v.\end{aligned}$$

**Notation:** From now on we denote with $\tilde{u}$ the velocity field of the fluid and with $u$ the associated vorticity defined by

$$u = \nabla \times \tilde{u} = \begin{pmatrix} \frac{\partial}{\partial x_2}\tilde{u}_3 - \frac{\partial}{\partial x_3}\tilde{u}_2 \\ \frac{\partial}{\partial x_3}\tilde{u}_1 - \frac{\partial}{\partial x_1}\tilde{u}_3 \\ \frac{\partial}{\partial x_1}\tilde{u}_2 - \frac{\partial}{\partial x_2}\tilde{u}_1 \end{pmatrix}. \tag{2.5}$$

To reconstruct the velocity field $\tilde{u}$ from the vorticity $u$ we use the Biot-Savart law

$$\tilde{u}(x,t) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x-y) \times u(y,t)}{|x-y|^3} dy, \ x \in \mathbb{R}^3. \tag{2.6}$$

In order to derive the vorticity formulation for the Navier-Stokes equations we use

$$\nabla \times \nabla \cdot (\tilde{u}\tilde{u}) = \nabla \cdot (u\tilde{u} - \tilde{u}u) \tag{2.7}$$

which implies

$$\nabla \times \nabla \cdot (\tilde{u}_c\tilde{u} + \tilde{u}\tilde{u}_c) = \nabla \cdot (u_c\tilde{u} + u\tilde{u}_c - \tilde{u}_c u - \tilde{u}u_c). \tag{2.8}$$

Therefore, we find

$$\frac{\partial}{\partial t}u - \nabla \cdot (u\tilde{u} - \tilde{u}u) + c\frac{\partial}{\partial x_1}u = \Delta u + \nabla \cdot (u_c\tilde{u} + u\tilde{u}_c - \tilde{u}_c u - \tilde{u}u_c), \tag{2.9}$$

where the space of divergence-free vector fields is invariant under the evolution of (2.9), i.e. additionally we assume $\nabla \cdot u = 0$.

Introducing

$$\begin{aligned} Bu &= \Delta u - c\frac{\partial}{\partial x_1}u, \\ 2Q(u,v) &= v\tilde{u} + u\tilde{v} - \tilde{v}u - \tilde{u}v, \end{aligned} \tag{2.10}$$

the vorticity formulation (2.9) can be written as

$$\frac{\partial}{\partial t}u = Bu + 2\nabla \cdot Q(u_c, u) + \nabla \cdot Q(u,u). \tag{2.11}$$

**Note 2.1** *The operator $B = \Delta - c\frac{\partial}{\partial x_1}$ is often called* Oseen operator [17].

# 3 Preliminary estimates

## 3.1 Function spaces and Fourier transform

As in [26] we define

**Definition 3.1** *We denote with $L^p(\Omega)$, $p \in [1, \infty]$ the Banach space of all Lebesgue measurable complex-valued functions $f : \Omega \subseteq \mathbb{R}^d \mapsto \mathbb{R}$, $d \in \mathbb{N}$, $\Omega$ an arbitrary domain, which have a finite norm given by*

$$\begin{aligned} \|f\|_{L^p} &= \left( \int\limits_{\Omega} |f(x)|^p dx \right)^{\frac{1}{p}}, \ 1 \leq p < \infty, \\ \|f\|_{L^\infty} &= ess\ sup_{x \in \Omega} |f(x)|. \end{aligned} \tag{3.1}$$

If a function $f$ is vector-valued, then the norm in $L^p$ is the norm of the absolute value $|f| = \sqrt{\sum_{j=1}^d f_j^2}$, which is the Euclidean norm in $\mathbb{R}^d$. Furthermore, we mention that the $L^p$-spaces are Banach spaces and for $p = 2$ the space $L^2$ is a Hilbert space. $L^p_m(\mathbb{R}^d)$, $p \in [1, \infty]$ is the spatially weighted Lebesgue space equipped with the norm $\|f\|_{L^p_m} = \|f\rho^m\|_{L^p}$, where $\rho(x) = \sqrt{1 + |x|^2}$ .
For the introduction of Sobolev spaces we introduce a multi-index $\alpha$ with $\alpha = (\alpha_1, \dots, \alpha_d), \alpha_j \geq 0,\ j = 1, \dots, d$ and $|\alpha| = \sum_{j=1}^d \alpha_j$. Furthermore we define for $x \in \mathbb{R}^d$

$$x^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_d^{\alpha_d} \tag{3.2}$$

and with $D^\alpha$ we denote the partial derivative

$$D^\alpha = \frac{\partial^\alpha}{\partial x^\alpha} = \frac{\partial^\alpha}{\partial x_1^{\alpha_1} \cdot \ldots \cdot \partial x_d^{\alpha_d}}. \tag{3.3}$$

With this we can introduce Sobolev spaces [19].

**Definition 3.2** *Let $\Omega \subseteq \mathbb{R}^d$, $d \geq 1$ be an arbitrary domain and $k \in \mathbb{N}$ and let $1 \leq p \leq \infty$. Then the **Sobolev space** $\mathbf{W^{k,p}(\Omega)}$ is the set of all distributions $u \in L^p(\Omega)$ such that $D^\alpha u \in L^p(\Omega)$ for all $|\alpha| \leq k$. In $W^{k,p}(\Omega)$ the norm is defined by*

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} \left( \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p}^p \right)^{\frac{1}{p}} & if \quad 1 \leq p < \infty \\ \max_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty} & if \quad p = \infty. \end{cases} \tag{3.4}$$

We remark that if $p = 2$ we can define an inner product by

$$(u, v)_k := \sum_{|\alpha| \le k} \int_{\Omega} D^{\alpha} u \overline{D^{\alpha} v} dx \tag{3.5}$$

and denote $W^{k,2}(\Omega)$ by $H^k(\Omega)$. $H^k(\Omega)$ is a Hilbert space. In all other cases the Sobolev spaces are only Banach spaces.

**Definition 3.3** *Let $f \in L^1(\mathbb{R}^d)$, then its* ***Fourier transform*** *is defined by*

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) \exp(-ix \cdot \xi) dx, \tag{3.6}$$

*the* ***inverse Fourier transform*** *is defined by*

$$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{f}(\xi) \exp(ix \cdot \xi) d\xi. \tag{3.7}$$

We mention that Fourier transform is continuous from $L^p, p \in [1, 2]$ to $L^q$, $q \in [2, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $p = 2$ Fourier transform is an isomorphism, i.e. we have $\|f\|_{L^2} = (2\pi)^{\frac{d}{2}} \|\widehat{f}\|_{L^2}$ (Plancherel)[4]. As a consequence, the Fourier transform is a continuous map from the Sobolev space $W^{k,p}$ into the weighted Lebesgue space $L^q_k$ with $1/p + 1/q = 1$ if $p \in [1, 2]$. Similarly, Fourier transform is a continuous mapping from the weighted Lebesgue space $L^q_k$ into the Sobolev space $W^{k,p}$ with $1/p + 1/q = 1$ if $q \in [1, 2]$. For $p = 2$, Fourier transform is an isomorphism between $H^k$ and $L^2_k$.

**Lemma 3.4** *For $p \ge r$ and $s > d\frac{p-r}{pr}$ we have the embedding $L^p_s(\mathbb{R}^d) \subset L^r(\mathbb{R}^d)$.*

**Proof.** We have with $\rho(x) = \sqrt{1 + |x|^2}$ and the Hölder inequality

$$\begin{aligned} \|f\|_{L^r} &= \|f \rho^s \rho^{-s}\|_{L^r} \le \|f \rho^s\|_{L^p} \|\rho^{-s}\|_{L^q}, \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \\ &= \|f\|_{L^p_s} \|\rho^{-s}\|_{L^q}. \end{aligned}$$

For $q = \infty$ it is obvious that $\|\rho^{-s}\|_{L^\infty} \le 1$ and we have with $r = p$ that $\|u\|_{L^p} \le \|u\|_{L^p_s}$ for $p \in [1, \infty]$. If $q \ne \infty$, we have

$$\|\rho^{-s}\|^q_{L^q} = \int_{\mathbb{R}^3} \frac{dx}{(1 + |x|^2)^{\frac{sq}{2}}} = \int_{|x| \le 1} \frac{dx}{(1 + |x|^2)^{\frac{sq}{2}}} + \int_{|x| > 1} \frac{dx}{(1 + |x|^2)^{\frac{sq}{2}}}.$$

Since the first integral is bounded we have

$$\int\limits_{|x|>1} \frac{dx}{(1+|x|^2)^{\frac{sq}{2}}} \le C \int\limits_1^\infty \frac{r^{d-1}}{(1+r^2)^{\frac{sq}{2}}} dr \le C \int\limits_1^\infty \frac{dr}{r^{sq-d+1}}.$$

The second integral is bounded for $sq - d + 1 > 1$ or $sq > d$. This yields $\|f\|_{L^r} \le C\|f\|_{L^p_s}$ for $s > d\frac{p-r}{pr}$. ■

## 3.2 The vorticity equation in Fourier space

After the definition of Fourier transform, we are able to formulate the vorticity equation in Fourier space. Applying standard rules of Fourier transform, equation (2.10) resp. (2.11) yields

$$\frac{\partial}{\partial t}\widehat{u} = \widehat{B}\widehat{u} + 2i\xi \cdot \widehat{Q}(\widehat{u}_c, \widehat{u}) + i\xi \cdot \widehat{Q}(\widehat{u}, \widehat{u}), \tag{3.8}$$

with

$$\begin{aligned} \widehat{B}\widehat{u} &= -|\xi|^2 - ic\xi_1, \\ 2\widehat{Q}(\widehat{u}, \widehat{v}) &= \widehat{v} * \widehat{\tilde{u}} + \widehat{u} * \widehat{\tilde{v}} - \widehat{\tilde{v}} * \widehat{u} - \widehat{\tilde{u}} * \widehat{v}, \end{aligned} \tag{3.9}$$

where $*$ denotes the convolution, i.e. $\widehat{u} * \widehat{v} = \int_{\mathbb{R}^d} \widehat{u}(\xi - \eta)\widehat{v}(\eta) d\eta$.

## 3.3 Reconstruction of the velocity from the vorticity

In the basic equation (2.9) for the evolution of the vorticity $u$ the velocity $\tilde{u}$ can be reconstructed via the Biot-Savart law (2.6) from the vorticity $u$. The following lemma allows to estimate the velocity $\tilde{u}$ in terms of the vorticity $u$ in Fourier space. For estimates in physical space see [11].

**Lemma 3.5** *Assume that* $\frac{1}{q} = \frac{1}{p} + \frac{1}{r}$ *with* $r \in [1, 3)$ *and* $p, q \in [1, \infty]$. *If* $\widehat{u} \in L^p(\mathbb{R}^3)^3 \cap L^q(\mathbb{R}^3)^3$ *then* $\widehat{\tilde{u}} \in L^q(\mathbb{R}^3)^3$ *and there exists a* $C > 0$ *such that*

$$\|\widehat{\tilde{u}}\|_{L^q} \le C(\|\widehat{u}\|_{L^p} + \|\widehat{u}\|_{L^q}). \tag{3.10}$$

*Moreover*

$$\|i\xi_j \widehat{\tilde{u}}\|_{L^q} \le C\|\widehat{u}\|_{L^q}. \tag{3.11}$$

**Proof.** The velocity $\tilde{u}$ is defined in terms of the vorticity $u$ by solving the equations

$$\nabla \times \tilde{u} = u \qquad \text{and} \qquad \nabla \cdot \tilde{u} = 0$$

for $u$ satisfying $\nabla \cdot u = 0$. This leads, in Fourier space, to

$$\begin{pmatrix} 0 & -i\xi_3 & i\xi_2 \\ i\xi_3 & 0 & -i\xi_1 \\ -i\xi_2 & i\xi_1 & 0 \\ i\xi_1 & i\xi_2 & i\xi_3 \end{pmatrix} \begin{pmatrix} \widehat{\tilde{u}}_1 \\ \widehat{\tilde{u}}_2 \\ \widehat{\tilde{u}}_3 \end{pmatrix} = \begin{pmatrix} \widehat{u}_1 \\ \widehat{u}_2 \\ \widehat{u}_3 \\ 0 \end{pmatrix}.$$

Multiplication from the left with the transposed matrix yields

$$-|\xi|^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \widehat{\tilde{u}}_1 \\ \widehat{\tilde{u}}_2 \\ \widehat{\tilde{u}}_3 \end{pmatrix} = \begin{pmatrix} 0 & i\xi_3 & -i\xi_2 & i\xi_1 \\ -i\xi_3 & 0 & i\xi_1 & i\xi_2 \\ i\xi_2 & -i\xi_1 & 0 & i\xi_3 \end{pmatrix} \begin{pmatrix} \widehat{u}_1 \\ \widehat{u}_2 \\ \widehat{u}_3 \\ 0 \end{pmatrix},$$

which is solved by

$$\begin{pmatrix} \widehat{\tilde{u}}_1 \\ \widehat{\tilde{u}}_2 \\ \widehat{\tilde{u}}_3 \end{pmatrix} = -\frac{1}{|\xi|^2} \begin{pmatrix} 0 & i\xi_3 & -i\xi_2 & i\xi_1 \\ -i\xi_3 & 0 & i\xi_1 & i\xi_2 \\ i\xi_2 & -i\xi_1 & 0 & i\xi_3 \end{pmatrix} \begin{pmatrix} \widehat{u}_1 \\ \widehat{u}_2 \\ \widehat{u}_3 \\ 0 \end{pmatrix}.$$

With Hölder's inequality we obtain

$$\left\| \begin{pmatrix} \widehat{\tilde{u}}_1 \\ \widehat{\tilde{u}}_2 \\ \widehat{\tilde{u}}_3 \end{pmatrix} \right\|_{L^q} \leq \left\| \chi_{\{|\xi|\leq 1\}} \frac{1}{|\xi|^2} \begin{pmatrix} 0 & i\xi_3 & -i\xi_2 & i\xi_1 \\ -i\xi_3 & 0 & i\xi_1 & i\xi_2 \\ i\xi_2 & -i\xi_1 & 0 & i\xi_3 \end{pmatrix} \right\|_{L^r} \left\| \begin{pmatrix} \widehat{u}_1 \\ \widehat{u}_2 \\ \widehat{u}_3 \\ 0 \end{pmatrix} \right\|_{L^p}$$

$$+ \left\| \chi_{\{|\xi|>1\}} \frac{1}{|\xi|^2} \begin{pmatrix} 0 & i\xi_3 & -i\xi_2 & i\xi_1 \\ -i\xi_3 & 0 & i\xi_1 & i\xi_2 \\ i\xi_2 & -i\xi_1 & 0 & i\xi_3 \end{pmatrix} \right\|_{L^\infty} \left\| \begin{pmatrix} \widehat{u}_1 \\ \widehat{u}_2 \\ \widehat{u}_3 \\ 0 \end{pmatrix} \right\|_{L^q},$$

with $\frac{1}{q} = \frac{1}{p} + \frac{1}{r}$. Hence it remains to estimate terms of the form

$$K_j^\infty(\xi) = \chi_{\{|\xi|>1\}} \frac{i\xi_j}{|\xi|^2}$$

in the space $L^\infty(\mathbb{R}^3)$ and

$$K_j(\xi) = \chi_{\{|\xi|\leq 1\}} \frac{i\xi_j}{|\xi|^2}$$

in the space $L^r(\mathbb{R}^3)$. The estimate for $K_j^\infty$ is obvious. For $K_j$ we have

$$\|K_j(\xi)\|_{L^r}^r = \int_{|\xi|\leq 1} \left| \frac{\xi_j}{|\xi|^2} \right|^r d\xi \underbrace{\leq}_{\text{Polar coordinates}} C \int_0^1 \frac{\rho^r}{\rho^{2r}} \rho^2 d\rho = \int_0^1 \frac{d\rho}{\rho^{r-2}},$$

which is bounded for $r < 3$.

Equation (3.11) follows from

$$i\xi_j \begin{pmatrix} \widehat{\tilde{u}}_1 \\ \widehat{\tilde{u}}_2 \\ \widehat{\tilde{u}}_3 \end{pmatrix} = -\frac{i\xi_j}{|\xi|^2} \begin{pmatrix} 0 & i\xi_3 & -i\xi_2 & i\xi_1 \\ -i\xi_3 & 0 & i\xi_1 & i\xi_2 \\ i\xi_2 & -i\xi_1 & 0 & i\xi_3 \end{pmatrix} \begin{pmatrix} \widehat{u}_1 \\ \widehat{u}_2 \\ \widehat{u}_3 \\ 0 \end{pmatrix}$$

and

$$\left\| i\xi_j \begin{pmatrix} \widehat{\widetilde{u}}_1 \\ \widehat{\widetilde{u}}_2 \\ \widehat{\widetilde{u}}_3 \end{pmatrix} \right\|_{L^q} \leq \left\| \frac{i\xi_j}{|\xi|^2} \begin{pmatrix} 0 & i\xi_3 & -i\xi_2 & i\xi_1 \\ -i\xi_3 & 0 & i\xi_1 & i\xi_2 \\ i\xi_2 & -i\xi_1 & 0 & i\xi_3 \end{pmatrix} \right\|_{L^\infty} \left\| \begin{pmatrix} \widehat{u}_1 \\ \widehat{u}_2 \\ \widehat{u}_3 \\ 0 \end{pmatrix} \right\|_{L^q}.$$

■

As a direct consequence of the last lemma 3.5 we get an analogous result in weighted Lebesgue spaces.

**Lemma 3.6** *Assume that $\frac{1}{q} = \frac{1}{p} + \frac{1}{r}$ with $r \in [1,3)$ and $p, q \in [1,\infty]$. If $\widehat{u} \in L^p(\mathbb{R}^3)^3 \cap L^q_s(\mathbb{R}^3)^3$ then $\widehat{\widetilde{u}} \in L^q_s(\mathbb{R}^3)^3$ and there exists a $C > 0$ such that*

$$\|\widehat{\widetilde{u}}\|_{L^q_s} \leq C(\|\widehat{u}\|_{L^p} + \|\widehat{u}\|_{L^q_s}). \tag{3.12}$$

*Moreover*

$$\|i\xi_j \widehat{\widetilde{u}}\|_{L^q_s} \leq C\|\widehat{u}\|_{L^q_s}. \tag{3.13}$$

## 3.4 Estimates for the bilinear term $\widehat{Q}(\widehat{u}, \widehat{v})$

**Lemma 3.7** *For $p \in (3/2, \infty]$ and every $s > 3\frac{p-1}{p}$ there exists a $C > 0$ such that for every $u \in L^p_s$*

$$\|\widehat{u} * \widehat{\widetilde{u}}\|_{L^p_s} \leq C\|\widehat{u}\|^2_{L^p_s}. \tag{3.14}$$

**Proof.** By Young's inequality, the Biot-Savart law lemma 3.5 with $1 = \frac{1}{q} + \frac{1}{r}$, where $r \in [1,3)$ which yields $q \in (3/2, \infty]$, and Sobolev's embedding $L^p_s \subset L^1 \cap L^q$ for $s > 3\frac{p-1}{p}$ and $p > q$ we have

$$\begin{aligned}
\|\widehat{u} * \widehat{\widetilde{u}}\|_{L^p_s} &\leq C(\|\widehat{u}\|_{L^p}\|\widehat{\widetilde{u}}\|_{L^1} + \|\xi^s\widehat{u}\|_{L^p}\|\widehat{\widetilde{u}}\|_{L^1} + \|\widehat{u}\|_{L^1}\|\xi^s\widehat{\widetilde{u}}\|_{L^p}) \\
&\leq C(\|\widehat{u}\|_{L^p}(\|\widehat{u}\|_{L^1} + \|\widehat{u}\|_{L^q}) + \|\xi^s\widehat{u}\|_{L^p}(\|\widehat{u}\|_{L^1} + \|\widehat{u}\|_{L^q}) \\
&\quad +(\|\widehat{u}\|_{L^1} + \|\widehat{u}\|_{L^q})\|\xi^{s-1}\widehat{u}\|_{L^p}) \\
&\leq C\|\widehat{u}\|^2_{L^p_s}.
\end{aligned}$$

■

**Lemma 3.8** *Let $q \leq p$ with $p \in (3/2, \infty]$ and $s > 3\frac{p-1}{p}$. Then there exists a $C > 0$ such that for every $\widehat{u} \in L^q_s$*

$$\|\widehat{u} * \widehat{\tilde{u}}\|_{L^p_s} \leq C\|\widehat{u}\|_{L^q_s}\|\widehat{u}\|_{L^p_s}. \tag{3.15}$$

**Proof.** With Young's inequality, lemma 3.5 and lemma 3.4 the proof is similar to the proof of the last lemma. ∎

**Lemma 3.9** *Let $p \in (3/2, \infty]$. For every $s > 3\frac{p-1}{p}$ there exists a $C > 0$ such that for every $\widehat{u}, \widehat{v} \in L^p_s$*

$$\|\widehat{Q}(\widehat{u}, \widehat{v})\|_{L^p_s} \leq C\|\widehat{u}\|_{L^p_s}\|\widehat{v}\|_{L^p_s}. \tag{3.16}$$

**Proof.** This is a direct consequence of lemma 3.7. ∎

**Lemma 3.10** *Let $q \leq p$ with $p \in (3/2, \infty]$ and $s > 3\frac{p-1}{p}$. Then there exists a $C > 0$ such that for every $\widehat{u} \in L^q_s, \widehat{v} \in L^p_s$*

$$\|\widehat{Q}(\widehat{u}, \widehat{v})\|_{L^p_s} \leq C\|\widehat{u}\|_{L^q_s}\|\widehat{v}\|_{L^p_s}. \tag{3.17}$$

**Proof.** This is a direct consequence of lemma 3.8. ∎

**Lemma 3.11** *Let $p, q, \tilde{p} \in [1, \infty]$ satisfy*

$$1 = \frac{1}{p} + \frac{1}{q}, \ \frac{1}{\tilde{q}} = \frac{1}{q} - \frac{1}{r^*}, \ r^* \in [1, 3), s > 3\frac{\tilde{q} - q}{q\tilde{q}}, \tilde{q} \geq q. \tag{3.18}$$

*Then there exists a $C > 0$ such that for every $\widehat{u} \in L^p_s$*

$$\|\widehat{u} * \widehat{\tilde{u}}\|_{L^\infty} \leq C\|\widehat{u}\|^2_{L^p_s}. \tag{3.19}$$

**Proof.** By Young's inequality with

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \qquad r = \infty, \tag{3.20}$$

the Biot-Savart law lemma 3.5 for $\frac{1}{\tilde{q}} = \frac{1}{p} - \frac{1}{r^*}$, $1 \leq \tilde{q} < p \leq \infty$, $r^* \in [1, 3)$, and Sobolev's embedding $L^p_s \subset L^{\tilde{q}}$, $L^{\tilde{q}}_s \subset L^q$ for $s > 3\frac{p-\tilde{q}}{p\tilde{q}}$, $s > 3\frac{\tilde{q}-q}{\tilde{q}q}$ and $p \geq \tilde{q} \geq q$ we have

$$\|\widehat{u} * \widehat{\tilde{u}}\|_{L^\infty} \leq \|\widehat{u}\|_{L^p}\|\widehat{\tilde{u}}\|_{L^q} \leq \|\widehat{u}\|_{L^p}\|\widehat{\tilde{u}}\|_{L^{\tilde{q}}_s} \leq \|\widehat{u}\|_{L^p}(\|\widehat{u}\|_{L^p} + \|\widehat{\tilde{u}}\|_{L^{\tilde{q}}})$$

$$\leq \quad \|\widehat{u}\|_{L^p_s}(\|\widehat{u}\|_{L^p_s} + \|\widehat{\tilde{u}}\|_{L^{\tilde{q}}}) \leq C\|\widehat{u}\|^2_{L^p_s}.$$

∎

**Lemma 3.12** *Let $p, q, \tilde{p} \in [1, \infty]$ and $s \geq 0$ satisfy (3.18). Then there exists a $C > 0$ such that for every $\widehat{u}, \widehat{v} \in L^p_s$*

$$\|\widehat{Q}(\widehat{u}, \widehat{v})\|_{L^\infty} \leq C\|\widehat{u}\|_{L^p_s}\|\widehat{v}\|_{L^p_s}. \tag{3.21}$$

**Proof.** This is a direct consequence of lemma 3.11. ∎

## 3.5 Estimates for the Oseen operator

In this section we show the invertibility of the linear operator $B$ defined (2.10).

**Lemma 3.13** *For $p < 4$ we have $\widehat{B}^{-1}i\xi_j : L^p \cap L^\infty \mapsto L^p$.*

**Proof:** We consider the equation $Bu = \Delta u - c\frac{\partial}{\partial x_1}u = \frac{\partial}{\partial x_j}f$. In Fourier space we have $\widehat{u} = -\frac{i\xi_j}{|\xi|^2+ic\xi_1}\widehat{f}$. For the $L^p$-norm we get

$$\|\widehat{u}\|_{L^p} \quad \leq \quad C\|\widehat{f}\|_{L^\infty}\int\limits_{|\xi|\leq 1}\left|\frac{i\xi_j}{|\xi|^2+ic\xi_1}\right|^p d\xi + C\|f\|_{L^p}\left\|\frac{i\xi_j}{|\xi|^2+ic\xi_1}\chi_{|\xi|\geq 1}\right\|_{L^\infty}$$

$\left\|\frac{i\xi_j}{|\xi|^2+ic\xi_1}\chi_{|\xi|\geq 1}\right\|_{L^\infty}$ is bounded for all $p \in [1, \infty)$, so we have to consider the first integral and we must distinguish between $j = 1$ and $j = 2, 3$. For $j = 1$ we have

$$\int\limits_{|\xi|\leq 1}\left|\frac{i\xi_1}{|\xi|^2+ic\xi_1}\right|^p d\xi \leq \int\limits_{|\xi|\leq 1}\left|\frac{i\xi_1}{(1-ic)\xi_1}\right|^p d\xi \leq \int\limits_{|\xi|\leq 1} d\xi \leq C$$

and the integral is bounded independent from $p$. For $j = 2, 3$ we get

$$\begin{aligned}
\int\limits_{|\xi|\leq 1}\left|\frac{i\xi_j}{|\xi|^2+ic\xi_1}\right|^p d\xi \quad &\leq \quad C\int_0^1\int_0^1\int_0^1\left|\frac{i\xi_j}{|\xi|^2+ic\xi_1}\right|^p d\xi_1 d\xi_2 d\xi_3 \\
&\leq \quad C\int_0^1\int_0^1\int_0^1\frac{|\xi_j|^p}{|\xi|^{2p}+|c\xi_1|^p}d\xi_1 d\xi_2 d\xi_3
\end{aligned}$$

$$
\begin{aligned}
&\underbrace{\leq}_{|\xi^*|^2=\xi_2^2+\xi_3^2} && C\int_0^1\int_0^1\int_0^1 \frac{|\xi_j|^p}{|\xi^*|^{2p}}\frac{1}{1+\frac{|c\xi_1|^p}{|\xi^*|^{2p}}}d\xi_1 d\xi_2 d\xi_3 \\
&\underbrace{\leq}_{y=\frac{|c\xi_1|}{|\xi^*|^2}} && C\int_0^1\int_0^1 \frac{|\xi_j|^p}{|\xi^*|^{2p-2}}d\xi_2 d\xi_3 \int_0^\infty \frac{1}{1+y^p}dy \\
&\leq && C\int_{|\xi^*|\leq\sqrt{2}} \frac{|\xi_j|^p}{|\xi^*|^{2p-2}}d\xi^* \leq C\int_0^{\sqrt{2}}\int_0^{2\pi} \frac{r^{p+1}}{r^{2p-2}}dr d\phi \\
&\leq && C\int_0^{\sqrt{2}} \frac{1}{r^{p-3}}dr < \infty
\end{aligned}
$$

for $p < 4$. ■

**Remark 3.1** *Since* $\frac{i\xi_1}{|\xi|^2+ic\xi_1}$ *is uniformly bounded,we additionally have* $\widehat{B}^{-1}i\xi_1 : L^p \mapsto L^p$ *for every* $p \in [1,\infty]$.

**Remark 3.2** *Later on we will use the following numerical values. From lemma 3.13 we have* $p = 4-\delta$ *and lemma 3.5 gives* $r \in [1,3)$ *and* $\tilde{q} = 12/7 + \mathcal{O}(\delta)$ *for a* $\delta > 0$ *small. Hölders inequality gives* $q = 4/3 + \mathcal{O}(\delta)$, *which is allowed by the Biot-Savart law with* $r \in [1,3)$.

# 4 The linearized problem

It is the purpose of this section to analyze the spectral properties of the linear operator

$$L\cdot = B\cdot + 2\nabla\cdot Q(u_c,\cdot), \tag{4.1}$$

with $B$ and $Q$ from equation (2.10). More precisely, we analyse the operator

$$\widehat{L}\cdot = \widehat{B}\cdot + 2i\xi\cdot \widehat{Q}(\widehat{u}_c,\cdot) \tag{4.2}$$

in the space $L^p_s$ with $p\in[1,\infty]$ and $s\geq 0$. The domain of definition is given by $L^p_{s+2}$.

By the theorem of Frechet-Kolmogorov [1, Theorem 2.26] and lemma 3.9 the operator $2i\xi\cdot\widehat{Q}(\widehat{u}_c,\cdot)$ is a relatively compact perturbation of the operator $\widehat{B}\cdot$ if $\widehat{u}_c\in L^{\tilde p}_s$ with $\tilde p<p$. Then by [12, p.136] the essential spectrum of the operator $\widehat{L}$ equals the essential spectrum

$$ess\ spec(\widehat{B}) = \{\lambda\in\mathbb{C} : \lambda = -|k|^2 - ick_1,\ k\in\mathbb{R}^3\} \tag{4.3}$$

of the operator $\widehat{B}$, i.e. the spectra of the two operators differ only by isolated eigenvalues.

## 4.1 Assumptions on the stationary solutions $u_c$

We assume that there is whole family of stationary solutions $\widehat{u}_c\in L^p_s,\ p\in[1,4)$ for all $\alpha\in[\alpha_c-\delta_0,\alpha_c+\delta_0]$ , which satisfies

(A0) $\lambda=0$ is not an eigenvalue of $\widehat{L}$ for all $\alpha\in[\alpha_c-\delta_0,\alpha_c+\delta_0]$.

(A1) For $\alpha=\alpha_c$ the operator $\widehat{L}$ possesses two eigenvalues $\lambda_0^\pm(\alpha)$, which satisfy

$$\lambda_0^\pm(\alpha_c) = \pm i\omega_0 \neq 0,\ \omega_0>0 \tag{4.4}$$

and

$$\frac{d}{d\alpha}Re(\lambda_0^\pm(\alpha))\bigg|_{\alpha=\alpha_c} > 0. \tag{4.5}$$

(A2) The points $\pm in\omega_0,\ n = 2, 3, \dots$ are not contained in the spectrum of $\widehat{L}$.

(A3) All the other eigenvalues of $\widehat{L}$ lie strictly bounded away from the imaginary axis in the left half plane for all $\alpha \in [\alpha_c - \delta_0, \alpha_c + \delta_0]$.

## 4.2 Estimates for the operator $\widehat{L}$

We reformulate the estimates for the operator $\widehat{B}$ from lemma 3.13 and remark 3.1 in the following theorem.

**Theorem 4.1** *For all $s \geq 0$ the following holds. For $p < 4$ and $j = 2, 3$ we have*

$$\widehat{B}^{-1} i\xi_j : L^p \cap L^\infty_s \mapsto L^p_s \tag{4.6}$$

*and for every $p \in [1, \infty]$ we have*

$$\widehat{B}^{-1} i\xi_1 : L^p_s \mapsto L^p_s. \tag{4.7}$$

Combining theorem 4.1 with the assumptions (A1)-(A3) allows us to prove a similar result for the operator $\widehat{L}$.

**Theorem 4.2** *Under the assumptions (A1)-(A3) for all $s \geq 0$ the following holds. For $p < 4$ and $j = 2, 3$ we have*

$$\widehat{L}^{-1} i\xi_j : L^p \cap L^\infty_s \mapsto L^p_s \tag{4.8}$$

*and for every $p \in [1, \infty]$ we have*

$$\widehat{L}^{-1} i\xi_1 : L^p_s \mapsto L^p_s. \tag{4.9}$$

**Proof.** We introduce

$$A\cdot = \hat{L}\cdot, \quad A_0\cdot = \widehat{B}\,\cdot \quad \text{and} \quad A_1\cdot = 2i\xi \cdot \widehat{Q}(\widehat{u}_c, \cdot), \tag{4.10}$$

such that $A = A_0 + A_1$. From

$$(A_0 + A_1)w = i\xi_j f \tag{4.11}$$

it follows

$$A_0(I + A_0^{-1}A_1)w = i\xi_j f \quad \text{resp.} \quad w = (I + A_0^{-1}A_1)^{-1}A_0^{-1} i\xi_j f. \tag{4.12}$$

The existence of $(I + A_0^{-1}A_1)^{-1}$ is established as follows. By the assumptions on $\widehat{u}_c$ the operator $A_0^{-1}A_1 : L_s^p \to L_s^p$ is compact. Hence $I + A_0^{-1}A_1$ is a Fredholm operator with index 0. Using assumption (A1), i.e. that the operator $A$ has no zero eigenvalue, shows that $Aw = 0$ possesses only the trivial solution $w = 0$, So $A_0^{-1}Aw = 0$ has also only the trivial solution, as well as $(I + A_0^{-1}A_1)w = 0$. Therefore, from the Fredholm property, the existence of $(I+A_0^{-1}A_1)^{-1} : L_s^p \to L_s^p$ follows. Using the boundedness of this expression and the estimates for $A_0$ yields directly the estimates for $A$, in detail

$$\|w\|_{L_s^p} \leq \|(I + A_0^{-1}A_1)^{-1}\|_{L_s^p \to L_s^p} \|A_0^{-1} i\xi_j f\|_{L_s^p}. \tag{4.13}$$

■

# 5 Occurrence of a Hopf bifurcation

In this section we prove theorem 1.2, i.e. the bifurcation of time-periodic solutions from the trivial solution $\widehat{u} = 0$ for $\alpha = \alpha_c$.

**Theorem 5.1** *If $\gamma \neq 0$ and if the assumptions (A0)-(A3) stated in section 4.1 are valid, then equation*

$$\frac{\partial}{\partial t}\widehat{u} = -(|\xi|^2 + ic\xi_1)\widehat{u} + 2i\xi \cdot \widehat{Q}(\widehat{u}_c, \widehat{u}) + i\xi \cdot \widehat{Q}(\widehat{u}, \widehat{u}) \tag{5.1}$$

*has a one-dimensional family of small time-periodic solutions $\widehat{u}_{per}$, i.e. there exists an $\epsilon_0 > 0$ such that for $\epsilon \in (0, \epsilon_0)$*

$$\widehat{u}_{per}(\xi, t) = \widehat{u}_{per}\left(\xi, t + \frac{2\pi}{\omega}\right) \tag{5.2}$$

*solves (5.1) for $\alpha = \alpha_c - sgn(\gamma)\epsilon$. Furthermore we have $\omega = \omega_0 + \mathcal{O}(\epsilon)$ and $\|u_{per}\|_{C_b^0\left(\mathbb{R}^3 \times \left[0, \frac{2\pi}{\omega}\right]\right)} = \mathcal{O}(\sqrt{\epsilon})$.*

**Proof.** The proof is given in the next subsections. ■

We look for $\frac{2\pi}{\omega}$-time-periodic solutions, with $\omega > 0$ close to $\omega_0$ of

$$\frac{\partial}{\partial t}\widehat{u} = \widehat{L}\widehat{u} + i\xi \cdot \widehat{Q}(\widehat{u}, \widehat{u}). \tag{5.3}$$

In order to simplify notation, we introduce $\bar{\widehat{u}}(\xi, \tau) = \widehat{u}\,(\xi, t)$ with $\tau = \frac{t}{\omega}$ and look for $2\pi$-time-periodic solutions of

$$\omega\frac{\partial}{\partial t}\widehat{u} = \widehat{L}\widehat{u} + i\xi \cdot \widehat{Q}(\widehat{u}, \widehat{u}), \tag{5.4}$$

where we write again $\widehat{u}$ for $\bar{\widehat{u}}$ and $t$ instead of $\tau$.

To prove the occurrence of a Hopf bifurcation we make the ansatz

$$\widehat{u}(\xi, t) \quad = \quad \sum_{k \in \mathbb{Z}} \widehat{u}_k(\xi) \exp(ikt) \tag{5.5}$$

for the vorticity.

**Remark 5.2** *With the vorticity in the form* $\widehat{u}(\xi,t) = \sum_{k\in\mathbb{Z}} \widehat{u}_n(\xi)\exp(int)$ *we can reconstruct the velocity* $\tilde{\widehat{u}}(\xi,t)$ *via the Biot-Savart law such that*

$$\tilde{\widehat{u}}(\xi,t) = \sum_{k\in\mathbb{Z}} \tilde{\widehat{u}}_k(\xi)\exp(ikt). \tag{5.6}$$

From equation (5.4) we obtain the infinite-dimensional system

$$(ik\omega I - \widehat{L})\widehat{u}_k \quad = \quad i\xi\cdot\widehat{Q}(\widehat{u},\widehat{u})_k, \tag{5.7}$$

where

$$\widehat{Q}(\widehat{u},\widehat{u})[\xi,t] = \sum_{k\in\mathbb{Z}} \widehat{Q}(\widehat{u},\widehat{u})_k[\xi]\exp(ikt), \tag{5.8}$$

i.e.

$$\widehat{Q}(\widehat{u},\widehat{u})_k[\xi] = \sum_{m\in\mathbb{Z}} \widehat{Q}(\widehat{u}_{k-m},\widehat{u}_m)[\xi]. \tag{5.9}$$

## 5.1 Functional analytic set-up

In order to solve (5.7), we introduce the space

$$\widehat{\mathcal{X}}^p_s := \{\widehat{u} = (\widehat{u}_n)_{n\in\mathbb{Z}} : \|\widehat{u}\|_{\widehat{\mathcal{X}}^p_s} < \infty\} \tag{5.10}$$

equipped with the norm

$$\|\widehat{u}\|_{\widehat{\mathcal{X}}^p_s} = \sum_{n\in\mathbb{Z}} \|\widehat{u}_n\|_{L^p_s}.$$

The counterpart in physical space is given by

$$\mathcal{X}^{s,p} := \{u = (u_n)_{n\in\mathbb{Z}} : \|u\|_{\mathcal{X}^{s,p}} < \infty\} \tag{5.11}$$

equipped with the norm

$$\|u\|_{\mathcal{X}^{s,p}} = \sum_{n\in\mathbb{Z}} \|u_n\|_{W^{s,p}}.$$

**Remark 5.3** *From the analytic properties of Fourier transform, i.e.* $\widehat{u}_n \in L^p_s$ *implies* $u_n \in W^{s,q}$*, if* $1/p + 1/q = 1$ *and* $q \in [1,2]$*. It follows that* $\widehat{u} \in \widehat{\mathcal{X}}^p_s$ *yields* $u \in \mathcal{X}^{s,q}$ *if* $1/p+1/q = 1$ *and* $q \in [1,2]$.

Moreover, we have

**Lemma 5.1** *The linear operator $J$ defined by*

$$(J\breve{u})(x,t) = u(x,t) := \sum_{l\in\mathbb{Z}} \breve{u}_l(x)\exp(ilt), \tag{5.12}$$

*where $\breve{u} = (\breve{u}_l)_{l\in\mathbb{Z}} \in \mathcal{X}^{s,p}$ is a smooth map from $\mathcal{X}^{s,p}$ into $C_b(\mathbb{R}^d \times [0,2\pi], \mathbb{C}^d)$, if $sp > d$.*

**Proof:** Sobolev's embedding theorem gives the continuous embedding $W^{s,p}(\mathbb{R}^d) \subset C_b(\mathbb{R}^d)$ for $sp > d$ with $\|u\|_{L^\infty} \leq C\|u\|_{W^{s,p}}$ and so

$$\begin{aligned} |u(x,t)| &\leq \sum_{l\in\mathbb{Z}} |\breve{u}_l(x)| \leq \sum_{l\in\mathbb{Z}} \sup_{y\in\mathbb{R}^d} |\breve{u}_l(y)| \\ &= \sum_{l\in\mathbb{Z}} \|\breve{u}_l\|_{L^\infty} \leq C\sum_{l\in\mathbb{Z}} \|\breve{u}_l\|_{W^{s,p}} = C\|\breve{u}\|_{\mathcal{X}^{s,p}}. \end{aligned}$$

■

The linear operator $\widehat{L}$ and the bilinear operator $\widehat{Q}$ act in the $\widehat{\mathcal{X}}^p_s$-space as follows.

**Lemma 5.2** *Let $\widehat{L}$ be a linear operator $\widehat{L}$ defined componentwise as $(\widehat{L}\widehat{\overline{u}})_l = \widehat{L}_l\widehat{\overline{u}}_l$ for $\widehat{\overline{u}} = (\widehat{\overline{u}}_l)_{l\in\mathbb{Z}}$. Then we have*

$$\|\widehat{L}\widehat{\overline{u}}\|_{\widehat{\mathcal{X}}^p_s} \leq \sup_{l\in\mathbb{Z}} \|\widehat{L}_l\|_{L^p_s\mapsto L^p_s} \|\widehat{\overline{u}}\|_{\widehat{\mathcal{X}}^p_s}. \tag{5.13}$$

**Proof:** We have

$$\begin{aligned} \|\widehat{L}\widehat{\overline{u}}\|_{\widehat{\mathcal{X}}^p_s} &= \sum_{l\in\mathbb{Z}} \|\widehat{L}_l\widehat{\overline{u}}_l\|_{L^p_s} \leq \sum_{l\in\mathbb{Z}} \|\widehat{L}_l\|_{L^p_s\mapsto L^p_s} \|\widehat{\overline{u}}_l\|_{L^p_s} \\ &\leq \sup_{l\in\mathbb{Z}} \|\widehat{L}_l\|_{L^p_s\mapsto L^p_s} \sum_{l\in\mathbb{Z}} \|\widehat{\overline{u}}_l\|_{L^p_s} \\ &\leq \sup_{l\in\mathbb{Z}} \|\widehat{L}_l\|_{L^p_s\mapsto L^p_s} \|\widehat{\overline{u}}_l\|_{\widehat{\mathcal{X}}^p_s}. \end{aligned}$$

■

**Lemma 5.3** *Let $s \geq 1$, $p = 4 - \delta$ with $\delta > 0$ small. Then there exists a $C > 0$ such that for all $\widehat{u}, \widehat{v} \in \widehat{\mathcal{X}}_s^p$*

$$\|\widehat{Q}(\widehat{u}, \widehat{v})\|_{\widehat{\mathcal{X}}_s^p} \leq C \|\widehat{u}\|_{\widehat{\mathcal{X}}_s^p} \|\widehat{v}\|_{\widehat{\mathcal{X}}_s^p}. \tag{5.14}$$

**Proof:** Let $\widehat{u} = (\widehat{u}_l)_{l \in \mathbb{Z}}$, $\widehat{v} = (\widehat{v}_l)_{l \in \mathbb{Z}} \in \widehat{\mathcal{X}}_s^p$. Using lemma 3.9 we find

$$\begin{aligned}
\|\widehat{Q}(\widehat{u}, \widehat{v})\|_{\widehat{\mathcal{X}}_s^p} &= \sum_{l \in \mathbb{Z}} \|(\widehat{Q}(\widehat{u}, \widehat{v}))_l\|_{L_s^p} = \sum_{l \in \mathbb{Z}} \| \sum_{j \in \mathbb{Z}} \widehat{Q}(\widehat{u}_{l-j}, \widehat{v}_j)\|_{L_s^p} \\
&\leq C \sum_{l \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \|\widehat{u}_{l-j}\|_{L_s^p} \|\widehat{v}_j\|_{L_s^p} \leq C \sum_{l \in \mathbb{Z}} \|\widehat{u}_l\|_{L_s^p} \sum_{j \in \mathbb{Z}} \|\widehat{v}_j\|_{L_s^p} \\
&= C \|\widehat{u}\|_{\widehat{\mathcal{X}}_s^p} \|\widehat{v}\|_{\widehat{\mathcal{X}}_s^p}.
\end{aligned}$$

■

## 5.2 The Reduction step

To prove the occurrence of a Hopf bifurcation we apply the Lyapunov-Schmidt method to equation (5.7).

From assumption (A1) we know that the operator $\widehat{L}$ possesses two purely imaginary eigenvalues $\lambda_0^{\pm}(\alpha)$. As a consequence there exist two projections $\widehat{P}_{\alpha,c}^{\pm}$, which are $L$-invariant, onto a ("center") subspace, spanned by the associated eigenfunction $\phi_\alpha^{\pm}$, given by

$$\widehat{P}_{\alpha,c}^{\pm} \widehat{u} = (\widehat{\phi}_\alpha^{\pm,*}, \widehat{u})_{L^2} \widehat{\phi}_\alpha^{\pm}. \tag{5.15}$$

Here $(\cdot,\cdot)_{L^2}$ denotes the inner product in $L^2$ and the $\widehat{\phi}_\alpha^{\pm,*}$ are the associated normalized eigenfunctions of the adjoint operator $\widehat{L}^*$. The projection on the bounded "stable" part is $\widehat{P}_{\alpha,s}^{\pm} = I - \widehat{P}_{\alpha,c}^{\pm}$ and we have by construction that the operators $\widehat{P}_{\alpha,c}^{\pm}$ and $\widehat{P}_{\alpha,s}^{\pm}$ commutate with $\widehat{L}$. With this knowledge we can decompose the system 5.7 in the following sense

$$\begin{aligned}
\widehat{u}_k &= -(\widehat{L} - ik\omega I)^{-1} P_k i\xi \cdot \widehat{Q}(\widehat{u}, \widehat{u}), \ k \neq \{-1, 0, 1\}, \\
\widehat{u}_{\pm 1,s} &= -(\widehat{L} \mp i\omega I)^{-1} P_k \widehat{P}_{\alpha,s}^{\pm} i\xi \cdot \widehat{Q}(\widehat{u}, \widehat{u}), \ k = \pm 1, \\
\widehat{u}_0 &= -L^{-1} P_0 i\xi \cdot \widehat{Q}(\widehat{u}, \widehat{u}),
\end{aligned}$$

$$\pm i\omega\widehat{u}_{\pm 1,c} = \widehat{L}\widehat{u}_{\pm 1,c} + P_k\widehat{P}^{\pm}_{\alpha,c} i\xi\cdot\widehat{Q}(\widehat{u},\widehat{u}),\ k=\pm 1, \tag{5.16}$$

where

$$P_k\widehat{u} = \frac{1}{2\pi}\int_0^{2\pi}\exp(-ikt)\widehat{u}(\xi,t)dt \tag{5.17}$$

is the projection on the $k$-th element of $\widehat{u}$.

With the help of the projections we have split $\widehat{u}_{\pm 1}$ in the last equation as $\widehat{u}_{\pm 1} = \widehat{u}_{\pm 1c} + \widehat{u}_{\pm 1s}$ with $\widehat{u}_{\pm 1c} = \widehat{P}^{\pm}_{\alpha,c}\widehat{u}_1$ and $\widehat{u}_{\pm 1s} = \widehat{P}^{\pm}_{\alpha,s}\widehat{u}_1$. As next step we must find bounds for the operators $-(\widehat{L}-ik\omega I)^{-1}, -(\widehat{L}-ik\omega I)^{-1}i\xi_j, (\widehat{L}\mp i\omega I)^{-1}\widehat{P}^{\pm}_{\alpha,s}$ and $(\widehat{L}\mp i\omega I)^{-1}\widehat{P}^{\pm}_{\alpha,s}i\xi_j$ in the Lebesgue space $L^p_s$ for a special selection for the values $s$ and $p$.

For the operators mentioned above we have the following lemma.

**Lemma 5.4** *Let $s > d/p$. Then there exists a $C > 0$ such that for $\omega$ close enough to $\omega_0$ the following holds*

$$\begin{aligned}
\|(\widehat{L}-ik\omega I)^{-1}\|_{L^p_s\mapsto L^p_s} &\leq C, \qquad k\in\mathbb{Z}\backslash\{-1,0,1\},\\
\left\|(\widehat{L}-ik\omega I)^{-1}i\xi_j\right\|_{L^p_s\mapsto L^p_s} &\leq C, \qquad k\in\mathbb{Z}\backslash\{-1,0,1\},\ j=1,2,3,\\
\|(\widehat{L}\mp i\omega I)^{-1}\widehat{P}_{\alpha,s}\|_{L^p_s\mapsto L^p_s} &\leq C,\\
\left\|(\widehat{L}\mp i\omega I)^{-1}\widehat{P}_{\alpha,s}i\xi_j\right\|_{L^p_s\mapsto L^p_s} &\leq C,\ j=1,2,3.
\end{aligned} \tag{5.18}$$

**Proof:** Since $B$ is a sectorial operator in $W^{k,p}$, also $L = B + 2\nabla\cdot Q$ is a sectorial operator in $W^{k,p}$ due to Henry [12] Theorem A.1. So $\widehat{B}$ and $\widehat{L} = \widehat{B} + 2i\xi\cdot\widehat{Q}$ are sectorial operators in $L^p_s$. Therefore for the invertibility of $L - ik\omega I$ and $\widehat{L} - ik\omega I,\ k\in\mathbb{Z}\backslash\{-1,0,1\}$, with the proposed estimates, it is sufficient that the spectrum has to be strictly bounded away from zero, and so the first estimate follows. The same arguments hold for the other three operators. ■

Next we have to consider the case $k = 0$. From (5.16) it is not clear that the equation

$$\widehat{u}_0 = -\widehat{L}^{-1}P_0 i\xi\cdot\widehat{Q}(\widehat{u},\widehat{u}) \tag{5.19}$$

is well defined for $\widehat{u}_0$ due to the continuous spectrum of $\widehat{B}$ up to the imaginary axis. Using the previous theorems 4.1, 4.2 shows the result.

Now redefine the first three equations of (5.16) as $F = F(\widehat{u}_c, \widehat{u}_s)$ by

$$\begin{aligned} F_k &= -\widehat{u}_k - (\widehat{L} - ik\omega I)^{-1} P_k i\xi \cdot \widehat{Q}(\widehat{u}, \widehat{u}),\ k \neq \{-1, 0, 1\}, \\ F_{\pm 1,s} &= -\widehat{u}_{\pm 1,s} - (\widehat{L} \mp i\omega I)^{-1} P_k \widehat{P}^{\pm}_{\alpha,s} i\xi \cdot \widehat{Q}(\widehat{u}, \widehat{u}),\ k = \pm 1, \\ F_0 &= -\widehat{u}_0 - L^{-1} P_0 i\xi \cdot \widehat{Q}(\widehat{u}, \widehat{u}), \end{aligned} \tag{5.20}$$

where $\widehat{u}_c = (\ldots, 0, \widehat{u}_{-1c}, 0, \widehat{u}_{1c}, 0, \ldots)$, $\widehat{u}_s = (\ldots, \widehat{u}_{-2}, \widehat{u}_{-1s}, \widehat{u}_0, \widehat{u}_{1s}, \widehat{u}_2, \ldots)$ and $\widehat{u} = \widehat{u}_c + \widehat{u}_s$.

In order to apply the implicit function theorem and to resolve $F(u_c, u_s) = 0$ with $F : \widehat{X}^p_s \times \widehat{\mathcal{X}}^p_s \to \widehat{\mathcal{X}}^p_s$ with respect to $\widehat{u}_s$ we have to prove $F(0,0) = 0$ and the invertibility of $D_{u_s}F(0,0) : \widehat{\mathcal{X}}^p_s \to \widehat{\mathcal{X}}^p_s$. The first condition is trivial. We find $D_{u_s}F(0,0) = -I$ since all other terms are quadratic. This can be seen as follows. In $L^p_s$ we have with the triangle inequality and lemma 5.4 for $k = \pm 1$

$$\begin{aligned} \left\| (\widehat{L} \mp i\omega I)^{-1} \widehat{P}^{\pm}_{\alpha,s} i\xi \cdot \widehat{Q}(\widehat{u}, \widehat{u})_{\pm 1} \right\|_{L^p_s} &\leq \sum_{j=1}^{3} \left\| (\widehat{L} \mp i\omega I)^{-1} \widehat{P}^{\pm}_{\alpha,s} i\xi_j \right\|_{L^p_s \mapsto L^p_s} \|\widehat{Q}(\widehat{u}, \widehat{u})\|_{\widehat{\mathcal{X}}^p_s} \\ &\leq C \|\widehat{u}\|^2_{\widehat{\mathcal{X}}^p_s}. \end{aligned}$$

For $k = 0$ we find

$$\|\widehat{L}^{-1} i\xi \cdot \widehat{Q}(\widehat{u}, \widehat{u})_0\|_{L^p_s} \leq \sum_{j=1}^{3} \left\| \widehat{L}^{-1} i\xi_j \right\|_{L^p_s \mapsto L^p_s} \|\widehat{Q}(\widehat{u}, \widehat{u})\|_{\widehat{\mathcal{X}}^p_s} \leq C \|\widehat{u}\|^2_{\widehat{\mathcal{X}}^p_s}.$$

For $k \neq \pm 1$ we have

$$\begin{aligned} \|((\widehat{L} - ik\omega I)^{-1} i\xi \cdot \widehat{Q}(\widehat{u}, \widehat{u})_k)_{k \in \mathbb{Z} \setminus \{\pm 1, 0\}}\|_{\widehat{\mathcal{X}}^p_s} &\leq \sup_{k \in \mathbb{Z} \setminus \{\pm 1, 0\}} \sum_{j=1}^{3} \left\| (\widehat{L} - ik\omega I)^{-1} i\xi_j \right\|_{L^p_s \mapsto L^p_s} \\ &\qquad \times \|\widehat{Q}(\widehat{u}, \widehat{u})\|_{\widehat{\mathcal{X}}^p_s} \\ &\leq C \|\widehat{u}\|^2_{\widehat{\mathcal{X}}^p_s}. \end{aligned}$$

Thus there exists a unique solution $\widehat{u}_s = \widehat{u}_s(\widehat{u}_c)$ of $F(\widehat{u}_c, \widehat{u}_s) = 0$ with $\widehat{u}_s : \widehat{\mathcal{X}}^p_s \mapsto \widehat{\mathcal{X}}^p_s$ satisfying $\|\widehat{u}_s(\widehat{u}_c)\|_{\widehat{\mathcal{X}}^p_s} \leq C \|\widehat{u}_c\|^2_{\widehat{\mathcal{X}}^p_s}$.

## 5.3 The Hopf Bifurcation

In this section we analyze the last equation of system (5.16), namely

$$\pm i\omega\widehat{u}_{\pm1,c} = \widehat{L}\widehat{u}_{\pm1,c} + P_k\widehat{P}^{\pm}_{\alpha,c} i\xi\cdot\widehat{Q}(\widehat{u},\widehat{u}),\ k=\pm1, \tag{5.21}$$

or equivalently

$$\begin{aligned} i\omega\widehat{u}_{1,c} &= \widehat{L}\widehat{u}_{1,c} + P_1\widehat{P}^{\pm}_{\alpha,c} i\xi\cdot\widehat{Q}(\widehat{u},\widehat{u}), \\ -i\omega\widehat{u}_{-1,c} &= \widehat{L}\widehat{u}_{-1,c} + P_{-1}\widehat{P}^{\pm}_{\alpha,c} i\xi\cdot\widehat{Q}(\widehat{u},\widehat{u}), \end{aligned} \tag{5.22}$$

with $\widehat{u} = \widehat{u}_c + \widehat{u}_s$ and $\widehat{u}_s = \widehat{u}_s(\widehat{u}_c)$. Since we have $\widehat{P}^{\pm}_{\alpha,c}\widehat{u} = (\widehat{\phi}^{\pm,*}_{\alpha},\widehat{u})_{L^2}\widehat{\phi}^{\pm}_{\alpha}$ we can write $\widehat{u}_{\pm1c} = A_{\pm1}\widehat{\phi}^{\pm}_{\alpha}$, where $A_{-1} = \bar{A}_1$. Using $\widehat{L}\widehat{\phi}^{\pm}_{\alpha} = \lambda^{\pm}_0(\alpha)\widehat{\phi}^{\pm}_{\alpha}$ gives

$$\begin{aligned} i\omega A_1\widehat{\phi}^{+}_{\alpha} &= \lambda^{+}_0(\alpha))A_1\widehat{\phi}^{+}_{\alpha} + f_1(A_1,A_{-1}), \\ -i\omega A_{-1}\widehat{\phi}^{-}_{\alpha} &= \lambda^{-}_0(\alpha)A_{-1}\widehat{\phi}^{-}_{\alpha} + f_{-1}(A_1,A_{-1}), \end{aligned} \tag{5.23}$$

with

$$f_{\pm1}(A_1,A_{-1}) = P_{\pm1}\widehat{P}^{\pm}_{\alpha,c} i\xi\cdot\widehat{Q}(A_1,A_{-1}), \tag{5.24}$$

where $\widehat{u}_c = \widehat{u}_c(A_1,A_{-1})$ and $\widehat{u}_s(\widehat{u}_c) = \widehat{u}_s(A_1,A_{-1})$.

System (5.23) can be rewritten as

$$\begin{aligned} g_1(\alpha-\alpha_c,\omega-\omega_0,A_1,A_{-1}) &= 0, \\ g_{-1}(\alpha-\alpha_c,\omega-\omega_0,A_1,A_{-1}) &= 0, \end{aligned} \tag{5.25}$$

with

$$g_{\pm1}(\alpha-\alpha_c,\omega-\omega_0,A_1,A_{-1}) = \mp i\omega A_{\pm1}\widehat{\phi}^{\pm}_{\alpha} + \lambda^{\pm}_0(\alpha)A_{\pm1}\widehat{\phi}^{\pm}_{\alpha} + f_{\pm1}(A_1,A_{-1}). \tag{5.26}$$

Since the original problem is invariant under $t\mapsto t+\varphi$, the reduced system is invariant under $A_{\pm1}\mapsto A_{\pm1}\exp(i\varphi)$. So $g_{\pm1}$ must have the form

$$A_1\tilde{g}_1(\alpha-\alpha_c,\omega-\omega_0,|A_1|^2) = 0,$$

$$A_{-1}\tilde{g}_{-1}(\alpha-\alpha_c, \omega-\omega_0, |A_1|^2) \quad = \quad 0 \tag{5.27}$$

with $\tilde{g}_{\pm 1}$ a smooth function in its arguments.

Introducing polar coordinates $A_{\pm 1} = r\exp(\pm i\varphi)$ yields

$$\begin{aligned} (\alpha-\alpha_c) + \gamma r^2 + \mathcal{O}(|\alpha-\alpha_c|^2 + |\omega-\omega_0|^2 + r^4) &= 0 \\ (\omega-\omega_0) + \mathcal{O}(r^2 + |\alpha-\alpha_c|^2 + |\omega-\omega_0|^2) &= 0, \end{aligned} \tag{5.28}$$

the well known equations for the occurrence of a Hopf bifurcation, so the proof of theorem 5.1 is complete.

# 6 Stability of the trivial solution

In this section we prove the nonlinear stability of the trivial solution $u \equiv 0$ of equation (2.9), resp. (3.8) in Fourier space, with respect to spatially localized perturbations, for parameter values $\alpha < \alpha_c$.
In the following let $1 \le q \le p \le \infty$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ and $d = 2, 3$. Then assume that

**(ASS1)** the analytic semigroup $e^{\widehat{L}t}$ generated by $\widehat{L}$ satisfies

$$\begin{aligned} \|e^{\hat{L}t}\|_{L^p_s \to L^q_s} &\le Ct^{-\frac{d}{2r}}, \\ \|e^{\hat{L}t}\xi_j\|_{L^p_s \to L^q_s} &\le Ct^{-\frac{1}{2}(\frac{d}{r}+1)}, \\ \|e^{\hat{L}t}\|_{L^p_s \to L^p_s} &\le C, \\ \|e^{\hat{L}t}\xi_j\|_{L^p_s \to L^p_s} &\le Ct^{-\frac{1}{2}}. \end{aligned}$$

In Section 6.2 we discuss the validity of this assumption.

## 6.1 The nonlinear stability

We prove

**Theorem 6.1** *Let $1 \le q \le p \le \infty$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, $r < d$ and $s > d\frac{p-1}{p}$. Assume that (ASS1) is true. For all $C_2 > 0$ there exists a $C_1 > 0$ such that the following holds. Let $\widehat{u}(0) \in L^p_s \cap L^q_s$ with $\|\widehat{u}(0)\|_{L^p_s \cap L^q_s} \le C_1$. Then for all $t > 0$ we have solutions $\widehat{u}$ of (3.8) satisfying*

$$\|\widehat{u}(t)\|_{L^p_s} + (1+t)^{\frac{d}{2r}} \|\widehat{u}(t)\|_{L^q_s} \le C_2. \tag{6.1}$$

**Proof.** Any mild solution of

$$\frac{\partial}{\partial t}\widehat{u} = \widehat{L}\widehat{u} + i\xi \cdot \widehat{Q}(\widehat{u}, \widehat{u}) \tag{6.2}$$

can be represented via the Duhamel formula

$$\widehat{u}(t) = \exp(\widehat{L}t)\widehat{u}^0 + \int_0^t \exp(\widehat{L}(t-\tau)) i\xi \widehat{Q}(\widehat{u}, \widehat{u})(\tau) d\tau. \tag{6.3}$$

Then define

$$
\begin{aligned}
\widehat{a}(t) &= \sup_{\tau\in[0,t]} \left((1+\tau)^{\frac{d}{2r}} \|\widehat{u}(\tau)\|_{L^q_s}\right), \\
\widehat{b}(t) &= \sup_{\tau\in[0,t]} \|\widehat{u}(\tau)\|_{L^p_s}.
\end{aligned} \tag{6.4}
$$

Using (ASS1) and lemma 3.9 we have for the $L^p_s$-norm,

$$
\begin{aligned}
\|\widehat{u}\|_{L^p_s} &\le \|\exp(\widehat{L}t)\|_{L^p_s\mapsto L^p_s}\|\widehat{u}^0\|_{L^p_s} + \sum_{j=1}^{d}\int_0^t \|\exp(\widehat{L}(t-\tau)\xi_j\|_{L^p_s\mapsto L^p_s}\|\widehat{Q}(\widehat{u},\widehat{u})\|_{L^p_s} d\tau \\
&\le C\left(\|\widehat{u}^0\|_{L^p_s} + \int_0^t (t-\tau)^{-\frac{1}{2}}\|\widehat{u}\|_{L^q_s}\|\widehat{u}\|_{L^p_s} d\tau\right) \\
&\le C\left(\widehat{b}(0) + \widehat{b}(t)\int_0^t (t-\tau)^{-\frac{1}{2}}\|\widehat{u}\|_{L^q_s} d\tau\right) \\
&\le C\left(\widehat{b}(0) + \widehat{b}(t)\widehat{a}(t)\int_0^t (t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{d}{2r}} d\tau\right) \\
&\le C\left(\widehat{b}(0) + \widehat{b}(t)\widehat{a}(t)\right),
\end{aligned}
$$

where $\int_0^t (t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{d}{2r}} d\tau < C$ for $\frac{d}{2r} > \frac{1}{2}$, i.e. $r < d$.

With the lemmas 3.7, 3.9 and (ASS1), we obtain for the $L^q_s$-norm that

$$
\begin{aligned}
(1+t)^{\frac{d}{2r}}\|\widehat{u}\|_{L^q_s} \le\ & (1+t)^{\frac{d}{2r}} \min\left\{\|\exp(\widehat{L}t)\|_{L^p_s\mapsto L^q_s}\|\widehat{u}^0\|_{L^p_s}, \|\exp(\widehat{L}t)\|_{L^q_s\mapsto L^q_s}\|\widehat{u}^0\|_{L^q_s}\right\} \\
&+(1+t)^{\frac{d}{2r}} \sum_{j=1}^{d}\int_0^t \min\left\{\|\exp(\widehat{L}(t-\tau)\xi_j\|_{L^p_s\mapsto L^q_s}\|\widehat{Q}(\widehat{u},\widehat{u})\|_{L^p_s},\right. \\
&\qquad\qquad \left.\|\exp(\widehat{L}(t-\tau)\xi_j\|_{L^q_s\mapsto L^q_s}\|\widehat{Q}(\widehat{u},\widehat{u})\|_{L^q_s}\right\} d\tau \\
\le\ & C\Bigg((1+t)^{\frac{d}{2r}} \min\left\{t^{-\frac{d}{2r}}\|\widehat{u}^0\|_{L^p_s}, \|\widehat{u}^0\|_{L^q_s}\right\} \\
&+(1+t)^{\frac{d}{2r}}\int_0^t \min\left\{(t-\tau)^{-\frac{1}{2}\left(\frac{d}{r}+1\right)}\|\widehat{u}\|_{L^p_s}\|\widehat{u}\|_{L^q_s}, (t-\tau)^{-\frac{1}{2}}\|\widehat{u}\|^2_{L^q_s}\right\} d\tau\Bigg) \\
\le\ & C\Bigg((\widehat{a}(0) + \widehat{b}(0))
\end{aligned}
$$

$$
\begin{aligned}
&+(1+t)^{\frac{d}{2r}}\widehat{b}(t)\int_0^t \min\left\{(t-\tau)^{-\frac{1}{2}\left(\frac{d}{r}+1\right)}\|\widehat{u}\|_{L^p_s},(t-\tau)^{-\frac{1}{2}}\|\widehat{u}\|_{L^q_s}\right\}\|\widehat{u}\|_{L^q_s}d\tau\Bigg)\\
\leq\ & C\Bigg((\widehat{a}(0)+\widehat{b}(0))+(\widehat{a}(t)+\widehat{b}(t))\widehat{a}(t))\times\\
&(1+t)^{\frac{d}{2r}}\int_0^t \min\left\{(t-\tau)^{-\frac{1}{2}\left(\frac{d}{r}+1\right)},(t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{d}{2r}}\right\}(1+\tau)^{-\frac{d}{2r}}d\tau\Bigg)\\
\leq\ & C\left(\widehat{a}(0)+\widehat{b}(0))+(\widehat{a}(t)+\widehat{b}(t))\widehat{a}(t)\right),
\end{aligned}
$$

where

$$
\min\left\{(1+t)^{\frac{d}{2r}}\int_0^t (t-\tau)^{-\frac{1}{2}\left(\frac{d}{r}+1\right)}(1+\tau)^{-\frac{d}{2r}}d\tau,\ (1+t)^{\frac{d}{2r}}\int_0^t (t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{d}{2r}}d\tau\right\}\leq C
$$

for $r<d$.

Altogether we have shown

$$
\begin{aligned}
\widehat{a}(t)\ &\leq\ C\left(\widehat{a}(0)+\widehat{b}(0))+(\widehat{a}(t)+\widehat{b}(t))\widehat{a}(t)\right),\\
\widehat{b}(t)\ &\leq\ C\left(\widehat{b}(0)+\widehat{b}(t)\widehat{a}(t)\right).
\end{aligned}
\tag{6.5}
$$

This implies [21] that for given $C_2>0$ we may choose $C_1>0$ such that $\widehat{a}(0)+\widehat{b}(0)\leq C_1$ leads to $\widehat{a}(t)+\widehat{b}(t)\leq C_2$ for all $t\in[0,\infty)$. ■

## 6.2 Estimates for the linear semigroup $e^{\widehat{B}t}$

In this section we look for assumptions on $\widehat{u}_c$ which validates the assumption (ASS1) for the linear semigroup $e^{\widehat{L}t}$. In order to do so we first prove the validity of assumption (ASS1) for the linear semigroup $e^{\widehat{B}t}$, i.e. we prove

**Lemma 6.1** *Let $1\leq q\leq p\leq\infty$, $\frac{1}{r}=\frac{1}{q}-\frac{1}{p}$ and $d=2,3$. Then the analytic semigroup $e^{\widehat{B}t}$ generated by $\widehat{B}(\xi)=-|\xi|^2-ic\xi_1$ satisfies*

$$\begin{aligned}
\|e^{\hat{B}t}\|_{L^p_s\to L^q_s} &\le Ct^{-\frac{d}{2r}},\\
\|e^{\hat{B}t}\xi_j\|_{L^p_s\to L^q_s} &\le Ct^{-\frac{1}{2}(\frac{d}{r}+1)},\\
\|e^{\hat{B}t}\|_{L^p_s\to L^p_s} &\le C,\\
\|e^{\hat{B}t}\xi_j\|_{L^p_s\to L^p_s} &\le Ct^{-\frac{1}{2}}.
\end{aligned}$$

**Proof.** In the sequel we use ([3]).

$$\begin{aligned}
\int_0^\infty \exp(-ax^2)dx &= \frac{\Gamma(\frac{1}{2})}{2\sqrt{a}} = \frac{1}{2}\sqrt{\frac{\pi}{a}},\\
\int_0^\infty x^n\exp(-ax^2)dx &= \frac{\Gamma(\frac{n+1}{2})}{2\sqrt{a^{n+1}}},\ a>0,\ n>-1.
\end{aligned} \tag{6.6}$$

Then we have

$$\begin{aligned}
\|\exp(\widehat{B}t)\widehat{f}\|_{L^q_s} &\le \|\exp(\widehat{B}t)\|_{L^r}\|\widehat{f}\|_{L^p_s} = \|\exp(-|\xi|^2t)\|_{L^r}\|\widehat{f}\|_{L^p_s}\\
&= \left(\int_{\mathbb{R}^d}\exp(-|\xi|^2tr)d\xi\right)^{\frac{1}{r}}\|\widehat{f}\|_{L^p_s} \le Ct^{-\frac{d}{2r}}\|\widehat{f}\|_{L^p_s}\\
&= Ct^{-\left(\frac{1}{q}-\frac{1}{p}\right)\frac{d}{2}}\|\widehat{f}\|_{L^p_s},
\end{aligned}$$

since

$$\int_{\mathbb{R}^d}\exp(-|\xi|^2a)d\xi = \prod_{j=1}^d\int_{-\infty}^\infty \exp(-a\xi_j^2)d\xi_j = 2^d\prod_{j=1}^d\int_0^\infty\exp(-a\xi_j^2)d\xi_j = \left(\frac{\pi}{a}\right)^{\frac{d}{2}}$$

with $a = tr$ and $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. For the second estimate let w.l.o.g $j = 1$. Then we have

$$\begin{aligned}
\|\exp(\widehat{B}t)\xi_1\widehat{f}\|_{L^q_s} &\le \|\exp(\widehat{B}t)\xi_1\|_{L^r}\|\widehat{f}\|_{L^p_s} = \|\exp(-|\xi|^2t)\xi_1\|_{L^r}\|\widehat{f}\|_{L^p_s}\\
&= \left(\int_{\mathbb{R}^d}\exp(-|\xi|^2tr)|\xi_1|^r d\xi\right)^{\frac{1}{r}}\|\widehat{f}\|_{L^p_s} \le Ct^{-\frac{d+r}{2r}}
\end{aligned}$$

$$= \quad Ct^{-\left(1+\frac{d}{q}-\frac{d}{p}\right)\frac{1}{2}},$$

since

$$\begin{aligned}
\int\limits_{\mathbb{R}^3} \exp(-|\xi|^2 a)|\xi_1|^r d\xi &= \int\limits_{-\infty}^{\infty} \exp(-\xi_1^2 a)|\xi_1|^r d\xi_1 \prod_{j=2}^{d} \int\limits_{-\infty}^{\infty} \exp(-\xi_j^2 a) d\xi_j \\
&= \frac{2\pi}{a} \int\limits_{0}^{\infty} \exp(-\xi_1^2 a)\xi_1^r d\xi_1 = \frac{\pi \Gamma\left(\frac{r+1}{2}\right)}{a^{\frac{r+d}{2}}}, \; a > 0, r > -1
\end{aligned}$$

with $a = tr$ and $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.

The third estimate follows from

$$\begin{aligned}
\|\exp(\widehat{B}t)\widehat{f}\|_{L^p_s} &\leq \|\exp(\widehat{B}t)\|_{L^\infty}\|\widehat{f}\|_{L^p_s} = \|\exp(-|\xi|^2 t)\|_{L^\infty}\|\widehat{f}\|_{L^p_s} \\
&= \sup_{\xi\in\mathbb{R}^d} |\exp(-|\xi|^2 t)|\|\widehat{f}\|_{L^p_s} \leq C\|\widehat{f}\|_{L^p_s}.
\end{aligned}$$

For the fourth estimate we remark that the function $g(x,t) = x\exp(-x^2 t)$ possesses extrema in $x = \pm\frac{1}{\sqrt{2t}}$ and so we have

$$\left| g(\pm\frac{1}{\sqrt{2t}}, t) \right| = \frac{1}{\sqrt{2e}} t^{-\frac{1}{2}}. \tag{6.7}$$

W.l.o.g. consider $j = 1$. Then we find

$$\begin{aligned}
\|\exp(\widehat{B}t)\xi_1\widehat{f}\|_{L^p_s} &\leq \|\exp(\widehat{B}t)\xi_1\|_{L^\infty}\|\widehat{f}\|_{L^p_s} = \|\exp(-|\xi|^2 t)\xi_1\|_{L^\infty}\|\widehat{f}\|_{L^p_s} \\
&\leq \frac{1}{\sqrt{2e}} t^{-\frac{1}{2}}\|\widehat{f}\|_{L^p_s} \leq C\|\widehat{f}\|_{L^p_s}.
\end{aligned}$$

■

## 6.3 Estimates for the linear semigroup $e^{\widehat{L}t}$ for small $u_c$

Next we prove that the same estimates hold for $\widehat{L}$ if $\widehat{u}_c$ is sufficient small.

**Theorem 6.2** *Let* $1 \leq q \leq p \leq \infty$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, $r < d$, $d = 2, 3$ *and* $s > d\frac{p-1}{p}$. *Then there exists a* $C > 0$ *such that the analytic semigroup generated by*

$$\widehat{L}\cdot = \widehat{B}\cdot + 2i\xi \cdot \widehat{Q}(\widehat{u}_c, \cdot) \tag{6.8}$$

*satisfies*

$$\begin{aligned} \|e^{\hat{L}t}\|_{L^p_s \to L^q_s} &\leq Ct^{-\frac{d}{2r}}, && (6.9)\\ \|e^{\hat{L}t}\xi_j\|_{L^p_s \to L^q_s} &\leq Ct^{-\frac{1}{2}\left(\frac{d}{r}+1\right)}, && (6.10)\\ \|e^{\hat{L}t}\|_{L^p_s \to L^p_s} &\leq C, && (6.11)\\ \|e^{\hat{L}t}\xi_j\|_{L^p_s \to L^p_s} &\leq Ct^{-\frac{1}{2}} && (6.12) \end{aligned}$$

*if* $\|\widehat{u}_0\|_{L^p_s \cap L^q_s} \leq C_0$.

**Proof.** We follow the scheme described in [21, p.293/294]. We consider the linear equation

$$\frac{\partial}{\partial t}\widehat{u} = \widehat{L}u = \widehat{B}\widehat{u} + 2i\xi \cdot \widehat{Q}(\widehat{u}_c, \widehat{u}).$$

Applying the variation of constant formula yields

$$\widehat{u}(t) = \exp(\widehat{B}t)\widehat{u}^0 + \int_0^t \exp(\widehat{B}(t-\tau))2i\xi \cdot \widehat{Q}(\widehat{u}_c, \widehat{u})(\tau)d\tau.$$

For the $L^p_s$-norm we get with lemma 6.1, lemma 3.10 and equation (6.4) that

$$\begin{aligned} \|\widehat{u}\|_{L^p_s} &\leq \|\exp(\widehat{B}t)\|_{L^p_s \mapsto L^p_s}\|\widehat{u}^0\|_{L^p_s} \\ &\quad + 2\sum_{j=1}^{d}\int_0^t \|\exp(\widehat{B}(t-\tau))\xi_j\|_{L^p_s \mapsto L^p_s}\|\widehat{Q}(\widehat{u}_c, \widehat{u})\|_{L^p_s} d\tau \\ &\leq C\left(\|\widehat{u}^0\|_{L^p_s} + \int_0^t (t-\tau)^{-\frac{1}{2}}\|\widehat{u}_c\|_{L^p_s}\|\widehat{u}\|_{L^q_s}\right) \\ &\leq C\left(\widehat{b}(0) + \|\widehat{u}_c\|_{L^p_s}\widehat{a}(t)\int_0^t (t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{d}{2r}}d\tau\right) \\ &\leq C(\widehat{b}(0) + \|\widehat{u}_c\|_{L^p_s}\widehat{a}(t)) \end{aligned}$$

for $r < d$ and with $\widehat{a}(t)$ from (6.4).

For the $L^q_s$-norm we get with $\widehat{a}(t)$, $\widehat{b}(t)$ from (6.4)

$$\begin{aligned}
(1+t)^{\frac{d}{2r}}\|u\|_{L^q_s} &\leq (1+t)^{\frac{d}{2r}} \min\Big\{\|\exp(\widehat{B}t)\|_{L^p_s\mapsto L^q_s}\|\widehat{u}^0\|_{L^p_s}, \|\exp(\widehat{B}t)\|_{L^q_s\mapsto L^q_s}\|\widehat{u}^0\|_{L^q_s}\Big\} \\
&\quad +2(1+t)^{\frac{d}{2r}} \sum_{j=1}^{d} \int_0^t \min\Big\{\|\exp(\widehat{B}(t-\tau)\xi_j\|_{L^p_s\mapsto L^q_s}\|\widehat{Q}(\widehat{u}_c,\widehat{u})\|_{L^p_s} \\
&\qquad \|\exp(\widehat{B}(t-\tau)\xi_j\|_{L^q_s\mapsto L^q_s}\|\widehat{Q}(\widehat{u}_c,\widehat{u})\|_{L^q_s}\Big\} d\tau \\
&\leq C\Bigg((1+t)^{\frac{d}{2r}} \min\Big\{t^{-\frac{d}{2r}}\|\widehat{u}^0\|_{L^p_s}, \|\widehat{u}^0\|_{L^q_s}\Big\}, \\
&\quad +(1+t)^{\frac{d}{2r}} \int_0^t \min\Big\{(t-\tau)^{-\frac{1}{2}\left(\frac{d}{r}+1\right)}\|\widehat{u}_c\|_{L^p_s}\|\widehat{u}\|_{L^q_s}, \\
&\qquad (t-\tau)^{-\frac{1}{2}}\|\widehat{u}_c\|_{L^q_s}\|\widehat{u}\|_{L^q_s}\Big\} d\tau\Bigg) \\
&\leq C\Bigg(\widehat{a}(0)+\widehat{b}(0)+(\|\widehat{u}_c\|_{L^p_s}+\|\widehat{u}_c\|_{L^q_s})\widehat{a}(t)) \\
&\times (1+t)^{\frac{d}{2r}} \int_0^t \min\Big\{(t-\tau)^{-\frac{1}{2}\left(\frac{d}{r}+1\right)}, (t-\tau)^{-\frac{1}{2}}\Big\}(1+\tau)^{-\frac{d}{2r}} d\tau\Bigg) \\
&\leq C(\widehat{a}(0)+\widehat{b}(0)+\|\widehat{u}_c\|_{L^p_s\cap L^q_s}\widehat{a}(t)),
\end{aligned}$$

where

$$(1+t)^{\frac{d}{2r}} \int_0^t \min\Big\{(t-\tau)^{-\frac{1}{2}\left(\frac{d}{r}+1\right)}, (t-\tau)^{-\frac{1}{2}}\Big\}(1+\tau)^{-\frac{d}{2r}} d\tau \leq C$$

for $r < d$.

Thus we have the estimates

$$\begin{aligned}
\widehat{a}(t) &\leq C(\widehat{a}(0)+\widehat{b}(0)+\|\widehat{u}_c\|_{L^p_s\cap L^q_s}\widehat{a}(t)), \\
\widehat{b}(t) &\leq C(\widehat{b}(0)+\|\widehat{u}_c\|_{L^p_s\cap L^q_s}\widehat{a}(t)).
\end{aligned} \tag{6.13}$$

Therefore, if $\|\widehat{u}_c\|_{L^p_s\cap L^q_s} > 0$ is sufficiently small, we can conclude

$$\begin{aligned} \widehat{a}(t) &\leq 2C\widehat{a}(0), \\ \widehat{b}(t) &\leq 2C\widehat{b}(0) \end{aligned} \tag{6.14}$$

for all $t \geq 0$. ■

**Remark 6.3** *Since $L^p_{\tilde{s}}$ can be embedded in $L^q_s$ if $\tilde{s} - s \geq \frac{d}{q} - \frac{d}{p}$ the assumption on $\widehat{u}_c$ can be relaxed to $\|\widehat{u}_c\|_{L^p_{\tilde{s}}} > 0$ is sufficiently small.*

## 6.4 Estimates for the resolvent $(\hat{B} - \lambda I)^{-1}$

Next we want to discuss the situation for non small $\widehat{u}_c$. We conclude the assumption (ASS1) from resolvent estimates for $\widehat{L}$. These resolvent estimates will allow in a subsequent section to prove the exchange of spectral stability in case of a Hopf bifurcation. In order to figure out the resolvent estimates for $\widehat{L}$, we discuss these estimates for $\widehat{B}$.

Since the operator $\widehat{B}$ is sectorial in every $L^q_s$ for $q \in [1, \infty]$ and $s \geq 0$ we have a $C > 0$ such that

$$\|(\widehat{B} - \lambda I)^{-1}\|_{L^q_s \mapsto L^q_s} \leq \frac{C}{|\lambda|} \tag{6.15}$$

for every $\lambda$ outside a sector containing the spectrum

$$\sigma(\widehat{B}) = \{\lambda = -|\xi|^2 - ic\xi_1 \in \mathbb{C} \mid \xi \in \mathbb{R}^d,\ d = 2, 3\} \tag{6.16}$$

of $\widehat{B}$. The polynomial decay rates of the associated semigroup $e^{\widehat{B}t}$, see lemma 6.1, are determined by the spectral properties near the imaginary axis. We have

**Theorem 6.4** *Let $1 \leq q \leq p \leq \infty$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ and $r > \frac{d-1}{2}$. Then there exists a $C > 0$ such that for all $\lambda \in i\mathbb{R}$ we have*

$$\|(\widehat{B} - \lambda I)^{-1}\|_{L^p_s \cap L^q_s \mapsto L^q_s} \leq \frac{C}{|\lambda|^{\frac{2r-d}{2r}}}. \tag{6.17}$$

**Proof.** Due to (3.9) we have that $(\widehat{B} - \lambda I)^{-1}$ is a multiplication operator given by

$$(\widehat{B} - \lambda I)^{-1} = \frac{1}{-|\xi|^2 - ic\xi_1 - \lambda}. \tag{6.18}$$

With Hölder's inequality we obtain

$$\begin{aligned}\|(\widehat{B}-\lambda I)^{-1}\widehat{f}\|_{L^q_s} &\leq \|(\widehat{B}-\lambda I)^{-1}\chi_{|\xi|\leq 1}\widehat{f}\|_{L^q_s} + \|(\widehat{B}-\lambda I)^{-1}\chi_{|\xi|\geq 1}\widehat{f}\|_{L^q_s}\\ &\leq \|(\widehat{B}-\lambda I)^{-1}\chi_{|\xi|\leq 1}\|_{L^r}\|\widehat{f}\|_{L^p_s} + \|(\widehat{B}-\lambda I)^{-1}\chi_{|\xi|\geq 1}\|_{L^\infty}\|\widehat{f}\|_{L^q_s}\end{aligned}$$

with $\frac{1}{q}=\frac{1}{r}+\frac{1}{p}$.

We first find

$$\|(\widehat{B}-\lambda I)^{-1}\chi_{|\xi|\geq 1}\|_{L^\infty} = \sup_{|\xi|\geq 1}\left(\frac{1}{||\xi|^2+ic\xi_1+\lambda|}\right) \leq C/(1+|\lambda|) \tag{6.19}$$

for all $\lambda \in i\mathbb{R}$. With polar coordinates and $\rho = \sigma\sqrt{|\lambda|}$ we find

$$\begin{aligned}\|(\widehat{B}-\lambda I)^{-1}\chi_{|\xi|\leq 1}\|^r_{L^r} &= \int\limits_{|\xi|\leq 1}\frac{d\xi}{||\xi|^2+ic\xi_1+\lambda|^r}\\ &\leq C\int\limits_{|\xi|\leq 1}\frac{d\xi}{|\xi|^{2r}+|\lambda|^r} \leq C\int\limits_0^1\frac{\rho^{d-1}}{\rho^{2r}+|\lambda|^r}d\rho\\ &\leq C\int\limits_0^{\frac{1}{\sqrt{|\lambda|}}}\frac{\sigma^{d-1}|\lambda|^{\frac{d-1}{2}}}{\sigma^{2r}|\lambda|^r+|\lambda|^r}\sqrt{|\lambda|}d\sigma\\ &= \frac{C}{|\lambda|^{r-\frac{d}{2}}}\int\limits_0^{\frac{1}{\sqrt{|\lambda|}}}\frac{\sigma^{d-1}}{\sigma^{2r}+1}d\sigma\\ &\leq \frac{C}{|\lambda|^{r-\frac{d}{2}}}\int\limits_0^{\infty}\frac{\sigma^{d-1}}{\sigma^{2r}+1}d\sigma\\ &\leq \frac{C}{|\lambda|^{r-\frac{d}{2}}},\end{aligned}$$

since $\int_0^\infty \frac{\sigma^{d-1}}{\sigma^{2r}+1}d\sigma < \infty$ for $r > \frac{d-1}{2}$. ■

Under the assumptions of the last theorem we have

**Lemma 6.2** *For $r > d-1$ there exists a $C>0$ such that for all $\lambda \in i\mathbb{R}$ we have*

$$\|(\widehat{B}-\lambda I)^{-1}\xi_j\|_{L^p_s\cap L^q_s\mapsto L^q_s} \leq \frac{C}{|\lambda|^{\frac{1}{2}-\frac{d}{2r}}},\ j=1,2,3. \tag{6.20}$$

**Proof.** Similar to the proof of the last lemma we get

$$\begin{aligned}
\|(\widehat{B}-\lambda I)^{-1}\xi_j\chi_{|\xi|\le 1}\|_{L^r}^r &= \int\limits_{|\xi|\le 1} \frac{|\xi_j|^r}{||\xi|^2+ic\xi_1+\lambda|^r}d\xi \le C\int\limits_0^1 \frac{\rho^r\rho^{d-1}}{\rho^{2r}+|\lambda|^r}d\rho \\
&\le C\int\limits_0^{\frac{1}{\sqrt{|\lambda|}}} \frac{\sigma^r|\lambda|^{\frac{r}{2}}\sigma^{d-1}|\lambda|^{\frac{d-1}{2}}}{\sigma^{2r}|\lambda|^r+|\lambda|^r}\sqrt{|\lambda|}d\sigma = \frac{C}{|\lambda|^{\frac{r-d}{2}}}\int\limits_0^{\frac{1}{\sqrt{|\lambda|}}} \frac{\sigma^{r+d-1}}{\sigma^{2r}+1}d\sigma \\
&\quad \frac{C}{|\lambda|^{\frac{r}{2}-1}}\int\limits_0^{\frac{1}{\sqrt{|\lambda|}}} \frac{\sigma^{r+d-1}}{\sigma^{2r}+1}d\sigma \le \frac{C}{|\lambda|^{\frac{r}{2}-1}}\int\limits_0^{\infty} \frac{\sigma^{r+d-1}}{\sigma^{2r}+1}d\sigma \\
&\le \frac{C}{|\lambda|^{\frac{r-d}{2}}}
\end{aligned}$$

since $\int_0^\infty \frac{\sigma^{r+1}}{\sigma^{2r}+1}d\sigma < \infty$ for $r > d-1$. ■

**Remark 6.5** *Using the representation*

$$\exp(\widehat{B}t) = \frac{1}{2\pi i}\int\limits_\Gamma \exp(\lambda t)(\widehat{B}-\lambda I)^{-1}d\lambda \tag{6.21}$$

*where $\Gamma$ is sketched in Figure 6.1, with the resolvent estimates from theorem 6.4 and lemma 6.2 gives the estimates for the linear semigroup from lemma 6.1.*

**Remark 6.6** *Due to the structure of the sectorial operator $\widehat{B}$, the contour $\Gamma$ can be chosen as follows*

$$\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 \tag{6.22}$$

*with*

$$\begin{aligned}
\Gamma_1 &= \{\lambda \in \mathbb{C} : \lambda = at - i,\ ; t \in (-\infty, 0]\}, \\
\Gamma_2 &= \{\lambda \in \mathbb{C} : \lambda = it,\ t \in [-1, 1]\}, \\
\Gamma_3 &= \{\lambda \in \mathbb{C} : \lambda = \bar{a}t + i,\ t \in [0, \infty]\}
\end{aligned}$$

*with $a \in \mathbb{C}$ suitable chosen.*

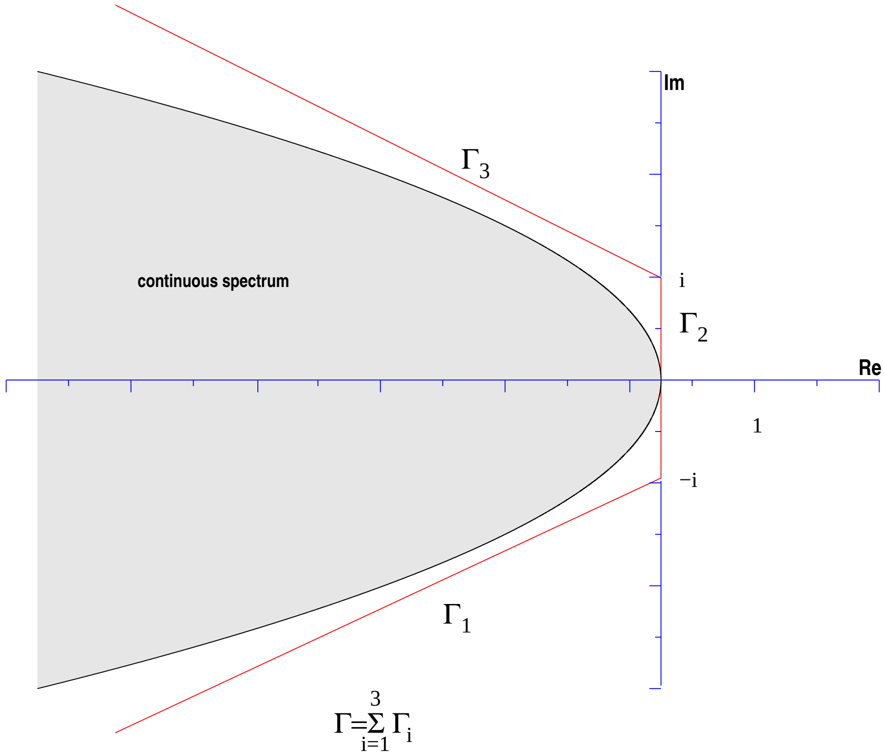



Figure 6.1: The contour $\Gamma$ for the Dunford integral (6.21)

## 6.5 Resolvent estimates for $\widehat{L}$

In this section we discuss assumptions for the resolvent which imply the validity of (ASS1).

For all values of $\alpha$ we assume that we have the resolvent estimate

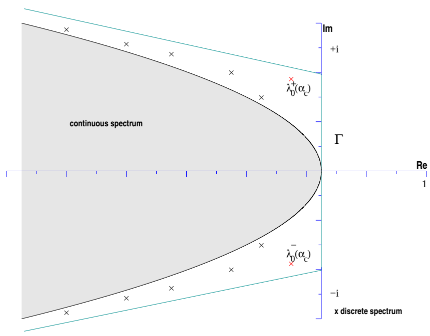
**(ASS2)** Let $1 \leq q \leq p \leq \infty$ and $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. Then there exists a $C > 0$ such that for all $\lambda \in \mathbb{C}$ we have

$$\|(\widehat{L} - \lambda I)^{-1}\|_{L^p_s \cap L^q_s \mapsto L^q_s} \leq \frac{C}{|\lambda|^{\frac{2r-d}{2r}}} \tag{6.23}$$

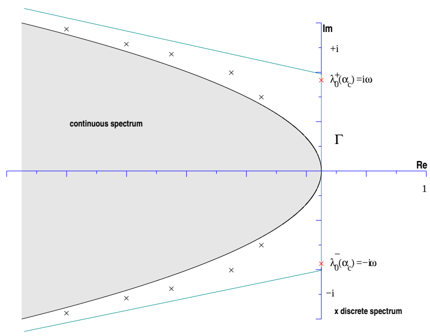
along the curve $\Gamma$ drawn in Figure 6.1.

Furthermore, we assume that except of two eigenvalues $\lambda_0^{\pm}(\alpha_c)$ the spectrum is located left of $\Gamma$. For $\alpha < \alpha_c$ the eigenvalues $\lambda_0(\alpha_c)^{\pm}$ are located left of the imaginary axis as shown in Figure 6.2 (a). For $\alpha = \alpha_c$ the eigenvalues $\lambda_0(\alpha_c)^{\pm} = \pm i\omega_0 \neq 0$ are on the imaginary axis (see Figure 6.2 (b)) and for $\alpha > \alpha_c$ the eigenvalues $\lambda_0(\alpha_c)$ are right of the imaginary axis, see Figure 6.2 (c). Then for $\alpha < \alpha_c$ (ASS2) implies the validity of (ASS1).

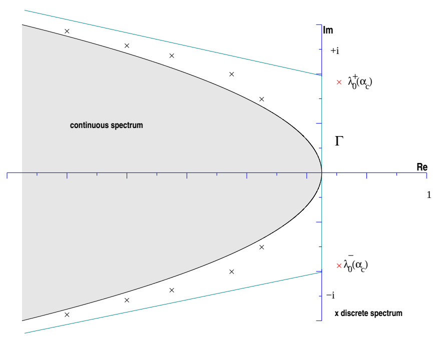
Figure 6.2: Localization of the discrete eigenvalues




(a) $\lambda < \lambda_c$




(b) $\lambda = \lambda_c$




(c) $\lambda > \lambda_c$

# 7 The exchange of spectral stability

In this section, similar to classical Hopf bifurcation, we prove the exchange of spectral stability under the assumption (ASS2). In the previous section we have established the dynamic stability of the trivial solution for $\alpha < \alpha_c$ and the occurrence of a Hopf bifurcation at $\alpha = \alpha_c$. Now we consider the spectral stability of the small bifurcating time-periodic solutions for $\alpha > \alpha_c$, i.e. we compute the Floquet exponents of the linearization around a bifurcating solution. In order to do so we use the so called principle of reduced stability [14].

We start again with equation (3.8)

$$\frac{\partial}{\partial t}\widehat{u} = \widehat{L}\widehat{u} + i\xi \cdot \widehat{Q}(\widehat{u}, \widehat{u}). \tag{7.1}$$

The linearization around the time-periodic solution $\widehat{u}_p(\xi, t)$ is given by

$$\frac{\partial}{\partial t}\widehat{v} = \widehat{L}\widehat{v} + 2i\xi \cdot \widehat{Q}(\widehat{u}_p, \widehat{v}). \tag{7.2}$$

The solutions are given by the Floquet ansatz

$$\widehat{v}(\xi, t) = \widehat{w}(\xi, t) \exp \lambda t, \tag{7.3}$$

where

$$\widehat{w}(\xi, t) = \widehat{w}(\xi, t + T) \tag{7.4}$$

is $T = 2\pi/\omega$-periodic. This yields

$$\frac{\partial}{\partial t}\widehat{w} = \widehat{L}\widehat{w} - \lambda\widehat{w} + 2i\xi \cdot \widehat{Q}(\widehat{u}_p, \widehat{w}). \tag{7.5}$$

We have to show that $\mathrm{Re}\lambda \leq 0$ in case of nontrivial solutions $\widehat{w}$. Due to the $2\pi/\omega$-periodicity we can use Fourier series. Analogous to section 5 about the Hopf bifurcation we consider equivalently $2\pi$ periodic solutions of

$$\omega\frac{\partial}{\partial t}\widehat{w} = (\widehat{L} - \lambda I)\widehat{w} + 2i\xi \cdot \widehat{Q}(\widehat{u}_p, \widehat{w}). \tag{7.6}$$

Introducing the Fourier series

$$\widehat{w}(\xi, t) = \sum_{k\in\mathbb{Z}} \widehat{w}_k(\xi) \exp(ikt),$$

$$\widehat{u}_p(\xi,t) \;=\; \sum_{k\in\mathbb{Z}} \widehat{u}_{k,p}(\xi)\exp(ikt) \tag{7.7}$$

for the solution $\widehat{w}$ of equation (7.6) and for the time-periodic solution $\widehat{u}_p(\xi,t)$ of equation (7.1) yields the infinite dimensional system

$$\begin{aligned}
\widehat{w}_k &= -(\widehat{L}-\lambda I - ik\omega I)^{-1}P_k 2i\xi\cdot\widehat{Q}(\widehat{u}_p,\widehat{w}),\ k\neq\{-1,0,1\},\\
\widehat{w}_{\pm 1,s} &= -(\widehat{L}-\lambda I \mp i\omega I)^{-1}P_k\widehat{P}^{\pm}_{\alpha,s}2i\xi\cdot\widehat{Q}(\widehat{u}_p,\widehat{w}),\ k=\pm 1,\\
\widehat{w}_0 &= -(\widehat{L}-\lambda I)^{-1}P_0 2i\xi\cdot\widehat{Q}(\widehat{u}_p,\widehat{w}),\\
\pm i\omega\widehat{w}_{\pm 1,c} &= (\widehat{L}-\lambda I)\widehat{w}_{\pm 1,c} + P_k\widehat{P}^{\pm}_{\alpha,c}2i\xi\cdot\widehat{Q}(\widehat{u}_p,\widehat{w}),\ k=\pm 1,
\end{aligned} \tag{7.8}$$

where

$$P_k\widehat{w} = \frac{1}{2\pi}\int_0^{2\pi}\exp(-ikt)\widehat{w}(\xi,t)dt \tag{7.9}$$

is the projection on the $k$-th component of $\widehat{w}$.

**Remark 7.1** *Note that equation* (7.8) *is linear w.r.t.* $\widehat{w}$.

Equation (7.8) can be rewritten as

$$\begin{aligned}
(\widehat{L}-ik\omega)\widehat{w}_k - 2i\xi\cdot\widehat{Q}(\widehat{u}_p,\widehat{w}_k) - \lambda\widehat{w}_k &= 0,\ k\neq\{-1,0,1\},\\
(\widehat{L}\mp i\omega)\widehat{w}_{\pm 1s} - 2iP^{\pm}_{\alpha,s}\xi\cdot\widehat{Q}(\widehat{u}_p,\widehat{w}_{\pm 1}) - \lambda\widehat{w}_{\pm 1s} &= 0,\ k=\pm 1,\\
\widehat{L}\widehat{w}_0 - 2i\xi\cdot\widehat{Q}(\widehat{u}_p,\widehat{w}_0) - \lambda\widehat{w}_0 &= 0,\\
(\widehat{L}\mp i\omega)\widehat{w}_{\pm 1c} - 2iP^{\pm}_{\alpha,c}\xi\cdot\widehat{Q}(\widehat{u}_p,\widehat{w}_{\pm 1}) - \lambda\widehat{w}_{\pm 1c} &= 0,\ k=\pm 1.
\end{aligned} \tag{7.10}$$

Since the continuous spectrum of $\widehat{L}-\lambda I$ lies in the left half-plane for $\mathrm{Re}\lambda\geq 0$, we have that lemma 5.4 holds for $\widehat{L}-\lambda I$ instead of $\widehat{L}$ too, and so the operator $\widehat{L}-\lambda I$ is also invertible for $\mathrm{Re}\lambda\geq 0$. Using the projections defined in section 5 we can rewrite the first three equation (7.8) as $F=F(\widehat{u}_c,\widehat{u}_s)=0$, where

$$\begin{aligned}
F_k &= -\widehat{w}_k - (\widehat{L}-\lambda I - ik\omega I)^{-1}P_k 2i\xi\cdot\widehat{Q}(\widehat{u}_p,\widehat{w}) = 0,\ k\neq\{-1,0,1\},\\
F_{\pm 1,s} &= -\widehat{w}_{\pm 1,s} - (\widehat{L}-\lambda I\mp i\omega I)^{-1}P_{\pm 1}\widehat{P}^{\pm}_{\alpha,s}2i\xi\cdot\widehat{Q}(\widehat{u}_p,\widehat{w}) = 0,\ k=\pm 1,\\
F_0 &= -\widehat{w}_0 - (\widehat{L}-\lambda I)^{-1}P_0 2i\xi\cdot\widehat{Q}(\widehat{u}_p,\widehat{w}) = 0
\end{aligned} \tag{7.11}$$

with

$\widehat{w}_c = (\ldots, 0, \widehat{w}_{-1c}, 0, \widehat{w}_{1c}, 0, \ldots)$, $\widehat{w}_s = (\ldots, \widehat{w}_{-2}, \widehat{w}_{-1s}, \widehat{w}_0, \widehat{w}_{1s}, \widehat{w}_2, \ldots)$ and $\widehat{w} = \widehat{w}_c + \widehat{w}_s$.
With the same arguments as in section 5 by lemma 3.9 we can conclude by the implicit function theorem and remark 7.1 that there exist a unique solution $\widehat{w}_s = \widehat{w}_s(\widehat{w}_c)$ which is linear w.r.t. $\widehat{w}_c$ satisfying $\|\widehat{w}_s\|_{\mathcal{X}_s^p} \leq C\|\widehat{u}_p\|_{\mathcal{X}_s^p}\|\widehat{w}_c\|_{\mathcal{X}_s^p}$.

Hence equation(7.10) can be reduced to

$$\begin{aligned} i\omega\widehat{w}_{1,c} &= (\widehat{L} - \lambda I)\widehat{w}_{1,c} + \widehat{P}^{\pm}_{\alpha,c}P_1 2i\xi \cdot \widehat{Q}(\widehat{u}_p, \widehat{w}), \\ -i\omega\widehat{w}_{-1,c} &= (\widehat{L} - \lambda I)\widehat{w}_{-1,c} + \widehat{P}^{\pm}_{\alpha,c}P_{-1} 2i\xi \cdot \widehat{Q}(\widehat{u}_p, \widehat{w}), \end{aligned} \tag{7.12}$$

with $\widehat{w} = \widehat{w}_c + \widehat{w}_s$ and $\widehat{w}_s = \widehat{w}_s(\widehat{w}_c)$. Since we know that $\widehat{P}^{\pm}_{\alpha,c}\widehat{w} = (\widehat{\phi}^{\pm *}_{\alpha}, \widehat{w})\widehat{\phi}^{\pm}_{\alpha}$ we introduce $c_{\pm 1}$ by $\widehat{w}_{\pm 1c} = c_{\pm 1}\widehat{\phi}^{\pm}_{\alpha}$. Using the fact that $\widehat{L}\widehat{\phi}^{\pm}_{\alpha} = \lambda_0(\alpha)\widehat{\phi}^{\pm}_{\alpha}$ gives

$$\begin{aligned} (\lambda_0(\alpha) - i\omega c_{-1} - \lambda)c_1 + h_1(\lambda, \alpha, \omega, c_1, c_{-1}) &= 0, \\ (\lambda_0(\alpha) + i\omega c_1 - \lambda)c_{-1} + h_{-1}(\lambda, \alpha, \omega, c_1, c_{-1}) &= 0, \end{aligned} \tag{7.13}$$

which is linear in $c_1, c_{-1}$ and nonlinear in the parameter $\alpha$, $\omega$ and $\lambda$ and where

$$\begin{aligned} h_1(\lambda, \alpha, \omega, c_1, c_{-1}) &= \widehat{P}^{\pm}_{\alpha,c}P_1 2i\xi \cdot \widehat{Q}(\widehat{u}_p, \widehat{w}), \\ h_{-1}(\lambda, \alpha, \omega, c_1, c_{-1}) &= \widehat{P}^{\pm}_{\alpha,c}P_{-1} 2i\xi \cdot \widehat{Q}(\widehat{u}_p, \widehat{w}). \end{aligned}$$

In order to analyze this two-dimensional system we remark that for $\lambda = 0$ it coincides with the linearization of the reduced bifurcation equation (5.25), $g_1(A_1, A_{-1}, \alpha, \omega) = 0$ around the bifurcating time-periodic solution. Thus (7.13) is of the form

$$\lambda c_1 + \frac{\partial}{\partial A_1} g_1(A_1, A_{-1}, \alpha, \omega)c_1 + \mathcal{O}(|\lambda|^2, |\alpha - \alpha_c|^2, |\omega - \omega_0|^2) = 0. \tag{7.14}$$

Thus for $\alpha$ close to $\alpha_c$ (7.14) can be resolved with respect to $\lambda$. We have two solutions, namely $\lambda = 0$ due to the fact that we have a whole family of time-periodic solutions and a negative eigenvalue, as usual in case of a supercritical Hopf bifurcation.

**Theorem 7.2** *For $\alpha - \alpha_c > 0$ sufficiently small, the bifurcating time-periodic solutions $\widehat{u}_p(t)$ of equation (7.1) are spectrally stable.*

# 8 Remarks about the two-dimensional case

In this section we consider the problems from above for the vorticity formulation of the Navier-Stokes equation in $\mathbb{R}^2$, i.e. we want to find out if a Hopf bifurcation theorem can be shown, if the trivial solution can be shown to be nonlinear stable and finally if an exchange of spectral stability holds in this case too.

## 8.1 The equations

We start with the equations from section 2 and use the notation therein, i.e. $\tilde{u}$ is the velocity and $u$ the vorticity. We consider equation (2.3)

$$\begin{aligned}
\frac{\partial}{\partial t}\tilde{u}(x,t) &= \Delta\tilde{u}(x,t) - c\frac{\partial}{\partial x_1}\tilde{u}(x,t) - (\tilde{u}_0(x)\cdot\nabla)\tilde{u}(x,t) - (\tilde{u}(x,t)\cdot\nabla)\tilde{u}_c(x) \\
&\quad -\nabla q(x,t) - (\tilde{u}(x,t)\cdot\nabla)\tilde{u}(x,t) \\
\nabla\cdot\tilde{u}(x,t) &= 0.
\end{aligned} \tag{8.1}$$

Defining the vorticity in $\mathbb{R}^2$ [10] as

$$u = \nabla\times\tilde{u} = \frac{\partial}{\partial x_1}\tilde{u}_2 - \frac{\partial}{\partial x_2}\tilde{u}_1 \tag{8.2}$$

yields the vorticity formulation

$$\frac{\partial}{\partial t}u = \Delta u - c\frac{\partial}{\partial x_1}u - (\tilde{u}\cdot\nabla)u - (\tilde{u}_c\cdot\nabla)u - (\tilde{u}\cdot\nabla)u_c, \tag{8.3}$$

where $\nabla\cdot u = \nabla\cdot(\nabla\times\tilde{u}) = 0$.

We note that the vorticity in two space dimensions is a scalar, while in $\mathbb{R}^3$ it is a vector. Furthermore, the velocity $\tilde{u}$ can be reconstructed from the vorticity via the Biot-Savart law

$$\tilde{u}(x) = \frac{1}{2\pi}\int\limits_{\mathbb{R}^2}\frac{(x-y)^{\perp}}{|x-y|^2}u(y)dy, \tag{8.4}$$

where $x^{\perp} = (-x_2, x_1)^T$. Some connections between $\tilde{u}$ and $u$ in terms of Lebesgue spaces are given in [10]. Using the fact that $\nabla\cdot u = \nabla\cdot u_c = 0$ and $\nabla\cdot\tilde{u} = \nabla\cdot\tilde{u}_c = 0$ we can rewrite equation (8.3) as

$$\frac{\partial}{\partial t}u = \Delta u - c\frac{\partial}{\partial x_1}u - \nabla\cdot(\tilde{u}u) - \nabla\cdot(\tilde{u}_c u) - \nabla\cdot(\tilde{u}u_c). \tag{8.5}$$

Introducing the operators

$$\begin{aligned} Bu &= \Delta u - c\frac{\partial}{\partial x_1}u, \\ 2Q(u,v) &= -(\tilde{u}v) - (\tilde{v}u), \end{aligned} \tag{8.6}$$

we can rewrite (8.5) as

$$\frac{\partial}{\partial t}u = Bu - 2\nabla \cdot Q(u_c, u) - \nabla \cdot Q(u,u). \tag{8.7}$$

As next step we transfer the last equation in Fourier space

$$\frac{\partial}{\partial t}\widehat{u} = \widehat{B}\widehat{u} - 2i\xi \cdot \widehat{Q}(\widehat{u}_c, \widehat{u}) - i\xi \cdot \widehat{Q}(\widehat{u}, \widehat{u}), \tag{8.8}$$

with

$$\begin{aligned} \widehat{B}\widehat{u} &= -(|\xi|^2 + ic\xi_1)u, \\ 2\widehat{Q}(\widehat{u}, \widehat{v}) &= -(\widehat{\tilde{u}} * \widehat{v}) - (\widehat{\tilde{v}} * \widehat{u}), \end{aligned} \tag{8.9}$$

where $*$ denotes again the convolution.

## 8.2 The connection between the velocity and vorticity in Fourier space

The velocity in terms of the vorticity is given by solving the system

$$\begin{aligned} \nabla \times \tilde{u} &= \frac{\partial}{\partial x_1}\tilde{u}_2 - \frac{\partial}{\partial x_2}\tilde{u}_1 = u, \\ \nabla \cdot \tilde{u} &= \frac{\partial}{\partial x_1}\tilde{u}_1 + \frac{\partial}{\partial x_2}\tilde{u}_2 = 0. \end{aligned} \tag{8.10}$$

In Fourier space we have

$$\begin{pmatrix} -i\xi_2 & i\xi_1 \\ i\xi_1 & i\xi_2 \end{pmatrix} \begin{pmatrix} \widehat{\tilde{u}}_1 \\ \widehat{\tilde{u}}_2 \end{pmatrix} = \begin{pmatrix} \widehat{u} \\ 0 \end{pmatrix}, \tag{8.11}$$

which is solved by

$$\begin{pmatrix} \widehat{\tilde{u}}_1 \\ \widehat{\tilde{u}}_2 \end{pmatrix} = -\frac{1}{|\xi|^2} \begin{pmatrix} -i\xi_2 & i\xi_1 \\ i\xi_1 & i\xi_2 \end{pmatrix} \begin{pmatrix} \widehat{u} \\ 0 \end{pmatrix}. \tag{8.12}$$

Applying the Hölder inequality yields

$$\begin{aligned} \left\| \begin{pmatrix} \widehat{\tilde{u}}_1 \\ \widehat{\tilde{u}}_2 \end{pmatrix} \right\|_{L^q} \leq & \left\| \chi_{\|\xi\| \leq 1} \frac{1}{|\xi|^2} \begin{pmatrix} -i\xi_2 & i\xi_1 \\ i\xi_1 & i\xi_2 \end{pmatrix} \right\|_{L^r} \left\| \begin{pmatrix} \widehat{u} \\ 0 \end{pmatrix} \right\|_{L^p} \\ & + \left\| \chi_{\|\xi\| \geq 1} \frac{1}{|\xi|^2} \begin{pmatrix} -i\xi_2 & i\xi_1 \\ i\xi_1 & i\xi_2 \end{pmatrix} \right\|_{L^\infty} \left\| \begin{pmatrix} \widehat{u} \\ 0 \end{pmatrix} \right\|_{L^q}, \end{aligned} \tag{8.13}$$

with $\frac{1}{q} = \frac{1}{p} + \frac{1}{r}$. From equation (8.13) we need estimates for

$$K_j^\infty(\xi) = \chi_{|\xi| \geq 1} \frac{i\xi_j}{|\xi|^2},\ j = 1, 2$$

in the space $L^\infty(\mathbb{R}^2)$ and

$$K_j(\xi) = \chi_{|\xi| \leq 1} \frac{i\xi_j}{|\xi|^2},\ j = 1, 2$$

in the space $L^r(\mathbb{R}^2)$. Analogous to three-dimensional case, the estimate for $K_j^\infty(\xi)$ is obvious. For $K_j(\xi)$ we have with polar coordinates

$$\|K_j(\xi)\|_{L^r}^r = \int_{|\xi| \leq 1} \left| \frac{\xi_j}{|\xi|^2} \right|^r d\xi \underbrace{\leq}_{\text{Polar coordinates}} C \int_0^1 \frac{\rho^r}{\rho^{2r}} \rho d\rho = \int_0^1 \frac{d\rho}{\rho^{r-1}},$$

which is bounded for $r < 2$. Furthermore if we multiply equation (8.12) with $i\xi_j$ we find

$$\left\| i\xi_j \begin{pmatrix} \widehat{\tilde{u}}_1 \\ \widehat{\tilde{u}}_2 \end{pmatrix} \right\|_{L^q} \leq \left\| \frac{i\xi_j}{|\xi|^2} \begin{pmatrix} -i\xi_2 & i\xi_1 \\ i\xi_1 & i\xi_2 \end{pmatrix} \right\|_{L^\infty} \left\| \begin{pmatrix} \widehat{u} \\ 0 \end{pmatrix} \right\|_{L^q}. \tag{8.14}$$

With these computations above we have the validity of a modified version of lemma 3.5 in two space dimensions.

**Lemma 8.1** *Assume that* $\frac{1}{q} = \frac{1}{p} + \frac{1}{r}$ *with* $r \in [1,2)$ *and* $p,q \in [1,\infty]$. *If* $\widehat{u} \in L^p(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$ *then* $\widehat{\widehat{u}} \in L^q(\mathbb{R}^2)^2$ *and there exists a* $C > 0$ *such that*

$$\|\widehat{\widehat{u}}\|_q \leq C(\|\widehat{u}\|_p + \|\widehat{u}\|_q). \tag{8.15}$$

*Moreover*

$$\|i\xi_j \widehat{\widehat{u}}\|_q \leq C\|\widehat{u}\|_q. \tag{8.16}$$

## 8.3 Estimates for the bilinear form $\widehat{Q}(\widehat{u},\widehat{v})$ and the Oseen Operator

The embedding lemma 3.4 is formulated in $\mathbb{R}^d$, i.e. also holds for $d = 2$. Then we have with lemma 8.1 that the results given in the lemmas 3.7-3.11 are all valid in two space dimensions too, if we keep in mind that we have to choose $p \in (2,\infty)$ and $s > 2\frac{p-1}{p}$. So we summarize

**Lemma 8.2**

- *Let* $p \in (2,\infty]$*and* $s > 2\frac{p-1}{p}$. *Then there exists a* $C > 0$ *such that for every* $u \in L^p_s$

$$\|\widehat{u} * \widehat{\widehat{u}}\|_{L^p_s} \leq C\|\widehat{u}\|^2_{L^p_s}. \tag{8.17}$$

- *Let* $q \leq p,\ p \in (2,\infty]$ *and* $s > 2\frac{p-1}{p}$. *Then there exists a* $C > 0$ *such that for every* $\widehat{u} \in L^q_s$

$$\|\widehat{u} * \widehat{\widehat{u}}\|_{L^p_s} \leq C\|\widehat{u}\|_{L^q_s}\|\widehat{u}\|_{L^p_s}. \tag{8.18}$$

- *Let* $p \in (2,\infty]$ *and* $s > 2\frac{p-1}{p}$. *Then there exists a* $C > 0$ *such that for every* $\widehat{u},\widehat{v} \in L^p_s$

$$\|\widehat{Q}(\widehat{u},\widehat{v})\|_{L^p_s} \leq C\|\widehat{u}\|_{L^p_s}\|\widehat{v}\|_{L^p_s} \tag{8.19}$$

.

- *Let $q \leq p$, $p \in (2, \infty]$ and $s > 2\frac{p-1}{p}$. Then there exists a $C > 0$ such that for every $\widehat{u} \in L^q_s, \widehat{v} \in L^p_s$*

$$\|\widehat{Q}(\widehat{u}, \widehat{v})\|_{L^p_s} \leq C\|\widehat{u}\|_{L^q_s}\|\widehat{v}\|_{L^p_s}. \tag{8.20}$$

- *Let $p, q, \tilde{q} \in [1, \infty]$ and $s \geq 0$ satisfying*

$$\begin{aligned} 1 &= \frac{1}{p} + \frac{1}{q}, \\ \frac{1}{\tilde{q}} &= = \frac{1}{q} - \frac{1}{r^*}, \\ s &> 2\frac{\tilde{q} - q}{q\tilde{q}}, \end{aligned} \tag{8.21}$$

  *with $p \geq \tilde{q}$, $p \geq q$ and $r^* \in [1, 2]$. Then there exists a $C > 0$ such that for every $\widehat{u} \in L^p_s$*

$$\|\widehat{u} * \widehat{\tilde{u}}\|_{L^\infty} \leq C\|\widehat{u}\|^2_{L^p_s}. \tag{8.22}$$

- *Let $p, q, \tilde{p} \in [1, \infty]$ and $s \geq 0$ satisfying (8.21). Then there exists a $C > 0$ such that for every $\widehat{u}, \widehat{v} \in L^p_s$*

$$\|\widehat{Q}(\widehat{u}, \widehat{v})\|_{L^\infty} \leq C\|\widehat{u}\|_{L^p_s}\|\widehat{v}\|_{L^p_s}. \tag{8.23}$$

In the next step we consider the invertibility of the operator $B$ in Fourier space. We get an analogous result as in lemma 3.13 and 3.1 for $p < 3$.

**Lemma 8.3** *If d=2 we have for $p < 3$ that $\widehat{B}^{-1} i\xi_2 : L^p \cap L^\infty \mapsto L^p$ and for $p \in [1, \infty]$ we have $\widehat{B}^{-1} i\xi_1 : L^p \mapsto L^p$ .*

**Remark 8.1** *With the lemmas above we get the following values. We have $p = 3 - \delta$, $r \in [1, 2)$ and $\tilde{q} = 6/5 + \mathcal{O}(\delta)$. By Hölder we find $q = 3/2 + \mathcal{O}(\delta)$, which contradicts the Biot-Savart law, where*

$$\frac{1}{q} = \frac{1}{p} + \frac{1}{r} \tag{8.24}$$

*leads to $r = 3$.*

Thus a Hopf bifurcation theorem with method described above cannot be established in $\mathbb{R}^2$. As a consequence the exchange of spectral stability is obsolete.

## 8.4 The nonlinear stability of the trivial solution in $\mathbb{R}^2$

The nonlinear stability of the trivial solution $u \equiv 0$ of equation (8.8) in $\mathbb{R}^2$ with respect to spatially localized perturbations can be proved in the same way as in the section 6. Only the values of some parameters will differ, since they depend on the space dimension. In section 6 theorem 6.1, 6.2 and 6.4 as well as the lemmas 6.1 and 6.2 are formulated for all $d \geq 2$. Therefore all results and assumptions from section 6 hold also in $\mathbb{R}^2$.

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# List of Figures

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