Sebastian Rudolph

Relational Exploration

Combining Description Logics and Formal Concept Analysis for Knowledge Specification

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Contents

Introduction

In the Palace of the Vatican, on a wall of a room called *stanza della segnatura*, the visitor encounters the fresco *The School of Athens* created by the famous renaissance painter Raphael around 1509. It shows renowned representatives of the ancient *artes liberales* led by philosophy. Right in the center of this masterpiece, one finds the certainly most brilliant Greek philosophers (besides Socrates): Plato and Aristotle. While Plato points up indicating his belief in the primacy of ideas situated in a higher world of forms, Aristotle holds his hand palm downwards suggesting a more grounded epistemic view, where real world facts (or in modern terms: the data) govern the way how concepts and classifications are formed. These two perspectives can be traced through the entire occidental philosophy occurring here and there in varying guises (cf. also the *problem of universals* or the bipolar setting *rationalism* vs. *empirism* brought up in the 19th century).
These antagonistic approaches personified by the two philosophers seem also characteristic for the underlying ways of thinking in two fields of knowledge processing and representation, we want to deal with in our work: *Formal Concept Analysis (FCA)* and *Description Logic (DL)*.

The – more Aristotelian – mathematical theory of Formal Concept Analysis came into being some twenty years ago as the attempt to model (hierarchies of) concepts in terms of lattice theory. It is based on the dualistic understanding of concepts as consisting of concept extent (i.e., all entities belonging to that concept) and concept intent (i.e., all attributes characterizing it).
FCA has proven solid in theory but also quite intuitive in representing conceptual knowledge, also for mathematically less skilled people. Thus, it has been successfully applied in various areas beyond mathematics. Besides this representational capabilities, algorithms assisting knowledge acquisition have been developed, implemented and used in practice.
In recent years, FCA has been developed further and extended by the project

of *contextual logic*, widening the scope of interest from *concepts* to *judgments*. In the course of these developments, the relational aspect has been increasingly emphasized.

Description Logics being a collective term for a family of knowledge representation formalisms are the result of a development starting in the 1970s with frame based systems and semantic networks.
The common underlying idea is to characterize classes of entities (resp. objects) of a domain and organize them hierarchically. In this, the similarities to FCA become apparent. However – thus identifying this approach as a Platonic one – for forming those hierarchies and characterizing classes by using so called terminologies, DL does not presuppose any knowledge about concrete domain entities. Moreover, from the very beginning, DL has put great emphasis on the relationships between the described classes – beyond the subclass-superclass relationship induced by the hierarchical ordering.
A guiding principle in this research was to combine logical expressiveness with computationally effective automated reasoning (as a minimal demand, reasoning should always be decidable).
In turn, this led to the development and implementation of computationally optimized reasoning algorithms for highly expressive logics, which is certainly one of the main reasons for the fact that DL formalisms are well established in practice and constitute the foundations of nowadays Semantic Web standards (cf. OWL – the web ontology language).

In this work, we want to exploit synergies between the two described research areas and show how results from either field can be mutually fruitful for both disciplines. We use DL formalisms for defining FCA attributes and FCA exploration techniques to obtain or refine DL Knowledge Representation specifications. More generally, DL exploits FCA techniques for interactive knowledge acquisition and FCA benefits from DL in terms of expressing relational knowledge.

Relational information can be found everywhere. Therefore, it is essential for instances dealing with knowledge – be it human beings or knowledge processing systems – not only to conceptually classify entities of a domain of discourse as isolated objects, but also to describe the inter*relation*ships among them. Consequently, relational properties are commonly used in human conceptual thinking

to define new concepts, as everyday terms like "wife", "mother", "son" (as well as all terms describing *relati*ves), "disciplinarian", or "neighbor" demonstrate.
Consequently, relational information is also a considerable fraction of human *background knowledge* which constitutes the basis of human communication, knowledge common to most people due to similar physical or sociocultural experiences. Yet, obviously, this kind of knowledge can not be presupposed in any kind of computer based knowledge processing system – it would have to be explicitly specified. Respective attempts, first addressed within the scope of AI research, anew gained importance as the Semantic Web idea evolved. In these areas, such specifications are usually referred to as ontologies.
Clearly, the process of conceptually specifying a domain cannot dispense of human contribution. However, although all information needed in order to describe a domain is in general implicitly present, its explicit formal specification may nevertheless be tedious for the expert. Moreover, it might remain unclear, whether a specification is complete, i.e., whether all specifiable information valid in the considered domain is indeed contained in the specification.
Hence, it would be beneficial to dispose of a method that organizes and structures the specification process by successively asking single questions to the domain expert in a way which minimizes the expert's effort (in particular, it does not ask redundant questions) and guarantees that the resulting specification will be complete in the sense stated above.
In our work, we present such an algorithm for *relational exploration*.

Chapter 1 introduces elementary notions of FCA with focus on attribute logic and the technique of attribute exploration which will play a major role in our further considerations.
Chapter 2 gives basic notions and definitions referring to a particular DL denoted by $\mathcal{FLE}$. Some of these notions are slightly adapted to a more FCA-apt point of view, some even uncommon to DL, but useful for developing our results.
Chapter 3 – certainly containing the deepest mathematical results of our work – deals with cumulated clauses, expressions commonly used in FCA to specify background knowledge. We present a deduction calculus for cumulated clauses on $\mathcal{FLE}$ and prove its soundness and completeness. Moreover, we show how the corresponding results can be even used to define an algorithm to decide entailment of this kind of cumulated clauses.
Chapter 4 introduces DL expressions as attributes into FCA and shows the cor-

respondence between DL class inclusions and implications on special formal contexts. Thus, it justifies the goal to use FCA techniques to obtain DL shaped axioms valid in an investigated domain. We provide the theoretical foundations for a successive exploration on $\mathcal{FLE}$.
Emerging from these results, a corresponding algorithm will be summarized and explicitly described in Chapter 5.
Chapter 6 addresses the question whether the previously described knowledge acquisition procedure terminates after finitely many steps. A necessary and sufficient criterion for this case will be provided.
Chapter 7 gives a simple mathematical example, demonstrating how the algorithm can be applied.
Chapter 8 tries to give an impression how the described methods and techniques can be applied in practice, namely in the field of ontology creation and refinement, by supplying a means for the structured search for domain axioms.
Chapter 9 finally sketches some possible ways how the results and methods achieved in this work can be generalized and extended by future (resp. already ongoing) research.

We try to give an intuitive understanding of our definitions, theorems, and proofs wherever possible. Nevertheless, we wanted to be absolutely precise in our proofs. Therefore, some passages might seem somewhat formal, technical, or too detailed. So, we beg your understanding for our decision to favor accurateness above smoothness.

Some last remarks concerning the layout of this work: As usual, Theorems and Definitions are differentiated from the regular text by italics. Additionally, we have put the proofs into a smaller font without serifs in order to assist orientation in the text. So, it should be possible to identify the function of a paragraph on the first glance and possibly skip passages that are obvious or too technical.

Chapter 1

Formal Concept Analysis and Attribute Exploration: Basic Notions

In Formal Concept Analysis (FCA), the terms *concept* and *conceptual hierarchy* are mathematically formulated, based on the philosophical ideas of intension and extension (as brought up by the Logic of Port-Royal, Pascal, and Leibniz). It thereby develops conceptual thinking in terms of lattice theory (for the classical representation, see [Bi67]). The results exert significant influence on the scientific areas of conceptual data analysis and knowledge processing. Many of them can be (and have been) implemented and used in practice. An especially useful technique is the attribute exploration algorithm (see e.g. [Ga84] and [Ga87]).
In this chapter, we will briefly sketch the basic terminology of FCA as well as known results that are important for our further work, mainly following the FCA standard reference [GaWi99b].

1.1 Closure Operators

Closure operators are nearly ubiquitous in FCA and constitute a basic notion being extensively used throughout this work. The closure of a set can be understood as a minimal extension of it in order to fulfill certain properties.

Definition 1.1 *Let M be an arbitrary set. A function $\mathfrak{cl} : \mathcal{P}(M) \to \mathcal{P}(M)$ will be called* CLOSURE OPERATOR *on M if it is*

1. EXTENSIVE, *i.e.*, $A \subseteq \mathfrak{cl}(A)$ *for all* $A \subseteq M$,

2. MONOTONE, *i.e.*, $A \subseteq B \Rightarrow \mathfrak{cl}(A) \subseteq \mathfrak{cl}(B)$ *for all* $A, B \subseteq M$, *and*

3. IDEMPOTENT, *i.e.*, $\mathfrak{cl}(\mathfrak{cl}(A)) = \mathfrak{cl}(A)$ *for all* $A \subseteq M$.

A set $A \subseteq M$ *will be called* CLOSED *(or* $\mathfrak{cl}$-CLOSED *in case of ambiguity), if* $\mathfrak{cl}(A) = A$.
The set of all closed sets $\{A \mid A = \mathfrak{cl}(A) \subseteq M\}$ *will be called* CLOSURE SYSTEM. ◇

It is easy to show that for an arbitrary closure system $\mathcal{S}$, the corresponding closure operator $\mathfrak{cl}$ can be reconstructed by

$$\mathfrak{cl}(A) = \bigcap_{B \in \mathcal{S}, A \subseteq B} B.$$

So there is a one-to-one connection between a closure operator and the accordant closure system.

1.2 Formal Contexts and Formal Concepts

Everything in Formal Concept Analysis starts from *formal contexts*. These mathematical structures are used in practice to describe various kinds of data. They can be visualized and understood as so called cross tables. For given *objects* and *attributes*, the cross table indicates which objects have which attributes. Mathematically, this correspondency is expressed by a binary relation:

Definition 1.2 *A* FORMAL CONTEXT $\mathbb{K}$ *is a triple* (G, M, I) *with*

- *an arbitrary set* G *called* OBJECTS,
- *an arbitrary set* M *called* ATTRIBUTES,
- *a relation* $I \subseteq G \times M$ *called* INCIDENCE RELATION

We read gIm *as: "object* g *has attribute* m*".* ◇

This basic data structure now can be used to define operations on sets of objects or attributes, respectively.

Definition 1.3 *Let* $\mathbb{K} = (G, M, I)$ *be a formal context. We define a function* $(.)^I : \mathcal{P}(G) \rightarrow \mathcal{P}(M)$ *with*

$$\widetilde{G}^I := \{m \mid gIm \ \text{ for all } g \in \widetilde{G}\}$$

for $\widetilde{G} \subseteq G$. *Furthermore, we use the same notation to define the function* $(.)^I$: $\mathcal{P}(M) \rightarrow \mathcal{P}(G)$ *where*

$$\widetilde{M}^I := \{g \mid gIm \ \text{ for all } m \in \widetilde{M}\}$$

for $\widetilde{M} \subseteq M$.
For convenience, we sometimes write g^I *instead of* $\{g\}^I$ *and* m^I *instead of* $\{m\}^I$. ◇

Applied to an object set, this function yields all attributes common to these objects; by applying it to an attribute set we get the set of all objects having those attributes. The following facts are consequences of the above definitions:

- $(.)^{II}$ is a closure operator on G as well as on M.
- For $A \subseteq G$, A^I is a $(.)^{II}$-closed set and dually
- for $B \subseteq M$, B^I is a $(.)^{II}$-closed set.

The next definition shows how a conceptual hierarchy can be built from a formal context.

Definition 1.4 *Given a formal context* $\mathbb{K} = (G, M, I)$ *a* FORMAL CONCEPT *is a pair* (A, B) *with* $A \subseteq G$, $B \subseteq M$, $A = B^I$, *and* $B = A^I$.
We call the set A EXTENT *and the set* B INTENT *of the concept* (A, B).
Let (A_1, B_1) *and* (A_2, B_2) *be formal concepts of a formal context. We call* (A_1, B_1) *a* SUBCONCEPT *of* (A_2, B_2) *(written:* $(A_1, B_1) \leq (A_2, B_2)$*) if* $A_1 \subseteq A_2$. *Then,* (A_2, B_2) *will be called* SUPERCONCEPT *of* (A_1, B_1). ◇

It is well known from FCA that the set of all formal concepts of a formal context together with the subconcept-superconcept-order form a complete lattice, the so called *concept lattice*. Infimum and supremum therein can be calculated by

$$\bigwedge_{t\in T}(A_t, B_t) = \left(\bigcap_{t\in T} A_t, \left(\bigcup_{t\in T} B_t\right)^{II}\right)$$

$$\bigvee_{t\in T}(A_t, B_t) = \left(\left(\bigcup_{t\in T} A_t\right)^{II}, \bigcap_{t\in T} B_t\right).$$

1.3 Implications

In this section, we will briefly introduce the basics of *attribute logic*. Given a set of attributes, *implications* on that set are logical expressions that can be used to describe certain attribute correspondencies which are valid for all objects of a formal context.

Definition 1.5 *Let M be an arbitrary set. An* IMPLICATION *on M is a pair (A, B) with $A, B \subseteq M$. To support intuition we write $A \rightarrow B$ instead of (A, B).*
We say an implication $A \rightarrow B$ HOLDS *for an attribute set C (also: C* RESPECTS *$A \rightarrow B$), if $A \not\subseteq C$ or $B \subseteq C$. Moreover, an implication $\mathfrak{i}$* HOLDS *(or: is* VALID*) in a formal context $\mathbb{K} = (G, M, I)$ if it holds for all sets g^I with $g \in G$. We then write $\mathbb{K} \models \mathfrak{i}$.*
The IMPLICATIONAL THEORY *$\mathfrak{Th}_{imp}(\mathbb{K})$ of a formal context $\mathbb{K} = (G, M, I)$ is the set of all implications that hold in $\mathbb{K}$.*
Given a set $A \subseteq M$ and a set $\mathfrak{I}$ of implications on M, we write $A^{\mathfrak{I}}$ for the smallest set that

- *contains A and*
- *respects all implications from $\mathfrak{I}$.*

(Since those two requirements are preserved under intersection, the existence of a smallest such set is assured).[1] ◇

[1] Note that this notation differs from that in [GaWi99b]. It has been chosen for better readability.

It is obvious that for any set $\mathfrak{I}$ of implications on M, $(.)^{\mathfrak{I}}$ is a closure operator on M. Furthermore, it can be easily shown that an implication $A \to B$ is valid in a formal context $\mathbb{K} = (G, M, I)$ exactly if $B \subseteq A^{II}$.

Definition 1.6 *We say an implication* $\mathfrak{i}$ FOLLOWS (SEMANTICALLY) *from a set* $\mathfrak{I}$ *of implications on* M*, if any subset of* M *that respects all implications from* $\mathfrak{I}$ *also respects* $\mathfrak{i}$*.* ◇

It is well known that an implication $\mathfrak{i}$ follows semantically from an implication set $\mathfrak{I}$ exactly if $\mathfrak{i}$ is derivable from $\mathfrak{I}$ via the Armstrong rules (see [Ar74]).[2]

Definition 1.7 *The* ARMSTRONG RULES *for implications on a set* M *are the following:*

$$\frac{}{X \to X} \quad \textit{identity}$$

$$\frac{X \to Y}{X \cup Z \to Y} \quad \textit{premise extension}$$

$$\frac{X \to Y \quad Y \cup Z \to W}{X \cup Z \to W} \quad \textit{substitution}$$

with $W, X, Y, Z \subseteq M$*.* ◇

In [DoGa84], it has been shown that semantical entailment of implications can be decided in linear time with respect to the cardinality of $\mathfrak{I}$. This also enables the fast calculation of $A^{\mathfrak{I}}$ for a given $A \subseteq M$. One natural question is that for a minimal set of implications that generates (in terms of semantical entailment) a given implicational theory (e.g. that of a formal context). The precise definition of that notion will be given by the next definition.

Definition 1.8 *An implication set* $\mathfrak{I}$ *will be called* NON-REDUNDANT*, if for any* $\mathfrak{i} \in \mathfrak{I}$ *we have that* $\mathfrak{i}$ *does not follow semantically from* $\mathfrak{I} \setminus \{\mathfrak{i}\}$*.*
An implication set $\mathfrak{I}$ *of a context* $\mathbb{K}$ *will be called* COMPLETE *if every implication valid in* $\mathbb{K}$ *follows semantically from* $\mathfrak{I}$*.*
$\mathfrak{I}$ *will be called an* IMPLICATIONAL BASE *of a formal context* $\mathbb{K}$ *if it is non-redundant and complete.* ◇

[2]We assume the reader to be familiar with the notion and notation of deduction rules.

In the sequel, a way to construct a canonical implicational base for a given formal context is described. The notion of *pseudo-intent* is needed for this.

Definition 1.9 *For a formal context $\mathbb{K} = (G, M, I)$, a set $P \subseteq M$ will be called* PSEUDO-INTENT *if $P^{II} \neq P$ and $Q^{II} \subseteq P$ holds for every pseudo-intent $Q \subsetneq P$.* ◇

Note that this definition is recursive. Since the set M is always assumed to be finite in the sequel, it is nevertheless correct.[3] With this notion, a canonical implicational base can be easily described (see [GuDu86] or [Du87]).

Theorem 1.10 *For a given formal context $\mathbb{K}$, the set*

$$\mathcal{L} := \{P \to P^{II} \mid P \text{ pseudo-intent}\}$$

is an implicational base.

This result justifies to call the implication set $\mathcal{L}$ STEM BASE of $\mathbb{K}$. This particular implicational base is successively constructed in the attribute exploration algorithm which will be described in the next section. Before, we will cite some results needed to justify this algorithm.

Lemma 1.11 *The set of all intents and pseudo-intents of a formal context constitutes a closure system.*

The corresponding closure operator $\mathfrak{cl}_Q$ for this closure system can be described by

$$\mathfrak{cl}_Q(P) := \bigcup_{n \in \mathbb{N}} P_n$$

with $P_0 := P$ and

$$P_{n+1} := P_n \cup \bigcup_{Pseudointent\ Q \subset P_n} Q^{II}.$$

Definition 1.12 *Let $(M, \prec)$ be an arbitrary linear strict order and $\mathfrak{cl}$ a closure operator on M. Then we define for $A, B \subseteq M$ and $m \in M$:*

$$A \oplus m := \mathfrak{cl}((A \cap \{a \mid a \prec m\}) \cup \{m\})$$
$$A <_m B :\Leftrightarrow m \in B \setminus A \text{ and } A \cap \{a \mid a \prec m\} = B \cap \{a \mid a \prec m\}$$
$$A < B :\Leftrightarrow A <_m B \text{ for one } m \in M$$

Obviously, $(\mathcal{P}_{\text{fin}}(M), <)$ is a linear strict order, which we will call the LECTIC ORDER. ◇

[3] However, for the applicability of this definition, also weaker conditions than finiteness are sufficient.

In the sequel, if we speak of the lectic order on subsets of M, we think of the lectic order with respect to an arbitrary (but fixed) linear strict order on M.
Note that – irrespective of how $\prec$ is chosen – $(\mathcal{P}_{\text{fin}}(M), <)$ is always a linearization of the strict order $(\mathcal{P}_{\text{fin}}(M), \subset)$, i.e., $A \subset B$ implies $A < B$ for any $A, B \subseteq M$.
The next lemma provides a way to enumerate all intents and pseudo-intents of a formal context in lectic order using the closure operator defined above.

Lemma 1.13 *The lectically smallest intent or pseudo-intent is $\emptyset$. Given a set P, the lectically next intent or pseudo-intent can be determined as follows: find the (with respect to $\prec$) maximal $m \in M$ such that $P \cap \{a \mid a \prec m\} = (P \oplus m) \cap \{a \mid a \prec m\}$ (where $\oplus$ is defined with respect to $\mathfrak{cl}_Q$). If this exists, $P \oplus m$ is the lectically next intent or pseudo-intent, otherwise there is no intent or pseudo-intent lectically greater than P.*

By defining the lectic order in the way we did, we ensure the following: When a certain set $A \in M$ is checked for $\mathfrak{cl}_Q$-closedness all pseudo-intents $P \subsetneq A$ have already been determined before. Thus the closedness checking can be done without recursion, if all previously determined pseudo-intents are stored.
The following lemma is essential for the attribute exploration technique as a means of acquiring information in an interactive process. It shows that the lectically first n pseudo-intents keep this property if the underlying context is modified (namely: extended) in a certain way.

Lemma 1.14 *Let $\mathbb{K} = (G, M, I)$ be a context and let $P_1, \ldots, P_n$ be the lectically first n pseudo-intents of $\mathbb{K}$. If we set $\widetilde{\mathbb{K}} := (G \cup \{g\}, M, I \cup (g \times \{\widetilde{M}\}))$ with $\widetilde{M} \subseteq M$ respecting all $P_i \to P_i^{II}$ with $1 \leq i \leq n$ then those P_i are the lectically first n pseudo-intents of $\widetilde{\mathbb{K}}$.*

1.4 Attribute Exploration

In this section, we describe the basic attribute exploration algorithm. The aim of this algorithm is to completely determine the implicational knowledge about a certain domain (also called the universe).
We presuppose an instance – called the expert – possessing complete knowledge about the universe and hence able to answer all questions about it. The attribute exploration algorithm can be seen as an organized way to acquire the knowledge by asking as few questions to the expert as possible.

Note that attribute exploration is based on a kind of *closed world assumption*: the expert is supposed to know all entities of the universe. Consequently, an entity added to the universe a posteriori could radically change its implicational theory.

1.4.1 Plain Attribute Exploration

The domain to explore is formalized as a formal context $\mathbb{K} = (U, M, I)$. Usually, it is not known completely in advance. However, possibly, some entities of the universe $g \in G_0 \subseteq U$ are already known, as well as their associated attributes g^I. Let $\underline{\mathbb{K}} := (G_0, M, J_0)$ with $J_0 = I \cap (G_0 \times M)$.
Now, we start enumerating the intents and pseudo-intents of $\underline{\mathbb{K}}$ as described in Lemma 1.13 and present for every such set S with $S \neq S^{J_0 J_0}$ the question "Does the implication $S \rightarrow S^{J_0 J_0}$ hold in the context $\mathbb{K} = (U, M, I)$?" to the human expert.
The expert might confirm this. In this case, S is a pseudo-intent of (U, M, I) and therefore $S \rightarrow S^{J_0 J_0}$ is archived as part of $\mathbb{K}$'s stem base $\mathcal{L}$. We proceed the enumeration with the pseudo-intent lectically next to S.
The other case would be that $S \rightarrow S^{J_0 J_0}$ does not hold in (U, M, I). But then there must exist a $g \in U$ with $S \in g^I$ and $S^{J_0 J_0} \notin g^I$. The expert is asked to input this g and g^I, which is used to update the context $\underline{\mathbb{K}}$ in the obvious way: we set $G_1 := G_0 \cup \{g\}$ and $J_1 := J_0 \cup (\{g\} \times g^I)$. We continue the enumeration with S (and J_1 instead of J_0). Notice that - thank to Lemma 1.14 - all information that has been explored so far remains valid.
This procedure terminates when all pseudo-intents and intents have been enumerated. Then, the implicational knowledge of the universe is completely acquired, i.e., the closure operator $(.)^{II}$ coincides with $(.)^{J_n J_n}$ and $(.)^{\mathcal{L}}$.
Figure 1.1 illustrates the interactive exploration process.

1.4.2 Background Knowledge

Now, we will describe how we can deal with background knowledge. In general, the term background knowledge denotes any information putting constraints on $\mathbb{K}$. In particular, this information may be non-implicational, which would prevent a direct incorporation into the stem base. Nevertheless, such knowledge can have an impact on the implicational theory, inasmuch as it could make implications

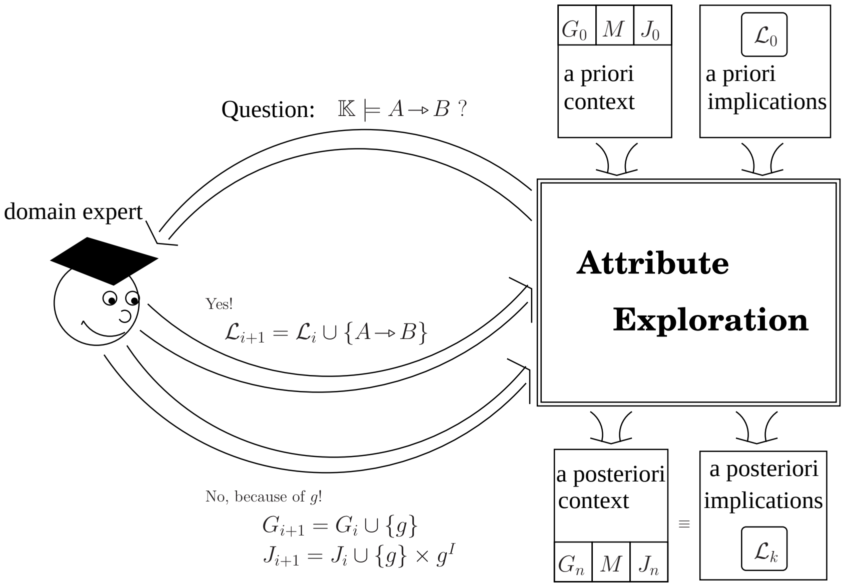

Figure 1.1: Scheme of the attribute exploration algorithm

follow from other implications even if they are not derivable using the Armstrong rules.

Example 1.15 Let $\mathbb{K} := (G, M, I)$ with $M := \{a, b, c, d\}$ and let be additionally known that for any $g \in G$ with gIa, we have gIb or gIc. Then, from the implication set $\{b \to d, c \to d\}$ follows semantically the implication $a \to d$ (since in every formal context with the above property whenever $b \to d$ and $c \to d$ hold also $a \to d$ holds).

To incorporate background knowledge we would need an instance that is capable of deciding whether an implication is entailed by the background knowledge and the implicational knowledge acquired so far. Any implication asked by the enumeration algorithm described above would then first be passed to this decision procedure. If the validity of this implication can be confirmed, it is just "silently" added to the stem base. Otherwise, the question will be directed to the expert. So in a certain way, the decision procedure is used as a strainer; only questions not decidable by means of the knowledge already specified are asked.

Now, we will present two different ways how this can be realized.

Test contexts.

First, observe that for $\mathbb{K}_1 = (G_1, M, I_1)$ and $\mathbb{K}_2 = (G_2, M, I_2)$ from $\{g^{I_1} \mid g \in G_1\} = \{g^{I_2} \mid g \in G_2\}$, it directly follows that $\mathfrak{Th}_{imp}(\mathbb{K}_1) = \mathfrak{Th}_{imp}(\mathbb{K}_2)$. Thus, the set $\{(\mathcal{M}, M, \ni) \mid \mathcal{M} \subseteq \mathcal{P}(M)\}$ comprises all contexts different with respect to their attribute logic. Background knowledge then can be used to decide, whether a certain context $(\mathcal{M}, M, \ni)$ possibly represents the universe (i.e., whether there is a $\mathbb{K} = (G, M, I)$ compatible with all our background knowledge and $\{g^I \mid g \in G\} = \mathcal{M}$). In this case, $\mathcal{M}$ will be called NON-CONTRADICTORY.

Example 1.16 Let M be a set of attributes with $m_1, m_2 \in M$ and let m_2 be "designed" to be the negation of m_1, i.e., an object g is stipulated to have an attribute m_2 exactly if it has not the attribute m_1. If this is the only restriction, $\{(\mathcal{M}, M, \ni)\}$ is non-contradictory if and only if for every $\widetilde{M} \in \mathcal{M}$ either $m_1 \in \widetilde{M}$ or $m_2 \in \widetilde{M}$.

Now, consider the TEST CONTEXT

$$\overline{\mathbb{K}} := (\bigcup\{\mathcal{M} \mid \mathcal{M} \text{ non-contradictory}\}, M, \ni).$$

Note that the object set of this context is not necessarily non-contradictory. However, it is easy to see that an implication on M holds in this context if and only if it holds in every context $(\mathcal{M}, M, \ni)$ with non-contradictory $\mathcal{M}$. Thus we know $\mathfrak{Th}_{imp}(\overline{\mathbb{K}}) \subseteq \mathfrak{Th}_{imp}(\mathbb{K})$. Every potential implication emerging from the algorithm can then be checked against $\overline{\mathbb{K}}$ and silently confirmed if found to be valid therein.[4]

Note that, if the implicational knowledge about U increases (i.e., if the expert confirms an implication during the exploration process), this influences the notion of non-contradictoriness and therefore the test context has to be recalculated.

This way to decide background knowledge consequences can be seen as an extensional one: it takes into account all "possible worlds" with respect to a set of constraints.

[4]On the first glance, this explanation may seem unnecessarily overcomplicated. In fact, if the non-contradictoriness of a context can be decided object-wise (i.e., for every object we can say whether it is "admissible" independently from the presence or absence of any other object), the test context can be defined directly. However, if the background knowledge contains restrictions as for instance "either all objects have the attribute m_1 or all objects have the attribute m_2", we have to proceed like this.

Reasoning procedures.

Another – intensional – way is to use a procedure that (given the complete background knowledge specified in some logical formalism) is capable of deciding whether this background knowledge together with the formerly acquired implications entails the asked implication. One common example for such a formalism are the so called cumulated clauses (a decision procedure for their entailment has been presented in [Kr98]). Depending on the interpretation of the attributes, also other algorithms (DL-reasoners, theorem provers, model checkers or logic programs to name just a few) could be used for this task. In Chapter 3, we will elaborate a way to provide such a method for cumulated clauses on a special kind of attributes.

Chapter 2

The Description Logic $\mathcal{FLE}$: Basic Notions

In this chapter, we will introduce the Description Logic notions necessary for our work – for a detailed and concise treatise see [Ba03]. In general, the term *Description Logic* comprises an amount of different formalisms which vary in expressiveness and decision procedure complexity. This allows to use a certain logic "tailored" to one's needs in practice. The common principle for DLs is to form complex concept descriptions out of simple ones using so called *concept constructors*.

DL formalisms are used to represent terminological knowledge of an application domain. Such a representation is usually called *knowledge base* and normally consists of two parts: a so called *TBox* introducing a *terminology* and an *ABox* containing *assertions*.

In the TBox, we can specify how concept descriptions are related to each other. This allows to define concepts by other concepts as well as to state constraints valid in the considered domain. The used formalism to do so are *general concept inclusion axioms* – rules that have to hold throughout the domain. Nevertheless, in the TBox, no propositions are made about specific entities of the domain.

Contrary, in the ABox, individuals are introduced and named. Furthermore one can specify, which concept descriptions they belong to. However, one basic maxim in DL is the *open world assumption*: the ABox is by no means thought of containing all entities of the described domain. It just demands the existence of some distinguished ones.

In our work, we focus on the rather simple description logic $\mathcal{FLE}$ which includes conjunction, existential and universal quantification. The reason to use this DL for our work (besides the necessity to keep expressiveness low for complexity reasons) is mainly psychological: We aim at an application where questions formulated in a DL are asked to a human expert. As has been shown by numerous studies (summarized in [Bo66]) a human's capability of learning and using a particular concept (this concept's *psychological complexity*) depends on its logical form. As a central result, conjunctive concepts have been found subjectively simpler than disjunctive ones. Likewise, negation has shown unfavorable in this regard. Hence, by using $\mathcal{FLE}$, the questions asked to the expert consist of comparatively intuitive concepts and can therefore be answered more easily. Moreover, they probably represent facts which are more "interesting" for humans.

Consequently, we use $\mathcal{FLE}$ concept descriptions to describe the entities of a certain domain. More precisely, we will employ the $\mathcal{FLE}$ formalism to define attributes from a given structure with unary and binary predicates. This is also the reason why we define notions of *subsumption* and *equivalence* with respect to one fixed *interpretation*. Apart from these special notions and the restricted language $\mathcal{FLE}^{\mathrm{norm}}$ introduced in Section 2.2, we just recall well known basic DL notions in this chapter.

2.1 The Language $\mathcal{FLE}$: Syntax and Semantics

In this section, we will present a way of constructing so called *concept descriptions* from two given attribute sets, which yields us the $\mathcal{FLE}$ language.

Definition 2.1 *Let $M_\mathcal{C}$ and $M_\mathcal{R}$ be arbitrary finite sets the elements of which we will call* CONCEPT NAMES[1] *and* ROLE NAMES*, respectively. By $\mathcal{FLE}(M_\mathcal{C}, M_\mathcal{R})$ (or shortly: $\mathcal{FLE}$ if there can be no confusion) we denote the set of* CONCEPT

[1] Whenever in this work we use the term *concept*, we refer to the notion used in Description Logic. If we want to refer to the meaning used in Formal Concept Analysis we use *formal concept*.

DESCRIPTIONS *being inductively defined as follows:*[2]

$$\begin{array}{rcl} M_{\mathcal{C}} \cup \{\top, \bot\} & \subseteq & \mathcal{FLE} \\ \mathtt{C}, \mathtt{D} \in \mathcal{FLE} & \Rightarrow & \mathtt{C} \sqcap \mathtt{D} \in \mathcal{FLE} \\ \mathtt{C} \in \mathcal{FLE}, \mathtt{R} \in M_{\mathcal{R}} & \Rightarrow & \exists\mathtt{R}.\mathtt{C} \in \mathcal{FLE} \\ \mathtt{C} \in \mathcal{FLE}, \mathtt{R} \in M_{\mathcal{R}} & \Rightarrow & \forall\mathtt{R}.\mathtt{C} \in \mathcal{FLE} \end{array}$$

◇

Note (since literature is not uniform in this regard) that our definition of $\mathcal{FLE}$ contains the bottom concept.

The next step after having defined the language of concept descriptions is to provide a semantics. In order to do this, we first define interpretations. In our case, an interpretation is formalized as *binary power context family.* Although not the usual way of representing an interpretation in DL, it is just a trivial reformulation and – in our point of view – facilitates definitions and aids conceptual thinking in this work.

Definition 2.2 *Let Δ be an arbitrary nonempty set called the* UNIVERSE. *The elements of the universe will be called* ENTITIES. *A* BINARY POWER CONTEXT FAMILY $\vec{\mathbb{K}}$ *on Δ is a pair $(\mathbb{K}_{\mathcal{C}}, \mathbb{K}_{\mathcal{R}})$ consisting of the formal contexts $\mathbb{K}_{\mathcal{C}} := (G_{\mathcal{C}}, M_{\mathcal{C}}, I_{\mathcal{C}})$ and $\mathbb{K}_{\mathcal{R}} := (G_{\mathcal{R}}, M_{\mathcal{R}}, I_{\mathcal{R}})$ with $G_{\mathcal{C}} = \Delta$ and $G_{\mathcal{R}} = \Delta \times \Delta$.*
Moreover, we assume $M_{\mathcal{C}}$ and $M_{\mathcal{R}}$ to be finite.
If for $\delta_1, \delta_2 \in \Delta$ we have $(\delta_1, \delta_2) I_{\mathcal{R}} \mathtt{R}$ for some $\mathtt{R} \in M_{\mathcal{R}}$, we call δ_2 the R-NEIGHBOR *of δ_1.* ◇

Note that our notion of binary power context family is a special case of the power context families used in FCA to encode relational structures (see e.g. [Wi97]). In binary power context families, only unary and binary relations can be expressed.[3]

[2]In DL terminology, it is usual to denote concept descriptions as well as role names by capital letters. We decided to abide by this convention, but – in order to avoid possible confusion (with other capital letters denoting sets) – we use typewriter font ($\mathtt{A}, \mathtt{B}, \mathtt{C}$) for $\mathcal{FLE}$ concept descriptions and calligraphic letters ($\mathcal{A}, \mathcal{B}, \mathcal{C}$) for sets of concept descriptions. Furthermore, we use the symbols $\exists$ and $\forall$ for "role quantification" to clearly distinguish them from the "ordinary" quantifiers $\exists$ and $\forall$ occurring in some proofs and definitions.

[3]Nevertheless, it is possible to deal with relations of higher arity by an adequate redefinition of concept and role names as well as the way models are interpreted. This technique is called *reification* (see [Ba03]).

Note that there is a trivial one-to-one correspondence to Kripke structures (being used as the usual models for modal logic, see e.g. [Bla01]) or labelled transition systems with attributes (LTSA, see [GaRu01]).
Next, we describe an extensional semantics for the above defined concept descriptions: for a given binary power context family $\vec{\mathbb{K}} = ((\Delta, M_\mathcal{C}, I_\mathcal{C}), (\Delta\times\Delta, M_\mathcal{R}, I_\mathcal{R}))$ we assign to each concept description $\mathtt{C} \in \mathcal{FLE}(M_\mathcal{C}, M_\mathcal{R})$ a set $A \in \mathcal{P}(\Delta)$ of entities[4] that "fulfill" this concept description.

Definition 2.3 *The* INTERPRETATION FUNCTION $[\![.]\!]_{\vec{\mathbb{K}}} : \mathcal{FLE}(M_\mathcal{C}, M_\mathcal{R}) \to \mathcal{P}(\Delta)$ *for a binary power context family* $\vec{\mathbb{K}}$ *on a universe* Δ *with attribute sets* $M_\mathcal{C}$ *and* $M_\mathcal{R}$ *is defined recursively as follows:*

$$\begin{array}{rcl}
[\![\top]\!]_{\vec{\mathbb{K}}} & := & \Delta \\
[\![\bot]\!]_{\vec{\mathbb{K}}} & := & \emptyset \\
[\![m]\!]_{\vec{\mathbb{K}}} & := & m^{I_\mathcal{C}} \text{ for all } m \in M_\mathcal{C} \\
[\![\mathtt{C} \sqcap \mathtt{D}]\!]_{\vec{\mathbb{K}}} & := & [\![\mathtt{C}]\!]_{\vec{\mathbb{K}}} \cap [\![\mathtt{D}]\!]_{\vec{\mathbb{K}}} \\
[\![\exists \mathtt{R}.\mathtt{C}]\!]_{\vec{\mathbb{K}}} & := & \{\delta_1 \in \Delta \mid \exists \delta_2 \in \Delta : (\delta_1, \delta_2) \in \mathtt{R}^{I_\mathcal{R}} \wedge \delta_2 \in [\![\mathtt{C}]\!]_{\vec{\mathbb{K}}}\} \text{ for } \mathtt{R} \in M_\mathcal{R} \\
[\![\forall \mathtt{R}.\mathtt{C}]\!]_{\vec{\mathbb{K}}} & := & \{\delta_1 \in \Delta \mid \forall \delta_2 \in \Delta : (\delta_1, \delta_2) \in \mathtt{R}^{I_\mathcal{R}} \to \delta_2 \in [\![\mathtt{C}]\!]_{\vec{\mathbb{K}}}\} \text{ for } \mathtt{R} \in M_\mathcal{R}.
\end{array}$$

For $\delta \in [\![\mathtt{C}]\!]_{\vec{\mathbb{K}}}$ *we will occasionally write* $\delta \models \mathtt{C}$ *and say* $\mathtt{C}$ *is* VALID *in* δ*. A whole set* $\mathcal{C}$ *of concept descriptions is* VALID *in* δ *(written:* $\delta \models \mathcal{C}$*) if* $\delta \models \mathtt{C}$ *for all* $\mathtt{C} \in \mathcal{C}$*.*
A concept description $\mathtt{D}$ $\vec{\mathbb{K}}$-SUBSUMES *a concept description* $\mathtt{C}$ *(write:* $\mathtt{C} \sqsubseteq_{\vec{\mathbb{K}}} \mathtt{D}$*) if* $[\![\mathtt{C}]\!]_{\vec{\mathbb{K}}} \subseteq [\![\mathtt{D}]\!]_{\vec{\mathbb{K}}}$*.*
A concept description $\mathtt{D}$ SUBSUMES *a concept description* $\mathtt{C}$ UNIVERSALLY *(write:* $\mathtt{C} \sqsubseteq \mathtt{D}$*) if* $\mathtt{C} \sqsubseteq_{\vec{\mathbb{K}}} \mathtt{D}$ *for all* $\vec{\mathbb{K}}$ *with attribute sets* $M_\mathcal{C}$ *and* $M_\mathcal{R}$*.*
Two concept descriptions $\mathtt{C}$ *and* $\mathtt{D}$ *are called* $\vec{\mathbb{K}}-$EQUIVALENT *(write:* $\mathtt{C} \equiv_{\vec{\mathbb{K}}} \mathtt{D}$*) if* $\mathtt{C} \sqsubseteq_{\vec{\mathbb{K}}} \mathtt{D}$ *and* $\mathtt{D} \sqsubseteq_{\vec{\mathbb{K}}} \mathtt{C}$ [5] *and* UNIVERSALLY EQUIVALENT *(write:* $\mathtt{C} \equiv \mathtt{D}$*) if this is the case for all* $\vec{\mathbb{K}}$ *with attribute sets* $M_\mathcal{C}$ *and* $M_\mathcal{R}$*.* ◇

Remark 2.4 It follows directly from the semantics definition that for any $\mathcal{FLE}$ concept descriptions $\mathtt{C}, \mathtt{D}, \mathtt{E}$ the composed concept descriptions $(\mathtt{C} \sqcap \mathtt{D}) \sqcap \mathtt{E}$ and $\mathtt{C} \sqcap (\mathtt{D} \sqcap \mathtt{E})$ are universally equivalent. The same holds for $\mathtt{C} \sqcap \mathtt{D}$ and $\mathtt{D} \sqcap \mathtt{C}$ as well as for $\mathtt{C} \sqcap \mathtt{C}$ and $\mathtt{C}$. In the sequel we will exploit these facts in several ways:

[4] Throughout this work, $\mathcal{P}$ will denote the powerset and $\mathcal{P}_{\text{fin}}$ the finite powerset.
[5] As an immediate consequence we have $\mathtt{C} \equiv_{\vec{\mathbb{K}}} \mathtt{D}$ exactly if $[\![\mathtt{C}]\!]_{\vec{\mathbb{K}}} = [\![\mathtt{D}]\!]_{\vec{\mathbb{K}}}$.

- We will omit all parentheses which are not necessary.
- We will make extensive use of the following abbreviation:

  Let $\mathcal{C} = \{\mathtt{C}_1, \ldots, \mathtt{C}_n\}$ be a finite set of $\mathcal{FLE}$ concept descriptions. Then the concept description $\mathtt{C}_1 \sqcap \ldots \sqcap \mathtt{C}_n$ will be abbreviated by $\bigsqcap \mathcal{C}$. We extend this definition in an intuitive way for $|\mathcal{C}| < 2$ by setting $\bigsqcap\{\mathtt{C}\} := \mathtt{C}$ and $\bigsqcap \emptyset := \top$. This "syntactic sugar" could then be directly incorporated into the semantics by adding

  $$[\![\bigsqcap \mathcal{C}]\!]_{\overline{\mathbb{K}}} := \bigcap_{\mathtt{C} \in \mathcal{C}} [\![\mathtt{C}]\!]_{\overline{\mathbb{K}}}$$

  to Definition 2.3.
- We will consider all concept descriptions which can be transformed into each other by the equivalences mentioned above as syntactically equal[6], i.e. we write for instance

  $$(\mathtt{C} \sqcap \mathtt{D}) \sqcap \mathtt{E} = (\mathtt{E} \sqcap \mathtt{C}) \sqcap \mathtt{D} = \bigsqcap\{\mathtt{C}, \mathtt{D}, \mathtt{E}\}.$$

A notion we will need in the sequel is the maximal role depth of an $\mathcal{FLE}$ concept description, indicating how deep the role quantifiers $\forall$ and $\exists$ are nested in that concept description. The formal definition is as follows:

Definition 2.5 *The* MAXIMAL ROLE DEPTH *of an $\mathcal{FLE}$ concept description is given by the function $\mathfrak{rd} : \mathcal{FLE} \to \mathbb{N}$ recursively defined as follows:*

$$\begin{aligned} \mathfrak{rd}(\top) &:= 0 \\ \mathfrak{rd}(\bot) &:= 0 \\ \mathfrak{rd}(m) &:= 0 \text{ for all } m \in M_{\mathcal{C}} \\ \mathfrak{rd}(\mathtt{C} \sqcap \mathtt{D}) &:= \max(\mathfrak{rd}(\mathtt{C}), \mathfrak{rd}(\mathtt{D})) \\ \mathfrak{rd}(\exists \mathtt{R}.\mathtt{C}) &:= \mathfrak{rd}(\mathtt{C}) + 1 \text{ for } \mathtt{R} \in M_{\mathcal{R}} \\ \mathfrak{rd}(\forall \mathtt{R}.\mathtt{C}) &:= \mathfrak{rd}(\mathtt{C}) + 1 \text{ for } \mathtt{R} \in M_{\mathcal{R}} \end{aligned}$$

For $n \in \mathbb{N}$ we define $\mathcal{FLE}_n := \{\mathtt{C} \mid \mathtt{C} \in \mathcal{FLE}, \mathfrak{rd}(\mathtt{C}) \leq n\}$. ◇

[6] We are aware that this argumentation is a bit inaccurate. Actually, we would have to define an equivalence relation, say $\approx$, on $\mathcal{FLE}$ capturing the universal equivalence of the concept descriptions mentioned above and subsequently consider the elements of the factorized structure $\mathcal{FLE}/_{\approx}$ as actual concept descriptions.

Remark 2.6 Due to the facts that we treat conjunctions as syntactically equal as explained in Remark 2.4 and we demand $M_{\mathcal{C}}$ and $M_{\mathcal{R}}$ to be finite, we can conclude that for any $n \in \mathbb{N}$ there are only finitely many concept descriptions with maximal role depth $\leq n$.

2.2 $\mathcal{FLE}^{\text{norm}}$ – Reduced, yet Complete

Consider a binary power context family $\overrightarrow{\mathbb{K}}$ and let δ be an entity from $\overrightarrow{\mathbb{K}}$. Then, knowing that the concept descriptions $\mathtt{C}$ and $\mathtt{D}$ are valid in δ, we automatically know that $\mathtt{C} \sqcap \mathtt{D}$ is valid in δ as well. Therefore, one could ask for a "test set" $\mathcal{S} \subset \mathcal{FLE}$ with the following property: Knowing for every concept description from $\mathcal{S}$ whether it is valid in δ allows to conclude for an arbitrary $\mathcal{FLE}$ concept description whether it is valid in δ. We will define a concept description set $\mathcal{FLE}^{\text{norm}}$ and show in the next theorem that it has this desired property.

Definition 2.7 *The set* $\mathcal{FLE}^{\text{norm}}$ *of* NORMALIZED $\mathcal{FLE}$ CONCEPT DESCRIPTIONS *is an* $\mathcal{FLE}$ *subset defined in the following way:*

$$\begin{array}{rcl} M_{\mathcal{C}} \cup \{\bot\} & \subseteq & \mathcal{FLE}^{\text{norm}} \\ \mathcal{C} \in \mathcal{P}_{\text{fin}}(\mathcal{FLE}^{\text{norm}}), \bot \notin \mathcal{C}, \mathtt{R} \in M_{\mathcal{R}} & \Rightarrow & \exists\mathtt{R}.\bigsqcap \mathcal{C} \in \mathcal{FLE}^{\text{norm}} \\ \mathtt{C} \in \mathcal{FLE}^{\text{norm}}, \mathtt{R} \in M_{\mathcal{R}} & \Rightarrow & \forall\mathtt{R}.\mathtt{C} \in \mathcal{FLE}^{\text{norm}} \end{array}$$

Additionally, for any $i \in \mathbb{N}$ *let* $\mathcal{FLE}_i^{\text{norm}} = \mathcal{FLE}^{\text{norm}} \cap \mathcal{FLE}_i$. ◇

Theorem 2.8 *For every* $\mathcal{FLE}$ *concept description* $\mathtt{C}$, *there is a set* $\mathcal{C}$ *of* $\mathcal{FLE}^{\text{norm}}$ *concept descriptions such that*

$$\mathtt{C} \equiv \bigsqcap \mathcal{C}.$$

Proof:

We define a function $\mathfrak{n} : \mathcal{FLE} \to \mathcal{P}(\mathcal{FLE}^{\text{norm}})$ in a recursive manner by

$$\begin{array}{lcl} \mathfrak{n}(\top) & = & \emptyset \\ \mathfrak{n}(\bot) & = & \{\bot\} \\ \mathfrak{n}(\mathtt{C}) & = & \{\mathtt{C}\} \text{ for } \mathtt{C} \in M_{\mathcal{C}} \\ \mathfrak{n}(\forall\mathtt{R}.\mathtt{C}) & = & \{\forall\mathtt{R}.\mathtt{D} \mid \mathtt{D} \in \mathfrak{n}(\mathtt{C})\} \\ \mathfrak{n}(\exists\mathtt{R}.\mathtt{C}) & = & \begin{cases} \{\bot\} \text{ if } \bot \in \mathfrak{n}(\mathtt{C}) \\ \{\exists\mathtt{R}.\bigsqcap \mathfrak{n}(\mathtt{C}) \text{ otherwise} \end{cases} \\ \mathfrak{n}(\bigsqcap \mathcal{C}) & = & \bigcup_{\mathtt{C} \in \mathcal{C}} \mathfrak{n}(\mathtt{C}) \end{array}$$

that calculates $\mathcal{C}$ from $\mathtt{C}$. We will prove $\mathtt{C} \equiv \bigsqcap \mathfrak{n}(\mathtt{C})$ via induction on the maximal role depth of $\mathtt{C}$:

- $[\![\top]\!]_{\overline{\mathbb{K}}} = \Delta = \bigcap \emptyset = [\![\bigsqcap\{\emptyset\}]\!]_{\overline{\mathbb{K}}}$
- $[\![\bot]\!]_{\overline{\mathbb{K}}} = \emptyset = \bigcap\{\emptyset\} = \bigcap [\![\bot]\!]_{\overline{\mathbb{K}}} = [\![\bigsqcap\{\bot\}]\!]_{\overline{\mathbb{K}}}$
- $[\![\mathtt{C}]\!]_{\overline{\mathbb{K}}} = [\![\bigsqcap\{\mathtt{C}\}]\!]_{\overline{\mathbb{K}}}$. (Remember that this was the reason, why we identified these expressions even syntactically in Remark 2.4.)
- $$\begin{aligned} & [\![\forall\mathtt{R.C}]\!]_{\overline{\mathbb{K}}} \\ = \; & \{\delta_1 \in \Delta \mid \forall \delta_2 \in \Delta : (\delta_1,\delta_2) \in \mathtt{R}^{I_\mathcal{R}} \to \delta_2 \in [\![\mathtt{C}]\!]_{\overline{\mathbb{K}}}\} \\ = \; & \{\delta_1 \in \Delta \mid \forall \delta_2 \in \Delta : (\delta_1,\delta_2) \in \mathtt{R}^{I_\mathcal{R}} \to \delta_2 \in [\![\bigsqcap \mathfrak{n}(\mathtt{C})]\!]_{\overline{\mathbb{K}}}\} \\ = \; & \{\delta_1 \in \Delta \mid \forall \delta_2 \in \Delta : (\delta_1,\delta_2) \in \mathtt{R}^{I_\mathcal{R}} \to \delta_2 \in \textstyle\bigcap_{\mathtt{D}\in\mathfrak{n}(\mathtt{C})} [\![\mathtt{D}]\!]_{\overline{\mathbb{K}}}\} \\ = \; & \{\delta_1 \in \Delta \mid \forall \delta_2 \in \Delta : (\delta_1,\delta_2) \in \mathtt{R}^{I_\mathcal{R}} \to \textstyle\bigwedge_{\mathtt{D}\in\mathfrak{n}(\mathtt{C})} \delta_2 \in [\![\mathtt{D}]\!]_{\overline{\mathbb{K}}}\} \\ = \; & \{\delta_1 \in \Delta \mid \forall \delta_2 \in \Delta : \textstyle\bigwedge_{\mathtt{D}\in\mathfrak{n}(\mathtt{C})} (\delta_1,\delta_2) \in \mathtt{R}^{I_\mathcal{R}} \to \delta_2 \in [\![\mathtt{D}]\!]_{\overline{\mathbb{K}}}\} \\ = \; & \{\delta_1 \in \Delta \mid \textstyle\bigwedge_{\mathtt{D}\in\mathfrak{n}(\mathtt{C})} \forall \delta_2 \in \Delta : (\delta_1,\delta_2) \in \mathtt{R}^{I_\mathcal{R}} \to \delta_2 \in [\![\mathtt{D}]\!]_{\overline{\mathbb{K}}}\} \\ = \; & \textstyle\bigcap_{\mathtt{D}\in\mathfrak{n}(\mathtt{C})} \{\delta_1 \in \Delta \mid \forall \delta_2 \in \Delta : (\delta_1,\delta_2) \in \mathtt{R}^{I_\mathcal{R}} \to \delta_2 \in [\![\mathtt{D}]\!]_{\overline{\mathbb{K}}}\} \\ = \; & \textstyle\bigcap_{\mathtt{D}\in\mathfrak{n}(\mathtt{C})} [\![\forall\mathtt{R.D}]\!]_{\overline{\mathbb{K}}} \\ = \; & [\![\bigsqcap \textstyle\bigcup_{\mathtt{C}\in\mathcal{C}} \{\forall\mathtt{R.D} \mid \mathtt{D} \in \mathfrak{n}(\mathtt{C})\}]\!]_{\overline{\mathbb{K}}} \end{aligned}$$
- $$\begin{aligned} & [\![\exists\mathtt{R.C}]\!]_{\overline{\mathbb{K}}} \\ = \; & \{\delta_1 \in \Delta \mid \exists \delta_2 \in \Delta : (\delta_1,\delta_2) \in \mathtt{R}^{I_\mathcal{R}} \wedge \delta_2 \in [\![\mathtt{C}]\!]_{\overline{\mathbb{K}}}\} \end{aligned}$$
  Case 1: $\bot \notin \bigcup_{\mathtt{C}\in\mathcal{C}} \mathfrak{n}(\mathtt{C})$
  $$\begin{aligned} = \; & \{\delta_1 \in \Delta \mid \exists \delta_2 \in \Delta : (\delta_1,\delta_2) \in \mathtt{R}^{I_\mathcal{R}} \wedge \delta_2 \in [\![\bigsqcap \textstyle\bigcup_{\mathtt{C}\in\mathcal{C}} \mathfrak{n}(\mathtt{C})]\!]_{\overline{\mathbb{K}}}\} \\ = \; & [\![\exists\mathtt{R}.\bigsqcap \textstyle\bigcup_{\mathtt{C}\in\mathcal{C}} \mathfrak{n}(\mathtt{C})]\!]_{\overline{\mathbb{K}}} \end{aligned}$$
  Case 2: $\bot \in \bigcup_{\mathtt{C}\in\mathcal{C}} \mathfrak{n}(\mathtt{C})$
  $$\begin{aligned} = \; & \{\delta_1 \in \Delta \mid \exists \delta_2 \in \Delta : (\delta_1,\delta_2) \in \mathtt{R}^{I_\mathcal{R}} \wedge \delta_2 \in \emptyset\} \\ = \; & \emptyset \\ = \; & [\![\bot]\!]_{\overline{\mathbb{K}}} \end{aligned}$$
- $$\begin{aligned} & [\![\bigsqcap \mathcal{C}]\!]_{\overline{\mathbb{K}}} \\ = \; & \textstyle\bigcap_{\mathtt{C}\in\mathcal{C}} [\![\mathtt{C}]\!]_{\overline{\mathbb{K}}} \\ = \; & \textstyle\bigcap_{\mathtt{C}\in\mathcal{C}} [\![\bigsqcap \mathfrak{n}(\mathtt{C})]\!]_{\overline{\mathbb{K}}} \\ = \; & \textstyle\bigcap_{\mathtt{C}\in\mathcal{C}} \bigcap\{[\![\mathtt{D}]\!]_{\overline{\mathbb{K}}} \mid \mathtt{D} \in \mathfrak{n}(\mathtt{C})\} \\ = \; & \textstyle\bigcap\{\bigcap\{[\![\mathtt{D}]\!]_{\overline{\mathbb{K}}} \mid \mathtt{D} \in \mathfrak{n}(\mathtt{C})\} \mid \mathtt{C} \in \mathcal{C}\} \\ = \; & \textstyle\bigcap \bigcup_{\mathtt{C}\in\mathcal{C}} \{[\![\mathtt{D}]\!]_{\overline{\mathbb{K}}} \mid \mathtt{D} \in \mathfrak{n}(\mathtt{C})\} \\ = \; & \in \textstyle\bigcap\{[\![\chi]\!]_{\overline{\mathbb{K}}} \mid \chi \in \bigcup_{\mathtt{C}\in\mathcal{C}} \mathfrak{n}(\mathtt{C})\} \\ = \; & [\![\bigsqcap \textstyle\bigcup_{\mathtt{C}\in\mathcal{C}} \mathfrak{n}(\mathtt{C})]\!]_{\overline{\mathbb{K}}} \end{aligned}$$

❑

Obviously, this theorem provides a way to check on the basis of the "test set" $\mathcal{FLE}^{\text{norm}}$ whether $\delta \models \mathtt{C}$ for any $\mathtt{C} \in \mathcal{FLE}$. This is the case exactly if $\mathfrak{n}(\mathtt{C}) \subseteq \mathcal{FLE}^{\text{norm}} \cap \{\mathtt{D} \mid \delta \models \mathtt{D}\}$.

This fact will prove helpful in the next sections since most of our considerations and proofs will have to deal only with $\mathcal{FLE}^{\text{norm}}$ but will propagate to whole $\mathcal{FLE}$.

Chapter 3

Cumulated Clauses on $\mathcal{FLE}^{\text{norm}}$

Cumulated clauses on attributes have been studied and used in FCA as a means of specifying knowledge about interrelationships between attributes (see [GaWi99a] and [Ga96]). A deduction calculus for cumulated clauses on logically opaque attributes has been presented in [Kr98]. In particular, they have been used to encode background knowledge for the attribute exploration process described in Section 1.4. It can be easily shown that in the case of propositional logic any logical formula can equivalently be expressed by a set of cumulated clauses on the atomic propositions.
In this chapter, we will consider cumulated clauses on $\mathcal{FLE}^{\text{norm}}$. The fact that attributes of this kind have an internal logical structure exerts influence on the corresponding clause logic. We will deal with these issues by presenting a sound and complete deduction calculus as well as a decision procedure for the entailment of cumulated clauses on $\mathcal{FLE}^{\text{norm}}$.

3.1 Definition and Deduction Calculus

This section introduces the notion *cumulated clause* being a generalization of implications as defined in Section 1.3. Furthermore, we will present a deduction calculus for cumulated clauses on $\mathcal{FLE}^{\text{norm}}$ concept descriptions and show its soundness and completeness.

Definition 3.1 *Given an arbitrary set M, a* CUMULATED CLAUSE *on M is an element from $\mathcal{CC}(M) := \mathcal{P}_{\text{fin}}(M) \times \mathcal{P}_{\text{fin}}\mathcal{P}_{\text{fin}}(M)$.*
To support intuition, we write $\mathcal{A} \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n\}$ instead of $(\mathcal{A}, \{\mathcal{B}_1, \ldots, \mathcal{B}_n\})$ for

$\mathcal{A}, \mathcal{B}_1, \ldots, \mathcal{B}_n \subseteq M$ and $n \in \mathbb{N}$.
A set $\mathcal{N} \subseteq M$ is said to RESPECT *a cumulated clause $\mathcal{A} \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n\}$ if $\mathcal{A} \not\subseteq \mathcal{N}$ or $\mathcal{B}_i \subseteq \mathcal{N}$ for some $1 \leq i \leq n$.*
A cumulated clause $\mathfrak{k}$ on $\mathcal{FLE}^{\text{norm}}$ is said to be VALID *in a binary power context family $\overrightarrow{\mathbb{K}}$ (also: $\overrightarrow{\mathbb{K}}$* RESPECTS *$\mathfrak{k}$, written: $\overrightarrow{\mathbb{K}} \models \mathfrak{k}$), if for every $\delta \in \Delta$, the set $\{\mathtt{C} \mid \delta \in [\![\mathtt{C}]\!]_{\overrightarrow{\mathbb{K}}}\}$ respects $\mathfrak{k}$.*[1]
If a cumulated clause $\mathfrak{k}$ is valid in every binary power context family that respects all cumulated clauses from a certain set $\mathfrak{K}$, we say $\mathfrak{k}$ FOLLOWS SEMANTICALLY *from $\mathfrak{K}$ (written $\mathfrak{K} \models \mathfrak{k}$).* ◇

In words, the validity of the cumulated clause $\mathcal{A} \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n\}$ in a power context family means that for every entity $\delta \in \Delta$ with $\delta \models \mathcal{A}$, we have additionally $\delta \models \mathcal{B}_1$ or $\delta \models \mathcal{B}_2$ ... or $\delta \models \mathcal{B}_n$. Obviously, for $n = 1$, the notion of a cumulated clause coincides with that of an implication.

Fact 3.2 *A cumulated clause $\mathcal{A} \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n\}$ is valid in a binary power context family $\overrightarrow{\mathbb{K}}$ iff*

$$\bigcap_{\mathtt{A} \in \mathcal{A}} [\![\mathtt{A}]\!]_{\overrightarrow{\mathbb{K}}} \subseteq \bigcup_{1 \leq i \leq n} \bigcap_{\mathtt{B} \in \mathcal{B}_i} [\![\mathtt{B}]\!]_{\overrightarrow{\mathbb{K}}}.$$

Proof:
We start with the definition of validity and show the equivalence to the statement above:

$$\begin{aligned}
&\forall \delta \in \Delta : \mathcal{A} \not\subseteq \{\mathtt{C} \mid \delta \in [\![\mathtt{C}]\!]_{\overrightarrow{\mathbb{K}}}\} \vee \textstyle\bigvee_{1 \leq i \leq n} \mathcal{B}_i \subseteq \{\mathtt{C} \mid \delta \in [\![\mathtt{C}]\!]_{\overrightarrow{\mathbb{K}}}\} \\
\iff &\forall \delta \in \Delta : \delta \notin \textstyle\bigcap_{\mathtt{A} \in \mathcal{A}} [\![\mathtt{A}]\!]_{\overrightarrow{\mathbb{K}}} \vee \bigvee_{1 \leq i \leq n} \delta \in \bigcap_{\mathtt{B} \in \mathcal{B}_i} [\![\mathtt{B}]\!]_{\overrightarrow{\mathbb{K}}} \\
\iff &\forall \delta \in \Delta : \delta \in \textstyle\bigcap_{\mathtt{A} \in \mathcal{A}} [\![\mathtt{A}]\!]_{\overrightarrow{\mathbb{K}}} \rightarrow \delta \in \bigcup_{1 \leq i \leq n} \bigcap_{\mathtt{B} \in \mathcal{B}_i} [\![\mathtt{B}]\!]_{\overrightarrow{\mathbb{K}}} \\
\iff &\textstyle\bigcap_{\mathtt{A} \in \mathcal{A}} [\![\mathtt{A}]\!]_{\overrightarrow{\mathbb{K}}} \subseteq \bigcup_{1 \leq i \leq n} \bigcap_{\mathtt{B} \in \mathcal{B}_i} [\![\mathtt{B}]\!]_{\overrightarrow{\mathbb{K}}}.
\end{aligned}$$

❑

In the next definition, we present a deduction calculus on $\mathcal{CC}(\mathcal{FLE}^{\text{norm}})$. Informally, we could call this rules the "logic of case distinction on $\mathcal{FLE}$" – disjunction is not allowed to construct new concepts but can be used to express alternative conclusions.

Definition 3.3 *The set $\mathcal{DR}$ of* DERIVATION RULES *consists of the following rules (with $\mathtt{A}, \mathtt{B}, \mathtt{C} \in \mathcal{FLE}^{\text{norm}}$ and $\mathcal{A}, \mathcal{B}_1, \ldots, \mathcal{B}_n, \mathcal{C}, \mathcal{D}_1, \ldots, \mathcal{D}_k \in \mathcal{P}_{\text{fin}}(\mathcal{FLE}^{\text{norm}})$)*

$$\frac{}{\{\bot\} \multimap \{\{\mathtt{A}\}\}} \textit{contradiction} \text{ (CONT)}$$

[1] In the sequel, when using the term cumulated clause, we always mean cumulated clause on $\mathcal{FLE}^{\text{norm}}$.

$$\frac{}{\mathcal{A} \multimap \{\mathcal{A}\}} \textit{identity (ID)}$$

$$\frac{\mathcal{A} \multimap \{\mathcal{B}_1, \dots, \mathcal{B}_n\}}{\mathcal{A} \cup \{\mathtt{C}\} \multimap \{\mathcal{B}_1, \dots, \mathcal{B}_n\}} \textit{premise extension (PE)}$$

$$\frac{\mathcal{A} \multimap \{\mathcal{B}_1, \dots, \mathcal{B}_n\}}{\mathcal{A} \multimap \{\mathcal{B}_1, \dots, \mathcal{B}_n, \mathcal{C}\}} \textit{conclusion extension (CE)}$$

$$\frac{\mathcal{A} \multimap \{\mathcal{B}_1, \dots, \mathcal{B}_n, \mathcal{C}\} \quad \mathcal{A} \cup \mathcal{C} \multimap \{\mathcal{D}_1, \dots, \mathcal{D}_k\}}{\mathcal{A} \multimap \{\mathcal{B}_1, \dots, \mathcal{B}_n, \mathcal{D}_1, \dots, \mathcal{D}_k\}} \textit{substitution (SUB)}$$

$$\frac{\mathcal{A} \multimap \{\mathcal{B}_1, \dots, \mathcal{B}_n\}}{[\exists \mathtt{R}]\mathcal{A} \multimap \{[\exists \mathtt{R}]\mathcal{B}_1, \dots, [\exists \mathtt{R}]\mathcal{B}_n\}} \exists\textit{-lifting (EL)}$$

$$\frac{}{[\exists \mathtt{R}]\mathcal{A} \cup \{\forall \mathtt{R.B}\} \multimap \{[\exists \mathtt{R}](\mathcal{A} \cup \{\mathtt{B}\})\}} \forall\textit{-propagation (AP)}$$

$$\frac{\mathcal{A} \multimap \{\mathcal{B}_1, \dots, \mathcal{B}_n, \mathcal{C}\}}{[\forall \mathtt{R}]\mathcal{A} \multimap \{[\exists \mathtt{R}]\mathcal{B}_1, \dots, [\exists \mathtt{R}]\mathcal{B}_n, [\forall \mathtt{R}]\mathcal{C}\}} \forall\textit{-lifting (AL)}$$

where

$$[\forall \mathtt{R}]\mathcal{A} := \{\forall \mathtt{R.A} \mid \mathtt{A} \in \mathcal{A}\}$$

and

$$[\exists \mathtt{R}]\mathcal{A} := \begin{cases} \{\bot\}, \textit{ if } \bot \in \mathcal{A} \\ \{\exists \mathtt{R}. \bigsqcap \mathcal{A}\} \textit{ otherwise.}^2 \end{cases}$$

Given a set $\mathfrak{K}$ of cumulated clauses, we denote by $\mathcal{DR}(\mathfrak{K})$ the smallest set of cumulated clauses containing $\mathfrak{K}$ and closed under the derivation rules above. For $\mathfrak{k} \in \mathcal{DR}(\mathfrak{K})$ we also write $\mathfrak{K} \vdash \mathfrak{k}$. ◇

The rules ID, PE, CE and SUB have already been described in [Kr98] as deduction rules for cumulated clauses on arbitrary sets. The CONT rule is valid in any formal system, where "universal falsehood" can be expressed by an attribute. EL, AP, and AL represent the interdependence of concept descriptions incorporating $\forall \mathtt{R}$ and $\exists \mathtt{R}$. While the first two are quite intuitive, the last one takes a little consideration.

[2] Note that in view of the semantics definition, we have $\bigsqcap([\forall \mathtt{R}]\mathcal{A}) \equiv \forall \mathtt{R}. \bigsqcap \mathcal{A}$ as well as $\bigsqcap([\exists \mathtt{R}]\mathcal{A}) \equiv \exists \mathtt{R}. \bigsqcap \mathcal{A}$. Moreover, $[\forall \mathtt{R}]\mathcal{C}$ and $[\exists \mathtt{R}]\mathcal{C}$ are subsets of $\mathcal{FLE}^{\text{norm}}$ if $\mathcal{C} \subseteq \mathcal{FLE}^{\text{norm}}$.

3.1.1 Soundness

The following theorem states the soundness of the presented deduction calculus with respect to the semantics defined above. As usual, proving soundness is much easier than showing completeness.

Theorem 3.4 *For $\mathfrak{K} \subseteq \mathcal{CC}(\mathcal{FLE}^{\text{norm}})$ and $\mathfrak{k} \in \mathcal{CC}(\mathcal{FLE}^{\text{norm}})$, we have*

$$\mathfrak{K} \vdash \mathfrak{k} \Longrightarrow \mathfrak{K} \models \mathfrak{k}.$$

Proof:
We will prove this by showing that for every derivation rule

$$\frac{\mathfrak{K}}{\mathfrak{k}} \in \mathcal{DR}$$

with $\mathfrak{K} \subseteq \mathcal{CC}(\mathcal{FLE}^{\text{norm}})$ and $\mathfrak{k} \in \mathcal{CC}(\mathcal{FLE}^{\text{norm}})$ holds that every binary power context family $\vec{\mathbb{K}}$ respecting all cumulated clauses from $\mathfrak{K}$ also respects $\mathfrak{k}$.

- contradiction
  Given an arbitrary $\vec{\mathbb{K}}$, because of $[\![\bot]\!]_{\vec{\mathbb{K}}} = \emptyset$, no $\delta \in \Delta$ fulfills $\delta \models \bot$. So, $\delta \models \bot \Rightarrow \delta \models \mathtt{A}$ is trivially true for every $\mathtt{A} \in \mathcal{FLE}^{\text{norm}}$.

- identity
  In every $\vec{\mathbb{K}}$ trivially holds $\delta \models \mathtt{A} \Rightarrow \delta \models \mathtt{A}$.

- premise extension
  Take a $\vec{\mathbb{K}}$ that respects $\mathcal{A} \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n\}$, i.e., (due to Fact 3.2)

  $$\bigcap_{\mathtt{A} \in \mathcal{A}} [\![\mathtt{A}]\!]_{\vec{\mathbb{K}}} \subseteq \bigcup_{1 \leq i \leq n} \bigcap_{\mathtt{B} \in \mathcal{B}_i} [\![\mathtt{B}]\!]_{\vec{\mathbb{K}}}.$$

  Obviously, $\bigcap_{\mathtt{A} \in \mathcal{A} \cup \{\mathtt{C}\}} [\![\mathtt{A}]\!]_{\vec{\mathbb{K}}} \subseteq \bigcap_{\mathtt{A} \in \mathcal{A}} [\![\mathtt{A}]\!]_{\vec{\mathbb{K}}}$ and thus

  $$\bigcap_{\mathtt{A} \in \mathcal{A} \cup \{c\}} [\![\mathtt{A}]\!]_{\vec{\mathbb{K}}} \subseteq \bigcup_{1 \leq i \leq n} \bigcap_{\mathtt{B} \in \mathcal{B}_i} [\![\mathtt{B}]\!]_{\vec{\mathbb{K}}}$$

  which means (again due to Fact 3.2) that $\mathcal{A} \cup \{\mathtt{C}\} \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n\}$ is valid in $\vec{\mathbb{K}}$.

- conclusion extension
  Take a $\vec{\mathbb{K}}$ that respects $\mathcal{A} \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n\}$, i.e., $\bigcap_{\mathtt{A} \in \mathcal{A}} [\![\mathtt{A}]\!]_{\vec{\mathbb{K}}} \subseteq \bigcup_{1 \leq i \leq n} \bigcap_{\mathtt{B} \in \mathcal{B}_i} [\![\mathtt{B}]\!]_{\vec{\mathbb{K}}}$. Obviously $\bigcup_{1 \leq i \leq n} \bigcap_{\mathtt{B} \in \mathcal{B}_i} [\![\mathtt{B}]\!]_{\vec{\mathbb{K}}} \subseteq \bigcup_{1 \leq i \leq n} \bigcap_{\mathtt{B} \in \mathcal{B}_i} [\![\mathtt{B}]\!]_{\vec{\mathbb{K}}} \cup \bigcap_{\mathtt{C} \in \mathcal{C}} [\![\mathtt{C}]\!]_{\vec{\mathbb{K}}}$ and thus

  $$\bigcap_{\mathtt{A} \in \mathcal{A}} [\![\mathtt{A}]\!]_{\vec{\mathbb{K}}} \subseteq \bigcup_{1 \leq i \leq n} \bigcap_{\mathtt{B} \in \mathcal{B}_i} [\![\mathtt{B}]\!]_{\vec{\mathbb{K}}} \cup \bigcap_{\mathtt{C} \in \mathcal{C}} [\![\mathtt{C}]\!]_{\vec{\mathbb{K}}}$$

  which means that $\mathcal{A} \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n, \mathcal{C}\}$ is valid in $\vec{\mathbb{K}}$.

- substitution
  If $\overrightarrow{\mathbb{K}}$ respects $\mathcal{A} \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n, \mathcal{C}\}$ and $\mathcal{A} \cup \mathcal{C} \multimap \{\mathcal{D}_1, \ldots, \mathcal{D}_k\}$, this can be expressed as

$$\bigcap_{\mathtt{A}\in\mathcal{A}} [\![\mathtt{A}]\!]_{\overrightarrow{\mathbb{K}}} \subseteq \bigcup_{1\le i\le n} \bigcap_{\mathtt{B}\in\mathcal{B}_i} [\![\mathtt{B}]\!]_{\overrightarrow{\mathbb{K}}} \cup \bigcap_{\mathtt{C}\in\mathcal{C}} [\![\mathtt{C}]\!]_{\overrightarrow{\mathbb{K}}}$$

  and

$$\bigcap_{\mathtt{A}\in\mathcal{A}\cup\mathcal{C}} [\![\mathtt{A}]\!]_{\overrightarrow{\mathbb{K}}} \subseteq \bigcup_{1\le i\le k} \bigcap_{\mathtt{D}\in\mathcal{D}_i} [\![\mathtt{D}]\!]_{\overrightarrow{\mathbb{K}}}.$$

  But then we can conclude:

$$\begin{aligned}
&\textstyle\bigcap_{\mathtt{A}\in\mathcal{A}} [\![\mathtt{A}]\!]_{\overrightarrow{\mathbb{K}}}\\
&\subseteq \textstyle\bigcup_{1\le i\le n}\bigcap_{\mathtt{B}\in\mathcal{B}_i} [\![\mathtt{B}]\!]_{\overrightarrow{\mathbb{K}}} \cup \big(\bigcap_{\mathtt{C}\in\mathcal{C}} [\![\mathtt{C}]\!]_{\overrightarrow{\mathbb{K}}} \cap \bigcap_{\mathtt{A}\in\mathcal{A}} [\![\mathtt{A}]\!]_{\overrightarrow{\mathbb{K}}}\big)\\
&= \textstyle\bigcup_{1\le i\le n}\bigcap_{\mathtt{B}\in\mathcal{B}_i} [\![\mathtt{B}]\!]_{\overrightarrow{\mathbb{K}}} \cup \bigcap_{\mathtt{C}\in\mathcal{A}\cup\mathcal{C}} [\![\mathtt{C}]\!]_{\overrightarrow{\mathbb{K}}}\\
&\subseteq \textstyle\bigcup_{1\le i\le n}\bigcap_{\mathtt{B}\in\mathcal{B}_i} [\![\mathtt{B}]\!]_{\overrightarrow{\mathbb{K}}} \cup \bigcup_{1\le i\le k}\bigcap_{\mathtt{D}\in\mathcal{D}_i} [\![\mathtt{D}]\!]_{\overrightarrow{\mathbb{K}}}
\end{aligned}$$

  which means that $\mathcal{A} \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n, \mathcal{D}_1, \ldots, \mathcal{D}_k\}$ is valid in $\overrightarrow{\mathbb{K}}$.

- $\exists$-lifting
  Let $\overrightarrow{\mathbb{K}}$ respect $\mathcal{A} \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n\}$, i.e.,

$$\bigcap_{\mathtt{A}\in\mathcal{A}} [\![\mathtt{A}]\!]_{\overrightarrow{\mathbb{K}}} \subseteq \bigcup_{1\le i\le n} \bigcap_{\mathtt{B}\in\mathcal{B}_i} [\![\mathtt{B}]\!]_{\overrightarrow{\mathbb{K}}}.$$

  Then we have

$$\begin{aligned}
&[\![\exists \mathrm{R}.\textstyle\bigsqcap\mathcal{A}]\!]_{\overrightarrow{\mathbb{K}}}\\
&= \{\delta \mid \exists\widetilde{\delta} : (\delta,\widetilde{\delta}) I_{\mathcal{R}}\mathrm{R} \wedge \widetilde{\delta} \in \textstyle\bigcap_{\mathtt{A}\in\mathcal{A}} [\![\mathtt{A}]\!]_{\overrightarrow{\mathbb{K}}}\}\\
&\subseteq \{\delta \mid \exists\widetilde{\delta} : (\delta,\widetilde{\delta}) I_{\mathcal{R}}\mathrm{R} \wedge \widetilde{\delta} \in \textstyle\bigcup_{1\le i\le n}\bigcap_{\mathtt{B}\in\mathcal{B}_i} [\![\mathtt{B}]\!]_{\overrightarrow{\mathbb{K}}}\}\\
&= \textstyle\bigcup_{1\le i\le n}\{\delta \mid \exists\widetilde{\delta} : (\delta,\widetilde{\delta}) I_{\mathcal{R}}\mathrm{R} \wedge \widetilde{\delta} \in \bigcap_{\mathtt{B}\in\mathcal{B}_i} [\![\mathtt{B}]\!]_{\overrightarrow{\mathbb{K}}}\}\\
&= \textstyle\bigcup_{1\le i\le n} [\![\exists \mathrm{R}.\bigsqcap\mathcal{B}_i]\!]_{\overrightarrow{\mathbb{K}}}
\end{aligned}$$

  which means that $[\exists\mathrm{R}]\mathcal{A} \multimap \{[\exists\mathrm{R}]\mathcal{B}_1, \ldots, [\exists\mathrm{R}]\mathcal{B}_n\}$.

- $\forall$-propagation
  For an arbitrary $\overrightarrow{\mathbb{K}}$ holds

$$\begin{aligned}
&[\![\exists \mathrm{R}.\textstyle\bigsqcap\mathcal{A}]\!]_{\overrightarrow{\mathbb{K}}} \cap [\![\forall\mathrm{R}.\mathtt{B}]\!]_{\overrightarrow{\mathbb{K}}}\\
&= \{\delta \mid \exists\widetilde{\delta} : (\delta,\widetilde{\delta}) I_{\mathcal{R}}\mathrm{R} \wedge \widetilde{\delta} \in \textstyle\bigcap_{\mathtt{A}\in\mathcal{A}} [\![\mathtt{A}]\!]_{\overrightarrow{\mathbb{K}}}\} \cap \{\delta \mid \forall\widetilde{\delta} : (\delta,\widetilde{\delta}) I_{\mathcal{R}}\mathrm{R} \to \widetilde{\delta} \in [\![\mathtt{B}]\!]_{\overrightarrow{\mathbb{K}}}\}\\
&\subseteq \{\delta \mid \exists\widetilde{\delta} : (\delta,\widetilde{\delta}) I_{\mathcal{R}}\mathrm{R} \wedge \widetilde{\delta} \in \textstyle\bigcap_{\mathtt{A}\in\mathcal{A}} [\![\mathtt{A}]\!]_{\overrightarrow{\mathbb{K}}} \wedge \widetilde{\delta} \in [\![\mathtt{B}]\!]_{\overrightarrow{\mathbb{K}}}\}\\
&= \{\delta \mid \exists\widetilde{\delta} : (\delta,\widetilde{\delta}) I_{\mathcal{R}}\mathrm{R} \wedge \widetilde{\delta} \in \textstyle\bigcap_{\mathtt{A}\in\mathcal{A}\cup\{\mathtt{B}\}} [\![\mathtt{A}]\!]_{\overrightarrow{\mathbb{K}}}\}\\
&= [\![\exists \mathrm{R}.\textstyle\bigsqcap\mathcal{A}\cup\{\mathtt{B}\}]\!]_{\overrightarrow{\mathbb{K}}}
\end{aligned}$$

  which means that $[\exists\mathrm{R}]\mathcal{A} \cup \{\forall\mathrm{R}.\mathtt{B}\} \multimap \{[\exists\mathrm{R}](\mathcal{A} \cup \{\mathtt{B}\})\}$.

- $\forall$-lifting
  Assume in $\overrightarrow{\mathbb{K}}$ holds $\mathcal{A} \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n, \mathcal{C}\}$ and therefore we have

$$\bigcap_{\mathtt{A}\in\mathcal{A}} [\![\mathtt{A}]\!]_{\overrightarrow{\mathbb{K}}} \subseteq \bigcup_{1\le i\le n} \bigcap_{\mathtt{B}\in\mathcal{B}_i} [\![\mathtt{B}]\!]_{\overrightarrow{\mathbb{K}}} \cup \bigcap_{\mathtt{C}\in\mathcal{C}} [\![\mathtt{C}]\!]_{\overrightarrow{\mathbb{K}}}$$

We now set $\Delta^* := \{\delta \mid \exists\widetilde{\delta} : (\delta,\widetilde{\delta})I_{\mathcal{R}}\mathtt{R} \wedge \bigvee_{1\le i\le n} \widetilde{\delta} \in \bigcap_{\mathtt{B}\in\mathcal{B}_i} [\![\mathtt{B}]\!]_{\overrightarrow{\mathbb{K}}}\}$ and thereby get $\Delta \setminus \Delta^* = \{\delta \mid \forall\widetilde{\delta} : (\delta,\widetilde{\delta})I_{\mathcal{R}}\mathtt{R} \rightarrow \bigwedge_{1\le i\le n} \widetilde{\delta} \notin \bigcap_{\mathtt{B}\in\mathcal{B}_i} [\![\mathtt{B}]\!]_{\overrightarrow{\mathbb{K}}}\}$ Then also holds

$$
\begin{aligned}
&\bigcap_{\mathtt{A}\in\mathcal{A}} [\![\forall\mathtt{R.A}]\!]_{\overrightarrow{\mathbb{K}}}\\
&= \{\delta \mid \forall\widetilde{\delta} : (\delta,\widetilde{\delta})I_{\mathcal{R}}\mathtt{R} \rightarrow \widetilde{\delta} \in \bigcap_{\mathtt{A}\in\mathcal{A}} [\![\mathtt{A}]\!]_{\overrightarrow{\mathbb{K}}}\}\\
&\subseteq \{\delta \mid \forall\widetilde{\delta} : (\delta,\widetilde{\delta})I_{\mathcal{R}}\mathtt{R} \rightarrow \widetilde{\delta} \in \bigcup_{1\le i\le n}\bigcap_{\mathtt{B}\in\mathcal{B}_i} [\![\mathtt{B}]\!]_{\overrightarrow{\mathbb{K}}} \cup \bigcap_{\mathtt{C}\in\mathcal{C}} [\![\mathtt{C}]\!]_{\overrightarrow{\mathbb{K}}}\}\\
&= (\Delta^* \cup (\Delta\setminus\Delta^*)) \cap \{\delta \mid \forall\widetilde{\delta} : (\delta,\widetilde{\delta})I_{\mathcal{R}}\mathtt{R} \rightarrow \widetilde{\delta} \in \bigcup_{1\le i\le n}\bigcap_{\mathtt{B}\in\mathcal{B}_i} [\![\mathtt{B}]\!]_{\overrightarrow{\mathbb{K}}} \cup \bigcap_{\mathtt{C}\in\mathcal{C}} [\![\mathtt{C}]\!]_{\overrightarrow{\mathbb{K}}}\}\\
&\subseteq \Delta^* \cup \big((\Delta\setminus\Delta^*) \cap \{\delta \mid \forall\widetilde{\delta} : (\delta,\widetilde{\delta})I_{\mathcal{R}}\mathtt{R} \rightarrow \widetilde{\delta} \in \bigcup_{1\le i\le n}\bigcap_{\mathtt{B}\in\mathcal{B}_i} [\![\mathtt{B}]\!]_{\overrightarrow{\mathbb{K}}} \cup \bigcap_{\mathtt{C}\in\mathcal{C}} [\![\mathtt{C}]\!]_{\overrightarrow{\mathbb{K}}}\}\big)\\
&\subseteq \Delta^* \cup \{\delta \mid \forall\widetilde{\delta} : (\delta,\widetilde{\delta})I_{\mathcal{R}}\mathtt{R} \rightarrow \widetilde{\delta} \in \bigcap_{\mathtt{C}\in\mathcal{C}} [\![\mathtt{C}]\!]_{\overrightarrow{\mathbb{K}}}\}\\
&= \bigcup_{1\le i\le n}\{\delta \mid \exists\widetilde{\delta} : (\delta,\widetilde{\delta})I_{\mathcal{R}}\mathtt{R} \wedge \widetilde{\delta} \in \bigcap_{\mathtt{B}\in\mathcal{B}_i} [\![\mathtt{B}]\!]_{\overrightarrow{\mathbb{K}}}\} \cup \{\delta \mid \forall\widetilde{\delta} : (\delta,\widetilde{\delta})I_{\mathcal{R}}\mathtt{R} \rightarrow \widetilde{\delta} \in \bigcap_{\mathtt{C}\in\mathcal{C}} [\![\mathtt{C}]\!]_{\overrightarrow{\mathbb{K}}}\}
\end{aligned}
$$

which means that $[\forall\mathtt{R}]\mathcal{A} \multimap \{[\exists\mathtt{R}]\mathcal{B}_1,\ldots,[\exists\mathtt{R}]\mathcal{B}_n,[\forall\mathtt{R}]\mathcal{C}\}$.

❑

3.1.2 Some Derivations

Lemma 3.5 *The following derivation rules can be deduced from $\mathcal{DR}$:*

$$\frac{\mathcal{A} \multimap \{\mathcal{B}_1,\ldots,\mathcal{B}_n\}}{\mathcal{A}\cup\mathcal{C} \multimap \{\mathcal{B}_1,\ldots,\mathcal{B}_n\}}\textit{cumulated premise extension}\ (\mathrm{PE}^*)$$

$$\frac{\mathcal{A} \multimap \{\mathcal{B}_1,\ldots,\mathcal{B}_n,\mathcal{C}\} \quad \mathcal{C} \multimap \{\mathcal{D}_1,\ldots,\mathcal{D}_k\}}{\mathcal{A} \multimap \{\mathcal{B}_1,\ldots,\mathcal{B}_n,\mathcal{D}_1,\ldots,\mathcal{D}_k\}}\textit{pure substitution}\ (\mathrm{SUB}^*)$$

$$\frac{\mathcal{A} \multimap \{\mathcal{B}_1,\ldots,\mathcal{B}_n\}}{\mathcal{A}\cup\mathcal{C} \multimap \{\mathcal{B}_1\cup\mathcal{C},\ldots,\mathcal{B}_n\cup\mathcal{C}\}}\textit{restriction}\ (\mathrm{RES})$$

$$\frac{\mathcal{A} \multimap \{\mathcal{B}_1,\ldots,\mathcal{B}_n\} \quad \mathcal{C} \multimap \{\mathcal{D}_1,\ldots,\mathcal{D}_m\}}{\mathcal{A}\cup\mathcal{C} \multimap \{\mathcal{B}_i\cup\mathcal{D}_j \mid 1\le i\le n, 1\le j\le m\}}\textit{distribution}\ (\mathrm{D})$$

$$\frac{}{\bot \multimap \{\mathcal{A}\}}\textit{cumulated contradiction}\ (\mathrm{CONT}^*)$$

$$\frac{}{[\forall\mathtt{R}]\mathcal{A}\cup[\exists\mathtt{R}]\mathcal{B} \multimap \{[\exists\mathtt{R}](\mathcal{A}\cup\mathcal{B})\}}\textit{cumulated } \forall\textit{-propagation}\ (\mathrm{AP}^*)$$

Proof:

The validity of these rules is proven by providing the $\mathcal{DR}$ proof trees that realize them.

1. cumulated premise extension
   Let $\mathcal{C} = \{\mathtt{C}_1,\ldots,\mathtt{C}_k\}$. Then we may iteratively add all elements of (the finite set) $\mathcal{C}$ to the premise by PE:

$$\cfrac{\cfrac{\cfrac{\mathcal{A} \multimap \{\mathcal{B}_1,\ldots,\mathcal{B}_n\}}{\mathcal{A}\cup\{\mathtt{C}_1\} \multimap \{\mathcal{B}_1,\ldots,\mathcal{B}_n\}}\mathrm{PE}}{\vdots}\mathrm{PE}}{\mathcal{A}\cup\{\mathtt{C}_1,\ldots,\mathtt{C}_k\} \multimap \{\mathcal{B}_1,\ldots,\mathcal{B}_n\}}\mathrm{PE}$$

2. pure substitution

$$\dfrac{\mathcal{A} \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n, \mathcal{C}\} \quad \dfrac{\mathcal{C} \multimap \{\mathcal{D}_1, \ldots, \mathcal{D}_k\}}{\mathcal{A} \cup \mathcal{C} \multimap \{\mathcal{D}_1, \ldots, \mathcal{D}_k\}}\text{PE*}}{\mathcal{A} \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n, \mathcal{D}_1, \ldots, \mathcal{D}_k\}}\text{SUB}$$

3. restriction

We start our derivation with

$$\dfrac{\dfrac{\mathcal{A} \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n\}}{\mathcal{A} \cup \mathcal{C} \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n\}}\text{PE*} \quad \dfrac{\dfrac{}{\mathcal{C} \cup \mathcal{B}_1 \multimap \{\mathcal{C} \cup \mathcal{B}_1\}}\text{ID}}{\mathcal{A} \cup \mathcal{C} \cup \mathcal{B}_1 \multimap \{\mathcal{C} \cup \mathcal{B}_1\}}\text{PE*}}{\mathcal{A} \cup \mathcal{C} \multimap \{\mathcal{B}_1 \cup \mathcal{C}, \mathcal{B}_2, \ldots, \mathcal{B}_n\}}\text{SUB}$$

and continue by successively "substituting" every $\mathcal{B}_i$ in the conclusion with $\mathcal{B}_i \cup \mathcal{C}$, finishing with

$$\dfrac{\mathcal{A} \cup \mathcal{C} \multimap \{\mathcal{B}_1 \cup \mathcal{C}, \ldots, \mathcal{B}_{n-1} \cup \mathcal{C}, \mathcal{B}_n\} \quad \dfrac{\dfrac{}{\mathcal{C} \cup \mathcal{B}_n \multimap \{\mathcal{C} \cup \mathcal{B}_n\}}\text{ID}}{\mathcal{A} \cup \mathcal{C} \cup \mathcal{B}_n \multimap \{\mathcal{C} \cup \mathcal{B}_n\}}\text{PE*}}{\mathcal{A} \cup \mathcal{C} \multimap \{\mathcal{B}_1 \cup \mathcal{C}, \ldots, \mathcal{B}_n \cup \mathcal{C}\}}\text{SUB}$$

4. distribution

We start the derivation with

$$\dfrac{\dfrac{\mathcal{A} \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n\}}{\mathcal{A} \cup \mathcal{C} \multimap \{\mathcal{B}_1 \cup \mathcal{C}, \ldots, \mathcal{B}_n \cup \mathcal{C}\}}\text{RES} \quad \dfrac{\mathcal{C} \multimap \{\mathcal{D}_1, \ldots, \mathcal{D}_m\}}{\mathcal{C} \cup \mathcal{B}_1 \multimap \{\mathcal{D}_1 \cup \mathcal{B}_1, \ldots, \mathcal{D}_m \cup \mathcal{B}_1\}}\text{RES}}{\mathcal{A} \cup \mathcal{C} \multimap \{\mathcal{B}_1 \cup \mathcal{D}_1, \ldots, \mathcal{B}_1 \cup \mathcal{D}_m, \mathcal{B}_2 \cup \mathcal{C}, \ldots, \mathcal{B}_n \cup \mathcal{C}\}}\text{SUB*}$$

and in the same way, step by step substitute every $\mathcal{B}_i \cup \mathcal{C}$ in the conclusion with $\mathcal{B}_i \cup \mathcal{D}_1, \ldots \mathcal{B}_i \cup \mathcal{D}_m$. So indeed, we end up with

$$\mathcal{A} \cup \mathcal{C} \multimap \{\mathcal{B}_i \cup \mathcal{D}_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}.$$

5. cumulated contradiction

Let $\mathcal{A} = \{\mathtt{A}_1, \ldots, \mathtt{A}_n\}$. Then

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\{\bot\} \multimap \{\{\mathtt{A}_1\}\}}\text{CONT} \quad \dfrac{}{\{\bot\} \multimap \{\{\mathtt{A}_2\}\}}\text{CONT}}{\{\bot\} \multimap \{\{\mathtt{A}_1, \mathtt{A}_2\}\}}\text{D}}{\vdots}\text{D}}{\{\bot\} \multimap \{\{\mathtt{A}_1, \ldots, \mathtt{A}_{n-1}\}\}}\text{D} \quad \dfrac{}{\{\bot\} \multimap \{\{\mathtt{A}_n\}\}}\text{CONT}}{\{\bot\} \multimap \{\{\mathtt{A}_1, \ldots, \mathtt{A}_n\}\}}\text{D}$$

6. cumulated $\mathbb{V}$-propagation

Again, let $\mathcal{A} = \{\mathtt{A}_1, \ldots, \mathtt{A}_n\}$. Furthermore, let $\mathcal{A}_i = \{\mathtt{A}_k \mid 1 \leq k \leq i\}$ for $i \leq n$. For any $i < n$, we have the following valid derivation tree:

$$\dfrac{\dfrac{}{\{\mathbb{V}\mathtt{R}.\mathtt{A}_{i+1}\} \cup [\mathbb{E}.\mathtt{R}](\mathcal{B} \cup \mathcal{A}_i) \multimap \{[\mathbb{E}\mathtt{R}](\mathcal{B} \cup \mathcal{A}_{i+1})\}}\text{AP}}{[\mathbb{V}\mathtt{R}](\mathcal{A} \setminus \mathcal{A}_i) \cup [\mathbb{E}\mathtt{R}](\mathcal{B} \cup \mathcal{A}_i) \multimap \{[\mathbb{V}\mathtt{R}](\mathcal{A} \setminus \mathcal{A}_{i+1}) \cup [\mathbb{E}\mathtt{R}](\mathcal{B} \cup \mathcal{A}_{i+1})\}}\text{RES}$$

By setting $\mathcal{C}_i = [\forall\text{R}](\mathcal{A} \setminus \mathcal{A}_i) \cup [\exists\text{R}](\mathcal{B} \cup \mathcal{A}_i)$, the derivation tree mentioned above justifies $\mathcal{C}_i \multimap \{\mathcal{C}_{i+1}\}$ for all $0 \leq i < n$. Now, we can successively apply pure substitution:

$$\dfrac{\dfrac{\dfrac{\mathcal{C}_0 \multimap \{\mathcal{C}_1\} \quad \mathcal{C}_1 \multimap \{\mathcal{C}_2\}}{\mathcal{C}_0 \multimap \{\mathcal{C}_2\}}\text{SUB*}}{\mathcal{C}_0 \multimap \{\mathcal{C}_{n-1}\}}\text{SUB*} \quad \mathcal{C}_{n-1} \multimap \{\mathcal{C}_n\}}{\mathcal{C}_0 \multimap \{\mathcal{C}_n\}}\text{SUB*}$$

The conclusion of this derivation tree is just $[\forall\text{R}]\mathcal{A} \cup [\exists\text{R}]\mathcal{B} \multimap \{[\exists\text{R}](\mathcal{A} \cup \mathcal{B})\}$.

❑

Example 3.6 Let $M_{\mathcal{C}} = \emptyset$ and $M_{\mathcal{R}} = \{\text{R}\}$. Now, consider the proposition: "Every entity has an R-neighbor or it has no R-neighbor." This proposition is clearly valid in any binary power context family. So, it should be the consequence of the empty set, even if no additional cumulated clauses are known to hold. Obviously, if an entity has an R-neighbor, it fulfills the $\mathcal{FLE}$ concept description $\exists\text{R}.\top$ which is another notation for $\exists\text{R}.\bigsqcap\emptyset$. By examining the definition, one also finds that an entity fulfills $\forall\text{R}.\bot$ exactly if it has no R-neighbor.
So, we know

$$\emptyset \models \emptyset \multimap \{\{\exists\text{R}.\textstyle\bigsqcap\emptyset\}, \{\forall\text{R}.\bot\}\}.$$

Now, if $\mathcal{DR}$ is complete (which has not been shown yet, but will be on the next pages), we should also have

$$\emptyset \vdash \emptyset \multimap \{\{\exists\text{R}.\textstyle\bigsqcap\emptyset\}, \{\forall\text{R}.\bot\}\}.$$

And in fact, taking into account that $[\forall\text{R}]\emptyset = \emptyset$ by definition, we can do the following derivation

$$\dfrac{\dfrac{\dfrac{}{\emptyset \multimap \{\emptyset\}}\text{ID}}{\emptyset \multimap \{\emptyset, \{\bot\}\}}\text{CE}}{\emptyset \multimap \{\{\exists\text{R}.\bigsqcap\emptyset\}, \{\forall\text{R}.\bot\}\}}\text{AL}$$

obtaining the expected result.

3.1.3 The Standard Model

After having shown the soundness of $\mathcal{DR}$, it remains to prove its completeness. This will be done in the following way: Given a set of cumulated clauses on $\mathcal{FLE}^{\text{norm}}$, we will define a particular binary power context family called the *standard model* which

- respects all the given clauses and
- respects just those clauses being derivable from the given ones via $\mathcal{DR}$.

As a consequence, the standard model can serve as a universal counterexample against the claim that any non-$\mathcal{DR}$-derivable clause holds in every binary power context family respecting the given clauses.
Note that the usual proof techniques for completeness from modal logic (see e.g. [Bla01] or [Po94]) using maximal consistent formula sets (also known as ultrafilters) is not applicable here, since they require that the considered logic is closed with respect to negation. This is not the case for $\mathcal{FLE}$.
For the same reason, adapting other calculi (like that for the multi-modal logic $K_{(m)}$ – see [Fi83]) and corresponding proofs to cumulated clauses on $\mathcal{FLE}$ cannot be easily realized.

Definition 3.7 *The* STANDARD MODEL $\overrightarrow{\mathbb{K}}(\mathfrak{K})$ *of a given set* $\mathfrak{K}$ *of cumulated clauses on* $\mathcal{FLE}^{\mathrm{norm}}$ *is the binary power context family*

$$\overrightarrow{\mathbb{K}} = (\mathbb{K}_{\mathcal{C}}, \mathbb{K}_{\mathcal{R}}) = ((\Delta, M_{\mathcal{C}}, I_{\mathcal{C}}), (\Delta \times \Delta, M_{\mathcal{R}}, I_{\mathcal{R}}))$$

defined as follows:

- *First, we set* $\overrightarrow{\mathbb{K}}^{(0)}(\mathfrak{K}) = ((\Delta^{(0)}, M_{\mathcal{C}}, I_{\mathcal{C}}^{(0)}), (\Delta^{(0)} \times \Delta^{(0)}, M_{\mathcal{R}}, I_{\mathcal{R}}^{(0)}))$ *with*
  - $\Delta^{(0)} := \{\mathcal{N} \subseteq \mathcal{FLE}^{\mathrm{norm}} \mid \mathcal{N}$ *respects all* $\mathfrak{k} \in \mathfrak{K}, \bot \notin \mathcal{N}\}$,
  - $\delta I_{\mathcal{C}}^{(0)} \mathtt{C} :\Leftrightarrow \mathtt{C} \in \delta$,
  - $(\delta_1, \delta_2) I_{\mathcal{R}}^{(0)} \mathtt{R} :\Leftrightarrow \exists\mathtt{R}.\bigsqcap \mathcal{C} \in \delta_1$ *for all finite* $\mathcal{C} \subseteq \delta_2$ *and* $\mathtt{C} \in \delta_2$ *for all* $\forall \mathtt{R}.\mathtt{C} \in \delta_1$.
- *From* $\overrightarrow{\mathbb{K}}^{(n)}(\mathfrak{K})$, *we determine* $\overrightarrow{\mathbb{K}}^{(n+1)}(\mathfrak{K})$ *by*
  - $\Delta^{(n+1)} := \Big\{\delta \in \Delta^{(n)} \mid \{\mathtt{C} \mid \forall \mathtt{R}.\mathtt{C} \in \delta\} = \bigcap_{(\delta,\widetilde{\delta}) I_{\mathcal{R}}^{(n)} \mathtt{R}} \widetilde{\delta}$ *and* $\{\mathcal{C} \mid \exists \mathtt{R}.\bigsqcap \mathcal{C} \in \delta\} = \bigcup_{(\delta,\widetilde{\delta}) I_{\mathcal{R}}^{(n)} \mathtt{R}} \mathcal{P}_{\mathrm{fin}}(\widetilde{\delta})$ *for all* $\mathtt{R} \in M_{\mathcal{R}}\Big\}$,
  - $I_{\mathcal{C}}^{(n+1)} := I_{\mathcal{C}}^{(0)} \cap (\Delta^{(n+1)} \times M_{\mathcal{C}})$,
  - $I_{\mathcal{R}}^{(n+1)} := I_{\mathcal{R}}^{(0)} \cap \big((\Delta^{(n+1)} \times \Delta^{(n+1)}) \times M_{\mathcal{R}}\big)$.
- *Finally, we set*

- $\Delta := \bigcap_{i\in\mathbb{N}} \Delta^{(i)}$,
- $I_{\mathcal{C}} := I_{\mathcal{C}}^{(0)} \cap (\Delta \times M_{\mathcal{C}})$,
- $I_{\mathcal{R}} := I_{\mathcal{R}}^{(0)} \cap (\Delta^2 \times M_{\mathcal{R}})$.

◇

Verbally: our standard model is approximated in a (possibly infinite) process, starting by taking as entities all those sets of $\mathcal{FLE}^{\text{norm}}$ concept descriptions that respect the given cumulated clauses $\mathfrak{K}$ and that do not contain $\bot$. The aim of the construction is to achieve that every entity fulfills exactly those $\mathcal{FLE}^{\text{norm}}$ concept descriptions from $\mathcal{FLE}^{\text{norm}}$ (semantically) that it contains (syntactically). To reach that goal, we successively delete those entities not "compatible" with their "role neighbors".[3] The final standard model can then be seen as the fixed point of this process. In the sequel, we will show that this construction indeed fulfills the intended purpose.

Lemma 3.8 *Let $\mathfrak{K}$ be a set of cumulated clauses on $\mathcal{FLE}^{\text{norm}}$ and $\overrightarrow{\mathbb{K}}(\mathfrak{K})$ the corresponding standard model. Then we have for every $\mathtt{D} \in \mathcal{FLE}^{\text{norm}}$ and every $\delta \in \Delta$*

$$\mathtt{D} \in \delta \Longleftrightarrow \delta \models \mathtt{D}.$$

Proof:
Obviously, for every $\delta \in \Delta$ from $\overrightarrow{\mathbb{K}}(\mathfrak{K})$ holds:

$$\{\mathcal{C} \mid \exists\mathtt{R}.\bigsqcap \mathcal{C} \in \delta\} = \bigcup\{\mathcal{P}_{\text{fin}}(\widetilde{\delta}) \mid (\delta, \widetilde{\delta}) I_{\mathcal{R}} \mathtt{R}\} \quad (*)$$

as well as

$$\{\mathtt{C} \mid \forall\mathtt{R}.\mathtt{C} \in \delta\} = \bigcap\{\widetilde{\delta} \mid (\delta, \widetilde{\delta}) I_{\mathcal{R}} \mathtt{R}\}. \quad (**)$$

We do now an induction over the maximal role depth of a concept description $\mathtt{D}$:

- Induction anchor: $\mathtt{D} \in \mathcal{FLE}_0^{\text{norm}}$.

  Then we have either $\mathtt{D} \in M_{\mathcal{C}}$ or $\mathtt{D} = \bot$. In the first case, we have $\mathtt{D} \in \delta$ iff $\delta I_{\mathcal{C}} \mathtt{D}$ by definition of the standard model. By the semantics definition, this is equivalent to $\delta \models \mathtt{D}$.

  Considering the second case, we find that $\bot \in \delta$ does not occur (due to the explicit exclusion of entities containing $\bot$ in the standard model definition). Likewise, $\delta \models \bot$ is never the case since $[\![\bot]\!]_{\overrightarrow{\mathbb{K}}} = \emptyset$. So, those both statements are trivially equivalent.

[3] This approach is related to Pratt's type elimination technique (see [Pra79]), originally used to decide satisfiability of modal formulae. However, contrary to his method, our standard model construction process will in general not terminate after finitely many steps.

- Induction step: $\mathtt{D} \in \mathcal{FLE}_n^{\mathrm{norm}}$, $n > 0$.

  Again, we have to distinguish two cases.

  First, assume $\mathtt{D} = \exists\mathtt{R}.\bigsqcap\mathcal{D}$ with $\mathcal{D} \subseteq \mathcal{FLE}_{n-1}^{\mathrm{norm}}$. Then the statement $\exists\mathtt{R}.\bigsqcap\mathcal{D} \in \delta$ is obviously equivalent to

  $$\mathcal{D} \in \{\mathcal{C} \mid \exists\mathtt{R}.\bigsqcap\mathcal{C} \in \delta\}$$

  and this because of (*) to

  $$\mathcal{D} \in \bigcup\{\mathcal{P}_{\mathrm{fin}}(\widetilde{\delta}) \mid (\delta,\widetilde{\delta})I_{\mathcal{R}}\mathtt{R}\}.$$

  So, we know that there exists an R-neighbor $\widetilde{\delta}$ of δ, which contains all concept descriptions from $\mathcal{D}$. Since $\mathcal{D} \subseteq \mathcal{FLE}_{n-1}^{\mathrm{norm}}$, we see by induction hypothesis that this is the case if and only if $\widetilde{\delta} \models \mathtt{E}$ for all $\mathtt{E} \in \mathcal{D}$. Subsequently, this is equivalent to

  $$\exists\widetilde{\delta} : (\delta,\widetilde{\delta})I_{\mathcal{R}}\mathtt{R} \wedge \widetilde{\delta} \in \bigcap_{\mathtt{E}\in\mathcal{D}} [\![\mathtt{E}]\!]_{\overrightarrow{\mathbb{K}}}$$

  and this (by the semantics definition) to

  $$\exists\widetilde{\delta} : (\delta,\widetilde{\delta})I_{\mathcal{R}}\mathtt{R} \wedge \widetilde{\delta} \in [\![\bigsqcap\mathcal{D}]\!]_{\overrightarrow{\mathbb{K}}}$$

  and finally to

  $$\delta \in [\![\exists\mathtt{R}.\bigsqcap\mathcal{D}]\!]_{\overrightarrow{\mathbb{K}}}$$

  which means just $\delta \models \exists\mathtt{R}.\bigsqcap\mathcal{D}$.

  It remains to consider the case $\mathtt{D} = \forall\mathtt{R}.\mathtt{E}$ with $\mathtt{E} \in \mathcal{FLE}_{n-1}^{\mathrm{norm}}$. Then $\forall\mathtt{R}.\mathtt{E} \in \delta$ can be written as

  $$\mathtt{E} \in \{\mathtt{C} \mid \forall\mathtt{R}.\mathtt{C} \in \delta\}$$

  which is due to (**) equivalent to

  $$\mathtt{E} \in \bigcap\{\widetilde{\delta} \mid (\delta,\widetilde{\delta})I_{\mathcal{R}}\mathtt{R}\}.$$

  Therefore knowing that all R-neighbors of δ contain E (which is an element of $\mathcal{FLE}_{n-1}^{\mathrm{norm}}$), we conclude by the induction hypothesis that this is equivalent to

  $$\forall\widetilde{\delta} : (\delta,\widetilde{\delta})I_{\mathcal{R}}\mathtt{R} \to \widetilde{\delta} \in [\![\mathtt{E}]\!]_{\overrightarrow{\mathbb{K}}}$$

  and by the semantics definition to

  $$\delta \in [\![\forall\mathtt{R}.\mathtt{E}]\!]_{\overrightarrow{\mathbb{K}}}$$

  which means just $\delta \models \forall\mathtt{R}.\mathtt{E}$.

  Note that all argumentations work in both directions. So indeed, the equivalence is assured.

❑

In the sequel, we will define the notion of a *homomorphism* from one binary power context family to another, being a mapping on the corresponding universes that preserves the (entity-wise) validity of all concept descriptions.

Definition 3.9 *Given two binary power context families $\vec{\mathbb{K}}_1$ (with universe Δ_1) and $\vec{\mathbb{K}}_2$ (with universe Δ_2) having identical sets of role and concept names, we call a function $f : \Delta_1 \to \Delta_2$ HOMOMORPHISM FROM $\vec{\mathbb{K}}_1$ TO $\vec{\mathbb{K}}_2$, if*

1. *for all $\delta \in \Delta_1$ and $\mathtt{C} \in \mathcal{FLE}$ we have $\delta \models \mathtt{C} \Longleftrightarrow f(\delta) \models \mathtt{C}$ and*
2. *for all $\delta, \widetilde{\delta} \in \Delta_1$ and $\mathtt{R} \in M_{\mathcal{R}}$ we have $(\delta, \widetilde{\delta}) I_{\mathcal{R}1}\mathtt{R}$ implies $(f(\delta), f(\widetilde{\delta})) I_{\mathcal{R}2}\mathtt{R}$.*

◇

Essentially, the next theorem shows that the standard model of a set $\mathfrak{K}$ of cumulated clauses is the most universal one: every model[4] respecting $\mathfrak{K}$ is contained in the standard model in a certain way. This "certain way" is formally captured in terms of the above defined notion of homomorphism.

Theorem 3.10 *Let $\mathfrak{K}$ be a set of cumulated clauses and*

$$\vec{\mathbb{K}}(\mathfrak{K}) = ((\Delta, I_{\mathcal{C}}, M_{\mathcal{C}}), (\Delta^2, I_{\mathcal{R}}, M_{\mathcal{R}}))$$

the corresponding standard model with universe Δ. Let furthermore

$$\vec{\mathbb{K}} = ((\widetilde{\Delta}, \widetilde{I}_{\mathcal{C}}, M_{\mathcal{C}}), (\widetilde{\Delta}^2, \widetilde{I}_{\mathcal{R}}, M_{\mathcal{R}}))$$

be a binary power context family that respects all cumulated clauses from $\mathfrak{K}$. Then the function $\varphi : \widetilde{\Delta} \to \mathcal{P}(\mathcal{FLE}^{\text{norm}})$ with $\varphi(\delta) = \{\mathtt{C} \in \mathcal{FLE}^{\text{norm}} \mid \delta \models \mathtt{C}\}$ is a homomorphism from $\vec{\mathbb{K}}$ to $\vec{\mathbb{K}}(\mathfrak{K})$.

Proof:
There are three facts to be shown:

(I) $\varphi(\delta) \in \Delta$ for all $\delta \in \widetilde{\Delta}$,

(II) for all $\delta \in \widetilde{\Delta}$ and $\mathtt{C} \in \mathcal{FLE}$ we have $\delta \models \mathtt{C} \Longleftrightarrow \varphi(\delta) \models \mathtt{C}$, and

(III) for all $\delta, \bar{\delta} \in \widetilde{\Delta}$ and $\mathtt{R} \in M_{\mathcal{R}}$ we have $(\delta, \bar{\delta}) \widetilde{I}_{\mathcal{R}}\mathtt{R}$ implies $(\varphi(\delta), \varphi(\bar{\delta})) I_{\mathcal{R}}\mathtt{R}$.

The proof of Fact I will be done by inductively showing that $\varphi(\delta)$ is contained in all $\Delta^{(i)}$ occurring in the construction of $\vec{\mathbb{K}}(\mathfrak{K})$ (and therefore also in $\Delta = \bigcap_{i \in \mathbb{N}} \Delta^{(i)}$).

[4] We use the term *model* synonymously for binary power context family.

- induction anchor: $\varphi(\delta) \in \Delta^{(0)}$.
Since $\overrightarrow{\mathbb{K}}$ respects $\mathfrak{K}$ by assumption and $[\![\bot]\!]_{\overrightarrow{\mathbb{K}}} = \emptyset$, we have that $\varphi(\delta) = \{\mathtt{C} \in \mathcal{FLE}^{\mathrm{norm}} \mid \delta \models \mathtt{C}\} \in \Delta^{(0)}$.
Furthermore, notice that $(\delta_1, \delta_2)\widetilde{I}_{\mathcal{R}}\mathtt{R}$ implies $\delta_1 \models \exists\mathtt{R}.\bigsqcap\mathcal{C}$ for all finite $\mathcal{C}$ with $\delta_2 \models \mathcal{C}$. It also implies $\delta_2 \models \mathtt{C}$ for all $\mathtt{C}$ with $\delta_1 \models \forall\mathtt{R}.\mathtt{C}$. Due to the definition of φ, we then have $(\varphi(\delta_1), \varphi(\delta_2))I_{\mathcal{R}}^{(0)}\mathtt{R}$. From this, we can conclude $(\varphi(\delta_1), \varphi(\delta_2))I_{\mathcal{R}}^{(i)}\mathtt{R}$ for any i as long as $\varphi(\delta_1), \varphi(\delta_2) \in \Delta^{(i)}$ (*).
- induction step: $\varphi(\delta) \in \Delta^{(i+1)}$.
Let $\models\!\!\approx$ be the restriction of $\models$ to $\mathcal{FLE}^{\mathrm{norm}}$, i.e., $\delta \models\!\!\approx \mathtt{C} :\Leftrightarrow \delta \models \mathtt{C}$ and $\delta \in \mathcal{FLE}^{\mathrm{norm}}$.
By induction hypothesis, we know that $\varphi(\widetilde{\delta}) \in \Delta^{(i)}$ for all $\widetilde{\delta} \in \widetilde{\Delta}$.
Note that then from (*), it follows that
$$(\delta, \widetilde{\delta})\widetilde{I}_{\mathcal{R}}\mathtt{R} \Rightarrow (\varphi(\delta), \varphi(\widetilde{\delta}))I_{\mathcal{R}}^{(i)}\mathtt{R}$$
which in turn implies
$$\{\varphi(\widetilde{\delta}) \mid (\delta, \widetilde{\delta})\widetilde{I}_{\mathcal{R}}\mathtt{R}\} \subseteq \{\bar{\delta} \mid (\varphi(\delta), \bar{\delta})I_{\mathcal{R}}^{(i)}\mathtt{R}\}$$
and therefore also
$$\{\mathcal{P}_{\mathrm{fin}}(\varphi(\widetilde{\delta})) \mid (\delta, \widetilde{\delta})\widetilde{I}_{\mathcal{R}}\mathtt{R}\} \subseteq \{\mathcal{P}_{\mathrm{fin}}(\bar{\delta}) \mid (\varphi(\delta), \bar{\delta})I_{\mathcal{R}}^{(i)}\mathtt{R}\}.$$
Now, we will show that both conditions for $\varphi(\delta)$ being in $\Delta^{(i+1)}$ are satisfied. First, we know
$$\begin{array}{ll} & \bigcap_{(\varphi(\delta),\bar{\delta})I_{\mathcal{R}}^{(i)}\mathtt{R}} \bar{\delta} \\ \subseteq & \bigcap_{(\delta,\widetilde{\delta})\widetilde{I}_{\mathcal{R}}\mathtt{R}} \varphi(\widetilde{\delta}) \\ = & \bigcap_{(\delta,\widetilde{\delta})\widetilde{I}_{\mathcal{R}}\mathtt{R}} \{\mathtt{C} \mid \widetilde{\delta} \models\!\!\approx \mathtt{C}\} \\ = & \{\mathtt{C} \mid \forall\widetilde{\delta} : (\delta, \widetilde{\delta})\widetilde{I}_{\mathcal{R}}\mathtt{R} \to \widetilde{\delta} \models\!\!\approx \mathtt{C}\} \\ = & \{\mathtt{C} \mid \delta \models\!\!\approx \forall\mathtt{R}.\mathtt{C}\} \\ = & \{\mathtt{C} \mid \forall\mathtt{R}.\mathtt{C} \in \varphi(\delta)\}. \end{array}$$
Conversely, by construction of $I_{\mathcal{R}}^{(i)}$, from $\forall\mathtt{R}.\mathtt{C} \in \varphi(\delta)$ it follows that $(\varphi(\delta), \bar{\delta})I_{\mathcal{R}}^{(0)}\mathtt{R} \Rightarrow \mathtt{C} \in \bar{\delta}$. Because of $I_{\mathcal{R}}^{(i)} \subseteq I_{\mathcal{R}}^{(0)}$, this implies $(\varphi(\delta), \bar{\delta})I_{\mathcal{R}}^{(i)}\mathtt{R} \Rightarrow \mathtt{C} \in \bar{\delta}$, from which we can immediately conclude $\mathtt{C} \in \bigcap_{(\varphi(\delta),\bar{\delta})I_{\mathcal{R}}^{(i)}\mathtt{R}} \bar{\delta}$. Therefore, we have
$$\{\mathtt{C} \mid \forall\mathtt{R}.\mathtt{C} \in \varphi(\delta)\} \subseteq \bigcap_{(\varphi(\delta),\bar{\delta})I_{\mathcal{R}}^{(i)}\mathtt{R}} \bar{\delta}.$$
Hence, the first condition is fulfilled.
Moreover, we have
$$\begin{array}{ll} & \bigcup_{(\varphi(\delta),\bar{\delta})I_{\mathcal{R}}^{(i)}\mathtt{R}} \mathcal{P}_{\mathrm{fin}}(\bar{\delta}) \\ \supseteq & \bigcup_{(\delta,\widetilde{\delta})\widetilde{I}_{\mathcal{R}}\mathtt{R}} \mathcal{P}_{\mathrm{fin}}(\varphi(\widetilde{\delta})) \\ = & \bigcup_{(\delta,\widetilde{\delta})\widetilde{I}_{\mathcal{R}}\mathtt{R}} \{\mathcal{C} \mid \widetilde{\delta} \models\!\!\approx \mathcal{C},\ \mathcal{C} \text{ finite}\} \\ = & \{\mathcal{C} \mid \mathcal{C} \text{ finite},\ \exists\widetilde{\delta} : (\delta, \widetilde{\delta})\widetilde{I}_{\mathcal{R}}\mathtt{R} \wedge \widetilde{\delta} \in \bigcap_{\mathtt{C}\in\mathcal{C}} [\![\mathtt{C}]\!]_{\widetilde{\mathbb{K}}}\} \\ = & \{\mathcal{C} \mid \delta \models\!\!\approx \exists\mathtt{R}.\bigsqcap\mathcal{C}\} \\ = & \{\mathcal{C} \mid \exists\mathtt{R}.\bigsqcap\mathcal{C} \in \varphi(\delta)\}. \end{array}$$

Vice versa, assume $\mathcal{C} \in \bigcup_{(\varphi(\delta),\bar{\delta})I_{\mathcal{R}}^{(i)}\mathtt{R}} \mathcal{P}_{\mathrm{fin}}(\bar{\delta})$. By $I_{\mathcal{R}}^{(i)} \subseteq I_{\mathcal{R}}^{(0)}$, we can deduce that $\mathcal{C} \in \bigcup_{(\varphi(\delta),\bar{\delta})I_{\mathcal{R}}^{(0)}\mathtt{R}} \mathcal{P}_{\mathrm{fin}}(\bar{\delta})$. This just means $\mathcal{C} \subseteq \bar{\delta}$ for some $\bar{\delta} \in \Delta$ with $(\varphi(\delta),\bar{\delta})I_{\mathcal{R}}^{(0)}\mathtt{R}$. By definition of $I_{\mathcal{R}}^{(0)}$, we know that then $\exists\mathtt{R}.\bigsqcap\mathcal{C}$ must be in $\varphi(\delta)$. So, this gives us

$$\bigcup_{(\varphi(\delta),\bar{\delta})I_{\mathcal{R}}^{(i)}\mathtt{R}} \mathcal{P}_{\mathrm{fin}}(\bar{\delta}) \subseteq \{\mathcal{C} \mid \exists\mathtt{R}.\bigsqcap\mathcal{C} \in \varphi(\delta)\}.$$

Thus, also the second condition is fulfilled.

By the two equalities shown above, we can conclude that $\varphi(\delta) \in \Delta^{(i+1)}$.

Fact II can be shown in the following way: Let $\delta \in \widetilde{\Delta}$ and $\mathtt{C} \in \mathcal{FLE}$. Then we know that $\delta \models \mathtt{C}$ if and only if $\delta \models \bigsqcap\mathfrak{n}(\mathtt{C})$ due to Theorem 2.8. Obviously, this is the case exactly if $\delta \models \mathfrak{n}(\mathtt{C})$ which (since $\mathfrak{n}(\mathtt{C}) \subseteq \mathcal{FLE}^{\mathrm{norm}}$ and by the definition of φ) coincides with $\mathfrak{n}(\mathtt{C}) \subseteq \varphi(\delta)$. Lemma 3.8 and Fact I of this theorem yield the equivalence to $\varphi(\delta) \models \mathfrak{n}(\mathtt{C})$ in the standard model. This means $\varphi(\delta) \models \bigsqcap\mathfrak{n}(\mathtt{C})$ and is equivalent to $\varphi(\delta) \models \mathtt{C}$. Since the argumentation works in both directions, the equivalence has been shown.

Fact III is an immediate consequence of (*) as stated above, since we now know that $\varphi(\delta) \in \Delta$ for all $\delta \in \widetilde{\Delta}$. ❑

A nice consequence of the preceding theorem is the fact that in order to decide whether a set of cumulated clauses entails another cumulated clause, one just has to take a look on the according standard model. This is formally specified in the following theorem.

Theorem 3.11 *Let $\mathfrak{K}$ be a set of cumulated clauses on $\mathcal{FLE}^{\mathrm{norm}}$ and let $\mathfrak{k}$ be a cumulated clause. Then*

$$\mathfrak{K} \models \mathfrak{k} \Longleftrightarrow \overrightarrow{\mathbb{K}}(\mathfrak{K}) \models \mathfrak{k}.$$

Proof:
"$\Longrightarrow$":
$\mathfrak{K} \models \mathfrak{k}$ by definition means that every binary power context family respecting all cumulated clauses also respects $\mathfrak{k}$. So, this in particular holds for $\overrightarrow{\mathbb{K}}(\mathfrak{K})$.
"$\Longleftarrow$":
Assume the contrary, i.e., there were a binary power context family $\overrightarrow{\mathbb{K}}$ with $\overrightarrow{\mathbb{K}} \models \mathfrak{K}$ but $\overrightarrow{\mathbb{K}} \not\models \mathfrak{k}$. However, this would imply that there is a δ in the universe of $\overrightarrow{\mathbb{K}}$ such that $\{\mathtt{C} \mid \delta \in [\![\mathtt{C}]\!]_{\overrightarrow{\mathbb{K}}}\}$ does not respect $\mathfrak{k}$. Using the preceding Theorem 3.10, we get that for the standard model entity $\varphi(\delta)$ the concept description set $\{\mathtt{C} \mid \varphi(\delta) \in [\![\mathtt{C}]\!]_{\overrightarrow{\mathbb{K}}(\mathfrak{K})}\}$ does not respect $\mathfrak{k}$ either and therefore conclude $\overrightarrow{\mathbb{K}}(\mathfrak{K}) \not\models \mathfrak{k}$. This contradicts our assumption. ❑

3.1.4 Realization Trees

Now, we will show that every cumulated clause $\mathfrak{k}$ valid in the standard model can be derived from $\mathfrak{K}$ by using $\mathcal{DR}$. This will be done in several steps:
First, we define a tree structure that – starting from a given set $\mathcal{A} \subseteq \mathcal{FLE}^{\text{norm}}$ – represents all "branching possibilities" of extending $\mathcal{A}$ in order to make it respect all cumulated clauses derivable from $\mathfrak{K}$. The basic idea of this construction thus resembles that of early tableau methods, originally used for deciding satisfiability in propositional logic (see [Be59] and [Sm68]) and also applied to modal logics (see [Kri63]). However, while finiteness of the tableau is essential for decidability problems, in our case the structure will be infinite in general.

Definition 3.12 *Given a set $\mathfrak{K}$ of cumulated clauses and a set $\mathcal{A} \subseteq \mathcal{FLE}^{\text{norm}}$, we call a structure $T_{\mathcal{A}}^{\mathfrak{K}} = (N, r, \prec, \epsilon)$* REALIZATION TREE *of $\mathcal{A}$ if*

- *N is an arbitrary set (the elements of N will be called* NODES*),*
  $r \in N$ (r will also be called the ROOT*),*
  $\prec \subseteq N \times N$ ($\prec$ will be called the SUCCESSOR RELATION*), and*
  ϵ is a function $N \to \mathcal{P}(\mathcal{FLE}^{\text{norm}})$,
- *$(N, \prec)$ is a tree with root r,*
- *$\epsilon(r) = \mathcal{A}$,*
- *a node $\nu \in N$ has successors (i.e., $\nu^{\prec} := \{\widetilde{\nu} \mid \nu \prec \widetilde{\nu}\}$ is nonempty), if and only if $\epsilon(\nu)$ does not respect all cumulated clauses from $\mathcal{DR}(\mathfrak{K})$. In this case, there is a cumulated clause $\mathfrak{k} = \mathcal{B} \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_n\} \in \mathcal{DR}(\mathfrak{K})$ (called* WITNESSING CLAUSE *of ν) with*
  - *$\mathcal{B} \subseteq \epsilon(\nu)$ and $\mathcal{C}_i \nsubseteq \epsilon(\nu)$ for some $i \in \{1 \ldots, n\}$ (i.e., $\epsilon(\nu)$ does not respect $\mathfrak{k}$),*
  - *$\mathfrak{k}$ is minimal with respect to the greatest role depth in $\mathcal{C}_1, \ldots, \mathcal{C}_n$, and*
  - *among those cumulated clauses fulfilling the two conditions above, $\mathfrak{k}$'s conclusion is minimal with respect to set inclusion,*

  such that $\nu^{\prec} = \{\nu_1, \ldots, \nu_n\}$ with $\epsilon(\nu_i) = \epsilon(\nu) \cup \mathcal{C}_i$.

Given such a realization tree, we call

- $(\nu_i)_{i \in \{0,\ldots,k\}}$ *a* FINITE PATH *if* $\nu_i \prec \nu_{i+1}$ *for all* $0 \leq i < k$ *and* ν_k *has no successors,*

- $(\nu_i)_{i \in \mathbb{N}}$ *an* INFINITE PATH *if* $\nu_i \prec \nu_{i+1}$ *for all* $i \in \mathbb{N}$*,*

- *a (finite ore infinite) path* (ν_i) COMPLETE *if* $\nu_0 = r$*,*

- $\mathcal{A} \subseteq \mathcal{FLE}^{\text{norm}}$ *a* LEAF *if* $\mathcal{A} = \epsilon(\nu)$ *for a* $\nu \in \widetilde{N}$ *that has no successors,*

- $\mathcal{A} \subseteq \mathcal{FLE}^{\text{norm}}$ *a* PSEUDOLEAF *if we have* $\mathcal{A} = \bigcup_{i \in \mathbb{N}} \epsilon(\nu_i)$ *for some infinite complete path* $(n_i)_{i \in \mathbb{N}}$*, and*

- $\mathcal{A} \subseteq \mathcal{FLE}^{\text{norm}}$ *a* QUASILEAF *if it is a leaf or pseudoleaf.*

◇

Next, we define the term *covering* of a realization tree, being a transversal of all complete paths in this tree.

Definition 3.13 *Given a realization tree* $T = (N, r, \prec, \epsilon)$*, a node set* $\widetilde{N} \subseteq N$ *will be called* COVERING *of* T *if every (finite or infinite) complete path* $r = \nu_0 \prec \nu_1 \prec \ldots$ *contains (at least) one element from* $\widetilde{N}$*.* ◇

Using the fact that a realization tree is finitely branching, we can show that every arbitrary covering contains a finite one.

Lemma 3.14 *For every covering* $\widetilde{N}$ *of a realization tree, there exists a finite covering* $N^{fin} \subseteq \widetilde{N}$*.*

Proof:
We let N^{fin} contain the minimal nodes from $\widetilde{N}$, i.e., all nodes ν fulfilling the condition that for the path $r = \nu_0 \prec \nu_1 \prec \ldots \prec \nu_k \prec \nu$ (the uniqueness of which is assured by the tree structure), we have $\nu_0, \nu_1, \ldots, \nu_k \notin \widetilde{N}$. We have to prove two propositions:

1. N^{fin} is a covering and
2. N^{fin} is finite.

1. Consider an arbitrary path $r = \nu_0 \prec \nu_1 \prec \ldots$ and suppose no node on it is in N^{fin}. But, due to the assumption, it contains at least one node from $\widetilde{N}$. Now, the minimal one (in the sense described above) of the nodes from $\widetilde{N}$ lying on the path has to be in N^{fin}. This gives a contradiction.

2. For a $\nu \in N$, let N^{fin}_{ν} be the set of all nodes from N^{fin} lying on paths going through ν.

   Now suppose N^{fin} is infinite. Then we construct an infinite complete path $\nu_0 \prec \nu_1 \prec \ldots$ as follows:
   We start with $\nu_0 = r$. Note that $N^{fin}_{\nu_0}$ is infinite (since it is equal to N^{fin}).
   For each ν_i (where we can presuppose that $N^{fin}_{\nu_i}$ is infinite - therefore, ν_i must have successors and cannot be in N^{fin} itself due to the definition of the latter via minimality), we consider all successors. Since there are only finitely many (due to the definition every realization tree is finitely branching), there must be (at least) one of them (say: $\widetilde{\nu}$) for which $N^{fin}_{\widetilde{\nu}}$ is infinite. Then we set $\nu_{i+1} := \widetilde{\nu}$.
   The path constructed in this way does not contain any element from N^{fin}. But, as we just have proven in (1), N^{fin} is a covering. So, we have a contradiction.

❑

In the sequel, we will show that for any cumulated clause "readable" from a realization tree endowed with a covering, we can construct a corresponding $\mathcal{DR}$ proof tree.

Lemma 3.15 *Let $\mathcal{A} \subseteq \mathcal{FLE}^{\text{norm}}$ and let $T^{\mathfrak{K}}_{\mathcal{A}}$ be a realization tree of $\mathcal{A}$. Let furthermore $\widetilde{N} \subseteq N$ be a covering of $T^{\mathfrak{K}}_{\mathcal{A}}$.*
Let now $\mathcal{C}_1, \ldots, \mathcal{C}_n \subseteq \mathcal{FLE}^{\text{norm}}$ be finite sets such that for every $\nu \in \widetilde{N}$, there is an $i \in \{1 \ldots n\}$ with $\mathcal{C}_i \subseteq \epsilon(\nu)$.
Then there is a finite $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathfrak{K} \vdash \mathcal{B} \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$.

Proof:
W.l.o.g., we can assume $\widetilde{N}$ to be finite due to Lemma 3.14.
We will prove the proposition by showing that for any node $\nu \in N$ where there is no $\widetilde{\nu} \in \widetilde{N}$ on the path from r to ν, there is a finite $\mathcal{B}_\nu \subseteq \epsilon(\nu)$ such that $\mathcal{B}_\nu \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$ is $\mathcal{DR}$-derivable. (For $\nu = r$ then follows the claimed result.)
(Note that every path starting from such a ν must contain an element from $\widetilde{N}$, for otherwise we could construct a path starting from r and containing no element from $\widetilde{N}$, which would contradict the precondition.)
Consider all paths starting from such a ν. For every such path $\nu = \nu_0 \prec \nu_1 \prec \ldots$, we can determine the smallest index i such that $\nu_i \in \widetilde{N}$. Among those path-wise smallest indices (there can be only finitely many due to the finiteness of $\widetilde{N}$), we select the greatest one, call it the TYPE of ν, and denote it by $\tau(\nu)$.
We will prove the proposition by induction over the type of the considered nodes.

- Induction anchor: $\tau(\nu) = 0$.
  Then we have $\nu \in \widetilde{N}$ and thus $\mathcal{C}_k \subseteq \epsilon(\nu)$ for some $k \in \{1, \ldots, n\}$. Clearly, $\mathfrak{K} \vdash \mathcal{C}_k \multimap \{\mathcal{C}_k\}$ due to the identity rule. By $(n-1)$fold application of conclusion extension, we can derive $\mathfrak{K} \vdash \mathcal{C}_k \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$. Thus, we have found an appropriate $\mathcal{B}_\nu$, namely $\mathcal{B}_\nu := \mathcal{C}_k$.

- Induction step: $\tau(\nu) > 0$.
  Then we have $\nu \notin \widetilde{N}$ and all successors $\nu_1, \ldots, \nu_k$ of ν are of type less than $\tau(\nu)$. Thus for every ν_i holds by induction hypothesis that there is a finite $\mathcal{B}_{\nu_i} \subseteq \epsilon(\nu_i)$ with $\mathfrak{K} \vdash \mathcal{B}_{\nu_i} \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$.
  Let now be $\mathcal{D} \multimap \{\mathcal{E}_1, \ldots, \mathcal{E}_k\}$ the witnessing clause of ν (and therefore in particular derivable from $\mathfrak{K}$). Then we know that $\mathcal{D} \subseteq \epsilon(\nu)$ and $\epsilon(\nu_i) = \epsilon(\nu) \cup \mathcal{E}_i$ for all $i \in \{1, \ldots, k\}$. Now, we can do the following for all ν_i:
  We define $\widetilde{\mathcal{B}}_{\nu_i} := \mathcal{B}_{\nu_i} \cap \epsilon(\nu)$. Then the cumulated clause $\widetilde{\mathcal{B}}_{\nu_i} \cup \mathcal{E}_i \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$ is either equal to $\mathcal{B}_{\nu_i} \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$ or can be derived from it by applying the cumulated premise extension rule. Then we can derive (with setting $\widetilde{\mathcal{B}} := \bigcup_{1 \leq j \leq k} \widetilde{\mathcal{B}}_{\nu_j}$):

$$\cfrac{\cfrac{\widetilde{\mathcal{B}}_{\nu_i} \cup \mathcal{E}_i \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}}{\widetilde{\mathcal{B}}_{\nu_i} \cup \mathcal{D} \cup \mathcal{E}_i \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}}\text{PE*}}{\widetilde{\mathcal{B}} \cup \mathcal{D} \cup \mathcal{E}_i \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}}\text{PE*}$$

  Using those clauses, we can do the following derivation (remember that the derivability of $\mathcal{D} \multimap \{\mathcal{E}_1, ..., \mathcal{E}_k\}$ can be presumed, as it is a witnessing clause):

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\mathcal{D} \multimap \{\mathcal{E}_1, ..., \mathcal{E}_k\}}{\widetilde{\mathcal{B}} \cup \mathcal{D} \multimap \{\mathcal{E}_1, ..., \mathcal{E}_k\}}\text{PE*} \quad \widetilde{\mathcal{B}} \cup \mathcal{D} \cup \mathcal{E}_1 \multimap \{\mathcal{C}_1, ..., \mathcal{C}_n\}}{\widetilde{\mathcal{B}} \cup \mathcal{D} \multimap \{\mathcal{C}_1, ..., \mathcal{C}_n, \mathcal{E}_2, ..., \mathcal{E}_k\}}\text{SUB}}{\vdots}\text{SUB}}{\widetilde{\mathcal{B}} \cup \mathcal{D} \multimap \{\mathcal{C}_1, ..., \mathcal{C}_n, \mathcal{E}_k\}}\text{SUB} \quad \widetilde{\mathcal{B}} \cup \mathcal{D} \cup \mathcal{E}_k \multimap \{\mathcal{C}_1, ..., \mathcal{C}_n\}}{\widetilde{\mathcal{B}} \cup \mathcal{D} \multimap \{\mathcal{C}_1, ..., \mathcal{C}_n\}}\text{SUB}$$

  So, we have $\mathfrak{K} \vdash \widetilde{\mathcal{B}} \cup \mathcal{D} \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$. But by construction, $\widetilde{\mathcal{B}} \cup \mathcal{D}$ is a subset of $\epsilon(\nu)$ and (as a union of finitely many finite sets) also finite. So, we can set $\mathcal{B}_\nu := \widetilde{\mathcal{B}} \cup \mathcal{D}$ and we are done.

❑

In the next lemma, we prove that any quasilieaf of a realization tree respects all clauses from $\mathcal{DR}(\mathfrak{K})$. The proof idea therein is to show that any such cumulated clause – if not respected "by accident" – will sooner or later become a witnessing clause in any path.
After this, we show that if $\mathcal{A}$ does not imply $\bot$, none of the realization tree nodes does contain it either.

Lemma 3.16 *Let $\mathfrak{K}$ be a set of cumulated clauses, $\mathcal{A} \subseteq \mathcal{FLE}^{\text{norm}}$ and let $T_{\mathcal{A}}^{\mathfrak{K}} = (N, r, \prec, \epsilon)$ be a corresponding realization tree. Then for every quasileaf Q of $T_{\mathcal{A}}^{\mathfrak{K}}$, we have that Q respects all clauses from $\mathcal{DR}(\mathfrak{K})$.*

Proof:
Let Q be the quasileaf and $\mathfrak{k} \in \mathcal{DR}(\mathfrak{K})$. We distinguish two cases:

- Q is a leaf with corresponding node ν. Suppose Q does not respect $\mathfrak{k}$. Then either $\mathfrak{k}$ fulfills the minimality conditions from the definition or there is a "smaller" $\widetilde{\mathfrak{k}} \in \mathcal{DR}(\mathfrak{K})$ that does. Thus, we have found a possible witnessing clause, which by definition forces ν to have successors. This contradicts our assumption.
- Q is a pseudoleaf with corresponding path $p := (\nu_i)_{i \in \mathbb{N}}$. Suppose Q does not respect $\mathfrak{k}$. Let k be the maximal role depth occurring in $\mathfrak{k}$. Now, we set $\widetilde{Q} = Q \cap \mathcal{FLE}_k^{\text{norm}}$. We know that $\widetilde{Q}$ is finite, since $\mathcal{FLE}_k^{\text{norm}}$ is finite (see Remark 2.6). Thus, there must exist a node ν_i in p, such that $\widetilde{Q} \subseteq \epsilon(\nu_i)$. Since ν_i is contained in an infinite path, it must have successors. But then, it has a witnessing clause $\widetilde{\mathfrak{k}} = \mathcal{A} \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n\}$. Let $\mathcal{B}_j$ be the set from the conclusion for which $\epsilon(\nu_{i+1}) = \epsilon(\nu_i) \cup \mathcal{B}_j$.

  The maximal role depth of $\mathcal{B}_j$ must be greater than k, since otherwise we had $\mathcal{B}_j \subseteq \widetilde{Q} \subseteq \epsilon(\nu_i)$ and thus, $\widetilde{\mathfrak{k}}$ would already be respected by $\epsilon(\nu_i)$, hence, it could not be a witnessing clause. Therefore, the maximal role depth of $\widetilde{\mathfrak{k}}$'s whole conclusion is greater than k. But then, $\widetilde{\mathfrak{k}}$ is not minimal as demanded in the definition, since the maximal role depth of $\mathfrak{k}$'s conclusion is less or equal k and thus definitely smaller. So, we have found a contradiction to the assumption that there is a $\mathfrak{k} \in \mathcal{DR}(\mathfrak{K})$ not respected by Q.

❑

Definition 3.17 *Let $\mathfrak{K}$ be a set of cumulated clauses. A set $\mathcal{A} \subseteq \mathcal{FLE}^{\text{norm}}$ will be called* CONSISTENT *with respect to $\mathfrak{K}$ if there is no finite set $\mathcal{A}^* \subseteq \mathcal{A}$ such that $\mathfrak{K} \vdash \mathcal{A}^* \multimap \{\{\bot\}\}$.* ◇

Lemma 3.18 *Let $\mathfrak{K}$ be a set of cumulated clauses and $\mathcal{A} \subseteq \mathcal{FLE}^{\text{norm}}$ be consistent with respect to $\mathfrak{K}$. For any realization tree $T_{\mathcal{A}}^{\mathfrak{K}} = (N, r, \prec, \epsilon)$ of $\mathcal{A}$ holds that $\epsilon(\nu)$ is consistent for all $\nu \in N$.*

Proof:
Assume the contrary.
By assumption, we have consistency of $\epsilon(r)$. So, if inconsistent nodes ν of $T_{\mathcal{A}}^{\mathfrak{K}}$ exist, there must be some among them, the predecessor $\widetilde{\nu}$ of which is still consistent. Assume ν to be such a minimal inconsistent node. Now, let $\mathfrak{k} = \mathcal{B} \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$ be the witnessing clause of $\widetilde{\nu}$.
First, note that $\mathfrak{K} \vdash \mathcal{C}_i \multimap \{\{\bot\}\}$ can not be true for *all* $1 \leq i \leq n$, since otherwise, we could derive

$$\cfrac{\cfrac{\mathcal{B} \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_n\} \quad \mathcal{C}_1 \multimap \{\{\bot\}\}}{\mathcal{B} \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_{n-1}, \{\bot\}\}}\text{SUB*} \atop \vdots}{}$$

$$\cfrac{\mathcal{B} \multimap \{\mathcal{C}_n, \{\bot\}\} \qquad \mathcal{C}_n \multimap \{\{\bot\}\}}{\mathcal{B} \multimap \{\{\bot\}\}}\text{SUB*}$$

contradicting the assumption that ν is a minimal inconsistent node. So, there must at least be one $1 \leq i \leq n$ such that $\mathcal{C}_i \multimap \{\{\bot\}\}$ is not valid. W.l.o.g., we assume $i = 1$.
Now, let $\mathcal{C}_n$ be the set with $\epsilon(\nu) = \epsilon(\widetilde{\nu}) \cup \mathcal{C}_n$. So, due to our assumption, we have

$$\mathfrak{K} \vdash \mathcal{B} \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$$

as well as

$$\mathfrak{K} \vdash \mathcal{D} \cup \mathcal{C}_n \multimap \{\{\bot\}\}$$

for a finite $\mathcal{D} \in \epsilon(\widetilde{\nu})$.
Then we can do the following derivation:

$$\dfrac{\dfrac{\dfrac{\mathcal{B} \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}}{\mathcal{D} \cup \mathcal{B} \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}}\text{PE*} \quad \dfrac{\mathcal{D} \cup \mathcal{C}_n \multimap \{\{\bot\}\}}{\mathcal{D} \cup \mathcal{B} \cup \mathcal{C}_n \multimap \{\{\bot\}\}}\text{PE*}}{\mathcal{D} \cup \mathcal{B} \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_{n-1}, \{\bot\}\}}\text{SUB} \quad \dfrac{}{\{\bot\} \multimap \{\mathcal{C}_1\}}\text{CONT*}}{\mathcal{D} \cup \mathcal{B} \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_{n-1}\}}\text{SUB*}$$

Yet, the maximal role depth of the conclusion of this new cumulated clause $\widetilde{\mathfrak{k}}$ (the derivability of which has just been shown) is less or equal to that of $\mathfrak{k}$ and furthermore, $\widetilde{\mathfrak{k}}$'s conclusion is contained in that of $\mathfrak{k}$. Therefore, $\mathfrak{k}$ cannot be the witnessing clause of $\widetilde{\nu}$ since the minimality conditions are violated. So, we have a contradiction to the prior assumption. ❑

3.1.5 Completeness

Exploiting the two preceding propositions, we now show that any quasileaf of a realization tree with consistent root is an entity of the corresponding standard model. The basic idea of this proof is to show that any such quasileaf "survives" all iterations done in the standard model construction.
Furthermore, we show that any standard model entity has a quasileaf as a subset as well.

Lemma 3.19 *Let $\mathfrak{K}$ be a set of cumulated clauses and $\mathcal{A} \subseteq \mathcal{FLE}^{\mathrm{norm}}$ consistent, let $T_{\mathcal{A}}^{\mathfrak{K}} = (N, r, \prec, \epsilon)$ be a corresponding realization tree and $\overrightarrow{\mathbb{K}}(\mathfrak{K})$ the corresponding standard model. Then the following two statements hold:*

1. *for all quasileafs Q of $T_{\mathcal{A}}^{\mathfrak{K}}$, we have $Q \in \Delta$ and*
2. *for all $\delta \in \Delta$ containing $\mathcal{A}$ and being minimal with respect to set inclusion, there is a quasileaf Q of $T_{\mathcal{A}}^{\mathfrak{K}}$ with $Q = \delta$.*

Proof:
(1):
We will prove inductively that $Q \in \Delta^{(n)}$ for all $n \in \mathbb{N}$.

Induction anchor: $n = 0$.
Obviously, $Q \in \Delta^{(0)}$, since Q respects $\mathcal{DR}(\mathfrak{K})$ (due to Lemma 3.16) and thus in particular $\mathfrak{K}$. Additionally, we know that Q is consistent (and therefore in particular $\bot \notin Q$) due to Lemma 3.18.
Induction step: $n > 0$.
Considering Q, we have to show that

$$\{\mathcal{C} \mid \exists\mathtt{R}.\mathcal{C} \in Q\} = \bigcup\{\mathcal{P}_{\text{fin}}(\widetilde{\delta}) \mid (Q, \widetilde{\delta}) I_{\mathcal{R}}^{(n-1)} \mathtt{R}\} \quad (*)$$

and

$$\{\mathtt{C} \mid \forall\mathtt{R}.\mathtt{C} \in Q\} = \bigcap\{\widetilde{\delta} \mid (Q, \widetilde{\delta}) I_{\mathcal{R}}^{(n-1)} \mathtt{R}\} \quad (**)$$

(*):

"$\supseteq$" Let $\mathcal{C}$ be a finite subset of an R-neighbor $\widetilde{\delta}$ of Q in $\overrightarrow{\mathbb{K}}^{(n-1)}$. Since by construction we have $I_{\mathcal{R}}^{(n-1)} \subseteq I_{\mathcal{R}}^{(0)}$, we also know that $(Q, \widetilde{\delta}) I_{\mathcal{R}}^{(0)} \mathtt{R}$. But in view of the definition of $I_{\mathcal{R}}^{(0)}$, we know that $\exists\mathtt{R}.\bigsqcap\mathcal{C}$ has to be in Q.

"$\subseteq$" By induction hypothesis, we can assume that $Q \in \Delta^{(n-1)}$. Let $\exists\mathtt{R}.\bigsqcap\mathcal{C} \in Q$. We now have to show that (in $\overrightarrow{\mathbb{K}}^{(n-1)}$) there is an R-neighbor of Q containing $\mathcal{C}$. Suppose there is no such successor. $(+)$
We set $\widetilde{\mathcal{C}} := \mathcal{C} \cup \{\mathtt{D} \mid \forall\mathtt{R}.\mathtt{D} \in Q\}$. $\widetilde{\mathcal{C}}$ is consistent because otherwise, Q would be inconsistent as the derivation

$$\dfrac{\dfrac{\widetilde{\mathcal{C}} \multimap \{\{\bot\}\}}{[\exists\mathtt{R}]\widetilde{\mathcal{C}} \multimap \{\{\bot\}\}}\text{EL} \qquad \dfrac{}{[\exists\mathtt{R}]\mathcal{C} \cup [\forall\mathtt{R}]\{\mathtt{D} \mid \forall\mathtt{R}.\mathtt{D} \in Q\} \multimap \{[\exists\mathtt{R}]\widetilde{\mathcal{C}}\}}\text{AP*}}{[\exists\mathtt{R}]\mathcal{C} \cup [\forall\mathtt{R}]\{\mathtt{D} \mid \forall\mathtt{R}.\mathtt{D} \in Q\} \multimap \{\{\bot\}\}}\text{SUB*}$$

immediately shows.
Now consider a realization tree $T_{\widetilde{\mathcal{C}}}^{\mathfrak{K}}$ of $\widetilde{\mathcal{C}}$ (whose quasileafs are all in $\Delta^{(n-1)}$ by induction hypothesis). Due to the assumption $(+)$, no quasileaf of $T_{\widetilde{\mathcal{C}}}^{\mathfrak{K}}$ is an R-neighbor of Q in $\overrightarrow{\mathbb{K}}^{(n-1)}$ (since each of them contains $\mathcal{C}$). But then (due to the definition of $I_{\mathcal{R}}^{(n-1)}$) no quasileaf of $T_{\widetilde{\mathcal{C}}}^{\mathfrak{K}}$ is an R-neighbor of Q in $\overrightarrow{\mathbb{K}}^{(0)}$. So each of these $T_{\widetilde{\mathcal{C}}}^{\mathfrak{K}}$-quasileafs must contradict one of the conditions for being an R-neighbor of Q in $\overrightarrow{\mathbb{K}}^{(0)}$. Obviously, every quasileaf $\widetilde{Q}$ of $T_{\widetilde{\mathcal{C}}}^{\mathfrak{K}}$ fulfills the condition that $\mathtt{C} \in \widetilde{Q}$ for all $\forall\mathtt{R}.\mathtt{C} \in Q$, since already $\widetilde{\mathcal{C}}$ contains all such $\mathtt{C}$. So, to fulfill our assumption $(+)$, every $T_{\widetilde{\mathcal{C}}}^{\mathfrak{K}}$-quasileaf $\widetilde{Q}$ must violate the other condition: it has to contain a finite set $\mathcal{D}_{\widetilde{Q}} \in \mathcal{FLE}^{\text{norm}}$ such that $\exists\mathtt{R}.\bigsqcap\mathcal{D}_{\widetilde{Q}} \notin Q$. $(++)$
For every $T_{\widetilde{\mathcal{C}}}^{\mathfrak{K}}$-quasileaf $\widetilde{Q}$, we find a node ν_p on each of its generating paths p with $\mathcal{D}_{\widetilde{Q}} \subseteq \epsilon(\nu_p)$. Taking for all quasileafs $\widetilde{Q}$ these nodes ν_p, we have found a covering N^* of $T_{\widetilde{\mathcal{C}}}^{\mathfrak{K}}$. Due to Lemma 3.14, we then also find a finite covering $\widetilde{N} \subseteq N^*$. For every $\widetilde{\nu} \in \widetilde{N}$, we choose an arbitrary path $\widetilde{p}$ containing $\widetilde{\nu}$. Let $\widetilde{p}$ generate $\widetilde{Q}$. Now, we again assign a finite $\mathcal{FLE}$ subset $\mathcal{D}_{\widetilde{\nu}}$ to each $\widetilde{\nu}$ by $\mathcal{D}_{\widetilde{\nu}} := \mathcal{D}_{\widetilde{Q}}$.
Now, let

$$\{\mathcal{D}_1, \ldots, \mathcal{D}_k\} := \{\mathcal{D}_{\widetilde{\nu}} \mid \widetilde{\nu} \in \widetilde{N}\}.$$

Using Lemma 3.15, it follows

$$\mathfrak{K} \vdash \mathcal{C}^* \multimap \{\mathcal{D}_1, \ldots, \mathcal{D}_k\}$$

for a finite $\mathcal{C}^* \subseteq \widetilde{\mathcal{C}}$.

Now, we can carry out the following derivation:

$$\dfrac{\dfrac{}{[\exists\mathtt{R}]\mathcal{C} \cup [\forall\mathtt{R}](\mathcal{C}^* \setminus \mathcal{C}) \multimap \{[\exists\mathtt{R}]\mathcal{C}^*\}}\text{AP*} \quad \dfrac{\mathcal{C}^* \multimap \{\mathcal{D}_1, \ldots, \mathcal{D}_k\}}{[\exists\mathtt{R}]\mathcal{C}^* \multimap \{[\exists\mathtt{R}]\mathcal{D}_1, \ldots, [\exists\mathtt{R}]\mathcal{D}_k\}}\text{EL}}{[\exists\mathtt{R}]\mathcal{C} \cup [\forall\mathtt{R}](\mathcal{C}^* \setminus \mathcal{C}) \multimap \{[\exists\mathtt{R}]\mathcal{D}_1, \ldots, [\exists\mathtt{R}]\mathcal{D}_k\}}\text{SUB*}$$

Due to the construction of $\mathcal{C}$ and $\mathcal{C}^*$, Q contains the premise of this cumulated clause. Furthermore, we know from Lemma 3.16 that Q has to respect all clauses from $\mathcal{DR}(\mathfrak{K})$. So, Q has to contain one element from $\{[\exists\mathtt{R}]\mathcal{D}_1, \ldots, [\exists\mathtt{R}]\mathcal{D}_k\}$ which contradicts the way they have been chosen in $(++)$.

So, our prior assumption $(+)$ must be false.

(**):

"$\subseteq$" Let $\mathtt{C}$ be a concept description for which $\forall\mathtt{R}.\mathtt{C} \in Q$. By definition of $I_{\mathcal{R}}^{(0)}$, we know that this implies $\mathtt{C} \in \widetilde{\delta}$ if $(Q, \widetilde{\delta}) I_{\mathcal{R}}^{(0)} \mathtt{R}$. So, we also know that $\mathtt{C} \in \bigcap\{\widetilde{\delta} \mid (Q, \widetilde{\delta}) I_{\mathcal{R}}^{(0)} \mathtt{R}\}$. From $I_{\mathcal{R}}^{(n-1)} \subseteq I_{\mathcal{R}}^{(0)}$, we can conclude that $\bigcap\{\widetilde{\delta} \mid (Q, \widetilde{\delta}) I_{\mathcal{R}}^{(0)} \mathtt{R}\} \subseteq \bigcap\{\widetilde{\delta} \mid (Q, \widetilde{\delta}) I_{\mathcal{R}}^{(n-1)} \mathtt{R}\}$ and therefore $\mathtt{C} \in \bigcap\{\widetilde{\delta} \mid (Q, \widetilde{\delta}) I_{\mathcal{R}}^{(n-1)} \mathtt{R}\}$.

"$\supseteq$" Let $\mathtt{C} \in \bigcap\{\widetilde{\delta} \mid (Q, \widetilde{\delta}) I_{\mathcal{R}}^{(n-1)} \mathtt{R}\}$. We have to show that $\forall\mathtt{R}.\mathtt{C} \in Q$.
Assume the contrary, i.e., $\forall\mathtt{R}.\mathtt{C} \notin Q$. Let $\mathcal{C} := \{\mathtt{D} \mid \forall\mathtt{R}.\mathtt{D} \in Q\}$. If $\mathcal{C}$ were inconsistent, we could immediately construct a contradiction by deriving

$$\dfrac{\dfrac{\mathcal{C} \multimap \{\{\bot\}\}}{[\forall\mathtt{R}]\mathcal{C} \multimap \{\{\forall\mathtt{R}.\bot\}\}}\text{AL} \quad \dfrac{\dfrac{\dfrac{}{\{\bot\} \multimap \{\{\mathtt{C}\}\}}\text{CONT}}{\{\forall\mathtt{R}.\bot\} \multimap \{\{\forall\mathtt{R}.\mathtt{C}\}\}}\text{AL}}{[\forall\mathtt{R}]\mathcal{C} \cup \{\forall\mathtt{R}.\bot\} \multimap \{\{\forall\mathtt{R}.\mathtt{C}\}\}}\text{PE*}}{[\forall\mathtt{R}]\mathcal{C} \multimap \{\{\forall\mathtt{R}.\mathtt{C}\}\}}\text{SUB}$$

which would force Q to contain $\forall\mathtt{R}.\mathtt{C}$. So, $\mathcal{C}$ has to be consistent.

Now, consider a realization tree $T_{\mathcal{C}}^{\mathfrak{K}}$ of $\mathcal{C}$ (remember that by induction hypothesis, all its quasileafs are in $\Delta^{(n-1)}$). We assign to each $T_{\mathcal{C}}^{\mathfrak{K}}$-quasileaf $\widetilde{Q}$ a finite concept description set $\mathcal{D}_{\widetilde{Q}} \subseteq \widetilde{Q}$ in the following way:

- For each quasileaf $\widetilde{Q}$ with $(Q, \widetilde{Q}) I_{\mathcal{R}}^{(n-1)} \mathtt{R}$, we set $\mathcal{D}_{\widetilde{Q}} := \{\mathtt{C}\}$ (this is correct, since $\mathtt{C}$ is contained in every $\mathtt{R}$-neighbor of Q in $\overrightarrow{\mathbb{K}}^{(n-1)}$).
- If a quasileaf $\widetilde{Q}$ is not an $\mathtt{R}$-neighbor of Q in $\overrightarrow{\mathbb{K}}^{(n-1)}$, it cannot be an $\mathtt{R}$-neighbor of Q in $\overrightarrow{\mathbb{K}}^{(0)}$ as well. Hence, it must violate one of the two conditions in the definition of $I_{\mathcal{R}}^{(0)}$. Obviously, every quasileaf $\widetilde{Q}$ of $T_{\mathcal{C}}^{\mathfrak{K}}$ fulfills the condition that $\mathtt{C} \in \widetilde{Q}$ for all $\forall\mathtt{R}.\mathtt{C} \in Q$, since already $\mathcal{C}$ contains all such $\mathtt{C}$. So, the second condition must be violated and thus there has to be a finite concept description set $\mathcal{E} \subseteq \widetilde{Q}$ with $\exists\mathtt{R}.\bigsqcap\mathcal{E} \notin Q$. Then we set $\mathcal{D}_{\widetilde{Q}} := \mathcal{E}$.

Now, since all those assigned concept description sets are finite, we find on every generating path p of a quasileaf $\widetilde{Q}$ a node ν_p for which already holds $\mathcal{D}_{\widetilde{Q}} \subseteq \epsilon(\nu_p)$. Collecting all those nodes, we get a covering N^* of $T_{\mathcal{C}}^{\mathfrak{K}}$. Due to Lemma 3.14, we find a finite covering $\widetilde{N} \subseteq N^*$.

For every $\widetilde{\nu} \in \widetilde{N}$, we choose an arbitrary path $\widetilde{p}$ containing $\widetilde{\nu}$. Let $\widetilde{p}$ generate $\widetilde{Q}$. Now, we again assign a finite $\mathcal{FLE}$ subset $\mathcal{D}_{\widetilde{\nu}}$ to each $\widetilde{\nu}$ by $\mathcal{D}_{\widetilde{\nu}} := \mathcal{D}_{\widetilde{Q}}$ Now, let

$$\{\mathcal{D}_1, \ldots, \mathcal{D}_k\} := \{\mathcal{D}_{\widetilde{\nu}} \mid \widetilde{\nu} \in \widetilde{N}\}.$$

By using Lemma 3.15, it follows

$$\mathfrak{K} \vdash \mathcal{C}^* \multimap \{\mathcal{D}_1, \ldots, \mathcal{D}_k\}$$

for a finite $\mathcal{C}^* \subseteq \mathcal{C}$. If $\{\mathtt{C}\}$ is not yet contained in $\{\mathcal{D}_1, \ldots, \mathcal{D}_k\}$, we may easily include it by one application of the conclusion extension rule. So, we get

$$\mathfrak{K} \vdash \mathcal{C}^* \multimap \{\{\mathtt{C}\}, \mathcal{E}_1, \ldots, \mathcal{E}_j\}$$

with $\exists \mathtt{R}.\bigsqcap \mathcal{E}_i \notin Q$ (as the $\mathcal{E}_i$ have been chosen).

But now, a single application of the $\forall$-lifting rule yields

$$\mathfrak{K} \vdash [\forall \mathtt{R}]\mathcal{C}^* \multimap \{\{\forall \mathtt{R}.\mathtt{C}\}, [\exists \mathtt{R}]\mathcal{E}_1, \ldots, [\exists \mathtt{R}]\mathcal{E}_j\}.$$

Since Q as a quasileaf of $T_{\mathcal{A}}^{\mathfrak{K}}$ has to respect all cumulated clauses of $\mathcal{DR}(\mathfrak{K})$ (due to Lemma 3.16) and we have $[\forall \mathtt{R}]\mathcal{C}^* \subseteq Q$ by construction, Q has to contain either $\forall \mathtt{R}.\mathtt{C}$ (which contradicts our first assumption) or one $[\exists \mathtt{R}]\mathcal{E}_i$ which contradicts the choice of the $\mathcal{E}_i$. So our assumption $\forall \mathtt{R}.\mathtt{C} \notin Q$ must be false.

(2)

Since we know that $\epsilon(r) = \mathcal{A}$, we also know $\epsilon(r) \subseteq \delta$.

Now, we construct a complete path $r = \nu_0 \prec \nu_1 \prec \ldots$ in $T_{\mathcal{A}}^{\mathfrak{K}}$ in the following way: If ν_i has no successors, we are done and have constructed a complete finite path. Otherwise, we select the node ν_{i+1} as follows: We presuppose that for a ν_i we have $\epsilon(\nu_i) \subseteq \delta$. Considering the witnessing clause $\mathfrak{k} = \mathcal{B} \multimap \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$ of ν_i in $T_{\mathcal{A}}^{\mathfrak{K}}$, we know that δ must respect $\mathfrak{k}$ due to the soundness of $\mathcal{DR}$. Furthermore, Lemma 3.8 assures the correspondence of (syntactic) containment and (semantic) validity of $\mathcal{FLE}^{\mathrm{norm}}$ concept descriptions in the standard model. Hence, since the premise of the witnessing clause is contained in $\epsilon(\nu_i)$ which in turn is a subset of δ, we have $\delta \models \mathcal{B}$ and therefore we have $\delta \models \mathcal{C}_k$ for some $k \in \{1, \ldots, n\}$. Now, we choose ν_{i+1} such that $\epsilon(\nu_{i+1}) = \epsilon(\nu_i) \cup \mathcal{C}_k$, thereby assuring $\epsilon(\nu_{i+1}) \subseteq \delta$.

Let Q be the quasileaf generated by the (finite or infinite) complete path $\nu_0 \prec \nu_1 \prec \ldots$. Due to the first part of this theorem, we know that $Q \in \Delta$. By construction, we also know that $\mathcal{A} \subseteq Q$ as well as $Q \subseteq \delta$. However, since δ is minimal with respect to set inclusion by assumption, we can conclude $Q = \delta$. ❑

Having established this correspondence between the standard model and realization trees, it is not difficult to prove that any cumulated clause valid in the standard model is $\mathcal{DR}$-derivable, which (as the subsequent corollary shows) gives us the completeness of $\mathcal{DR}$.

Theorem 3.20 *Let $\mathfrak{K}$ be a set of cumulated clauses and let $\mathfrak{k}$ be a cumulated clause. Then,*

$$\overrightarrow{\mathbb{K}}(\mathfrak{K}) \models \mathfrak{k} \Longrightarrow \mathfrak{K} \vdash \mathfrak{k}.$$

Proof:
Let $\mathfrak{k} = \mathcal{A} \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n\}$. Consider a realization tree $T_{\mathcal{A}}^{\mathfrak{K}}$ of $\mathcal{A}$. From Theorem 3.19, we know that for each quasileaf Q of $T_{\mathcal{A}}^{\mathfrak{K}}$ holds $Q \in \Delta$. From $\mathcal{A} \subseteq Q$ and using Lemma 3.16, we can conclude $\mathcal{B}_i \subseteq Q$ for some $i \in \{1, \ldots, n\}$.
Since all $\mathcal{B}_i$ are finite, we find on every complete path a node ν for which already holds $\mathcal{B}_i \subseteq \epsilon(\nu)$ for some $\mathcal{B}_i$. This means that we have found a covering N^* of $T_{\mathcal{A}}^{\mathfrak{K}}$, that due to Lemma 3.14 can be minimized to a finite covering $\widetilde{N} \subseteq N^*$. In view of Lemma 3.15 we then get $\mathfrak{K} \vdash \mathcal{A}^* \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n\}$ for some $\mathcal{A}^* \subseteq \mathcal{A}$ and consequently by cumulated premise extension $\mathfrak{K} \vdash \mathcal{A} \multimap \{\mathcal{B}_1, \ldots, \mathcal{B}_n\}$. ❑

Corollary 3.21 *The deduction calculus $\mathcal{DR}$ for cumulated clauses on $\mathcal{FLE}^{\mathrm{norm}}$ is sound and complete.*

Proof:
Soundness has been shown by Theorem 3.4. Completeness also follows directly from the preceding theorem: If a cumulated clause $\mathfrak{k}$ is valid in all power context families that respect a set $\mathfrak{K}$ of cumulated clauses, it is in particular valid in $\overrightarrow{\mathbb{K}}(\mathfrak{K})$. But then it is derivable. ❑

3.2 A Decision Procedure

In this section, we describe a way to decide whether a cumulated clause on $\mathcal{FLE}^{\mathrm{norm}}$ is a semantic consequence of a finite set of cumulated clauses. This will be done by the construction of a "finite version" of the standard model where the maximal role depth of the involved concept descriptions is restricted. Hence, this model can be computed in finitely many steps.

Definition 3.22 *Let $\mathfrak{K}$ be a set of cumulated clauses on $\mathcal{FLE}_k^{\mathrm{norm}}$ with $k \in \mathbb{N}$. The k-LIMITED STANDARD MODEL $\overrightarrow{\mathbb{K}}_k(\mathfrak{K})$ is the binary power context family $\overrightarrow{\mathbb{K}}_k(\mathfrak{K}) = (\mathbb{K}_{\mathcal{C}}, \mathbb{K}_{\mathcal{R}}) = ((\Delta, M_{\mathcal{C}}, I_{\mathcal{C}}), (\Delta \times \Delta, M_{\mathcal{R}}, I_{\mathcal{R}}))$ defined as follows:*

- *First, we set $\overrightarrow{\mathbb{K}}^{(0)}(\mathfrak{K}) = ((\Delta^{(0)}, M_{\mathcal{C}}, I_{\mathcal{C}}^{(0)}), (\Delta^{(0)} \times \Delta^{(0)}, M_{\mathcal{R}}, I_{\mathcal{R}}^{(0)}))$ with*
  - $\Delta^{(0)} := \{\mathcal{N} \subseteq \mathcal{FLE}_k^{\mathrm{norm}} \mid \mathcal{N}$ *respects all* $\mathfrak{k} \in \mathfrak{K}, \bot \notin \mathcal{N}\}$,
  - $\delta I_{\mathcal{C}}^{(0)} \mathtt{C} :\Leftrightarrow \mathtt{C} \in \delta$,
  - $(\delta_1, \delta_2) I_{\mathcal{R}}^{(0)} \mathtt{R} :\Leftrightarrow \exists \mathtt{R}. \bigsqcap \mathcal{C} \in \delta_1$ *for all* $\mathcal{C} \subseteq \delta_2 \cap \mathcal{FLE}_{k-1}^{\mathrm{norm}}$ *and* $\mathtt{C} \in \delta_2$ *for all* $\forall \mathtt{R}.\mathtt{C} \in \delta_1$.
- *From $\overrightarrow{\mathbb{K}}_k^{(n)}(\mathfrak{K})$, we determine $\overrightarrow{\mathbb{K}}_k^{(n+1)}(\mathfrak{K})$ by*

- $\Delta^{(n+1)} := \Big\{\delta \in \Delta^{(n)} \mid \{\mathtt{C} \mid \forall\mathtt{R.C} \in \delta\} = \mathcal{FLE}_{k-1}^{\text{norm}} \cap \bigcap_{(\delta,\widetilde{\delta}) I_{\mathcal{R}}^{(n)} \mathtt{R}} \widetilde{\delta}$ *and* $\{\mathcal{C} \mid \exists\mathtt{R}.\bigsqcap \mathcal{C} \in \delta\} = \bigcup_{(\delta,\widetilde{\delta}) I_{\mathcal{R}}^{(n)} \mathtt{R}} \mathcal{P}(\widetilde{\delta} \cap \mathcal{FLE}_{k-1}^{\text{norm}})$ *for all* $\mathtt{R} \in M_{\mathcal{R}}\Big\}$,
- $I_{\mathcal{C}}^{(n+1)} := I_{\mathcal{C}}^{(0)} \cap \Delta^{(n+1)} \times M_{\mathcal{C}}$,
- $I_{\mathcal{R}}^{(n+1)} := I_{\mathcal{R}}^{(0)} \cap (\Delta^{(n+1)} \times \Delta^{(n+1)}) \times M_{\mathcal{R}}$.

- *Now, we set*
  - $\Delta := \bigcap_{i \in \mathbb{N}} \Delta^{(i)}$,
  - $I_{\mathcal{C}} := I_{\mathcal{C}}^{(0)} \cap \Delta \times M_{\mathcal{C}}$, *and*
  - $I_{\mathcal{R}} := I_{\mathcal{R}}^{(0)} \cap \Delta^2 \times M_{\mathcal{R}}$.

◇

Note that due to the finiteness of $\Delta^{(0)}$, the fact $\Delta^{(0)} \supseteq \Delta^{(1)} \supseteq \Delta^{(2)} \supseteq \ldots$, and knowing that from $\Delta^{(i)} = \Delta^{(i+1)}$ directly follows $\Delta^{(i)} = \Delta^{(j)}$ for any $j \geq i$, we know that

- $\Delta = \Delta^{(i)}$ for some i as well as
- computing the sequence $\Delta^{(0)}, \Delta^{(1)}, \Delta^{(2)}, \ldots$, it can be decided when this i has been reached.

This ensures that $\overrightarrow{\mathbb{K}}_k(\mathfrak{K})$ is computable.

Lemma 3.23 *Let $\mathfrak{K}$ be a set of cumulated clauses on $\mathcal{FLE}_k^{\text{norm}}$ and $\overrightarrow{\mathbb{K}}_k(\mathfrak{K})$ the corresponding k-limited standard model. Then, we have for every $\mathtt{D} \in \mathcal{FLE}_k^{\text{norm}}$ and every $\delta \in \Delta$*

$$\mathtt{D} \in \delta \Longleftrightarrow \delta \models \mathtt{D}.$$

Proof:
Obviously, for every $\delta \in \Delta$ from $\overrightarrow{\mathbb{K}}(\mathfrak{K})$ holds:

$$\{\mathcal{C} \mid \exists\mathtt{R}.\bigsqcap \mathcal{C} \in \delta\} = \bigcup \{\mathcal{P}(\widetilde{\delta} \cap \mathcal{FLE}_{k-1}^{\text{norm}}) \mid (\delta,\widetilde{\delta}) I_{\mathcal{R}} \mathtt{R}\} \qquad (*)$$

as well as

$$\{\mathtt{C} \mid \forall\mathtt{R.C} \in \delta\} = \mathcal{FLE}_{k-1}^{\text{norm}} \cap \bigcap \{\widetilde{\delta} \mid (\delta,\widetilde{\delta}) I_{\mathcal{R}} \mathtt{R}\}. \qquad (**)$$

We do now an induction over the maximal role depth of a concept description D:

- Induction anchor: $\mathtt{D} \in \mathcal{FLE}_0^{\text{norm}}$.
  Then we have either $\mathtt{D} \in M_{\mathcal{C}}$ or $\mathtt{D} = \bot$. In the first case, we have $\mathtt{D} \in \delta$ if and only if $\delta I_{\mathcal{C}} \mathtt{D}$ by definition of the standard model. In view of the semantics definition, this is equivalent to $\delta \models \mathtt{D}$.
  Considering the second case, we find that $\bot \in \delta$ does not occur (due to the explicit exclusion of entities containing $\bot$ in the standard model definition) as well as $\delta \models \bot$ is never the case since $[\![\bot]\!]_{\vec{\mathbb{K}}} = \emptyset$. So, those both statements are trivially equivalent.
- Induction step: $\mathtt{D} \in \mathcal{FLE}_n^{\text{norm}}$, $0 < n \leq k$.
  Again, we have to distinguish two cases.
  First, assume $\mathtt{D} = \exists\mathtt{R}.\bigsqcap\mathcal{D}$ with $\mathcal{D} \subseteq \mathcal{FLE}_{n-1}^{\text{norm}}$. Then, the statement $\exists\mathtt{R}.\bigsqcap\mathcal{D} \in \delta$ is obviously equivalent to

$$\mathcal{D} \in \{\mathcal{C} \mid \exists\mathtt{R}.\bigsqcap\mathcal{C} \in \delta\}$$

and this - because of (*) - to

$$\mathcal{D} \in \bigcup\{\mathcal{P}(\widetilde{\delta} \cap \mathcal{FLE}_{k-1}^{\text{norm}}) \mid (\delta, \widetilde{\delta}) I_{\mathcal{R}} \mathtt{R}\}.$$

So, we know that there exists an R-neighbor $\widetilde{\delta}$ of δ, which contains all concept descriptions from $\mathcal{D}$. Since $\mathcal{D} \subseteq \mathcal{FLE}_{n-1}^{\text{norm}}$, we see by induction hypothesis that this is the case exactly if $\widetilde{\delta} \models \mathtt{E}$ for all $\mathtt{E} \in \mathcal{D}$. Subsequently, this is equivalent to

$$\exists\widetilde{\delta} : (\delta, \widetilde{\delta}) I_{\mathcal{R}} \mathtt{R} \wedge \widetilde{\delta} \in \bigcap_{\mathtt{E} \in \mathcal{D}} [\![\mathtt{E}]\!]_{\vec{\mathbb{K}}}$$

and this (by the semantics definition) to

$$\exists\widetilde{\delta} : (\delta, \widetilde{\delta}) I_{\mathcal{R}} \mathtt{R} \wedge \widetilde{\delta} \in [\![\bigsqcap\mathcal{D}]\!]_{\vec{\mathbb{K}}}$$

and finally to

$$\delta \in [\![\exists\mathtt{R}.\bigsqcap\mathcal{D}]\!]_{\vec{\mathbb{K}}}$$

which means just $\delta \models \exists\mathtt{R}.\bigsqcap\mathcal{D}$.

It remains to consider the case $\mathtt{D} = \forall\mathtt{R}.\mathtt{E}$ with $\mathtt{E} \in \mathcal{FLE}_{n-1}^{\text{norm}}$. Then, $\forall\mathtt{R}.\mathtt{E} \in \delta$ can be written as

$$\mathtt{E} \in \{\mathtt{C} \mid \forall\mathtt{R}.\mathtt{C} \in \delta\}$$

which is due to (**) equivalent to

$$\mathtt{E} \in \bigcap\{\widetilde{\delta} \mid (\delta, \widetilde{\delta}) I_{\mathcal{R}} \mathtt{R}\}.$$

Therefore knowing that all R-neighbors of δ contain E (which is an element of $\mathcal{FLE}_{n-1}^{\text{norm}}$), we conclude by the induction hypothesis that this is equivalent to

$$\forall\widetilde{\delta} : (\delta, \widetilde{\delta}) I_{\mathcal{R}} \mathtt{R} \rightarrow \widetilde{\delta} \in [\![\mathtt{E}]\!]_{\vec{\mathbb{K}}}$$

and by the semantics definition to

$$\delta \in [\![\forall\mathtt{R}.\mathtt{E}]\!]_{\vec{\mathbb{K}}}$$

which means just $\delta \models \forall\mathtt{R}.\mathtt{E}$.

Note that all argumentations work in both directions. So indeed, the equivalence is assured.

❑

In the sequel, we will show a correspondence between the k-limited standard model and the standard model from section 3.1.

Theorem 3.24 *Let $\mathfrak{K}$ be a set of cumulated clauses on $\mathcal{FLE}_k^{\text{norm}}$ for some $k \in \mathbb{N}$. Let*

$$\overrightarrow{\mathbb{K}}(\mathfrak{K}) = ((\Delta, I_{\mathcal{C}}, M_{\mathcal{C}}), (\Delta^2, I_{\mathcal{R}}, M_{\mathcal{R}}))$$

and

$$\overrightarrow{\mathbb{K}}_k(\mathfrak{K}) = ((\widetilde{\Delta}, \widetilde{I}_{\mathcal{C}}, M_{\mathcal{C}}), (\widetilde{\Delta}^2, \widetilde{I}_{\mathcal{R}}, M_{\mathcal{R}})).$$

Then

$$\widetilde{\Delta} = \{\delta \cap \mathcal{FLE}_k \mid \delta \in \Delta\}.$$

Proof:
"$\subseteq$"
$\overrightarrow{\mathbb{K}}_k(\mathfrak{K})$ respects all clauses from $\mathfrak{K}$ due to the definition of $\widetilde{\Delta}$ (via $\widetilde{\Delta}^{(0)}$) and Lemma 3.23. Hence, Theorem 3.10 is applicable. So, we get for all $\widetilde{\delta} \in \widetilde{\Delta}$

$$\varphi(\widetilde{\delta}) \in \Delta$$

and therefore also

$$\varphi(\widetilde{\delta}) \cap \mathcal{FLE}_k^{\text{norm}} \in \{\delta \cap \mathcal{FLE}_k^{\text{norm}} \mid \delta \in \Delta\}.$$

On the other hand, from the definition of φ and Lemma 3.23, it follows

$$\varphi(\widetilde{\delta}) \cap \mathcal{FLE}_k^{\text{norm}} = \{\mathtt{C} \in \mathcal{FLE}_k^{\text{norm}} \mid \widetilde{\delta} \models \mathtt{C}\} = \{\mathtt{C} \in \mathcal{FLE}_k^{\text{norm}} \mid \mathtt{C} \in \widetilde{\delta}\} = \widetilde{\delta}.$$

This yields

$$\widetilde{\delta} \in \{\delta \cap \mathcal{FLE}_k^{\text{norm}} \mid \delta \in \Delta\}.$$

"$\supseteq$"
First, we will show two helpful facts:

- For all $\delta \in \Delta$, we have $\delta \cap \mathcal{FLE}_k^{\text{norm}} \in \widetilde{\Delta}^{(0)}$.

  By construction, we know that δ has to be in $\Delta^{(0)}$. If so, it has to respect all cumulated clauses from $\mathfrak{K}$ and must not contain $\bot$. Since $\mathfrak{K}$ is taken from $\mathcal{CC}(\mathcal{FLE}_k^{\text{norm}})$, we can conclude that the same conditions hold for $\delta \cap \mathcal{FLE}_k^{\text{norm}}$ and hence $\delta \cap \mathcal{FLE}_k^{\text{norm}} \in \widetilde{\Delta}^{(0)}$.

- For all $\delta, \widetilde{\delta} \in \Delta$ with $(\delta, \widetilde{\delta}) I_{\mathcal{R}} \mathtt{R}$, we have $(\delta \cap \mathcal{FLE}_k^{\text{norm}}, \widetilde{\delta} \cap \mathcal{FLE}_k^{\text{norm}}) \widetilde{I}_{\mathcal{R}}^{(0)} \mathtt{R}$.

  From $(\delta, \widetilde{\delta}) I_{\mathcal{R}} \mathtt{R}$, it follows that $\exists \mathtt{R}. \bigsqcap \mathcal{C} \in \delta$ for all finite $\mathcal{C} \subseteq \widetilde{\delta}$ as well as $\mathtt{C} \in \widetilde{\delta}$ for all $\forall \mathtt{R}.\mathtt{C} \in \delta$ by definition. This obviously directly implies $\exists \mathtt{R}. \bigsqcap \mathcal{C} \in \delta \cap \mathcal{FLE}_k^{\text{norm}}$ for all $\mathcal{C} \subseteq \widetilde{\delta} \cap \mathcal{FLE}_{k-1}^{\text{norm}}$ and $\mathtt{C} \in \widetilde{\delta} \cap \mathcal{FLE}_k^{\text{norm}}$ for all $\forall \mathtt{R}.\mathtt{C} \in \delta \cap \mathcal{FLE}_k^{\text{norm}}$. Since we also know by the preceding fact that $\delta \cap \mathcal{FLE}_k^{\text{norm}} \in \widetilde{\Delta}^{(0)}$ and $\widetilde{\delta} \cap \mathcal{FLE}_k^{\text{norm}} \in \widetilde{\Delta}^{(0)}$, we can conclude $(\delta \cap \mathcal{FLE}_k^{\text{norm}}, \widetilde{\delta} \cap \mathcal{FLE}_k^{\text{norm}}) \widetilde{I}_{\mathcal{R}}^{(0)} \mathtt{R}$.

Now, we consider the inclusion to show. Assume the contrary, i.e., for a certain $\delta \in \Delta$, there is no counterpart in $\widetilde{\Delta}$ coinciding with δ on $\mathcal{FLE}_k^{\mathrm{norm}}$. However, certainly $\delta \cap \mathcal{FLE}_k^{\mathrm{norm}} \in \widetilde{\Delta}^{(0)}$, since δ respects all clauses from $\mathfrak{K}$ and $\bot \notin \delta$ (these are necessary conditions for $\delta \in \Delta$ by definition). Then, $\delta \cap \mathcal{FLE}_k^{\mathrm{norm}} \in \widetilde{\Delta}^{(h)}$ and $\delta \cap \mathcal{FLE}_k^{\mathrm{norm}} \notin \widetilde{\Delta}^{(h+1)}$ for some $h \in \mathbb{N}$. We consider a δ with minimal h.
So, we know that $\{\bar{\delta} \cap \mathcal{FLE}_k^{\mathrm{norm}} \mid \bar{\delta} \in \Delta\} \subseteq \widetilde{\Delta}^{(h)}$. By the second of the facts shown above and the definition of $\widetilde{I}_{\mathcal{R}}^{(h)}$, we have also $(\delta \cap \mathcal{FLE}_k^{\mathrm{norm}}, \widetilde{\delta} \cap \mathcal{FLE}_k^{\mathrm{norm}})\widetilde{I}_{\mathcal{R}}^{(h)}\mathtt{R}$ for all $\delta, \widetilde{\delta} \in \Delta$ with $(\delta, \widetilde{\delta})I_{\mathcal{R}}\mathtt{R}$.
Thus, we can conclude:

$$\begin{aligned} & \{\mathcal{C} \mid \exists\mathtt{R}\textstyle\bigsqcap\mathcal{C} \in \delta \cap \mathcal{FLE}_k^{\mathrm{norm}}\} \\ = & \{\mathcal{C} \cap \mathcal{FLE}_{k-1}^{\mathrm{norm}} \mid \exists\mathtt{R}\textstyle\bigsqcap\mathcal{C} \in \delta\} \\ = & \textstyle\bigcup_{(\delta,\widetilde{\delta})I_{\mathcal{R}}\mathtt{R}} \mathcal{P}_{\mathrm{fin}}(\widetilde{\delta} \cap \mathcal{FLE}_{k-1}^{\mathrm{norm}}) \\ \subseteq & \textstyle\bigcup_{(\delta,\widetilde{\delta})\widetilde{I}_{\mathcal{R}}^{(h)}\mathtt{R}} \mathcal{P}_{\mathrm{fin}}(\widetilde{\delta} \cap \mathcal{FLE}_{k-1}^{\mathrm{norm}}) \end{aligned}$$

and by construction of $\widetilde{I}_{\mathcal{R}}^{(h)}$ via $\widetilde{I}_{\mathcal{R}}^{(0)}$,

$$\bigcup_{(\delta,\widetilde{\delta})\widetilde{I}_{\mathcal{R}}^{(h)}\mathtt{R}} \mathcal{P}_{\mathrm{fin}}(\widetilde{\delta} \cap \mathcal{FLE}_{k-1}^{\mathrm{norm}}) \subseteq \{\mathcal{C} \mid \exists\mathtt{R}\bigsqcap\mathcal{C} \in \delta \cap \mathcal{FLE}_k^{\mathrm{norm}}\}$$

as well as

$$\begin{aligned} & \{\mathtt{C} \mid \forall\mathtt{R}.\mathtt{C} \in \delta \cap \mathcal{FLE}_k^{\mathrm{norm}}\} \\ = & \textstyle\bigcap_{(\delta,\widetilde{\delta})I_{\mathcal{R}}\mathtt{R}} \widetilde{\delta} \cap \mathcal{FLE}_{k-1}^{\mathrm{norm}} \\ \supseteq & \textstyle\bigcap_{(\delta,\widetilde{\delta})\widetilde{I}_{\mathcal{R}}^{(h)}\mathtt{R}} \widetilde{\delta} \cap \mathcal{FLE}_{k-1}^{\mathrm{norm}} \end{aligned}$$

and by construction of $\widetilde{I}_{\mathcal{R}}^{(h)}$ via $\widetilde{I}_{\mathcal{R}}^{(0)}$,

$$\{\mathtt{C} \mid \forall\mathtt{R}.\mathtt{C} \in \delta \cap \mathcal{FLE}_k^{\mathrm{norm}}\} \subseteq \bigcap_{(\delta,\widetilde{\delta})\widetilde{I}_{\mathcal{R}}^{(h)}\mathtt{R}} \widetilde{\delta} \cap \mathcal{FLE}_{k-1}^{\mathrm{norm}}.$$

But then, the facts

$$\{\mathcal{C} \mid \exists\mathtt{R}\bigsqcap\mathcal{C} \in \delta \cap \mathcal{FLE}_k^{\mathrm{norm}}\} = \bigcup_{(\delta,\widetilde{\delta})\widetilde{I}_{\mathcal{R}}^{(h)}\mathtt{R}} \mathcal{P}_{\mathrm{fin}}(\widetilde{\delta} \cap \mathcal{FLE}_{k-1}^{\mathrm{norm}})$$

and

$$\{\mathtt{C} \mid \forall\mathtt{R}.\mathtt{C} \in \delta \cap \mathcal{FLE}_k^{\mathrm{norm}}\} = \bigcap_{(\delta,\widetilde{\delta})\widetilde{I}_{\mathcal{R}}^{(h)}\mathtt{R}} \widetilde{\delta} \cap \mathcal{FLE}_{k-1}^{\mathrm{norm}}.$$

imply by definition $\delta \cap \mathcal{FLE}_k^{\mathrm{norm}} \in \widetilde{\Delta}^{(h+1)}$. This contradicts our assumption. ❑

Corollary 3.25 *Let $\mathfrak{K}$ be a set of cumulated clauses on $\mathcal{FLE}^{\mathrm{norm}}$ and let $\mathfrak{k}$ be a cumulated clause on $\mathcal{FLE}^{\mathrm{norm}}$. Let $k \in \mathbb{N}$ be a number equal or greater than the greatest role depth occurring in $\mathfrak{K} \cup \{\mathfrak{k}\}$. Then*

$$\mathfrak{K} \models \mathfrak{k} \Longleftrightarrow \overrightarrow{\mathbb{K}}_k(\mathfrak{K}) \models \mathfrak{k}.$$

Proof:
Let Δ be the universe of $\overrightarrow{\mathbb{K}}(\mathfrak{K})$, let $\widetilde{\Delta}$ be the universe of $\overrightarrow{\mathbb{K}}_k(\mathfrak{K})$, and let $\mathfrak{k} = A \multimap \{B_1, \ldots, B_k\}$. Due to Theorem 3.11, we know that $\mathfrak{K} \models \mathfrak{k}$ if and only if $\overrightarrow{\mathbb{K}}(\mathfrak{K}) \models \mathfrak{k}$. By definition, this is equivalent to the statement $A \subseteq \{\mathtt{C} \mid \delta \in [\![\mathtt{C}]\!]_{\overrightarrow{\mathbb{K}}(\mathfrak{K})}\} \to \bigvee_{1 \leq i \leq n} B_i \subseteq \{\mathtt{C} \mid \delta \in [\![\mathtt{C}]\!]_{\overrightarrow{\mathbb{K}}(\mathfrak{K})}\}$ for every $\delta \in \Delta$. From Theorem 3.8, it follows that $A \subseteq \delta \to \bigvee_{1 \leq i \leq n} B_i \subseteq \delta$ for every $\delta \in \Delta$, and due to Theorem 3.24 and the choice of k, this statement is valid for all $\delta \in \widetilde{\Delta}$ as well. Now, from Lemma 3.23 follows the equivalence to $A \subseteq \{\mathtt{C} \mid \delta \in [\![\mathtt{C}]\!]_{\overrightarrow{\mathbb{K}}_k(\mathfrak{K})}\} \to \bigvee_{1 \leq i \leq n} B_i \subseteq \{\mathtt{C} \mid \delta \in [\![\mathtt{C}]\!]_{\overrightarrow{\mathbb{K}}_k(\mathfrak{K})}\}$ for every $\delta \in \widetilde{\Delta}$ which by definition just means $\overrightarrow{\mathbb{K}}_k(\mathfrak{K}) \models \mathfrak{k}$. ❑

We are aware that a decision procedure based on the k-limited standard model will be quite inefficient or even wholly infeasible in practice, at least if implemented straightforward.

However, we consider this approach nevertheless interesting due to several (theoretical and practical) reasons:

- It seems to be "dual" (in an informal sense) to the well known tableau based decision procedures. Both methods try to construct a model fulfilling all desired properties. DL tableau algorithms start with what is absolutely necessary and successively extend the model-to-be by individuals forced to exist by the descriptions. They end up either with a clash (contradiction) or a minimal model.

  In contrast, the method presented in this section starts with virtually "everything" and prunes this structure by successively deleting "invalid" entities until nothing remains (in case of unsatisfiability) or a model is obtained. The result is *the* maximal model in the sense elaborated in Theorem 3.10.

- In the scientific field of model checking, there have been achievements in finding ways to economically specify large entity sets with attributes. One promising approach are ordered binary decision diagrams (OBDDs) (see [Br92] for a survey) which have already proven useful in applications (as comprehensively described in [HR00]) even closely related to modal logic. If it were possible to encode the k-limited standard model (resp. the intermediate structures used to construct it) as OBDD and to find a way to execute the described construction steps implicitly on this OBBD, the proposed method could turn out to be not as infeasible for practical cases. However, this is speculative and according evidence has still to be supplied.

- If the above mentioned problems could be overcome, one advantage of this approach in comparison to the ad-hoc-construction done in tableau algorithms is that the model for a fixed set of general concept inclusion axioms has to be constructed only once (and used for all queries – for a bounded (and a-priori known) maximal role depth).

Chapter 4

Complete Attribute Exploration on $\mathcal{FLE}$

In this chapter, we provide the theoretical background for an exploration algorithm on $\mathcal{FLE}$ concept descriptions (a similar procedure for $\mathcal{EL}$ has been presented in [Ru03] and the extension to $\mathcal{FLE}$ sketched in [Ru04]). The intended purpose is to accumulate all information about a binary power context family necessary to decide whether any $\mathcal{FLE}$ subsumption statement of a certain maximal role depth is valid therein. Moreover, this should be done as efficient as possible. In more detail, we will proceed as follows:

Given a binary power context family and a set of concept descriptions, we define an *$\mathcal{FLE}$-context* that mirrors the validity of these concept descriptions for all entities of the binary power context family. We show that implications in the $\mathcal{FLE}$-context coincide with $\overrightarrow{\mathbb{K}}$-subsumption statements in the binary power context family $\overrightarrow{\mathbb{K}}$. This motivates that attribute exploration on particular $\mathcal{FLE}$-contexts will be used to achieve the goal depicted above.

We propose to proceed stepwise, i.e., we collect all information expressible by concept descriptions with role depth of at most $i \in \mathbb{N}$ and increment i thereafter. As we will show, this gives the opportunity to reduce the set of attributes of the context to explore. Additionally, we show how exactly the collected information (actually consisting only of implications on a rather restricted $\mathcal{FLE}$ subset) can be used to decide an arbitrary $\mathcal{FLE}$ subsumption.

4.1 $\mathcal{FLE}$-Contexts

On the basis of a binary power context family, we can define for an arbitrary set of $\mathcal{FLE}$ concept descriptions a corresponding formal context which states for every entity from the underlying universe which of the concept descriptions are valid in it.[1]

Definition 4.1 *Given a binary power context family $\vec{\mathbb{K}} = (\mathbb{K}_\mathcal{C}, \mathbb{K}_\mathcal{R})$ on a universe Δ and a set $M \subseteq \mathcal{FLE}(M_\mathcal{C}, M_\mathcal{R})$, the corresponding $\mathcal{FLE}$-CONTEXT is defined in the following way:*

$$\mathbb{K}_{\mathcal{FLE}}(M) := (\Delta, M, I) \text{ with } \delta I m :\Leftrightarrow \delta \in [\![m]\!]_{\vec{\mathbb{K}}}.$$

◇

The next theorem shows that implications in $\mathcal{FLE}$-contexts coincide with $\vec{\mathbb{K}}$-subsumption statements on the described binary power context family $\vec{\mathbb{K}}$.

Theorem 4.2 *Let $\vec{\mathbb{K}}$ be an arbitrary binary power context family and $\mathbb{K}_{\mathcal{FLE}}(M)$ a corresponding $\mathcal{FLE}$-context. Then for $\mathcal{C}, \mathcal{D} \subseteq M$, the implication*

$$\mathcal{C} \rightarrow \mathcal{D}$$

holds in $\mathbb{K}_{\mathcal{FLE}}$ if and only if

$$\bigsqcap \mathcal{C} \sqsubseteq_{\vec{\mathbb{K}}} \bigsqcap \mathcal{D}.$$

Proof:

$$\begin{aligned}
&\mathbb{K}_{\mathcal{FLE}} \models \mathcal{C} \rightarrow \mathcal{D} \\
\Rightarrow\quad & \forall \delta \in \Delta : \mathcal{C} \in \delta^I \rightarrow \mathcal{D} \in \delta^I \\
\Rightarrow\quad & \forall \delta \in \Delta : \textstyle\bigwedge_{\mathtt{C} \in \mathcal{C}} \delta I \mathtt{C} \rightarrow \bigwedge_{\mathtt{D} \in \mathcal{D}} \delta I \mathtt{D} \\
\Rightarrow\quad & \forall \delta \in \Delta : \textstyle\bigwedge_{\mathtt{C} \in \mathcal{C}} \delta \in [\![\mathtt{C}]\!]_{\vec{\mathbb{K}}} \rightarrow \bigwedge_{\mathtt{D} \in \mathcal{D}} \delta \in [\![\mathtt{D}]\!]_{\vec{\mathbb{K}}} \\
\Rightarrow\quad & \forall \delta \in \Delta : \delta \in \textstyle\bigcap_{\mathtt{C} \in \mathcal{C}} [\![\mathtt{C}]\!]_{\vec{\mathbb{K}}} \rightarrow \delta \in \bigcap_{\mathtt{D} \in \mathcal{D}} [\![\mathtt{D}]\!]_{\vec{\mathbb{K}}} \\
\Rightarrow\quad & \forall \delta \in \Delta : \delta \in [\![\textstyle\bigsqcap \mathcal{C}]\!]_{\vec{\mathbb{K}}} \rightarrow \delta \in [\![\bigsqcap \mathcal{D}]\!]_{\vec{\mathbb{K}}} \\
\Rightarrow\quad & [\![\textstyle\bigsqcap \mathcal{C}]\!]_{\vec{\mathbb{K}}} \subseteq [\![\bigsqcap \mathcal{D}]\!]_{\vec{\mathbb{K}}} \\
\Rightarrow\quad & \textstyle\bigsqcap \mathcal{C} \sqsubseteq_{\vec{\mathbb{K}}} \bigsqcap \mathcal{D}
\end{aligned}$$

❑

[1] Of course, this kind of contexts can be constructed not only by taking $\mathcal{FLE}$ concept descriptions as attributes. In principle, concept descriptions of any description logic can be used (as it is done for $\mathcal{ALC}$ in [Pre00]; also in [Ba95] and [BS04], similar constructions have been used). However in this work, we focus on $\mathcal{FLE}$ and the subsequent results are specific to that formalism. The possibility of extending the results to more expressive description logics will be briefly discussed in Section 9.5.

Remark 4.3 Due to Definition 3.1, we additionally have for any binary power context family $\overrightarrow{\mathbb{K}}$ with a corresponding $\mathcal{FLE}$-context $\mathbb{K}_{\mathcal{FLE}}(M)$ and $\mathcal{C}, \mathcal{D} \in M$

$$\mathbb{K}_{\mathcal{FLE}}(M) \models \mathcal{C} \rightarrow \mathcal{D} \iff \overrightarrow{\mathbb{K}} \models \mathcal{C} \multimap \{\mathcal{D}\}$$

as a trivial consequence. Thus, when discussing semantic entailment in an $\mathcal{FLE}$-context, we can use appropriate results from Chapter 3.

In the sequel, we will exploit this correspondence in the following way: employing the FCA exploration method allows us to collect all information about a (not explicitly given) binary power context family expressible by $\overrightarrow{\mathbb{K}}$-subsumption statements on $\mathcal{FLE}_i$ for a certain role depth i.

In order to achieve this, one could simply explore the context $\mathbb{K}_{\mathcal{FLE}}(\mathcal{FLE}_i)$. Yet, the complexity of the exploration algorithm is exponential with respect to the number of attributes and a lot of the $\mathcal{FLE}_i$ concept descriptions are (even universally) equivalent to the conjunctions of others and therefore dispensable. Thus, it is essential to see how the set of attributes can be reduced without losing the above mentioned property.

Theorem 2.8 shows that $\mathcal{FLE}_i^{\text{norm}}$ would be such an attribute set: to check $\mathtt{C} \sqsubseteq_{\overrightarrow{\mathbb{K}}} \mathtt{D}$ with $\mathtt{C}, \mathtt{D} \in \mathcal{FLE}_i$ one would just have to look whether $\mathfrak{n}(\mathtt{C}) \rightarrow \mathfrak{n}(\mathtt{D})$ is valid in $\mathbb{K}_{\mathcal{FLE}}(\mathcal{FLE}_i^{\text{norm}})$.

In the following, we will show that one can do still much better by proceeding iteratively: Starting with role depth 0, explore an according context, and then exploit the information gathered so far for defining a reduced set of attributes for the next context with incremented role depth.

4.2 Empiric Attribute Reduction

Based on a binary power context family, we define a sequence of particular $\mathcal{FLE}$-contexts. The attribute set of a context in this sequence depends on the implicational theory of the preceding context.

Definition 4.4 *Let $\overrightarrow{\mathbb{K}}$ be a binary power context family. We define the sequence $(\mathbb{K}_i)$ of formal contexts by*

$$\begin{array}{lll} M_0 & := & M_{\mathcal{C}} \cup \{\bot\}, \\ \mathbb{K}_i & := & \mathbb{K}_{\mathcal{FLE}}(M_i) = (\Delta, M_i, I_i), \\ M_{i+1} & := & M_0 \\ & & \cup\{\forall R.\mathtt{C} \mid R \in M_{\mathcal{R}}, \mathtt{C} \in M_i\} \\ & & \cup\{\exists R.\bigsqcap \mathcal{C} \mid R \in M_{\mathcal{R}}, \mathcal{C} \textit{ concept intent of } \mathbb{K}_i, \bot \notin \mathcal{C}\} \\ & & \textit{for } i \geq 0 \end{array}$$

◇

Now, we show a way how the validity of any $\overrightarrow{\mathbb{K}}$-subsumption statement on $\mathcal{FLE}_i$ can be checked by using just the attribute sets (M_i) as well as the corresponding closure operators $(.)^{I_i I_i}$ on that sets (which could e.g. be represented by the according stem bases). First, we will define functions that provide for any $\mathcal{FLE}_i$ concept description $\mathtt{C}$ a set of attributes $\mathcal{C} \subseteq M_i$ such that for any entity δ of the underlying universe we have $\delta \models \mathtt{C}$ iff $\delta \models \mathcal{C}$.

Definition 4.5 *Let $\overrightarrow{\mathbb{K}}$ be a binary power context family and the corresponding sequences $(M_i), (\mathbb{K}_i)$ defined as above. Given the according sequence $\mathfrak{cl}_0, \ldots, \mathfrak{cl}_n$ of closure operators (i.e., $\mathfrak{cl}_i(\mathcal{C}) = \mathcal{C}^{I_i I_i}$ for $\mathcal{C} \subseteq M_i$), we define two sequences of functions $\tau_i : \mathcal{FLE}_i \to \mathcal{P}(\mathcal{FLE}_i)$ and $\bar{\tau}_i : \mathcal{FLE}_i \to \mathcal{P}(\mathcal{FLE}_i)$ recursively:*

$$\begin{array}{lll} \bar{\tau}_i(\mathtt{C}) & = & \mathfrak{cl}_i(\tau_i(\mathtt{C})) \\ \tau_i(\mathtt{C}) & = & \{\mathtt{C}\} \textit{ for } \mathtt{C} \in M_0 \\ \tau_i(\bigsqcap \mathcal{C}) & = & \bigcup\{\tau_i(\mathtt{C}) \mid \mathtt{C} \in \mathcal{C}\} \\ \tau_i(\forall \mathtt{R}.\mathtt{C}) & = & [\forall \mathtt{R}]\bar{\tau}_{i-1}(\mathtt{C}) = \{\forall \mathtt{R}.\widetilde{\mathtt{C}} \mid \widetilde{\mathtt{C}} \in \bar{\tau}_{i-1}(\mathtt{C})\} \\ \tau_i(\exists \mathtt{R}.\mathtt{C}) & = & [\exists \mathtt{R}]\bar{\tau}_{i-1}(\mathtt{C}) = \begin{cases} \{\bot\} & \textit{if } \bot \in \bar{\tau}_{i-1}(\mathtt{C}), \\ \{\exists \mathtt{R}.\bigsqcap \bar{\tau}_{i-1}(\mathtt{C})\} & \textit{otherwise.} \end{cases} \end{array}$$

◇

Note that by this definition, we also have $\bar{\tau}_i(\top) = \bar{\tau}_i(\bigsqcap \emptyset) = \mathfrak{cl}_i(\emptyset)$. Next, we have to show that the functions just defined behave in the desired way. The following lemma assures that $\bar{\tau}_i$ and τ_i indeed map to M_i.

Lemma 4.6 *Suppose $\mathtt{C} \in \mathcal{FLE}_i$. Then we have $\tau_i(\mathtt{C}) \subseteq M_i$ and $\bar{\tau}_i(\mathtt{C}) \subseteq M_i$.*

Proof:
Obviously, $\bar{\tau}_i(\mathtt{C}) \subseteq M_i$ whenever $\tau_i(\mathtt{C}) \subseteq M_i$. We show the latter by induction on the role depth considering four cases:

- $\mathtt{C} \in M_\mathcal{C} \cup \{\bot\}$. Then by definition $\mathtt{C} \in M_i$.
- $\mathtt{C} = \exists\mathtt{R}.\widetilde{\mathtt{C}}$. If $\bot \in \bar{\tau}_{i-1}(\widetilde{\mathtt{C}})$, we get $\tau_i(\exists\mathtt{R}.\widetilde{\mathtt{C}}) = [\exists\mathtt{R}]\bar{\tau}_{i-1}(\mathtt{C}) = [\exists\mathtt{R}]M_{i-1} = \{\bot\} \subseteq M_i$.
  Now suppose $\bot \notin \bar{\tau}_{i-1}(\widetilde{\mathtt{C}})$. As immediate consequence of the induction hypothesis we have $\bar{\tau}_{i-1}(\widetilde{\mathtt{C}}) \subseteq M_{i-1}$. Since $\bar{\tau}_{i-1}$ gives a closed set with respect to $\mathfrak{cl}_{i-1}$, we have also $\exists\mathtt{R}.\bigsqcap\bar{\tau}_{i-1}(\widetilde{\mathtt{C}}) \in M_i$, as a look to the constructive definition of M_i immediately shows. Therefore, $\tau_i(\exists\mathtt{R}.\mathtt{C}) = \{\exists\mathtt{R}.\bigsqcap\bar{\tau}_{i-1}(\widetilde{\mathtt{C}})\} \subseteq M_i$
- $\mathtt{C} = \forall\mathtt{R}.\widetilde{\mathtt{C}}$. Again, our induction hypothesis yields $\bar{\tau}_{i-1}(\widetilde{\mathtt{C}}) \subseteq M_{i-1}$ which implies $\{\forall\mathtt{R}.\widetilde{\mathtt{C}} \mid \widetilde{\mathtt{C}} \in \bar{\tau}_{i-1}(\mathtt{C})\} \subseteq M_i$ due to the definition of M_i and therefore also $\tau_i(\forall\mathtt{R}.\mathtt{C}) = \{\forall\mathtt{R}.\widetilde{\mathtt{C}} \mid \widetilde{\mathtt{C}} \in \bar{\tau}_{i-1}(\mathtt{C})\} \subseteq M_i$.
- $\mathtt{C} = \bigsqcap\widetilde{\mathcal{C}}$. W.l.o.g., we presuppose that there is no conjunction outside the quantifier range in any $\widetilde{\mathtt{C}} \in \widetilde{\mathcal{C}}$. So we have $\tau_i(\widetilde{\mathtt{C}}) \subseteq M_i$ due to the three cases above, and subsequently also $\tau_i(\bigsqcap\mathcal{C}) = \left(\bigcup\{\tau_i(\widetilde{\mathtt{C}}) \mid \mathtt{C} \in \mathcal{C}\}\right) \subseteq M_i$. ❑

The next lemma and theorem show that in our fixed interpretation $\overrightarrow{\mathbb{K}}$, for any concept description $\mathtt{C} \in \mathcal{FLE}_i$, the entity sets fulfilling $\mathtt{C}$ on the one hand and $\bar{\tau}_i(\mathtt{C})$ as well as $\tau_i(\mathtt{C})$ on the other hand coincide.

Lemma 4.7 *For any $\mathcal{C} \subseteq M_i$, we have $\bigsqcap\mathcal{C} \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap\mathfrak{cl}_i(\mathcal{C})$.*

Proof:
$[\![\bigsqcap\mathcal{C}]\!]_{\overrightarrow{\mathbb{K}}} = \bigcap\{[\![\mathtt{C}]\!]_{\overrightarrow{\mathbb{K}}} \mid \mathtt{C} \in \mathcal{C}\} = \bigcap\{\mathtt{C}^{I_i} \mid \mathtt{C} \in \mathcal{C}\} = \mathcal{C}^{I_i} = \mathcal{C}^{I_iI_iI_i} = \mathfrak{cl}_i(\mathcal{C})^{I_i} = \bigcap\{\mathtt{C}^{I_i} \mid \mathtt{C} \in \mathfrak{cl}_i(\mathcal{C})\} = \bigcap\{[\![\mathtt{C}]\!]_{\overrightarrow{\mathbb{K}}} \mid \mathtt{C} \in \mathfrak{cl}_i(\mathcal{C})\} = [\![\bigsqcap\mathfrak{cl}_i(\mathcal{C})]\!]_{\overrightarrow{\mathbb{K}}}$. ❑

Theorem 4.8 *Let $\mathtt{C} \in \mathcal{FLE}_i$. Then $\mathtt{C} \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap\tau_i(\mathtt{C}) \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap\bar{\tau}_i(\mathtt{C})$.*

Proof:
The second equivalence is a direct consequence of Lemma 4.7. We show the first one again via induction on the role depth:

- $\mathtt{C} \in M_\mathcal{C} \cup \{\bot\}$. Then, we trivially have $[\![\mathtt{C}]\!]_{\overrightarrow{\mathbb{K}}} = [\![\bigsqcap\{\mathtt{C}\}]\!]_{\overrightarrow{\mathbb{K}}}$.
- $\mathtt{C} = \exists\mathtt{R}.\widetilde{\mathtt{C}}$. By induction hypothesis, we get $[\![\widetilde{\mathtt{C}}]\!]_{\overrightarrow{\mathbb{K}}} = [\![\bigsqcap\bar{\tau}_{i-1}(\widetilde{\mathtt{C}})]\!]_{\overrightarrow{\mathbb{K}}}$, therefore $[\![\exists\mathtt{R}.\widetilde{\mathtt{C}}]\!]_{\overrightarrow{\mathbb{K}}} = [\![\exists\mathtt{R}.\bigsqcap\bar{\tau}_{i-1}(\widetilde{\mathtt{C}})]\!]_{\overrightarrow{\mathbb{K}}}$ which by definition equals $[\![\bigsqcap\tau_i(\exists\mathtt{R}.\widetilde{\mathtt{C}})]\!]_{\overrightarrow{\mathbb{K}}}$.
- $\mathtt{C} = \forall\mathtt{R}.\widetilde{\mathtt{C}}$. Again, by induction hypothesis, we get $[\![\widetilde{\mathtt{C}}]\!]_{\overrightarrow{\mathbb{K}}} = [\![\bigsqcap\bar{\tau}_{i-1}(\widetilde{\mathtt{C}})]\!]_{\overrightarrow{\mathbb{K}}} = \bigcap\{[\![\mathtt{D}]\!]_{\overrightarrow{\mathbb{K}}} \mid \mathtt{D} \in \bar{\tau}_{i-1}(\widetilde{\mathtt{C}})\}$. Now, observe that the statement $(\delta,\widetilde{\delta})I_\mathcal{R}\mathtt{R} \to \widetilde{\delta} \in [\![\widetilde{\mathtt{C}}]\!]_{\overrightarrow{\mathbb{K}}}$ is equivalent to $\bigwedge_{\mathtt{D}\in\bar{\tau}_{i-1}(\widetilde{\mathtt{C}})}\left((\delta,\widetilde{\delta})I_\mathcal{R}\mathtt{R} \to \widetilde{\delta} \in [\![\mathtt{D}]\!]_{\overrightarrow{\mathbb{K}}}\right)$ and thus $[\![\forall\mathtt{R}.\widetilde{\mathtt{C}}]\!]_{\overrightarrow{\mathbb{K}}} = \{\delta \mid (\delta,\widetilde{\delta})I_\mathcal{R}\mathtt{R} \to \widetilde{\delta} \in \bigcap\{[\![\widetilde{\mathtt{C}}]\!]_{\overrightarrow{\mathbb{K}}}\}\} = \{\delta \mid \bigwedge_{\mathtt{D}\in\bar{\tau}_{i-1}(\widetilde{\mathtt{C}})}\delta \in [\![\forall\mathtt{R}.\mathtt{D}]\!]_{\overrightarrow{\mathbb{K}}}\} = \bigcap\{[\![\forall\mathtt{R}.\mathtt{D}]\!]_{\overrightarrow{\mathbb{K}}} \mid \mathtt{D} \in \bar{\tau}_{i-1}(\widetilde{\mathtt{C}})\} = [\![\bigsqcap\{\forall\mathtt{R}.\mathtt{D} \mid \mathtt{D} \in \bar{\tau}_{i-1}(\widetilde{\mathtt{C}})\}]\!]_{\overrightarrow{\mathbb{K}}}$ which by definition is just $[\![\bigsqcap\tau_i(\forall\mathtt{R}.\mathtt{C})]\!]_{\overrightarrow{\mathbb{K}}}$.

- $\mathtt{C} = \bigsqcap \widetilde{\mathcal{C}}$. Again, we can presume no conjunction outside the quantifier range in any $\widetilde{\mathtt{C}} \in \widetilde{\mathcal{C}}$. Then $[\![\bigsqcap \widetilde{\mathcal{C}}]\!]_{\overrightarrow{\mathbb{K}}} = \bigcap\{[\![\widetilde{\mathtt{C}}]\!]_{\overrightarrow{\mathbb{K}}} \mid \widetilde{\mathtt{C}} \in \widetilde{\mathcal{C}}\} = \bigcap\{[\![\bigsqcap \tau_i(\widetilde{\mathtt{C}})]\!]_{\overrightarrow{\mathbb{K}}} \mid \widetilde{\mathtt{C}} \in \widetilde{\mathcal{C}}\}$ because of the cases shown before. Now, this is obviously the same as $\bigcap\{[\![\mathtt{D}]\!]_{\overrightarrow{\mathbb{K}}} \mid \mathtt{D} \in \bar{\tau}_i(\widetilde{\mathtt{C}}), \widetilde{\mathtt{C}} \in \widetilde{\mathcal{C}}\} = [\![\bigsqcap(\bigcup\{\bar{\tau}_i(\widetilde{\mathtt{C}}) \mid \widetilde{\mathtt{C}} \in \widetilde{\mathcal{C}}\})]\!]_{\overrightarrow{\mathbb{K}}} = \tau_i(\bigsqcap \widetilde{\mathcal{C}})$. ❑

Using these propositions, we can easily provide a method to check – using only the closure operators $\mathfrak{cl}_0, \ldots, \mathfrak{cl}_i$ – the validity of any $\overrightarrow{\mathbb{K}}$-subsumption statement on $\mathcal{FLE}_i$ with respect to a fixed (but not explicitly known) binary power context family $\overrightarrow{\mathbb{K}}$. It suffices to apply $\bar{\tau}_i$ on both sides and then check for inclusion.

Corollary 4.9 *Let* $\mathtt{C}_1, \mathtt{C}_2 \in \mathcal{FLE}_i$. *Then* $\mathtt{C}_1 \sqsubseteq_{\overrightarrow{\mathbb{K}}} \mathtt{C}_2$ *if and only if* $\bar{\tau}_i(\mathtt{C}_2) \subseteq \bar{\tau}_i(\mathtt{C}_1)$.

Proof:
Due to Theorem 4.8, $\mathtt{C}_1 \sqsubseteq_{\overrightarrow{\mathbb{K}}} \mathtt{C}_2$ is equivalent to $\bigsqcap \bar{\tau}_i(\mathtt{C}_1) \sqsubseteq_{\overrightarrow{\mathbb{K}}} \bigsqcap \bar{\tau}_i(\mathtt{C}_2)$. According to Lemma 4.6, we have $\bar{\tau}_i(\mathtt{C}_1) \subseteq M_i$ and $\bar{\tau}_i(\mathtt{C}_2) \subseteq M_i$. In view of Theorem 4.2, this means the same as the validity of the implication $\bar{\tau}_i(\mathtt{C}_1) \rightarrow \bar{\tau}_i(\mathtt{C}_2)$ in $\mathbb{K}_i$. Now, since the application of $\bar{\tau}$ always gives a closed set with respect to $\mathbb{K}_i$, this is equivalent to $\bar{\tau}_i(\mathtt{C}_2) \subseteq \bar{\tau}_i(\mathtt{C}_1)$. ❑

Finally, consider the function τ_i from Definition 4.5. It is easy to see that for any $\mathtt{C} \in M_{i-1}$ by calculating $\tau_i(\mathtt{C})$ we get a singleton set $\{\mathtt{D}\}$ with $\mathtt{D} \in M_i$. We then have even $\mathtt{C} \equiv_{\overrightarrow{\mathbb{K}}} \mathtt{D}$. For the sake of readability we will just write $\mathtt{D} = (\mathtt{C})_i$. Roughly spoken, $\mathtt{D}$ is just the "equivalent M_i-version" of $\mathtt{C}$. Note that evaluating τ_i does not need the closure operator $\mathfrak{cl}_i$ but only $\mathfrak{cl}_0, \ldots, \mathfrak{cl}_{i-1}$.

4.3 Directly Derivable Implications

Aiming at a stepwise exploration process proceeding from role depth i to role depth $i+1$, we are now interested in which implications valid in $\mathbb{K}_{i+1}$ can be directly derived from those holding in $\mathbb{K}_i$.

Lemma 4.10 *Let* $\overrightarrow{\mathbb{K}}$ *be a binary power context family,* $\mathtt{R} \in M_{\mathcal{R}}$, $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}, \mathcal{B} \subseteq M_i$, $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$, *and let* $\mathcal{A} \rightarrow \mathcal{B}$ *be an implication that holds in* $\mathbb{K}_i$. *Then the following implications are valid in* $\mathbb{K}_{i+1}$*:*

1. $\{\bot\} \rightarrow M_{i+1}$
2. $\{(\mathtt{A})_{i+1} \mid \mathtt{A} \in \mathcal{A}\} \rightarrow \{(\mathtt{B})_{i+1} \mid \mathtt{B} \in \mathcal{B}\}$
3. $[\exists\mathtt{R}]\mathfrak{cl}_i(\mathcal{A}) \rightarrow [\exists\mathtt{R}]\mathfrak{cl}_i(\mathcal{B})$
4. $[\forall\mathtt{R}]\mathcal{A} \rightarrow [\forall\mathtt{R}]\mathcal{B}$
5. $[\forall\mathtt{R}]\mathcal{A}_1 \cup [\exists\mathtt{R}]\mathfrak{cl}_i(\mathcal{A}_2) \rightarrow [\exists\mathtt{R}]\mathfrak{cl}_i(\mathcal{B})$

Proof:
That all concept descriptions used in this description are in M_i is assured by their construction (cf. Definition 4.4).
The soundness of (1) can be seen as an immediate consequence of the cumulated contradiction rule in $\mathcal{DR}$ (see Lemma 3.5).
(2) is just the repetition of already known implications in terms of the new attribute set. Formally, it can be justified as follows: $\mathbb{K}_i \models \mathcal{A} \rightarrow \mathcal{B}$ implies $\bigsqcap \mathcal{A} \sqsubseteq_{\overrightarrow{\mathbb{K}}} \bigsqcap \mathcal{B}$ due to Theorem 4.2. Furthermore, as a consequence of Corollary 4.9 we have $\bigsqcap \mathcal{A} \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap\{(\mathtt{A})_{i+1} \mid \mathtt{A} \in \mathcal{A}\}$ and $\bigsqcap \mathcal{B} \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap\{(\mathtt{B})_{i+1} \mid \mathtt{B} \in \mathcal{B}\}$. This implies $\bigsqcap\{(\mathtt{A})_{i+1} \mid \mathtt{A} \in \mathcal{A}\} \sqsubseteq_{\overrightarrow{\mathbb{K}}} \bigsqcap\{(\mathtt{B})_{i+1} \mid \mathtt{B} \in \mathcal{B}\}$ and (again by Theorem 4.2) consequently $\mathbb{K}_{i+1} \models \{(\mathtt{A})_{i+1} \mid \mathtt{A} \in \mathcal{A}\} \rightarrow \{(\mathtt{B})_{i+1} \mid \mathtt{B} \in \mathcal{B}\}$.
For showing (3), note that by Corollary 4.9, $\mathbb{K}_i \models \mathcal{A} \rightarrow \mathcal{B}$ implies $\mathfrak{cl}_i(\mathcal{B}) = \bar{\tau}_i(\bigsqcap \mathcal{B}) \subseteq \bar{\tau}_i(\bigsqcap \mathcal{A}) = \mathfrak{cl}_i(\mathcal{A})$. Therefore, we can derive

$$\cfrac{\cfrac{\cfrac{}{\mathfrak{cl}_i(\mathcal{B}) \multimap \{\mathfrak{cl}_i(\mathcal{B})\}}\text{ID}}{\mathfrak{cl}_i(\mathcal{A}) \multimap \{\mathfrak{cl}_i(\mathcal{B})\}}\text{PE}}{[\exists R]\mathfrak{cl}_i(\mathcal{A}) \multimap \{[\exists R]\mathfrak{cl}_i(\mathcal{B})\}}\text{EL}$$

and are done.
The validity of (4) is a one-step consequence of $\mathcal{A} \rightarrow \mathcal{B}$ by using the $\forall$-lifting deduction rule from $\mathcal{DR}$.
Now, consider (5). Let $\bar{\mathcal{A}}_2 := \mathfrak{cl}_i(\mathcal{A}_2)$ as well as $\bar{\mathcal{B}} := \mathfrak{cl}_i(\mathcal{B})$. Then we can deduce

$$\cfrac{\cfrac{}{[\forall.\mathtt{R}]\mathcal{A}_1 \cup [\exists.\mathtt{R}]\bar{\mathcal{A}}_2 \multimap \{[\exists.\mathtt{R}](\bar{\mathcal{A}}_2 \cup \mathcal{A}_1)\}}\text{AP*} \quad \cfrac{\cfrac{\mathcal{A}_2 \cup \mathcal{A}_1 \multimap \{\bar{\mathcal{B}}\}}{\bar{\mathcal{A}}_2 \cup \mathcal{A}_1 \multimap \{\bar{\mathcal{B}}\}}\text{PE}}{[\exists.\mathtt{R}](\bar{\mathcal{A}}_2 \cup \mathcal{A}_1) \multimap \{[\exists.\mathtt{R}]\bar{\mathcal{B}}\}}\text{EL}}{[\forall.\mathtt{R}]\mathcal{A}_1 \cup [\exists.\mathtt{R}]\bar{\mathcal{A}}_2 \multimap \{[\exists.\mathtt{R}]\bar{\mathcal{B}}\}}\text{SUB}$$

❑

Naturally, we are interested in a small set of implications generating all those implications stated above by the Armstrong rules. Here, we provide such a small representation and prove that every implication from Lemma 4.10 can be derived from it.

Definition 4.11 *Let $\mathcal{L}_i$ be the stem base of $\mathbb{K}_i$. Then we define the implication set $\lambda(\mathcal{L}_i)$ as follows*

$$\begin{aligned}
\lambda(\mathcal{L}_i) :=\; & \\
& \big\{\{\bot\} \rightarrow M_{i+1}\big\} \\
\cup & \big\{\{(\mathtt{A})_{i+1} \mid \mathtt{A} \in \mathcal{A}\} \rightarrow \{(\mathtt{B})_{i+1} \mid \mathtt{B} \in \mathcal{B}\} \mid \mathcal{A} \rightarrow \mathcal{B} \in \mathcal{L}_i\big\} \\
\cup & \big\{\{\forall \mathtt{R}.\mathtt{A} \mid \mathtt{A} \in \mathcal{A}\} \rightarrow \{\forall \mathtt{R}.\mathtt{B} \mid \mathtt{B} \in \mathcal{B}\} \mid \mathcal{A} \rightarrow \mathcal{B} \in \mathcal{L}_i\big\} \\
\cup & \big\{\{\exists \mathtt{R}.\bigsqcap \mathcal{A}^{\mathcal{L}_i}\} \rightarrow \{\exists \mathtt{R}.\bigsqcap \mathcal{B}^{\mathcal{L}_i}\} \mid \mathcal{B}^{\mathcal{L}_i} \subsetneq \mathcal{A}^{\mathcal{L}_i} \subseteq M_i, \nexists \mathcal{C}^{\mathcal{L}_i} : \mathcal{A}^{\mathcal{L}_i} \subsetneq \mathcal{C}^{\mathcal{L}_i} \subsetneq \mathcal{B}^{\mathcal{L}_i}\big\} \\
\cup & \big\{\{\exists \mathtt{R}.\bigsqcap \mathcal{A}, \forall \mathtt{R}.\mathtt{A}\} \rightarrow \{\exists \mathtt{R}.\bigsqcap(\mathcal{A} \cup \{\mathtt{A}\})^{\mathcal{L}_i}\} \mid \mathcal{A} = \mathcal{A}^{\mathcal{L}_i} \subseteq M_i, \mathtt{A} \in M_i \backslash \mathcal{A}\big\}
\end{aligned}$$

◇

Lemma 4.12 *Every implication from Lemma 4.10 can be derived from $\lambda(\mathcal{L}_i)$ using the Armstrong rules.*

Proof:

1. The implication $\{\bot\} \rightarrow M_{i+1}$ is contained in $\lambda(\mathcal{L}_i)$.

2. If $\mathcal{A} \rightarrow \mathcal{B}$ is valid in $\mathbb{K}_i$, it is also Armstrong-derivable from $\mathcal{L}_i$. If we take this derivation and exchange every attribute $\mathtt{C}$ with $(\mathtt{C})_i$, we obviously obtain a valid Armstrong-derivation of $\{(\mathtt{A})_{i+1} \mid \mathtt{A} \in \mathcal{A}\} \rightarrow \{(\mathtt{B})_{i+1} \mid \mathtt{B} \in \mathcal{B}\}$ from $\lambda(\mathcal{L}_i)$ because $\lambda(\mathcal{L}_i)$ contains the "$i+1$-version" of $\mathcal{L}_i$.

3. From $\mathbb{K}_i \models \mathcal{A} \rightarrow \mathcal{B}$ follows $\mathcal{B}^{\mathcal{L}_i} \subseteq \mathcal{A}^{\mathcal{L}_i}$. If $\mathcal{B}^{\mathcal{L}_i} = \mathcal{A}^{\mathcal{L}_i}$, we can immediately Armstrong-derive $[\exists\mathtt{R}]\mathcal{A} \rightarrow [\exists\mathtt{R}]\mathcal{B}$ by the identity rule. If $\mathcal{B}^{\mathcal{L}_i} \subsetneq \mathcal{A}^{\mathcal{L}_i}$, we either have $\nexists \mathcal{C}^{\mathcal{L}_i} : \mathcal{A}^{\mathcal{L}_i} \subsetneq \mathcal{C}^{\mathcal{L}_i} \subsetneq \mathcal{B}^{\mathcal{L}_i}$, which would mean that $[\exists\mathtt{R}]\mathcal{A}^{\mathcal{L}_i} \rightarrow [\exists\mathtt{R}]\mathcal{B}^{\mathcal{L}_i}$ is explicitly included in $\lambda(\mathcal{L}_i)$. Otherwise (remember that M_i is finite, therefore there are only finitely many $\mathcal{L}_i$-closed sets), we can find a finite sequence $\mathcal{A}^{\mathcal{L}_i} = \mathcal{C}_0 \subsetneq \ldots \subsetneq \mathcal{C}_n = \mathcal{B}^{\mathcal{L}_i}$ of $\mathcal{L}_i$-closed sets with $\nexists \mathcal{C}^{\mathcal{L}_i} : \mathcal{C}_k \subsetneq \mathcal{C}^{\mathcal{L}_i} \subsetneq \mathcal{C}_{k+1}$ for any k with $0 \leq k < n$. Then, all $[\exists\mathtt{R}]\mathcal{C}_k^{\mathcal{L}_i} \rightarrow [\exists\mathtt{R}]\mathcal{C}_{k+1}^{\mathcal{L}_i}$ are contained in $\lambda(\mathcal{L}_i)$ and can be successively combined via the Armstrong-substitution rule which ends up with an Armstrong-derivation of $[\exists\mathtt{R}]\mathcal{A}^{\mathcal{L}_i} \rightarrow [\exists\mathtt{R}]\mathcal{B}^{\mathcal{L}_i}$ from $\lambda(\mathcal{L}_i)$.

4. This can be shown in an analogous manner to case 2. If $\mathcal{A} \rightarrow \mathcal{B}$ is valid in $\mathbb{K}_i$, it is also Armstrong-derivable from $\mathcal{L}_i$. If we take this derivation and exchange every attribute C with $\forall\mathtt{R}.C$, we obviously obtain a valid derivation of $\{\forall\mathtt{R.A} \mid \mathtt{A} \in \mathcal{A}\} \rightarrow \{\forall\mathtt{R.B} \mid \mathtt{B} \in \mathcal{B}\}$ because $\lambda(\mathcal{L}_i)$ contains the "$\forall$-version" of $\mathcal{L}_i$.

5. Due to the fact that $(.)^{\mathcal{L}_i}$ is a closure operator, we find

$$\begin{array}{rrll} & \mathcal{C} & \subseteq \mathcal{C}^{\mathcal{L}_i} & \text{extensive} \\ \Rightarrow & \mathcal{C} \cup \mathcal{D} & \subseteq \mathcal{C}^{\mathcal{L}_i} \cup \mathcal{D} & \\ \Rightarrow & (\mathcal{C} \cup \mathcal{D})^{\mathcal{L}_i} & \subseteq (\mathcal{C}^{\mathcal{L}_i} \cup \mathcal{D})^{\mathcal{L}_i} & \text{monotone} \end{array}$$

and

$$\begin{array}{rrll} & \mathcal{D} & \subseteq (\mathcal{C} \cup \mathcal{D})^{\mathcal{L}_i} & \text{extensive} \\ \Rightarrow & (\mathcal{C} \cup \mathcal{D})^{\mathcal{L}_i} \cup \mathcal{D} & = (\mathcal{C} \cup \mathcal{D})^{\mathcal{L}_i} & (*) \end{array}$$

$$\begin{array}{rrll} & \mathcal{C} & \subseteq \mathcal{C} \cup \mathcal{D} & \\ \Rightarrow & \mathcal{C}^{\mathcal{L}_i} & \subseteq (\mathcal{C} \cup \mathcal{D})^{\mathcal{L}_i} & \text{monotone} \\ \Rightarrow & \mathcal{C}^{\mathcal{L}_i} \cup \mathcal{D} & \subseteq (\mathcal{C} \cup \mathcal{D})^{\mathcal{L}_i} \cup \mathcal{D} & \\ \Rightarrow & \mathcal{C}^{\mathcal{L}_i} \cup \mathcal{D} & \subseteq (\mathcal{C} \cup \mathcal{D})^{\mathcal{L}_i} & \text{due to } (*) \\ \Rightarrow & (\mathcal{C}^{\mathcal{L}_i} \cup \mathcal{D})^{\mathcal{L}_i} & \subseteq (\mathcal{C} \cup \mathcal{D})^{\mathcal{L}_i \mathcal{L}_i} & \text{monotone} \\ \Rightarrow & (\mathcal{C}^{\mathcal{L}_i} \cup \mathcal{D})^{\mathcal{L}_i} & \subseteq (\mathcal{C} \cup \mathcal{D})^{\mathcal{L}_i} & \text{idempotent,} \end{array}$$

hence: $(\mathcal{C}^{\mathcal{L}_i} \cup \mathcal{D})^{\mathcal{L}_i} = (\mathcal{C} \cup \mathcal{D})^{\mathcal{L}_i}$ for all $\mathcal{C}, \mathcal{D} \subseteq M_i$ (**).
Now, let $\mathcal{A}_1 = \{\mathtt{A}_1, \ldots, \mathtt{A}_n\}$ and let $\mathcal{C}_k := \mathcal{A}_2 \cup \{\mathtt{A}_j \mid j \leq k\}$ for $0 \leq k \leq n$. Then, for all $k = 0, \ldots, n-1$, the implication

$$\mathtt{i}_k := \{\forall\mathtt{R.A}_{k+1}\} \cup [\exists\mathtt{R}]\mathcal{C}_k^{\mathcal{L}_i} \rightarrow [\exists\mathtt{R}](\mathcal{C}_k^{\mathcal{L}_i} \cup \{\mathtt{A}_{k+1}\})^{\mathcal{L}_i}$$

(where – due to (**) – the conclusion equals $[\exists R]\mathcal{C}_{k+1}^{\mathcal{L}_i}$) is either contained in $\lambda(\mathcal{L}_i)$ or can be directly Armstrong-derived using identity and premise extension (if $A_{k+1} \in \mathcal{C}_k^{\mathcal{L}_i}$). If we iteratively combine those i_k by Armstrong-substitution, we get

$$[\forall R]\mathcal{A}_1 \cup [\exists R]\mathcal{A}_2^{\mathcal{L}_i} \rightarrow [\exists R](\mathcal{A}_1 \cup \mathcal{A}_2)^{\mathcal{L}_i}$$

Furthermore, we know from case 3 that for any $\mathcal{A} \rightarrow \mathcal{B}$ valid in $\mathbb{K}_i$, $[\exists R]\mathcal{A}^{\mathcal{L}_i} \rightarrow [\exists R]\mathcal{B}^{\mathcal{L}_i}$ is derivable from $\lambda(\mathcal{L}_i)$. Therefore a last single Armstrong substitution gives

$$[\forall R]\mathcal{A}_1 \cup [\exists R]\mathcal{A}_2^{\mathcal{L}_i} \rightarrow [\exists R]\mathcal{B}^{\mathcal{L}_i}.$$

❑

However, notice that this set $\lambda(\mathcal{L}_i)$ of a-priori implications is not complete in the sense that every implication on M_{i+1} derivable from the implicational base $\mathcal{L}_i$ on M_i by using $\mathcal{DR}$ can be deduced from $\lambda(\mathcal{L}_i)$ using the Armstrong rules.

Example 4.13 Let $M_{\mathcal{C}} = \{A, B, C\}$, $M_{\mathcal{R}} = \{R_1, R_2\}$ and $\mathcal{L}_1$ consist of the following implications:

$$
\begin{array}{lrcl}
i_1 := & \{\exists R_1.\bigsqcap\emptyset, \exists R_2.\bigsqcap\emptyset\} & \rightarrow & \{A\} \\
i_2 := & \{\exists R_1.\bigsqcap\emptyset, \forall R_2.\bot\} & \rightarrow & \{B\} \\
i_3 := & \{\exists R_1.\bigsqcap A\} & \rightarrow & \{C\} \\
i_4 := & \{\exists R_1.\bigsqcap B\} & \rightarrow & \{C\}
\end{array}
$$

Then, the implication $\{\exists R_1.\exists R_1.\bigsqcap\emptyset\} \rightarrow \{C\}$ is a semantic consequence of $\mathcal{L}_1$ justified by the derivation

$$
\begin{array}{lll}
\emptyset \multimap \{\emptyset\} & & \text{ID} \\
\emptyset \multimap \{\emptyset, \{\bot\}\} & & \text{CE} \\
\emptyset \multimap \{\{\exists R_2.\bigsqcap\emptyset\}, \{\forall R_2.\bot\}\} & & \text{AL} \\
\{\exists R_1.\bigsqcap\emptyset\} \multimap \{\{\exists R_2.\bigsqcap\emptyset, \exists R_1.\bigsqcap\emptyset\}, \{\forall R_2.\bot, \exists R_1.\bigsqcap\emptyset\}\} & & \text{RES} \\
\{\exists R_1.\bigsqcap\emptyset\} \multimap \{\{A\}, \{\forall R_2.\bot, \exists R_1.\bigsqcap\emptyset\}\} & i_1 & \text{SUB*} \\
\{\exists R_1.\bigsqcap\emptyset\} \multimap \{\{A\}, \{B\}\} & i_2 & \text{SUB*} \\
\{\exists R_1.\exists R_1.\bigsqcap\emptyset\} \multimap \{\{\exists R_1.A\}, \{\exists R_1.B\}\} & & \text{EL} \\
\{\exists R_1.\exists R_1.\bigsqcap\emptyset\} \multimap \{\{C\}, \{\exists R_1.B\}\} & i_3 & \text{SUB*} \\
\{\exists R_1.\exists R_1.\bigsqcap\emptyset\} \multimap \{\{C\}\} & i_4 & \text{SUB*}
\end{array}
$$

but not Armstrong-derivable from $\lambda(\mathcal{L}_1)$.

This example emphasizes that a desirable property has not been achieved: while exploring the formal context $\mathbb{K}_{i+1}$, the attribute exploration algorithm might

come up with potential implications that are necessarily valid in all binary power context families that deliver $\mathcal{L}_i$ as stem base of $\mathbb{K}_i$, even if we provide $\lambda(\mathcal{L}_i)$ as a-priori implicational knowledge.
However, there are two reasons not to be too sad about this:

- Naturally, one could determine (a representation of) all M_{i+1}-implications semantically entailed by $\mathcal{L}_i$: since M_{i+1} is finite, the set of all implications is finite as well. Hence, we could check in finite time, which ones are necessary consequences of $\mathcal{L}_i$ by applying the procedure described in Section 3.2 or a DL reasoner. However, this approach is obviously "brute force" and would be algorithmically costly (even if optimized). So $\lambda(\mathcal{L}_i)$ contains just those implications that can be calculated directly from $\mathcal{L}_i$ with minimal effort.

- Furthermore, as we will point out in Chapter 5, in the intended application, every potentially valid implication "asked" by the exploration algorithm will first be passed to an automatic decision procedure, which would recognize its derivability from $\mathcal{L}_i$ and tacitly confirm it. Therefore, if the attribute exploration method is supplemented by a decision algorithm for semantic entailment of implications, no such redundant question will be presented to the expert. Moreover, proceeding like this structures the way hypothetical implications are brought up and thereby minimizes the number of calls to the decision procedure (more than ever in comparison with the complete "a-priori scan" sketched above).

Chapter 5

Algorithm Description

After all necessary theoretical considerations, we will now sketch the entire relational exploration algorithm.
Essentially, there are three instances involved:

1. The *attribute exploration algorithm* has been described in Section 1.4. It organizes the question-and-answer process and makes sure that the implicational theory on the used attributes is completely determined.

2. A *decision procedure* capable of deciding whether an $\mathcal{FLE}^{\mathrm{norm}}$ subsumption statement is valid in all binary power context families fulfilling a given set of $\mathcal{FLE}^{\mathrm{norm}}$ subsumption statements and possibly some additional predetermined restrictions (background knowledge).

   This could be the decision procedure described in Section 3.2, in which case the background knowledge would consist of cumulated clauses on $\mathcal{FLE}^{\mathrm{norm}}$.

   This could as well be any DL reasoner – as e.g. FaCT (see [Ho99]) or RACER (see [HM01]) – being the much more practical choice due to

   - their high optimization with respect to time costs and
   - the much greater variety of background knowledge that can be used due to the greater expressiveness of the kind of DL most reasoners are based on.

3. The *expert* knowing the universe which has to be explored and therefore capable of answering all $\overrightarrow{\mathbb{K}}$-subsumption questions asked by the exploration algorithm.

As indicated before, it will be an iteratively organized process where the maximal role depth of the considered concept descriptions will be successively incremented. Every single step of this procedure is subdivided into three phases: attribute generation, background knowledge explication, and (semi-)interactive exploration.

5.1 Attribute Generation

In this phase, we stipulate the attribute set $M_i \subseteq \mathcal{FLE}^{\text{norm}}$ based on the information collected in the previous exploration steps. If $i = 0$, we simply set $M_0 := M_{\mathcal{C}} \cup \{\bot\}$. Otherwise, we use the closure operator $\mathfrak{cl}_{i-1}$ represented by the implicational base $\mathcal{L}_{i-1}$ explored in the previous step in order to generate an empirically reduced set of attributes. The new set of attributes then comprises:

- all primitive concept descriptions $M_{\mathcal{C}}$ as well as the $\bot$-concept,
- for every concept description $\mathtt{C} \in M_{i-1}$, the all-quantified versions $\forall\mathtt{R}.\mathtt{C}$ for every $\mathtt{R} \in M_{\mathcal{R}}$, and
- for every set of concept descriptions $\mathcal{C} \subseteq M_{i-1}$ that is closed with respect to $\mathfrak{cl}_{i-1}$ and does not contain $\bot$, the existentially quantified conjunction $\exists\mathtt{R}.\bigsqcap\mathcal{C}$ for every $\mathtt{R} \in M_{\mathcal{R}}$.

As Theorem 4.8 formally shows, this set is roughly spoken still sufficient to comprehensively "talk about" the considered binary power context family in $\mathcal{FLE}$-terms.

5.2 Background Knowledge Explication

After having stipulated the attribute set for the current exploration step, we can determine the implications that can be added as a-priori knowledge. In order to do that, we exploit the previous implicational base $\mathcal{L}_{i-1}$, and determine the set of prior implications in the following way:

- add
  $\{\bot\} \to M_i$,
- for every implication $\mathcal{A} \to \mathcal{B}$ from $\mathcal{L}_{i-1}$, add
  $\{\widetilde{\mathtt{A}} \mid \tau_i(\mathtt{A}) = \{\widetilde{\mathtt{A}}\}, \mathtt{A} \in \mathcal{A}\} \to \{\widetilde{\mathtt{B}} \mid \tau_i(\mathtt{B}) = \{\widetilde{\mathtt{B}}\}, \mathtt{B} \in \mathcal{B}\}$,

- for every implication $\mathcal{A} \to \mathcal{B}$ from $\mathcal{L}_{i-1}$, add
$\{\forall\mathtt{R.A} \mid \mathtt{A} \in \mathcal{A}\} \to \{\forall\mathtt{R.B} \mid \mathtt{B} \in \mathcal{B}\}$,

- for all $\mathfrak{cl}_{i-1}$-closed sets $\mathcal{A}, \mathcal{B} \subseteq M_{i-1}$ with $\mathcal{A} \subsetneq \mathcal{B}$ where there is no $\mathfrak{cl}_{i-1}$-closed set $\mathcal{C}$ with $\mathcal{A} \subsetneq \mathcal{C} \subsetneq \mathcal{B}$, add
$\{\exists\mathtt{R}.\bigsqcap\mathcal{A}\} \to \{\exists\mathtt{R}.\bigsqcap\mathcal{B}\}$, and

- for every $\mathfrak{cl}_i$-closed set $\mathcal{A} \subseteq M_{i-1} \setminus \mathcal{A}$ and every concept description $\mathtt{A} \in M_{i-1}$, add
$\{\exists\mathtt{R}.\bigsqcap\mathcal{A}, \forall\mathtt{R.A}\} \to \{\exists\mathtt{R}.\bigsqcap\mathfrak{cl}_{i-1}(\mathcal{A} \cup \{\mathtt{A}\})\}$.

In doing this, we deliver implicational knowledge that trivially follows from former exploration steps prior to engaging in the next interactive exploration phase. The "observable behavior" of the system (i.e., the questions asked to the human expert) would be the same without that preparation, since the used decision procedure would automatically answer questions concerning this kind of knowledge. However, providing this knowledge in advance obviously reduces the number of calls to the decision procedure, which are assumed to be costly.

5.3 Interactive Exploration

After all these preparations, the actual exploration process as described in Section 1.4 takes place on the attribute set M_i.
Assume, the algorithm comes up with a hypothetical implication

$$\mathcal{A} \to \mathcal{B}.$$

This question has to be interpreted in the following way:

> "Does every entity δ from the universe Δ of the considered binary power context family that fulfills all concept descriptions from $\mathcal{A}$ also fulfill every concept description from $\mathcal{B}$?"

The first thing to do is to check this question against the facts already known. For, if the implication in question would be valid in any model compatible with all our predetermined background knowledge and the implicational knowledge acquired so far, it would be valid also in the very considered model. This can be decided by any subsumption decision algorithm for some description logic

containing $\mathcal{FLE}$ and general concept inclusion axioms. So, we query the decision algorithm whether $\bigsqcap \mathcal{A} \sqsubseteq \bigsqcap \mathcal{B}$ can be inferred from the specified knowledge.
Now, assume that the decision algorithm can prove that $\mathcal{A} \rightarrow \mathcal{B}$ is a consequence of the facts already known. In this case, the question of the exploration algorithm is tacitly (i.e., without bothering the expert) answered with "yes" and the exploration continues.
If the validity of the implication cannot be proven from the known facts, the question will be passed to the domain expert. (S)he has to decide whether the asked implication is true for all entities of the considered universe. If this is the case, the implication is confirmed to the exploration algorithm and added to the domain specification (the TBox, respectively). If not, (s)he provides a counterexample δ with $\delta \models \mathcal{A}$ but $\delta \not\models \mathcal{B}$ (which could also be added to the ABox) and the exploration process will be continued.
At the end of this phase, we have an implicational base $\mathcal{L}_i$ for M_i and thus a means to decide any $\overrightarrow{\mathbb{K}}$-subsumption statement on $\mathcal{FLE}_i$ as we have shown in Theorem 4.8. Furthermore, $\mathcal{L}_i$ is a representation of the closure operator $\mathfrak{cl}_i$ and thus provides all information necessary for the next exploration step.

Chapter 6

Termination

Although the exploration process just described will have to be stopped after few steps in most practical cases due to the drastic increase of time costs, at least from the theoretical point of view the question emerges, whether and under which circumstances the proposed algorithm terminates, i.e., all information necessary to decide any $\overrightarrow{\mathbb{K}}$-subsumption statement on $\mathcal{FLE}$ (of arbitrary role depth) has been acquired. We will show that this is the case precisely if the following property is fulfilled:

Definition 6.1 *Let $\overrightarrow{\mathbb{K}}$ be a binary power context family and let $(\mathbb{K}_i)$ be the according sequence of formal contexts as defined in Definition 4.4. Furthermore, let $(\mathcal{L}_i)$ be the corresponding sequence of implicational bases.*
$\overrightarrow{\mathbb{K}}$ will be called FINITELY $\mathcal{FLE}$-CHARACTERIZABLE *if there is an $n \in \mathbb{N}$ such that the mapping $F_n : \{\mathcal{A}^{\mathcal{L}_n} \mid \mathcal{A} \subseteq M_n\} \to \{\mathcal{B}^{\mathcal{L}_{n+1}} \mid \mathcal{B} \subseteq M_{n+1}\}$ with $F_n(\mathcal{A}) := \bar{\tau}_{n+1}(\bigsqcap \mathcal{A})$ is a bijection between the $\mathcal{L}_n$-closed subsets from M_n and the $\mathcal{L}_{n+1}$-closed subsets from M_{n+1}.* ◇

In the subsequent theorems, we show that this criterion is sufficient by providing a way to decide whether any $\overrightarrow{\mathbb{K}}$-subsumption statement on $\mathcal{FLE}$ holds by using just the implicational bases $\mathcal{L}_0, \ldots, \mathcal{L}_{n+1}$

Theorem 6.2 *Let $\overrightarrow{\mathbb{K}}$ be a finitely $\mathcal{FLE}$-characterizable binary power context family and n be the natural number for which F_n is a bijection. Then*

1. *For any $\mathcal{B} = \mathcal{B}^{\mathcal{L}_{n+1}} \subseteq M_{n+1}$ and $\mathcal{A} = F_n^{-1}(\mathcal{B})$ we have $\bigsqcap \mathcal{A} \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap \mathcal{B}$.*
2. *For any $\mathtt{C} \in \mathcal{FLE}_{n+1}$ we have $\mathtt{C} \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap F_n^{-1}(\bar{\tau}_{n+1}(\mathtt{C}))$.*

Proof:
Because $\mathcal{A} \subseteq M_n$, we know $\bigsqcap \mathcal{A} \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap \bar{\tau}_{n+1}(\bigsqcap \mathcal{A})$ due to Theorem 4.8. By definition of F_n, we see that the right hand side of the equivalence is just $\bigsqcap F_n(\mathcal{A})$. Since $F_n(\mathcal{A}) = \mathcal{B}$, we are done. The second proposition can then be proven as follows: We know $\mathtt{C} \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap \bar{\tau}_{n+1}(\mathtt{C})$ by Theorem 4.8. From the first part of this theorem it follows that $\bigsqcap(\bar{\tau}_{n+1}(\mathtt{C})) \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap F_n^{-1}(\bar{\tau}_{n+1}(\mathtt{C}))$. ❑

This theorem provides a way to "shrink" an $\mathcal{FLE}_{n+1}$ concept description to maximal role depth n preserving its semantics with respect to $\overrightarrow{\mathbb{K}}$. But – exploiting this fact – we can do even more: for any concept description $\mathtt{C} \in \mathcal{FLE}$ (i.e., of arbitrary role depth), we find an "empirically equivalent"[1] concept description $\widetilde{\mathtt{C}} \in \mathcal{FLE}_n$ by applying the function $\pi : \mathcal{FLE} \to \mathcal{FLE}_n$ with[2]

$$\begin{array}{rcl}
\mathtt{D} & \mapsto & \mathtt{D} \text{ for all } \mathtt{D} \in M_{\mathcal{C}} \cup \{\bot\} \\
\exists \mathtt{R}.\mathtt{D} & \mapsto & \begin{cases} \bigsqcap[\exists \mathtt{R}](\bar{\tau}_{n-1}(\mathtt{D})) \text{ if } \exists \mathtt{R}.\mathtt{D} \in \mathcal{FLE}_n, \\ \bigsqcap F_n^{-1}([\exists \mathtt{R}](\bar{\tau}_n(\pi(\mathtt{D}))))^{\mathcal{L}_{n+1}} \text{ otherwise.} \end{cases} \\
\forall \mathtt{R}.\mathtt{D} & \mapsto & \begin{cases} \bigsqcap[\forall \mathtt{R}]\bar{\tau}_{n-1}(\mathtt{D}) \text{ if } \forall \mathtt{R}.\mathtt{D} \in \mathcal{FLE}_n, \\ \bigsqcap F_n^{-1}([\forall \mathtt{R}]\bar{\tau}_n(\pi(\mathtt{D})))^{\mathcal{L}_{n+1}} \text{ otherwise.} \end{cases} \\
\bigsqcap \mathcal{D} & \mapsto & \bigsqcap\{\pi(\mathtt{D}) \mid \mathtt{D} \in \mathcal{D}\}.
\end{array}$$

Theorem 6.3 *Let $\overrightarrow{\mathbb{K}}$ be a finitely $\mathcal{FLE}$-characterizable binary power context family. Then for any $\mathtt{C} \in \mathcal{FLE}$ we have $\pi(\mathtt{C}) \in \mathcal{FLE}_n$ and $\pi(\mathtt{C}) \equiv_{\overrightarrow{\mathbb{K}}} \mathtt{C}$.*

Proof:
Let $n \in \mathbb{N}$ be the smallest natural number for which F_n is a bijection. This proof will be done by induction on the maximal role depth of $\mathtt{C}$. We have to consider the following cases:

- $\mathtt{C} \in M_{\mathcal{C}} \cup \{\bot\}$.
  This is trivial: $\mathtt{C} \equiv_{\overrightarrow{\mathbb{K}}} \mathtt{C} = \pi(\mathtt{C})$.
- $\mathtt{C} = \exists \mathtt{R}.\mathtt{D} \in \mathcal{FLE}_n$.
  Applying Theorem 4.8 yields $\mathtt{D} \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap \bar{\tau}_{n-1}(\mathtt{D})$, directly implying $\exists \mathtt{R}.\mathtt{D} \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap[\exists \mathtt{R}]\bar{\tau}_{n-1}(\mathtt{D}) = \pi(\mathtt{C})$. Since $\bar{\tau}_{n-1}(\mathtt{D}) \subseteq \mathcal{FLE}_{n-1}$, we also have $\pi(\mathtt{C}) \in \mathcal{FLE}_n$.
- $\mathtt{C} = \exists \mathtt{R}.\mathtt{D} \notin \mathcal{FLE}_n$.
  By induction hypothesis, $\mathtt{D} \equiv_{\overrightarrow{\mathbb{K}}} \pi(\mathtt{D})$ and $\pi(\mathtt{D}) \in \mathcal{FLE}_n$. Theorem 4.8 gives us $\pi(\mathtt{D}) \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap \bar{\tau}_n(\pi(\mathtt{D}))$. From this, we conclude $\exists \mathtt{R}.\mathtt{D} \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap[\exists \mathtt{R}]\bar{\tau}_n(\pi(\mathtt{D}))$. Notice that the equivalence's right hand side is in M_{n+1} due to Theorem 4.6 and the definition of M_{n+1}. By applying Lemma 4.7, we get $\bigsqcap[\exists \mathtt{R}]\bar{\tau}_n(\pi(\mathtt{D})) \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap([\exists \mathtt{R}]\bar{\tau}_n(\pi(\mathtt{D})))^{\mathcal{L}_{n+1}}$ and by Theorem 6.2 we have $\bigsqcap([\exists \mathtt{R}]\bar{\tau}_n(\pi(\mathtt{D})))^{\mathcal{L}_{n+1}} \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap F_n^{-1}([\exists \mathtt{R}]\bar{\tau}_n(\pi(\mathtt{D})))^{\mathcal{L}_{n+1}} = \pi(\mathtt{C})$. So we have shown $\mathtt{C} \equiv_{\overrightarrow{\mathbb{K}}} \pi(\mathtt{C})$. The application of F_n^{-1} assures $\pi(\mathtt{C}) \in \mathcal{FLE}_n$.

[1] i.e., $\overrightarrow{\mathbb{K}}$-equivalent
[2] In this notation, $(.)^{\mathcal{L}}$ binds stronger than F_n^{-1}, $\bar{\tau}_n$, $\bar{\tau}_{n-1}$, $[\forall \mathtt{R}]$, and $[\exists \mathtt{R}]$.

- $\texttt{C} = \forall\texttt{R.D} \in \mathcal{FLE}_n$.
Applying Theorem 4.8 yields $\texttt{D} \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap \bar{\tau}_{n-1}(\texttt{D})$, directly implying $\forall\texttt{R.D} \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap[\forall\texttt{R}]\bar{\tau}_{n-1}(\texttt{D}) = \pi(\texttt{C})$. Since $\bar{\tau}_{n-1}(\texttt{D}) \subseteq \mathcal{FLE}_{n-1}$, we also have $\pi(\texttt{C}) \in \mathcal{FLE}_n$.

- $\texttt{C} = \forall\texttt{R.D} \notin \mathcal{FLE}_n$.
By induction hypothesis, $\texttt{D} \equiv_{\overrightarrow{\mathbb{K}}} \pi(\texttt{D})$ and $\pi(\texttt{D}) \in \mathcal{FLE}_n$. Theorem 4.8 gives us $\pi(\texttt{D}) \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap \bar{\tau}_n(\pi(\texttt{D}))$. From this, we conclude $\forall\texttt{R.D} \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap[\forall\texttt{R}]\bar{\tau}_n(\pi(\texttt{D}))$. Notice that $[\forall\texttt{R}]\bar{\tau}_n(\pi(\texttt{D})) \subseteq M_{n+1}$ due to Theorem 4.6 and the definition of M_{n+1}. By applying Lemma 4.7, we get $\bigsqcap[\forall\texttt{R}]\bar{\tau}_n(\pi(\texttt{D})) \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap([\forall\texttt{R}]\bar{\tau}_n(\pi(\texttt{D})))^{\mathcal{L}_{n+1}}$ and by the first proposition of Theorem 6.2 we have $\bigsqcap([\forall\texttt{R}]\bar{\tau}_n(\pi(\texttt{D})))^{\mathcal{L}_{n+1}} \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap F_n^{-1}([\forall\texttt{R}]\bar{\tau}_n(\pi(\texttt{D})))^{\mathcal{L}_{n+1}} = \pi(\texttt{C})$. So we have shown $\texttt{C} \equiv_{\overrightarrow{\mathbb{K}}} \pi(\texttt{C})$. The application of F_n^{-1} assures $\pi(\texttt{C}) \in \mathcal{FLE}_n$.

- $\texttt{C} = \bigsqcap \mathcal{D}$.
W.l.o.g., we can assume that every $\texttt{D} \in \mathcal{D}$ has no conjunction outside the range of a quantifier, thus, one of the cases above is applicable. Therefore, we know $\pi(\texttt{D}) \in \mathcal{FLE}_n$ and $\texttt{D} \equiv_{\overrightarrow{\mathbb{K}}} \pi(\texttt{D})$ for every $\texttt{D} \in \mathcal{D}$. This implies $\bigsqcap \mathcal{D} \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap\{\pi(\texttt{D}) \mid \texttt{D} \in \mathcal{D}\} = \pi(\texttt{C})$ as well as $\pi(\texttt{C}) \in \mathcal{FLE}_n$.

❑

In words, the π function just realizes the following transformation: beginning from "inside" the concept expression $\texttt{C}$, subformulae having maximal role depth of $n+1$ are substituted by $\overrightarrow{\mathbb{K}}$-equivalent ones with smaller role depth. When applied iteratively, this results in a concept description $\widetilde{\texttt{C}}$ from $\mathcal{FLE}_n$ that is $\overrightarrow{\mathbb{K}}$-equivalent to the original one. The validity of this concept description can now be checked by the method described in the preceding section.
It remains to show that the above mentioned bijection property is also a necessary criterion. This is a direct consequence from the next lemma.

Lemma 6.4 *Let $\overrightarrow{\mathbb{K}}$ be a binary power context family that is not finitely $\mathcal{FLE}$-characterizable. Then there exists no $n \in \mathbb{N}$ such that the set of valid $\overrightarrow{\mathbb{K}}$-subsumption statements on $\mathcal{FLE}_n$ determines the validity for all $\overrightarrow{\mathbb{K}}$-subsumption statements on whole $\mathcal{FLE}$.*

Proof:
Assume the contrary. Let then $\mathfrak{I}$ be the set of all $\mathcal{FLE}_n$ subsumption statements valid in $\overrightarrow{\mathbb{K}}$, coded as cumulated clauses on $\mathcal{FLE}^{\text{norm}}$. Now, consider the n-limited standard model $\overrightarrow{\mathbb{K}}_n(\mathfrak{I})$ as described in Definition 3.22. As a direct consequence of Lemma 3.23, it satisfies exactly those subsumption statements from $\mathfrak{I}$. Furthermore, by construction, $\overrightarrow{\mathbb{K}}_n(\mathfrak{I})$ (precisely: the underlying universe $\Delta_{\overrightarrow{\mathbb{K}}_n(\mathfrak{I})}$) is finite. We now consider the sequence $(\widetilde{\mathbb{K}}_k)$ defined for $\overrightarrow{\mathbb{K}}_n(\mathfrak{I})$ (as described in Definition 4.4) and can conclude that for all $k \in \mathbb{N}$, the number of formal concepts of any $\mathbb{K}_k$ is bounded by $2^{|\Delta_{\overrightarrow{\mathbb{K}}_n(\mathfrak{I})}|}$. On the other hand, for the sequence $(\mathbb{K}_k)$ defined for $\overrightarrow{\mathbb{K}}$, every concept lattice has more formal concepts than its predecessor. Therefore, the concept lattices for $\widetilde{\mathbb{K}}_k$ and $\mathbb{K}_k$ cannot be isomorphic for all k (in fact, they are certainly not isomorphic for every $k > 2^{|\Delta_{\overrightarrow{\mathbb{K}}_n(\mathfrak{I})}|}$). Thus, the implicational theories of $\widetilde{\mathbb{K}}_k$ and $\mathbb{K}_k$ are not equal. So, we have found two models the

implicational theories of which coincide on $\mathcal{FLE}_n$ yet differ on $\mathcal{FLE}$. This clearly contradicts the assumption. ❑

It is easy to show that being finitely $\mathcal{FLE}$-characterizable as termination criterion is equivalent to the finiteness of $\mathcal{FLE}/\equiv_{\overrightarrow{\mathbb{K}}}$, which is trivially fulfilled, if Δ is finite.

Chapter 7

A Small Example

After having presented the algorithm in theory, we will provide an easy example for this method in order to show what type of information we can expect from it. The advantage of choosing an example from mathematics is that the danger of diverging opinions about the correct answer to a question is rather low – apart from open problems.

So, let the considered universe Δ be the natural numbers including zero. Furthermore, let $M_{\mathcal{C}}$ and $M_{\mathcal{R}}$ be defined as shown in Figure 7.1 on the left. Carrying out the exploration on $\mathbb{K}_0$ (where the attributes M_0 are just the elements from $M_{\mathcal{C}}$ plus $\perp$) we get the implicational base $\mathcal{L}_0$ shown in Figure 7.1 on the right. This first exploration step is essentially "ordinary" attribute exploration, since no interrelations between domain entities (mediated by roles or binary predicates, respectively) have been taken into account by now. This changes now.

We generate the attribute set M_1 for the next exploration step as follows: First, we reuse all attributes from M_0, second, we take the conjunction over any $\mathcal{L}_0$-closed subset of $M_0 \setminus \{\perp\}$ preceded by an existential quantifier, and third, we include all combinations of a universal quantifier and one attribute from M_0. Figure 7.2 lists the attributes from M_1.

Then, we generate a-priori knowledge for the second exploration step. First, we use the information collected so far. When proceeding from the first ($i = 0$) to the second ($i = 1$) step, we simply can use $\mathcal{L}_0$ as additional a priori information without further adaption because $M_0 \subseteq M_1$. Furthermore, applying the deduc-

C ∈ $M_\mathcal{C}$	name	$\mathtt{C}^{I_\mathcal{C}}$
Ev	even	$\{2n \mid n \in \mathbb{N}\}$
Od	odd	$\{2n+1 \mid n \in \mathbb{N}\}$
Pr	prime	$\{n \geq 2 \mid kl = n \Rightarrow k \in \{1,n\}\}$
E0	equals zero	$\{0\}$
E1	equals one	$\{1\}$
E2	equals two	$\{2\}$
G2	greater than two	$\{n \in \mathbb{N} \mid n \geq 3\}$

R ∈ $M_\mathcal{R}$	name	$\mathtt{R}^{I_\mathcal{R}}$
s	successor	$\{(n, n+1) \mid n \in \mathbb{N}\}$
p	predecessor	$\{(n+1, n) \mid n \in \mathbb{N}\}$
d	divisor	$\{(m,n) \mid \exists k \in \mathbb{N} : m = kn\}$
m	multiple	$\{(n,m) \mid \exists k \in \mathbb{N} : m = kn\}$

{E0}	→	{Ev}
{E1}	→	{Od}
{E2}	→	{Ev,Pr}
{Ev, Pr}	→	{E2}
{Od, Pr}	→	{G2}
{Pr, G2}	→	{Od}
{Ev, Od}	→	{⊥}
{G2, E0}	→	{⊥}
{G2, E1}	→	{⊥}
{E0, E2}	→	{⊥}

Figure 7.1: Attributes $M_\mathcal{C}$, $M_\mathcal{R}$ and definition of the incidence relations $I_\mathcal{C}$, $I_\mathcal{R}$ for the example and the implicational base $\mathcal{L}_0$ resulting from the first exploration step.

G2	Pr	Od	Ev	E1	E0	E2	⊥
∃s.⊤	∃s.G2	∃s.Pr	∃s.Od	∃s.Ev	∃s.(Od ⊓ G2)	∃s.(Od ⊓ E1)	
∃p.⊤	∃p.G2	∃p.Pr	∃p.Od	∃p.Ev	∃p.(Od ⊓ G2)	∃p.(Od ⊓ E1)	
∃m.⊤	∃m.G2	∃m.Pr	∃m.Od	∃m.Ev	∃m.(Od ⊓ G2)	∃m.(Od ⊓ E1)	
∃d.⊤	∃d.G2	∃d.Pr	∃d.Od	∃d.Ev	∃d.(Od ⊓ G2)	∃d.(Od ⊓ E1)	
∃s.(Ev ⊓ G2)	∃s.(Ev ⊓ E0)	∃s.(Od ⊓ G2 ⊓ Pr)	∃s.(Ev ⊓ Pr ⊓ E2)				
∃p.(Ev ⊓ G2)	∃p.(Ev ⊓ E0)	∃p.(Od ⊓ G2 ⊓ Pr)	∃p.(Ev ⊓ Pr ⊓ E2)				
∃m.(Ev ⊓ G2)	∃m.(Ev ⊓ E0)	∃m.(Od ⊓ G2 ⊓ Pr)	∃m.(Ev ⊓ Pr ⊓ E2)				
∃d.(Ev ⊓ G2)	∃d.(Ev ⊓ E0)	∃d.(Od ⊓ G2 ⊓ Pr)	∃d.(Ev ⊓ Pr ⊓ E2)				
∀s.G2	∀s.Pr	∀s.Od	∀s.Ev	∀s.E1	∀s.E0	∀s.E2	∀s.⊥
∀p.G2	∀p.Pr	∀p.Od	∀p.Ev	∀p.E1	∀p.E0	∀p.E2	∀p.⊥
∀m.G2	∀m.Pr	∀m.Od	∀m.Ev	∀m.E1	∀m.E0	∀m.E2	∀m.⊥
∀d.G2	∀d.Pr	∀d.Od	∀d.Ev	∀d.E1	∀d.E0	∀d.E2	∀d.⊥

Figure 7.2: Attributes M_1 for the second exploration step.

tion consequences mentioned in Section 4.3 we can add numerous implications for instance:

- $\{\bot\} \rightarrow M_1$,
- $\{\exists \mathtt{s}.(\mathtt{Od} \sqcap \mathtt{G2} \sqcap \mathtt{Pr})\} \rightarrow \{\exists \mathtt{s}.(\mathtt{Od} \sqcap \mathtt{G2})\}$,
- $\{\exists \mathtt{s}.\mathtt{Pr}, \forall \mathtt{s}.\mathtt{G2}\} \rightarrow \{\exists \mathtt{s}.(\mathtt{Od} \sqcap \mathtt{G2} \sqcap \mathtt{Pr})\}$, and
- $\{\forall \mathtt{p}.\mathtt{Ev}, \forall \mathtt{p}.\mathtt{Od}\} \rightarrow \{\forall \mathtt{p}.\bot\}$.

After these preparations, the next exploration step is invoked. We visualize its result by the according concept lattice in Figure 7.3.
As an example, we will now demonstrate how to check the validity of the $\overrightarrow{\mathbb{K}}$-subsumption statement

$$\mathtt{Pr} \sqcap \exists \mathtt{s}.(\mathtt{Od} \sqcap \mathtt{Pr}) \sqsubseteq_{\overrightarrow{\mathbb{K}}} \mathtt{E2},$$

verbally: "is two the only prime number having an odd prime successor?" Now, we carry out the necessary calculations and find

$$\begin{aligned} & \tau_1(\mathtt{Pr} \sqcap \exists \mathtt{s}.(\mathtt{Od} \sqcap \mathtt{Pr})) \\ = \; & \tau_1(\mathtt{Pr}) \cup \tau_1(\exists \mathtt{s}.(\mathtt{Od} \sqcap \mathtt{Pr})) \\ = \; & \tau_1(\mathtt{Pr}) \cup [\exists \mathtt{s}](\bar{\tau}_0(\mathtt{Od} \sqcap \mathtt{Pr})) \\ = \; & \tau_1(\mathtt{Pr}) \cup [\exists \mathtt{s}](\tau_0(\mathtt{Od} \sqcap \mathtt{Pr}))^{\mathcal{L}_0} \\ = \; & \tau_1(\mathtt{Pr}) \cup [\exists \mathtt{s}](\tau_0(\mathtt{Od}) \cup \tau_0(\mathtt{Pr}))^{\mathcal{L}_0} \\ = \; & \{\mathtt{Pr}\} \cup [\exists \mathtt{s}]\{\mathtt{Od}, \mathtt{Pr}\}^{\mathcal{L}_0} \\ = \; & \{\mathtt{Pr}\} \cup [\exists \mathtt{s}]\{\mathtt{Od}, \mathtt{Pr}, \mathtt{G2}\} \\ = \; & \{\mathtt{Pr}, \exists \mathtt{s}.(\mathtt{Od} \sqcap \mathtt{Pr} \sqcap \mathtt{G2})\} \end{aligned}$$

as well as

$$\tau_1(\mathtt{E2}) = \{\mathtt{E2}\}.$$

When applying the $\mathcal{L}_1$-closure to both sets to obtain the values for $\bar{\tau}_1$ (the result is to large to be displayed here but can be derived from the line diagram in Figure 7.3), we find the outcomes even identical. Thus, in view of Corollary 4.9, the validity of our hypothetical $\overrightarrow{\mathbb{K}}$-subsumption statement can be confirmed.
Finally, we deal with the question whether the exploration algorithm terminates in our case after some step. This has to be denied for the following reason. Consider the infinite sequence $\mathtt{E0}$, $\exists \mathtt{p}.\mathtt{E0}$, $\exists \mathtt{p}.\exists \mathtt{p}.\mathtt{E0}, \ldots$. Every concept description in this sequence is satisfied by exactly one natural number. Moreover, these

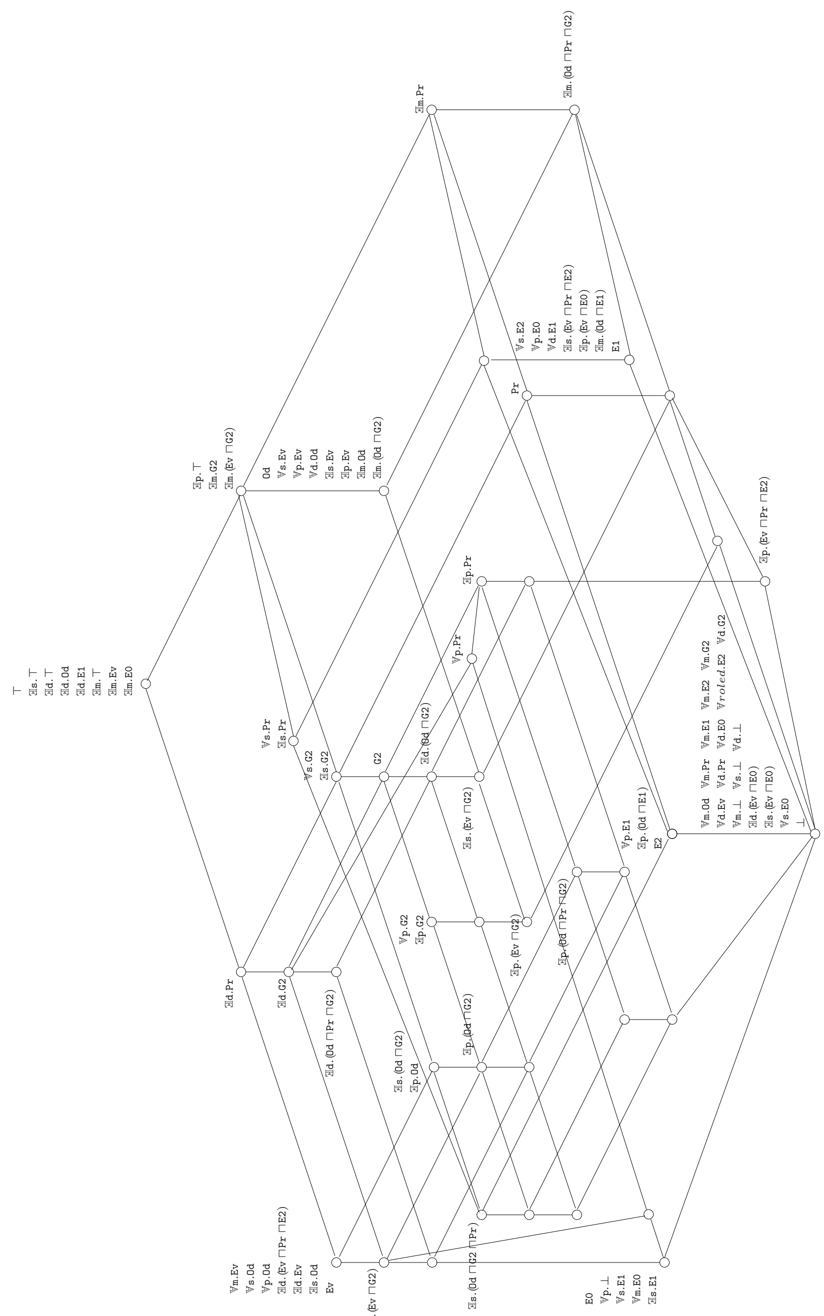

Figure 7.3: Concept lattice from the second exploration step representing the implicational knowledge in $\mathbb{K}_1$.

numbers are all pairwise different. Therefore, every concept description of the sequence is in another $\equiv_{\overrightarrow{\mathbb{K}}}$-equivalence class, thus $\mathcal{FLE}/\equiv_{\overrightarrow{\mathbb{K}}}$ is infinite. Hence, the algorithm does not terminate due to the concluding remark in Chapter 6.

Chapter 8

Ontology Refinement

As already stated, we think the proposed algorithm could be very helpful in designing conceptual descriptions of world aspects. Markup ontologies are a very popular example for this. So, in this chapter, we will briefly introduce ontologies, refer to the current standards, and describe how the proposed methods can be helpful in this area.

8.1 On Ontologies

The Web provides a huge amount of documents and services and is still rapidly increasing. But, due to its necessarily decentral and non-hierarchic organization, the search for the desired piece of information has become a difficult and tedious task for human users. Therefore, powerful tools for information retrieval on the Web are indispensable to fully exploit its potential. However, the overwhelming majority of Web documents has been designed for human consumers, i.e., amongst others

- the information is given in natural language and
- it relies on background knowledge which is shared by (or even commonplace to) most humans.

Thus, most of the information retrieval tools provided today (or at least those most widely used) are restricted to keyword searches.
While one could try to overcome the first problem with natural language processing technologies (which is in our opinion not developed enough to be successful in

this task and would anyway leave the language-inherent ambiguities unresolved) the second one would still persist.
One alternative would be to provide machine-readable descriptions of the content and capabilities of Web resources and thus making the semantics of the data more explicit and thus better accessible for non-human agents. This aim constitutes the idea of the SEMANTIC WEB:

> *The Semantic Web is a vision for the future of the Web in which information is given explicit meaning, making it easier for machines to automatically process and integrate information available on the Web.* ([He04])

Due to [Gr93], an *ontology* is “an explicit specification of a conceptualization.” Indeed, the basic ideas strongly resemble the conceptual thinking usual in DL and FCA.
In [NoMG01], Noy and McGuinness enumerate purposes for developing an ontology. Due to them, an ontology should assist

- sharing common understanding of the structure of information among people or software agents,
- enabling reuse of domain knowledge,
- making domain assumptions explicit,
- separating domain knowledge from the operational knowledge, and
- analyzing domain knowledge.

To achieve these goals, a language is needed, wherein the meaning of the terminology used in Web documents can be formally described. It must be able to specify class hierarchies as well as information about relations between classes.
In recent years, Description Logics have been highly influential to the development of logic standards for the semantic web (see e.g. [Sa03]). They have been used as formal base for defining OWL – the web ontology language (see [MGvH04]).
OWL has been designed to offer more facilities for expressing meaning and semantics than previous standard content specification languages (as XML, RDF, and RDF-S). Moreover, due to its close relationship to DL, well-tried optimized DL reasoners can be used to deal with inference questions.

8.2 Structured Search for New Ontology Axioms

We are confident that FCA can contribute to the development and refinement of ontologies (see also [GaSt03] for another application of FCA to ontology related problems). Here, we will describe how the algorithm proposed in the previous chapters can be used to construct or refine an ontology by an organized search for new general concept inclusion axioms of a certain shape (namely those expressible by $\mathcal{FLE}$ concept constructors).

Clearly, the description logics OWL is based on are much more complex than $\mathcal{FLE}$. Nonetheless, our algorithm is still applicable as long as there are complete reasoning algorithms for deciding subsumption (as for instance the FaCT or RACER, both capable of reasoning in $\mathcal{SHIQ}(D)$ - see [HST00]). All information beyond $\mathcal{FLE}$ is treated as background knowledge and "hidden" from the exploration algorithm.

After stipulating the names of concepts and roles, the next step in designing or refining an ontology would be to define axioms or rules stating how the specified concepts (resp. classes) are interrelated. The exploration algorithm can support this tedious and error-prone task by guiding the expert. Every potential axiom the algorithm comes up with will first be passed to the employed reasoning algorithm. If this axiom can be proven based on the knowledge already present in the OWL domain specification, it will be confirmed to the algorithm, if not, the human expert has to be asked. If (s)he judges the rule to be generally valid in the considered domain, a genuinely new axiom[1] has been found and can be incorporated into the ontology description. Otherwise, the expert has to enter a counterexample, which violates the hypothetical axiom. If the ontology description is meant to contain information about individuals, this counterexample can be added to it as well.

One advantage of applying this technique is the guarantee that all axioms expressible as subsumption statements on $\mathcal{FLE}$ with a certain role depth will certainly be found and specified.

In turn, we want to reply to a possible remark from the point of view of DL: one could object that sometimes or even most times ontologies are designed for several different domains, such that an expert would not want to commit himself to one

[1] i.e., one not already logically entailed by the present specification

specific domain, as it seems necessary when applying this algorithm. However, from the mathematical point of view, this is not a severe problem: we just take the disjoint union of all domains we want to describe as reference domain of our exploration. A rule would be valid in this "superdomain" if and only if it is valid in all of the original domains.

Consequently, we are very confident that an implementation of this algorithm could be a very helpful tool in order to build and refine domain descriptions – not only for working with ontologies. As there is a strong relationship between DL and modal logic (which in turn can be enriched by temporal and epistemic features), describing discrete dynamic systems and multi agent systems are further promising potential applications – see Chapter 9 for an outlook.

Chapter 9

Perspectives

In this chapter, we will briefly point out directions for further research and sketch possible applications.
On one hand, several extensions or modifications of the presented exploration algorithm itself appear possible. Sections 9.1 to 9.5 will discuss these theoretical issues.
On the other hand, we will argue that possible applications of the described technique are not restricted to ontologies. In Section 9.6, we sketch two further scenarios where the algorithm could be used.

9.1 Exploiting Knowledge about Roles

Sometimes, certain properties of the roles are known in advance. This knowledge can be used to state additional implications on M_i before starting exploration. In the sequel, we state implications to be added for some "popular" role properties:

- If $\mathtt{R}$ is reflexive, the implications

  $$\{\mathbb{V}\mathtt{R}.\mathtt{C}\} \rightarrow \{(\mathtt{C})_i\}$$

  for any $\mathtt{C} \in M_{i-1}$ as well as

  $$\{(\mathtt{C})_i \mid \mathtt{C} \in \mathcal{C}\} \rightarrow [\exists\mathtt{R}]\mathcal{C}$$

  for any $\mathfrak{cl}_{i-1}$-closed $\mathcal{C} \subseteq M_{i-1}$ are valid in $\mathbb{K}_i$.

- If $\mathtt{R}$ is symmetric, the implications

  $$\{(\mathtt{C})_i \mid \mathtt{C} \in \mathcal{C}\} \rightarrow [\mathbb{V}\mathtt{R}][\exists\mathtt{R}]\mathcal{C}$$

for any $\mathfrak{cl}_{i-2}$-closed $\mathcal{C} \subseteq M_{i-2}$ and

$$[\exists \mathtt{R}]\mathfrak{cl}_{i-1}(\forall \mathtt{R}.\mathtt{C}) \to \{(\mathtt{C})_i\}$$

for any $\mathtt{C} \in M_{i-2}$ hold in $\mathbb{K}_i$.

- If $\mathtt{R}$ is transitive, the implications

$$\{\forall \mathtt{R}.(\mathtt{C})_{i-1}\} \to \{\forall \mathtt{R}.\forall \mathtt{R}.\mathtt{C}\}$$

for any $\mathtt{C} \in M_{i-2}$ and

$$[\exists \mathtt{R}]\mathfrak{cl}_{i-1}([\exists \mathtt{R}]\mathcal{C}) \to [\exists \mathtt{R}]\mathfrak{cl}_{i-1}\{(\mathtt{C})_{i-1} \mid \mathtt{C} \in \mathcal{C}\}$$

for any $\mathfrak{cl}_{i-2}$-closed $\mathcal{C} \subseteq M_{i-2}$ are valid in $\mathbb{K}_i$.

- If $\mathtt{R}$ is functional[1] the implications

$$[\forall \mathtt{R}]\mathcal{C} \to [\exists \mathtt{R}]\mathcal{C}$$

as well as

$$[\exists \mathtt{R}]\mathcal{C} \to [\forall \mathtt{R}]\mathcal{C}$$

for any $\mathfrak{cl}_{i-1}$-closed $\mathcal{C} \subseteq M_{i-1}$ are valid in $\mathbb{K}_i$.[2]

The case of role inclusion can be treated similarly. If we know in advance that for two roles $\mathtt{R}_1, \mathtt{R}_2 \in M_{\mathcal{R}}$ we have $\mathtt{R}_1^{I_{\mathcal{R}}} \subseteq \mathtt{R}_2^{I_{\mathcal{R}}}$, we can obviously add

$$\{\forall \mathtt{R}_2.\mathtt{C}\} \to \{\forall \mathtt{R}_1.\mathtt{C}\}$$

for any $\mathtt{C} \in M_{i-1}$ as well as

$$\{[\exists \mathtt{R}_1]\mathcal{C}\} \to \{[\exists \mathtt{R}_2]\mathcal{C}\}$$

for any $\mathfrak{cl}_{i-1}$-closed $\mathcal{C} \subseteq M_{i-1}$ to the a-priori implications for $\mathbb{K}_i$.

[1]A role $\mathtt{R}$ is called *functional* if for every $\delta \in \Delta$, there is exactly one $\widetilde{\delta} \in \Delta$ with $(\delta, \widetilde{\delta}) I_{\mathcal{R}} \mathtt{R}$.

[2]In this last case, we can even modify the way attributes are generated for the exploration of $\mathbb{K}_i$. Obviously, for a functional role $\mathtt{R}$ holds $\forall \mathtt{R}.\mathtt{C} \equiv \exists \mathtt{R}.\mathtt{C}$ for any concept description $\mathtt{C}$. Thus we can work without all the existentially quantified attributes and just use the universally quantified ones (omitting $\forall \mathtt{R}.\bot$ as well, since we have $\forall \mathtt{R}.\bot \equiv_{\overrightarrow{\mathbb{K}}} \exists \mathtt{R}.\bot \equiv \bot$).

9.2 Alternative Ways of Coding

Certainly, $\mathcal{FLE}$-contexts do not represent the only way to encode – and consequently explore – (binary) relational information. Subsequently, we will sketch some alternative ways to do so.

9.2.1 Role contexts

One of those possibilities has been described in [GaRu01]. We will shortly sketch a generalized version of this approach.

Definition 9.1 *Let $\overrightarrow{\mathbb{K}}$ be a binary power context family, $\mathtt{R} \in M_{\mathcal{R}}$, and $\mathcal{C}$ a set of concept descriptions of some DL. The* $\mathtt{R}$-CONTEXT, *$\mathbb{K}_{\mathtt{R}}(\mathcal{C})$ is defined by*

$$\mathbb{K}_{\mathtt{R}} := (\{(\delta_1, \delta_2) \mid (\delta_1, \delta_2) I_{\mathcal{R}} \mathtt{R}\}, \mathcal{C} \times \{1, 2\}, I_{\mathtt{R}})$$

where

$$(\delta_1, \delta_2) I_{\mathtt{R}} (\mathtt{C}, i) :\Leftrightarrow \delta_i \models \mathtt{C}.$$

◇

In words, the object set of this formal context consists of those entity pairs (δ_1, δ_2) that are "connected" by role $\mathtt{R}$ and for every concept description $\mathtt{D} \in \mathcal{C}$ the context states, whether $\mathtt{D}$ is valid in δ_1 and whether it is valid in δ_2.
It is rather easy to establish a direct correspondence between simple implications[3] valid in an $\mathtt{R}$-context and $\overrightarrow{\mathbb{K}}$-subsumption statements:

Lemma 9.2 *Let $\overrightarrow{\mathbb{K}}$ be a binary power context family, $\mathtt{R} \in M_{\mathcal{R}}$, and $\mathcal{C}$ a set of concept descriptions of some DL. Then for $\mathcal{D}, \mathcal{E} \subseteq \mathcal{C}$ and $\mathtt{C} \in \mathcal{C}$ we have*

$$\mathcal{D} \times \{1\} \cup \mathcal{E} \times \{2\} \to \{(\mathtt{C}, 1)\} \iff \bigsqcap \mathcal{D} \sqcap \exists \mathtt{R}. \bigsqcap \mathcal{E} \sqsubseteq_{\overrightarrow{\mathbb{K}}} \mathtt{C} \quad \textit{and}$$

$$\mathcal{D} \times \{1\} \cup \mathcal{E} \times \{2\} \to \{(\mathtt{C}, 2)\} \iff \bigsqcap \mathcal{D} \sqcap \exists \mathtt{R}. \bigsqcap (\mathcal{E} \cup \{\neg \mathtt{C}\}) \sqsubseteq_{\overrightarrow{\mathbb{K}}} \bot$$

provided, $\neg\mathtt{C}$ can be described in the DL used.

[3] An implication $A \to B$ is called simple, if $|B| = 1$. Trivially, an arbitrary implication $A \to B$ can be equivalently expressed by $|B|$ simple implications.

9.2.2 Binary Rule Exploration

$\mathcal{FLE}$ is less expressive than the description logic $\mathcal{ALC}$ and $\mathcal{ALC}$ can be seen as a fragment of first order logic with at most binary predicates and at most 2 variables (see [Ba03]). In FCA, there have been other approaches to deal with first order logic fragments, e.g. rule exploration (see [Zi91]) where the Horn logic of a domain is explored. So another alternative way for exploring a binary power context family could be described as follows:

Definition 9.3 *Let $\vec{\mathbb{K}}$ be a binary power context family and X an arbitrary set, called variables. We define the* RULE CONTEXT *:*

$$\mathbb{K}_X := (\Delta^X, M_X, I_X)$$

with

$$\begin{aligned} M_X &:= (M_\mathcal{C} \times X) \cup (M_\mathcal{R} \times X^2) \cup \{\bot\}, \\ f I_X(\mathtt{A}, x) &\Leftrightarrow f(x) I_\mathcal{C} \mathtt{A}, \\ f I_X(\mathtt{R}, x, y) &\Leftrightarrow (f(x), f(y)) I_\mathcal{R} \mathtt{R}, \text{ and} \\ (f, \bot) &\notin I_X. \end{aligned}$$

◇

In words, the objects of our context are not entities from the universe but variable assignments. The attributes represent predicate logic literals (where primitive concepts are interpreted as unary predicates and primitive roles as binary predicates). The incidence relation tells, whether a literal is true in $\vec{\mathbb{K}}$ with respect to the variable assignment. By exploring this context (where we can exploit natural symmetries induced by permutations on the variable set), we get an implicational base for the logic of Horn clauses (see [Zi91] for a thorough treatise).
This is a rather general approach. Depending on the cardinality of X, a lot of different facts about the binary power context family can be expressed by implications on M_X, for instance (let $x, y, z \in X$):

- R is reflexive
  $\emptyset \to \{(\mathtt{R}, x, x)\}$

- R is symmetric
  $\{(\mathtt{R}, x, y)\} \to \{(\mathtt{R}, y, x)\}$

- $\mathtt{R}$ is transitive
  $\{(\mathtt{R}, x, y), (\mathtt{R}, y, z)\} \rightarrow \{(\mathtt{R}, x, z)\}$

- $\mathtt{R}_1 \subseteq \mathtt{R}_2$
  $\{(\mathtt{R}_1, x, y)\} \rightarrow \{(\mathtt{R}_2, x, y)\}$

- every statement expressible by any $\mathtt{R}$-context: if

$$\mathbb{K}_{\mathtt{R}} \models \mathcal{A} \times \{1\} \cup \mathcal{B} \times \{2\} \rightarrow \mathcal{C} \times \{1\} \cup \mathcal{D} \times \{2\}$$

  then

$$\begin{aligned} &\mathbb{K}_X \models \\ &\quad \{(\mathtt{R}_1, x, y)\} \cup \{(\mathtt{C}, x) \mid \mathtt{C} \in \mathcal{A}\} \cup \{(\mathtt{C}, y) \mid \mathtt{C} \in \mathcal{B}\} \\ &\qquad \rightarrow \{(\mathtt{C}, x) \mid \mathtt{C} \in \mathcal{C}\} \cup \{(\mathtt{C}, y) \mid \mathtt{C} \in \mathcal{D}\}. \end{aligned}$$

Remark 9.4 While the relational exploration algorithm presented in this work does not cover all kind of information collectable by rule exploration, the opposite does not hold either: For instance, the $\mathbb{K}_X$-implication

$$\{(\mathtt{R}_1, x, y), (\mathtt{R}_2, y, x)\} \rightarrow \{\bot\}$$

has no $\mathbb{K}_{\mathcal{FLE}}$ counterpart (one would need a much more expressive DL including full Boolean role constructors for this). Vice versa, the $\mathbb{K}_{\mathcal{FLE}}$ implication $\{\forall \mathtt{R}.\mathtt{C}\} \rightarrow \{\exists \mathtt{R}.\mathtt{D}\}$ cannot be expressed in a rule context with arbitrary many variables.

9.3 Partial or Uncertain Knowledge

Obviously, not in every case, the expert has complete knowledge about the universe. Many different approaches have been proposed to incorporate partial or uncertain information into FCA (see [Bu91], [Ga96], [Ho01]). We will sketch an approach related to [Ga96].

Definition 9.5 *A* PARTIAL FORMAL CONTEXT $\mathbb{K}^?$ *is a quadruple* $(G, M, I^\square, I^\diamond)$ *where* $(G, M, I^\square)$ *and* $(G, M, I^\diamond)$ *are formal contexts and* $I^\square \subseteq I^\diamond$. *The operators* $(.)^{I^\square}$ *and* $(.)^{I^\diamond}$ *are defined as usual.* ◇

The meaning of this definition is the following: $gI^{\Box}m$ means, it is certain that object g has the attribute m, while $gI^{\Diamond}m$ means, it is possible that object g has the attribute m or – in other words – it is not certain that object g does not have the attribute m. An intuitive visualization would be a cross table having crosses where $gI^{\Box}m$, blanks where not $gI^{\Diamond}m$ and question marks anywhere else.

Definition 9.6 *A formal context $\mathbb{K} = (G, M, I)$ will be called* COMPLETION *of a partial formal context $\mathbb{K}^? = (G, M, I^{\Box}, I^{\Diamond})$, if $I^{\Box} \subseteq I \subseteq I^{\Diamond}$.* ◇

Note that every formal context in the original sense can be considered as a particular partial formal context with $I^{\Box} = I^{\Diamond} = I$.

Definition 9.7 *Let G, M be arbitrary but fixed sets. We define the* INFORMATION ORDER *on the set of all partial formal contexts with object set G and attribute set M as follows:*

$$\mathbb{K}_1^? \trianglelefteq \mathbb{K}_2^? :\Longleftrightarrow I_1^{\Box} \subseteq I_2^{\Box} \wedge I_2^{\Diamond} \subseteq I_1^{\Diamond}$$

◇

It is obvious that this definition of the information order directly corresponds to the notion of completion: if $\mathbb{K}_1^? \trianglelefteq \mathbb{K}_2^?$, then every completion of $\mathbb{K}_2^?$ is a completion of $\mathbb{K}_1^?$.

Definition 9.8 *An implication $A \to B$ with $A, B \subseteq M$ is* POSSIBLY VALID *(also* HOLDS POSSIBLY*) in a partial context $\mathbb{K}^?$ if it is valid in (at least) one completion of $\mathbb{K}^?$.* ◇

Clearly, if we consider a partial formal context with $I^{\Box} = I^{\Diamond}$, the notions of validity and possible validity coincide.

Theorem 9.9 *Let $\mathbb{K}^?$ be a partial formal context. Then for all $A, B \subseteq M$, we have that $A \to B$ holds possibly in $\mathbb{K}^?$ exactly if $B \subseteq A^{I^{\Box}I^{\Diamond}}$.*

Proof:

"⇒" If the implication $A \to B$ is possibly valid in $\mathbb{K}^?$, there is a completion $\mathbb{K} = (G, M, I)$, where it is valid. But then in $\mathbb{K}$ we have also $B \subseteq A^{II}$. So, it suffices to show that $A^{II} \subseteq A^{I^{\Box}I^{\Diamond}}$. Due to $I^{\Box} \subseteq I$ we have $A^{I^{\Box}} \subseteq A^{I}$ and due to $I \subseteq I^{\Diamond}$ this implies $A^{II} \subseteq A^{I^{\Box}I^{\Diamond}}$. So we are done.

"⇐" Suppose $B \subseteq A^{I^\Box I^\Diamond}$. Obviously, $\widehat{\mathbb{K}} := (G, M, I_A)$ with $I_A = I^\Box \cup (I^\Diamond \cap (A^{I^\Box} \times M))$ is a completion of $\mathbb{K}^?$.

If we consider that

$$gI_Aa \Leftrightarrow gI^\Box a \vee (gI^\Diamond a \wedge (\forall \widetilde{a} \in A:\ gI^\Box \widetilde{a}))$$

and additionally assume $a \in A$, we find the second part of the disjunction to be a special case of the first and conclude $gI_Aa \Leftrightarrow gI^\Box a$. This immediately implies $A^{I_A} = A^{I^\Box}$.(*)

Again considering the equivalence above, now assuming $g \in A^{I^\Box}$ and a being arbitrary, the second part of the conjunction in parentheses is true due to our assumption. Hence, the first part of the disjunction implies the second one (because $I^\Box \subseteq I^\Diamond$) and therefore $gI_Aa \Leftrightarrow gI^\Diamond a$. This in turn gives $A^{I^\Box I_A} = A^{I^\Box I^\Diamond}$. Together with (*), this directly implies $A^{I_A I_A} = A^{I^\Box I^\Diamond}$ which allows to conclude $B \subseteq A^{I_A I_A}$. Therefore, $A \rightarrow B$ holds in $\widehat{\mathbb{K}}$. Yet, $\widehat{\mathbb{K}}$ is a completion of $\mathbb{K}^?$, thus, $A \rightarrow B$ holds possibly in $\mathbb{K}^?$.

❑

By now, we have defined the formalization of the idea that some implication "might hold" in a partial formal context. The contrary would be that an implication "does certainly not hold" – as a means to witness this, we introduce the notion of a guaranteed counterexample.

Definition 9.10 *Let $\mathbb{K}^? = (G, M, I^\Box, I^\Diamond)$ be a partial formal context and $\mathfrak{i} = A \rightarrow B$ an implication on the attribute set. We will call an object $g \in G$* GUARANTEED COUNTEREXAMPLE *for $\mathfrak{i}$ if $A \subseteq g^{I^\Box}$ and $B \not\subseteq g^{I^\Diamond}$.* ◇

Theorem 9.11 *An implication $\mathfrak{i} = A \rightarrow B$ holds possibly in a partial formal context $\mathbb{K}^? = (G, M, I^\Box, I^\Diamond)$ if and only if $\mathbb{K}^?$ contains no guaranteed counterexample for $\mathfrak{i}$.*

Proof:

By Theorem 9.9, we have that $A \rightarrow B$ holds possibly in a partial formal context exactly if $B \subseteq A^{I^\Box I^\Diamond}$. Then we can conclude

$$\begin{aligned} & B \subseteq A^{I^\Box I^\Diamond} \\ \Leftrightarrow\ & A^{I^\Box} \subseteq B^{I^\Diamond} \\ \Leftrightarrow\ & \neg\exists g \in G : g \in A^{I^\Box} \wedge g \notin B^{I^\Diamond} \\ \Leftrightarrow\ & \neg\exists g \in G : A \subseteq g^{I^\Box} \wedge B \not\subseteq g^{I^\Diamond} \end{aligned}$$

which just means, that $\mathbb{K}^?$ does not contain a guaranteed counterexample for $A \rightarrow B$. ❑

Definition 9.12 *A partial formal context $\mathbb{K}^? = (G, M, I^\Box, I^\Diamond)$ will be called* CONSISTENT *with an implication set $\mathfrak{I}$ if it does not contain a guaranteed counterexample for any of the implications from $\mathfrak{I}$.* ◇

Note that this property is equivalent to the existence of a completion of $\mathbb{K}^?$ that respects all implications from $\mathfrak{I}$.

Definition 9.13 *Let $\mathbb{K}^? = (G, M, I^\Box, I^\Diamond)$ be a partial formal context and $\mathfrak{I}$ be a set of implications on M. An object $g \in G$ will be called $\mathfrak{I}$-REVISED, if*

- $g^{I^\Box} = (g^{I^\Box})^{\mathfrak{I}}$ *and*
- $(g^{I^\Box} \cup \{m\})^{\mathfrak{I}} \subseteq g^{I^\Diamond}$ *for all* $m \in g^{I^\Diamond}$

A partial formal context $\mathbb{K}^?$ will be called $\mathfrak{I}$-REVISED if all its objects $g \in G$ are $\mathfrak{I}$-revised. ◇

Furthermore, it is not difficult to show that for every partial formal context $\mathbb{K}^? = (G, M, I^\Box, I^\Diamond)$ compatible with an implication set $\mathfrak{I}$, there is exactly one (with respect to $\trianglelefteq$) minimal $\mathfrak{I}$-revised partial formal context $\mathbb{K}^?_{\mathfrak{I}}$ with $\mathbb{K}^? \trianglelefteq \mathbb{K}^?_{\mathfrak{I}}$. Then, every completion of $\mathbb{K}^?$ respecting all implications from $\mathfrak{I}$ is also a completion of $\mathbb{K}^?_{\mathfrak{I}}$.

In words, $\mathfrak{I}$-revising a context $\mathbb{K}^?_{\mathfrak{I}}$ is a way to make a partial context $\mathbb{K}^?$ more specific using knowledge about valid implications while preserving all its “completion potential”.

Having these notions at hand, the attribute exploration algorithm described in Section 1.4 can be adapted to handle incomplete knowledge as follows:

- We operate on the partial context $\underline{\mathbb{K}}^?$.
- The algorithm comes up with implications that are possibly valid in $\underline{\mathbb{K}}^?$. The expert knows, whether such an implication $\mathfrak{i}$ is valid in the explored universe or not.
  - If (s)he confirms the implication, it is added to the implicational base $\mathcal{L}$ and the partial context $\underline{\mathbb{K}}^?$ will be substituted by its $\mathcal{L}$-revised version.
  - If (s)he denies it, (s)he has to provide a counterexample g. However, the counterexample does not need to be completely specified. It just needs to be a guaranteed counterexample for $\mathfrak{i}$. The $\mathcal{L}$-revised version of g will be added to $\underline{\mathbb{K}}^?$.

This extension of the original algorithm gives some advantages for the relational exploration proposed in this work: when proceeding from $\mathbb{K}_i$ to $\mathbb{K}_{i+1}$ we can not only "reuse" $\mathcal{L}_i$ but also the counterexample set G_i entered while exploring $\mathbb{K}_i$ (or already present before as a-priori knowledge). As described in Section 4.2, for any attribute $\mathtt{C} \in M_i$ there exists an attribute $(\mathtt{C})_{i+1} \in M_{i+1}$ with $\mathtt{C} \equiv_{\overline{\mathbb{K}}} (\mathtt{C})_{i+1}$. So we can use the set G_i with

- $g^{I_{i+1}^{\square}} := \{(m)_{i+1} \mid m \in g^{I_i^{\square}}\}$ and
- $g^{I_{i+1}^{\Diamond}} := M_{i+1} \setminus \{(m)_{i+1} \mid m \in (M_i \setminus g^{I_i^{\Diamond}})\}$

(respectively their $\lambda(\mathcal{L}_i)$-revised versions) as a-priori objects for the exploration of $\mathbb{K}_{i+1}$. In fact, this is the way how we can make the algorithm "remember" the denied implications from previous exploration steps.

A further notion worth considering in this regard would be that of a *partial binary power context family* where $\mathbb{K}_{\mathcal{C}}$ and $\mathbb{K}_{\mathcal{R}}$ are partial formal contexts. This corresponds to databases with incomplete information. The connection of partial binary power context family and partial $\mathcal{FLE}$-contexts would have to be studied.

9.4 Permutations on Attributes

Sometimes, a domain of discourse is known to have symmetries that can be expressed by permutations (resp. automorphisms) on the attributes. Consider the following "toy example":

Example 9.14 Imagine, we would like to build an ontology referring to terms about genealogy. We might stipulate the following concept and role names:

$$\begin{aligned} M_{\mathcal{C}} &= \{\mathtt{Female}, \mathtt{Male}, \mathtt{Parent}, \mathtt{Mother}, \mathtt{Father}, \\ &\quad \mathtt{Grandparent}, \mathtt{Grandmother}, \mathtt{Grandfather}\} \\ M_{\mathcal{R}} &= \{\mathtt{HasChild}, \mathtt{HasSon}, \mathtt{HasDaughter}, \mathtt{HasParent}\} \end{aligned}$$

Now, we would easily find the following true statements:

$$\exists\mathtt{HasSon}.\top \sqsubseteq \exists\mathtt{HasChild}.\mathtt{Male}$$

$$\exists\mathtt{HasDaughter}.\top \sqsubseteq \exists\mathtt{HasChild}.\mathtt{Female}$$

Note that one of these statements can be constructed out of the other one by substituting all male terms by female terms and vice versa. Formally, we could define two permutations $\sigma_{\mathcal{C}} : M_{\mathcal{C}} \to M_{\mathcal{C}}$ and $\sigma_{\mathcal{R}} : M_{\mathcal{R}} \to M_{\mathcal{R}}$ with

$$\sigma_{\mathcal{C}} : \begin{cases} \texttt{Male} & \mapsto \texttt{Female} \\ \texttt{Female} & \mapsto \texttt{Male} \\ \texttt{Parent} & \mapsto \texttt{Parent} \\ \texttt{Mother} & \mapsto \texttt{Father} \\ \texttt{Father} & \mapsto \texttt{Mother} \\ \texttt{Grandparent} & \mapsto \texttt{Grandparent} \\ \texttt{Grandmother} & \mapsto \texttt{Grandfather} \\ \texttt{Grandfather} & \mapsto \texttt{Grandmother} \end{cases}$$

and

$$\sigma_{\mathcal{R}} : \begin{cases} \texttt{HasChild} & \mapsto \texttt{HasChild} \\ \texttt{HasSon} & \mapsto \texttt{HasDaughter} \\ \texttt{HasDaughter} & \mapsto \texttt{HasSon} \\ \texttt{HasParent} & \mapsto \texttt{HasParent} \end{cases}$$

that describe this substitution on the concept and role names. This can be canonically extended to $\mathcal{FLE}$ by defining $\sigma : \mathcal{FLE} \to \mathcal{FLE}$ with:

$$\begin{aligned} \sigma(\texttt{C}) &= \sigma_{\mathcal{C}}(\texttt{C}) \text{ for } \texttt{C} \in M_{\mathcal{C}} \cup \bot \\ \sigma(\textstyle\prod \mathcal{C}) &= \textstyle\bigcup \{\sigma(\texttt{C}) \mid \texttt{C} \in \mathcal{C}\} \\ \sigma(\forall \texttt{R}.\texttt{C}) &= \forall \sigma_{\mathcal{R}}(\texttt{R}).\sigma(\texttt{C}) \\ \sigma(\exists \texttt{R}.\texttt{C}) &= \exists \sigma_{\mathcal{R}}(\texttt{R}).\sigma(\texttt{C}). \end{aligned}$$

In general, there could be more than one such truth preserving permutation on $\mathcal{FLE}$. Knowing that the permutations $\sigma_1, \ldots, \sigma_n$ behave in that way, the same would be true for any element of the permutation group generated by $\{\sigma_1, \ldots, \sigma_n\}$. In [Ga90] it has been described how such information (given as generating set of the group of truth preserving permutations on the attribute set) can be effectively incorporated in the exploration process resulting in a reduction of questions asked to the expert. In our opinion, the example above substantiates the idea that an integration of this feature into the relational exploration algorithm presented in this work could be quite useful in practice.

9.5 Using more Expressive DLs

As already mentioned in Section 8.2, if DL reasoners are employed, the utilizable background knowledge is not restricted to $\mathcal{FLE}$, but can be expressed in any formalism the reasoner is able to deal with.

However, in some cases, it could be desirable to explore the complete knowledge not only in terms of $\mathcal{FLE}$ subsumption statements but also of some more expressive DL (as, say, $\mathcal{ALE}$, $\mathcal{ALC}$, or $\mathcal{ALN}$). The necessary adaptations respectively the overall applicability of our approach depend significantly on the expressivity. In $\mathcal{ALE}$, atomic negation (written as $\neg\mathtt{A}$ for $\mathtt{A} \in M_{\mathcal{C}}$) is allowed in addition to the usual $\mathcal{FLE}$ syntax. An adaption to this case not to difficult. We just sketch the changes to be made in the major definitions and theorems:

- For $\mathtt{A} \in M_{\mathcal{C}}$, add the deduction rules

  $$\frac{}{\emptyset \multimap \{\{\mathtt{A}\}, \{\neg\mathtt{A}\}\}}\textit{tertium non datur } (\text{TND})$$

  and

  $$\frac{}{\{\mathtt{A}, \neg\mathtt{A}\} \multimap \{\{\bot\}\}}\textit{exclusiveness } (\text{EXC})$$

  to the deduction calculus $\mathcal{DR}$ in Definition 3.3.

- The standard model from Definition 3.7 has to be minimally changed by setting

  $$\begin{aligned}\Delta^{(0)} := \ & \{\mathcal{N} \subseteq \mathcal{FLE}^{\text{norm}} \mid \mathcal{N} \text{ respects all } \mathfrak{k} \in \mathfrak{K}, \\ & \text{ for every } \mathtt{A} \in M_{\mathcal{C}}, \text{ either } \mathtt{A} \in \mathcal{N} \text{ or } \neg\mathtt{A} \in \mathcal{N}, \\ & \bot \notin \mathcal{N}\}.\end{aligned}$$

  Mark that the “or” in the second line has indeed to be exclusive. With this, all results concerning the standard model and completeness from Chapter 3 do easily propagate to $\mathcal{ALE}$.

- In Chapter 4, we have to change Definition 4.4 as follows:

  $$M_0 := \{\mathtt{A}, \neg\mathtt{A} \mid \mathtt{A} \in M_{\mathcal{C}}\} \cup \{\bot\}.$$

  Furthermore we can add

  $$\{\mathtt{A}, \neg\mathtt{A}\} \rightarrow \{\bot\}$$

  for $\mathtt{A} \in M_{\mathcal{C}}$ to the immediately valid implications in Lemma 4.10 and Definition 4.11, respectively.

  With this minor changes, all results from Chapter 4 remain valid.

- The algorithm description in Chapter 5 and the considerations with respect to termination in Chapter 6 can stay unabridged for $\mathcal{ALE}$.

As to DLs including disjunction (like $\mathcal{ALC}$ or $\mathcal{ALN}$), the case is different. In particular, the distinction cumulated clause vs. implication would be futile in such a DL, since any cumulated clause can be transformed into an implication by incorporating the conclusion disjunction into the attributes.

In the case of $\mathcal{ALN}$, we even lose the property described in 2.6: For every $i > 0$, we have infinitely many semantically distinct $\mathcal{ALN}$ concept descriptions of maximal role depth i. Thus, a stepwise procedure as presented in Chapter 4 would have to be based on another "measure" on the concept descriptions.

In general, DLs equally expressive as (or more expressive than) $\mathcal{ALC}$ would force an even more drastical blowup in the attribute numbers from one exploration step to the next. Moreover, an according exploration algorithm would present questions to the expert that are mainly difficult to understand and cope with (see the according remark at the beginning of Chapter 2).

Thus, we think that it might be a better strategy to stick to $\mathcal{FLE}$ as "exploration language" and treat information of a more complicated structure as static background knowledge.

9.6 Applications apart from Ontologies

Depending on how the roles are interpreted, various applications of the proposed techniques seem possible also in areas hardly connected to ontologies. We will name just two: process exploration and epistemic exploration.

9.6.1 Process Exploration

We could interpret the entities of our universe Δ as possible states of a system. $M_{\mathcal{C}}$ could be interpreted as properties which those states can have or not. $M_{\mathcal{R}}$ could be seen as repertoire of actions. $(\delta_1, \delta_2) I_{\mathcal{R}} \mathtt{R}$ would mean that the system can be transferred from state δ_1 to δ_2 by action $\mathtt{R}$. Then, $\overrightarrow{\mathbb{K}}$-subsumption statements in some DL would be just a description of the system's dynamic behavior.

	Playing	Silent	FastForward	FastRewind	NoMovement	$\perp$
	×					
		×	×			
		×		×		
		×			×	

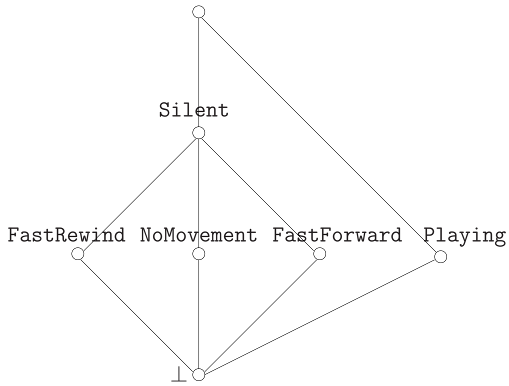

Figure 9.1: Context $\mathbb{K}_0$ and according concept lattice.

Example 9.15 Imagine, we would like to specify a cassette recorder. Let

$$M_{\mathcal{C}} := \{\texttt{Playing}, \texttt{Silent}, \texttt{FastForward}, \texttt{FastRewind}, \texttt{NoMovement}\}$$

describe the possible attributes of a state and

$$M_{\mathcal{R}} := \{ ▶, ⏩, ⏪, ■, ⌛\}$$

be the actions to be taken by pressing a corresponding button or just waiting some time. The formal context in Fig. 9.1 describes the directly "observable" states the device can be in.

All actions except ⌛ are functional. So, as explained in Section 9.1, we can work without all existentially quantified attributes and $\forall$⌛$.\perp$ (as we know $\forall$⌛$.\perp \equiv_{\mathbb{K}}$ $\exists$⌛$.\perp \equiv \perp$). The reduced set M_1 is shown in Figure 9.2. In the sequel, we substitute the attribute names $\exists$⌛.(FastForward⊓Silent), $\exists$⌛.(FastRewind⊓Silent), and $\exists$⌛.(NoMovement ⊓ Silent) by their $\overrightarrow{\mathbb{K}}$-equivalent variants $\exists$⌛.FastForward, $\exists$⌛FastRewind, and $\exists$⌛.NoMovement.[4]

Then, $\mathbb{K}_1$ has to be explored; see Figure 9.3 for the interactions taking place. For better readability, the concept names have been abbreviated and some redundant attributes deleted from the implications. Therefore, collecting the confirmed implications does not yield a genuine stem base as described by Duquenne and Gigue, nevertheless it is an implicational base.

[4] In general, in concept descriptions of the shape $\exists \mathtt{R}.\bigsqcap \mathcal{C}$, the set $\mathcal{C}$ could be substituted by a subset $\mathcal{D} \subseteq \mathcal{C}$ with $\bigsqcap \mathcal{D} \sqsubseteq_{\overrightarrow{\mathbb{K}}} \bigsqcap \mathcal{C}$ for better readability.

⊥	Playing	Silent	FastForward	FastRewind	NoMovement
▶.Playing	▶.Silent	▶.FastForward	▶.FastRewind	▶.NoMovement	
⏩.Playing	⏩.Silent	⏩.FastForward	⏩.FastRewind	⏩.NoMovement	
⏪.Playing	⏪.Silent	⏪.FastForward	⏪.FastRewind	⏪.NoMovement	
■.Playing	■.Silent	■.FastForward	■.FastRewind	■.NoMovement	
∀⧗.Playing	∀⧗.Silent	∀⧗.FastForward	∀⧗.FastRewind	∀⧗.NoMovement	
∀⧗.⊥					
∃⧗.⊤	∃⧗.(FastForward⊓Silent)				
∃⧗.Playing	∃⧗.(FastRewind⊓Silent)				
∃⧗.Silent	∃⧗.(NoMovement⊓Silent)				

Figure 9.2: Reduced attribute set M_1.

Figure 9.4 shows the concept lattice generated by the acquired implicational base. Many facts can be directly read therefrom, e.g. that in any case after some waiting, the device will be silent (∃⧗.Silent $\equiv_{\overrightarrow{\mathbb{K}}} \top$) or that, if the device is silent, it will be so further on, if no action (except waiting) is taken (Silent $\equiv_{\overrightarrow{\mathbb{K}}}$ ∀⧗.Silent).
Before continuing with greater role depths, note the following: Obviously, many attributes are found to be redundant in the previous step (for instance all attributes at the bottom of the diagram which are $\overrightarrow{\mathbb{K}}$-equivalent to ⊥). Thus, it is possible to carry on with a set of attributes containing no attributes which are $\overrightarrow{\mathbb{K}}$-equivalent to a conjunction of others.[5] Figure 9.5 shows the concept lattice corresponding to the implicational base found in the exploration step belonging to maximal role depth 2 with just the "relevant" attributes.

Summa summarum, this example shows that the proposed exploration technique can also be used to describe discrete dynamic systems. In particular, it could assist a system engineer to specify the dynamic behavior of a system that (s)he is just developing.[6] This might help avoiding that during specification certain cases are not taken care of.

[5]This directly corresponds to column reducing the context $\mathbb{K}_1$ after finishing the according exploration step and taking the reduced context's attribute set as new set M_1 for the generation of M_2.

[6]In fact, when the termination criterion from Chapter 6 is reached in step n, the corresponding n-limited standard model corresponds to a – generally non-deterministic – automaton with exactly the specified behavior.

No	asked implication	answer
1	∅ → ⊥	no: s_1
2	∅ → {Pl, ▶.Pl, ■.Si, ■.NM, ▶▶.FF, ▶▶.Si, ◀◀.FR, ◀◀.Si, ∃⌛.Pl, ∃⌛.NM, ∃⌛.Si, ∃⌛.⊤}	no: s_2
3	∅ → {▶.Pl, ■.Si, ■.NM, ▶▶.FF, ▶▶.Si, ◀◀.FR, ◀◀.Si ∃⌛.NM, ∃⌛.Si, ∃⌛.⊤}	no: s_3
4	∅ → {■.Si, ■.NM, ▶▶.Si, ◀◀.FR, ◀◀.Si, ∃⌛.NM, ∃⌛.Si, ∃⌛.⊤}	no: s_4
5	∅ → {■.Si, ■.NM, ▶▶.Si, ◀◀.Si, ∃⌛.NM, ∃⌛.Si, ∃⌛.⊤}	yes
6	{Pl} → {▶.Pl, ▶▶.FF, ◀◀.FR, ∃⌛.Pl}	yes
7	{Si} → {NM, ∀⌛.NM, ∀⌛.Si}	no: s_5
8	{Si} → {∀⌛.Si}	yes
9	{FF} → {▶.Pl, ▶▶.FF, ◀◀.FR, ∃⌛.FF}	yes
10	{FR} → {⊥}	no: s_6
11	{FR} → {FR, ▶.Pl, ▶▶.FF, ◀◀.FR, ∃⌛.FR}	yes
12	{NM} → {∀⌛.NM}	yes
13	{■.Si, ▶.Pl} → {▶▶.FF}	yes
14	{▶.Si} → {▶.NM, ▶▶.NM, ◀◀.FR}	yes
15	{▶▶.FF} → {▶.Pl}	yes
16	{▶▶.FR} → {⊥}	yes
17	{▶▶.NM} → {▶.NM, ▶.Si, ◀◀.FR}	yes
18	{◀◀.FF} → {⊥}	yes
19	{◀◀.NM} → {Si, NM, ▶.Pl, ▶▶.FF}	yes
20	{∀⌛.⊥} → {⊥}	yes
21	{∀⌛.Pl} → {⊥}	yes
22	{∀⌛.Si} → {Si}	yes
23	{∀⌛.FF} → {⊥}	yes
24	{∀⌛.FR} → {⊥}	yes
25	{∀⌛.NM} → {NM}	yes
26	{∃⌛.Pl} → {Pl}	yes
27	{∃⌛.FF} → {FF}	yes
28	{∃⌛.FR} → {FR}	yes

$s_1^{I_1}$ = {Pl, ▶.Pl, ■.Si, ■.NM, ▶▶.FF, ▶▶.Si, ◀◀.FR, ◀◀.Si, ∃⌛.NM, ∃⌛.Si, ∃⌛.⊤}
$s_2^{I_1}$ = {Si, NM, ▶.Pl, ■.Si, ■.NM, ▶▶.FF, ▶▶.Si, ◀◀.FR, ◀◀.Si, ∃⌛.NM, ∃⌛.Si, ∃⌛.⊤, ∀⌛.Si}
$s_3^{I_1}$ = {▶.NM, ▶.Si, ■.Si, ■.NM, ▶▶.NM, ▶▶.Si, ◀◀.FR, ◀◀.Si, ∃⌛.NM, ∃⌛.Si, ∃⌛.⊤}
$s_4^{I_1}$ = {Si, NM, ▶.Pl, ■.Si, ■.NM, ▶▶.FF, ▶▶.Si, ◀◀.NM, ◀◀.Si, ∃⌛.NM, ∃⌛.Si, ∃⌛.⊤}
$s_5^{I_1}$ = {FF, ▶.Pl, ■.Si, ■.NM, ▶▶.FF, ▶▶.Si, ◀◀.FR, ◀◀.Si, ∃⌛.FF, ∃⌛.NM, ∃⌛.Si, ∃⌛.⊤}
$s_6^{I_1}$ = {FR, ▶.Pl, ■.Si, ■.NM, ▶▶.FF, ▶▶.Si, ◀◀.FR, ◀◀.Si, ∃⌛.FR, ∃⌛.NM, ∃⌛.Si, ∃⌛.⊤}

Figure 9.3: Protocol of the exploration step for role depth 1.

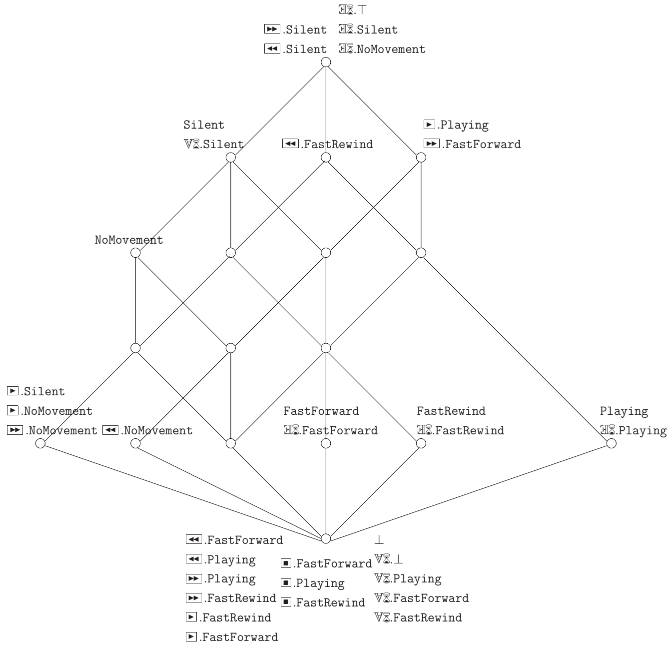

Figure 9.4: Concept lattice representing $\mathcal{L}_1$.

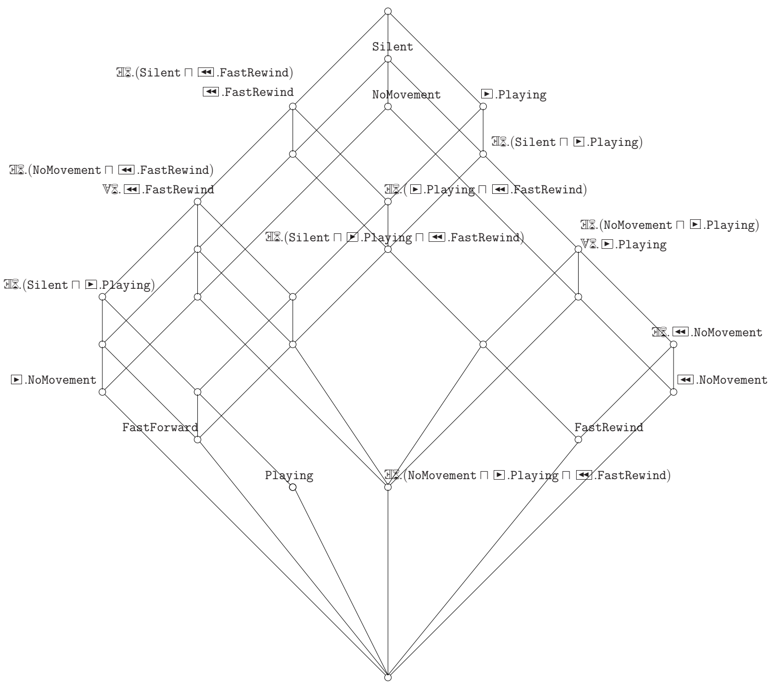

Figure 9.5: Concept lattice representing $\mathcal{L}_2$.

9.6.2 Epistemic Exploration

It is well known that there is a tight connection between DL and modal logics (see [Sch91]). (Multi-)modal logic has also been used to describe epistemic phenomena, i.e., settings where we have agents that know something about their environment or each other. Here, we will shortly explain the basic ideas for this (formulated in DL-style notation).
Let $\mathfrak{A}$ be a set of agents. In a binary power context family $\overrightarrow{\mathbb{K}}$, interpret

- Δ as a set of states the world can be in,
- $M_{\mathcal{C}}$ as a set of facts that might hold in a certain world state or not, and
- $M_{\mathcal{R}} := \mathfrak{A}$ as the agents' "indiscernibility" relations, connecting world states that cannot be distinguished by the respective agent. Obviously, all role names from $M_{\mathcal{R}}$ then have to be interpreted as equivalence relations.

If we see $\overrightarrow{\mathbb{K}}$ in this way, the composed concept descriptions would have to be understood as follows: $\forall \mathtt{A.C}$ would be read as "agent $\mathtt{A}$ knows that $\mathtt{C}$" and $\exists \mathtt{A.C}$ would mean "agent $\mathtt{A}$ considers $\mathtt{C}$ possible".
Hereby, it becomes apparent that the provided methods could also be applied to cases where aspects of communication and belief of several agents are considered. Again, relational exploration could help in specifying (resp. axiomatizing) such a setting.

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Facing the growing amount of information in today‘s society, the task of specifying human knowledge in a way that can be unambiguously processed by computers becomes more and more important.

Two acknowledged fields in this evolving scientific area of *Knowledge Representation* are *Description Logics* (DL) and *Formal Concept Analysis* (FCA). While DL concentrates on characterizing domains via logical statements and inferring knowledge from these characterizations, FCA builds conceptual hierarchies on the basis of present data.

This work introduces *Relational Exploration*, a method for acquiring complete relational knowledge about a domain of interest by successively consulting a domain expert without ever asking redundant questions. This is achieved by combining DL and FCA: DL formalisms are used for defining FCA attributes while FCA exploration techniques are deployed to obtain or refine DL knowledge specifications.

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