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A numerical solution method for an infinitesimal elasto-plastic Cosserat model

Patrizio Neff, Krzysztof Chełmiński , Wolfgang Müller, and Christian Wieners

Abstract. We present a finite element implementation of a Cosserat elasto-plastic model allowing for non-symmetric stresses and we provide a numerical analysis of the introduced time-incremental algorithm. The model allows the use of standard tools from convex analysis as known from classical Prandtl-Reuss plasticity. We derive the dual stress formulation and show that for vanishing Cosserat couple modulus $\mu_c \to 0$ the classical problem with symmetric stresses is approximated. Our numerical results testify to the robustness of the approximation. Notably, for positive couple modulus $\mu_c > 0$ there is no need for a safe-load assumption. For small μ_c the response is numerically indistinguishable from the classical response.

1. Introduction

This article addresses a finite element implementation and the numerical analysis of *geometrically linear* generalized continua of *Cosserat micropolar* type for elasto-plasticity. General continuum models involving *independent rotations* as additional degrees of freedom have been first introduced by the Cosserat brothers [**12**].

Their development has been largely forgotten for decades only to be rediscovered in the beginning of the sixties [**56, 32, 1, 20, 18, 67, 68, 31, 41, 63, 69**]. At that time theoretical investigations of non-classical continuum theories were the main motivation [**37**]. The Cosserat concept has been generalized in various directions, for an overview of these so called *microcontinuum* theories look at [**19, 21, 7**].

Among the first contributions extending the Cosserat framework to infinitesimal elasto-plasticity we should mention [**62, 40, 6**]. More recent infinitesimal elasto-plastic formulations have been investigated in [**14, 16, 35, 58**]. These models directly comprise joint elastic and plastic Cosserat effects. Lately, the models have been extended to a finite elasto-plastic setting as well, see e.g. [**30, 59, 60, 61, 64, 29, 22**] and references therein. Most of these extensions directly comprise joint elastic and plastic Cosserat effects as well but we pretend that their physical and mathematical significance is at present much more difficult to asses than models where Cosserat effects are restricted to the elastic response of the material [**22**] and references therein. We will investigate a model of the second type which has been introduced in [**50, 48**] in a finite strain framework. A geometrical linearization of this model has been investigated in [**52, 54**] and is shown to be well-posed also in the rate-independent limit for both quasistatic and dynamic processes.

Apart from the theoretical development, the Cosserat type models are today increasingly advocated as a means to regularize the pathological mesh size dependence of localization computations where shear failure mechanisms [**11, 44, 42, 5, 4**] play a dominant role, for applications in plasticity see the non-exhaustive list [**35, 16, 58, 13, 15, 14**]. The occurring

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mathematical difficulties reflect the physical fact that upon localization the validity limit of the classical continuum models is reached. In models without any internal length the deformation should be homogeneous on the scale of a representative volume element of the material [**43**].

The incorporation of a length scale, which is natural in a Cosserat theory, has the potential to remove the mesh sensitivity. The presence of the internal length scale causes the localization zones to have finite width. However, the actual length scale of a material is difficult to establish experimentally and theoretically [**38**] and remains basically an open question as is the determination of other additionally appearing material constants in the Cosserat framework. It is also not entirely clear, how the shear band width depends on the characteristic length.

For the older mathematical analysis of infinitesimal, linearly elastic Cosserat micropolar models the reader may consult [**17, 34, 25, 26**]. Existence results for a geometrically exact elastic Cosserat model are obtained in [**45**].

As far as classical rate-independent (perfect) elasto-plasticity is concerned we remark that global existence for the displacement has been shown only in a very weak, measure-valued sense, while the stresses could be shown to remain in $L^2(\Omega)$, provided a safe-load condition is assumed. For this results we refer for example to [**3, 10, 66**]. If hardening or viscosity is added, then global classical solution are found see e.g. [**2, 9, 8**], already without safe-load assumption. A complete theory for the classical rate-independent case remains, however, elusive, see also the remarks in [**10**].

While the infinitesimal Cosserat micropolar elasto-plasticity model in its various versions is interesting mathematically in its own right we concentrate in this contribution on the *regularizing properties* for positive Cosserat couple modulus $\mu_c > 0$ of the model presented in [**52**]. We emphasize that our non-dissipative formulation seems to provide just the necessary amount of regularization missing in classical perfect plasticity. By looking at the Cosserat couple modulus μ_c as a regularizing parameter instead of a material parameter we avoid the problematic issue of identifying this parameter in a physically reasonable way. Indeed in [**47**] and [**51, 46, 49**] it is argued that this parameter must be set to zero, if treated as a material parameter. This is at variance with practically all the previous literature in this field. It does not, however, mean that the Cosserat model has lost its physical significance; rather it calls for a geometrically exact treatment as e.g. done in [**55**].

Our contribution is organized as follows: first, we recall the linearized elasto-plastic Cosserat model introduced in [**50, 48**] and investigated mathematically in [**52, 54**]. We then reformulate the setting in the discrete finite element spaces together with a backward Euler discretization of the plastic flow rule in time. We show that all classical concepts from convex analysis apply to the classical stress part and note that the time incremental problem still has variational structure. The incremental problem is shown to have unique minimizers (in the case where Dirichlet data for both displacements and microrotations are assumed), and we prove finite element convergence for standard finite element approximations. Formulating the dual problem in terms of stresses and microrotations we prove that for $\mu_c \to 0$ classical perfect plasticity is approximated. This result for the dual, time-incremental problem complements a similar approximation result for the primal, time-continuous formulation obtained in [**53**].

This is complemented by a Newton-type algorithm for the computation of minimizers of the convex primal problem. Since the first variation of the primal functional is Lipschitz continuous, standard semi-smooth Newton methods can be applied, where the generalized derivative is given by the well-known consistent tangent in classical plasticity. In the last section, numerical experiments confirm our analytical results (even for more general boundary conditions) and show that our Cosserat plasticity model is more regular than Prandtl-Reuss plasticity and convergent to the classical model for $\mu_c \to 0$.

2. An infinitesimal elasto-plastic Cosserat model

In this section we recall the specific infinitesimal elasto-plastic Cosserat model which is analyzed in **[52]**, and we derive a discrete formulation. This section does not contain new results; it serves for the clear definition of the problem and for the introduction of the notation.

Data. Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be the reference configuration (an open and connected set in $\mathbb{R}^d$ with piecewise smooth Lipschitz boundary), and let $\Gamma_D \cup \Gamma_N = \partial\Omega$ be a decomposition of the boundary. We fix a time interval $[0, T]$.

The problem depends on the following data: a prescribed displacement vector

$$\mathbf{u}_D \colon \Gamma_D \times [0, T] \longrightarrow \mathbb{R}^d$$

for the essential boundary conditions on Γ_D, and a load functional

$$\ell(t, \mathbf{v}) = \int_\Omega \mathbf{b}(t) \cdot \mathbf{v}\, d\mathbf{x} + \int_{\Gamma_N} \mathbf{t}_N(t) \cdot \mathbf{v}\, d\mathbf{a}, \qquad t \in [0, T],$$

depending on body force densities

$$\mathbf{b} \colon \Omega \times [0, T] \longrightarrow \mathbb{R}^d$$

and traction force densities

$$\mathbf{t}_N \colon \Gamma_N \times [0, T] \longrightarrow \mathbb{R}^d\ .$$

We start with the initial state $\mathbf{u}_D(0) = \mathbf{0}$, $\mathbf{b}(0) = \mathbf{0}$, $\mathbf{t}_N(0) = \mathbf{0}$.

The material is described by a linear elastic response depending on the Lamé constants $\lambda, \mu > 0$. Furthermore, we consider materials which allow for independent infinitesimal microrotations $\bar{A} \in \mathfrak{so}(d)$, where $\mathfrak{so}(d) = \{\boldsymbol{\tau} \in \mathbb{R}^{d,d} \colon \boldsymbol{\tau}^T = -\boldsymbol{\tau}\}$ is the Lie-algebra of skew-symmetric matrices.

The symmetric matrices are denoted by $\mathrm{Sym}(d) = \{\boldsymbol{\tau} \in \mathbb{R}^{d,d} \colon \boldsymbol{\tau}^T = \boldsymbol{\tau}\}$. We have the orthogonality relation

$$\bar{A} : \boldsymbol{\tau} = 0, \qquad \bar{A} \in \mathfrak{so}(d), \quad \boldsymbol{\tau} \in \mathrm{Sym}(d)\ ,$$

with respect to the inner product $A : B = \sum_{i,j=1}^d A_{ij} B_{ij}$ for $A, B \in \mathbb{R}^{d,d}$. Moreover, we use the norm $|B| = \sqrt{B : B}$ and the decomposition $B = \mathrm{sym}(B) + \mathrm{skew}(B)$ with $\mathrm{sym}(B) = \frac{1}{2}(B + B^T)$ and $\mathrm{skew}(B) = \frac{1}{2}(B - B^T)$.

The coupling of the skew-symmetric part of the displacement gradient $D\mathbf{u}$ and the microrotation $\bar{A}$ is determined by a parameter $\mu_c \geq 0$, called the *Cosserat couple modulus*, and the internal length scale of the microrotations is described by a parameter $L_c > 0$ describing an internel length. On the Dirichlet boundary we prescribe

$$\bar{A}_D \colon \Gamma_D \times [0, T] \longrightarrow \mathfrak{so}(d)\ .$$

For the formulation of consistent boundary conditions we may assume that the prescribed displacement vector $\mathbf{u}_D$ is extended into Ω such that $\bar{A}_D(\mathbf{x}, t) = \mathrm{skew}(D\mathbf{u}_D(\mathbf{x}, t))$ is well-defined.

Finally, inelastic material behavior is modeled by a convex function

$$\phi \colon \mathrm{Sym}(d) \longrightarrow \mathbb{R}\,,$$

determining the convex set $\mathbf{K} = \{\boldsymbol{\tau} \in \mathrm{Sym}(d) \colon \phi(\boldsymbol{\tau}) \leq 0\}$ of admissible (symmetric elastic) stresses. We assume that ϕ is smooth for $\boldsymbol{\tau} \neq \mathbf{0}$, and we assume $\phi(\mathbf{0}) < 0$.

The basic example is the von Mises flow rule $\phi(\boldsymbol{\tau}) = |\,\mathrm{dev}(\boldsymbol{\tau})| - K_0$ for a given constant $K_0 > 0$. Here, $\mathrm{dev} \colon \mathrm{Sym}(d) \longrightarrow \mathrm{Sym}(d)$ is the projection orthogonal to the isotropic operator $\frac{1}{d}\mathbf{I} \otimes \mathbf{I} \colon \mathrm{Sym}(d) \longrightarrow \mathrm{Sym}(d)$ defined by $(\frac{1}{d}\mathbf{I} \otimes \mathbf{I}) : \boldsymbol{\tau} = \frac{1}{d}\mathrm{tr}(\boldsymbol{\tau})\mathbf{I}$, i. e., we have $\mathrm{dev} = \mathbb{1} - \frac{1}{d}\mathbf{I} \otimes \mathbf{I}$, i. e., $\mathrm{dev}\, B = B - \frac{1}{d}\mathrm{tr}(B)\mathbf{I}$, where $\mathbb{1}$ denotes the identity map in $\mathbb{R}^{d,d}$ and $\mathbf{I}$ is the identity tensor in $\mathbb{R}^{d,d}$.

The equations of the infinitesimal elasto-plastic Cosserat model. We want to determine displacements

$$\mathbf{u}\colon \bar{\Omega} \times [0,T] \longrightarrow \mathbb{R}^d \ ,$$

in general *non-symmetric* Cauchy-stresses

$$\boldsymbol{\sigma}\colon \Omega \times [0,T] \longrightarrow \mathbb{R}^{d,d} \ ,$$

(skew-symmetric infinitesimal) microrotations

$$\bar{A}\colon \Omega \times [0,T] \longrightarrow \mathfrak{so}(d) \ ,$$

(symmetric infinitesimal) plastic strains (no plastic spin: $\operatorname{skew}(\boldsymbol{\varepsilon}_p) = 0$)

$$\boldsymbol{\varepsilon}_p\colon \Omega \times [0,T] \longrightarrow \operatorname{Sym}(d) \ ,$$

(with initial state $\boldsymbol{\varepsilon}_p(0) = \mathbf{0}$), and a plastic multiplier

$$\Lambda\colon \Omega \times [0,T] \longrightarrow \mathbb{R}$$

satisfying the essential boundary conditions

$$\begin{aligned} \mathbf{u}(\mathbf{x},t) &= \mathbf{u}_D(\mathbf{x},t), \qquad (\mathbf{x},t) \in \Gamma_D \times [0,T] \ , \\ \bar{A}(\mathbf{x},t) &= \bar{A}_D(\mathbf{x},t), \qquad (\mathbf{x},t) \in \Gamma_D \times [0,T] \ , \end{aligned}$$

the constitutive relation

$$\begin{aligned} \boldsymbol{\sigma}(\mathbf{x},t) &= 2\mu\big(\operatorname{sym}(D\mathbf{u}(\mathbf{x},t)) - \boldsymbol{\varepsilon}_p(\mathbf{x},t)\big) + \lambda \operatorname{div}(\mathbf{u})(\mathbf{x},t)\mathbf{I} \\ &\qquad + 2\mu_c\big(\operatorname{skew}(D\mathbf{u}(\mathbf{x},t)) - \bar{A}(\mathbf{x},t)\big), \qquad (\mathbf{x},t) \in \Omega \times [0,T] \ , \end{aligned}$$

the equilibrium equations

$$\begin{aligned} -\operatorname{div}\boldsymbol{\sigma}(\mathbf{x},t) &= \mathbf{b}(\mathbf{x},t), \qquad (\mathbf{x},t) \in \Omega \times [0,T] \ , \\ \boldsymbol{\sigma}(\mathbf{x},t)\mathbf{n}(\mathbf{x}) &= \mathbf{t}_N(\mathbf{x},t), \qquad (\mathbf{x},t) \in \Gamma_N \times [0,T] \ , \\ -\mu L_c^2 \Delta \bar{A}(\mathbf{x},t) &= \mu_c\big(\operatorname{skew}(D\mathbf{u}(\mathbf{x},t)) - \bar{A}(\mathbf{x},t)\big), \qquad (\mathbf{x},t) \in \Omega \times [0,T] \ , \\ D\bar{A}(\mathbf{x},t) \cdot \mathbf{n}(\mathbf{x}) &= \mathbf{0}, \qquad (\mathbf{x},t) \in \Gamma_N \times [0,T] \end{aligned}$$

(where $\mathbf{n}(\mathbf{x})$ denotes the outer unit normal vector), the flow rule

$$\frac{d}{dt}\boldsymbol{\varepsilon}_p(\mathbf{x},t) = \Lambda(\mathbf{x},t) D\phi\big(T_E(\mathbf{x},t)\big) \ , \qquad (\mathbf{x},t) \in \Omega \times [0,T] \ , \tag{1}$$

depending (only) on (the symmetric elastic Eshelby stress tensor T_E)

$$T_E(\mathbf{x},t) = 2\mu\big(\operatorname{sym}(D\mathbf{u}(\mathbf{x},t)) - \boldsymbol{\varepsilon}_p(\mathbf{x},t)\big), \qquad (\mathbf{x},t) \in \Omega \times [0,T] \ ,$$

and the complementary conditions (Karush-Kuhn-Tucker)

$$\Lambda(\mathbf{x},t)\phi\big(T_E(\mathbf{x},t)\big) = 0, \ \Lambda(\mathbf{x},t) \ge 0, \ \phi\big(T_E(\mathbf{x},t)\big) \le 0, \ (\mathbf{x},t) \in \Omega \times [0,T] \ . \tag{2}$$

Here and in the following, for $\boldsymbol{\tau} \in \operatorname{Sym}(d)$ the derivative $D\phi(\boldsymbol{\tau}) \in \operatorname{Sym}(d)$ is represented in $\operatorname{Sym}(d)$ such that for the Gâteaux derivative $D\phi(\boldsymbol{\tau})[\boldsymbol{\eta}]$ holds

$$D\phi(\boldsymbol{\tau}) : \boldsymbol{\eta} = D\phi(\boldsymbol{\tau})[\boldsymbol{\eta}] = \lim_{h \longrightarrow 0} \frac{1}{h}\big(\phi(\boldsymbol{\tau} + h\boldsymbol{\eta}) - \phi(\boldsymbol{\tau})\big), \qquad \boldsymbol{\eta} \in \operatorname{Sym}(d) \ .$$

REMARK 1. *For given material history $\boldsymbol{\varepsilon}_p(t)$ at fixed time t, the displacement and the microrotation are determined by minimizing the total energy*

$$I(\mathbf{u}, \bar{A}, \boldsymbol{\varepsilon}_p) = \mathcal{E}(D\mathbf{u}, \bar{A}, \boldsymbol{\varepsilon}_p) - \ell(t, \mathbf{u}) \ ,$$

where the corresponding elastic free energy is given by

$$\begin{aligned} \mathcal{E}(D\mathbf{u}, \bar{A}, \boldsymbol{\varepsilon}_p) &= \mu \int_\Omega \big|\operatorname{sym}(D\mathbf{u}) - \boldsymbol{\varepsilon}_p\big|^2 \, d\mathbf{x} + \frac{\lambda}{2}\int_\Omega \operatorname{tr}(D\mathbf{u})^2 \, d\mathbf{x} \\ &\qquad + \mu_c \int_\Omega |\operatorname{skew}(D\mathbf{u}) - \bar{A}|^2 \, d\mathbf{x} + \mu L_c^2 \int_\Omega |D\bar{A}|^2 \, d\mathbf{x} \ . \end{aligned} \tag{3}$$

3. The discrete elasto-plastic Cosserat model

Discretization in space. Let h be a mesh size parameter, let $\mathbf{V}_h \subset C^{0,1}(\Omega, \mathbb{R}^d)$ be a finite element space, and set

$$\mathbf{V}_h(\mathbf{u}_D) = \{\mathbf{v} \in \mathbf{V}_h \colon \mathbf{v}(\mathbf{x}) = \mathbf{u}_D(\mathbf{x}) \text{ for } \mathbf{x} \in D_h\},$$

where $D_h \subset \Gamma_D$ is the set of all nodal points on Γ_D.

Analogously, let $W_h \subset C^{0,1}(\Omega, \mathfrak{so}(d))$ be another finite element space, and let

$$W_h(A_D) = \{B \in W_h \colon B(\mathbf{x}) = A_D(\mathbf{x}) \text{ for } \mathbf{x} \in D_h'\}$$

where $D_h' \subset \Gamma_D$ is the set of all nodal points on Γ_D of W_h.

Let $\Xi_h \subset \Omega$ be quadrature points and let $\omega_{\boldsymbol{\xi}}$ be corresponding quadrature weights such that

$$\int_\Omega \mathbf{v} \cdot \mathbf{w} \, d\mathbf{x} = \sum_{\boldsymbol{\xi} \in \Xi_h} \omega_{\boldsymbol{\xi}} \, \mathbf{v}(\boldsymbol{\xi}) \cdot \mathbf{w}(\boldsymbol{\xi}) \,, \qquad \mathbf{v}, \mathbf{w} \in \mathbf{V}_h \ .$$

We set $\boldsymbol{\Lambda} = \{\Lambda \colon \Xi_h \longrightarrow \mathbb{R}\}$, $\boldsymbol{\Sigma}_h = \{\boldsymbol{\tau} \colon \Xi_h \longrightarrow \mathbb{R}^{d,d}\}$ and $\mathbf{E}_h^p = \{\boldsymbol{\tau} \colon \Xi_h \longrightarrow \mathfrak{sl}(d) \cap \mathrm{Sym}(d)\}$, where $\mathfrak{sl}(d) = \{\boldsymbol{\tau} \in \mathbb{R}^{d,d} \colon \operatorname{tr}(\boldsymbol{\tau}) = 0\}$ is the Lie algebra of trace-free matrices.

In our notation the integral is used also for the finite sums

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon} \, d\mathbf{x} \ := \sum_{\boldsymbol{\xi} \in \Xi_h} \omega_{\boldsymbol{\xi}} \, \boldsymbol{\sigma}(\boldsymbol{\xi}) : \boldsymbol{\varepsilon}(\boldsymbol{\xi}), \qquad \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \boldsymbol{\Sigma}_h \ .$$

The semi-discrete equations of the elasto-plastic Cosserat model. Determine

displacements	$\mathbf{u} \colon [0,T] \longrightarrow \mathbf{V}_h$ with $\mathbf{u}(t) \in \mathbf{V}_h(\mathbf{u}_D(t))$ for $t \in [0,T]$,
microrotations	$\bar{A} \colon [0,T] \longrightarrow W_h$ with $\bar{A}(t) \in W_h\big(\bar{A}_D(t)\big)$ for $t \in [0,T]$,
stresses	$\boldsymbol{\sigma} \colon [0,T] \longrightarrow \boldsymbol{\Sigma}_h$,
plastic strains	$\boldsymbol{\varepsilon}_p \colon [0,T] \longrightarrow \mathbf{E}_h^p$,
plastic multiplier	$\Lambda \colon [0,T] \longrightarrow \boldsymbol{\Lambda}$,

satisfying the constitutive relation

$$\boldsymbol{\sigma}(\boldsymbol{\xi}, t) = 2\mu\big(\operatorname{sym}(D\mathbf{u}(\boldsymbol{\xi}, t)) - \boldsymbol{\varepsilon}_p(\boldsymbol{\xi}, t)\big) + \lambda \operatorname{div}(\mathbf{u})(\boldsymbol{\xi}, t)\mathbf{I} + 2\mu_c(\operatorname{skew}(D\mathbf{u})(\boldsymbol{\xi}, t) - \bar{A}(\boldsymbol{\xi}, t))$$

for $(\boldsymbol{\xi}, t) \in \Xi_h \times [0,T]$ (using $\boldsymbol{\sigma}(\boldsymbol{\xi}, t) := \boldsymbol{\sigma}(t)(\boldsymbol{\xi})$ for $\boldsymbol{\sigma}(t) \in \boldsymbol{\Sigma}_h$), the equilibrium equations

$$\int_\Omega \boldsymbol{\sigma}(t) : D\mathbf{v} \, d\mathbf{x} = \ell(t, \mathbf{v}), \qquad t \in [0,T], \quad \mathbf{v} \in \mathbf{V}_h(\mathbf{0}) \ ,$$

$$\mu L_c^2 \int_\Omega D\bar{A}(t) \cdot D\bar{B} \, d\mathbf{x} = \mu_c \int_\Omega \big(\operatorname{skew}(D\mathbf{u}(t)) - \bar{A}(t)\big) : \bar{B} \, d\mathbf{x}, \quad t \in [0,T], \ \bar{B} \in W_h(\mathbf{0})$$

(where $D\bar{A} \cdot D\bar{B} = \sum_{ijk} \partial_i A_{jk} \partial_i B_{jk}$), the flow rule

$$\frac{d}{dt} \boldsymbol{\varepsilon}_p(\boldsymbol{\xi}, t) = \Lambda(\boldsymbol{\xi}, t) D\phi\big(T_E(\boldsymbol{\xi}, t)\big) \ , \qquad (\boldsymbol{\xi}, t) \in \Xi_h \times [0,T] \ ,$$

depending on

$$T_E(\boldsymbol{\xi}, t) = 2\mu\big(\operatorname{sym}(D\mathbf{u}(\boldsymbol{\xi}, t)) - \boldsymbol{\varepsilon}_p(\boldsymbol{\xi}, t)\big) \qquad (\boldsymbol{\xi}, t) \in \Xi_h \times [0,T] \ ,$$

and the complementary conditions (Karush-Kuhn-Tucker)

$$\Lambda(\boldsymbol{\xi}, t)\phi\big(T_E(\boldsymbol{\xi}, t)\big) = 0, \quad \Lambda(\boldsymbol{\xi}, t) \geq 0, \quad \phi\big(T_E(\boldsymbol{\xi}, t)\big) \leq 0, \qquad (\boldsymbol{\xi}, t) \in \Omega \times [0,T] \ .$$

Discretization in time. The model of incremental infinitesimal Cosserat plasticity is obtained by a decomposition

$$0 = t_0 < t_1 < \cdots < t_N = T$$

of the time interval and the backward Euler scheme: for $n = 1, 2, 3, ...$, the next increment depends on the material history described by $\boldsymbol{\varepsilon}_p^{n-1}$ (where $\boldsymbol{\varepsilon}_p^0 = \mathbf{0}$ at $t_0 = 0$ is given), the new load $\ell^n[\mathbf{v}] = \ell(t_n, \mathbf{v})$ and the new Dirichlet boundary values $\mathbf{u}_D^n = \mathbf{u}_D(t_n)$ and $\bar{A}_D^n = \bar{A}_D(t_n)$. We compute the displacement $\mathbf{u}^n \in \mathbf{V}_h(\mathbf{u}_D^n)$ satisfying the essential boundary conditions, the stress $\boldsymbol{\sigma}^n \in \boldsymbol{\Sigma}_h$, the microrotation $\bar{A}^n \in W_h(\bar{A}_D^n)$, the plastic strain $\boldsymbol{\varepsilon}_p^n \in \mathbf{E}_h^p$, and the plastic multiplier $\Lambda^n \in \boldsymbol{\Lambda}$ satisfying the constitutive relation

$$\begin{aligned}\boldsymbol{\sigma}^n(\boldsymbol{\xi}) \;&=\; 2\mu\big(\operatorname{sym}(D\mathbf{u}^n(\boldsymbol{\xi})) - \boldsymbol{\varepsilon}_p^n(\boldsymbol{\xi})\big) + \lambda \operatorname{div}(\mathbf{u}^n)(\boldsymbol{\xi})\mathbf{I} \\ &\qquad + 2\mu_c\big(\operatorname{skew}(D\mathbf{u}^n(\boldsymbol{\xi})) - \bar{A}^n(\boldsymbol{\xi})\big), \qquad \boldsymbol{\xi} \in \Xi_h\,, \end{aligned} \tag{4}$$

the equilibrium equations

$$\int_\Omega \boldsymbol{\sigma}^n : D\mathbf{v}\, d\mathbf{x} \;=\; \ell^n[\mathbf{v}], \qquad \mathbf{v} \in \mathbf{V}_h(\mathbf{0}) \tag{5a}$$

$$\mu L_c^2 \int_\Omega D\bar{A}^n \cdot D\bar{B}\, d\mathbf{x} \;=\; \mu_c \int_\Omega \big(\operatorname{skew}(D\mathbf{u}^n) - \bar{A}^n\big) : \bar{B}\, d\mathbf{x}, \quad \bar{B} \in W_h(\mathbf{0})\,, \tag{5b}$$

the flow rule

$$\frac{1}{t_n - t_{n-1}} \Big(\boldsymbol{\varepsilon}_p^n(\boldsymbol{\xi}) - \boldsymbol{\varepsilon}_p^{n-1}(\boldsymbol{\xi})\Big) = \Lambda^n(\boldsymbol{\xi}) D\phi\big(T_E^n(\boldsymbol{\xi})\big)\,, \qquad \boldsymbol{\xi} \in \Xi_h\,,$$

depending on

$$T_E^n(\boldsymbol{\xi}) = 2\mu\big(\operatorname{sym}(D\mathbf{u}^n(\boldsymbol{\xi})) - \boldsymbol{\varepsilon}_p^n(\boldsymbol{\xi})\big), \qquad \boldsymbol{\xi} \in \Xi_h\,, \tag{6}$$

and the complementary conditions (Karush-Kuhn-Tucker)

$$\Lambda^n(\boldsymbol{\xi})\phi\big(T_E^n(\boldsymbol{\xi})\big) = 0, \quad \Lambda^n(\boldsymbol{\xi}) \geq 0, \quad \phi\big(T_E^n(\boldsymbol{\xi})\big) \leq 0, \qquad \boldsymbol{\xi} \in \Xi_h\,.$$

Since the problem is rate-independent, rescaling of the time parameter does not affect the model. Thus, we define $\gamma^n = 2\mu(t_n - t_{n-1})\Lambda^n \in \boldsymbol{\Lambda}$, i. e., the flow rule has the form

$$\boldsymbol{\varepsilon}_p^n(\boldsymbol{\xi}) = \boldsymbol{\varepsilon}_p^{n-1}(\boldsymbol{\xi}) + \frac{\gamma^n(\boldsymbol{\xi})}{2\mu} D\phi\big(T_E^n(\boldsymbol{\xi})\big)\,, \qquad \boldsymbol{\xi} \in \Xi_h\,. \tag{7}$$

Together with (4), (5) and (6), we can state the fully discrete elasto-plastic Cosserat problem: for given $\boldsymbol{\varepsilon}_p^{n-1} \in \mathbf{E}_h^p$ find $\boldsymbol{\sigma}^n, T_E^n \in \boldsymbol{\Sigma}_h$, $\mathbf{u}^n \in \mathbf{V}_h(\mathbf{u}_D^n)$, $\bar{A}^n \in W_h(\bar{A}_D^n)$, and $\gamma^n \in \boldsymbol{\Lambda}$ such that

$$T_E^n(\boldsymbol{\xi}) = 2\mu\Big(\operatorname{sym}(D\mathbf{u}^n(\boldsymbol{\xi})) - \boldsymbol{\varepsilon}_p^{n-1}(\boldsymbol{\xi})\Big) - \gamma^n(\boldsymbol{\xi}) D\phi\big(T_E^n(\boldsymbol{\xi})\big) \qquad \boldsymbol{\xi} \in \Xi_h\,, \tag{8a}$$

$$\phi\big(T_E^n(\boldsymbol{\xi})\big) \leq 0, \quad \gamma^n(\boldsymbol{\xi})\phi\big(T_E^n(\boldsymbol{\xi})\big) = 0, \quad \gamma^n(\boldsymbol{\xi}) \geq 0, \qquad \boldsymbol{\xi} \in \Xi_h\,, \tag{8b}$$

$$\boldsymbol{\sigma}^n(\boldsymbol{\xi}) = T_E^n(\boldsymbol{\xi}) + \lambda \operatorname{div}(\mathbf{u}^n)(\boldsymbol{\xi})\mathbf{I} + 2\mu_c\big(\operatorname{skew}(D\mathbf{u}^n(\boldsymbol{\xi})) - \bar{A}^n(\boldsymbol{\xi})\big), \quad \boldsymbol{\xi} \in \Xi_h, \tag{8c}$$

$$\int_\Omega \boldsymbol{\sigma}^n : D\mathbf{v}\, d\mathbf{x} = \ell^n[\mathbf{v}], \qquad \mathbf{v} \in \mathbf{V}_h(\mathbf{0}) \tag{8d}$$

$$\mu L_c^2 \int_\Omega D\bar{A}^n \cdot D\bar{B}\, d\mathbf{x} = \mu_c \int_\Omega \big(\operatorname{skew}(D\mathbf{u}^n) - \bar{A}^n\big) : \bar{B}\, d\mathbf{x}, \qquad \bar{B} \in W_h(\mathbf{0})\,. \tag{8e}$$

Then, for the next time step $\boldsymbol{\varepsilon}_p^n$ is determined by the discrete flow rule (7).

4. The closest point projection

The algorithmic treatment of the incremental plasticity problem relies on equivalent characterizations obtained by the closest point projection of arbitrary stresses to the admissible stresses. Since the set of admissible stresses is convex, the computation of the projection onto this set is a standard problem in convex optimization.

Projection onto the set of admissible stresses. Let $P_{\mathbf{K}}\colon \mathrm{Sym}(d) \longrightarrow \mathbf{K}$ be the orthogonal projection onto the convex set of admissible stresses $\mathbf{K} = \{\boldsymbol{\tau} \in \mathrm{Sym}(d)\colon \phi(\boldsymbol{\tau}) \le 0\}$ with respect to the norm $|\boldsymbol{\tau}| = \sqrt{\boldsymbol{\tau} : \boldsymbol{\tau}}$. Note that $\mathbf{0} \in \mathbf{K}$ (since $\phi(\mathbf{0}) < 0$). We assume that ϕ is smooth for $\boldsymbol{\tau} \neq 0$.

LEMMA 2. *For given $\boldsymbol{\theta} \in \mathrm{Sym}(d)$ the projection $T = P_{\mathbf{K}}(\boldsymbol{\theta}) \in \mathbf{K}$ is uniquely determined by the solution $(T,\gamma) \in \mathrm{Sym}(d) \times \mathbb{R}$ of the KKT-system*

$$0 = T - \boldsymbol{\theta} + \gamma D\phi(T) \ , \tag{9a}$$

$$0 \ge \phi(T), \quad \gamma\phi(T) = 0, \quad \gamma \ge 0. \tag{9b}$$

PROOF. The constraint minimization problem

$$T \in \mathrm{Sym}(d)\colon \qquad \frac{1}{2}\,|T - \boldsymbol{\theta}|^2 = \min \qquad \text{subject to } \phi(T) \le 0$$

satisfies the Slater condition (since $\phi(\mathbf{0}) < 0$). Thus, the minimizer is characterized by a saddle point $(T,\gamma) \in \mathrm{Sym}(d) \times \mathbb{R}$ of the Lagrange functional

$$L(T,\gamma) = \frac{1}{2}\,|T - \boldsymbol{\theta}|^2 + \gamma\,\phi(T) \ ,$$

and the corresponding KKT-system (9). □

The convex potential. The corresponding convex potentials are denoted by

$$\varphi_{\mathbf{K}}(\boldsymbol{\theta}) = \frac{1}{2}\,|\boldsymbol{\theta} - P_{\mathbf{K}}(\boldsymbol{\theta})|^2 \ , \qquad \psi_{\mathbf{K}}(\boldsymbol{\theta}) = \frac{1}{2}\,|\boldsymbol{\theta}|^2 - \frac{1}{2}\,|\boldsymbol{\theta} - P_{\mathbf{K}}(\boldsymbol{\theta})|^2 \ ,$$

for $\boldsymbol{\theta} \in \mathrm{Sym}(d)$. Note that we have

$$D\varphi_{\mathbf{K}}(\boldsymbol{\theta})[\boldsymbol{\eta}] = \big(\boldsymbol{\theta} - P_{\mathbf{K}}(\boldsymbol{\theta})\big) : \boldsymbol{\eta} \ , \qquad \boldsymbol{\theta}, \boldsymbol{\eta} \in \mathrm{Sym}(d) \ . \tag{10}$$

LEMMA 3. *The functional $\psi_{\mathbf{K}}(\cdot)$ is convex, non-negative, and we have*

$$D\psi_{\mathbf{K}}(\boldsymbol{\theta})[\boldsymbol{\eta}] = P_{\mathbf{K}}(\boldsymbol{\theta}) : \boldsymbol{\eta} \ , \qquad \boldsymbol{\theta},\ \boldsymbol{\eta} \in \mathrm{Sym}(d) \ . \tag{11}$$

PROOF. The orthogonal projection $P_{\mathbf{K}}$ is uniquely characterized by

$$\big(\boldsymbol{\theta} - P_{\mathbf{K}}(\boldsymbol{\theta})\big) : \big(\boldsymbol{\eta} - P_{\mathbf{K}}(\boldsymbol{\theta})\big) \le 0, \qquad \boldsymbol{\theta} \in \mathrm{Sym}(d),\ \boldsymbol{\eta} \in \mathbf{K} \ . \tag{12}$$

Inserting (10) gives $D\psi_{\mathbf{K}}(\boldsymbol{\theta})[\boldsymbol{\eta}] = \boldsymbol{\theta} : \boldsymbol{\eta} - D\varphi_{\mathbf{K}}(\boldsymbol{\theta})[\boldsymbol{\eta}] = P_{\mathbf{K}}(\boldsymbol{\theta}) : \boldsymbol{\eta}$, and we obtain from (12) for $\boldsymbol{\theta}, \boldsymbol{\eta} \in \mathrm{Sym}(d)$

$$\begin{aligned}
&\psi_{\mathbf{K}}(\boldsymbol{\theta}) - \psi_{\mathbf{K}}(\boldsymbol{\eta}) - D\psi_{\mathbf{K}}(\boldsymbol{\eta})[\boldsymbol{\theta} - \boldsymbol{\eta}] \\
&\quad = P_{\mathbf{K}}(\boldsymbol{\theta}) : \boldsymbol{\theta} - \frac{1}{2}P_{\mathbf{K}}(\boldsymbol{\theta}) : P_{\mathbf{K}}(\boldsymbol{\theta}) - P_{\mathbf{K}}(\boldsymbol{\eta}) : \boldsymbol{\eta} + \frac{1}{2}P_{\mathbf{K}}(\boldsymbol{\eta}) : P_{\mathbf{K}}(\boldsymbol{\eta}) - P_{\mathbf{K}}(\boldsymbol{\eta}) : (\boldsymbol{\theta} - \boldsymbol{\eta}) \\
&\quad = \big(P_{\mathbf{K}}(\boldsymbol{\theta}) - P_{\mathbf{K}}(\boldsymbol{\eta})\big) : \big(\boldsymbol{\theta} - P_{\mathbf{K}}(\boldsymbol{\theta})\big) + \frac{1}{2}|P_{\mathbf{K}}(\boldsymbol{\theta}) - P_{\mathbf{K}}(\boldsymbol{\eta})|^2 \ge \frac{1}{2}|P_{\mathbf{K}}(\boldsymbol{\theta}) - P_{\mathbf{K}}(\boldsymbol{\eta})|^2.
\end{aligned}$$

Thus, $D\psi_{\mathbf{K}}(\boldsymbol{\eta})[\boldsymbol{\theta} - \boldsymbol{\eta}] \le \psi_{\mathbf{K}}(\boldsymbol{\theta}) - \psi_{\mathbf{K}}(\boldsymbol{\eta})$, i. e., $D\psi_{\mathbf{K}}$ is monotone and therefore $\psi_{\mathbf{K}}$ is convex. Finally, since $\mathbf{0} \in \mathbf{K}$ we have

$$\psi_{\mathbf{K}}(\boldsymbol{\theta}) = \frac{1}{2}|\boldsymbol{\theta}|^2 - \frac{1}{2}|\boldsymbol{\theta} - P_{\mathbf{K}}(\boldsymbol{\theta})|^2 \ge \frac{1}{2}|\boldsymbol{\theta}|^2 - \frac{1}{2}|\boldsymbol{\theta} - \mathbf{0}|^2 = 0 \ .$$

□

Example. We consider the evaluation of the projection for the classical von Mises flow rule $\phi(T) = |\operatorname{dev}(T)| - K_0$ for given (yield stress) $K_0 > 0$. For $\boldsymbol{\theta} \in \mathbf{K}$ we have $T = \boldsymbol{\theta}$ and $\gamma = 0$. Otherwise, the KKT-system

$$\begin{aligned} 0 &= T - \boldsymbol{\theta} + \gamma \frac{\operatorname{dev}(T)}{|\operatorname{dev}(T)|}, \\ 0 &= |\operatorname{dev}(T)| - K_0 \end{aligned}$$

has a unique solution, and using $\operatorname{dev}(\boldsymbol{\theta}) = \Big(1 + \dfrac{\gamma}{|\operatorname{dev}(T)|}\Big)\operatorname{dev}(T)$ we obtain

$$\begin{aligned} P_{\mathbf{K}}(\boldsymbol{\theta}) &= \boldsymbol{\theta} - \max\big\{0, |\operatorname{dev}(\boldsymbol{\theta})| - K_0\big\} \frac{\operatorname{dev}(\boldsymbol{\theta})}{|\operatorname{dev}(\boldsymbol{\theta})|}, \\ \gamma &= \max\big\{0, |\operatorname{dev}(\boldsymbol{\theta})| - K_0\big\}, \\ \psi_{\mathbf{K}}(\boldsymbol{\theta}) &= \begin{cases} \frac{1}{2}|\boldsymbol{\theta}|^2 & |\operatorname{dev}(\boldsymbol{\theta})| \le K_0, \\ \frac{1}{2}\Big(\frac{1}{d}\operatorname{tr}(\boldsymbol{\theta})^2 + 2K_0|\operatorname{dev}(\boldsymbol{\theta})| - K_0^2\Big) & |\operatorname{dev}(\boldsymbol{\theta})| > K_0, \end{cases} \end{aligned}$$

see **[57, 23, 24]**. Defining $m(s) = \max\{0, s\}$ and using

$$\partial m(s) = \begin{cases} 1 & s > 0 \\ [0,1] & s = 0 \\ 0 & s < 0 \end{cases}$$

we obtain for the multi-valued derivative of the projection

$$\begin{aligned} \partial P_{\mathbf{K}}(\boldsymbol{\theta})[\boldsymbol{\tau}] &= \boldsymbol{\tau} - \partial m(|\operatorname{dev}(\boldsymbol{\theta})| - K_0)\frac{\operatorname{dev}(\boldsymbol{\theta}):\operatorname{dev}(\boldsymbol{\tau})}{|\operatorname{dev}(\boldsymbol{\theta})|}\frac{\operatorname{dev}(\boldsymbol{\theta})}{|\operatorname{dev}(\boldsymbol{\theta})|} \\ &\quad - m(|\operatorname{dev}(\boldsymbol{\theta})| - K_0)\Big(\frac{\operatorname{dev}(\boldsymbol{\tau})}{|\operatorname{dev}(\boldsymbol{\theta})|} - \frac{\operatorname{dev}(\boldsymbol{\theta}):\operatorname{dev}(\boldsymbol{\tau})}{|\operatorname{dev}(\boldsymbol{\theta})|}\frac{\operatorname{dev}(\boldsymbol{\theta})}{|\operatorname{dev}(\boldsymbol{\theta})|^2}\Big), \end{aligned}$$

i. e.,

$$\begin{aligned} \partial P_{\mathbf{K}}(\boldsymbol{\theta}) &= \mathbb{1} - \partial m(|\operatorname{dev}(\boldsymbol{\theta})| - K_0)\frac{\operatorname{dev}(\boldsymbol{\theta})}{|\operatorname{dev}(\boldsymbol{\theta})|} \otimes \frac{\operatorname{dev}(\boldsymbol{\theta})}{|\operatorname{dev}(\boldsymbol{\theta})|} \\ &\quad - \frac{m(|\operatorname{dev}(\boldsymbol{\theta})| - K_0)}{|\operatorname{dev}(\boldsymbol{\theta})|}\Big(\big(\mathbb{1} - \tfrac{1}{d}\mathbf{I}\otimes\mathbf{I}\big) - \frac{\operatorname{dev}(\boldsymbol{\theta})}{|\operatorname{dev}(\boldsymbol{\theta})|} \otimes \frac{\operatorname{dev}(\boldsymbol{\theta})}{|\operatorname{dev}(\boldsymbol{\theta})|}\Big). \end{aligned} \tag{13}$$

For the special choice $m'(s) \in \partial m(s)$ defined by

$$m'(s) = \begin{cases} 1 & s > 0 \\ 0 & s \le 0 \end{cases}$$

we obtain the following realization $\mathbb{C}(\boldsymbol{\theta}) \in \partial P_{\mathbf{K}}(\boldsymbol{\theta})$ for the consistent tangent defined by

$$\mathbb{C}(\boldsymbol{\theta}) = \begin{cases} \mathbb{1} & |\operatorname{dev}(\boldsymbol{\theta})| \le K_0, \\ \frac{1}{d}\mathbf{I}\otimes\mathbf{I} + \frac{K_0}{|\operatorname{dev}(\boldsymbol{\theta})|}\Big(\big(\mathbb{1} - \frac{1}{d}\mathbf{I}\otimes\mathbf{I}\big) - \frac{\operatorname{dev}(\boldsymbol{\theta})}{|\operatorname{dev}(\boldsymbol{\theta})|}\otimes\frac{\operatorname{dev}(\boldsymbol{\theta})}{|\operatorname{dev}(\boldsymbol{\theta})|}\Big) & |\operatorname{dev}(\boldsymbol{\theta})| > K_0. \end{cases}$$

Since $\mathbb{C}(\boldsymbol{\theta}) \in \partial^2\psi_{\mathbf{K}}(\boldsymbol{\theta})$ is the second variation of the convex function $\psi_{\mathbf{K}}(\cdot)$, the consistent tangent $\mathbb{C}(\boldsymbol{\theta})$ is positive semi-definite. Moreover, we have $\mathbb{C}(\boldsymbol{\theta}) : \operatorname{dev}(\boldsymbol{\theta}) = \mathbf{0}$, i. e., $\mathbb{C}(\boldsymbol{\theta})$ is not positive definite. Furthermore, $\psi_{\mathbf{K}}(\cdot)$ is not strictly convex and it is of asymptotic linear growth **[57]**.

Note that $m(\cdot)$ and therefore $P_{\mathbf{K}}$ is semi-smooth satisfying

$$\sup_{A\in\partial P_{\mathbf{K}}(\boldsymbol{\theta}+\delta\boldsymbol{\tau})} \big|P_{\mathbf{K}}(\boldsymbol{\theta}+\delta\boldsymbol{\tau}) - P_{\mathbf{K}}(\boldsymbol{\theta}) - \delta A[\boldsymbol{\tau}]\big| = o(\delta).$$

As a consequence, the nonlinear problem which will be studied in the next section is semi-smooth as well. Thus, the convergence analysis for generalized Newton methods **[36]** can be applied.

5. Variational formulation of the discrete elasto-plastic Cosserat model

Depending on $\mathbf{u}^n$ and $\boldsymbol{\varepsilon}_p^{n-1}$ we define the trial stress

$$\boldsymbol{\theta}^n(\boldsymbol{\xi}) = 2\mu\big(\operatorname{sym}(D\mathbf{u}^n(\boldsymbol{\xi})) - \boldsymbol{\varepsilon}_p^{n-1}(\boldsymbol{\xi})\big) \qquad \boldsymbol{\xi} \in \Xi_h \ , \tag{14}$$

i. e.,

$$T_E^n(\boldsymbol{\xi}) = \boldsymbol{\theta}^n(\boldsymbol{\xi}) - \gamma^n(\boldsymbol{\xi}) D\phi\big(T_E^n(\boldsymbol{\xi})\big) \qquad \boldsymbol{\xi} \in \Xi_h \ .$$

LEMMA 4. *The system* (8) *for the discrete elasto-plastic Cosserat model is equivalent to the following nonlinear variational problem:*
for given $\boldsymbol{\varepsilon}_p^{n-1}$ *find* $(\mathbf{u}^n, \bar{A}^n) \in \mathbf{V}_h(\mathbf{u}_D^n) \times W_h(\bar{A}_D^n)$ *such that*

$$\int_\Omega P_{\mathbf{K}}\big(2\mu(\operatorname{sym}(D\mathbf{u}^n) - \boldsymbol{\varepsilon}_p^{n-1})\big) : D\mathbf{v}\, d\mathbf{x} + \lambda \int_\Omega \operatorname{div}(\mathbf{u}^n) \operatorname{div}(\mathbf{v})\, d\mathbf{x}$$
$$+ 2\mu_c \int_\Omega \big(\operatorname{skew}(D\mathbf{u}^n) - \bar{A}^n\big) : D\mathbf{v}\, d\mathbf{x} = \ell^n[\mathbf{v}], \qquad \mathbf{v} \in \mathbf{V}_h(\mathbf{0}) \ , \tag{15a}$$

$$\mu L_c^2 \int_\Omega D\bar{A}^n \cdot D\bar{B}\, d\mathbf{x} = \mu_c \int_\Omega \big(\operatorname{skew}(D\mathbf{u}^n) - \bar{A}^n\big) : \bar{B}\, d\mathbf{x}, \qquad \bar{B} \in W_h(\mathbf{0}) \ . \tag{15b}$$

PROOF. Inserting Lemma 2 we obtain directly that (8a), (8b) is equivalent to

$$T_E^n(\boldsymbol{\xi}) = P_{\mathbf{K}}\big(\boldsymbol{\theta}^n(\boldsymbol{\xi})\big), \qquad \boldsymbol{\xi} \in \Xi_h \ .$$

Then, (8c) gives

$$\boldsymbol{\sigma}^n(\boldsymbol{\xi}) = P_{\mathbf{K}}\big(\boldsymbol{\theta}^n(\boldsymbol{\xi})\big) + \lambda \operatorname{div}(\mathbf{u}^n)(\boldsymbol{\xi})\mathbf{I} + 2\mu_c\big(\operatorname{skew}(D\mathbf{u}^n(\boldsymbol{\xi})) - \bar{A}^n(\boldsymbol{\xi})\big), \quad \boldsymbol{\xi} \in \Xi_h,$$

and (15) follows from (8d) and (8e). □

It is important to observe that the weak form of the incremental Cosserat problem still has a variational structure in the following sense.

LEMMA 5. *Any minimizer* $(\mathbf{u}^n, \bar{A}^n) \in \mathbf{V}_h(\mathbf{u}_D^n) \times W_h(\bar{A}_D^n)$ *of the functional*

$$I_{\text{incr}}^n(\mathbf{u}, \bar{A}) \quad = \quad \mathcal{E}_{\text{incr}}(D\mathbf{u}, \bar{A}, \boldsymbol{\varepsilon}_p^{n-1}) - \ell^n[\mathbf{u}] \tag{16}$$

solves the nonlinear variational problem (15)*. Here* $\mathcal{E}_{\text{incr}}$ *denotes the free energy of the incremental problem defined by*

$$\mathcal{E}_{\text{incr}}(D\mathbf{u}, \bar{A}, \boldsymbol{\varepsilon}_p) \quad = \quad \frac{1}{2\mu} \int_\Omega \psi_{\mathbf{K}}\big(2\mu(\operatorname{sym}(D\mathbf{u}) - \boldsymbol{\varepsilon}_p)\big)\, d\mathbf{x} + \frac{\lambda}{2} \int_\Omega \operatorname{tr}(D\mathbf{u})^2\, d\mathbf{x}$$
$$+ \mu_c \int_\Omega |\operatorname{skew}(D\mathbf{u}) - \bar{A}|^2\, d\mathbf{x} + \mu L_c^2 \int_\Omega |D\bar{A}|^2\, d\mathbf{x} . \tag{17}$$

Note that for the first time step $n = 1$ and $\boldsymbol{\varepsilon}_p^0 = \mathbf{0}$, $\mu_c = 0$, $L_c = 0$ the functional $I_{\text{incr}}^1(\mathbf{u}, \bar{0})$ reduces to the primal plastic functional of static perfect plasticity (Hencky plasticity) [**65, 57, 23, 24**].

PROOF. Any minimizer is a critical point, i. e., $DI_{\text{incr}}^n(\mathbf{u}, \bar{A}) = 0$. From (11) we obtain for the first variation of I_{incr}^n with respect to $\mathbf{u}$

$$D_1 I_{\text{incr}}^n(\mathbf{u}, \bar{A})[\mathbf{v}] \quad = \quad \frac{1}{2\mu} \int_\Omega P_{\mathbf{K}}\big(2\mu(\operatorname{sym}(D\mathbf{u}) - \boldsymbol{\varepsilon}_p^{n-1})\big) : \big(2\mu \operatorname{sym}(D\mathbf{v})\big)\, d\mathbf{x}$$
$$+ \lambda \int_\Omega \operatorname{div}(\mathbf{u}) \operatorname{div}(\mathbf{v})\, d\mathbf{x} + 2\mu_c \int_\Omega \big(\operatorname{skew}(D\mathbf{u}) - \bar{A}\big) : \operatorname{skew}(D\mathbf{v})\, d\mathbf{x} - \ell^n[\mathbf{v}]$$

which proves (15a); analogously $D_2 I_{\text{incr}}^n(\mathbf{u}^n, \bar{A}^n)[\bar{B}] = 0$ gives (15b). □

6. Analysis of the discrete elasto-plastic Cosserat model

For the analysis of the model we restrict ourselves to the pure Dirichlet problem with homogeneous boundary conditions $\mathbf{u}_D \equiv \mathbf{0}$ and $\bar{A}_D \equiv \mathbf{0}$ on $\Gamma_D = \partial\Omega$, to linear finite elements $\mathbf{V}_h$, W_h, and to mid-point quadrature on the triangles or tetrahedra. Thus, we can identify the discrete stress space $\mathbf{\Sigma}_h$ with element-wise constant functions in $L_2(\Omega, \mathbb{R}^{d,d})$. By abuse of notation we set $\mathbf{V}_h := \mathbf{V}_h(\mathbf{0})$ and $W_h := W_h(\mathbf{0})$, so that we have $\mathbf{V}_h \subset \mathbf{V}$ and $W_h \subset W$, where $\mathbf{V} = H_0^1(\Omega, \mathbb{R}^d)$ and $W = H_0^1(\Omega, \mathfrak{so}(d))$. The norm in $L_2(\Omega)$ is denoted by $\|\cdot\|$.

Due the the boundary conditions, Poincaré constants C_0, C_1 exists such that

$$\begin{aligned} \|\mathbf{v}\| &\leq C_0\, \|D\mathbf{v}\|, \quad \mathbf{v} \in \mathbf{V}\,, \\ \|\bar{B}\| &\leq C_1\, \|D\bar{B}\|, \quad \bar{B} \in W\,, \end{aligned} \tag{18}$$

and, a constant $C_3 > 0$ exists such that

$$\|D\mathbf{v}\|^2 \leq C_3\, \Big(\|\operatorname{div}\mathbf{v}\|^2 + \|\operatorname{curl}\mathbf{v}\|^2\Big), \qquad \mathbf{v} \in \mathbf{V}\,, \tag{19}$$

see **[27]**. Note that we have

$$\|\operatorname{div}(\mathbf{v})\|^2 + \|\operatorname{curl}(\mathbf{v})\|^2 = \|\operatorname{tr}(D\mathbf{v})\|^2 + \|\operatorname{skew}(D\mathbf{v})\|^2\,. \tag{20}$$

Finally, we define $\|\ell^n\|_{\mathbf{V}'} = \sup\limits_{\|D\mathbf{v}\|=1} |\ell^n[\mathbf{v}]|$.

THEOREM 6. *The functional $I^n_{\text{incr}} \colon \mathbf{V} \times W \longrightarrow \mathbb{R}$ in (16) is uniformly convex and bounded from below satisfying*

$$I^n_{\text{incr}}(\mathbf{u}, \bar{A}) \geq \mu_c c_4\, \Big(\|D\mathbf{u}\|^2 + \|D\bar{A}\|^2\Big) - \frac{1}{4\mu_c c_4}\|\ell^n\|^2_{\mathbf{V}'}\,. \tag{21}$$

Moreover, for $\mu_c \in (0, \mu]$ the constant c_4 is independent of μ_c.

PROOF. In the first step, we show that the symmetric bilinear form

$$\begin{aligned} b[(\mathbf{u}, \bar{A}), (\mathbf{v}, \bar{B})] \;=\; &\lambda \int_\Omega \operatorname{div}(\mathbf{u}) \operatorname{div}(\mathbf{v})\, d\mathbf{x} + 2\mu L_c^2 \int_\Omega D\bar{A} \cdot D\bar{B}\, d\mathbf{x} \\ &+ 2\mu_c \int_\Omega \big(\operatorname{skew}(D\mathbf{u}) - \bar{A}\big) : \big(\operatorname{skew}(D\mathbf{v}) - \bar{B}\big)\, d\mathbf{x} \end{aligned}$$

induces a coercive quadratic form. Inserting $\|\operatorname{skew}(D\mathbf{u}) - \bar{A}\|^2 \geq (1-\alpha)\, \|\operatorname{skew}(D\mathbf{u})\|^2 + (1 - \frac{1}{\alpha})\, \|\bar{A}\|^2$, with $\alpha \in (0,1)$ yields together with (19)

$$\begin{aligned} b[(\mathbf{u}, \bar{A}), (\mathbf{u}, \bar{A})] \;\geq\; &\lambda\, \|\operatorname{div}(\mathbf{u})\|^2 + 2\mu L_c^2 \|D\bar{A}\|^2 \\ &+ 2\mu_c(1-\alpha)\, \|\operatorname{skew}(D\mathbf{u})\|^2 + 2\mu_c\Big(1 - \frac{1}{\alpha}\Big)\, \|\bar{A}\|^2 \\ \geq\; &\lambda\, \|\operatorname{div}(\mathbf{u})\|^2 + \mu L_c^2 \|D\bar{A}\|^2 \\ &+ 2\mu_c(1-\alpha)\, \|\operatorname{skew}(D\mathbf{u})\|^2 + \Big(2\mu_c\Big(1 - \frac{1}{\alpha}\Big) + \frac{\mu L_c^2}{C_1^2}\Big)\, \|\bar{A}\|^2\,, \end{aligned}$$

which gives for $1 > \alpha > \frac{\mu_c}{\mu + \frac{\mu L_c^2}{C_1^2}} \geq \frac{\mu_c}{\mu_c + \frac{\mu L_c^2}{C_1^2}}$ and $\mu_c \in (0, \mu]$

$$b[(\mathbf{u}, \bar{A}), (\mathbf{u}, \bar{A})] \;\geq\; 4\mu_c c_4\, \big(\|D\mathbf{u}\|^2 + \|D\bar{A}\|^2\big)$$

using (20), where c_4 depends on λ, μ and L_c but c_4 is independent of μ_c. Thus, $I^n_{\text{incr}}(\cdot)$ is uniformly convex since $\psi_{\mathbf{K}}(\cdot)$ is convex, $b[\cdot,\cdot]$ is coercive and $\ell^n[\cdot]$ is linear. Finally, we obtain the assertion from $\psi_{\mathbf{K}} \geq 0$ and

$$\begin{aligned} I^n_{\text{incr}}(\mathbf{u},\bar{A}) &\geq \frac{1}{2}b[(\mathbf{u},\bar{A}),(\mathbf{u},\bar{A})] - \ell^n(\mathbf{u}) \\ &\geq 2\mu_c c_4\left(\|D\mathbf{v}\|^2 + \|D\bar{A}\|^2\right) - \frac{1}{4\mu_c c_4}\|\ell^n\|^2_{\mathbf{V}'} - \mu_c c_4\,\|D\mathbf{u}\|^2 . \end{aligned}$$

□

In the pure Dirichlet case this gives directly for the norm

$$\|(\mathbf{u},\bar{A})\|_{\mathbf{V}\times W} = \left(\|D\mathbf{u}\|^2 + \|D\bar{A}\|^2\right)^{1/2}$$

the following a priori bound for the solution.

COROLLARY 7. *The minimization problem* (16) *has a unique solution* $(\mathbf{u}^n_h, \bar{A}^n_h) \in \mathbf{V}_h \times W_h$ *of the discrete incremental problem, which is uniquely characterized by the nonlinear variational problem* (15)*, i. e.,*

$$DI^n_{\text{incr}}(\mathbf{u}^n_h, \bar{A}^n_h)[\mathbf{v}_h, \bar{B}_h] = 0, \qquad (\mathbf{v}_h, \bar{B}_h) \in \mathbf{V}_h \times W_h \ , \tag{22}$$

and which solves the discrete system (8)*. Moreover, we have the a priori bound*

$$\|(\mathbf{u}_h, \bar{A}_h)\|^2_{\mathbf{V}\times W} \leq \frac{2}{\mu_c c_4}\left(\|\ell^n\|^2_{\mathbf{V}'} + 2\mu\,\|\boldsymbol{\varepsilon}_p^{n-1}\|^2\right) . \tag{23}$$

Analogously, a unique solution $(\mathbf{u}^n, \bar{A}^n) \in \mathbf{V} \times W$ of the incremental problem in the continuous spaces exists, satisfying

$$DI^n_{\text{incr}}(\mathbf{u}^n, \bar{A}^n)[\mathbf{v}, \bar{B}] = 0, \qquad (\mathbf{v}, \bar{B}) \in \mathbf{V} \times W \ . \tag{24}$$

THEOREM 8. *We have*

$$\|(\mathbf{u}-\mathbf{u}_h, \bar{A}-\bar{A}_h)\|_{\mathbf{V}\times W} \leq \frac{C_5}{\mu_c} \inf_{(\mathbf{v}_h,\bar{B}_h)\in\mathbf{V}_h\times W_h} \|(\mathbf{u}-\mathbf{v}_h, \bar{A}-\bar{B}_h)\|_{\mathbf{V}\times W} . \tag{25}$$

Again C_5 *is independent of* $\mu_c \in (0,\mu]$.

PROOF. Since the projection $P_{\mathbf{K}}$ is non-expansive, the derivative $DI^n_{\text{incr}}(\cdot)$ is uniformly Lipschitz continuous satisfying

$$\|DI^n_{\text{incr}}(\mathbf{u},\bar{A}) - DI^n_{\text{incr}}(\mathbf{v},\bar{B})\|_{\mathbf{V}\times W} \leq C_6\ \|(\mathbf{u},\bar{A}) - (\mathbf{v},\bar{B})\|_{\mathbf{V}\times W} \tag{26}$$

with $C_6 > 0$ independent of $\mu_c \in (0,\mu]$. Since $I^n_{\text{incr}}(\cdot)$ is uniformly convex and inserting (22), (24), (26) gives

$$\begin{aligned} &\mu_c c_4\ \|(\mathbf{u}^n - \mathbf{u}^n_h, \bar{A}^n - \bar{A}^n_h)\|^2_{\mathbf{V}\times W} \\ &\quad \leq DI^n_{\text{incr}}(\mathbf{u}^n,\bar{A}^n)[\mathbf{u}^n - \mathbf{u}^n_h, \bar{A}^n - \bar{A}^n_h] - DI^n_{\text{incr}}(\mathbf{u}^n_h,\bar{A}^n_h)[\mathbf{u}^n - \mathbf{u}^n_h, \bar{A}^n - \bar{A}^n_h] \\ &\quad = -DI^n_{\text{incr}}(\mathbf{u}^n_h,\bar{A}^n_h)[\mathbf{u}^n - \mathbf{v}_h, \bar{A}^n - \bar{B}_h] \\ &\quad = DI^n_{\text{incr}}(\mathbf{u}^n,\bar{A}^n)[\mathbf{u}^n - \mathbf{v}_h, \bar{A}^n - \bar{B}_h] - DI^n_{\text{incr}}(\mathbf{u}^n_h,\bar{A}^n_h)[\mathbf{u}^n - \mathbf{v}_h, \bar{A}^n - \bar{B}_h] \\ &\quad \leq \|DI^n_{\text{incr}}(\mathbf{u}^n,\bar{A}^n) - DI^n_{\text{incr}}(\mathbf{u}^n_h,\bar{A}^n_h)\|_{\mathbf{V}\times W}\|(\mathbf{u}^n - \mathbf{v}_h, \bar{A}^n - \bar{B}_h)\|_{\mathbf{V}\times W} \\ &\quad \leq C_6\ \|(\mathbf{u}^n - \mathbf{u}^n_h, \bar{A}^n - \bar{A}^n_h)\|_{\mathbf{V}\times W}\ \|(\mathbf{u}^n - \mathbf{v}_h, \bar{A}^n - \bar{B}_h)\|_{\mathbf{V}\times W} . \end{aligned}$$

□

REMARK 9. *Here, we do not discuss the convergence in time. Formally, the problem has the same structure as considered in* [**33**, Chap. 13] *for the generalized stresses in the case of plasticity with hardening. Thus, we expect also first order convergence in* $\Delta t \to 0$ *for the implicit Euler method. Our numerical experiments show that for realistic time step sizes the spatial error is dominant and therefore there is no need for higher order methods in time.*

7. The dual problem of the elasto-plastic Cosserat model

The uniform convexity and the finite element convergence estimate derived in the previous section deteriorate in the limit $\mu_c \longrightarrow 0$. This reflects the missing regularity of the primal solution of the limiting model of perfect plasticity. Thus, in this section we derive the dual formulation in order to study the dependence of the solution on μ_c. For simplicity, we study $d = 3$.

Let $(\mathbf{u}^n, \bar{A}^n, \boldsymbol{\sigma}^n)$ be the solution of the incremental problem. From (8c) and and (8d) we obtain $\operatorname{skew}(\boldsymbol{\sigma}^n) = 2\mu_c\big(\operatorname{skew}(D\mathbf{u}^n) - \bar{A}^n\big)$ and $\int_\Omega \boldsymbol{\sigma}^n : D\mathbf{u}^n\, d\mathbf{x} = \ell^n[\mathbf{u}^n]$. Now, (8e) and

$$
\begin{aligned}
\int_\Omega \operatorname{skew}(\boldsymbol{\sigma}^n) : D\mathbf{u}^n\, d\mathbf{x} &= \int_\Omega \operatorname{skew}(\boldsymbol{\sigma}^n) : (\operatorname{skew}(D\mathbf{u}^n) - \bar{A}^n)\, d\mathbf{x} + \int_\Omega \operatorname{skew}(\boldsymbol{\sigma}^n) : \bar{A}^n\, d\mathbf{x} \\
&= 2\mu_c \int_\Omega |\operatorname{skew}(D\mathbf{u}^n) - \bar{A}^n|^2\, d\mathbf{x} + 2\mu L_c^2 \int_\Omega |D\bar{A}^n|^2\, d\mathbf{x}
\end{aligned}
$$

yields for the primal functional (16)

$$
\begin{aligned}
I^n_{\mathrm{incr}}(\mathbf{u}^n, \bar{A}^n) &= \mathcal{E}_{\mathrm{incr}}(D\mathbf{u}^n, \bar{A}^n, \boldsymbol{\varepsilon}_p^{n-1}) - \ell^n[\mathbf{u}^n] \\
&= \frac{1}{2\mu}\int_\Omega \psi_{\mathbf{K}}(\boldsymbol{\theta}^n)\, d\mathbf{x} + \frac{\lambda}{2}\int_\Omega \operatorname{tr}(D\mathbf{u}^n)^2\, d\mathbf{x} \\
&\quad + \mu_c \int_\Omega |\operatorname{skew}(D\mathbf{u}^n) - \bar{A}^n|^2\, d\mathbf{x} + \mu L_c^2 \int_\Omega |D\bar{A}^n|^2\, d\mathbf{x} - \ell^n[\mathbf{u}^n] \\
&= \frac{1}{2\mu}\int_\Omega \psi_{\mathbf{K}}(\boldsymbol{\theta}^n)\, d\mathbf{x} + \frac{\lambda}{2}\int_\Omega \operatorname{tr}(D\mathbf{u}^n)^2\, d\mathbf{x} \\
&\quad + \frac{1}{2}\int_\Omega \operatorname{skew}(\boldsymbol{\sigma}^n) : D\mathbf{u}^n\, d\mathbf{x} - \int_\Omega \boldsymbol{\sigma}^n : D\mathbf{u}^n\, d\mathbf{x} \\
&= \frac{1}{2\mu}\int_\Omega \psi_{\mathbf{K}}(\boldsymbol{\theta}^n)\, d\mathbf{x} + \frac{\lambda}{2}\int_\Omega \operatorname{tr}(D\mathbf{u}^n)^2\, d\mathbf{x} \\
&\quad - \frac{1}{2}\int_\Omega \operatorname{skew}(\boldsymbol{\sigma}^n) : D\mathbf{u}^n\, d\mathbf{x} - \int_\Omega \operatorname{sym}(\boldsymbol{\sigma}^n) : D\mathbf{u}^n\, d\mathbf{x}\ .
\end{aligned}
$$

Again using (14) and (16) we obtain

$$
\begin{aligned}
\operatorname{sym}(\boldsymbol{\sigma}^n) &= P_{\mathbf{K}}(\boldsymbol{\theta}^n) + \lambda \operatorname{div}(\mathbf{u}^n)\mathbf{I}\ , \\
\operatorname{dev}(\operatorname{sym}(\boldsymbol{\sigma}^n)) &= \operatorname{dev}(P_{\mathbf{K}}(\boldsymbol{\theta}^n))\ , \\
P_{\mathbf{K}}(\boldsymbol{\theta}^n) &= \operatorname{dev}(\operatorname{sym}(\boldsymbol{\sigma}^n)) + \frac{2\mu}{3}\operatorname{tr}(D\mathbf{u}^n)\mathbf{I}\ , \\
\psi_{\mathbf{K}}(\boldsymbol{\theta}^n) &= \frac{1}{2}\,|\boldsymbol{\theta}^n|^2 - \frac{1}{2}\,|\boldsymbol{\theta}^n - P_{\mathbf{K}}(\boldsymbol{\theta}^n)|^2\ = P_{\mathbf{K}}(\boldsymbol{\theta}^n) : \boldsymbol{\theta}^n - \frac{1}{2}\,|P_{\mathbf{K}}(\boldsymbol{\theta}^n)|^2 \\
&= 2\mu \operatorname{dev}(\operatorname{sym}(\boldsymbol{\sigma}^n)) : (D\mathbf{u}^n - \boldsymbol{\varepsilon}_p^{n-1}) + \frac{2\mu^2}{3}\operatorname{tr}(D\mathbf{u}^n)^2 - \frac{1}{2}\,|\operatorname{dev}(\operatorname{sym}(\boldsymbol{\sigma}^n))|^2\ , \\
\operatorname{tr}(\boldsymbol{\sigma}^n)\mathbf{I} : D\mathbf{u}^n &= \frac{1}{2\mu + 3\lambda}\operatorname{tr}(\boldsymbol{\sigma}^n)^2 = (2\mu + 3\lambda)\operatorname{tr}(D\mathbf{u}^n)^2\ ,
\end{aligned}
$$

and together with

$$
\int_\Omega \operatorname{skew}(\boldsymbol{\sigma}^n) : D\mathbf{u}^n\, d\mathbf{x} = \frac{1}{2\mu_c}\int_\Omega \operatorname{skew}(\boldsymbol{\sigma}^n) : \operatorname{skew}(\boldsymbol{\sigma}^n)\, d\mathbf{x} + 2\mu L_c^2 \int_\Omega D\bar{A}^n \cdot D\bar{A}^n\, d\mathbf{x}\ ,
$$

we finally obtain

$$
\begin{aligned}
&I^n_{\mathrm{incr}}(\mathbf{u}^n,\bar A^n)\\
&=\frac{1}{2\mu}\int_\Omega \psi_{\mathbf{K}}(\boldsymbol{\theta}^n)\,d\mathbf{x}+\frac{\lambda}{2}\int_\Omega \mathrm{tr}(D\mathbf{u}^n)^2\,d\mathbf{x}-\frac12\int_\Omega \mathrm{skew}(\boldsymbol{\sigma}^n):D\mathbf{u}^n\,d\mathbf{x}-\int_\Omega \mathrm{sym}(\boldsymbol{\sigma}^n):D\mathbf{u}^n\,d\mathbf{x}\\
&=\int_\Omega \mathrm{dev}(\mathrm{sym}(\boldsymbol{\sigma}^n)):(D\mathbf{u}^n-\varepsilon_p^{n-1})\,d\mathbf{x}+\frac{2\mu+3\lambda}{6}\int_\Omega \mathrm{tr}(D\mathbf{u}^n)^2\,d\mathbf{x}\\
&\qquad-\frac{1}{4\mu}\int_\Omega |\,\mathrm{dev}(\mathrm{sym}(\boldsymbol{\sigma}^n))|^2\,d\mathbf{x}-\frac12\int_\Omega \mathrm{skew}(\boldsymbol{\sigma}^n):D\mathbf{u}^n\,d\mathbf{x}-\int_\Omega \mathrm{sym}(\boldsymbol{\sigma}^n):D\mathbf{u}^n\,d\mathbf{x}\\
&=-\int_\Omega \boldsymbol{\sigma}^n:\varepsilon_p^{n-1}\,d\mathbf{x}-\frac{1}{4\mu}\int_\Omega |\,\mathrm{dev}(\mathrm{sym}(\boldsymbol{\sigma}^n))|^2\,d\mathbf{x}+\frac16\int_\Omega \mathrm{tr}(\boldsymbol{\sigma}^n)\mathbf{I}:D\mathbf{u}^n\,d\mathbf{x}\\
&\qquad-\frac12\int_\Omega \mathrm{skew}(\boldsymbol{\sigma}^n):D\mathbf{u}^n\,d\mathbf{x}-\frac13\int_\Omega \mathrm{tr}(\boldsymbol{\sigma}^n)\mathbf{I}:D\mathbf{u}^n\,d\mathbf{x}\\
&=-\int_\Omega \boldsymbol{\sigma}^n:\varepsilon_p^{n-1}\,d\mathbf{x}-\frac{1}{4\mu}\int_\Omega |\,\mathrm{dev}(\mathrm{sym}(\boldsymbol{\sigma}^n))|^2\,d\mathbf{x}-\frac{1}{6(2\mu+3\lambda)}\int_\Omega \mathrm{tr}(\boldsymbol{\sigma}^n)^2\,d\mathbf{x}\\
&\qquad-\frac{1}{4\mu_c}\int_\Omega |\,\mathrm{skew}(\boldsymbol{\sigma}^n)|^2\,d\mathbf{x}-\mu L_c^2\int_\Omega |D\bar A^n|^2\,d\mathbf{x}\ .
\end{aligned}
$$

Thus, we define the *dual functional*

$$
\begin{aligned}
\mathcal{D}^n_{\mathrm{incr}}(\boldsymbol{\sigma},\bar A)\ &=\ \int_\Omega \boldsymbol{\sigma}:\varepsilon_p^{n-1}\,d\mathbf{x}+\frac{1}{4\mu}\int_\Omega |\,\mathrm{dev}(\mathrm{sym}(\boldsymbol{\sigma}))|^2\,d\mathbf{x}+\frac{1}{6(2\mu+3\lambda)}\int_\Omega \mathrm{tr}(\boldsymbol{\sigma})^2\,d\mathbf{x}\\
&\quad+\frac{1}{4\mu_c}\int_\Omega |\,\mathrm{skew}(\boldsymbol{\sigma})|^2\,d\mathbf{x}+\mu L_c^2\int_\Omega |D\bar A|^2\,d\mathbf{x}\ . \qquad (27)
\end{aligned}
$$

Standard calculus in convex analysis yields the following lemma.

LEMMA 10. *Let* $\varepsilon_p^{n-1}\in L_2(\Omega,\mathrm{Sym}(3))$ *with* $\mathrm{tr}(\varepsilon_p^{n-1})=0$ *be given.*
Then, $(\boldsymbol{\sigma}^n,\bar A^n)\in L_2(\Omega,\mathbb{R}^{3,3})\times W$ *is uniquely determined by minimizing the dual functional* (27) *subject to the plastic constraint*

$$\phi\big(\mathrm{sym}(\boldsymbol{\sigma}^n)\big)\le 0 \qquad (28)$$

and the equilibrium constraint

$$\int_\Omega \boldsymbol{\sigma}^n:D\mathbf{v}\,d\mathbf{x}=\ell^n[\mathbf{v}],\qquad \mathbf{v}\in\mathbf{V} \qquad (29)$$

$$2\mu L_c^2\int_\Omega D\bar A^n\cdot D\bar B\,d\mathbf{x}=\int_\Omega \mathrm{skew}(\boldsymbol{\sigma}^n):\bar B\,d\mathbf{x},\qquad \bar B\in W\ . \qquad (30)$$

From Theorem 6 we obtain, that for $\mu_c>0$ always a primal solution and therefore also a dual solution exists. For the limit case $\mu_c=0$, we have to assume that the convex set

$$\mathbf{K}^n=\Big\{\boldsymbol{\tau}\in L^2(\Omega,\mathrm{Sym}(3)):\qquad \boldsymbol{\tau}\in\mathbf{K}\text{ a. e. in }\Omega,\ \int_\Omega \boldsymbol{\tau}:D\mathbf{v}\,d\mathbf{x}=\ell^n[\mathbf{v}],\ \mathbf{v}\in\mathbf{V}\Big\}$$

is not empty (weak safe-load assumption).

LEMMA 11. *If* $\boldsymbol{\eta}^n\in\mathbf{K}^n$ *exists, we have*

$$\|\,\mathrm{skew}(\boldsymbol{\sigma}^n)\|^2\ \le\ 4\mu_c\,\mathcal{D}^n_{\mathrm{incr}}(\boldsymbol{\eta}^n,\mathbf{0}) \qquad (31)$$

and

$$\|D\bar A^n\|^2\ \le\ \frac{C_1^2}{\mu^2L_c^4}\,\mu_c\,\mathcal{D}^n_{\mathrm{incr}}(\boldsymbol{\eta}^n,\mathbf{0})\ . \qquad (32)$$

PROOF. Since $(\boldsymbol{\eta}^n,\mathbf{0})$ is admissible, we obtain the first assertion from

$$D^n_{\mathrm{incr}}(\boldsymbol{\eta}^n,\mathbf{0})\ge D^n_{\mathrm{incr}}(\boldsymbol{\sigma}^n,\bar A^n)=D^n_{\mathrm{incr}}(\mathrm{sym}(\boldsymbol{\sigma}^n),\bar A^n)+\frac{1}{4\mu_c}\int_\Omega |\,\mathrm{skew}(\boldsymbol{\sigma}^n)|^2\,d\mathbf{x} \qquad (33)$$

since $D^n_{\rm incr}(\mathrm{sym}(\boldsymbol{\sigma}^n), \bar{A}^n) \geq 0$. Testing with $B = \bar{A}^n$ we obtain from (30)

$$2\mu L_c^2 \, \|D\bar{A}^n\|^2 \leq \|\,\mathrm{skew}(\boldsymbol{\sigma}^n)\| \, \|\bar{A}^n\| \leq C_1 \, \|\,\mathrm{skew}(\boldsymbol{\sigma}^n)\| \, \|D\bar{A}^n\| \, ,$$

where the second inequality follows from (18). This gives directly the second assertion by inserting the first statement. □

Now, we denote the incremental dual solution for fixed $\boldsymbol{\varepsilon}_p^{n-1} \in L_2(\Omega, \mathrm{Sym}(3))$ by $(\boldsymbol{\sigma}^n_{\mu_c}, \bar{A}^n_{\mu_c})$. If $\mathbf{K}^n \neq \emptyset$, a unique dual solution of perfect plasticity $\boldsymbol{\sigma}^n_0 \in \mathbf{K}^n$ defined by

$$\mathcal{D}^n_{\rm incr}(\boldsymbol{\sigma}^n_0, \mathbf{0}) \leq \mathcal{D}^n_{\rm incr}(\boldsymbol{\tau}, \mathbf{0}), \qquad \boldsymbol{\tau} \in \mathbf{K}^n$$

exists **[65]**. Equivalently, $\boldsymbol{\sigma}^n_0 \in \mathbf{K}^n$ is characterized by

$$d[\boldsymbol{\sigma}^n_0, \boldsymbol{\sigma}^n_0 - \boldsymbol{\tau}] \;\; \leq \;\; \int_\Omega (\boldsymbol{\sigma}^n_0 - \boldsymbol{\tau}) : \boldsymbol{\varepsilon}_p^{n-1} \, d\mathbf{x}, \qquad \boldsymbol{\tau} \in \mathbf{K}^n \tag{34}$$

with the symmetric bilinear form

$$d[\boldsymbol{\sigma}, \boldsymbol{\tau}] \;\; = \;\; \frac{1}{2\mu} \int_\Omega \mathrm{dev}(\mathrm{sym}(\boldsymbol{\sigma})) : \mathrm{dev}(\mathrm{sym}(\boldsymbol{\tau})) \, d\mathbf{x} + \frac{1}{6(2\mu + 3\lambda)} \int_\Omega \mathrm{tr}(\boldsymbol{\sigma}) \, \mathrm{tr}(\boldsymbol{\tau}) \, d\mathbf{x} \; .$$

Now, we study the limit $\mu_c \longrightarrow 0$.

THEOREM 12. *If $\boldsymbol{\eta}^n \in \mathbf{K}^n$ exists, we have*

$$\lim_{\mu_c \longrightarrow 0} \|\,\mathrm{sym}(\boldsymbol{\sigma}^n_{\mu_c}) - \boldsymbol{\sigma}^n_0\| = 0 \; . \tag{35}$$

PROOF. The proof follows the general framework of penalty solutions of variational inequalities **[28**, Chap. I.7**]**.

Let μ_c^j be a decreasing sequence with $\mu_c^j \longrightarrow 0$. Since $\mathrm{sym}(\boldsymbol{\sigma}^n_{\mu_c^j})$ is uniformly bounded by (33), we can extract a subsequence (again denoted by $\mathrm{sym}(\boldsymbol{\sigma}^n_{\mu_c^j})$) which is weakly convergent to $\hat{\boldsymbol{\sigma}}^n \in L^2(\Omega, \mathrm{Sym}(3))$. We have $\boldsymbol{\sigma}^n_{\mu_c^j} \in \mathbf{K}$ and therefore $\hat{\boldsymbol{\sigma}}^n \in \mathbf{K}$ a. e. in Ω. Moreover, for $\mathbf{v} \in \mathbf{V}$ we obtain from (29) and (31)

$$\begin{aligned} \int_\Omega \hat{\boldsymbol{\sigma}}^n : D\mathbf{v} \, d\mathbf{x} - \ell^n[\mathbf{v}] \;\; &= \;\; \lim_{j \longrightarrow \infty} \int_\Omega \mathrm{sym}(\boldsymbol{\sigma}^n_{\mu_c^j}) : D\mathbf{v} \, d\mathbf{x} - \ell^n[\mathbf{v}] \\ &= \;\; \lim_{j \longrightarrow \infty} \int_\Omega \mathrm{skew}(\boldsymbol{\sigma}^n_{\mu_c^j}) : D\mathbf{v} = 0 \; . \end{aligned}$$

Thus, we have $\hat{\boldsymbol{\sigma}}^n \in \mathbf{K}^n$.

For all $\boldsymbol{\tau} \in \mathbf{K}^n$ the pair $(\boldsymbol{\tau}, \mathbf{0})$ is admissible for the incremental dual Cosserat problem satisfying (29), (30), which gives

$$\begin{aligned} d[\mathrm{sym}(\boldsymbol{\sigma}^n_{\mu_c^j}), \mathrm{sym}(\boldsymbol{\sigma}^n_{\mu_c^j}) - \boldsymbol{\tau}] + \frac{1}{2\mu_c} \int_\Omega |\,\mathrm{skew}(\boldsymbol{\sigma}^n_{\mu_c^j})|^2 \, d\mathbf{x} + 2\mu L_c^2 \int_\Omega |D\bar{A}^n_{\mu_c^j}|^2 \, d\mathbf{x} \\ \leq \int_\Omega (\boldsymbol{\sigma}^n_{\mu_c^j} - \boldsymbol{\tau}) : \boldsymbol{\varepsilon}_p^{n-1} \, d\mathbf{x} \; . \end{aligned} \tag{36}$$

Passing to the limit yields

$$d[\hat{\boldsymbol{\sigma}}^n, \hat{\boldsymbol{\sigma}}^n] \leq \liminf_{j \longrightarrow \infty} d[\mathrm{sym}(\boldsymbol{\sigma}^n_{\mu_c^j}), \mathrm{sym}(\boldsymbol{\sigma}^n_{\mu_c^j})] \leq d[\hat{\boldsymbol{\sigma}}^n, \boldsymbol{\tau}] + \int_\Omega (\hat{\boldsymbol{\sigma}}^n - \boldsymbol{\tau}) : \boldsymbol{\varepsilon}_p^{n-1} \, d\mathbf{x} \; .$$

Since $\boldsymbol{\sigma}^n_0 \in \mathbf{K}^n$ is uniquely characterized by (34), this proves that the weak limit solves the dual problem of perfect plasticity, i e., $\hat{\boldsymbol{\sigma}}^n = \boldsymbol{\sigma}^n_0$.

Finally, we show strong convergence. Inserting $\boldsymbol{\tau} = \boldsymbol{\sigma}^n_0$ in (36) gives

$$d[\mathrm{sym}(\boldsymbol{\sigma}^n_{\mu_c^j}), \mathrm{sym}(\boldsymbol{\sigma}^n_{\mu_c^j})] \leq d[\mathrm{sym}(\boldsymbol{\sigma}^n_{\mu_c^j}), \boldsymbol{\sigma}^n_0] + \int_\Omega (\boldsymbol{\sigma}^n_{\mu_c^j} - \boldsymbol{\sigma}^n_0) : \boldsymbol{\varepsilon}_p^{n-1} \, d\mathbf{x} \; ,$$

which gives

$$\begin{aligned} d[\mathrm{sym}(\boldsymbol{\sigma}^n_{\mu_c^j}) - \boldsymbol{\sigma}^n_0, \mathrm{sym}(\boldsymbol{\sigma}^n_{\mu_c^j}) - \boldsymbol{\sigma}^n_0] &= d[\mathrm{sym}(\boldsymbol{\sigma}^n_{\mu_c^j}), \mathrm{sym}(\boldsymbol{\sigma}^n_{\mu_c^j})] \\ &\quad - d[\mathrm{sym}(\boldsymbol{\sigma}^n_{\mu_c^j}), \boldsymbol{\sigma}^n_0] - d[\boldsymbol{\sigma}^n_0, \mathrm{sym}(\boldsymbol{\sigma}^n_{\mu_c^j})] + d[\boldsymbol{\sigma}^n_0, \boldsymbol{\sigma}^n_0] \\ &\leq d[\mathrm{sym}(\boldsymbol{\sigma}^n_{\mu_c^j}), \boldsymbol{\sigma}^n_0] + \int_\Omega (\boldsymbol{\sigma}^n_{\mu_c^j} - \boldsymbol{\sigma}^n_0) : \varepsilon_p^{n-1}\, d\mathbf{x} \\ &\quad - d[\mathrm{sym}(\boldsymbol{\sigma}^n_{\mu_c^j}), \boldsymbol{\sigma}^n_0] - d[\boldsymbol{\sigma}^n_0, \mathrm{sym}(\boldsymbol{\sigma}^n_{\mu_c^j})] + d[\boldsymbol{\sigma}^n_0, \boldsymbol{\sigma}^n_0]\ . \end{aligned}$$

The limit on the right hand side is well defined, and we obtain

$$\lim_{j \longrightarrow \infty} d[\mathrm{sym}(\boldsymbol{\sigma}^n_{\mu_c^j}) - \boldsymbol{\sigma}^n_0, \mathrm{sym}(\boldsymbol{\sigma}^n_{\mu_c^j}) - \boldsymbol{\sigma}^n_0] = 0\,.$$

This proves the assertion since $d[\cdot,\cdot]$ is an inner product in $L_2(\Omega, \mathrm{Sym}(3))$. □

REMARK 13. *Our numerical experiments suggest that we have*

$$\|\,\mathrm{sym}(\boldsymbol{\sigma}^n_{\mu_c}) - \boldsymbol{\sigma}^n_0\| \;=\; O(\sqrt{\mu_c})\ .$$

We surmise that in general additional regularity assumptions are required to obtain an explicit estimate of this type. In principle, such an estimate could then be used for estimating perfect plasticity in the form

$$\|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_{0,h}\| \;\leq\; \|\boldsymbol{\sigma}_0 - \mathrm{sym}(\boldsymbol{\sigma}_{\mu_c})\| + \|\,\mathrm{sym}(\boldsymbol{\sigma}_{\mu_c}) - \mathrm{sym}(\boldsymbol{\sigma}_{\mu_c,h})\| + \|\,\mathrm{sym}(\boldsymbol{\sigma}_{\mu_c,h}) - \boldsymbol{\sigma}_{0,h}\|$$

by a suitable choice of $\mu_c = \mu_c(h)$. Note that an analogous result is obtained in **[57]** *for the static case by approximating perfect plasticity with plasticity with hardening for vanishing hardening modulus. It is not clear if such a result can be transferred to the quasi-static case where in general the plastic strain is not smooth enough.*

8. Numerical solution algorithm

We formulate a semi-smooth Newton method for the nonlinear variational problem

$$(\mathbf{u}^n, \bar{A}^n) \in \mathbf{V}_h(\mathbf{u}^n_D) \times W_h(\bar{A}^n_D): \qquad F^n(\mathbf{u}^n, \bar{A}^n) = 0$$

in every time step n, where F^n is the first variation of I^n_{incr} defined by

$$F^n(\mathbf{u}, \bar{A})[\mathbf{v}, \bar{B}] = DI^n_{\mathrm{incr}}(\mathbf{u}, \bar{A})[\mathbf{v}, \bar{B}], \qquad (\mathbf{v}, \bar{B}) \in \mathbf{V}_h(\mathbf{0}) \times W_h(\mathbf{0})$$

(cf. Lemma 5). The functional F^n is semi-smooth, and the variation $\partial F^n = \partial^2 I^n_{\mathrm{incr}}$ is symmetric and multi-valued. Thus, the corresponding semi-smooth Newton method can be formally written as

$$0 \in F^n(\mathbf{u}^{n,k}, \bar{A}^{n,k}) + \partial F^n(\mathbf{u}^{n,k}, \bar{A}^{n,k})[\mathbf{u}^{n,k+1} - \mathbf{u}^{n,k}, \bar{A}^{n,k+1} - \bar{A}^{n,k}]\ .$$

Since I^n_{incr} is convex and $F^n = DI^n_{\mathrm{incr}}$ is Lipschitz continuous, this can be analyzed within the framework of generalized Newton methods **[36]**.

We consider the special case of the von Mises flow rule by inserting (13). This results in the following realization of the semi-smooth Newton method.

S0) We start for $t_0 = 0$ with $\boldsymbol{\varepsilon}_p^0 = \mathbf{0}$, and we set $n = 1$.

S1) Choose $\mathbf{u}^{n,0}$, $\bar{A}^{n,0}$, and set $k = 0$. Then, set Dirichlet boundary values $\mathbf{u}^{n,0}(\mathbf{x}) = \mathbf{u}_D^n(\mathbf{x})$ for all $\mathbf{x} \in D$ and $\bar{A}^{n,0}(\mathbf{x}) = \bar{A}_D^n(\mathbf{x})$ for all $\mathbf{x} \in D'$.

S2) Compute for every integration point $\boldsymbol{\xi} \in \boldsymbol{\Xi}_h$

$$\begin{aligned}
\boldsymbol{\theta}^{n,k}(\boldsymbol{\xi}) &= 2\mu\big(\operatorname{sym}(D\mathbf{u}^{n,k}(\boldsymbol{\xi})) - \boldsymbol{\varepsilon}_p^{n-1}(\boldsymbol{\xi})\big)\\
\boldsymbol{\eta}^{n,k}(\boldsymbol{\xi}) &= \frac{\operatorname{dev}(\boldsymbol{\theta}^{n,k}(\boldsymbol{\xi}))}{|\operatorname{dev}(\boldsymbol{\theta}^{n,k}(\boldsymbol{\xi}))|}\\
T_E^{n,k}(\boldsymbol{\xi}) &= \boldsymbol{\theta}^{n,k}(\boldsymbol{\xi}) - \max\big\{0, |\operatorname{dev}(\boldsymbol{\theta}^{n,k}(\boldsymbol{\xi}))| - K_0\big\}\,\boldsymbol{\eta}^{n,k}(\boldsymbol{\xi})\\
\boldsymbol{\sigma}^{n,k}(\boldsymbol{\xi}) &= T_E^{n,k}(\boldsymbol{\xi}) + \lambda \operatorname{div}(\mathbf{u}^n)(\boldsymbol{\xi})\mathbf{I} + 2\mu_c\big(\operatorname{skew}(D\mathbf{u}^n(\boldsymbol{\xi})) - \bar{A}^n(\boldsymbol{\xi})\big)\ .
\end{aligned}$$

S3) Compute the residual

$$\begin{aligned}
F^{n,k}[\mathbf{v}, \bar{B}] &= \int_\Omega \boldsymbol{\sigma}^{n,k} : D\mathbf{v}\, d\mathbf{x} - \ell^n[\mathbf{v}]\\
&\quad + 2\mu L_c^2 \int_\Omega D\bar{A}^{n,k} \cdot D\bar{B}\, d\mathbf{x} - 2\mu_c \int_\Omega \big(\operatorname{skew}(D\mathbf{u}^{n,k}) - \bar{A}^{n,k}\big) : \bar{B}\, d\mathbf{x}\ .
\end{aligned}$$

If $\|F^{n,k}\|$ is small enough, set $\mathbf{u}^n = \mathbf{u}^{n,k}$, $\bar{A}^n = \bar{A}^{n,k}$,

$$\boldsymbol{\varepsilon}_p^n(\boldsymbol{\xi}) = \boldsymbol{\varepsilon}_p^{n-1}(\boldsymbol{\xi}) + \frac{\max\big\{0, |\operatorname{dev}(\boldsymbol{\theta}^{n,k}(\boldsymbol{\xi}))| - K_0\big\}}{2\mu}\,\boldsymbol{\eta}^{n,k}(\boldsymbol{\xi})$$

and go to the next time step $n := n + 1$ in S1).

S4) Compute for every integration point $\boldsymbol{\xi} \in \boldsymbol{\Xi}_h$ the consistent linearization

$$\mathbb{C}^{n,k}(\boldsymbol{\xi}) = \begin{cases} \mathbb{1} & |\operatorname{dev}(\boldsymbol{\theta}^{n,k}(\boldsymbol{\xi}))| \le K_0 \\ \frac{1}{d}\mathbf{I}\otimes\mathbf{I} + \frac{K_0}{|\operatorname{dev}(\boldsymbol{\theta}^{n,k}(\boldsymbol{\xi}))|}\Big(\big(\mathbb{1} - \frac{1}{d}\mathbf{I}\otimes\mathbf{I}\big) - \boldsymbol{\eta}^{n,k}(\boldsymbol{\xi})\otimes\boldsymbol{\eta}^{n,k}(\boldsymbol{\xi})\Big) & |\operatorname{dev}(\boldsymbol{\theta}^{n,k}(\boldsymbol{\xi}))| > K_0 \end{cases}$$

and define the bilinear form

$$\begin{aligned}
a^{n,k}[(\mathbf{w}, \bar{C}), (\mathbf{v}, \bar{B})] &= 2\mu \int_\Omega \operatorname{sym}(D\mathbf{w}) : \mathbb{C}^{n,k} : \operatorname{sym}(D\mathbf{v})\, d\mathbf{x}\\
&\quad + \lambda \int_\Omega \operatorname{div}(\mathbf{w}) \operatorname{div}(\mathbf{v})\, d\mathbf{x}\\
&\quad + 2\mu_c \int_\Omega \big(\operatorname{skew}(D\mathbf{w}) - \bar{C}\big) : \big(\operatorname{skew}(D\mathbf{v}) - \bar{B}\big)\, d\mathbf{x}\\
&\quad + 2\mu L_c^2 \int_\Omega D\bar{C} \cdot D\bar{B}\, d\mathbf{x}\ .
\end{aligned}$$

S5) Compute $(\mathbf{w}^{n,k}, \bar{C}^{n,k}) \in \mathbf{V}_h(\mathbf{0}) \times W(\mathbf{0})$ solving the linear variation problem

$$a^{n,k}[(\mathbf{w}^{n,k}, \bar{C}^{n,k}), (\mathbf{v}, \bar{B})] = -F^{n,k}[\mathbf{v}, \bar{B}], \qquad (\mathbf{v}, \bar{B}) \in \mathbf{V}_h(\mathbf{0}) \times W_h(\mathbf{0})\ ;$$

set $\mathbf{u}^{n,k+1} = \mathbf{u}^{n,k} + \mathbf{w}^{n,k+1}$, $\bar{A}^{n,k+1} = \bar{A}^{n,k} + \bar{C}^{n,k+1}$, and go to the next iteration step $k := k + 1$ in S2).

In general, we cannot guarantee that the iteration is globally convergent without damping. Therefore, the algorithm can be extended by a time stepping control which adapts the time step such that the nonlinear iteration is convergent within a small number of steps.

9. Numerical experiments

In this section we discuss numerical test calculations for the infinitesimal elasto-plastic Cosserat model. Therefore, we study a benchmark problem in perfect plasticity (see [**39**]). The computations are realized in the finite element code M++ [**70**] supporting parallel multigrid methods.

The geometry and the boundary conditions are illustrated in Fig. 1: let $\Omega = (0,10) \times (0,10) \setminus B_1(10,0)[\mathrm{mm}^2]$ be a quarter of a rectangle with a hole. The Dirichlet data arising by symmetry are given by

$$u_1(10,x_2) = 0, \quad \bar{A}(10,x_2) = \mathbf{0}, \quad x_2 \in (1,10),$$
$$u_2(x_1,0) = 0, \quad \bar{A}(x_1,0) = \mathbf{0}, \quad x_1 \in (0,9)\,,$$

and the load functional given by

$$\ell(t,\mathbf{v}) = 100\;t \int_0^{10} \mathbf{v}(x_1,10)\,dx_1\;.$$

depends linearly on the loading parameter $t \geq 0$.

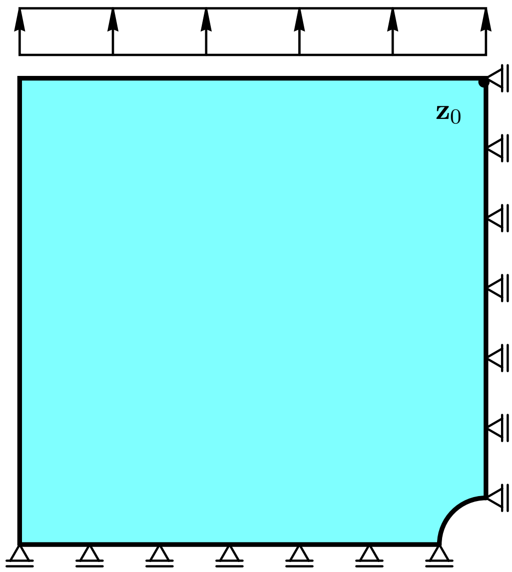

Figure 1: Geometry and boundary conditions for the test problem. For the numerical comparisons, we evaluate the vertical displacement at the upper right corner $\mathbf{z}_0 = (10,10)$.

The material parameters for this benchmark are given in Tab. 1, and for the internal length parameter L_c in the Cosserat model we choose a small value in order to keep the elastic Cosserat effect small.

Poisson ratio	ν	0.29
Young modulus	E	206900.00 [N/mm^2]
yield stress	K_0	450.00 [N/mm^2]
Cosserat internal length parameter	L_c	1/48 [mm]
Cosserat couple modulus	μ_c	0,...,10μ [N/mm^2]

Table 1: Parameters for the Cosserat model for infinitesimal perfect plasticity with von Mises yield criterion. For various test computations a different Cosserat couple modulus is used. In the algorithms we use the Lamé parameter $\mu = E/(2(1+\nu))$ and $\lambda = E\nu/((1+\nu)(1-2\nu))$ and the compression modulus $\kappa = \frac{2}{3}\mu + \lambda$.

We use bilinear finite elements on quadrilaterals. The coarse mesh on level 0 (cf. Fig. 2) is refined uniformly. For the numerical results we use up to 3 151 875 unknowns on refinement level 6. In Tab. 2 and 3 we test the convergence of the discrete model with respect to space and time. Results for the deformations, the microrotations, the plastic strain are illustrated in Fig. 2, 3 and 4.

level	1	2	3	4	5	6
unknowns	3267	12675	49923	198147	789507	3151875
$h_{\max}$	1.48	0.78	0.40	0.20	0.10	0.05
$u_2(\mathbf{z}_0, 1)$	0.0046533	0.0046550	0.0046554	0.0046555	0.0046556	0.0046556
$u_2(\mathbf{z}_0, 3)$	0.0140210	0.0140295	0.0140317	0.0140324	0.0140325	0.0140325
$u_2(\mathbf{z}_0, 4)$	0.0190866	0.0191066	0.0191124	0.0191138	0.0191142	0.0191143
$u_2(\mathbf{z}_0, 4.25)$	0.0208431	0.0208952	0.0209105	0.0209145	0.0209155	0.0209158
$u_2(\mathbf{z}_0, 4.5)$	0.0240566	0.0243134	0.0243963	0.0244190	0.0244249	0.0244263
$u_2(\mathbf{z}_0, 4.73)$	0.0384216	0.0514969	0.0723472	0.0944045	0.1075278	0.1123997

Table 2: Convergence with respect to the mesh size h for $\mu_c = \mu$ for a fixed time series with $\Delta t_{\max} = 0.25$. The vertical displacement component u_2 is evaluated at the point $\mathbf{z}_0 = (10, 10)^T$ for different loads.

level	2	3	4	5
$\Delta t_{\max} = 1.0$	0.0243135	0.0243963	0.0244190	0.0244249
$\Delta t_{\max} = 0.5$	0.0243135	0.0243963	0.0244190	0.0244249
$\Delta t_{\max} = 0.25$	0.0243134	0.0243963	0.0244190	0.0244249
$\Delta t_{\max} = 0.125$	0.0243134	0.0243962	0.0244189	0.0244248
$\Delta t_{\max} = 0.0625$	0.0243132	0.0243960	0.0244187	0.0244246

Table 3: Convergence in h and $\Delta t_{\max}$ for $\mu_c = \mu$ and $t = 4.5$. By comparing the convergence rate in time and space we conclude that the error in space is dominating. Moreover, comparing with the results in Tab. 2 we observe that the results on level 4 are correct up to 4 digits.

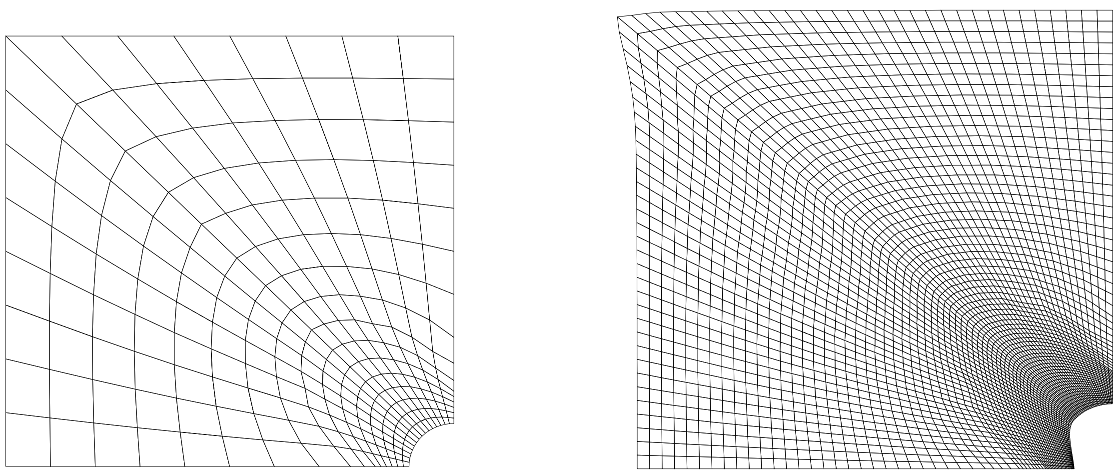

Figure 2: Coarse mesh (level 0, 256 quadrilaterals) for the benchmark problem (left) and deformed mesh on level 2 (4096 quadrilaterals) for the Cosserat model with $\mu_c = \mu$ at $t = 4.9$ (right).

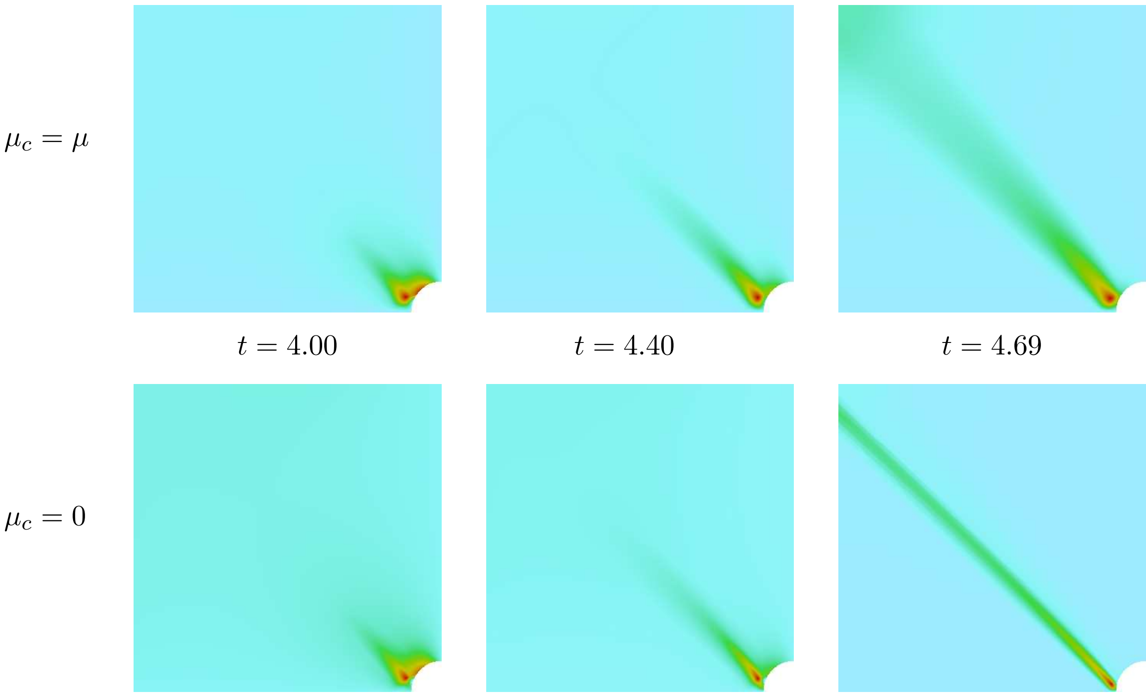

Figure 3: The distribution of the Cosserat microrotations $|\bar{A}|$ for $\mu_c = \mu$ is compared (on refinement level 4) with the continuum rotations $(1/2)(D_{12}\mathbf{u} - D_{21}\mathbf{u})$ for the model of perfect plasticity ($\mu_c = 0$). Due to the symmetry boundary conditions, the microrotations are zero on the right and the lower boundary.

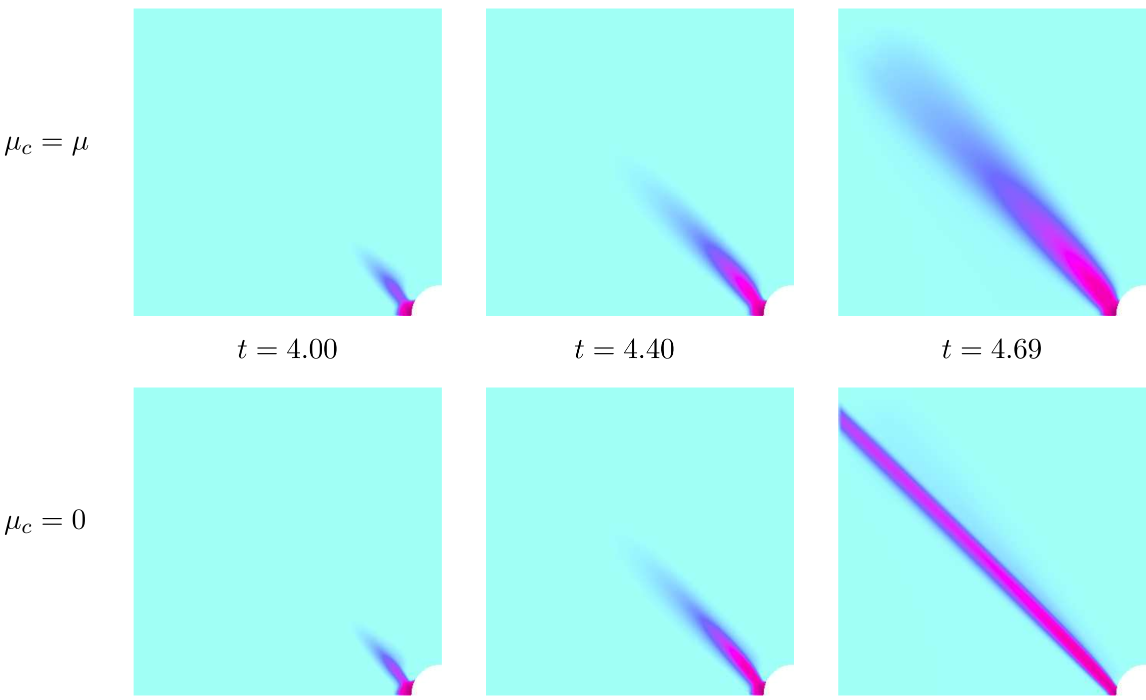

Figure 4: Distribution of the effective plastic strain for the Cosserat model with $\mu_c = \mu$ and for Prandtl-Reuss ($\mu_c = 0$) on refinement level 4.

Finally, we test the limit behavior for $\mu_c \longrightarrow 0$. The load displacement curve in Fig. 5 shows the regularization effect of the Cosserat model, and in Tab. 4 the convergence to perfect plasticity is tested. The numerical results clearly confirm theoretical results in Th. 12.

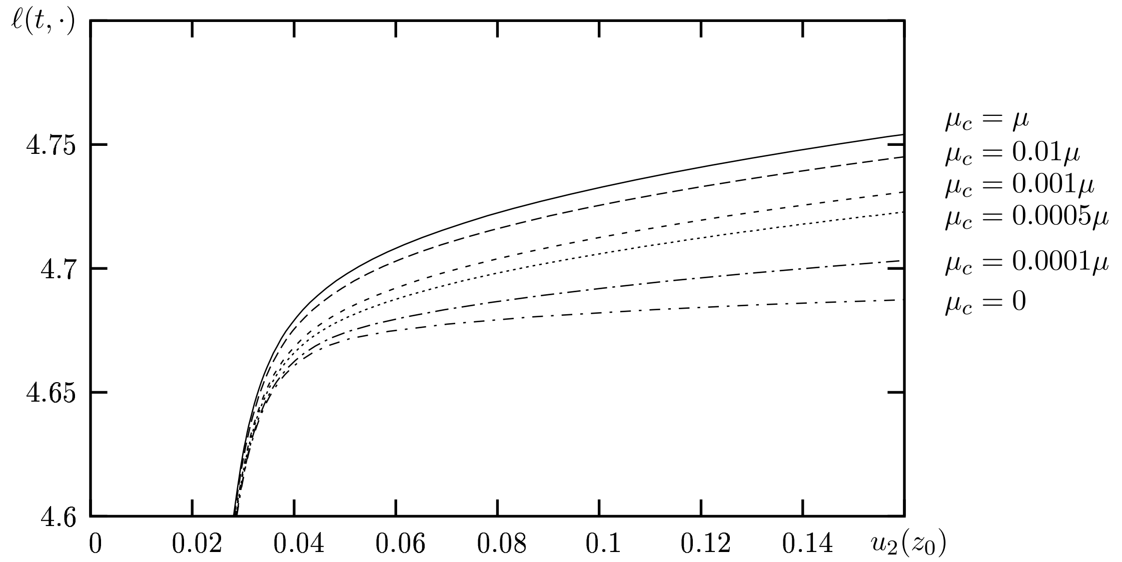

Figure 5: Load-displacement curve on refinement level 4 for different $\mu_c \in [0, \mu]$. The displacement $\mathbf{u} = (u_1, u_2)$ is evaluated at the point $\mathbf{z}_0 = (10, 10)^T$. For $t < 4.5$ there is nearly no significant difference in the solutions.

μ_c/μ	$u_2(\mathbf{z}_0, 1)$	$u_2(\mathbf{z}_0, 3)$	$u_2(\mathbf{z}_0, 4)$	$u_2(\mathbf{z}_0, 4.4)$	$u_2(\mathbf{z}_0, 4.6)$
1	0.004655	0.014032	0.019113	0.022586	0.028123
0.1	0.004655	0.014032	0.019114	0.022592	0.028158
0.01	0.004655	0.014032	0.019117	0.022608	0.028262
0.0016	0.004655	0.014033	0.019119	0.022633	0.028450
0.0008	0.004655	0.014033	0.019120	0.022641	0.028527
0.0004	0.004655	0.014033	0.019120	0.022647	0.028592
0.0002	0.004655	0.014033	0.019121	0.022652	0.028641
0.0001	0.004655	0.014033	0.019121	0.022655	0.028673
0	0.004655	0.014033	0.019121	0.022659	0.028720

Table 4: Convergence for $\mu_c \longrightarrow 0$ the displacement evaluated at the point $\mathbf{z}_0 = (10, 10)^T$ with $\Delta t_{\max} = 0.0625$ on level 4. For small values of μ_c we observe linear convergence of the Cosserat model to perfect plasticity.

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