The Pricing of Real Options: An Overview

Diploma Thesis

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This diploma thesis is dedicated to my parents Ilse and Gerrit.

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When I started to research for this diploma thesis last year, I did not know much about option pricing in general, and nearly nothing about the pricing of real options. That this has - hopefully - changed in the meantime is rather the result of successful cooperation and generous support than my personal merit.

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The *experiment* of writing this paper in English significantly increased the number of potential readers. Nevertheless, it contained the hazard of reduced clarity since my native language is German. In this context, I wish to express my sincere thanks to Herbert John Raths, Hamburg, who invested a lot of time and effort to increase the quality of this thesis by helpful advice and stimulating discussions.

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Christian von Drathen,

London, November, 1990.

The Pricing of Real Options: An Overview

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1. Introduction and Summary

A contingent claim is a security whose value depends on those of other, more basic underlying variables. In recent years contingent claims have increased in importantance in the field of finance. Options and Futures are now actively traded on many exchanges. Financial intermediates and their corporate clients regularly issue forward contracts and swaps. Other more specialized contingent claims often form part of a bond or stock issue. Even stock itself can be regarded as a contingent claim on the value of the underlying firm's assets.

Often, the variables underlying the contingent claims are the prices of traded securities. For example, a stock option constitutes a claim contingent upon the the price of the stock. Contingent claims, however, can be a function of almost any variable "... from the price of hogs to the amount of snow falling at a certain ski resort." (Hull (1989) p. 1). This paper concentrates on contingent claims, whose underlying variables are values of either real investments or factors that determine their value. If these contingent claims are options, they are called real options.

Introduction of different types of real options, and their applications facilitates the presentation of models currently available for quantitative evaluation. It turns out that the applications have a high empirical relevance, where uncertainty about future events is involved. In particular, for capital budgeting decisions concerning operating flexibility or growth opportunities, contingent claims analysis (CCA) proves to be an invaluable tool.

Two main parts of this paper reflect the relative importance of applications and models. Part I treats the basic concepts necessary for minimal understanding of the pricing of real options. It addresses those unfamiliar with the analysis of contingent claims. Part II, the intellectual bulk of this paper, deals with models to evaluate real options.

In part I, as a necessary prerequisite Option Pricing Theory (OPT) and Contingent Claims Analysis (CCA) are introduced with a demonstration of the main concepts to be applied to simple financial options and securities encountered in corporate capital structure. The next step in the assessment of real options, is to introduce options for specific situations. They provide an impression of how to derive more complex real option models. Finally, different real options themselfes are structured and individually introduced. Readers already familiar with CCA can skip this part.

Chapter 2 explains the basic concepts of OPT and CCA. Besides the option terminology and general relations for option prices, three different approaches to derive an exact option pricing formula are presented. The approaches rely solely on arbitrage arguments, that is a portfolio of securities which is continuously immunized against risk must earn the risk-free rate of return.

Since OPT covers only the narrow field of traded financial options, the general character of CCA is demonstrated via an example of corporate liabilities. It turns out, that all corporate liabilities are contingent claims upon the value of firm's assets, where the risk of bankrupcy is taken into account. An extension in this chapter shows that the value of all contingent claims can be described by a generalized partial differential equation (PDE). The equation is subject to a few specific functions to the claim, but presents a framework to evaluate most contingent claims, nevertheless.

Finally, chapter 2 also shows how to relax several standard assumptions in order to arrive at specially tailored models. In addition to different types of stock price movements, special options - such as compound options, options to exchange one risky asset for another, and options on the minimum and the maximum of two assets - are presented. They prove useful for understanding the process of deriving models for real options.

Chapter 3 demonstrates how to use CCA for the assessment of further corporate liabilities and introduces the different categories of real options. The corporate liabilities addressed are subordinated debt, warrants, convertibles, and callable convertibles. It turns out, that for some securities there are representations by simple options, but others can only be evaluated with their own uniquely tailored model. The financial options treated in this section constitute a part of corporate capital structure and typically appear on the liabilities side of corporate balance sheets.

On contrary, most real options are part of the assets side of corporate balance sheets. They assess management's flexibility to react as uncertainty over future events resolves. Real options fall into three groups: Operating options, investment opportunities, and project financings. Operating options represent management's flexibility to react to changes in profitability by altering the operation process. The means to achieve this are the options to temporarily shut down, to abandon, or to switch to a set of more profitable input or output factors. Investment opportunities can either be complete in themselves or entail further investment opportunities. Finally, project financing applies when investments contain both financial and real options. In particular, application of these models allow to consider the effects of operating flexibility, growth opportunities, and bankrupcy risk on all components of project's capital structure.

Chapter 4 introduces valuation methods for option pricing models that do not have closed-form, or analytical solutions. For a few simple options, analytical solutions exist for their price.

For many realistic financial and real options, however, they do not. In such cases numerical approximation procedures can be employed. One distinguishes between two categories of techniques: First, methods that approximate the value of the underlying variable directly, and second, methods that approximate the PDE. The chapter briefly introduces the properties of both methods and ends with a short discussion of efficiency, accuracy, and stability of the approximation results.

Part II concentrates on the characteristics of models that evaluate real options. It includes an appendix that shows how to use numerical procedures to evaluate four specific real options.

Chapter 5 summarizes empirical experience with recent real option approaches, introduces the particular problems of evaluating real options, presents models for operating options, investment opportunities, and project financings, and concludes with a critical summary of the chapter.

There is significant empirical evidence, that CCA reveals a greater part of real investment value than traditional static method, such as Net Present Value (NPV) and Discounted Cash Flow (DCF) analysis would suggest. Even though both static methods and CCA yield identical results under certainty, CCA is able to assess the value contingent on flexibility to react when future is uncertain. Consequently, CCA delivers systematically higher estimates for the real investment value when future is uncertain. Extra value traditional static approaches generally do not recognize.

Problems arise when CCA is applied to real options because eligible pricing models assume that traded risky assets are concerned. They can be put into a portfolio which can be immunized against risk by an appropriate strategy. Usually, the underlying risky assets are real investment projects, which are rarely traded. To still use the risk immunization strategy, one assumes that a hypothetical financial asset exists, which has exactly the same properties as the non-traded asset. Even though this assumption is sufficient to arrive at a suitable real option pricing model, several important parameters such as risk-adjusted discount rates and expected growth must additionally be estimated.

Among the models evaluating operating options, the cases of operations with temporary shut down, abandonment, and operations with both temporary shut down and abandonment receive consideration. The section on investment opportunities treats single stage, two stage, and multi stage growth options. It goes into greater detail for two other applications concerning construction time flexibility and the flexibility to meet growing demand. The section on project financing presents an example of a project which contains both financial and real options. A purely theoretical discussion is near to impossible, because of the nature of this category.

In summary, the models to evaluate real options have greatly different qualities. A few of them are very simple and limited and therefore give only a rough idea of the influence of uncertainty on real investments. Others focus on very specific properties of real investments and thus reveal a very detailed picture with a narrow frame. Moreover, nearly all real option pricing models are faced with considerable difficulties concerning the estimation of risk-adjusted discount and growth rates. Although some models have defects, the advanced ones prove more accurate that state-of-the-art static approaches such as NPV and DCF techniques, nevertheless.

PART I: INTRODUCTION TO OPTION PRICING

2. Option Pricing and Contingent Claims Analysis

2.1 Introduction

Both Option Pricing Theory (OPT) and Contingent Claims Analaysis (CCA) generally serve to determine the price of securities whose pattern of payoffs depends upon the price of other securities. The intellectual bulk of this paper, real options, builds on the basic concepts of these approaches.

Option Pricing Theory represents the intellectual starting point of all models, receives mention in section 2.2 of this chapter. The discussion concentrates on three different approaches to option pricing. All three approaches are built upon the central perception, that option prices can be determined through construction of riskless portfolios.

The binomial model approach of Cox, Ross, and Rubinstein is presented first. The binomial model formulates option pricing in a discrete-time framework and is the easiest way to understand the properties of option pricing methods. The well-known Black-Scholes option pricing formula is treated second. Unlike the binomial model, the Black-Scholes model works within a continuous-time framework. It is more difficult to understand as a result, but represents the way most option models are developed. The third model, the risk-neutral approach of Cox and Ross presents an alternative to Black and Scholes's way to derive option prices in a continuous-time framework.

Originally, OPT focused on the narrow sector of stock options. There is, however, a variety of securities that also depends on the price of other securities. This variety of securities is summarzed under the term contingent claims. Black and Scholes were the first to point out, that all corporate liabilities are in fact contingent claims. Section 2.3 looks on a Contingent Claims Analysis of corporate liabilities like equity, risky debt, and loan guarantees. Further, the section shows, that the price of all contingent cllaims is ruled by a generalized partial differential equation (PDE), as presented by Merton (1977). To demonstrate the concept of generalized PDEs, the last part of the section shows how to allow for dividends, coupon payments, and early exercise for corporate liabilities and other securities.

The last part of this chapter, section 2.4., presents four extensions of OPT and CCA which prove useful for special situations. The first extension focuses on the price evolution process of the underlying security. Pure jump and mixed diffusion-jump processes are introduced as alternatives to the standard pure diffusion process. A second extension shows how to assess a compound option - that is an option on an option. Compound options are useful for pricing options on stock of leveraged firms. The third extension treats the option to exchange one risky asset for another. Formally, it amounts to both put and call options with uncertain exercise price. Finally, the fourth extension presents the formulas for the value of call options on either the minimum or the maximum of two risky assets. The structure of the options in the last three extensions resembles that of real options.

To facilitate comparison to the original texts, variables were sometimes directly adopted. Due to the compilation of various models from independent sources, the letters employed overlap on occasion. One should not confuse variables among models.

2.2 Option Pricing Theory

Definitions

The following paragraphs define the terminology necessary for minimal understanding of option pricing.

An *option* is a contract which gives its owner the right to trade a fixed number of shares of a specified common stock at a fixed price within a particular period of time.

A *call* option gives the right to buy the shares whereas a *put* option gives the right to sell the shares.

Translating the right into action is referred to as *exercising* the option. The specified stock is known as the *underlying security*. The fixed price is termed the *striking price* or *exercise price*. The end of the period interval is called the *maturity date* or the *expiration date*. The individual who creates and issues an option is termed *seller* or *writer*. The individual who purchases an option is called the *holder* or *buyer*. The market price of the option is the *option price* or *premium.*

American type options contain the right to exercise during the whole time period up to expiration whereas for *European* type options, this right is limited to maturity itself.

Some race track terms have slipped into the options vocabulary: An option *finishes in-the-money*; if it has a positive value at expiration. It *finishes out-of-the-money* if its expercise value is negative at expiration. Before expiration, options would be *in-the-money*, *at-the-money*, or *out-of-the-money*, if they, when exercised immediately, resulted in a positive, zero, or negative value, respectively.

Finally, four terms from portfolio strategy remain to be introduced. A *hedge portfolio* is a riskless portfolio with respect to changes in the price of its components. Holding a *long* position of a security is a strategy that involves owning the security itself and leads to profits when prices increase. Holding a *short* position of a security is equivalent to selling a security that is not owned. It involves borrowing the security and is profitable when prices fall. *Arbitrage* is a portfolio strategy that generates riskless profits with zero net investment. Arbitrage is only possible in market disequilibrium and consists of buying underpriced securities in return for overpriced ones with the same payoffs.

Preference and Distribution-Free Results

Several rational restrictions on the prices of put and call options exist.

To represent the mechanisms of options, let c(S, T, X) and p(S, T, X) be the current value of a European call and put option, respectively. They are a function of the stock price, S, the time to maturity of the option, T, and the exercise price, X. The time to maturity, T, is the difference between the current date, t, and the expiration date, t*, of the option, or T = t* - t. Further, let C(S, T, X) and P(S, T, X) denote the present value of an American call and put option, respectively.

If options differ only in stock price, time to maturity, and exercise price, and moreover pay no dividends, then it is possible to derive rational bounds on the option price. The bounds rely only on the assumption, that investors prefer more wealth to less. Since neither a particular risk-preference structure of investors nor a specific distribution of future stock prices has to be stipulated, the boundaries can be derived with disregard to preferences and stock price distributions. As a consequence, the bounds are generally valid and must apply to all options.

The boundaries in option prices can be summarized as follows (see Merton (1973)):

B.1 Due to limited liability the option value is never negative:

$C(S, T, X) \geq 0 \qquad c(S, T, X) \geq 0$

$P(S, T, X) \geq 0 \qquad p(S, T, X) \geq 0$

B.2 At maturity, the value of an option is equivalent to its actual exercise value:

$C(S, 0, X) = c(S, 0, X) = \max(S - X, 0)$

$P(S, 0, X) = p(S, 0, X) = \max(X - S, 0)$

B.3 Before maturity, European options are at least worth the present value of the final payoff:

$c(S, T, X) = \max(S - X\,B(T), 0)$

$p(S, T, X) = \max(X\,B(T) - S, 0)$

where B(T) is the present value of a risk-free zero bond expiring at T = 0, with B(0) = 1.

B.4 American options are at least as valuable as European options with identical terms. However, Merton (1973) demonstrates that before maturity, the value of an unexercised American option always exceeds the payoff from immediate exercise. It follows that it never pays to exercise an American call prematurely and that the value of American and European calls is identical.

$C(S, T, X) = c(S, T, X)$	on a non-dividend paying stock
$C(S, T, X) \geq c(S, T, X)$	on a dividend paying stock
$P(S, T, X) \geq p(S, T, X)$	in all cases

B.5 The upper bound of the call or put value is the stock or exercise price, respectively.

$$c(S, T, X) \leq C(S, T, X) \leq S$$

$$p(S, T, X) \leq P(S, T, X) \leq X$$

B.6 European options with longer life are never worth less than otherwise identical options with shorter life:

$$c(S, T_1, X) \leq c(S, T_2, X) \qquad \text{where } T_1 < T_2$$

$$p(S, T_1, X) \leq p(S, T_2, X)$$

B.7 European calls with lower exercise price are never worth less than otherwise identical calls with higher exercise price. For European puts, the relationship is reciprocal:

$$c(S, T, X_1) \geq c(S, T, X_2) \qquad \text{where } X_1 < X_2$$

$$p(S, T, X_1) \leq p(S, T, X_2)$$

B.8 Incremental increases of European call (put) prices never exceed (fall below) incremental increases (decreases) of stock prices:

$$0 \leq c_S(S, T, X) \leq 1$$

$$0 \geq p_S(S, T, X) \geq -1$$

The suffix denotes the first order partial derivative with repect to the stock price, S.
B.9 The value of European call and put options is a convex function of the stock price:

$$c_{SS}(S, T, X) \geq 0$$

$$p_{SS}(S, T, X) \geq 0$$

The suffix denotes the second order partial derivative with repect to the stock price, S.

B.10 Calls and puts are connected to underlying stocks and riskless bonds. For European options a relation, called the put-call parity, must hold (see Stoll (1969)).

$$c(S, T, X) = S + p(S, T, X) - X\,B(T)$$

The preference and distribution free results must hold for all option pricing models. The bounds sufficiently describe a stylized evolution of European options value. The trend in call prices varying stock prices is illustrated in Figure 1.

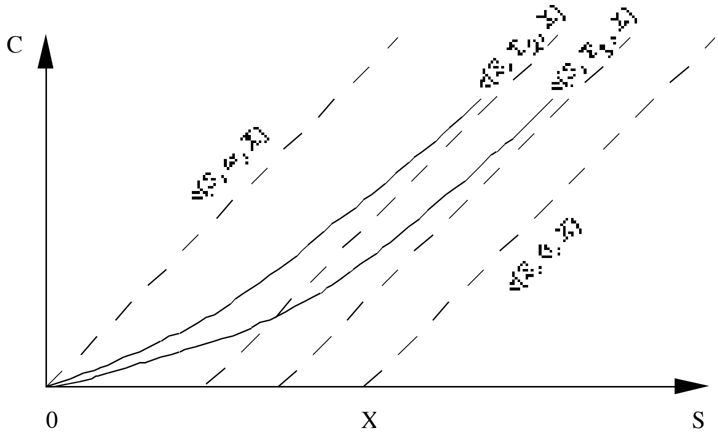


Figure 1

Discrete-Time Approach

This intuitive option pricing model builts upon several idealizing assumptions:

A.1 Capital markets are free of transaction costs and taxes.

A.2 Borrowing and lending occurs at a single risk-free rate, that is both known and constant.

A.3 There are no restrictions on borrowing, lending, or short sales.

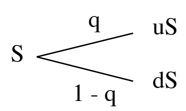
A.4 Capital markets do not allow riskless arbitrage opportunities.

A.5 The underlying security neither pays dividends nor provides other distributions.

Assumptions A.1 - A.4 imply frictionless capital markets, a standard asumption in contemporary finance literature. Assumption A.5 simplifies the derivation of the model. Succeeding sections show how to relax it.

The discrete time approach of Cox, Ross, and Rubinstein (1979) constitutes an intuitive model for the pricing of European call and put options. Since stock prices are supposed to follow a multiplicative binomial movement - hence the name binomial model. Binomial movement means that stock prices make certain jumps after discrete time intervals. After a certain time interval, Δt, the current stock price, S, can either climb to uS with probability, q, or fall to dS with probability, 1 - q. The multiplicative factor, u, stands for the magnitude of upward movements in the stock price and the factor, d, represents the size of downward movements. Graphically, the binomial stock price movement looks like.

(1)



In order to prevent arbitrage, the periodical risk-free discount factor, r', imposes a restriction on the magnitude of multiplicative factors: u > r' > d > 0.

Consider a rudimentary European call option with exercise price, X, that expires after Δt, the length of one binomial period. Obviously, the underlying stock and with it the option can have only two possible outcomes at expiration of the call. For this case, Cox, Ross, and Rubinstein show that it is possible to create a replication portfolio of risk-free zero bonds and the underlying stock, that has exactly the same payoff as the rudimentary call. By setting the payoffs at maturity equal, the current value of the portfolio determines the unknown current value of the call, c(S, t).

To calculate this value, determine first the payoff from the call at maturity. Since the stock price can have only two possible outcomes, the call value must be either $c(uS, t + \Delta t) = \max(uS - X, 0)$ or $c(dS, t + \Delta t) = \max(dS - X, 0)$.

(2) $c(S, t) \begin{cases} \xrightarrow{q} c(uS, t + \Delta t) = \max(uS - X, 0) \\ \xrightarrow[1-q]{} c(dS, t + \Delta t) = \max(dS - X, 0) \end{cases}$

To form a replication portfolio, buy x shares of stock and finance the purchase by selling riskless zero bonds. The current value of the bonds (shares) is B (xS), but they have to be paid back for $B\, r'^{\Delta t}$. So the value of the portfolio evolves according to

(3) $xS + B \begin{cases} \xrightarrow{q} x\, uS + B\, r'^{\Delta t} \\ \xrightarrow[1-q]{} x\, dS + B\, r'^{\Delta t} \end{cases}$

In order to reach identical outcomes, set the portfolio's payoff equal to the call's payoff.

(4)
$$x\, uS + B\, r'^{\Delta t} = c(uS, t + \Delta t)$$
$$x\, dS + B\, r'^{\Delta t} = c(dS, t + \Delta t)$$

Solving for x and B leads to the proportions of shares and debt in the replication portfolio.

(5)
$$x = \frac{c(u\,S, t + \Delta t) - c(d\,S, t + \Delta t)}{(u - d)\,S}$$
$$B = \frac{u\,c(d\,S, t + \Delta t) - d\,c(u\,S, t + \Delta t)}{(u - d)\,r'}$$

It is easy to verify, that a portfolio consisting of x shares of stock financed by zero bonds worth B has exactly the same payoff as the call option. Perfect substitutes must have the same price, since arbitrage is otherwise possible. Current replication portfolio prices are identical to current call price. The current value of the rudimentary call option is therefore:

(6)
$$c(S, t) = x\,S + B = \frac{p\,c(u\,S, t + \Delta t) + (1 - p)\,c(d\,S, t + \Delta t)}{r'}$$

where $p = \frac{r' - d}{u - d}$

If investors have a risk-neutral risk preference structure, i.e. are indifferent to risk, then the factor, p, has the properties of a probability and is called hedging probability. In which case, current option prices can be interpreted as the expected future option value, discounted back at the risk-free rate.

The single time interval analysis can easily be extended to a multi interval model. Every period, starting with the maturity date, the portfolio proportions are revised and discounted to the next, more recent time step. Cox, Ross, and Rubinstein show that the extension results in a binomial model for an arbitrary number, n, of discrete time intervals.

$$(7) \qquad c(S,t) = \left[\sum_{j=0}^{n} \frac{n!}{j!\,(n-j)!}\, p^{j}(1-p)^{(n-j)} \max\left(u^{j} d^{(n-j)} S - X, 0\right) \right] / (r')^{n}$$

An important advantage of the formula, the probability of up- and downward movements, q, has become irrelevant. Disagreement among investors about the probability of increases and decreases in stock prices, does not result in disagreement upon current option prices.

Another advantage is, that in order to derive the model, no particular risk preference structure had to be stipulated. This means, that option prices do not contain risk premiums. Consequently, it is not necessary to employ procedures to estimate risk-adjusted discount rates. Unlike option pricing models, many other valuation models in modern finance theory must incorporate the disadvantage of having to use these procedures.

The origin of the two advantages lies in the fact, that investor's expectations and risk preferences are already reflected in current stock prices. Therefore, the quality of stock prices is of major importance. Only if stock prices are result of frictionless financial markets, can option pricing models yield reliable results. As partial equilibrium models, they are only able to describe the pricing relationship between stock and options. They are not able to provide asset prices on a stand alone basis and also fail to work if the underlying stock price is not the market price of a financial security.

Nevertheless, Part II of this paper demonstrates how modified option pricing models apply to optionlike assets, if market prices of traded financial securities are inavailable.

Continuous-Time Approach

This subsection demonstrates how to derive a formula for the call option pricing problem when underlying stock prices follow a continuous movement.

The length of the discrete time intervals, Δt, in the previous approach was not limited. If Δt becomes large, say a day, then the binomial process allows only two different stock prices each day. It is obvious, that this no longer is an appropriate model for actual stock prices. In turn, if Δt becomes very small, i.e. approaches zero, then the discrete binomial process becomes continuous in the limit (see Cox, Ross, and Rubinstein (1979) p. 246-255). Infintely small time intervals cause an infinite number of binomial steps. Unfortunately, an infinite number of binomial steps is computationally intractable, so that it is easier to replace the discrete time assumption by its continuous time equivalent:

A.6 Trading takes place continuously.

A.7 Stock prices evolve according to a continuous stochastic differential equation of the form:

$$(1) \qquad \frac{dS}{S} = \mu\, dt + \sigma\, dz$$

where dS / S is the instantaneous rate of return on the stock, μ is the expected rate of change in the stock price, dt is a small increment of calendar time, σ is the instantaneous standard deviation of the rate of change in the stock price, and dz is a standard Wiener process with

$$(2) \qquad dz = \varepsilon \sqrt{t^* - t}$$

where ε is a random sample from a standardized normal distribution, t* is the maturity date, and t is the current date.

Generally, the stochastic differential equation (1) is termed an Itô process (see for example McKean (1965)). In the special case, where μ and σ are constants, it is called geometric Brownian motion (see for example Hull (1989) p. 69 ff).

An Itô process implies that the rate of return on a stock consists of two components: A deterministic drift term μ dt, and a normally distributed stochastic term σ dz. The influence of the stochastic term increases with $\sqrt{(t^* - t)}$ in order to reflect increasing uncertainty due to increasing forecasting horizons.

Generally, stock prices that follow geometric Brownian motions are distributed lognormally (see for example Hull (1989) p. 83 ff).

Figure 2 depicts a sample stock price path and Figure 3 shows the lognormal probability distribution of stock prices that follow geometric Brownian motions.

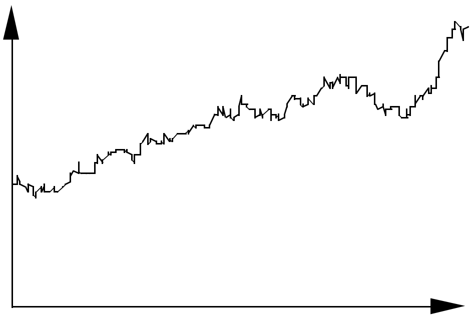


Figure 2

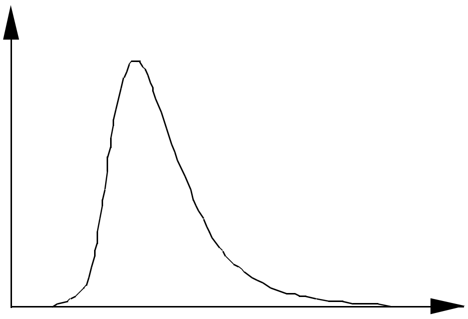


Figure 3

With the additional assumptions A.6 and A.7, it is possible to derive a continuous time call option pricing formula, the Black-Scholes model.

For this purpose, consider a portfolio consisting of a long position of stock, S, and a short position of European calls, c, on the stock. The value of the portfolio, V, can be expressed as:

(3) $V = Q^s S + Q^c C$

where Q^s (Q^c) denotes the quantity of shares (calls). The instantaneous change in the portfolio value is then

(4) $dV = Q^s dS + Q^c dc$

where dV, dS, and dc are the instantaneous changes in portfolio, stock, and call value, respectively.

Since the option price is a function of current stock prices, its movement over time must consequently be related to stock price movements. For stock prices following a geometric Brownian motion, Black and Scholes (1973) demonstrate that Itô's lemma (see for example McKean (1965)) can be used to derive a stochastic differential equation which desribes the instantaneous change in the call value

(5) $$dc = c_S ds + c_t dt + \frac{1}{2} \sigma^2 S^2 c_{SS} dt$$

where c_s, c_t, and c_{ss} are first and second order partial derivatives with respect to S and t. Substituting into equation (4) leads to

$$(6) \qquad dV = (Q^s + c_S Q^c)\, dS + Q^c \left[c_t + \frac{1}{2} \sigma^2 S^2 c_{SS} \right] dt$$

The only stachastic term on the right-hand-side is the first, dS. The rest of the return on the portfolio is deterministic. In order to eliminate stochastic returns, that is to create a riskless hedge, one can set $Q^S = 1$ and $Q^c = -1 / c_S$. If this risk immunization strategy is conducted continuously over time, the portfolio will remain instantaneously riskless. To avoid riskless arbitrage, the portfolio must therefore earn the risk-free rate.

$$(7) \qquad \frac{dV}{V} = r\ dt$$

Substituting (3) and (6) into the (7), and rearranging leads to the following PDE, that implicitly describes the value of the call.

$$(8) \qquad \frac{1}{2} \sigma^2 S^2 c_{SS} + r S c_S + c_t - r c = 0$$

The PDE is defined for the stock price-time space R = { (S, t') | 0 ≤ S ≤ ∞, t ≤ t' ≤ t* } where t is the current date, and t* is the maturity date of the option. In order to solve the PDE for the current call value, it is necessary to specify three boundaries of R. They follow from the preference and distribution free results presented earlier in this chapter. The payoff at maturity, t*, specifies the terminal boundary.

$$(9a) \quad c(S, t^*) = \max (S - X, 0)$$

A call is worthless if the stock price is zero. So the lower boundary is

$$(9b) \quad c(0, t) = 0$$

As stock prices approach infinity, the increments of change in call and stock price become equal. Therefore the upper boundary is

$$(9c) \qquad \lim_{S \to \infty} c_S(S, t) = c_S(\infty, t) = 1$$

Having specified the three boundary conditions, Black and Scholes solve the PDE by transforming the problem into the well-known heat exchange equation of physics. The solution is the Black-Scholes formula for the value of a European call option.

(10) $c(S, t) = S\ N_1(d_1) - X e^{-r(t^* - t)} N_1(d_2)$

where $d_1 = \dfrac{\ln(S/X) + r(t^* - t)}{\sigma\sqrt{t^* - t}} + \dfrac{1}{2}\sigma\sqrt{t^* - t}$

$d_2 = d_1 - \sigma\sqrt{t^* - t}$

$N_1(.)$ = Univariate cumulative standard normal function

The call value can be regarded as the stock price, S, minus the discounted value of the exercise price, X. However, each component is weighted by a probability. The probability, $N_1(d_1)$, is the inverse of the hedge ratio, and tells the investor how many call options ought to be written against one share of stock in order to eliminate instantaneous stock price risk. The term $N_1(d_2)$ can be regarded as the probability that the option will finish in-the-money.

The sensitivity of the call value to changes in the relevant parameters is captured by corresponding first order partial derivatives (see for example Jarrow and Rudd (1983) p. 117).

(11a) $c_S = N_1(d_1) \qquad > 0$

(11b) $c_t = -X e^{-r(t^* - t)} \left[r N_1(d_2) + \dfrac{1}{2}\dfrac{\sigma}{\sqrt{t^* - t}} N_1'(d_2) \right] \qquad < 0$

(11c) $c_X = -e^{-r(t^* - t)} N_1(d_2) \qquad < 0$

(11d) $c_\sigma = S\sqrt{t^* - t}\ N_1'(d_2) \qquad > 0$

(11e) $c_r = X e^{-r(t^* - t)} (t^* - t)\ N_1(d_2) \qquad > 0$

where $N_1'(.)$ is the univariate standard normal density function.

The value of a European put option can easily be determined with the put-call parity. Since the call price follows from the Black-Scholes formula, the put price is

(12) $p(S, t) = -S\ N_1(d_1) + X e^{-r(t^* - t)} N_1(-d_2)$

where d_1 and d_2 are defined as for (10).

Risk-Neutral Approach

The hedge portfolios in the discrete- and continuous-time approach generally do not attribute to investors any risk premium, because the portfolios are completely riskless. Even if investors have different attitudes toward risk, they will still agree upon the expected return for the hedge, the risk-free rate. Since this argument is valid for arbitrary risk preference structures, it is admissible to assume a particular one, the risk-neutral structure. In a risk-neutral world, any financial asset, no matter how risky, is expected to earn the risk-free rate. As a result, the current price of the option is the expected payoff at maturity, discounted back to the present at the risk-free rate. Cox and Ross (1976) show, that this approach applies to all options that can be incorporated into a hedge portfolio.

For a European call, the current value is identical to the expected value at maturity discounted back to the present at the risk-free rate.

(1) $c(S, t) = e^{-r\,(t^* - t)}\, E(c(S, t^*))$

When stock prices follow geometric Brownian motions, they are distributed lognormally at maturity. Letting L' (.) denote a lognormal density function, the call price can be expressed as.

(2) $c(S, t) = e^{-r(t^* - t)} \int_X (S^* - X)\, L'(S^*)\, dS$

To solve the integral, Smith (1976) uses the following theorem, which also can be applied to a broad set of related problems.

<u>Theorem</u>

If L'(S*) is a lognormal density function with

$$Q = \begin{cases} 0, & \text{if } S^* > \square X \\ \square S^* - \square X, & \text{if } \square X \geq S^* > \square X \\ 0, & \text{if } S^* < \square X \end{cases}$$

then

$$. \qquad E(Q) \equiv \int_{\square X}^{\square X} (\square S^* \square \square X)\ L'(S^*)\ dS^*$$

$$= e^{\square T} \square S \left[N_1 \left(\frac{\ln (S / \square X) + (\square + \square^2 / 2)\, T}{\square \sqrt{T}} \right) \square N_1 \left(\frac{\ln (S / \square X) + (\square + \square^2 / 2)\, T}{\square \sqrt{T}} \right) \right]$$

$$\square \square X \left[N_1 \left(\frac{\ln (S / \square X) + (\square \square \square^2 / 2)\, T}{\square \sqrt{T}} \right) \square N_1 \left(\frac{\ln (S / \square X) + (\square \square \square^2 / 2)\, T}{\square \sqrt{T}} \right) \right]$$

where □, □, □ and □ are arbitrary parameters, T the time to maturity, i.e. $T = t^* - t$, and □ the expected rate of return in S, i.e. $E(S^* / S) = e^{\square T}$.

Smith shows that for $\square = \square = e^{-r\,(t^*-t)}$, $\square = 1$, $\square = \infty$, and $\square = r$, the above theorem yields the Black-Scholes call formula.

2.3 Contingent Claims Ananlysis

2.3.1 Corporate Liabilities as Contingent Claims

The basic option pricing model, presented in the previous section, can be extended to price option like assets, or contingent claims. Contingent claims are defined as securities, whose value depends upon the price of at least one other security. The theory leading to pricing formulas is therefore called Contingent Claims Analysis (CCA). Options on stock are a subgroup of contingent claims and Option Pricing Theory (OPT) is a subset of CCA.

Black and Scholes (1973) were the first to recognize that all corporate liabilites are contingent claims upon the value of firm's assets. If a firm goes bankrupt because all assets have become worthless, then all liabilities such as equity and debt will also become worthless.

To be more specific, consider a stock corporation with a capital structure consisting of shareholder's equity and a single issue of zero coupon bonds, maturing at time t*. Assume that the current market value of assets is V(t) and that the current value of equity and debt, is E(V, t) and D(V, t), respectively. Suppose further, that the firm cannot pay any dividends until after the face value of the bonds, X, has been paid off.

From a contingent claims perspective, two different things can happen at maturity. First, the firm is able to cover outstanding debt payments by selling some or all of its assets. Second, the outstanding debt exceeds the value of assets. The firm goes bankrupt and as a result it winds up being liquidated. In the first case, V(t*) > X, debt holders receive the face value of the bonds, X, and equity is worth, V(t*) - X. In the second case debt holders receive the firm, worth V(t*), and equity is worthless. Due to limited liability, equity holders are not obliged to make up for the shortfall of debt.

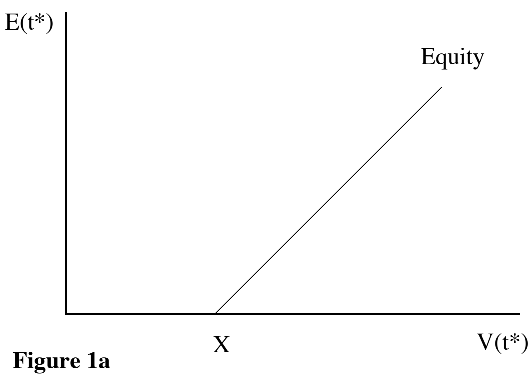


Figure 1a

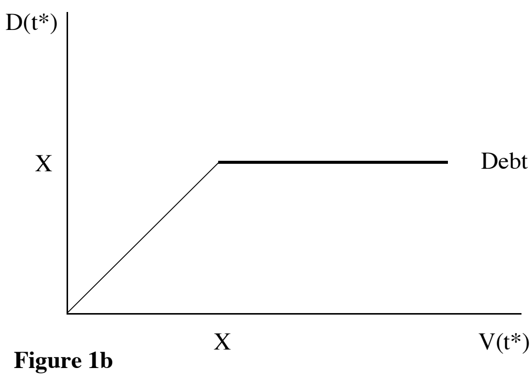


Figure 1b

Figure 1a and Figure 1b depict the respective payoffs for equity and debt at maturity. In the presence of zero coupon debt, equity resembles a call option on the value of firm's assets, V, with exercise price, X, and maturity, t*.

(1) $E(V, t^*) = \max(V - X, 0)$

Call value, c, and equity value, E, are identical at maturity, and consequently their value must also be identical prior to maturity.

(2) $E(V, t) = c(V, t)$

In turn, Figure 1b shows that at maturity the value of risky zero coupon debt is

(3) $D(V, t^*) = \min(V, X)$

In effect, debt holders own the firm's assets, but have granted equity holders a call option to buy assets back at maturity of debt. Another interpretation is obtained by using the put-call parity in combination with the identity of assets and liabilities, i.e. $V(t) = E(V, t) + D(V, t)$.

(4) $P(V, t) = E(V, t) - V + X e^{-r (t^* - t)}$

$= - D(V, t) + X e^{-r (t^* - t)}$

or $D(V, t) = X e^{-r (t^* - t)} - p(V, t)$

This reinterpretation shows, that risky debt equals the difference between riskfree debt and the value of a put option on the firm's assets that can be exercised by shareholders. The put option acts as insurance against default. It exactly corrects for the sum that would be lost in case of bankrupcy. So the put premium, p(V, t), is equivalent to the value of a loan guarantee, G(V, t), that makes the bond riskless. Figure 1c depicts the payoff of the loan guarantee at maturity.

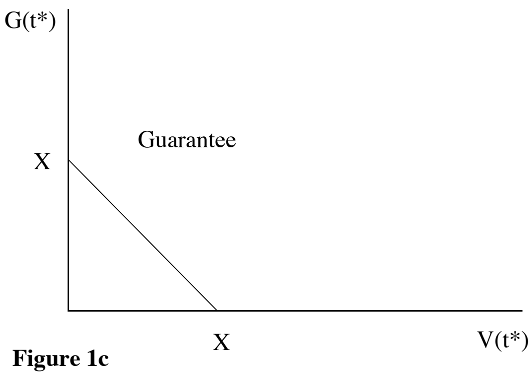


Figure 1c

Rearranging equation (4) leads to the value of equity in presence of risky zero coupon debt.

(5) $E(V, t) = V + P(V, t) - X e^{-r (t^* - t)}$

The higher risk, uncertainty that is, the higher put option value and the higher equity value. This is perhaps the reason why shareholders are ready to accept high risk projects when firms are near bankrupcy. In this case, shareholders profit from good outcomes of the project but leave bad outcomes to debt holders.

The Generalized PDE for Contingent Claims

To generalize Black and Scholes's model, the assumption that firms pay no dividends and that corporate liabilities can only be zero bonds is rather restrictive. Furthermore, there are several corporate liabilities that can be exercised before maturity.

Including dividends, coupon payments, and allowing early exercise leads to greater detail, however, also to a loss of generality. In the above subsection, the value of liabilities was derived without employing a particular evaluation model. In this subsection, a specific valuation model must be used to capture the increased complexity of contingent claims when dividends, coupons, and early exercise are considered.

Merton (1977) shows a very general approach to determining the price of one security whose value is a function of another security. He shows, that any contingent claim can be described by a specific PDE which is subject to specific boundary conditions. Although Merton derives the generalized PDE by usual arbitrage arguments, he shows that the PDE remains valid even if institutional restrictions prohibit arbitrage. This means, that this method allows to evaluate assets which are not explicitly traded.

The generalized PDE does not always have closed form solutions. Therefore, it is sometimes necessary to employ numerical techniques to approximate the implicit value of a contingent claim.

The derivation of the generalized PDE is built upon assumptions A.1 - A.7, one relaxes assumption A.5 by allowing underlying securities to distribute a cash dividend yield, □(V, t). Furthermore, it is explicitly allowed that the magnitude of dividends is a function of underlying security value, V, and calendar time, t. The geometric Brownian motion for the value of the underlying security is

(1) $dV = (\square - \square(V, t))\, V\, dt + \square\, V\, dz$

where □ is the expected rate of change in V, □ the instantaneous standard deviation of the rate of change in V, and dz a standard Wiener process.

Consider a second risky asset, W, whose value depends upon the value of the underlying security, V. This second asset is the contingent claim and has the following properties: First, security holders receive a payout, $\square_W(V, t)$. Second, the claim takes the value of known functions at specific bounds: If the underlying security value reaches a lower boundary V^l, i.e. $V(t) = V^l(t)$, then claim value follows from function f_1, i.e. $W(V^l(t), t) = f_1(V^l(t), t)$. Similarily, at an upper boundary, $V^u(t)$, claim value is determined by function f_2, i.e. $W(V^u(t), t) = f_2(V^u(t), t)$. At maturity, claim value is given by function f_3, i.e $W(V, t^*) = f_3(V, t^*)$.

If the function for the contingent claim value is twice differentiable with respect to V, and differentiable with respect to t, then a PDE for W(V, t) can be derived by forming a hedge portfolio consisting of underlying security, contingent claim, and riskless bonds. Merton (1977) shows that this portfolio is instantaneously riskless and that it therefore must earn the risk-free rate. Rearranging leads to the generalized PDE for the value of arbitrary contingent claims

$$\frac{1}{2}\sigma^2 V^2 W_{VV} + (r - \delta(V, t))\, V W_V + W_t - r W + \delta_W(V, t) = 0 \qquad (2)$$

which is subject to the previously defined boundary conditions

$$W(V^l(t), t) = f_1(V^l(t), t) \qquad (3a)$$
$$W(V^u(t), t) = f_2(V^u(t), t) \qquad (3b)$$
$$W(V, t) = f_3(V, t^*) \qquad (3c)$$

In general, the functions f_1, f_2, f_3, δ, and δ_W required to solve for W can be deduced from the particular terms of the contingent claim under consideration.

In order to demonstrate the use of the generalized PDE, consider again a European call option, c, on a stock that pays no dividends. Without dividend payments, $\delta = 0$. Since option holders are not entitled to any payments from the firm, $\delta_c = 0$. Boundary conditions (9a) - (9c) in the continuous-time subsection describe the call value at the boundaries. The generalized PDE for the call becomes identical to Black-Scholes's PDE. It follows, that Black-Scholes's approach is a special case of Merton's model.

Consider risky zero coupon debt, D, of a firm that pays no dividends and that has no other outstanding liabilities. Without dividend payments, $\delta = 0$. For zero coupon payments, $\delta_D = 0$. If the firm's assets are worthless, debt is worthless as well. If assets become very valuable, risky debt will have the same value as risk-free bonds with the same terms. At maturity, debt will be worth the lesser of asset's value or debt's face value. Formally, the boundary conditions become

(4a) $D(0, t) = f_1(0, t) = 0$

(4b) $D(\infty, t) = f_2(\infty, t) = X e^{-r (t^* - t)}$

(4c) $D(V, t^*) = f_3(V, t^*) = \min (V, X)$

Solving the PDE subject to the boundary conditions leads to identical results as already demonstrated in (4) in the previous subsection, only that Black-Scholes put formula replaced P(V, t).

If the firm pays a continuous dividend yield, □, and fixed cash coupon payments, c, it turns out that □ = (c / V) + □ and $□_D$ = c. The equation for the upper boundary condition (4b) then becomes

(5) $D(\infty, t) = (c / r) (1 - e^{-r (t^* - t)}) + X e^{-r (t^* - t)}$

In this case, no analytical solution exist for the PDE. Numerical approximations must be employed.

The possibility of early exercise can generally be captured by modified boundary conditions. For callable coupon bonds, there exists an upper boundary in firm value at which it is optimal for the firm to redeem bonds. According to Brennan and Schwartz (1977) the optimal call policy is calling the bond whenever its market value reaches the call price, $D^c(t)$. For a callable coupon bond, the upper boundary condition (4b) should therefore be replaced by the free boundary condition

(6a) $D(V^c(t), t) = D^c(t)$

(6b) $D_V(V^c(t), t) = 0$

The free boundary of the firm value is $V^c(t)$. If the firm value reaches this critical value, the firm will redeem the bonds optimally.

Since $V^c(t)$ follows implicitly, it is necessary to introduce a fourth condition, the Merton-Samuelson (see Samuelson (1965) and Merton (1973)) high contact condition (6b). The high-contact condition defines the slope in the free boundary. If the condition is specified accurately, the schedule of critical firm values, $V^c(t)$, maximizes the value of the contingent claim.

Problems of this variety, where free boundaries are specified and contingent claims have limited life, have no closed-form solution in general. Therefore, numerical approximation procedures must be employed which solve for the contingent claim's value together and the optimal exercise schedule simultaneously.

Assume that the callable bond is guaranteed. In this case, the problem can be specified to derive a solution for the value of the loan guarantee, G. The firm must still pay a continuous dividend yield so that □ = (c / V) + □. Since the loan guarantee has no payout, $□_G$ = 0. The boundary conditions for the loan guarantee are.

(7a) $G(0, t) = (c / r) (1 - e^{-r (t^* - t)}) + X e^{-r (t^* - t)}$

(7b) $G(V^c(t), t) = 0$

(7c) $G_V(V^c(t), t) = 0$

(7d) $G(V, t^*) = \max (X - V, 0)$

Again numerical approximation procedures must be used to determine the value of the loan guarantee, since no analytical solution can be obtained. From the discussion in the previous subsection, it follows that a loan guarantee resembles to a put option, which covers the cash shortfall in case of default. For this reason, loan guarantee value should roughly equal the difference between risky callable coupon bonds and their riskless equivalents.

Up to this point, continuous dividend and coupon payments were considered. This is rather unrealistic, because most payments occur discretely, that is in lump sums at specific dates. To bring the model closer to reality by capturing the discrete nature of cash payments, additional free boundaries have to be introduced.

Consider an arbitrary contingent claim, W, upon a dividend paying stock, S. In this case, □ = $□_W$ = 0 resolves the payout function. It is well known, that from cum-dividend date, t^-, to ex-dividend date, t^+, the stock price falls approximately by the amount of the dividend payment, d. It may prove worthwile to exercise the contingent claim immediately before or after the dividend payment.

For American call options, C, it sometimes pays to exercise immediately before the dividend payment, or at the cum-dividend date, t^-. In this case, the free boundary is

(8) $C(S(t^-), t^-) = \max (0, S(t^-) - X, C(S(t^-) - d, t^+))$

Whenever exercise value, $S(t^-) - X$, exceeds expected call value, $C(S(t^-) - d, t^+)$, the option will be exercised.

For American put options, P, exercise sometimes pays immediately after dividend payments, or at the ex-dividend date, t^+.

(9) $P(S, t^+) = \max (0, X - S, P(S, t^+))$

Within the range of these assumptions, the generalized PDE can be used to formulate the valuation problem for arbitrary contingent claims. As demonstrated, continuous and discrete dividend payments as well as early exercise can be successfully considered.

2.4 Extensions of CCA

Alternative Stochastic Processes

The first extension relaxes the assumption that the underlying security follows a continuous diffusion process, or equivalently that future security prices are distributed lognormally.

This can be achieved by introducing alternative stochastic processes, that generate the unexpected changes in underlying security prices. In fact, there are several other models describing stock price movements, but this subsection concentrates only on the most intuitive ones, the pure jump process and the mixed diffusion-jump process.

The pure diffusion process, or geometric Brownian motion, which was already introduced is the first of two general classes of continuous time processes. Figure 1a depicts a sample stock price path that follows a pure diffusion process. In this sample, the drift factor, □, is positive since prices increase on the average.

The pure jump process represents the second class of stochastic processes. These processes capture sudden jumps in stock prices upon arrival of unexpected important information. Since jumps can have a considerable influence on the value of derivative securities, it is important to reflect this possibility.

Cox and Ross (1976) show that the price for securities that evolves according to a pure jump process can be described by the following stochastic differential equation.

$$(1) \qquad \frac{dS}{S} = \square \; dt + (k \,\square\, 1) \; dq$$

where dq is a random Poisson variable that either takes the value 1 if a jump occurs, or the value 0 if no jump occurs. The probability of a jump, a Poisson event, is (□ dt), and the probability that no jump occurs is (1 - □ dt). The nonnegative constant, □, determines the frequency of jumps and is termed Poisson parameter. The other non-negative parameter, k, determines the amplitude of the jump.

In the pure jump process, price changes are determined by two components, a drift term □ dt and a stochastic term with magnitude (k - 1). At Poisson events, prices change by $e^{(k - 1)}$. If $k > 1$, each jump increases the underlying security prices. In turn, for $k < 1$ jumps decrease prices. Cox and Ross (1976) show that for constant jump amplitude, k, it is possible to create a riskless hedge which helps to solve the option pricing problem.

Figure 1b illustrates a sample stock price path, that follows a pure jump process. Apparently, the drift term, μ, is negative because stock prices decrease continuously, and the amplitude, k , is constant and positive since prices increase with each jump by a fixed percentage.

Usually in practice, stock prices can make large discrete jumps, and therefore pure diffusion processes do not accurately reflect their properties. The Black-Scholes model, which assumes that stock prices follow pure diffusions, is not a correct specification of real world option pricing problems. However, empirical tests reveal that pricing errors due to deviating stock price movements are usally small (see Hull (1989) p. 314-319).

Comparing the theoretical stock price movements in Figure 1a and 1b with empirical real world stock price charts shows, that theoretical specifications differ from empirical observations. Pure diffusion, as in Figure 1a, does not capture discrete jumps and pure jumps as in Figure 1b, do not consider stochastic changes between the jumps.

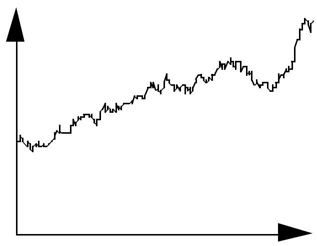


Figure 1a

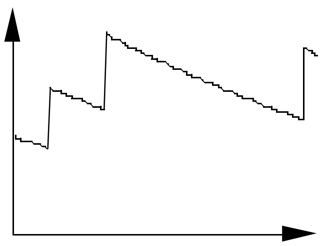


Figure 1b

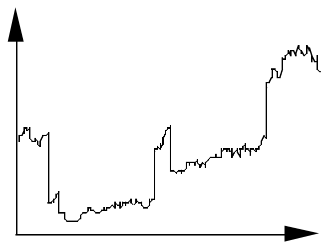


Figure 1c

To overcome these disadvantages, Merton (1976) introduces a combination of both processes, the mixed diffusion-jump process.

(2) $$\frac{dS}{S} = \mu\, dt + \sigma\, dz + (k - 1)\, dq$$

where the amplitude, k, is distributed lognormally, and the random variables dz, dq, and k are assumed to be independent.

A mixed diffusion-jump process leads to a lognormal distribution of future stock prices as long as no jumps occur. Figure 1c depicts the path of a sample stock price, that follows a mixed diffusion-jump process. The magnitude of the jumps no longer turns constant and price movement between the jumps becomes stochastic.

Merton (1976) shows that creating a riskless portfolio, immune to continuous and discrete changes in stock prices, is impossible. However, if the jump component of the changes is assumed to be unsystematic and diversifiable, risk-neutral valuation arguments can be used to find an analytical solution to the option pricing problem.

Compound Options

The second extension of CCA concentrates on leverage effects on option prices.

As pointed out by Black and Scholes (1973), equity of a leveraged firm equals a call option on the firm's assets. Therefore, a call option on stock of a leveraged firm is equivalent to an option on an option, that is an compound option.

Accordingly, the Black-Scholes option model holds only for the special case of unleveraged firms. For options on leveraged firms, Black-Scholes specifications are incorrect.

Unlike for debt-free firms, for leveraged firms the stock price volatility is inversely related to the stock price level. As firm value falls, the market value of shareholder's equity usually decreases more than the market value of issued debt. Consequently, the risk of owning the stock rises, inducing increased stock price volatility. Stock price volatility becomes a function of the stock price. Obviously, this contradicts Black and Scoles's assumption that stock prices follow standard geometric Brownian motions where volatility is constant.

Geske (1979) shows that the compound option pricing problem can be solved with risk-neutral arguments, also that the Black-Scholes model is a special case of his approach.

In order to derive Geske's compound option model, first consider the Black-Scholes model for the value of equity, S, of a leveraged firm with assets worth V financed by a single issue of zero coupon debt with face value, X_2, due at t_2*. The value of assets is uncertain, although assumed to vary over time according to a geometric Brownian motion. As already mentioned, equity in this situation amounts to a Black-Scholes call on asset value.

$$(1) \qquad S(V,t) = V\,N_1(d_1) - X_2\,e^{-r(t_2^*-t)}\,N_1(d_2)$$

where $d_1 = \dfrac{\ln(V/X_2) + r(t_2^*-t)}{\sigma\sqrt{t_2^*-t}} + \dfrac{1}{2}\sigma\sqrt{t_2^*-t}$

$$d_2 = d_1 - \sigma\sqrt{t_2^*-t}$$

Suppose a European call, C, written on equity, S, with exercise price, X_1, and maturity t_1*, where $t_1* \leq t_2*$. At t_1* the call value depends upon the value of equity. So the call value at maturity can be written as

$$(2) \qquad C(S(V, t_1^*), t_1^*) = \max\,(S(V, t_1^*) - X_1, 0)$$

According to risk-neutral valuation arguments, as presented by Cox and Ross (1976), the current value of the call is the expected payoff discounted back to the present at the risk-free rate.

$$\text{(3)} \quad C(S(V, t), t) = e^{-r\,(t_1{}^* - t)} \max\,(S(V, t_1{}^*) - X_1, 0)$$

Substitute $S(V, t_1{}^*)$ by equation (1), take the conditional density function of stock prices at $t_1{}^*$, and solve for the call value. According to Geske (1979), the solution to the problem of pricing a European call on another European call, or a compound option , is

$$\text{(4)} \quad \begin{aligned} C(V, t) \;=\;& V\, N_2\,(b_1, d_1;\, \rho) \\ & - X_2\, e^{-r\,T_2}\, N_2\,(b_2, d_2;\, \rho) \\ & - X_1\, e^{-r\,T_1}\, N_1\,(b_1) \end{aligned}$$

where
$$b_1 = \frac{\ln\,(V / V^c) + r\,T_1}{\sigma \sqrt{T_1}} + \frac{1}{2}\,\sigma \sqrt{T_1}$$

$$b_2 = b_1 - \sigma \sqrt{T_1}$$

$$\rho = \sqrt{T_1 / T_2}$$

The function $N_2(b_i, d_i;\, \rho)$ is the cumulative bivariate standard normal function with b_i and d_i as upper integral limits and coefficient of correlation ρ. The free boundary $V^c(t_1{}^*)$ is the critical firm value for which the call is at-the-money, i.e. $S(V^c(t_1{}^*), t_1{}^*) = X_1$.

Options to Exchange one Asset for Another

The next extension considers exercise prices, that vary randomly over time.

It can be shown, that a call option with randomly varying exercise price is equivalent to the option to exchange one asset for another. Margrabe (1978) derives a closed-form analytical solution to this problem and demonstrates that the Black-Scholes formula is a special case of his approach.

To derive the model, consider a European option to exchange a risky asset V, for another risky asset H at maturity. Both assets are assumed to follow geometric Brownian motions with

$$\text{(1a)} \qquad \frac{dV}{V} = \mu_V \, dt + \sigma_V \, dz_V$$

$$\text{(1b)} \qquad \frac{dH}{H} = \mu_H \, dt + \sigma_H \, dz_H$$

where μ_V is the expected rate of change in V, σ_V the instantaneous standard deviation of the rate of change in V, and dz_V a standard Wiener process, generating the unexpected changes in V. The parameters for H are defined similarily. The instantaneous coefficient of correlation between dz_V and dz_H is ρ.

Margrabe shows, that a riskless portfolio consisting of a long position in the option and short positions in assets V and H can be formed if a suitable hedge strategy is employed. Since riskless portfolios must earn the risk-free rate, he is able to derive a PDE that describes the value of the European option to exchange one asset for another.

$$\text{(2)} \qquad \frac{1}{2}\sigma_V^2 V^2 W_{VV} + \frac{1}{2}\sigma_H^2 H^2 W_{HH} + \rho\sigma_V\sigma_H V H W_{VH} + W_t = 0$$

subject to the boundary conditions

(3a) $W(0, H, t) = 0$

(3b) $W_V(\infty, H, t) = 1$

(3c) $W(V, H, t) = \max(V - H, 0)$

The boundary conditions state that the value of the exchange option is zero, when the underlying asset value is zero; the increase in option value equals the increase in the underlying security 's value, when the latter grow very large; and the payoff at maturity is the larger of zero and the difference between asset V and H.

Margrabe shows that the following analytically solves the European exchange option pricing problem.

(4) $$W(V, H, t) = V\ N_1(d_1) - H\ N_1(d_2)$$

where $$d_1 = \frac{\ln (V/H)}{\sigma\sqrt{t^* - t}} + \frac{1}{2}\sigma\sqrt{t^* - t}$$

$$d_2 = d_1 - \sigma\sqrt{t^* - t}$$

$$\sigma^2 = \sigma_V^2 - 2\rho\sigma_V\sigma_H + \sigma_H^2$$

The exchange option blatantly resembles to the Black-Scholes call formula. However, the risk has two sources and discounting is no longer necessary. If H is the present value of riskless zero coupon debt with face value, X, then the option to exchange two risky assets is identical to Black and Scholes call formula.

Options on the Minimum and the Maximum of two Assets

The last extension of CCA in this chapter treats European call options on the minimum of two assets. Further, it will illuminate the use of this option to value an option on the maximum of two assets.

Stultz (1982) and Johnson (1981) independently derive a solution to the above problem and show that Margrabe's (1978) exchange option model is a special case of their approaches.

To derive the model, consider a European call option, M, to receive the minimum of two assets, V and H, upon payment of the exercise price, X, at the maturity date, t*. Both asset prices are assumed to follow geometric Brownian motions, as introduced in the previous subsection. Stultz shows that a self financing portfolio strategy (see Harrison and Kreps (1979)) containing a riskless bond and assets V and H, enables one to replicate the option's payoff. Since at maturity, the self financing portfolio value is identical to the option's value, it follows that the current value of this portfolio must be identical to the unknown current value of the option.

According to Stulz, the value of the portfolio and the option's value can be described by the following PDE

(1) $$\frac{1}{2}\sigma_V^2 V^2 M_{VV} + \frac{1}{2}\sigma_H^2 H^2 M_{HH} + \rho\sigma_V\sigma_H V H M_{VH} + r V M_V + r H M_H - r M - M_t = 0$$

subject to three boundary conditions

(2a) $M(0, H, t) = 0$

(2b) $M(V, 0, t) = 0$

(2c) $M(V, H, t^*) = \max(\min(V, H) - X, 0)$

The boundary conditions state that the option is worthless whenever one of the assets is worthless, and that the payoff at maturity equals the difference between the minimum of the two assets and the exercise price or zero - whichever is larger.

The PDE can be solved by using the risk-neutral approach of Cox and Ross (1976). Stultz shows, that the value of the European call option on the minimum of two risky assets is

(3) $$M(V, H, t) = H\, N_2\left(a_1, c; (\rho\sigma_V - \sigma_H)/\sigma\right)$$
$$+\ V\, N_2\left(b_1, d; (\rho\sigma_H - \sigma_V)/\sigma\right)$$
$$-\ X e^{-rT} N_2\left(a_2, b_2; \rho\right)$$

where
$$a_1 = \frac{\ln(H/X) + rT}{\sigma_H \sqrt{T}} + \frac{1}{2}\sigma_H\sqrt{T}$$
$$a_2 = a_1 - \sigma_H\sqrt{T}$$
$$b_1 = \frac{\ln(V/X) + rT}{\sigma_V \sqrt{T}} + \frac{1}{2}\sigma_V\sqrt{T}$$
$$b_2 = b_1 - \sigma_V\sqrt{T}$$
$$c = \frac{\ln(V/H) + rT}{\sigma \sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}$$
$$d = \frac{\ln(H/V) + rT}{\sigma \sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}$$

The value of the call on the maximum of two assets can be derived by arbitrage arguments. Stultz shows, that the value of a European call option, MX, on the maximum of two risky assets is

(4) $$MX(V, H, t) = c(V, t) + c(H, t) - M(V, H, t)$$

where c(A, t) is a European call on asset A with the same terms as the option on the minimum of two assets, M.

The extensions presented in this section not only prove to be useful to assess options for special situations, but also demonstrate roughly the methods how real option models in part II will be derived.

3. The Theory of Financial and Real Options

3.1 Introduction

The last chapter laid the basis for the evaluation of options and contingent claims. This chapter moves on to treat further options embedded in financial securities and introduces the most important real options.

In general, financial options are contained in most financial instruments that appear on the liabilities side of corporate balance sheets. A more recent string of reseach on CCA developed approaches to evaluating real options. Real options concern primarily the asset side of corporate balance sheets and assess flexibility which is inherent in many real investment projects. For many applications, real options have a significant impact on firm and project value and therefore affect capital budgeting decisions. In particular, when uncertainty about future events is high, real option pricing models supply a valuable tool for capital budgeting decisions.

Real options can roughly be structured into three groups: Operating options, investment opportunities and project financing. Operating options treat flexibility which is inherent in real investment projects already operating. Examples are temporary shut down, abandonment, and switching options. Investment opportunities that arise for a firm can either be undertaken or not. The flexibility to invest into projects not yet carried out is also a valuable real option. Finally, project financing presents a loose framework that allows modeling individual projects when both real and financial options are involved. Interrelations between the different types of options can be especially well considered with this method.

3.2 Financial options

In section 2.3, several financial securites such as equity, risky debt, and loan guarantees were introduced to demonstrate their option character. In this section the analysis will be extended to further corporate liabilities such as subordinated debt, warrants, convertible bonds, and callable convertible bonds.

For corporate securities, the possibility of default is an important source of risk. Since default is ruled by the value of firm's assets, the underlying variable for the above contingent claims is the asset value, or equivalently the firm value. Of course, there may be other sources of uncertainty that influence the value of corporate securities such as interest rates, exchange rates, and inflation risk. For simplicity, it will be assumed that the other sources of risk are negligable.

Junior Debt

At first consider subordinated, or junior debt (see Black and Cox (1976)). Junior debt is distinct from senior debt solely by the fact that it has subordinated lien on the firm's assets in case of default. Assume that the firm's assets have a current value of V(t), and that the firm is financed by equity, E(V, t), senior debt, S(V, t), and junior debt, J(V, t). Debt consists of zero bonds that pay a face value of X_S and X_J, respectively, at maturity t* . If firm's assets are insufficient to cover senior debt at t*, i.e. $V(t^*) < X_S$, the firm gets liquidated and senior debt receives the assets worth V(t*). Junior debt and equity receive nothing. When firm's assets can satisfy senior, but not junior debt, the firm also gets liquidated and senior debt receives S. Junior debt gets $V(t^*) - X_S$, and equity holders are left empty-handed. However, if the firm is successful and asset value exceeds the face value of outstanding debt, then senior and junior debt receive X_S and X_J, respectively, and equity equals $V(t^*) - X_S - X_J$.

Compared to the situation in section 2.3, where a firm was financed solely by equity and risky debt, the value of equity is still equivalent to a European call option on the firm's assets, but now with exercise price, $X_S + X_J$. The value of senior debt is not affected by junior debt , i.e. $S(V, t) = X_S\, e^{-r(t^* - t)} - P(V, t; X_S)$. The accounting identity V = E + S + J together with the put-call parity leads to the value of junior debt

$$(1) \qquad \begin{aligned} J(V, t; X_J) &= V(t) - S(V, t; X_S) - C(V, t; X_S + X_J) \\ &= C(V, t; X_S) - C(V, t; X_S + X_J) \end{aligned}$$

The value of junior debt is equivalent to the difference between two European call options on the firm's assets with exercise price X_S and $X_S + X_J$, respectively. The payoff at maturity for equity, junior debt, and senior debt is depicted in Figure 1a, 1b and 1c, respectively.

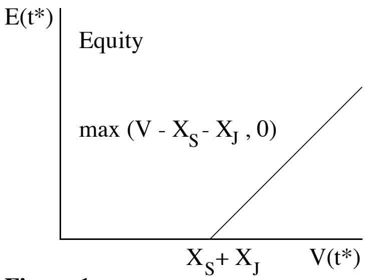


Figure 1a

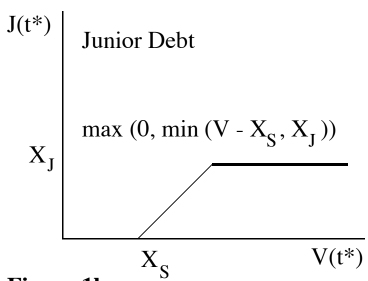


Figure 1b

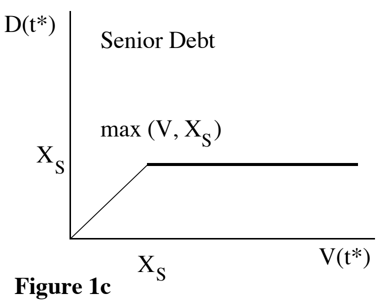


Figure 1c

Warrants

Further corporate securities will be considered, for example, warrants (see Smith (1976)). Warrants are identical to call options except that they are issued by the stock company itself, rather than by an independent market participant. When warrants are exercised, new shares of stock are issued and the striking price paid for them becomes part of the firm's value, V(t).

Imagine a firm with a capital structure containing N shares of common stock and n warrants each of which comprises the right to buy one share for an exercise price of X at maturity t*. If the warrants are exercised, the firm value will increase by nX. In this case, a single warrant is worth $[(V(t) + nX) / (n + N)] - X$ if exercised. Accordingly, the value of all warrants, W, is $n(V(t) - NX) / (n + N)$ which remains positive for $V(t) > NX$. The warrant value, W, can be regarded as a fraction of $n / (n + N)$ European calls on the firm value with exercise price NX. Consequently, equity has a value which is equivalent to the difference between firm value and option value granted to warrant holders. The payoff of equity and warrant at maturity is shown in Figure 2.

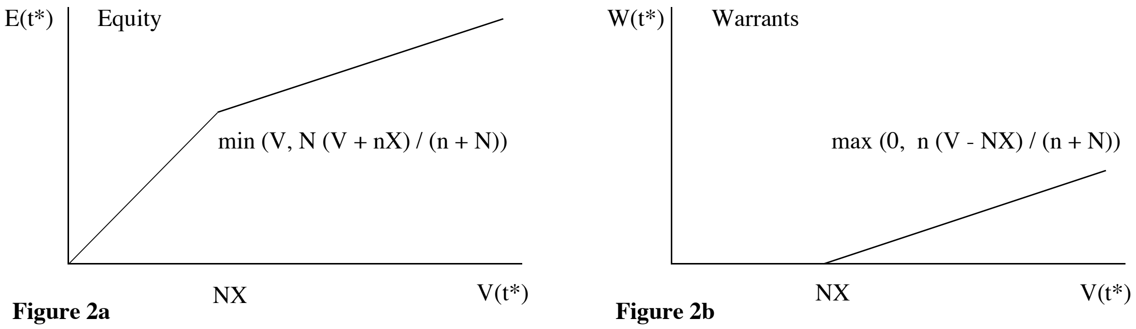


Figure 2a **Figure 2b**

Convertibles

Convertible bonds (convertibles for short) represent a kind of combination of warrants and risky debt (see Ingersoll (1977)). The convertible contract specifies that at his choice, the creditor can either receive the face value of the bond or one share of stock. As before, if assets, $V(t)$, are insufficient to cover debt repayments at maturity, the firm gets liquidated, creditors receive the firm, and shareholders are left empty-handed.

Assume now that the firm is financed by N shares of stock and n convertibles, each a zero bond with face value, X. Conversion takes place at maturity, t*, when profit exceeds the cost, i.e. when $V(t) / (n + N) > X$ or $V(t) > (n + N)X$. In this case, the total value of the convertibles, CD, is $nV / (n + N)$. It follows, that total value of convertibles equals the sum of a zero bond with face value nX and a European call to receive $nV / (n + N)$ at an exercise price of nX.

$$(2) \qquad CD(V, t; nX) = D(V, t; nX) + nC(V, t; (n + N)X) / (n + N)$$
$$= V - C(V, t; nX) + nC(V, t; (n + N)X) / (n + N)$$

Equity's value therefore equals a call on the firm's assets minus the conversion option granted to creditors. Formally, this is $E(V, t; nX) = C(V, t; nX) - nC(V, t; (n + N) X) / (n + N)$. Figure 3 shows the value of equity and convertible debt at maturity.

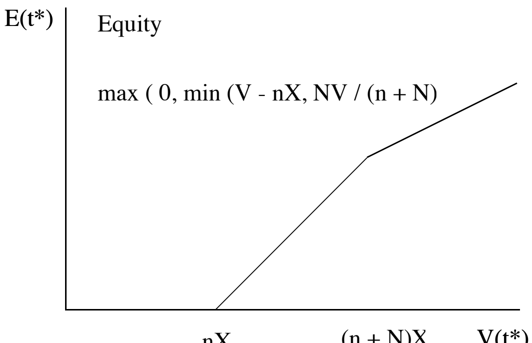


Figure 3a

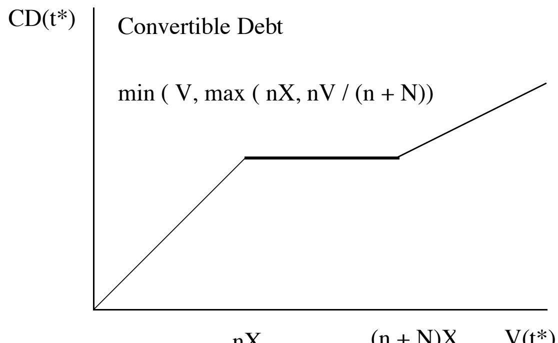


Figure 3b

Callable Convertibles

To evaluate callable convertible debt, the features of callable bonds and convertible bonds must be united. For callable bonds, equity amounts an American call on the firm's assets, because shareholders now have the right to redeem bonds at any time up to maturity. In additon, creditors have the right to convert bonds into shares at any time up to maturity - in particular, when shareholder decide to redeem. Brennan and Schwartz (1977) show that the firm's optimal call strategy is to redeem if the bonds' market price reaches the call price. The optimal conversion strategy, however, is not as simple and thus the solution to the valuation problem cannot be obtained by a representation incorporating simple options. Instead, the generalized PDE requires numerical methods to evaluate callable convertible debt.

Consider a firm that has a capital structure consisting of N shares of stock and n callable convertible bonds, each containing the right to convert the face value, X, into one share until maturity t*. Moreover, assume that shareholders receive discrete dividends and debt holders get discrete coupon payments. As demonstrated in section 2.3, components of corporate capital structure are contingent upon the value of corporate assets, V, and obey the generalized PDE. In particular, this is the case for the value of callable convertibles, W.

(3) $$\frac{1}{2}\sigma^2 V^2 W_{VV} + r V W_V + W_t - r W = 0$$

In this equation the payout function, $\delta = \delta_W = 0$, because free boundary conditions take dividends and coupon payment into consideration.

The total value of debt cannot exceed the value of corporate assets. As a result, one upper boundary condition for the value of the callable convertible is

(4a) $W(V, t) \leq V / n$

If corporate assets are worthless, the firm's debt is also worthless. Therefore, a lower boundary condition is given by

(4b) $W(0, t) = 0$

Further, the value of immediate conversion of a callable convertible limits the bond's minimum value. The second lower boundary condition is

(4c) $W(V, t) \geq V / (n + N)$

At maturity of debt, equityholder's right to redeem early turns worthless and the value of a callable convertible equals the value of a simple convertible. Therefore, the terminal boundary condition is the payoff of a simple convertible at maturity.

(4d) $W(V, t^*) = \min (V / n, \max (X, V / (n + N)))$

The optimal call policy is to exercise if the value of unexercised bonds reaches the current call price, $K^c(t)$. The following condition defines the optimal call schedule, $V^c(t)$.

(4e) $W(V^c(t), t) = K^c(t)$

If the firm chooses to call the bonds for redemption, creditors can either redeem at the current call price or convert in return for the current stock price. If they react rational, they will maximize their value. Thus, at the free boundary, $V^c(t)$, the callable convertible is worth

(4f) $W(V^c(t), t) = \max (K^c(t), V / (n + N))$

Figure 4 illustrates the boundary conditions for the callable convertible. The shaded area represents the region where neither call nor conversion take place.

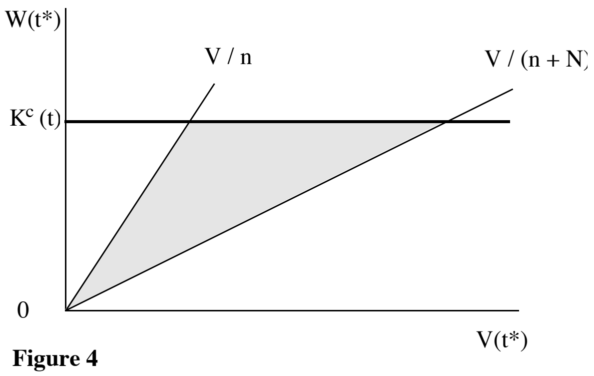


Figure 4

Further, the bond will be called at a coupon date t^- if the sum of the bond's value and coupon payment, c, at t^+ exceeds the call price at t^-

(4g) $W(V, t^-) = \min (K^c(t^-), W(V - c, t^+) + c / n)$

For a bondholder, conversion is optimal, once the value of the unconverted bond reaches the conversion value V / (n + N). However, similar to American call options on dividend-paying stocks, converting the bond never leads to optimal results except immediately before a dividend payment or at maturity. With a dividend payment, d, at t^+ the value of the callable convertible at t^- is

(4h) $W(V, t^-) = \max (V / (n + N), W(V - d, t^+))$

The solution to the PDE (3) subject to the simple boundary conditions (4a) - (4d) and the free boundary conditions (4e) - (4h) can be obtained with the numerical approximation techniques as demonstrated in chapter 4.

3.3 Real Options

Introduction

The previous section demonstrate how to evaluate several financial options which are embedded in financial components of the liabilities side of corporate balance sheets. Real options are involved in real investments, available to the firm and usually appear on the assets side of corporate balance sheets. Similar to financial options, they evaluate flexibility. This section demonstrates how to classify real options, and introduces each option type individually.

Real options fall roughly into three groups: Operating options, investment opportunities, and project financing.

Operating options, the first group, make up part of real investments that are already operating. This group includes shut down, abandonment, and switch options. Investment opportunities, the second group, are options to undertake investments that have yet to be carried out. Since most investment opportunities allow future growth, these are termed growth options. The third group is more general and concentrates on various combinations of options inherent to real investments. Project financing facilitates investigaton of the financial effects and interralations of financial and real options.

Operating Options

The temporary shut down option belongs to the group of operating options. It provides the firm with the choice either to operate or to halt operations temporarily. This option amounts to the opportunity to operate if economic conditions are favourable, e.g. if operating revenues exceed cost. When economic conditions worsen, the firm discountinues operations to avoid losses, operations resume, once conditions recover. Figure 1 illustrates this approach within a decision tree built upon a simplified binomial economy. Rising and falling branches represent good and bad outcomes, respectively. At each decision node, the firm can decide how to react in response to changed economic conditions.

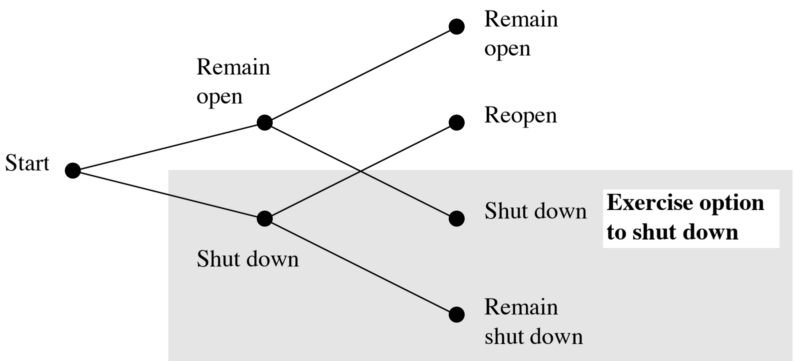


Figure 1

If the bad outcome turns up at the end of the first period, the firm may decide to close the facility to avoid further losses. If favourable economic conditions follow the first bad outcome, the firm may decide to resume operations. If unfavourable economic conditions follow the first bad outcome, operations will remain closed.

The opportunity to shut down temporarily can help to avoid losses, and thus can have significant value. This is particularily true when high uncertainty about future economic conditions is involved.

The abandonment option is another operating option. Similar to the shut down option, the abandonment option limits potential losses from operations. However, unlike temporary shut down, once the abandonment option has been exercised, it is not possible to resume operations. In other words, abandonment is irreversible. In exercising the option, management abandons the project by selling it in return for its salvage value. Abandonment value equals the extra value inherent in the option to bail out of a project. The following decision tree illustrates the abandonment option.

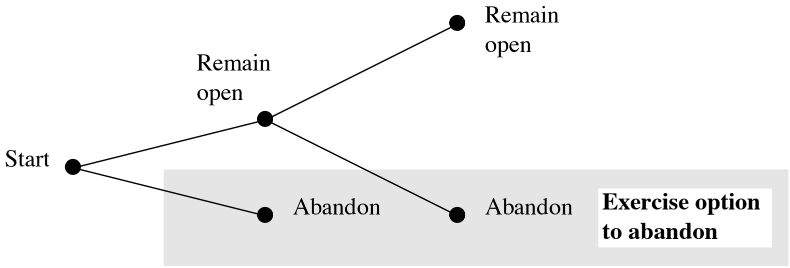


Figure 2

If a bad outcome turns up at the end of the first or second period, the firm may decide to abandon the project in return for the salvage value. The abandonment option can also be regarded as insurance against severe drops in project value. If project value becomes too low, the project can be sold for its salvage value to avoid further losses. With the possibility to abandon, project value can never fall below salvage value, and therefore the project is never worthless unless salvage value drops to zero.

Temporary shut down and abandonment options can be combined. If economic conditions turn bad, the firm may first decide to shut down temporarily, reserving the option to reopen. If conditions become worse, management may finally choose to abandon forever. The following picture depicts a stylized decision tree for a facility that incorporates both temporary shut down and abandonment.

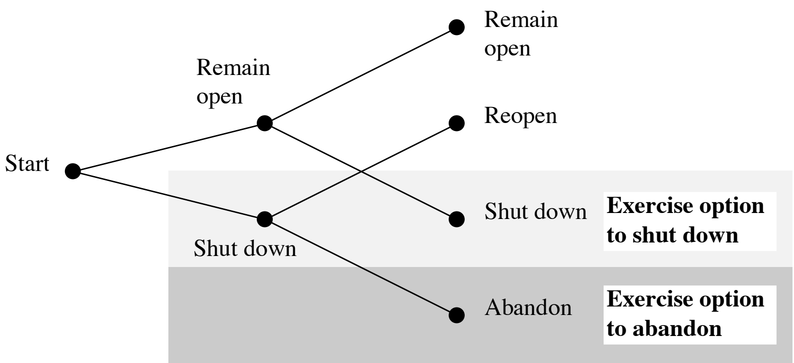


Figure 3

Figure 3's interpretation is analogous to that of the previous two figures.

The option to switch various inputs or outputs is the last operating option. A real investment, that allows one to fall back upon alternative inputs or outputs, is worth more than a similar investment, that is fixed to a single input and a single output. Whenever economic conditions favour some particular input or output factors, the former project benefits by switching to the most profitable combination of input and output. The latter project, however, remains fixed to its crucial input and output and cannot react to conditions favouring other factors. As an example of alternative input factors, consider power plants offering the flexibility to fire oil, coal, or gas. The installation of multi use firing technologies could be more costly, but the value of the flexibility to chose the cheapest input may be outweighted by this. As an example of alternative output factors suppose a flexible manufacturing system (FMS) that can easily be switched from one product to another as output prices and demand changes. To demonstrate the switching option, consider the decision trees for two alternative production factors A and B.

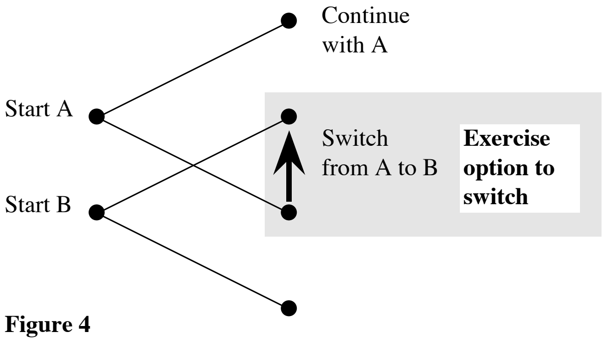


Figure 4

Assume that current economic conditions favour production factor A which is therefore used at the moment. If after one period a bad outcome turns up for A, and a good outcome for B, the firm may decide to switch from A to B to profit from the new situation.

Investment Opportunities

Investment opportunities, the second group of real options, consider investment projects that have yet to be carried out. An important feature of investment opportunities is that some of them allow the firm to expand as future economic conditions turn out to be profitable. Socalled growth options capture the growth aspect of investment opportunities.

A single stage growth option is the most basic case. After a certain period of time, the firm must make a final decision whether to invest or not. As an example consider R&D expenditures for new products. On their own, these expenditures are unprofitable but they may allow the firm to sell a new product in an attractive market. In this case, R&D is an option to produce the new product at a future date. The following decision tree illustrates the single stage growth option.

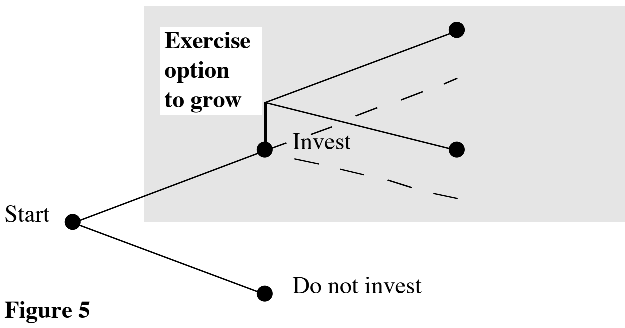


Figure 5

If a good outcome turns up at the end of the first period, the firm may exercise its growth option to realize the real investment project. If the outcome is bad, the firm may not establish the project, because it would not be profitable. The growth option would then expire unexercised.

Another class of growth options contains options on further options. These multi stage growth options contain a chain of single stage growth options successively contingent upon one another. The last growth option is contingent upon the real investment. These investment opportunities are termed multi stage growth options.

As an example, take a firm that is considering development of a new product A, with which it plans on entering an attractive market. This is only a single stage growth option. Suppose that a successful launch of A offers the possibility to launch a second product B. Without development and introduction of the first product A, the second product B cannot be launched. The whole investment opportunity is then a two stage growth option. With further products each depending upon sucessful launch of its predecessor, the investment opportunity becomes a multi stage growth option. The following figure depicts the two stage growth option.

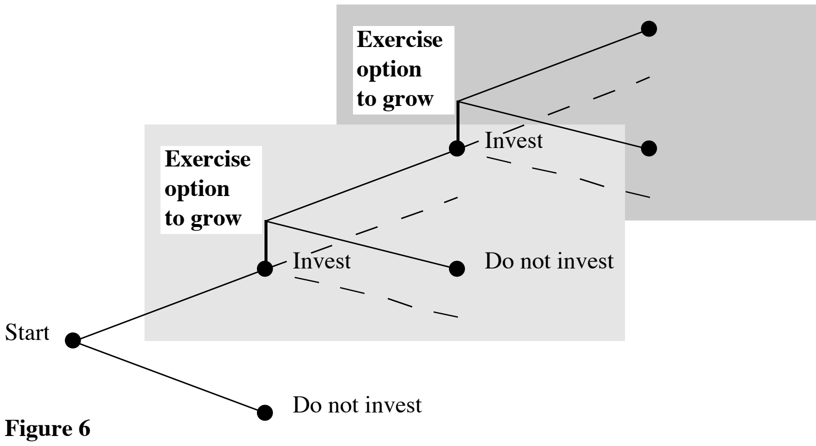


Figure 6

If a good outcome turns up after one period, the firm may decide to exercise the first growth option, that is to invest. If another good outcome turns up, the firm may exercise the second growth option that was provided by exercising the first one. In general, this approach to investment evaluation applies to all sequential investment projects.

Other applications in the category of investment opportunities consider the extend to which flexibility indirectly affects investment value. Examples are construction time flexibility for sequential investments and flexibility to meet growing demand with alternative production technologies.

Project Financing

Project financing, the third group of real options, differs form operating options and investment opportunities for two reasons. First, it does not only consider real options but also financial options. Second, it does not only regard single, or arrays of homogeneous options. It is able to handle all kinds of completely different options and their interrelations. Project financing is a framework concentrating on single, separable investment projects and models the various options that are contained in financing, investment, and operations. Since project financing can be quite complex it is not possible to illustrate it with a single decision tree. It consists of several trees that are generally interrelated in a complex manner. Unlike the other approaches, it is therefore necessary to develop and individually tailored model reflecting the specific structure of the investment project.

4. Numerical Methods to Calculate Option Value

4.1 Introduction

The last chapters presented the foundations of Option Pricing Theory (OPT) and Contingent Claims Analysis (CCA). The generalized PDE for contingent claims was derived, and it turned out that the claims differed only in payout functions and boundary conditions. Depending on the particular payout function and boundary condition, some PDEs did or did not have closed-form, analytical solutions. For a European call option on a non-dividend paying stock, Black and Scholes (1973) derived an analytical solution. In many other realistic situations, however, analytical solutions do not currently exist and analysis must resort to other methods. These methods are numerical procedures that either approximate the value of the corresponding PDE or the underlying security directly. The procedures that approximate the value of the underlying security involve binomial or lattice approaches, and Monte-Carlo simulation. Among the methods that approximate the actual PDE, two forms of the finite difference method receive introduction. For four particular applications in real option pricing, appendices A - D describe the procedures in more detail.

4.2 Models that Approximate the Value of the Underlying Security

Binomial Approximation

The binomial or lattice method approximates the stochastic process of the underlying security directly and uses risk-neutral valuation arguments to evaluate derivative securities. Depending on how the limits are set, the binomial distribution converges to either a Normal or a Poisson distribution. Thus, binomial approximation can be used for pure diffusion, pure jump, or jump-diffusion models (see Geske and Shastri (1985) p. 49). To illustrate the basic idea of this approach consider as underlying security a non-dividend paying stock, whose value follows a binomial process, i.e. the stock price, S, either climbs to uS or falls to dS after one period. For convenience define d = 1 / u, so that the evolution of S is symmetrical. The binomial process for the value of the underlying security then takes the form of a cone with a defined terminal distribution of stock prices.

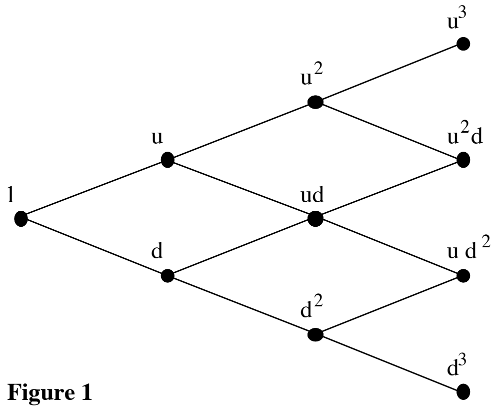


Figure 1

The probabilities of the up and down movement and the appropriate discount rate follow from a hedge strategy developed by Cox, Ross, and Rubinstein (1979). With this method it is possible to determine the distribution of stock prices at the maturity date of the derivative security. With the terminal boundary condition of the derivative security it is the possible to determine the current price of the claim via risk-neutral evaluation arguments.

Suppose now, that in period 2, the underlying security pays a constant dividend rate, □. Then the binomial process of the stock, which is net of dividends, evolves according to

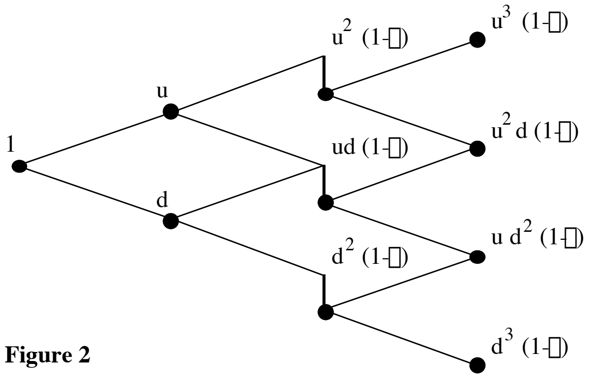


Figure 2

Again, it is relatively easy to determine the current price of derivative securities from the terminal stock price distribution.

Since dividends that are dependent upon the stock price level are not always realistic, assume now that during period 2 the underlying security pays a fixed cash dividend, D.

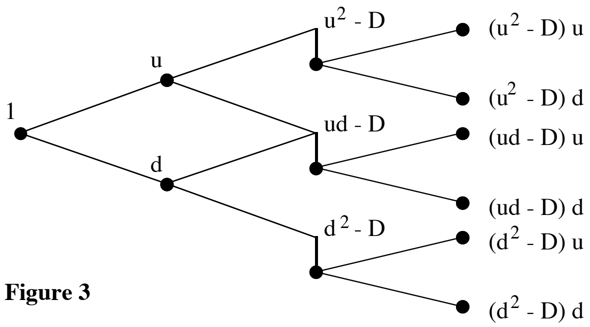


Figure 3

A fixed cash dividend leads to an increased number of possible final outcomes. Thus, the constant dividend yield approach is computationally more efficient than the fixed cash dividend model because fewer calculations have to be carried out to determine the final stock price distribution.

The solution procedure for all cases is a dynamic backward programming algorithm, that begins with the payoff of the claim at maturity and through all binomial states recursively determines the current value of the derivative security.

The payoff of an American put option, for example, is P(S, t*) = max (0, X - S) at maturity and

(1) $P'(S, t^* - 1) = \max(0, (p\, P(uS, t^*) + (1 - p)\, P(dS, t^*)) / (1 +$

immediately before maturity. American options can, however, be exercised before maturity and the option value at (t* -1) is the maximum of (X - S), the value of immediate exercise, and equation (1), the value of the option if unexercised

(2) $P(S, t^* - 1) = \max(X - S, P'(S, t^* - 1))$

Current stock prices, S, exercise prices, X, and discount rates, r, are known. The probability determining up and down movements, p, and their magnitudes, u and d, follow from the hedge strategy. As a result, the recursive procedure can sucessively evaluate the American put for periods t*, t* - 1, t* - 2, ... until the present value of the put in period t is reached.

The procedure can be used to evaluate any contingent claim for which a final payoff function is specified and whose value follows a pure jump, a jump-diffusion, or a pure diffusion process. In particular, it is possible to include several dividend payments, rights offers, etc.

Monte-Carlo Simulation

Valuation according to the Monte-Carlo simulation method (see Boyle (1977)) relies on the fact, that when a riskless hedge can be formed, any contingent claim can be evaluated simply by discounting the expected final payoff at the risk-free rate. The expected final payoff can be described by a probability distribution of expected stock prices. Since stock prices evolve according to the underlying stochastic process, a sample of stock price paths allows approximation of the probability distribution of stock prices at maturity of the contingent claim. The Monte-Carlo simulation depends critically upon the number of simulation paths, N. Generally, the accuracy increases with 1 / N. The computation cost doubles whereas the error diminishes by 70% (see Geske and Shastri (1985) p. 51).

The method is simple and flexible in the sense that it can easily handle a wide variety of different stochastic processes. In particular, it can handle complex payout and exercise contingencies. However, for American options a probability distribution must be approximated for each time step as opposed to just one at maturity in the case of European options.

For the special case that the stock price movement follows a geometric Brownian motion, it can be demonstrated that the stock price at maturity is distributed lognormally (see for example Jarrow and Rudd (1983) p. 89-91). Thus, a sample for the future stock price, S, at time, t, can be described by

$$S(t) = S(0)\, e^{(r - (1/2)\sigma^2)\, t + \sigma\, dz} \qquad (1)$$

where dz is a standard Wiener process. With a sample of future stock prices, S(t), the expected distribution can be approximated. With this distribution, the payoff can be determined and then discounted to yield the present value of the contingent claim.

4.3 Methods that Approximate the Value of the PDE

Finite Difference Methods

Unlike methods that estimate the value of the underlying security directly, finite difference methods solve the PDE. Among these approaches, finite difference methods are the most common. They use equations based on finite differences that approximate the continuous partial derivatives in the PDE. These equations allow to estimate the claim's sensitivity to discrete changes in the state variables. The finite difference approximation is therefore one of the most general valuation methods in option pricing. Nearly all contingent claims can at least be specified by their PDE and boundary conditions. By evaluating these PDEs subject to their boundary conditions, one can arrive at numerical approximations for most options.

In general, the approximations are performed in a space-time hyperspace whose dimension depends upon the number of stochastic variables in the problem. The simplest case with just one stochastic variable, i.e. price of the underlying security, is termed one-dimensional. In this case, the stock price-time space is reduced into a set of points in the (S, t) plane given by stock price, $S = i\,\Delta S$, and calendar time, $t = j\,\Delta t$, where $i = 0, 1, 2 \ldots m$, and $j = 0, 1, 2, \ldots n$. The division into discrete points results into a grid whose mesh size is determined by the increments ΔS and Δt.

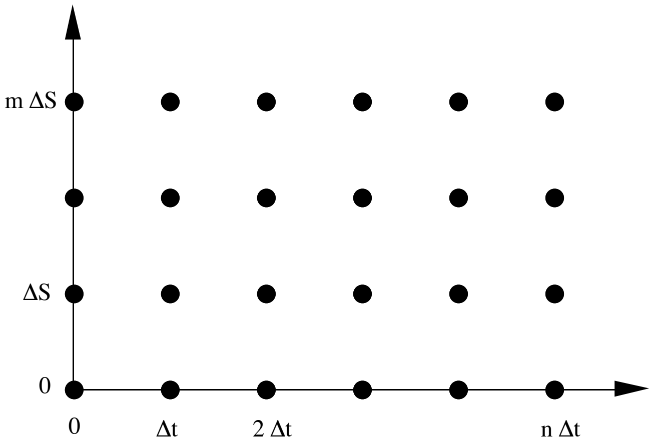


Figure 1

The size of the increments is chosen to be small enough to ensure accurate, stable, and efficient convergence to the solution.

Usually, one is interested in the current value of the contingent claim which is the edge of the price time space for $j = 0$. The terminal boundary condition at maturity of the claim supplies the other limit of the time dimension and determines the value at the edge of the stock price-time space for $j = n$. The price dimension has a lower boundary condition which describes the claim's value for i = 0. The upper limit of the price dimension describes the value of the contingent claim as the value of the underlying security approaches infinity.

The finite difference method replaces partial differences by finite difference equations which fill the stock price-time space successively, to obtain the current price of the contingent claim. There are, however, several ways to estimate the changes in the value of the derivative security with respect to stock price and time. For the difference equations, forward and backward differences most often apply (see Brennan and Schwartz (1978)).

Forward, or explicit, differences determine the unknown price in terms of already known prices. For backward, or implicit, differences, a set of simultaneous equations must be solved because unknown prices determine the value of the known price. The direction of determination for both methods can be illustrated as follows.

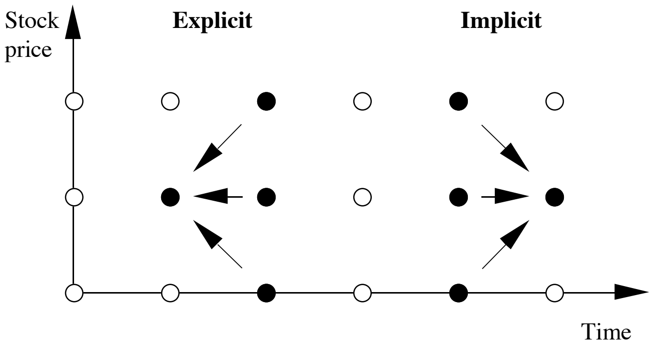


Figure 2

Formally, the generalized PDE of the contingent claim to be evaluated (see 2.3) can be rewritten using a log-transformation of the form X = ln S.

(1) $$\frac{1}{2}\sigma^2 W_{XX} + \left[r - \delta - \frac{1}{2}\sigma^2\right] W_X + W_t - r W + \delta_W = 0$$

With this transformation the PDE becomes independent of the level of stock prices.

The finite difference method discretizes the stock-price time space to an array of discrete points, so that the partial derivatives can be replaced by finite differences. With W(X, t) = W(i ΔX, j Δt) = W(i, j) the substitutions

(2a) $$W_X(i, j) = \frac{W(i+1, *) - W(i-1, *)}{2\Delta X} + O(\Delta X^2)$$

(2b) $$W_{XX}(i, j) = \frac{W(i+1, *) - 2\,W(i, *) + W(i-1, *)}{\Delta X^2} + O(\Delta X^2)$$

(2c) $$W_t(i, j) = \frac{W(i, j+1) - W(i, j)}{\Delta t} + O(\Delta t)$$

where O(.) represents the order of errors in the approximations.

For the explicit method, the time index * is set to j + 1. For the implicit method * = j. The explicit finite difference method uses the known prices at time j + 1 to calculate the price at j, whereas the implicit method solves a system of simultaneous equations to obtain the price at j.

The Explicit Finite Difference Method

For the explicit version, the discrete form of the PDE for a generalized contingent claim becomes a difference equation

$$\text{(3)} \qquad W(i\,,j) = \frac{1}{1 + r\,\Delta t}\,[\,a\,W(i-1,\,j+1) + b\,W(i,\,j+1) + c\,W(i+1,\,j+1) + \square_W\,]$$

where

$$a = \frac{\Delta t}{2\,\Delta X}\left[\frac{\sigma^2}{\Delta X} - \left(r - \frac{1}{2}\,\sigma^2\right)\right]$$

$$b = 1 - \frac{\Delta t\,\sigma^2}{\Delta X^2}$$

$$c = \frac{\Delta t}{2\,\Delta X}\left[\frac{\sigma^2}{\Delta X} + \left(r - \frac{1}{2}\,\sigma^2\right)\right]$$

Brennan and Schwartz (1978) demonstrate that the coefficients a, b, and c have the properties of probabilities in a trinomial process which formally is identical to a pure jump process as discussed by Coy and Ross (1976).

The price of the contingent claim at time j can be regarded as its expected value at time j + 1 discounted at the risk-free rate. With a terminal, upper, and lower boundary one can successively fill the stock price-time space starting with the terminal boundary and progressing until the border containing the current prices has been reached.

However, for contingent claims that allow early exercise, further free boundaries must be introduced. If the value of early exercise exceeds the value of maintaining the option, the claim is usually exercised and the free boundary has been reached. Accordingly, the free boudary represents the optimal exercise schedule and must be determined jointly with the solution to the pricing problem.

The Implicit Finite Difference Method

The difference equation for the implicit version of the finite difference method is similar to (3), yet defines the claim's price at time j + 1 in terms of j, only the direction in time is reversed.

$$(4)\qquad W(i\,,j+1) = \frac{1}{1+r\Delta t}\left[\,a\,W(i-1,j) + b\,W(i,j) + c\,W(i+1,j) + \square_W\right]$$

where

$$a = \frac{\Delta t}{2\,\Delta X}\left[-\frac{\sigma^2}{\Delta X} + \left(r - \frac{1}{2}\sigma^2\right)\right]$$

$$b = 1 + \frac{\Delta t\,\sigma^2}{\Delta X^2}$$

$$c = \frac{\Delta t}{2\,\Delta X}\left[-\frac{\sigma^2}{\Delta X} - \left(r - \frac{1}{2}\sigma^2\right)\right]$$

Again, the three boundary conditions are sufficient to solve for the current option price. In contrast to the explicit version W(i, j) can not be determined directly, rather follows implicitly from the following system of linear equations.

$$(5)\qquad \begin{bmatrix} b & c & 0 & \dots & 0 \\ a & b & c & & \dots \\ 0 & a & b & c & 0 \\ \dots & & a & b & c \\ 0 & \dots & 0 & a & b \end{bmatrix} \begin{bmatrix} W(1,j) \\ W(2,j) \\ \dots \\ \\ W(n,j) \end{bmatrix} = \begin{bmatrix} W(1,j) - a\,W(0,j) \\ W(2,j) \\ W(3,j) \\ \dots \\ W(n,j) - c\,W(n+1,j) \end{bmatrix}$$

Usually, Gaussian elimination allows one to solve for the matrix inversion (see Brennan and Schwartz (1978)) and yields an (n−1)-vector that can be used to successively determine the columns in the stock-price time space, starting with the terminal boundary.

Unlike the explicit version, it is not easy to consider early exercise. In this case there is no way to directly test whether the free boundary has been reached, and only complex search algorithms can help to find the boundaries.

Efficiency, Accuracy, and Stability

Among Monte-Carlo, binomial, and finite difference approaches that solve numerically for the value of contingent claims several other techniques apply. Examples are analytical approximations (Roll (1977), Geske (1979), and Whaley (1981); Geske and Johnson (1984)) and numerical integration (Parkinson (1977)). The most common methods, however, are binomial and finite difference techniques. So these approaches will briefly be discussed with respect to accuracy, stability, and efficiency (see Geske and Shastri (1985)).

The measure of accuracy is determined by the approximation error. If u(i, j) is an exact solution and u'(i, j) is an approximation of the solution, then the approximation error is defined as e(i, j) = u(i, j) - u'(i ,j) at the point $X = i\ \Delta X$, $t = j\ \Delta t$. If the error becomes smaller with each time step then the method converges. If the method converges with finite error the solution is not accurate. If the method does not converge, i.e. the error becomes larger, then the solution is not stable. Efficiency can be measured in terms of computation time per option calculated.

The approximation error has an upper bound which is limited by the error order function $O(\Delta X^2 + \Delta t)$. For well defined problems the order function yields approximation errors that depend on the problems input data. Problem that incorporate free boundaries are not well defined in the boundaries and hence no error proxies are obtainable.

If larger steps ΔX, and Δt are employed, less computations are necessary and efficiency increases but approximation error may also increase. Increasing errors, however, cause instability. A technique is stable when the solution converges, or when unbounded results become impossible. In order to prevent unbounded solutions specific conditions can be imposed. For the explicit finite difference method it is therefore necessary (see Brennan and Schwartz (1978)) that a, b, c ≥ 0, which leads to the conditions $\Delta X \leq \square^2 / \mathrm{abs}(r - \square^2 / 2 - \square_i)$ and $\Delta t \leq \square^2 / (r - \square^2 / 2 - \square_i)^2$. For the implicit finite difference method the condition is $\Delta X \leq \square^2 / \mathrm{abs}(r - \square^2 / 2 - \square_i)$.

Geske and Shastri (1985) find out that the log-transformed explicit finite difference method dominates the other methods in terms of efficiency when option prices for a range of stock prices have to be calculated. The binomial method is the most efficient technique when the option price for a single stock price have to be calculated. The implicit finite difference method is dominated by the explicit version in terms of efficiency but is the most accurate technique.

PART II: MODELS TO EVALUATE REAL OPTIONS

Chapter 5. Models to Evaluate Real Options

Structure

The discussion of models to evaluate real options constitutes the intellectual bulk of this paper. Development of these models required rethinking such basic, broad concepts as the origin of value and the nature and influence of uncertainty. Half the sections of this chapter deal directly with real options; the other half provides background information crucial for the models in context.

Section 5.1 briefly introduces real options typically encountered in real investment projects. Empirical relevance of real option pricing techniques and general difficulties associated with them are presented in section 5.2 . Sections 5.3 - 5.5 concentrate on recent models to evaluate real options, such as operating options, investment opportunities, and project financing. Finally, section 5.6 summarizes the models and supplies a brief, critical commentary.

5.1 Introduction

Real options fall roughly into three groups: Operating options, investment opportunities, and project financing

Operating options, the first group, are an integral part of already existing real investments.

The temporary shut down option and the abandonment option are the most important ones. Both are an significant source of value. They represent the firm's opportunity to interrupt operations, which turn out to be unprofitable. McDonald and Siegel (1985) discuss a basic shut down option, and treat the right to operate during a particular period as a European option to exchange input for ouput. In a more recent approach, Dixit (1989) concentrates on market entry and exit decisions for firms in respect to their industries by using a framework of shut down and reopening options. Myers and Majd (1983) present a model for abandonment options, which regards abandonment as insurance against severe drops in project value. If project value sinks too low, the project can be sold for its salvage value. McDonald and Siegel (1986) also briefly present a powerful approach to abandonment value, which follows from their approach to growth options, a topic already covered by investment opportunities. Real investments already in operation, seldom offer just a single operating option. Most often, projects contain several operating options that can be contingent upon each other. Brennan and Schwartz (1985) arrive at an assessment of a project, by incorporating a set of abandonment, shut down and reopening options inherent in the project.

Other common operating options are options to maintain the scale and life of a project, and options to switch various inputs or outputs. For these options, no formula has yet been found although the models of Stultz (1982) and Margrabe (1978) might lead to a solution.

Investment opportunities, the second group of real options, evaluate real investment projects that have yet to be carried out.

Growth options constitute the most important members of this group. They offer an exclusive right to establish future real investments, that are contingent upon project value and cost. Kemna (1987) presents the simplest case, which is a European call on the project value with predetermined, finite life. McDonald and Siegel (1986) present a very powerful model for American growth options, with infinite or even stochastic life. The basic case, the single stage growth option, provides the firm with the right to undertake a single investment. Some growth options, however, have a more complex structure. Two stage options incorporate two single stage options. At maturity of the first option, the firm may decide whether or not to *buy* the second growth option, and at maturity of the second option, the firm may decide whether or not to undertake the underlying real investment. In fact, the two stage growth option is a compound option. The scheme can be extended to multi stage options, where each stage is contingent upon the decision at maturity of its predecessor. The last option is always a claim upon the real investment. In Kemna (1987) a simple two stage growth option turns up. Pindyck (1988b) combines multi stage growth and operating options to derive a microeconomic model for the value of entire firms. His approach follows from Myers's (1977) supposition that growth options constitute a large portion of firm's value.

Unlike growth options, several other applications for evaluation do not consider the entire investment opportunity, rather investigate the value of indirect effects. Majd and Pindyck (1987) show how to assess construction time flexibility for sequential investments. Pindyck (1988a) evaluates alternative production technologies with different cost functions and different degrees of flexibility in meeting constantly increasing demand.

The term project financing summarizes the third group of real options. Project financing combines real and financial options, by uniting real options usually found at the asset side of corporate balance sheets with financial options from the liabilities side. The approach considers

Table 1	Overview of Real Options					
Group	**Subgroup**	**Model**	**Option Model**	**Life**	**Exercise**	**Solution**
Operating Option	Shut down	McDonald/Siegel (1985)	Simple European Exchange	Finite	-	Closed
	Shut down	Dixit (1989)	Compound American Call	Infinite	Rule	Semi-closed
	Abandon	Myers/Majd (1983)	Simple American Exchange	Finite	Schedule	Numerical
	Abandon	McDonald/Siegel (1986)	Simple American Exchange	Uncertain	Rule	Closed
	Shut down & abandon	Brennan/Schwartz (1985)	Compound American Call	Finite	Schedule	Numerical
	Shut down & abandon	Brennan/Schwartz (1985)	Compound American Call	Infinite	Rule	Semi-closed
Investment Opportunity	Single stage growth	Kemna (1987)	Simple European Call	Finite	-	Closed
	Single stage growth	McDonald/Siegel (1986)	Simple American Exchange	Infinite	Rule	Closed
	Single stage growth	McDonald/Siegel (1986)	Simple American Exchange	Uncertain	Rule	Closed
	Single stage growth	Brennan/Schwartz (1985)	Simple American Exchange	Finite	Schedule	Numerical
	Single stage growth	Pindyck (1988b)	Simple American Call	Infinite	Rule	Semi-closed
	Two stage growth	Kemna (1987)	Compound European Call	Finite	Rule	Semi-closed
	Multi stage growth	Pindyck (1988b)	Compound American Call	Infinite	Rule	Semi-closed
	Other application	Majd/Pindyck (1987)	Compound American Call	Finite	Schedule	Numerical
	Other application	Pindyck (1988a)	Simple American Call	Infinite	Rule	Closed
Project Financing		Mason/Merton (1985)	American Call/Put	Finite	Schedule	Numerical

interrelations between shut down, abandonment, growth, and the risk of bankrupcy on components in corporate capital structure. Project financing, as demonstrated by Mason and Merton (1985), provides a consistent framework to evaluate complex financial and operating agreements especially in large scale projects.

Table 1 contains an overview of the 16 models treated in this chapter. *Option Model* summarizes the way authors interpret the respective option. *Life* indicates the maturity or life of the option. Since many options can be exercised prematurely, *Exercise* shows if the model provides a simple exercise rule, or a schedule for optimal exercise decisions. The last column indicates whether there is an analytical, closed-form solution, a semi-closed form that needs iterative solution procedures, or if the model can only be solved with numerical approximation techniques.

A short summary first introduces each of the models in this chapter, each is then formulated, and an example or a short case study wraps things up. Some approaches include extensions of the model.

5.2 Empirical Relevance and Problems

Introduction and Summary

Traditional project evaluation methods are usually static since they consider future events as if they were certain. Future events are usually *uncertain*, and traditional methods are thus unable to assess flexibility to react as uncertainty resolves. The flexibility to react upon changes in the operating environment can represent considerable value. Projects containing flexibility must be worth more than static methods would imply. The flexibility inherent in real investment projects is summarized under the term real options, which Contingent Claims Analysis (CCA) can assess. The next subsection therefore briefly summarizes recent real investment valuations where traditional methods have been compared to CCA. These valuations show, that the empirical relevance of real options is significant. If uncertainty about future events is high, static methods generally underestimate the *true* value of real investments. In turn, if the future is certain, both static and CCA methods yield identical results. In reality, the future is usually uncertain and thus CCA models generally prove superior to static models.

Despite the fact that CCA captures extra value, which goes unrecognized by traditional methods, CCA approaches also have their shortcoming. Unlike similar approaches to financial instruments, CCA approaches for real investments are more complex and difficult to apply. The second subsection shows, that difficulties arise for two main reasons. First, real investments often are not traded assets. For this reason the common strategy of deriving the option value from a riskless trading strategy, cannot be realized. To overcome this restriction, asset pricing procedures can be employed, but they share the disadvantage that risk-adjusted discount rates must be estimated. The second reason for difficulties follows from the finding, that real investment value can grow at a rate which differs from the rate of an otherwise identical financial investment. For this reason, it is necessary to estimate the expected rate of growth for real investments. As a consequence, real option value varies not only due to different attitudes towards risk, but also due to different expectations of future growth rates.

Nevertheless, CCA represents a major breakthrough for the evaluation of real investments and proves very useful when estimation procedures are carried out carefully.

Empirical Relevance

Unlike CCA, common Net Present Value (NPV) and Discounted Cash Flow (DCF) methods ignore the flexibility incorporated in many real investments . Therefore this section presents several empirical investigations, which demonstrate how CCA outperforms NPV and DCF approaches when flexibility of response to uncertain future events is involved.

Usually, static NPV or DCF techniques are used to determine the value of investment projects. When future events can be regarded as certain, both NPV and DCF approaches deliver identical results to those calculated by CCA methods. If the future is uncertain and projects contain flexibity to react to changes, traditional techniques fail to yield reliable results. As example consider stock options, which comprise the flexibilty to react as uncertainty resolves. In this case, NPV and DCF's method of first forecasting future cash flows and then discounting them at a properly rsik-adjusted rate is impossible. The first step is feasible, but messy. Performing the second step proves impossible, because with varying values of the underlying security the option's risk and the discount rate along with it undergoes continuous changes (see Brealey and Myers (1988) p. 485). It is noteworthy, that static methods are generally unable to capture the value of flexibility and therefore underestimate the *true* value of many real investments.

If traditional techniques are used nevertheless, the results will be incorrect: "... we have repeatedly noted the failure of traditional capital budgeting techniques to properly take the value of these [operating] options into account. Although ignoring a single operating option may not introduce an important error in a project's evaluation, the cumulative error ignoring all the operating options embedded in that project can cause a significant underestimate of its value." (Mason and Merton (1985) p. 36 - 37).

Empirical work confirms Mason and Merton's conjecture, and moreover reveals that even ignoring single real options can cause severe errors in project evaluation.

For 15 selected companies in five U.S. industries, Kester (1984) estimated the value of growth options as a component of total market value. He found that ratio to be 50% or more in the majority of the cases. Furthermore, it turns out to be less than 50% in industries with low demand uncertainty (i.e. tires and rubber, food processing) but between 60% and 80% in industries with high demand uncertainty (i.e. electronics, computers).

In their investigation, Paddock, Siegel, amd Smith (1988) evaluated offshore petroleum leases and compared them with industry bids for these properties. They used a CCA approach, since petroleum leases can be regarded as compound options. Whereas industry bids were usually calculated with DCF techniques, CCA proved more accurate and eliminated overvaluation as well as undervaluation of claims. Furthermore, CCA was able to provide an investment timing schedule.

Pindyck (1988a) estimated the value of flexibility to meet increasing demand for electric power plants. Comparing a large, inflexible, coal-fired powerplant to a system of flexible, oil-fired turbine generators he found out that the flexible turbine generators are still economically favourable if they exceed the cost of the inflexible coal-fired plant by 60%. This means that flexibility to meet growing demand, or real option value increased total investment value by up to 60%.

A recent McKinsey (1989) analysis showed that in mineral mining, operating flexibility can account for a considerable part of a project's value. Over 50% of a coal mine lease's value hinged upon the right to either defer development of the mine to a later date or to halt development completely. The option to defer is equivalent to a growth option and traditional techniques completely ignored this.

Another McKinsey (1990) analysis revealed that in the aircraft industry contractual operating lease cancellation clauses - a type of abandonment option granted to customers - can account for as much as 80% of the aircraft price. Without knowing it, the aircraft company was foresaking this amount as a seemingly *free* service to its customers.

In a valuation of heavy oil reserves, Copeland, Murrin, and Koller (1990) reported that the estimated value of the property increased by more than 20% over former NPV calculations when growth options were taken into consideration. Unlike CCA values, the NPV figures diverged greatly due to differing expectations of future oil prices. In addition, CCA was able to supply guidelines for investment timing decisions.

These examples show , that the value represented by flexibility, real options that is can constitute a significant part of total real investment value. It therefore seems promising, to augment state-of-the-art capital budgeting methods with models that assess the value of unrecognized flexibility.

Problems

Even though CCA models for real investments prove superior to static methods, their application is not as straightforward as that of similar CCA approaches for financial investments.

As mentioned in previous chapters, any fundamental PDE for contingent claims is derived from arbitrage arguments that involve strategies using *traded* financial assets. Obviously, assets and options incorporated in real investments seldom are explicitly traded. So it seems rather questionable, that riskless hedge strategies still hold for portfolios of real investments consisting of non-traded assets.

However, "all capital budgeting procedures have as a common objective the estimation of the price that an asset or project would have if it *were* traded. Thus, for example, a standard discounted cash flow analysis uses as a discount rate the equilibrium expected return required on a traded security in the same risk class as the nontraded project. Because the absence of arbitrage is a necessary condition for equilibrium prices, the no-arbitrage price of an option on a traded security must be the equilibrium price of an option on a corresponding nontraded project." (Mason and Merton (1985) p. 38).

Therefore, it is adequate to use a single asset or a portfolio of traded financial assets, instead of the nontraded asset, as long as both are perfectly correlated. Since finding traded assets that are perfectly correlated with the specific nontraded asset can be rather tedious, it is advisable to use asset pricing models like the Capital Asset Pricing Model (CAPM), to estimate the expected market equilibrium rate of return for the unknown financial asset. For the portfolio strategy it is not necessary, that an explicit traded finacial asset be found. Instead an estimate of the market equilibrium rate of return sufficies.

Compared to the pricing of financial options, the estimation of the market equilibrium rate of return is a major drawback. For deriving the price of financial options, the risk-free rate was fully adequate.

Even with the help of asset pricing models, the whole strategy breaks down, if the real investment opportunity under consideration changes the opportunity set of available investments. In this case existing assets do not span the investment and it becomes impossible to establish a perfect correlation with financial assets. As an alternative, Bertola (1987) shows how to use dynamic programming approaches that work without the spanning assumption. "Moreover, the resulting error from using the option model in such rare cases would be no different from the one arising from the standard procedure." (Mason and Merton (1985) p. 39).

Usually, financial assets grow at a market determined rate of return. Real assets, however, often grow at a rate that is different from the rate of return that accrues to financial assets that are perfectly correlated with real investments (see McDonald and Siegel (1984)).

This may seem surprising, but as an example consider a real investment into pencils (see McDonald and Siegel (1985) p. 338). Suppose pencils can be produced instantaneously as needed for consumption at constant marginal cost, that is the price of pencils is nonstochastic and constant. The rate of return on a real investment into pencils is thus zero. In contrast, the rate of return on a financial investment with the same nonstochastic character is the risk-free rate. The same principle applies, if future prices of pencils are uncertain. The reason for below-equilibrium rates of return follows from the fact, that due to competitive markets the price of asstes destinated for consumption grows at a lower rate than the price of otherwise identical financial assets. This is the reason why storage of goods, that is an investment into assets destinated for consumption, involves opportunity cost, which is the difference to alternative financial investments. It follows, that in a portfolio strategy where financial assets are used to mimic the behavior of nonfinancial assets, the natural differences between both returns have to be taken into account.

In order to determine these differences, the expected rate of growth of the nonfinancial asset's value must be estimated. This, however, is a difficult task and represents the second drawback of real option pricing compared to financial option pricing. For financial options, it is not necessary to estimate growth rates of the underlying asset, because these expectations are already reflected in current prices of the underlying security.

Fortunately, for some real assets, future contracts are available that incorporate market's expectations about the rate of growth in the value of these assets. Brennan and Schwartz (1985), McDonald and Siegel (1985) and Kemna (1987) use this approach to overcome subjective estimates of growth rates.

As demonstrated, the only threatening problems arise from the estimation of risk-adjusted returns and expected growth rates of the real assets. Nevertheless, real option pricing models are valid from the theoretical point of view and can be proved rigorously.

5.3 Operating Options

5.3.1 Operations with Temporary Shut Down

Introduction and Summary

A temporary shut down option comprises the right to produce during a certain period but also to stop operations during that same period. McDonald and Siegel (1985) develop an analytical model to evaluate basic operating options, in which firms can suspend operations, if operating revenues fall below variable costs. Formally, a production facility provides the firm a series of exchange options. During each specific time period, the firm can choose whether or not to exchange variable costs (input) for operating revenues (output). Since both input and output vary randomly over time, the exchange option resembles a call option with uncertain exercise price (see Margrabe (1978) and Fischer (1978)). McDonald and Siegel's model is indeed simple yet ignores a few important factors: First, shut down and resumption incur no costs. Second, neither fixed costs during operations nor maintenance costs during suspension are considered. Third, the facility's physical life is determined exogenously and hence is independent of frequency and intensity of usage. Nevertheless the model adequately captures the main effects of uncertainty on investment decisions.

The Model

Denoting periodical per unit output and input prices with P_t and C_t, the firm either provides a specific profit, P_t - C_t, during each period, t, or shuts down when P_t - C_t is negative to avoid losses. Thus, per unit profits in any period t are identical to the payoff from, W, a maturing European call option to exchange C_t in return for P_t.

(1) $$W(P_t, C_t, t) = \max(P_t - C_t, 0)$$

Subject to current forecasts of P_t and C_t the per unit present value, V(T), of a facility with a physical life, T, is the sum of all periodical operating options during its life.

(2) $$V(T) = \int_0^T W(P_t, C_t, t)\, dt$$

To derive an analytical solution for the value of the operating option, W, before maturity, assume that P and C vary randomly according to a geometric Brownian motion.

(3a) $$\frac{dP}{P} = \mu_P \, dt + \sigma_P \, dz_P$$

(3b) $$\frac{dC}{C} = \mu_C \, dt + \sigma_C \, dz_C$$

where μ_p is the expected rate of change in P, σ_p the instantaneous standard deviation of the rate of change in P, and dz_p a standard Wiener process for P. The variables for C are defined similarly. The instantaneous coefficient of correlation between dz_p and dz_c is ρ_{pc}.

The next step towards a solution to the value of the operating option when shut down is possible would be to form a riskless hedge portfolio. If P and C were the prices of financial assets, the procedure would be straightforward. In the above case, P and C are most often the prices of non-financial, or even non-traded assets. Since there is no way to build a riskless portfolio out of non-traded assets, one usually corrects for this by using financial assets that are perfectly correlated with P and C. It can be tedious finding such an asset. Since only the assets' return is of interest, an asset pricing model, e.g. the CAPM, is well suited and sufficient to determine the expected rate of return.

The expected CAPM-rate of return, α_i', from a hypothetical financial asset which is perfectly correlated with i = P, C is

(4) $$\alpha_i' = r + \phi \, \rho_{im} \, \sigma_i$$

where r is the risk-free rate of return, ϕ the market price of risk, and ρ_{im} the instantaneous correlation coefficient between the return from the financial asset, i, and the market portfolio, m.

Given i and assuming that sufficient information exists to calculate α_i', one could form an appropriate hedge portfolio immediately, if the difference between the returns on financial and nonfinancial assets did not merit consideration.

Consistent with accepted theory, investors are risk averse and demand an appropriately risk-adjusted return for investments into risky assets. In general, financial assets are able to provide investors with this premium. Nonfinancial assets, such as investments into consumer goods, fail to provide this premium. This distinction may seem astonishing but is easily to understand. The only profit from holding an investment into products that are for consumption stems from possible price increases which, in competitive industries, are much lower than returns from financial assets. It follows that if financial assets are used to mimic nonfinancial assets, the difference between their returns must also be accounted for.

One defines

(5) $$\delta_i = \alpha_i' - \mu_i$$

where δ_i represents the opportunity cost of holding nonfinancial asset i compared to a financial asset which is perfectly correlated with that asset.

With these extensions one can finally form a riskless hedge portfolio whose return must be the risk-free rate, r. McDonald and Siegel show that the value of the shut down option must obey the following partial differential equation (PDE).

$$(6) \qquad W_u = r\,W - (r - \delta_P)\,P\,W_P - (r - \delta_C)\,C\,W_C - \frac{1}{2}\left(\sigma_P^2 P^2 W_{PP} + \sigma_C^2 C^2 W_{CC} + 2\,\sigma_P\,\sigma_C\,\rho_{PC}\,P\,C\,W_{PC}\right)$$

subject to the terminal boundary condition (1). Calendar time is u and time to expiration is t..

In their article McDonald and Siegel do not provide the other boundary conditions. However, they show that the solution to the PDE is

$$(7) \qquad W(P_0, C_0, t) = P_0\, e^{-\delta_P t} N(d_1) - C_0\, e^{-\delta_C t} N(d_2)$$

where

$$(8) \qquad d_1 = \frac{\ln(P_0 / C_0) + (\delta_C - \delta_P)}{\sigma \sqrt{t}} + \frac{1}{2}\sigma\sqrt{t}$$

$$d_2 = d_1 - \sigma\sqrt{t}$$

$$\sigma^2 = \sigma_P^2 + \sigma_C^2 - 2\,\sigma_P\,\sigma_C\,\rho_{PC}$$

When $\delta_p = 0$, the formula is identical to Fischer's (1978) model for a European call with stochastic exercise price. When $\delta_p = \delta_c = 0$ the formula is equivalent to Margrabe's (1978) model for the option to exchange one asset for another.

If there are future contracts with delivery date, t, on either P, C, or both the estimation of δ_p and δ_c becomes unnecessary with the substitutions

$$(9) \qquad P_o^t = e^{(r - \delta_P)\,t}\, P_o \qquad \text{and} \qquad (10) \qquad C_o^t = e^{(r - \delta_C)\,t}\, C_o$$

where P_o^t and C_o^t are the current prices for futures contract for delivery of, respectively, one unit of output and input at time t.

Example

As case assume that both current input and output prices are equal, e.g. P = C = 1, so that the option is at-the-money. Further, let the rates of opportunity cost be $\delta_P = \delta_C = 5\%$, the instantaneous risk be $\sigma_P = \sigma_C = 10\%$, and the instantaneous coefficient of correlation be $\rho = 0$. Finally, suppose that the operation option comprises the right to produce one unit of output in t = 10 years.

Table 1 Value of Shut Down Option in percent of P and C					
Opportunity Cost □c	Opportunity Cost, □p				
	0,0%	**5,0%**	**10,0%**	**15,0%**	**20,0%**
0,0%	23.65%	5.46%	-1.52%	-3.28%	-2.98%
5,0%	38.58%	14.34%	3.31%	-0.92%	-1.99%
10,0%	52.69%	23.40%	8.70%	2.01%	-0.56%
15,0%	65.14%	31.96%	14.19%	5.28%	1.22%
20,0%	75.43%	39.51%	19.38%	8.61%	3.20%

Table 1 shows the sensitivity of the value of the shut down option as a percentage of total input and output costs for different levels of □P and □C. Under these assumptions, the shut down option is worth 14.34% of total input and output cost. The higher the opportunity cost of input, □C, the lower the present value of future input cost, and the higher the option value. The higher opportunity cost of output, □P, the lower the present value of future revenues, and the lower the value of the shut down option. The table reveals that the option is quite sensitive to changes in the opportunity costs, so that accurate values of W necessitate precise estimates of □P and □C.

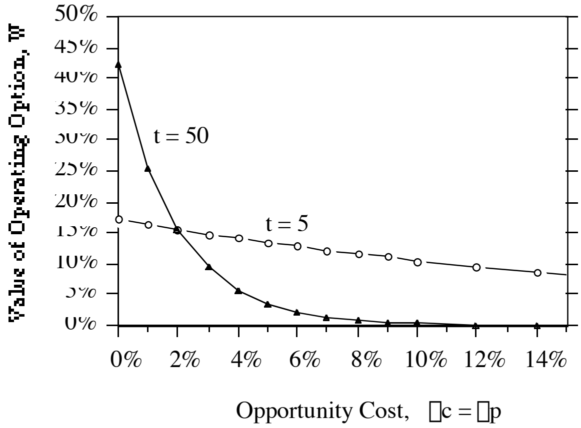


Figure 1 demonstrates that the higher uncertainty, i.e. standard deviation, of future prices the higher the value of the option to shut down. Figure 2 reveals that the shut down option's value approaches zero as the time to maturity becomes large. Longer life usually leads to increased value when European options on stock are concerned; later maturity, however, reduces the value of future profits. Here, the reduction in future profits for long maturities dominates the increase in value of the option. Both figure 1 and figure 2 confirm that a precise determination of opportunity costs is essential, since option value is quite sensitive to them.

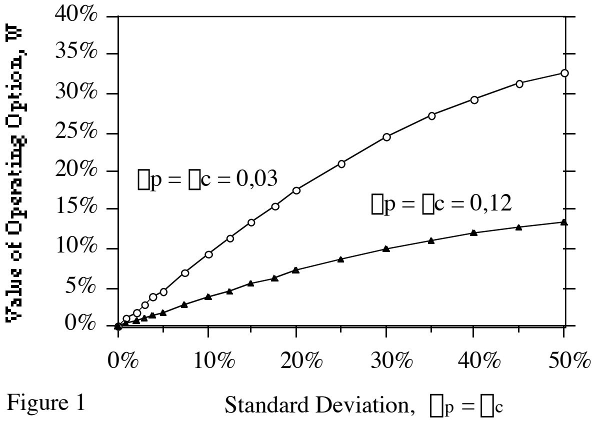


Figure 1

Figure 3 depicts the dependance of the option's value on the coefficient of correlation. The option's value increases as the price movement of input and output becomes more opposite. The reason for this behaviour lies in the fact, that the more negative the correlation, the higher total variance of both procresses in (8).

Introduction and Summary

Temporary shut down options do not only apply to production activities but also to that presence of firms in diverse markets, which can also be temporarily suspended. Dixit (1989) uses a CCA approach to analyse optimal market entry and exit decisions of firms. Explicitly taking entry and exit cost into account, he closely follows the approach of Brennan and Schwartz (1985).

Via the term *hysteresis*, Dixit explains the nature of the problem. Hysteresis is defined as the failure of a phenomenon to reverse itself when the underlying cause is reversed. For example, assume a foreign company that entered the U.S.-market mainly as a result of appreciated dollar exchange rates. If the company withdrew from the U.S.-market when the dollar fell to its original level, the firm's distribution network and brand recognition would disintegrate quite rapidly. It would have to be rebuilt should the company decide to reenter when the dollar reappreciated again. Thus, the company will not exit immediately when the dollar falls to its original low level and variable costs are left uncovered partially. The effect - presence in the U.S.-market - is not reversed if the underlying cause - favourable exchange rates - is reversed.

Allowing output prices to vary randomly over time, idle firm's and active firm's can be regarded as American call options on each other. Besides evaluating the interrelation of assets, Dixit determines pairs of critical or *trigger* prices that induce firms to change their state from idle to active or the reverse. He finds that entry triggers exceed the sum of variable costs and interest on sunk entry costs, whereas exit triggers are less than the difference between variable cost and interest on sunk exit costs. Even for low levels of risk and small sunk costs, these amounts are found to be significant. These triggers, thus, quantify the hysteresis he observes. Further, this analysis demonstrates that Marshallian decision rules to enter a new market if full costs are covered, and to exit if variable cost are no longer covered, are not optimal when future output prices are uncertain.

The Model

A firm is defined in this model by its exclusive access to production technology. Thus, this model assumes operating options to have an unlimited life, unaffected by competitors. After investing a certain amount, k, which is a sunk cost, the firm can become immediately active. The firm can then produce one unit of output each period at a variable cost, w. To cease activity it must pay a lump exit cost, l. Along with, ▯, the firm's discount rate, w, l, and k are constants. The market price of one unit of output, P, evolves according to a geometric Brownian motion.

$$(1) \qquad \frac{dP}{P} = \mu\, dt + \sigma\, dz$$

where μ is the expected rate of change in P, σ the instantaneous standard deviation of the changes in P, and dz a standard Wiener process.

The current output price, P, and a binary variable, indicating an active firm (1) or an idle firm (0), comprise the two state variables of the problem. In state (P, 0) the firm can decide to either continue being idle or to enter the market. In state (P, 1) it can either continue being active or exit the market. $V^0(P)$ and $V^1(P)$ describe the respective NPV of an initially idle and active firm, starting with price, P, and a subsequent value-maximizing policy of switching between both operating states. Dixit shows that $V^0(P)$ and $V^1(P)$ must satisfy the following ordinary differential equations

$$(2a) \qquad \frac{1}{2}\sigma^2 P^2 V^0_{PP} + \mu P V^0_P - \rho V^0 = 0$$

$$(2b) \qquad \frac{1}{2}\sigma^2 P^2 V^1_{PP} + \mu P V^1_P - \rho V^1 = w - P$$

subject to the boundary conditions

$$(3a) \qquad V^0(P_H) = V^1(P_H) - k$$

$$(3b) \qquad V_P{}^0(P_H) = V_P{}^1(P_H)$$

$$(3c) \qquad V^1(P_L) = V^0(P_L) - l$$

$$(3d) \qquad V_P{}^1(P_L) = V_P{}^0(P_L)$$

Condition (3a) defines the higher price trigger, P_H, where an idle firm can switch to the active state at a cost k. (3c) defines the lower price trigger, P_L, where the firm can switch to the idle state at a cost l. (3b) and (3d) are the value maximizing Merton-Samuelson high contact conditions (see Samuelson (1965) and Merton (1973)) for P_H and P_L, respectively.

The solutions to (2a) and (2b) are easy to obtain because both are linear and have the same homogeneous part.

$$(4a) \qquad V^0(P) = B\,P^{\beta}$$

$$(4b) \qquad V^1(P) = A\,P^{-\alpha} + \frac{P}{\rho - \mu} - \frac{w}{\rho}$$

where

(5a) $\square = \square \frac{(1 \square m) \square \sqrt{(1 \square m)^2 + 4\,r}}{2}$

(5b) $\square = \frac{(1 \square m) + \sqrt{(1 \square m)^2 + 4\,r}}{2}$

$m = 2\,\square / \square^2$, and $r = 2\,\square / \square^2$.

The value of $V^0(P)$ results exclusively from the option to enter, whereas the value of $V^1(P)$ consists of the option to exit plus operating profits minus operating costs. The last two terms of (4b) represent the firm's value if abandonment were impossible.

The constants A and B together with the triggers P_H and P_L follow from the boundary conditions (3a) - (3d) and are determined completely by the following system of non-linear equations

(6a) $A\,P_L^{\square\square} + \frac{P_L}{\square\square\square} \square \frac{w}{\square} = B\,P_L^{\square} \square C$

(6b) $A\,P_H^{\square\square} + \frac{P_H}{\square\square\square} \square \frac{w}{\square} = B\,P_H^{\square} + k$

(6c) $\square A \square\, P_L^{\square\square\square 1} + \frac{1}{\square\square\square} = B \square P_L^{\square\square 1}$

(6d) $\square A \square\, P_L^{\square\square\square 1} + \frac{1}{\square\square\square} = B \square P_H^{\square\square 1}$

With the results, Dixit is able to show that the trigger prices satisfy the following conditions

(7a) $P_H > w + \square k = W_H$
(7b) $P_L < w - \square l = W_L$

Compared to standard Marshallian theory where full costs are entry trigger, i.e. $W_H = w + p\,k$, and variable costs serve as exit trigger, i.e. $W_L = w - p\,l$, the span between P_H and P_L is always wider for Dixit's approach. Thus, firms that follow Marshallian theory enter and exit too often. Differences between Dixit's and Marshall's approach are due solely to the inability of the latter to capture the essence of uncertain prices. If the risk associated with future prices reduces to zero, then both models become identical, i.e. if $\square = 0$, then $P_H = w + p\,k$ and $P_L = w + p\,l$.

Example

For demonstrating purposes (see Dixit (1989) p. 630-634) assume variable cost w = 1, entry cost k = 4, exit cost l = 0, discount rate □ = 0.025, standard deviation □ = 0.1 , and expected rate of change change □ = 0. For these figures the triggers become

(8a) $1.4667 = P_H > w + □ k = W_H = 1.1$

(8b) $0.7657 = P_L < w - □ l = W_L = 1$

Entry is not optimal unless prices increase to 1.4667 times variable cost, or 33% more than full cost. Exit is not optimal unless prices decrease to 0.7657 times variable cost, or 24% less than variable cost. It shows that hysteresis leads to a much wider span between P_H and P_L than the span between Marshallian triggers W_H and W_L.

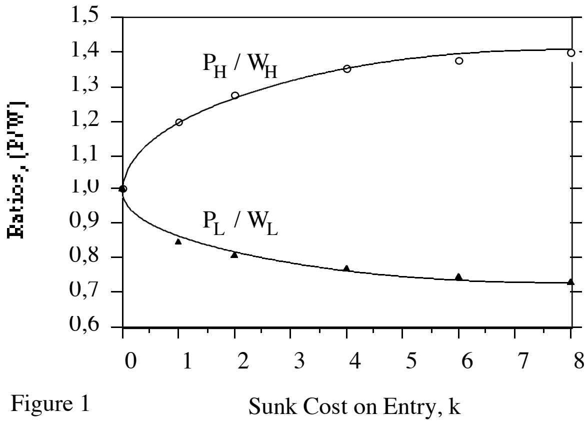


Figure 1

The figures show the effects of changes in k, and □ on the ratios (P_H / W_H) and (P_L / W_L), which is the relative error of Marshallian theory under uncertainty plus one. Figure 1 depicts the effect of varying sunk costs, k, on the trigger prices. At k = 0 the curves emerge from the common limiting value with slopes of +∞ and -∞ respectively. Thus, the effect of hysteresis is extremely strong even for small sunk costs k. Figure 2 shows the effect of changes in the output price risk, □. For small values of □ hysteresis remains significant. Decreasing discount rates, □, and increasing expected rates of change in outut prices, □, reduce both triggers.

Extensions

Apart from the standard assumption, that output prices follow geometric Brownian motions, other stochastic processes can also be modelled. Assuming that prices show a tendency towards some predictable long-term equilibrium level, P*, though they might fluctuate in response to short-term influences, leads to a mean reverting process

(9) $$dP = \eta (P^* - P)\, dt + \sigma P\, dz$$

where η is the mean reversion parameter. Including this stochastic process the differential equations for $V^0(P)$ and $V^1(P)$ turn into

(10a) $$\frac{1}{2} \sigma^2 P^2 V^0_{PP} + \eta (P^* - P)\, V^0_P - \rho V^0 = 0$$

(10b) $$\frac{1}{2} \sigma^2 P^2 V^1_{PP} + \eta (P^* - P)\, V^1_P - \rho V^0 = w - P$$

These equations do not have closed form solutions and must be approximated by numerical methods.

Intuitively, it seems that the effect of mean reversion can only lead to a wider span between the trigger prices, P_H and P_L. Since prices rebound to their long time average, P*, both, high and low prices, are likely to remain shorter than for a geometric Brownian motion. Consequently, investors will be more reluctant before entering or exiting the market. This behavior is reflected by a wider span in trigger prices.

Further insight can be achieved, if output rate is no longer assumed to be constant. Suppose that production follows a Cobb-Douglas function with decreasing marginal returns. If there are fixed costs, f, then operating profit is p^θ - f, with $\theta < 1$. Defining $Y = p^\theta$ and using Itô's lemma yields

(11) $$dY = \theta P^{\theta - 1} dP + \frac{1}{2} \theta (\theta - 1) P^{\theta - 2} \sigma^2 P^2 dt$$

$$\frac{dY}{Y} = \theta \left[\alpha + \frac{(\theta - 1)\, \sigma^2}{2} \right] dt + \theta \sigma\, dz$$

$$= \alpha'\, dt + \sigma'\, dz$$

where α' and σ' are constants. This equation has the same form as (1). Hence, substituting α by α' and σ by σ' allows use of the previous equations to solve for $V^0(P)$ and $V^1(P)$ assuming a production technology that follows a Cobb-Douglas function.

Finally, instead of a risk-neutral preference structure the firm can have a risk-averse one. Dixit shows that in this case α just has to be replaced by $r - \delta$, where $\delta = \mu - \alpha$, the estimated opportunity cost of holding the output, as opposed to a perfectly correlated financial asset. With this substitution the differential equations for the value of active and idle firms become

(12a) $\frac{1}{2}\sigma^2 P^2 V^0_{PP} + (r - \delta)\, P\, V^0_P - r V^0 = 0$

(12b) $\frac{1}{2}\sigma^2 P^2 V^1_{PP} + (r - \delta)\, P\, V^1_P - r V^1 = w - P$

The solution to (12a) and (12b) resembles (2a) and (2b).

This approach does not necessitate that the underlying asset be traded. If, however, it is a traded asset, e.g. an output commodity market price, then matters simplify and estimates are no longer necessary. Assuming that the asset pays a continuous dividend or convenience yield rate, δ can be directly set to this rate. Dixit proves that the respective differential equations can be derived from a replicating portfolio strategy.

5.3.2 Abandonment

Introduction and Summary

Abandonment options represent their owner's possible right to stop operations forever. Unlike temporary shut down options, they do not allow for resumption of production. Myers and Majd (1983) present a model to evaluate abandonment options, where they treat these options as American puts on the project with an exercise price equivalent to the salvage value. Since project and salvage value vary randomly over time, abandonment is a kind of insurance that project value can never drop below salvage value. Myers and Majd assume that the project can be abandoned at any time during its life and that project and salvage value correspond to prices of traded financial assets. Further, they explicitly allow cash flow from operations, the discount rate and the standard deviation to be functions of project value and calendar time. Using these assumptions, they are able to show how to reach a numerical solution for the value of the abandonment option together with an optimal abandonment decision rule.

The Model

For the application of contingent claims valuation techniques, one must specify the cash flow generating process of the project as a function of the project value. Therefore,

(1) $$C(t) = V(t)\,\delta(V, t)$$

where C(t) is the cash flow, V(t) the project value, and $\delta(V, t)$ is the payout ratio at time t.

In order to capture the stochastic nature of project and salvage value, their incremental changes are assumed to follow geometric Brownian motions

(2a) $$\frac{dP}{P} = (\alpha_P - \delta_P)\,dt + \sigma_P\,dz_P$$

(2b) $$\frac{dS}{S} = (\alpha_S - \delta_S)\,dt + \sigma_S\,dz_S$$

where P is the project value, α_p the expected rate of change in P, δ_p the project's payout ratio, σ_p the instantaneous standard deviation of change in P, and dz_p a standard Wiener process generating the unexpected changes in P. The variables for the salvage value, S, are similarily defined. The instantaneous coeffincient ρ correlates project and salvage value.

By forming a portfolio of the project value, salvage value, and riskless Treasury Bills with interest rate r, Myers and Majd replicate the payoff from the abandonment option. They show that the value of the abandonment option, A(P, S, t), is the solution to the following PDE

$$\text{(3)} \qquad \frac{1}{2}\sigma_P^2 P^2 A_{PP} + \rho \sigma_P \sigma_S P\, S\, A_{PS} + \frac{1}{2}\sigma_S^2 S^2 A_{SS} + (r - \delta_P)\, P\, A_P + (r - \delta_S)\, S\, A_S - r A + A_t = 0$$

subject to the boundary conditions

(4a) $A(P^c(t), S, t) = \max(S - P, 0)$

(4b) $A_p(P^c(t), S, t) = 1$

(4c) $A(\infty, S, t) = 0$

(4d) $A(P, S, t^*) = \max(S - P, 0)$

Boundary conditions (4a) and (4b) define a schedule of critical project values $P^c(t)$ for which abandonment is optimal. (4b) is the value-maximizing Merton-Samuelson high contact condition. (4c) states that for high project values, the proportion of abandonment options becomes negligible. (4d) means that at the end of the project's physical life, t*, the abandonment option has only the value of immediate exercise.

For three reasons this specification of the abandonment option does not allow a closed-form solution: First, salvage value is uncertain and related in a complex manner to an uncertain project value. Second, future payout is uncertain and can depend on any of time, project, or salvage value in a complex manner. Third, as an American option, abandonment may occur before the end of the physical life. The abandonment schedule can only be determined simultaneously with the abandonment value.

Through four steps, the abandonment option as specified under (3) and (4) can be transformed into a more tractable form, which treats the ratio of project to salvage value rather than their absolute counterparts: First, introduce a new state variable, the ratio of project to salvage value, i.e. X = P / S. Second, let the standard deviation of X be $\sigma_x^2 = \sigma_p^2 + \sigma_s^2 - 2\, \rho\, \sigma_p\, \sigma_s$. Third, set the exercise price to unity. Fourth, replace the riskless rate r by δ_s, the payout ratio of salvage.

Under these conditions the abandonment option G(X, t) = A(P, S, t) / S, where X = P / S, must satisfy the following PDE

$$\text{(5)} \qquad \frac{1}{2}\sigma_X^2 X^2 G_{XX} + (\delta_S - \delta_P)\, X\, G_X - \delta_S G + G_t = 0$$

subject to the boundary conditions

(6a) $G(X^c(t), t) = \max(1 - X, 0)$

(6b) $G_X(X^c(t), t) = 1$

(6c) $G(\infty, t) = 0$

(6d) $G(X, t^*) = \max(1 - X, 0)$

Although no closed-form solution is known for this problem either, an explicit finite difference approximation can be used to solve for the value of the abandonment option, G, together with the optimal exercising schedule $X^c(t)$. Appendix A introduces an explicit finite difference approach that shows how to approximate the value of abandonment options.

Example

Assume the following numbers: In t = 0, current project and salvage value are P = $ 100 mill. and S = $ 50 mill., so the ratio X = P / S = 2. Further, assume that P and S are market prices of financial assets and suppose that the changes in the prices of P and S are independent of each other, i.e. $\rho = 0$.. If P and S are not market prices of financial assets, then δ_p and δ_s together with an appropriately risk-adjusted rate of return from P and S must be considered in an extension of the above model. As base case, let annual payout ratios be $\delta_p = 6\%$, $\delta_s = 7\%$, and annual standard deviations be $\sigma_p = \sigma_s = 20\%$ which are reasonable parameter values .The physical life of the facility is set to t* = 70 years.

Table 1	Abandonment Value as Percentage of Salvage Value							
Ratio of Values,	Calendar Time, t							
X = P / S	**0**	**10**	**20**	**30**	**40**	**50**	**60**	**70**
2,000	18,0%	17,9%	17,9%	17,7%	17,2%	15,6%	10,8%	0,0%
1,732	20,0%	20,0%	19,9%	19,8%	19,3%	17,9%	13,3%	0,0%
1,500	22,3%	22,3%	22,2%	22,1%	21,7%	20,4%	16,2%	0,0%
1,299	24,9%	24,8%	24,8%	24,7%	24,3%	23,2%	19,4%	0,0%
1,125	27,7%	27,7%	27,6%	27,5%	27,2%	26,3%	23,0%	0,0%
0,974	30,9%	30,8%	30,8%	30,7%	30,5%	29,7%	26,9%	2,6%
0,843	34,4%	34,4%	34,3%	34,3%	34,1%	33,5%	31,3%	15,7%
0,730	38,3%	38,3%	38,3%	38,2%	38,1%	37,6%	36,0%	27,0%
0,633	42,6%	42,6%	42,6%	42,6%	42,5%	42,2%	41,1%	36,8%
0,548	47,5%	47,5%	47,5%	47,5%	47,4%	47,2%	46,6%	45,2%
0,474	52,9%	52,9%	52,9%	52,9%	52,9%	52,8%	52,6%	52,6%
0,411	58,9%	58,9%	58,9%	58,9%	58,9%	58,9%	58,9%	58,9%
0,356	64,4%	64,4%	64,4%	64,4%	64,4%	64,4%	64,4%	64,4%
0,308	69,2%	69,2%	69,2%	69,2%	69,2%	69,2%	69,2%	69,2%
0,267	73,3%	73,3%	73,3%	73,3%	73,3%	73,3%	73,3%	73,3%
0,231	76,9%	76,9%	76,9%	76,9%	76,9%	76,9%	76,9%	76,9%
0,200	80,0%	80,0%	80,0%	80,0%	80,0%	80,0%	80,0%	80,0%

Table 1 shows abandonment value as percentage of salvage value for varying ratios X = P / S, and calendar time, t. The entries were calculated with the numerical procedure described in appendix A. The solid line represents the optimal exercise schedule. If the ratio of project to salvage value falls below the line at time of consideration, the firm should optimally abandon. For example, if project value in t = 0 accounts to less than 0.474 times salvage value, exercise of the abandonment option becomes optimal. Compared with static valuation methods that recomend abandoment when project equals salvage value, this approach shows that abandonment is not optimal unless project value falls to about half its salvage value. Since project values vary stochastically, there is always a good chance for low project values to rebound to higher levels. So the project will not be abandonned unless project value is considerably lower than salvage value. For the critical ratio of 0.474 the abandonment option is worth 52.9% of salvage value. For the current ratio in the example of X = 2, the value of the abandonment option is 18.0% of salvage value, that is $ 9 mill. .

Table 2 Abandonment Value as Percentage of Salvage Value for a Ratio of X = 2

Variance Rate, σ^2	Coefficient of Correlation, ρ				
	0,9	**0,5**	**0**	**-0,5**	**-0,9**
0,00	10,5%	10,5%	10,5%	10,5%	10,5%
0,04	2,3%	10,4%	18,0%	23,8%	27,4%
0,08	5,2%	15,2%	23,8%	30,2%	34,3%
0,12	9,2%	19,8%	28,5%	34,9%	38,8%
0,16	13,7%	23,8%	32,4%	38,5%	42,6%

Table 2 depicts the sensitivity of this percentage to changes in standard deviation and correlation. For the base case and a ratio of X = 2 in t = 0, the abandonment option is worth 18% of its salvage value. The higher the uncertainty about future prices, and the more negative the correlation between project and salvage value, the higher the resulting value of the abandonment option turns out to be.

5.3.3 Operations with Temporary Shut Down and Abandonment

Introduction and Summary

In general, real investments involve both temporary shut down options and abandonment options. Either one alone is usually insufficient for an accurate analysis of real investments. Brennan and Schwartz (1985) present an approach to evalutate real investment projects that involves both options. Depending upon output prices the project can either be shut down temporarily, until prices recover, or even be abandoned forever. Although the model was initially designed for analyzing natural resource mines, it applies to all industries that produce a single traded good, e.g. crude oil, minerals, or other commodities. The model explicitly considers managerial control over output rates in response to varying output prices. Further, it accounts for variations in risk and discount rate whether due to depletion of the resource, output prices, or both. In an extension, Brennan and Schwartz develop a rule for determining optimal price conditions, for installing the investment project, that is exercising a growth option.

The Model

Assume that the project under consideration produces a single homogenous commodity whose output market price, S, follows a geometric Brownian motion.

$$(1) \qquad \frac{dS}{S} = \square\, dt + \square\, dz$$

where □ is the expected rate of change in S, □ the instantaneous standard deviation in the rate of change of S, and dz is an increment in a standard Wiener process.

Further, suppose that futures contracts on the output commodity are available. Let F(S, □) be the futures price at time t for one unit of commodity to be delivered at time T, where □ = T - t, and let c be the constant, continuous net convenience yield rate of the output commodity.

The net convenience yield rate accrues to the owner of a physical commodity but not to the owner of a futures contract. The yield stems from that profit, which results from temporary local shortages of a commodity and the subsequent sale of the physical commodity a an atypically high price. One could also think of the yield as the opportunity cost of retaining the capability of keeping a production process running.

According to Itô's lemma, the change in the futures price is then

$$(2) \qquad dF = S\, F_S\, (\square + c - \square)\, dt + \square\, S\, F_S\, dz$$

The total value of the project, H, not only depends on current output prices, S, but also on project's lifetime capacity, Q, calendar time, t, project's current state, j, and its operating policy, □.

The relevant state variables are S, Q, and t. Lifetime capacity, Q, directly depends on the intesity of usage, or the current output rate, q. The output rate, q, itself is a function of S, Q, and t. Project's current state, j, can either be operating (j = 1) or inactive (j = 0), were inactive means either closed temporarily or abandoned permanently. The operating policy, $\square$, is described by a function ruling output rates, q(S, Q, t), and three critical output commodity prices. The three critical prices are $S_1(Q, t)$, the price at which an operating facility is closed either temporarily or permanently; $S_2(Q, t)$, the price at which a temporarily closed facility is reopened; and $S_0(Q, t)$, the price at which a facility is abandoned.

By Itô's lemma the project's total value, H(S, Q, t; j, $\square$), changes according to

(3) $$dH = H_S\, dS + H_Q\, dQ + H_t\, dt + \frac{1}{2} H_{SS} (dS)^2$$

By definition, dQ = - q dt where q is again the current commodity output rate. During operations q is variable free of cost within the limits of q^+ and q^-. Temporarily closed projects still incur periodical maintenance costs, unless they have been abandoned. Activating and closing down facilities also incurs costs. If the project is operating, it is possible to inactivate it for a sunk cost of $K_1(Q, t)$. In turn, if the project is not operating, it can be activated for a sunk cost of $K_2(Q, t)$.

Before establishing a riskless portfolio strategy, it is important to consider the cash payouts from the project. The after-tax cash flow from the project is

(4) $$q(S - A) - M(1 - j) - \square_j H - T$$

where A(q, Q, t) is the average cash cost rate when operating, M(t) the after-tax fixed-cost maintenance rate when closed, $\square_j$ the tax rate on the value of the project depending on the current state j, and T(q, Q, S, t) are total income tax and royalties when operating.

The combination of a long position of the project and a short position of (H_s / F_s) future contracts on the output, forms a riskless hedge portfolio.

Brennan and Schwartz show that a riskless position must earn the risk-free rate and thus under the value-maximizing operating policy $\square$* = {q*, So*, S1*, S2*} the value of the project when open, V(S, Q, t), and when closed, W(S, Q, t) satisfies the following PDEs

(5a) $$\max_{q \,\square\, [q^+, q^-]} \left[\frac{1}{2} \square^2 S^2 V_{SS} + (\square \square c)\, C\, V_S \,\square\, q V_Q + V_t + q(S \,\square\, A) \,\square\, T \,\square\, (\square + \square_1) V \right] = 0$$

(5b) $$\frac{1}{2} \square^2 S^2 W_{SS} + (\square \square c)\, S\, W_S + W_t \,\square\, M \,\square\, (\square + \square_0) W = 0$$

subject to the boundary conditions

(6a) $W(S_0^*, Q, t) = 0$

(6b) $W_S(S_0^*, Q, t) = 0$

(6c) $V(S_1^*, Q, t) = \max\left(W(S_1^*, Q, t) - k_1(Q, t), 0\right)$

(6d) $V_S(S_1^*, Q, t) = \begin{cases} W_S(S_1^*, Q, t) & \text{if } V(S_1^*, Q, t) > 0 \\ 0 & \text{if } V(S_1^*, Q, t) = 0 \end{cases}$

(6e) $W(S_2^*, Q, t) = V(S_2^*, Q, t) - k_2(Q, t)$

(6f) $W_S(S_2^*, Q, t) = V_S(S_2^*, Q, t)$

(6g) $V(S, 0, t) = 0$

(6h) $W(S, 0, t) = 0$

Conditions (6a) and (6g), (6h) state that the value of the project is zero when abandoned or when the lifetime capacity is exhausted. (6c) and (6e) define the critical output prices at which closing and reopening occurs. Finally, conditions (6b), (6d), and (6f) follow directly from the Merton-Samuelson high contact condition that maximizes the value of the project subject to the critical commodity prices. The critical output commodity prices themselves follow implicitly from the boundary conditions and must be determined jointly with the solution of V and W.

The solution to (5a) and (5b) depends on three state variables, S, Q, and t, when it possibly only need depend on two state variables. If the rate of inflation, π, is constant over time, the solution procedure can be simplified. In this case the third state variable, t, is no longer necessay because deflated parameters sufficiently describe the problem.

(7a) $a(q, Q) = A(q, Q, t)\, e^{-\pi t}$

(7b) $f = M(t)\, e^{-\pi t}$

(7c) $k_1(Q) = K_1(Q, t)\, e^{-\pi t}$

(7d) $k_2(Q) = K_2(Q, t)\, e^{-\pi t}$

(7e) $v(S, Q) = V(S, Q, t)\, e^{-\pi t}$

(7f) $w(S, Q) = W(S, Q, t)\, e^{-\pi t}$

Since in either case the problem is too complex for analytical solutions, it must be solved numerically. Appendix B shows an explicit finite difference approach to solve for an investment project where inflation is constant and the output rate is fixed to $q = q^+ = q^-$.

Example

Consider a hypothetical copper mine (see Brennan and Schwartz (1985) p. 147-150). The mine inventory which is exogenously determined to be 150 mill. lbs. represents the lifetime capacity, Q. Since the output rate of the mine when operating is constrainted to be q* = 10 mill. lbs. per year, Q translates into a potential to produce continuously for 15 years. Table 1 (from Brennan and Schwartz (1985) p. 148) contains further relevant data.

Table 1 Data for Hypothetical Copper Mine
Mine:
Inventory (Q): 150 mill. lbs.
Output rate (q*): 10 mill. lbs. per year
Average cost of production a(q*, Q): $ 0.50 per lb.
Cost of opening and closing (k_1, k_2): $ 200,000
Maintenance costs (f): $ 500,000 per year
Inflation rate (□): 8% per year
Copper:
Convenience yield (c): 1% per year
Price variance (□^{2}): 8% per year
Taxes:
Real estate (□$_1$, □$_2$): 2% per year
Income (t_2): 50%
Interest rate (□): 10% per year

Brennan and Schwartz's CCA reveals that for an inventory equivalent to 15 years of production and a production cost of 50 cents per lb., the cost of opening the mine leads to suboptimal returns from beginning production unless copper prices reach a level of 76 cents per lb. . However, for a mine that is already operating, temporarily shutting down production leads to suboptimal returns unless the copper price falls to 44 cents per lb. . Finally, the mine should not be abandoned until output prices fall to 20 cents per lb. .

Table 2 Value of Copper Mine for Different Copper Prices

Copper Price ($ / per lb.) (1)	Mine Value ($ mill.) Open (2)	Mine Value ($ mill.) Closed (3)	Value of Fixed-Output Rate Mine ($ mill.) (4)	Value of Closure Option ($ mill.) (5)	Risk (6)	Value of Mine under Certainty, □^{2} = 0 (7)
0,30	1,25	1,45	0,38	1,07		0,00
0,40	4,15	4,35	3,12	1,23		0,00
0,50	7,95	8,11	7,22	0,89	0,75	1,85
0,60	12,52	12,49	12,01	0,51	0,66	7,84
0,70	17,56	17,38	17,19	0,37	0,59	13,87
0,80	22,88	22,68	22,61	0,27	0,54	19,91
0,90	28,38	28,18	28,18	0,20	0,50	25,94
1,00	34,01	33,81	33,85	0,16	0,47	31,98

In Table 2 (from Brennan and Schwartz (1985) p. 149), columns 2 and 3 list the present value of the mine for different copper prices respectively when open or closed. The value of a mine without operating options, or a mine that is required to operate at a rate of 10 mill. lbs. per year no matter how low copper prices fall, appears in column 4. Column 5 shows the value of the operating options which is the difference between the maxium of column 2 or 3, and column 4. Operating options increase in value as prices fall, much like a put option. Column 6 shows the instantaneous risk of the mine, which Brennan and Schwartz define as (V_S / V) when open and (W_S / V) when closed. Since the risk of the mine varies randomly with copper prices, the instantaneous rate of return required by risk-averse investors must take these variations into account.

Using a constant discount rate as implied by traditional NPV analysis proves to be an error in the light of CCA. Column 7 represents the value of the mine under certainty, that is for a zero standard deviation of output prices. Under certainty it is never optimal to close the mine once it is open. Opening the mine, however, is optimal as soon as copper prices exceed full cost. Under certainty operating options are worthless.

Extension

Closed-form solutions can be obtained by making further simplifications. Allowing the lifetime capacity to be unlimited, that is not allowing the project to deteriorate, reduces the set of state variables. Under this assumption Q is no longer a relevant state variable and both V and W depend only on one single state variable, i.e. current output prices, S. The advantage of this simplification occurs in (5a) and (5b). They are no longer partial differentials but rather reduce to ordinary differentials equations, which have closed-form solutions. Further, allowing the maintenance of a closed facility to be costless, eliminates the need to abandon. In other words S_0* becomes zero. As a result abandoning is never optimal, which limits the set of possible solutions for the value of a closed facility to non-negative values, i.e. w(S) ≥ 0. Finally, one assumes operations to proceed at a single constant rate, q*, in order to avoid any maximization as in (5a).

Under the above assumptions, the following system of ordinary differential equations describes the value of the project when open, v, and when closed, w, for a uniform tax rate □.

(8a) $$\frac{1}{2} \square^2 S^2 v_{SS} + (r \square c) S v_S + m S \square n \square (r + \square) v = 0$$

(8b) $$\frac{1}{2} \square^2 S^2 w_{SS} + (r \square c) S w_S + \square (r + \square) w = 0$$

where m = q* (1 - t_1) (1 - t_2) and n = q* a (1 - t_2). Equations (8a) and (8b) are subject to boundary conditions (6c) - (6f) with the state variables Q and t omitted.

Brennan and Schwartz show, that the analytic solution to (8a) and (8b) is

(9a) $$w(S) = \beta_1 S^{\gamma_1}$$

(9b) $$v(S) = \beta_2 S^{\gamma_2} + \frac{m\,S}{\lambda + c} - \frac{n}{r + \lambda}$$

where

(10a) $$\gamma_1 = \alpha_1 + \alpha_2$$

(10b) $$\gamma_2 = \alpha_1 - \alpha_2$$

(10c) $$\alpha_1 = \frac{1}{2} - \frac{r - c}{\sigma^2}$$

(10d) $$\alpha_2 = \sqrt{\alpha_1^2 + \frac{2\,(r + \lambda)}{\sigma^2}}$$

The value of a closed facility, w, consists solely of a set of options to reopen with infinite duration. The value of an operating facility, v, is composed of a set of options to close with infinite duration and operating revenues minus full costs . Under the condition that the project cannot be shut down, the last two terms in (9b) represent the value of v.

The constants β_1 and β_2 are the result of

(11a) $$\beta_1 = \frac{d\,S_2^*\,(\gamma_2 - 1) + b\,\gamma_2}{(\gamma_2 - \gamma_1)\,S_2^{*\,\gamma_1}}$$

(11b) $$\beta_2 = \frac{d\,S_2^*\,(\gamma_2 - 1) + b\,\gamma_1}{(\gamma_2 - \gamma_1)\,S_1^{*\,\gamma_2}}$$

(11c) $$S_1^* = S_2^*\,x$$

(11d) $$S_2^* = \frac{\gamma_2\,(e - b\,x^{\gamma_1})}{(x^{\gamma_1} - x)}\;d(\gamma_2 - 1)$$

where x solves the non-linear equation

(11e) $$\frac{(x^{\gamma_2} - x)\,(\gamma_1 - 1)}{\gamma_1\,(e - b\,x^{\gamma_2})} = \frac{(x^{\gamma_1} - x)\,(\gamma_2 - 1)}{\gamma_2\,(e - b\,x^{\gamma_1})}$$

Figure 1 depicts the result to (9a) and (9b). The dotted line represents an operating facility, v, without operating options, i.e. $\beta_2 = 0$. For very low prices, i.e. S < S_1*, the facility is worth more closed than open. At S_1* it pays to close an operating facility at a cost k_1. For increasing prices, however, the value of the operating facility increases faster than the value of a closed one and at $S = S_2^*$ it pays to reopen at a cost k_2. For very high prices the effect of shut down options becomes negligable and the project's value approaches its static NPV, the value under certainty.

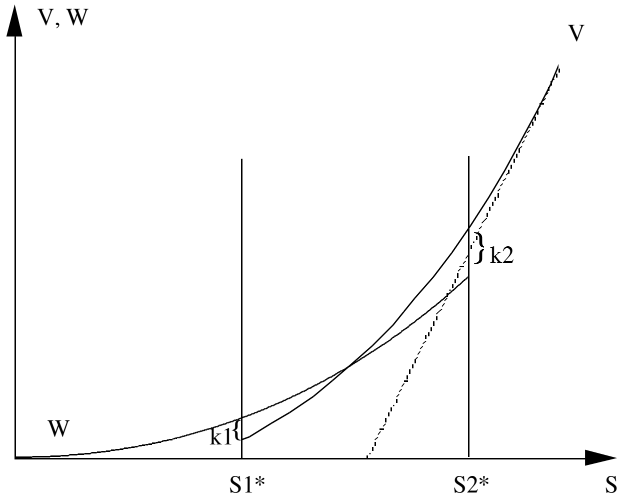


Figure 1

Extension

Until now, only the valuation of the facility itself has been considered. The investment decision not only incorporates the value of the facility, but also compares this value with the initial investment, I(S, Q, t), needed to install the facility. Assuming that construction time lags can be neglected, the problems consists of maximizing the following NPV.

(12) $NPV(S, Q, t) = V(S, Q, t) - I(S, Q, t)$

A naive investor would immediately start with construction, if the NPV were positive. Postponing construction, however, can increase the entire project value since NPV varies stochastically with output prices. The probability that future NPV will wind up higher than current NPV is positive. The dynamic aspect of this timing opportunity resembles the factors which go into determining an optimal strategy for exercising an American option with uncertain exercise price. The right to establish the project is equivalent to the option, the value of the facility comprises the underlying security, and the initial investment is the exercise price.

Define X(S, Q, t) for $0 \leq t \leq t^*$ as the value of the right to establish the project, such that X satisfies the following PDE

(13) $$\frac{1}{2}\square^2 S^2 X_{SS} + (\square\square c)\, S\, X_S + X_t \square (\square + \square)\, X = 0$$

subject to the boundary conditions

(14a) $X(0, Q, t) = 0$

(14b) $X(S, Q, t^*) = 0$

(14c) $X(S^c(t), Q, t) = V(S^c(t), Q, t) - I(S^c(t), Q, t)$

(14d) $X_S(S^c(t), Q, t) = V_S(S^c(t), Q, t) - I_S(S^c(t), Q, t)$

Conditions (14a) and (14b) mean that the investment opportunity looses its value as output prices fall to zero or as the right expires at time t*. (14c) and (14d) define the critical schedule of output prices $S^c(t)$ at which establishment of the investment is optimal. In addition, (14d) constitutes the value-maximizing Merton-Samuelson high contact condition.

If the lifetime capacity of the project, Q, depends upon the amount of initial investment instead of being exogenously given, an additional boundary condition has to be introduced in order to determine the optimal project capacity, Q*.

(14e) $V_Q(S^c(t), Q^*, t) = I_Q(S^c(t), Q^*, t)$

The value of the right to establish the project, X, follows from the solution of (13) subject to the boundary conditions (14a) - (14e). A series of optimal exercise prices, $S^c(t)$, is obtained from (14c) and (14d) . The optimal amount to invest follows from (14e). Since the boundary conditions of (13) involve the value of the established facility, V(S, Q, t), it is necessary to solve for (5a) and (5b) first. Analytical solutions do not exist, so numerical procedures must be applied here.

5.4 Investment Opportunities

5.4.1 Single Stage Growth Options

Introduction and Summary

Investments into key technologies and R&D expenditures constitute an important basis for future expansion of any firm. Although such investments most often have negative NPVs, decision makers employ intuition instead of quantitative techniques and include them in their budget nonetheless to induce growth. In contrast to NPV techniques, this growth option is explicitly recognized by CCA. A single opportunity to grow characterizes the single stage option. Kemna (1987) describes a simple approach to a single stage growth option by treating the potential investment opportunity as a standard European call option to investment at a future date. The present value of the completed project is the underlying security. That of future investment expenditures is the exercise price. Growth options are equivalent to options to defer the decision to undertake investment projects under the condition that investment is irreversible. Since project value varies randomly over time, deferral can increase project value. The effect is usally advantageous, since losses are limited. Kemna shows that her model is formally identical to Black's (1976) solution for the value of a call option on commodity futures. A brief case study from the offshore oil industry clarifies the application of Kemna's model.

The Model

The value of the completed project, V, is assumed to evolve according to a geometric Brownian motion

$$(1) \qquad \frac{dV}{V} = (\alpha - \delta)\, dt + \sigma\, dz$$

where α is the expected rate of return from owning a completed project including an appropriate risk premium, δ the opportunity cost rate of delaying completion of the project, σ the instantaneous standard deviation of the rate of change in V, and dz a standard Wiener process generating the unexpected changes in V.

Assuming that the riskiness of the project relies upon a single, specific factor price, e.g. a natural resource spot price, and that futures contracts are available on that factor, it is possible to build a riskless hedge portfolio with a long position in the growth option, W(V, t), and a short position of (W_V / F_V) units of a synthetic futures contract, F(V, t), on the cash flows from the completed project. Synthetic futures contracts on cash flows and futures contracts on the commodity are implicitly assumed to be perfectly correlated. With this strategy the position is completely riskless and must therefore earn the risk-free rate of return. Kemna shows, that the value of the growth option, W, must satisfy the following PDE

$$\frac{1}{2}\sigma^2 V^2 W_{VV} + (r - \delta)\, V W_V + W_t - r W = 0 \tag{2}$$

subject to the boundary conditions

$$W(0, t) = 0 \tag{3a}$$

$$W_V(\infty, t) = 1 \tag{3b}$$

$$W(V, t^*) = \max\,(V - K, 0) \tag{3c}$$

Boundary condition (3a) says that if the underlying project value is zero, the growth option is also worthless. (3b) states that for high project values, the rate of increase of project value and option value become equal. (3c) means that at maturity, t*, the project is either established for a profit of V - K, where K is the present value of investment cost, or the growth option expires unexercised.

Following Merton (1973) the problem can be solved to yield the following analytical solution

$$W(V, t) = V e^{-\delta (t^* - t)} N(d_1) - K e^{-r (t^* - t)} N(d_2) \tag{4}$$

where

$$d_1 = \frac{\ln (V / K) + (r - \delta)\,(t^+ - t)}{\sigma \sqrt{(t^* - t)}} + \frac{1}{2}\sigma\sqrt{(t^* - t)} \tag{5a}$$

$$d_2 = d_1 - \sigma\sqrt{(t^* - t)} \tag{5b}$$

Substituting $V e^{-\delta (t^* - t)} = F e^{-r (t^* - t)}$ in (6) leads to Black's (1976) formula for the value of a commodity option.

Estimating the opportunity cost of delaying completion of the project, δ, can be tedious. Usually, asset pricing models, e.g. the CAPM, must be employed to find a market determined risk premium. Alternatively, futures contracts combined with the above substitution provides an equally accurate yet more direct and economical approach.

Example

A short case study (see Kemna (1987) p. 166-167) from the offshore oil industry shows the impact of single stage growth options on total project value. In this study, a multinational oil company is faced with the problem of determining an optimal timing decision for the development of an already explored offshore oil field.

At the end of the exploration phase, the company has one among three choices: First, it can refuse to develop and return the right to prospect to the local government. Second, it may start development of the oil field immediately. Third, it can extend the exploration phase at a predetermined extra cost and postpone the final decision to a later date. The objective rests in finding the value-maximizing policy among the three choices. The first two alternatives can sufficiently be assessed with standard capital budgeting techniques. The third alternative, however, can only be evaluated with CCA approaches.

Assuming that the current present value of a completed project, the developed oil reserve, equals the present value of investment expenditures, i.e. V(t) - K(t) = 0, management should not develop the reserve immediately.

But what exactly should management do, if the final decision can be postponed for 2 years at an extra cost of 20% of overall investment expenditure ? How does investment's value depend on uncertainty, opportunity cost, and the allowed duration of deferral ?

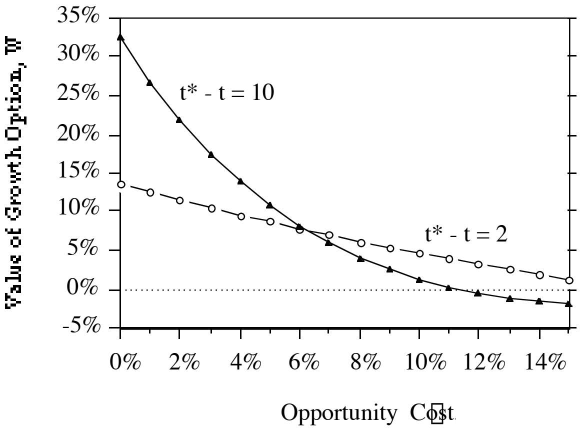
For a real risk-free rate of 5%, table 1 depicts the percentage of investment expenditures, to which the value of the option to grow (deferral option that is) amounts, dependent on uncertainty, □, and opportunity cost, □, of delaying the project's completion.

Table 1 Total Investment Value in % of K

	□			
□	**0**	**0,05**	**0,1**	**0,15**
0,1	9,6	3,1	-0,3	-1,6
0,2	14,3	8,2	4	1,2
0,3	19,4	13,3	8,6	5,1
0,4	29,6	23,1	17,8	13,4

Similar to European calls, the value of this growth option increases as the output price uncertainty, □, increases. The higher the opportunity cost of delaying development, □, the lower the project value. The negative values for several scenarios result from the extra cost of extending exploration. This extra cost can be regarded as the price to buy the growth option.

The allowed duration of delay also plays an important role in managements final decision. Figure 1 depicts the value of the growth option, W, for varying times to maturity and reveals that in particular high opportunity cost, i.e. □ = 0.12, imply low option values. Usually European call options increase in value for longer maturities. But for very long maturities, this effect is offset by the effect of discounting the payoff at maturity at the opportunity cost, □.



Depending on the actual combination of □ and □, the present value of the third alternative, to extend exploration and not reach final decision until a later date, can now be compared to the other two alternatives on a consistent, quantitative foundation to derive an overall optimal investment decision.

Introduction and Summary

McDonald and Siegel (1986) develop a different approach to evaluating single-step growth options. In addition to Kemna's (1987) assumptions they grant their growth option an infinite life and a random future exercise price. Further, their growth option can be exercised at any time. Along with option's value McDonald and Siegel determine an optimal decision rule for its exercise. Both have analytical solutions. The assumption that growth options last forever is sometimes unrealistic, so they derive an extension of their model, that considers growth options with uncertain, finite lives. In addition, they show that the same analysis also applies to optimal scrapping decisions.

According to the classical NPV investment decision rule, investors establish a project when the present value of the installed project exceeds the present value of investment cost. As long as capital is completely reversible, as in most financial investments, this strategy maximizes investors' value because the investment can always be sold at market prices. Unlike financial investments, however, most real investments are irreversible. Once specialized capacity is installed, it cannot necessarily be used for alternative purposes. Thus, investment cost are most likely sunk cost and real investments have considerable opportunity costs - those of *killing* a growth option. Consequently for real investments, the classical NPV criterion is incomplete and usually incorrect. In contrast, the investment decision rule is similar to an optimal exercising policy for an American call option with unceratin exercise price. Exercise (investment) is not optimal unless the value of the underlying security (value of installed project) exceeds the sum of exercise price (value of investment costs) and the current option value (value of growth option).

The growth option, also equivalent to an option to defer an investment where investment is irreversible, can add significant value to a project. According to this model and reasonable parameter values, investment should sometimes be defered unless the present value of benefits is twice the present value of investment costs.

The Model

The market value of the installed project, V, and the market value of investment costs, F, are assumed to vary randomly according to a geometric Brownian motion

(1a) $$\frac{dV}{V} = \square_V dt + \square_V dz_V$$

(1b) $$\frac{dF}{F} = \square_F dt + \square_F dz_F$$

where $\square_V$ is the expected rate of change in V, $\square_V$ the instantaneous standard deviation of the rate of change in V, and dz_V is a standard Wiener process for V. The variables for F are analoguously defined. The instantaneous coefficient of correlation between dz_V and dz_F is $\square_{VF}$.
To be able to reach analytical solutions, assume for the moment that the growth option has unlimited life, i.e. the investment opportunity lasts forever. As a direct result calendar time loses its relevance as a state variable. The optimal exercise timing problem thus consists of finding a real number, C*, such that whenever (V / F) ≥ C* the option is exercised. For arbitrary numbers C the value of the growth option, X, is the option's expected payoff at the uncertain exercise date discounted to t = 0.

(2) $$X = E_o((V_{t'} \square F_{t'})\, e^{\square\square t'})$$

where t' is the date (V / F) first reaches the boundary C, i.e. $(V_{t'} / F_{t'}) = C$, $\square$ an appropriately risk-adjusted discount rate, and $E_0(.)$ is the expected value at time zero in light of all current information. The number C* is chosen so as to maximize present value at time zero of the expected payoff. Since V and F are homogenous of degree zero, and $V_{t'} - F_{t'} = F_{t'}\,(C^* - 1)$, the problem reduces to

(3) $$X = (C^* \square 1)\, E_o(F_{t'}\, e^{\square\square t'})$$

For convenience define $E_0(F_{t'}\, e^{-\square t'}) = L(V_0, F_0, t = 0)$. From the definiton, it follows that L earns the risk-adjusted discount rate $\square$, i.e. $L_t = \square L$. McDonald and Siegel show that L must satisfy the following PDE

(4) $$\frac{1}{2}\square_V^2 V^2 L_{VV} + \frac{1}{2}\square_F^2 F^2 L_{FF} + \square_{VF}\square_V\square_F V F L_{VF} + \square_V V L_V + \square_F F L_F \square \square L = 0$$

subject to the boundary conditions

(5a) $$L(V, F, t) = F \qquad \text{if } (V / F) = C^*$$

(5b) $$\lim_{(V/F)\square 0} L(V, F, t) = 0$$

Condition (5a) says that once exercised, L is worth F. (5b) means that as V nears worthlessness L drops to zero.

McDonald and Siegel do not provide the other boundary conditions necessary to solve for (4) and the optimal exercise rule, C*. They do, however, show that the discount rate, $\square$, is

(6) $$\delta = \varepsilon\,\delta_v + (1 - \varepsilon)\,\delta_f + (1/2)\;\varepsilon(\varepsilon - 1)\,\sigma^2$$

with a variance in the rate of change in (V / F) of

(7) $$\sigma^2 = \sigma_v^{\,2} + \sigma_f^{\,2} - 2\,\rho_{vf}\,\sigma_v\,\sigma_f$$

whereby

(8) $$\varepsilon = \sqrt{\left(\frac{\delta_F - \delta_V}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\,\delta_F}{\sigma^2}} + \left(\frac{1}{2} - \frac{\delta_F - \delta_V}{\sigma^2}\right)$$

The constant δ_V represents the opportunity cost of retaining the growth option instead of receiving the benefits from the completed project, whereas δ_F constitutes the interest gain by defering exercise of the growth option. Thus, $\delta_V = \mu_V' - a_V$ and $\delta_F = \mu_F' - \alpha_F$, where μ_V' and μ_F' are appropriately risk-adjusted discount rates for portfolios of traded assets that respectively correlate perfectly with V and F. In general, appropriately risk-adjusted discount rates can be determined with an asset pricing model such as the CAPM.

With the results embedded in equations (6), (7), and (8), McDonald and Siegel demonstrate that the value of the growth option obeys the following equations.

(9) $$X = (C^* - 1)\,F_0\left(\frac{V_0 / F_0}{C^*}\right)^{\varepsilon}$$

(10) $$C^* = \frac{\varepsilon}{\varepsilon - 1}$$

where V_0 and F_0 are the respective current market values of V and F.

Extension

In several competitive industries the assumption that growth options, i.e. investment opportunities, have unlimited lives is rather unrealistic. In fact, deferral of investments can have disastrous consequences. Tough competitors and technical revolutions can reduce the value of future investments, and in the worst case future investment value can even drop to zero. Thus, assume a growth option, X*, with limited life whose length is uncertain. To account for a possible sudden drop of investment value to zero, assume that V follows a mixed Poisson-Wiener process instead of a geometric Brownian motion as in (1a)

(11) $$\frac{dV}{V} = \alpha_V\,dt + \sigma_V\,dz + dq$$

where $$dq = \begin{cases} -1 & \text{with probability } \lambda\,dt \\ 0 & \text{with probability } 1 - \lambda\,dt \end{cases}$$

The occurrence of the Poisson event, i.e. dq = -1, forces the process to stop, i.e. V = 0, thereby terminating the option's life. First occurrences of Poisson events with parameter λ are distributed exponentially. Thus, the expected life of the growth option , E(t), and the variance thereof, Var(t), are

(12a) $E(t) = 1 / \lambda$ (12b) $Var(t) = 1 / \lambda^2$

The present value of the expected payoff with uncertain expiration date is

$$(13) \qquad X^* = \int_0^\infty \lambda e^{-\lambda \tau} d\tau$$

According to Merton (1971) this can be integrated by parts

$$(14) \qquad X = \max_{C_{t'}} \{ E_o(F_{t'}(C_{t'} - 1) e^{-(\mu + \lambda) t'}) \}$$

$$= (C^* - 1) E_o(F_{t'} e^{-(\mu + \lambda) t'})$$

which is identical to (3), the problem with geometric Brownian motion, except that the discount rate μ has been replaced by $(\mu + \lambda)$. Substituting $(\mu + \lambda)$ for μ in (3) must therefore yield a solution for (14).

$$(15) \qquad \varepsilon = \sqrt{\left[\frac{\delta_F - \delta_V}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2(\delta_F + \lambda)}{\sigma^2}} + \left[\frac{1}{2} - \frac{\delta_F - \delta_V}{\sigma^2}\right]$$

Note, that (9) and (10) are used again to determine the value of the growth option, X*. Further, both models are identical when $\lambda = 0$.

Example

To demonstrate both models numerically, let $\sigma_V^2 = \sigma_F^2 = 4\%$ and $\delta_V = \delta_F = 10\%$. Both are reasonable estimates for U.S. stock companies, according to McDonald and Siegel. For normalization, set V = F = 1. For the moment let $\lambda = 0$, and $\rho = 0$.

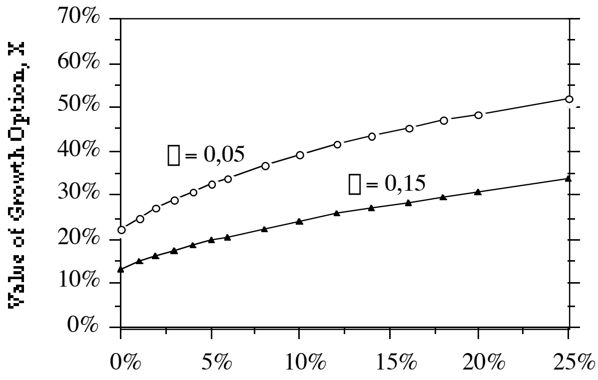


Figure 1 Variance, □f^2= □v^2

In the base case growth option value accounts for 23% of the whole investment. Figure 1 shows the effect of risk, i.e. □$_V^2$ = □$_f^2$, on the value of the growth option for different opportunity cost □$_V$ = □$_f$ = □. The higher the risk and the lower the opportunity cost, the higher the resulting value of the growth option. Figure 2 depicts the influence of expected life on growth option value, and therewith demonstrates the crucial fact of this model. The value increases as expected life increases.

Table 1 (from McDonald and Siegel (1986) p. 720) shows C*, the ratios of (V / F) at which exercise of the growth option is optimal. For the base case, exercise should occur if the completed project value, V, is 1.56 times the investment cost, F. C* never falls below 1, and for many resonable parameter values it remains well above 2. It follows that sometimes investment should be postponed until the present value of benefits equals twice the initial cost of investment.

Table 2 Value of Benefits Relative to Cost at which Investment is Optimal									
□v		**0,05**			**0,10**			**0,25**	
□vf	**-0,5**	**0**	**0,5**	**-0,5**	**0**	**0,5**	**-0,5**	**0**	**0,5**
□v², □f²									
0,01	2,50	2,35	2,18	1,47	1,37	1,25	1,09	1,06	1,03
0,02	2,91	2,64	2,35	1,72	1,56	1,37	1,18	1,12	1,06
0,04	3,65	3,17	2,64	2,13	1,86	1,56	1,34	1,24	1,12
0,10	5,65	4,56	3,41	3,19	2,62	2,00	1,77	1,54	1,29
0,20	8,77	6,70	4,56	4,79	3,73	2,62	2,44	2,00	1,54
0,30	11,83	8,77	5,65	6,34	4,79	3,19	3,07	2,44	1,77
□									
0,00	3,65	3,17	2,64	2,13	1,86	1,56	1,34	1,24	1,12
0,05	2,50	2,23	1,92	1,86	1,67	1,44	1,32	1,23	1,12
0,10	2,10	1,90	1,67	1,72	1,56	1,37	1,30	1,22	1,12
0,25	1,67	1,54	1,40	1,51	1,40	1,27	1,27	1,19	1,11
□f									
0,01	2,31	1,89	1,46	1,64	1,43	1,22	1,25	1,17	1,08
0,05	2,85	2,38	1,86	1,83	1,58	1,32	1,28	1,19	1,10
0,10	3,65	3,17	2,64	2,13	1,86	1,56	1,34	1,24	1,12
0,25	6,42	5,96	5,49	3,35	3,09	2,81	1,62	1,49	1,33

Extension

By reinterpreting variables, both models can be used for optimal scrapping decisions, i.e. abandonment of projects. By interpreting F as the market value of capital in place, and V as the project's scrapping value, i.e. salvage value, both models provide the value of the abandonment option, X ' , together with an optimal scrapping rule. This approach resembles to Myers and Majd's (1983) formulation of the abandoment problem.

The value of the abandonment option, X ', is

(16) $$X' = (c^* \square 1)\, F_0 \left[\frac{V_0 / F_0}{{}^*c} \right]^{(1 \square \square)}$$

(17) $$c^* = \frac{(\square \square 1)}{\square}$$

where c* is the optimal exercise rule c*, or the critical ratio of (F / V) that induces abandonment. The variable □ must be chosen according to the underlying stochastic process for V, (1a) or (11). The result that c* never exceeds 1. Hence, the firm waits to abandon until the value of capital in place falls below the salvage value by a prescribed amount - by waiting the firm profits from increases in V - F, while enjoying protection against losses.

5.4.2 Two Stage Growth Option

Introduction and Summary

Kemna (1987) not only determines an approach for a single stage growth option (see 5.4.1) but also for a two stage growth option. The single stage option consists of a single growth opportunity, whereas the two stage option consists of two interdependent, single stage growth opportunities. The second opportunity depends on realization of the first. The case of a two stage growth option is similar to a compound call option (see Geske (1979)). A case study from the oil industry follows the model to illustrate the two stage growth option.

The Model

Regarding the value of a realized project, i.e. the present value of future cash flows from operations, as a futures contract, the single stage growth option amounts to a simple European call option on a futures contract. Setting F equal to the stochastic value of the completed project, and K to the constant cost of installing the project, and evaluating both at the expiration date, t*, the value of the single step growth option can be determined with Black's (1976) formula for the value of a European call on a futures contract. Letting $W_1(F, t)$ be the value of a single stage growth option,

(1) $$W_1(F, t) = F\, e^{-r(t^*-t)} N(d_1) - K\, e^{-r(t^*-t)} N(d_2)$$

where

(2a) $$d_1 = \frac{\ln(F/K)}{\sigma\sqrt{(t^*-t)}} + \frac{1}{2}\sigma\sqrt{(t^*-t)}$$

(2b) $$d_2 = d_1 - \sigma\sqrt{(t^*-t)}$$

Extending the model by a second phase includes a second decision moment, and assuming this second phase is contingent upon a positive decision during the first decision moment, one creates a two stage growth option. For example, after a research phase, management may decide to begin development. If so, at the end of the development phase, management would in turn, have to decide whether to begin production.

After the first decision moment, that is once a decision has been reached which leads to the project's continuation, the two stage option becomes identical to a single stage option. Thus, it is an option on an option upon a futures contract, or a compound option upon a futures contract. According to Kemna, this two stage option can be evaluated by a modified version of

Geske's (1979) compound option formula. Let t_1^* (t_2^*) be the decision moment at the end of phase 1 (phase 2), K_1 (K_2) be the exercise price of continuing the project at t_1^* (t_2^*), and V^c be the lowest value of V for which it is still optimal to continue at t_1^*. The value of the two stage growth option is then

$$(3)\qquad \begin{aligned} W_2(F, t) = {} & F\, e^{-r(t_2^* - t)} N_2(b_1, a_1; \rho) \\ & - K_2\, e^{-r(t_2^* - t)} N_2(b_2, a_2; \rho) \\ & - K_1\, e^{-r(t_1^* - t)} N(b_2) \end{aligned}$$

where

$$(4a)\qquad \rho = \sqrt{(t_1^* - t) / (t_2^* - t)}$$

$$(4b)\qquad a_1 = \frac{\ln(F / K_2)}{\sigma\sqrt{(t_2^* - t)}} + \frac{1}{2}\sigma\sqrt{(t_2^* - t)}$$

$$(4c)\qquad a_2 = a_1 - \sigma\sqrt{(t_2^* - t)}$$

$$(4d)\qquad b_1 = \frac{\ln(F / F^c)}{\sigma\sqrt{(t_1^* - t)}} + \frac{1}{2}\sigma\sqrt{(t_1^* - t)}$$

$$(4e)\qquad b_2 = b_1 - \sigma\sqrt{(t_1^* - t)}$$

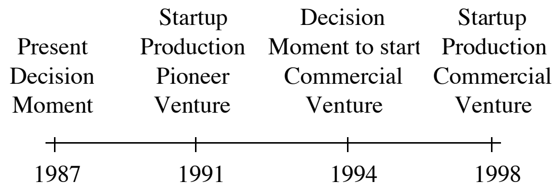
The critical value V^c has to be determined iteratively. Accordingly, the model has merely a semi-analytic solution.

Example

The growth option to be considered here is a manufacturing Pioneer Venture from the petrochemical industry (see Kemna (1987) p. 168-174). High capital expenditures and low cash flows characterize the project. Although in and of itself unattractive, the project mainly aims to develop and boast a new technology for the sake of maintaining the current market position. Only from this strategic point of view, does Pioneer Venture have any benefit. By investing in Pioneer Venture, the manufacturer puts itself in a position to invest in a future Commercial Venture, which is approximately five times as large as Pioneer Venture. In Pioneer Venture, future uncertainty stems mainly from the uncertainty of future oil prices.

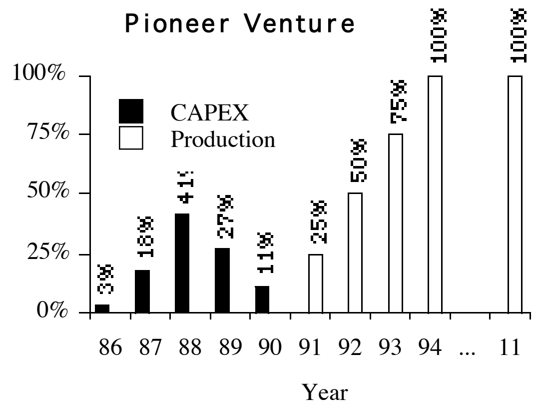
The main question reduces to whether negative NPVs of Pioneer Venture are offset by growth option value.

Assuming for the moment that Pioneer Venture comprises a single stage project, the following planning situation would be encountered (current date = 1987).



The expiration date, t*, of the single stage growth option is 1994, the riskless rate, r, is 2% in real terms, and the futures price, F, and exercise price, K, the 1994 value of the completed Commercial Venture and the cost necessary to establish it.

Production and capital expenditures (CAPEX) of Pioneer and Commercial Venture are distributed according to the following figures.



The CAPEX of 3% for Pioneer Venture in 1986 are sunk costs and hence irrelevant. The uncertain 1994 value of Commercial Venture, F, is estimated to be $ 1000 mill. . The certain value of construction cost, K, for Commercial Venture in 1994 is $ 1000 mill. Thus, the single stage growth option is currently at-the-money, and it follows that current NPV is zero. Since future oil price volatility, ▯, can not be directly observed, one employs annual estimates of 15%, 20%, and 25%. The current NPV of Pioneer Venture, that is the price to buy the growth option, is negative $ 200 mill. Thus, according to traditional NPV analysis, overall project value, including Pioneer and Commercial Venture, is negative $ 200 mill., and the company should stop the project immediately. NPV ignores growth option - mistakenly.

Using a single stage growth approach, results differ markedly from NPV analysis. For a future oil price volatility of 15%, the single stage growth option is worth $ 137 mill., but costs are $ 200 mill. to buy. Thus, the net value of the overall project totals to negative $ 63 mill.. The net value for a future oil price volatility of 20% annually is negative $ 19 mill. and for a 25% annual uncertanty, plus \$ 25 mill.. It follows, that is only pays to initiate Pioneer Venture, if uncertainty about future oil prices is high.

Extending the decision problem to a two stage growth option, leads to an even greater value of the investment opportunity. Assume now that Pioneer Venture can be stopped after one year, i.e. $t_1^* = 1988$, the first year's CAPEX is \$ 98 mill., and that after one year, Pioneer Venture's NPV rests at negative \$ 90 mill., i.e. K_1 = \$ 90 mill.. The other parameters remain unaltered, i.e. $t_2^* = 1994$, F = \$ 1000 mill., K_2 = \$ 1000 mill., and r = 2%.

Table 1 summarizes the results for both approaches.

Table 1 Value of Growth Options (in \$ mill.)

□	**0,15**	**0,2**	**0,25**
Single Stage Option, W_1	137 - 200 = -63	181 - 20 = -19	225 - 200 = 25
Two Stage Option, W_2	57 - 98 = -41	98 - 98 = 0	141 - 98 = 43
Critical Project Value, F^c	812	730	653

Compared to the single stage approach, the value of Pioneer Venture increases significantly as a two stage option. As multi stage option it would be even higher. Nevertheless, the favourableness of the project relies on high oil price volatility. If future oil prices were certain, growth options would be worthless. Pioneer Venture's value would be equal to its NPVof negative \$ 200 mill., and neither Pioneer nor Commercial Venture would ever be undertaken.

The critical value, F^c, indicates the lowest acceptable 1988 value of Commercial Vanture, for which it pays to proceed with phase two. If, for example, the value of Commercial Venture fell below \$ 812 mill. in 1988 and future oil price uncertainty were □ = 15%, then Pioneer Venture should be stopped.

5.4.3 Multi Stage Growth Option

Introduction and Summary

In a microeconomic model, Pindyck (1988b) picks up Myers' s (1977) supposition, that the value of any firm is mainly attributable to two sources, operating and growth options. Operating options determine the value of capital in place whereas growth options both reflect the value of future investments and equal the opportunity cost of investing in irreversible investments. Pindyck focuses mainly on capacity choice, utilization, and firm value in terms of homogenous, marginal investment decisions. Since each marginal investment buys a growth option on the next unit of capacity, this can be thought of as a multi stage growth option. The operating options treated here are temporary shut down options. For simplification it will be assumed that investment is incremental, no delivery lags exist for installing investments, only one source of uncertainty exists, and that facilities do not no depreciate. Pindyck concludes from the model that when investments are irreversible, capacity installation is smaller, and also that firm value is largely attributable to growth options. Finally, he extends his model to a form which resembles to the single stage growth options of McDonald and Siegel (1986).

The Model

In the model, a linear price-demand function confronts firms

(1) $P(\square, Q) = \square(t) - \square Q$

where P is the output price, Q the output quantity, □ an industry specific parameter (for price takers □ = 0), and □(t) the exogenously determined price shift parameter, that varies randomly according to the following geometric Brownian motion.

(2) $\frac{d\square}{\square} = \square\, dt + \square\, dz_V$

where □ is the expected rate of change in □, □ the instantaneous standard deviation in the rate of change in □, and dz a standard Wiener process.

With a specified amount of installed capacity, K, the value of the firm, W, consists of

(3) $W = V(K, \square) + F(K, \square)$

where V(K, □) embodies the cumulative value of all operating options and F(K, □) the cumulative value of all growth options.

Since capacity can be installed incrementally, (3) can be rewritten with respect to the capacity already installed, K, to reflect this.

$$(4) \qquad W = \int_0^K \Delta V(□, □)\, d□ + \int_K^\infty \Delta F(□, □)\, d□$$

where $\Delta V(□, □)$ represents both the value of the marginal operating option provided that capacity □ is already installed and the present value of incremental profits generated by d□. The function $\Delta F(□, □)$ represents both the present value of the marginal growth option given that capacity □ is already installed and the present value of the opportunity cost of increasing capacity by d□. In other words, the value of the firm consists of an infinite number of marginal operating options plus an infinite number of marginal growth options.

The optimal capacity K*(□), or the value-maximizing amount of capital in place, follows from a common microeconomic investment decision rule: Beginning with zero, capacity is installed sequentially until expected cash flows from that marginal unit of capacity, i.e. the value of the marginal operating option, equal total cost of that unit. Total cost of that unit equals the sum of installation cost, k, and opportunity cost for investing, i.e. the value of the marginal growth option. Thus, the optimality condition is

$$(5) \qquad \Delta V(K^*, □) = k + \Delta F(K^*, □)$$

Per definition each unit of capacity enables the firm to produce one unit of output per period assuming that the firm starts with zero capacity, capacity can be instantly installed, and the capacity does not depreciate over time.

Total operating cost, C, is a function of current output, Q.

$$(6) \qquad C(Q) = c_1 Q + (1/2)\, c_2 Q^2$$

where c_1 and c_2 are production technology-specific factors. In general, c_1 and c_2 can be zero, but if □= 0, it is necessary that $c_2 > 0$ in order to restrict the firm's size.

Pindyck presupposes that a portfolio of traded assets exists, which is perfectly correlated with the demand shift parameter, □, so that an appropriately risk adjusted discount rate, □, can be determined from the CAPM. Future output can then be discounted appropriately. Nonetheless □ is expected to grow at a rate of □. Thus, the effective discount rate is □= □□□.

The value of $\Delta V(□, □)$ evaluated at □ = K, is equivalent to an incremental project that has a periodic output of one unit at a cost of $(2□ + c_2)K + c_1$ and can be shut down if output prices, □(t), fall below per unit cost. According to this reinterpretation of the pricing problem, the price of such a project is □(t). A portfolio of a long position in the project and a short position of $\Delta V_{\square}$ units of output is riskless and earns the risk-free rate. By Itô's lemma, Pindyck shows that ΔV must satisfy the following ordinary differential equation

(7) $$\frac{1}{2}\square^2\square^2\Delta V_{\square\square} + (r\square\square)\square\Delta V_{\square} + j(\square\square(2\square + c_2)K\square c_1)\square r\Delta V = 0$$

subject to the boundary conditions

(8a) $$\Delta V(K, 0) = 0$$

(8b) $$\lim_{\square\square\infty} \Delta V(K, \square) = \frac{\square}{\square}\square\frac{((2\square + c_2)K + c_1)}{r}$$

(8c) $$\lim_{\square\square\infty} \Delta V_{\square}(K, \square) = \frac{1}{\square}$$

where the binary variable, j, indicates whether ΔV is currently used (1) or not (0), i.e. whether output price □(t) > (2□+ c_2)K+ c_1. Condition (8a) says that ΔV is worthless when □(t) is zero. (8b) means that for high prices, operating option becomes negligable and the value of capacity approaches its certainty equivalent. Thus, costs - which are certain - are discounted at the risk-free rate and future benefits - which are uncertain - are discounted at a risk-adjusted rate. (8c) states that for high ouput prices, the value of installed capacity increases accordingly to a perpetual annuity.

Pindyck shows that the solution for the value of the marginal operating option equals

(9) $$\Delta V(K, \square) = \begin{cases} b_1\square^{\square_1} & \text{if } \square < ((2\square + c_2)K + c_1 \\ b_2\square^{\square_2} + \frac{\square}{\square}\square\frac{((2\square + c_2)K + c_1)}{r} & \text{if } \square \geq ((2\square + c_2)K + c_1 \end{cases}$$

where

(10a) $$\square_1 = \square\frac{r\square\square\square\square^2/2}{\square^2} + \frac{1}{\square^2}\sqrt{(r\square\square\square\square^2/2)^2 + 2r\square^2} \quad > 1$$

(10b) $$\square_2 = \square\frac{r\square\square\square\square^2/2}{\square^2}\square\frac{1}{\square^2}\sqrt{(r\square\square\square\square^2/2)^2 + 2r\square^2} \quad < 0$$

(10c) $$b_1 = \frac{r\square\square_2(r\square\square)}{r\square(\square_1\square\square_2)}(((2\square + c_2)K + c_1)^{(1\square\square_1)} \quad > 0$$

(10d) $$b_2 = \frac{r\square\square_1(r\square\square)}{r\square(\square_1\square\square_2)}(((2\square + c_2)K + c_1)^{(1\square\square_2)} \quad > 0$$

If output prices fall below operating cost, i.e. □(t) < (2□+ c_2)K + c_1, the marginal unit of capacity is not utilized. Thus, $b_1\theta^{\beta_1}$ is the option not to utilize marginal capacity. If output prices are above operating costs, the marginal unit will be utilized and $b_2\theta^{\beta_2}$ is the value of the marginal option to shut down temporarily at any future date, the second term in (9) is the present value of future benefits, and the third term represents the present value of future costs.

By using Itô's lemma and forming a riskless portfolio, Pindyck shows that the value of the marginal growth option, $\Delta F(K, \theta)$, must satisfy the following ordinary differential equation

$$(11) \qquad \frac{1}{2}\sigma^2\theta^2\Delta F_{\theta\theta} + (r-\delta)\theta\Delta F_{\theta} - r\Delta F = 0$$

subject to the boundary conditions

$$(12a) \qquad \Delta F(K, 0) = 0$$

$$(12b) \qquad \Delta F(K, \theta^*) = \Delta V(K, \theta^*) - k$$

$$(12c) \qquad \Delta F_{\theta}(K, \theta^*) = \Delta V_{\theta}(K, \theta^*)$$

Boundary condition (12a) says that for output prices of zero the marginal growth option is worthless. (12b) defines the critical output price, $\theta^*(K)$, for which it is optimal for the firm to exercise the marginal growth option at cost k in order to obtain a marginal operating option. (12c) is the value maximizing Merton-Samuelson high contact condition for $\theta^*(K)$.

Pindyck proves that the solution to this problem is the value of the marginal growth option.

$$(13) \qquad \Delta F(K, \theta) = \begin{cases} a\,\theta^{\beta_1} & \text{if } \theta < \theta^*(K) \\ \Delta V(K, \theta) - k & \text{if } \theta \geq \theta^*(K) \end{cases}$$

where

$$(14) \qquad a = \frac{\beta_2 b_2}{\beta_1}(\theta^*)^{(\beta_2-\beta_1)} + \frac{1}{\delta\beta_1}(\theta^*)^{(1-\beta_1)} \qquad > 0$$

Above $\theta^*(K)$, an increase in capacity becomes optimal. The optimal exercise point follows implicitly from a non-linear equation that must be solved iteratively

$$(15) \qquad \frac{b_2(\beta_1-\beta_2)}{\beta_1}(\theta^*)^{\beta_2} + \frac{(\beta_1-1)}{\delta\beta_1}\theta^* - \frac{(2\gamma + c_2)K + c_1}{r} - k = 0$$

If output prices fall below the optimal exercise point, i.e. $\theta(t) < \theta^*(K)$, it does not pay to increase capacity and thus for this price $a\theta^{\beta_1}$ represents the value of the option to install marginal capacity at any future date. If, however, output prices exceed the optimal exercise point, it is optimal to exercise the marginal growth option in order to increase current capacity. The function ΔV(K, □) - k determines the net gain from this undertaking.

Via the function □*(K), a firm knows when conditions are optimal for investment. To determine the proper extend of investment, the firm must know the optimal capacity, K*(Q), which follows from the numerical solution to the nonlinear equation

$$(16) \qquad \frac{r □ □_1 (r □ □)}{r □□_1} □^{□_2} ((2 □ + c_2)\, K^* + c_1)^{(1 □□_2)} □ \frac{(2 □ + c_2)\, K^* + c1}{r} + \frac{□_1 □ 1}{□□_1} □ □ k = 0$$

Example

For a basic numerical example (see Pindyck (1988b) p. 974-975), assume that the risk-free rate, r, is 5%, that the risk-adjusted rate of return on output, □, equals 5%, that the cost function's constants, c_1 and c_2, both equal zero, that the price-demand function's constant, □, equals 0.5, and that the standard deviation of output prices, □, takes values between 0% and 40%. Further, installed capacity, K, is 1 unit, the price to install a further unit of capacity, k, is 10, and output price, □, has a current value of 2.

Figure 1 depicts the value of the marginal operating option, ΔV(K, □), for varying □. It looks just like the value of a call option. In, fact it is nothing else as an infinite number of European call options with infinite maturity.

Figure 2 delineates the value of the marginal growth option for varying □. The optimal exercise price, □*, is indicated by "+". When □ exceeds this point, the firm should increase capacity.

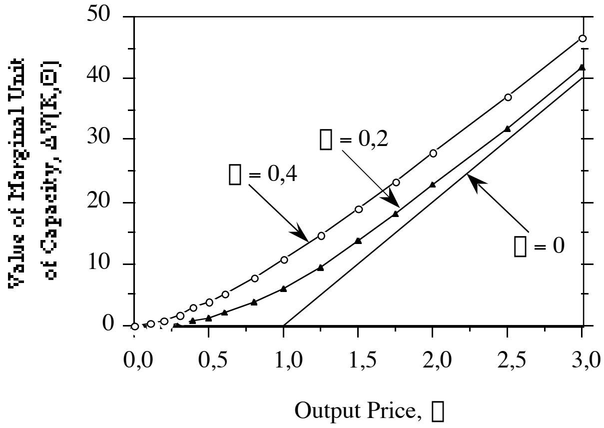


Figure 3 illustrates the optimal capacity, K*, as a function of □. Interestingly, the higher the uncertainty about future prices, □, the lower the optimal capacity.

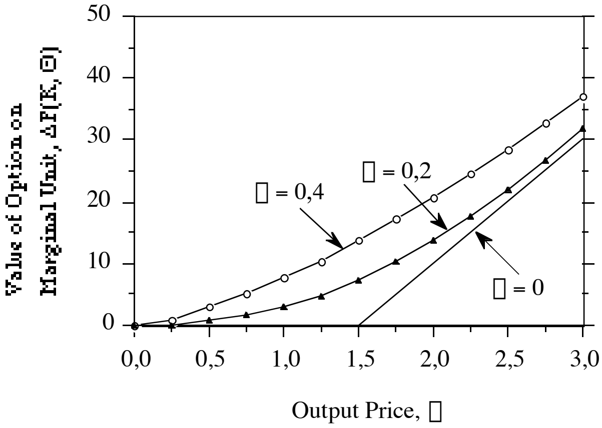


Figure 4 shows the net value of the marginal operating option, ΔV(K, □) - k, and the value of the marginal growth option, ΔF(K, □), as a function of K. If the opportunity costs of investing, i.e. the opportunity cost of exercising the growth option were ignored, the firm would overinvest, i.e. K = 2.3. Incorporating the opportunity cost reveals a much lower optimal capacity of K* = 0.67.

Table 1 Value of the Firm

□	□	K*	V(K*)	F(K*)	Value
0	**0,5**	0	0	0	0
	1	0,5	7,5	0	7,5
	2	1,5	37,5	0	37,5
	3	12,5	87,5	0	87,5
	4	23,5	157,5	0	157,5
0,2	**0,5**	0	0	3,1	3,1
	1	0,04	0,8	13,5	14,3
	2	0,67	24	49,2	73,2
	3	1,37	67,1	94,6	161,7
	4	2,09	134,1	143,7	277,8
0,4	**0,5**	0	0	25,8	25,8
	1	0	0	69,7	69,7
	2	0,15	5,9	182,6	188,7
	3	0,64	36,2	307,2	343,4
	4	1,22	91,3	427,5	518,8

Table 1 (see Pindyck (1987) p. 980) shows the optimal capacity, K*, the value of operating options (capital in place), V(K*), the value of growth options, F(K*), and the total value of the firm for different levels of uncertainty, □, and output prices, □. The higher the uncertainty, the lower optimal capacity and value of operating options, but the higher value of growth options.

The higher the output price, the higher optimal capacity and value of the firm. A decrease in value of operating options is overcompensated by an increase of growth option value when uncertainty grows. Thus, it stands to reason, that total firm value increases as uncertainty becomes larger. Another interesting implication is that even for moderate levels of price uncertainty (σ = 20% annually is not unusual for U.S. stock companies) the proportion of growth options in total firm value plays a rather dominant role. This theoretical result seems to confirm earlier empirical findings (see Kester (1984)).

Extension

Since the assumption that capacity can be expanded incrementally is somewhat unrealistic for real investments, Pindyck extends his model to the other extreme. Assume now, that investment occurs in one large discrete amount and that the firm can only establish a single project and must decide when and how large to build it. Under these conditions the growth option is no longer a multi stage option but reduces to a single stage option similar to the one in McDonald and Siegel (1986).

Once built, the project value equals the sum of all operating options

(17) $$V(K, \theta) = \int_0^K \Delta V(\nu, \theta)\, d\nu$$

But before capacity is installed, the project's value consists exclusively of one growth option, $G(K, \theta)$, to build the plant at a future date. Pindyck reasons that the value of the single stage growth option must obey the following ordinary differential equation

(18) $$\frac{1}{2}\sigma^2\theta^2 G_{\theta\theta} + (r - \delta)\,\theta\, G_{\theta} - r G = 0$$

subject to the boundary conditions

(19a) $G(K, 0) = 0$

(19b) $G(K^*, \theta^*) = V(K^*, \theta^*) - k\,K^*$

(19c) $G_{\theta}(K^*, \theta^*) = V_{\theta}(K^*, \theta^*)$

(19d) $V_K(K^*, \theta^*) - k = 0$

Boundary condition (19a) states that the growth option is worthless when output prices are zero. (19b) defines the optimal exercise point, $K = K^*$ and $\theta = \theta^*$, where growth options can be exchanged for operating options at cost $k\,K^*$. (19c) and (19d) are value-maximizing Merton-Samuelson high contact conditions for the optimal exercise point.

The solution to the single step growth option pricing problem has the form

(20) $$G(K, Q) = a\square^{\square_1}$$

Pindyck does not explain the solution of (20), but an analytical result can generally be obtained. Thus, the coefficient a and the critical values K* and $\square$* can be determined from the boundary conditions (19b) and (19c) whereas $\square_1$ follows directly from (10a).

5.4.4 Other Applications for Investment Opportunities

Flexibility to Meet Growing Demand

Introduction and Summary

Growth option models generally look at the investment opportunity as a whole. In contrast, there are other applications that focus on particular aspects of investment opportunities that indirectly affect investment value. Pindyck (1988a) develops a contingent claims approach for evaluating the degree of flexibility of alternative production technologies. He defines flexibility as the firm's ability to satisfy increasing demand. In order to determine the opportunity cost of inflexibility, the technology under consideration is compared to an ideal production technology that has maximum flexibility to meet increasing demand. Compared with the maximum flexibility alternative, an investment into inflexible technology is like the exercise of an American call option. Pindyck provides an analytical solution for the option value and an optimal decision rule for its exercise. Originally developed to evaluate alternative designs of electric power plants, the model applies to a broad set of related problems. The approach resembles a single stage growth option with infinite life.

The Model

In order to capture the cost structure of the particular system, a detailed cost-benefit analysis between the production technology under consideration in relation to the most flexible alternative must be conducted. The difference in the benefits has the form

(1) $$B(t) = A_1 + A_2 P(t)$$

where A_1 and A_2 are constants that depend upon the various parameters of the production technologies, and P(t) is the stochastic market price of one input or output factor that is responsable for differences in the benefit of both technologies.

The price, P , is assumed to evolve according to a geometric Brownian motion

(2) $$\frac{dP}{P} = \square\, dt + \square\, dz$$

where $\square$ is the expected rate of change in P, $\square$ the instantaneous standard deviation of the rate of change in P, and dz is a standard Wiener process generating the unexpected changes in P.

Let K be the present value of investment cost to install the considered production technology, define r as the risk-free rate, and let $\square = \square - \square$ be the effective discount rate where $\square$ is the risk-adjusted discount rate of a financial asset that is perfectly correlated with P.

If the option is exercised, i.e. inflexible production technology is installed, then installation cost, K, yields a benefit of $B(t) = A_1 + A_2 P(t)$. This is equivalent to paying $X = (K - A_1) / A_2$ in return for P(t).

To solve for the value of this modified option, f(P), and the optimal exercise price, P*, form a portfolio of a long position of the option and a short position of f_P units of the factor. Since this portfolio is riskless it earns the risk-free rate. Pindyck demonstrates that the value of the option, f, satisfies the ordinary differential equation

$$(3) \qquad \frac{1}{2}\sigma^2 P^2 f_{PP} + (r - \delta) P f_P - r f = 0$$

subject to the boundary conditions

$$(4a) \qquad f(0) = 0$$

$$(4b) \qquad f(P^*) = P - X$$

$$(4c) \qquad F_P(P^*) = 1$$

Boundary condition (4a) says that for a factor price of zero, the option is worthless. (4b) defines the optimal exercise price, that is exercise the option pays (P - X). (4c) is the value maximizing Merton-Samuelson high-contact condition for the optimal exercise price.

Pindyck derives the following solution for the value of the option

$$(5) \qquad f(P) = aP^{\beta}$$

where

$$(6a) \qquad \beta = \frac{1}{2} - \frac{r - \delta}{\sigma^2} + \sqrt{\left[\frac{r - \delta}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2r}{\sigma^2}}$$

$$(6b) \qquad a = \frac{P^* - X}{(P^*)^{\beta}}$$

$$(6c) \qquad P^* = \frac{\beta X}{\beta - 1}$$

Instead of defining the project in terms of paying X and receiving P, the variables can be substituted in terms of paying $A_2 X = K - A_1$ and receiving $A_2 P = B(t) - A_1$. For this approach, the modified option, F(P), to install production technology is worth

$$(7) \qquad F(P) = A_2 (P^* - X) (P / P^*)^{\beta}$$

and with $P = P^*$, $B = B^* = A_1 + A_2 P^*$

(8) $$B^{*} = \frac{A_1 + \beta(K - A_1)}{\beta - 1}$$

or

(9) $$\left[\frac{B}{K}\right]^{*} = \frac{\beta - (A_1 / K)}{\beta - 1}$$

Example

To illustrate the model, assume a case, where evaluating the flexibility of a large coal-fired power plant relative to small oil combined cycle plants (see Pindyck (1988a) p. 3-5) is of interest. Under the convenient assumption that oil and coal prices are perfectly correlated, it is possible to consider a single index, P, for the randomly fluctuating fuel price in U.S.-dollar per MMBTU.

High capital cost, low fuel cost, and a commitment for a large amount of capacity characterize the coal plant, whereas the oil combined cycle plants are specified by low capital cost, high fuel cost, and capacity incrementally installed as needed, i.e. maximum flexibility to meet future demand.

Supposing that economic conditions favour a system of oil combined cycle plants, the two main questions reduce to: 1) how far must the fuel price rise until the coal plant becomes favourable ? 2) what is the opportunity cost of relative inflexibility for the coal plant ?

The coal plant with total capacity of 800 MW would incur a capital cost of \$ 1200 per kW or total installation cost of \$ 960 mill., maintenance cost of \$ 20 per kW and year, and a fuel cost of \$ 33.35 per ton of coal or 14.53 mills per kWh. The system of small oil combined cycle plants would incur capital cost of \$ 500 per kW, maintenance cost of \$ 15 per kW and year, and fuel cost of \$ 17 per barrel of oil or 26.88 mills per kWh. The initial system capacity of 6000 MW is covered by existing power plants and grows at an annual rate of 2.5%. The fuel price P has an expected annual growth rate of 3%, yet fluctuates at a standard deviation of 10% per year (Pindyck estimates that the annual standard deviation of the oil price has been 17% during the period of 1965-86). Finally, an annual real discount rate of 6% applies.

The coal plant would cause a present value of capital and maintenance cost of \$ 1186 mill. and a current NPV of fuel cost of \$ 1982 mill. The system of oil combined cycle plants would produce a present value of capital and maintenance cost of \$ 524 mill. and a current NPV of fuel cost of $ 3676 mill. .Thus the present value of cost for the coal-fired plant is $ 3168 mill. and that for the oil-fired system is $ 4200 mill. . Thus, for a traditional NPV investment decision regime, a utility would install the inflexible coal plant. But due to different levels of flexibility, this does not amount to a comparison on a *apples-to-apples* basis.

Table 1 Decision to Build a Large Coal-Fired Power Plant

System Characteristics		
Initial Capacity		6000 MW
Projected Growth		2.5%
Discount Rate		6.0% Real
Alternative Plant Characteristics	Coal Plant	Oil Combined Cycle Plant
Size	800 MWe	Any Size
Capital Cost	$ 1200/kW	$ 500/kW
Maintenance	$ 20/kWYear	$ 15/kWYear
Fuel	$ 33.35/Ton	$ 17.00/Barrel
	14.53 Mills/kWh	26.88 Mills/kWh
Real Fuel Escalation	3.0%	3.0%
Fuel Price Uncertainty	±10%/Year	±10%/Year
Expected Generating Cost	Coal Plant	Oil Combined Cycle Plant
Capital plus O&M	$ 1186 mill.	$ 524 mill.
Fuel	$ 1982 mill.	$ 3676 mill.
Total	$ 3168 mill.	$ 4200 mill.
Current Ratio of Cost	1,30	
Critical Ratio of Cost	1,32	

CCA analysis, on the other hand, takes different degrees of flexibility to meet growing demand into consideration and reveals that the coal plant has an opportunity cost of relative inflexibility of $ 1062 mill. and that current fuel prices must climb another 1.75% for the coal plant to become economic. In other words, the current ratio of cost of 1.30 must climb to 1.32 before the coal plant becomes favourable.

Construction Time Flexibility

Introduction and Summary

Until now, it has been assumed that real investments can be established in zero construction time. In fact, this is quite unrealistic, because many large scale investment opportunities take years before their construction is complete. Majd and Pindyck (1987) show how to evaluate real investment opportunities that have a minimum construction time during which original investment decisions can be revised. Upon arrival of new information the project's construction schedule can be accelerated, decelerated, or even stopped in midstream. This construction flexibility represents a valuable option. Majd and Pindyck use a CCA framework to provide both, a numerical procedure to evaluate construction time flexibility and an optimal decision rule concerning construction speed. For reasonable parameter values, they show that traditional NPV techniques can lead to significant errors.

The analysis concentrates on real investment projects with the following characteristics: First, investment decisions and corresponding construction expenditures occur sequentially. Second, due to technical limitations, there is a maximum speed at which construction can proceed - it takes time to build. Third, investors do not receive any cash flows until the project is complete. Fourth, at any instant construction can be halted or resumed at no extra cost. Fifth, investments are completely irreversible.

Under these assumptions, the investment opportunity amounts to a compound call option on the value of a completed project. Each unit of construction expenditures buys an option on the next unit. The last option in succession is a claim to the completed project. In fact, an investment opportunity with these characteristics combines features of single and multi stage growth options. Single stage because the underlying security is a single completed project and multi stage since investment expenditures occur incrementally.

The investment opportunity can only be evaluated jointly with an optimal exercise rule, or a plan that determines when and how much to spent during each construction phase. The decision rule together with the value of the investment opportunity depends upon the current value of a completed project. Since no analytic solution is available for this problem, appendix C demonstrates a finite difference approach that approximates the value of construction time flexibility.

The Model

Assume to be the only source of uncertainty, the value of the completed project, V, is further assumed to vary randomly according to a geometric Brownian motion.

$$(1) \qquad \frac{dV}{V} = (\square\,\square\,\square)\ dt + \square\ dz$$

where □ is the expected rate of return from owning a completed project, □ the expected rate of opportunity cost from delaying completion, □ the instantaneous standard deviation of the rate of change in V, and dz a standard Wiener process. The risk-free rate is r.

If V is the price of a traded asset, then □ is the appropriately risk-adjusted discount rate for V. If V is not a market price, then □ is the risk-adjusted discount rate for a traded financial asset that is perfectly correlated with V. If the asset can be stored, □ is the net convenience yield rate. In any other case □ must be estimated. The stochastic differential (1) describes the value of a completed project. This should not be confused with the value of an operating project. For an operating project, □ represents the payout ratio, which in general is much higher than in this case.

The present value of remaining construction expenditures is assumed to be constant and known in advance. Nonetheless, the value of the project under construction is always uncertain unless the project is completed. Due to a limited maximum rate of construction, there is always a time interval before completion during which the project value can change. In general, a greater maximum construction rate amounts to greater flexibility of the construction schedule to respond to fluctuating values of the completed project.

The optimal investment rule is contingent upon two state variables, the market value of a complete project, V, and the present value of remaining construction expenditures, K. The rule lays down at what speed construction shall proceed. Since no adjustment costs are associated with changes in the level of construction progress, the rule is simple. If the current value of the completed project is higher than a cutoff value, $V \geq V^*(K)$, investment proceeds at the maximum rate, k. In turn, if $V < V^*(K)$ then further investments are immediately stopped, until V recovers. The optimal cutoff value, $V^*(K)$, is determined jointly with the value of the investment opportunity.

For convenience define the value of the investment opportunity as F(V, K), if construction proceeds at a maximum speed, i.e. $V \geq V^*(K)$. In contrast, f(V, K) defines the value of the investment opportunity if construction is halted, i.e. $V < V^*(K)$.

Following Merton's (1977) approach to evaluate general contingent claims, Majd and Pindyck show that F and f satisfy the differential equations

(2a) $\frac{1}{2}\sigma^2 V^2 F_{VV} + (r - \delta) V F_V - r F - k F_K - k = 0$

(2b) $\frac{1}{2}\sigma^2 V^2 f_{VV} + (r - \delta) V f_V - r f = 0$

subject to the boundary conditions

(3a) $F(V, 0) = V$

(3b) $\lim_{V \to \infty} F_V(V, K) = e^{-\delta K / k}$

(3c) $f(0, K) = 0$

(3d) $f(V^*, K) = F(V^*, K)$

(3e) $f_V(V^*, K) = F_V(V^*, K)$

Equation (2a) is a PDE that has no known closed-form solution whereas (2b) is an ordinary differential equation that can be solved analytically. Only (2a) depends on k. Since for (2) numerical solution procedures must be employed, it follows that k can be a function of K, that is, the maximum speed of construction can depend on the stage of the project.

Boundary condition (3a) says that the value of the investment opportunity at the end of construction equals the value of the completed project. (3b) accounts for the fact, that for large values of V, the construction flexibility becomes negligable and the opportunity's value increases by a rate that is adjusted to opportunity costs from delaying completion. (3c) states that the investment opportunity is worthless if V is zero. (3d) and (3e) define the optimal cutoff value, where (3e) is the value-maximizing Merton-Samuelson high contact condition.

The analytical solution to (2b) is the value, f, of the investment opportunity when construction is halted

(4) $f(V, K) = a V^{\beta}$

where $\beta = \frac{-(r - \delta - \sigma^2/2) + \sqrt{(r - \delta - \sigma^2/2)^2 + 2 r \sigma^2}}{2}$

The coefficient a must be determined jointly with the numerical solution of F via the shared boundary conditions (3d) and (3e). They can be simplified to

(5) $F(V^*, K) = V^* / \beta \, F_V(V^*, K)$

Appendix C presents a numerical procedure that approximates the value of F by the explicit finite difference method.

Example

Consider a project with total investment expenditures of K = $ 6 mill., a maximum construction rate of k = $ 1 mill. per year, and thus a construction time of at least 6 years (see Majd and Pindyck (1978), p. 17-25). The real interest rate is r = 2%. The value of the completed project follows (1) with a rate of opportunity cost of □ = 6% for delaying completion and an instantaneous standard deviation of □ = 20% annually.

Table 1 Value of Sequential Real Investment Project (in $ mill.)

Value of completed Project, V	NPV for K = 6	Total remaining investment, K: **6**	**5**	**4**	**3**	**2**	**1**	**0**
42,52	23,95	23,95	26,69	29,56	32,57	35,73	39,04	42,52
36,60	19,83	19,83	22,31	24,91	27,63	30,48	33,46	36,60
31,50	16,28	16,28	18,54	20,90	23,37	25,96	28,66	31,50
27,11	13,23	13,23	15,30	17,46	19,71	22,07	24,53	27,11
23,33	10,60	10,60	12,51	14,49	16,56	18,72	20,98	23,33
20,08	8,34	8,34	10,10	11,94	13,85	15,84	17,92	20,08
17,29	6,39	6,39	8,04	9,74	11,52	13,36	15,29	17,29
14,88	4,72	4,72	6,25	7,85	9,51	11,23	13,02	14,88
12,81	3,28	3,28	4,72	6,22	7,78	9,39	11,07	12,81
11,02	2,00	2,09	3,40	4,82	6,29	7,81	9,39	11,02
9,49	0,97	1,27	2,28	3,62	5,01	6,45	7,94	9,49
8,17	0,05	0,78	1,39	2,58	3,91	5,28	6,70	8,17
7,03	-0,75	0,47	0,85	1,68	2,96	4,27	5,63	7,03
6,05	-1,43	0,29	0,51	1,03	2,14	3,41	4,71	6,05
5,21	-2,02	0,18	0,31	0,63	1,44	2,66	3,91	5,21
4,48	-2,53	0,11	0,19	0,38	0,88	2,02	3,23	4,48
3,86	-2,96	0,07	0,12	0,23	0,53	1,46	2,64	3,86
3,32	-3,34	0,04	0,07	0,14	0,33	0,99	2,14	3,32
2,86	-3,66	0,02	0,04	0,09	0,20	0,60	1,70	2,86
2,46	-3,94	0,01	0,03	0,05	0,12	0,37	1,33	2,46
2,12	-4,17	0,01	0,02	0,03	0,07	0,22	1,01	2,12
1,82	-4,38	0,01	0,01	0,02	0,04	0,14	0,73	1,82
1,57	-4,56	0,00	0,01	0,01	0,03	0,08	0,44	1,57
1,35	-4,71	0,00	0,00	0,01	0,02	0,05	0,27	1,35
1,16	-4,84	0,00	0,00	0,00	0,01	0,03	0,16	1,16
1,00	-4,96	0,00	0,00	0,00	0,01	0,02	0,10	1,00
0,00	-5,65	0,00	0,00	0,00	0,00	0,00	0,00	0,00

Table 1 shows the value of the investment opportunity, F, as a function of the value of the completed project, V, and total remaining investment, K. The entries have been calculated with the procedure in appendix C. The cutoff value, V*(K), is underlined. For example, a project with remaining investment outlays of K = $ 4 mill. has a cutoff value V* = $ 7.03 mill. . In other words, if the project is worth $ 7.03 mill. or more, it pays to invest, otherwise construction profits from being halted until V recovers. The investment opportunity itself is worth $ 1.68 mill. at that instant. The table can be used to make optimal investment decisions during the whole construction period. If investors decided to spend $ 1 mill., the next interesting column would be K = $ 3 mill., etc.

The second column of Table 1 depicts the value of a simple NPV approach to the investment opportunity. Since NPV is characterized by total neglect of construction flexibility, it treats construction as if it would always proceed at the maximum rate. The table shows that NPV never exceeds CCA figures. NPV does not recognize an important source of value when project value is low. Furthermore, NPV approves of construction for values that are not optimal. According to NPV analysis, this investment is realized if the value of a completed project is at least $ 8.17 mill. CCA analysis, however, does not approve construction unless project value equals at least $ 11.02 mill.

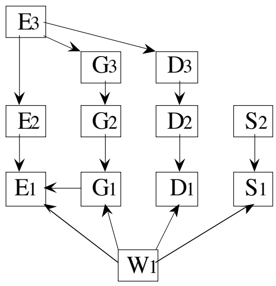


Figure 1 depicts the value of the investment opportunity, F, as a function of the maximum construction rate, k. Many investment projects can be built with alternative construction technologies. They differ in flexibility in terms of maximum construction rate. Technologies offering higher flexibility are usually more expensive, so that higher cost must be compared to increases in the value of the investment opportunity due to higher flexibilty. The value of F increases as k increases. However, the marginal value of F falls as k increases. The value-maximizing degree of flexibiltity can be obtained, when k is increased until the marginal value of increased flexibility, ($\Delta F/\Delta k$), equals the marginal cost to increase flexibility.

Figure 2 shows the cutoff value, V*, as a function of the opportunity cost, □. As □ increases from 1% to 6% the cutoff value reduces significantly. However, if □ is increased further, the movement of the cutoff value critically depends upon the magnitude of k. For k = 2, the cutoff value remains low for further increases of □, but for k = 0.5, the cutoff value rises again. Hence, for projects with small flexibility (long minimum construction phases), the knowledge of the opportunity cost □ is crucial to the investment decision.

Table 2	Optimal Current Cutoff Value, V*			
Standard Deviation	Annual Rate of Opportunity Cost δ			
σ	**0,03**	**0,06**	**0,09**	**0,12**
0,1	9,49	8,17	9,49	11,02
0,2	14,88	11,02	11,02	12,81
0,3	23,33	17,29	14,88	14,88
0,4	49,40	23,33	20,08	20,08
0,5	77,47	42,52	31,50	27,11

The sensitivity of the investment decision rule is summarized in Table 2 where the optimal cutoff value, V*, is illustrated for different values of σ and δ. V* increases as σ rises, that is, the higher the risk, the lower the incentive to invest. For δ, two reciprocal effects are encountered: Increasing opportunity costs from delaying completion increase the incentive to invest immediately and thus reduce the cutoff value. But the higher δ, the higher the value of foregone profits during construction is. The value of the investment opportunity is lower, the incentive to invest is lower, the cutoff value is higher. For σ = 0.1 and σ = 0.2, the latter effect dominates. An increase in δ from 0.06 to 0.12 increases the cutoff value.

5.5 Project Financing

Introduction and Summary

For major long term investments, interactive effects between financial and real options can be quite strong when uncertainty about project value is involved. Project financing takes both finanical and real options, into account and represents a further step towards overcoming methodical deficiencies of traditional NPV techniques. Project financing was first employed to evaluate natural resource developments (see Mason and Merton (1985)) and alternative types of energy production (see Baldwin, Mason, and Ruback (1983)). Where scale of operations and financings were large, project life was long and uncertainty about future prices was high. For single companies, it is often difficult to decide whether to participate in such a project or not. The present value of invested equity does not only depend on project profitability alone, but also on a variety of contractual agreements with other parties. These agreements include financial questions, such as capital structure, debt indentures, repayment, and loan or price guarantees, as well as growth and operating options. Each party, like investors, banks, or governments, is quite interested in understanding the consequences of these agreements on the value of their investment.

Even though evaluations of project financings can become quite complex, CCA offers a general framework to derive an individually tailored model, that generates numerical approximations of the value of the project's financial components. As a result, it is not only possible to assess traded assets, e.g. equity, but also to determine the price of non-traded asset such as government price guarantees.

The Model

Every investment project is financed under different circumstances, with different overall goals, and accordingly, through different means. For projects with more than one paricipant, the circumstances, goals, and means of each individual profoundly affect those of the group. Every project financing package differs in terms of financial and operating options. As a result, each project financing must be individually modelled. No global standard formula for a project's value exists. Guidelines applicable to any project, on the other hand, do exist.

Following Merton's (1977) approach, one must first identify all relevant contingent claims. Thereafter, one can determine the specific boundary conditions and payout functions from the project and the financial components. The final step, sucessively calculating the value of all financial components by numerical methods, follows in cours.

It is crucial to the success of this approach, that uncertainty about future project values stem from a single, observable source. It is assumed, that project value, V, follows a geometric Brownian motion, as thoroughly explained in section 2.3. .

(1) $$\frac{dV}{V} = (\alpha - \delta)\, dt + \sigma\, dz$$

Since each financial component is contingent upon project value, the components' value can be described by a generalized PDE. The generalized PDE for arbitrary financial claims, W, is

(2) $$\frac{1}{2}\sigma^2 V^2 W_{VV} + (r - \delta)\, V W_{VV} + W_t - r W + \delta_W = 0$$

subject to three claim-specific boundary conditions. The function δ describes the net payout of the project, and δ_W accounts for the specific pay-in / pay-out situation of the claim. Since numerical procedures solve for the value of the claim, the parameters σ^2, r, δ, and δ_W can be continuous or discrete functions of project value, V, and calendar time, t.

Example

Mason and Merton (1985), p. 39-49, present a hypothetical case study, whose purpose is to decide, whether a frim should join a consortium, that has the opportunity to develop a large mineral resource base. Mason and Merton identify all relevant contingent claims and specify boundary conditions and payout functions, to tailor the guidelines to the project under consideration. Appendix D outlines a numerical method, which can be employed to calculate all financial components' value, in succession.

Equity, E, senior debt, D, and subordinated junior debt, J, finance the project. For the junior debt, equityholders issue an irrevocable loan guarantee, G, which is satisfied under all circumstances. Junior debt is riskless, because it is guaranteed. Further, the host government provides a price guarantee, S, for a certain period of operations. Three distinct phases characterize the project, a construction phase until time t_c, a first operating phase under host goverment's price guarantee until time t_p, and a second operating phase without price guarantee which lasts at least until time t_d, the date at which the last debt is repaid.

During phase one, the construction period, total investment outlays are predetermined by the schedule I(t). Of this amount, equityholders provide $I_E(t)$ and junior and senior debt finance the remainder.

Until construction is complete, the project generates no cash flows. Therefore, $\square_1(V, t) = 0$, and abandonment is determined from the value of the incomplete project, W. If the value of the incomplete project falls to zero, equityholders will abandon, i.e. $W(V^c(t)) \leq 0$. Thus, $V^c(t)$ is the schedule of critical project values that affects all financial components.

The corresponding pay-in / pay-out functions for equity, loan guarantee, senior debt, price guarantee, and value of incomplete project during phase one are

(3) $\square_{E1} = - I(t) - G1(V, t)$

(4) $\square_{G1} = 0$

(5) $\square_{D1} = DPS(t)$

(6) $\square_{S1} = 0$

(7) $\square_{W1} = - I(t)$

Equityholders not only pay for construction according to $I_E(t)$, but also provide the loan guarantee to junior debt during this phase which is reflected in (3).

The boundary conditions for equity, loan guarantee, senior debt, price guarantee, and value of incomplete project during phase one are

(8a) $E1(V^c(t), t) = \max (W(V^c(t), t) - DPS(t) - BPS(t) - DPJ(t) - BPJ(t), 0)$

(8b) $E1_V(\infty, t) = 1$

(8c) $E1(V, t_c) = E2(V, t_c)$

(9a) $G1(V^c(t), t) = \max (\min (DPJ(t) + BPJ(t), DPJ(t) + BPJ(t) + DPS(t) + BPS(t) - W1(V^c(t), t)), 0)$

(9b) $G1_V(\infty, t) = 0$

(9c) $G1(V, t_c) = G2(V, t_c)$

(10a) $D1(V^c(t), t) = \min (W1(V^c(t), t)), BPS(t) + DPS(t))$

(10b) $D1(\infty, t) = RBS(t)$

(10c) $D1(V, t_c) = D2(V, t_c)$

(11a) $S1(V^c(t), t) = 0$

(11b) $S1_V(\infty, t) = 0$

(11c) $S1(V, t_c) = S2(V, t_c)$

(12a) $W1(V^c(t), t) = 0$

(12b) $W1_V(\infty, t) = 1$

(12c) $W1(V, t_c) = V$

The terminal conditions of E_1, G_1, G_1 and S_1 are the result of calculations in phase two.

In phase two, the first operation period, the cash flow accrues to all parties - appropriate from contractual agreements. The first share is paid to senior debt, then junior debt is served, and finally equityholders receive the remainder as a dividend. If cash flow is insufficient to cover payments due to senior debt, DPS(t), then equityholders can either choose to make up the cash flow shortfall or to abandon the project. In case of abandonment, principal to senior (junior) debt, BPS (BPJ), and all debt payments are immediately due. If cash flow is insufficient to cover payments due to junior debt, DPJ(t), then the loan guarantee must make up the cash flow shortfall. During the first operation phase, government's price guarantee takes the form of guaranteed minimum cash flows, CF(t), that are assumed to cover at least DPS(t).

In the second phase cash flows from the project are ($\square_2(V, t)\ V$). During this phase, a minimum cash flow, CF(t), guarantees at least all debt payments due to senior debt, so abandonment is impossible and $V^c(t) = 0$.

The corresponding pay-in / pay-out functions for equity, loan guarantee, senior debt, and price guarantee during phase two are

(13) $\square_{E2} = \max(\max(\square_2 V, CF(t)) - DPS(t) - DPJ(t)), 0)$

(14) $\square_{G2} = \max(DPS(t) + DPJ(t) - \max(CF(t), \square_2 V), 0)$

(15) $\square_{D2} = DPS(t)$

(16) $\square_{S2} = \max(CF(t) - \square_2 V, 0)$

The boundary conditions for equity, loan guarantee, senior debt, and price guarantee during phase two are

(17a) $E2(0, t) = PVE(t)$

(17b) $E2_V(\infty, t) = 1$

(17c) $E2(V, t_p) = E3(V, t_p)$

(18a) $G2(0, t) = PVG(t)$

(18b) $G2_V(\infty, t) = 0$

(18c) $G2(V, t_p) = G3(V, t_p)$

(19a) $D2(0, t) = PVD(t)$

(19b) $D2(\infty, t) = RBS(t)$

(19c) $D2(V, t_p) = D3(V, t_p)$

(20a) $S2(0, t) = PVS(t)$

(20b) $S2_V(\infty, t) = 0$

(20c) $S2(V, t_p) = 0$

where PVE(t), PVG(t), PVD(t), and PVS(t) are the present values under certainty for equity, loan guarantee, senior debt, and price guarantee, i.e. the present values of max (CF(t) - DPS(t) - DPJ(t), 0), of max(DPS(t) + DPJ(t) - CF(t), 0) plus BPJ(t_p), of DPS(t), and of CF(t) evaluated out to t_p, respectively. The terminal boundary conditions for E_2, G_2, and D_2 are the result of the respective calculations in phase three, which assures continuity at the border between phases.

In phase three, the second operation period, cash flows from the project are ($\square_3$(V, t) V). Depending upon the amount of cash flows, there are three possible payment situations for equity and loan guarantee: 1.) If cash flow is less than payments currently due to senior debt, then equity can choose to make up the shortfall or to abandon; loan guarantors must settle the payments due to junior debt. 2.) If cash flow is sufficient to satisfy payments currently due to senior, but not also junior debt, then equityholders do not pay or receive anything; loan guarantors must settle the payments due to junior debt. 3.) If cash flow is more than payments currently due to both senior and junior debt, then equity receives the remainder as dividend; loan guarantors do not have to pay anything.

Thus, the pay-in / pay-out functions for equity, loan guarantee, and senior debt during phase three are

(21) $\square_{E3}$ = max (min ($\square_3$ V - DPS(t), 0), $\square_3$ V - DPS(t) - DPJ(t))

(22) $\square_{G3}$ = max (min (DPJ(t), DPJ(t) + DPS(t) - $\square_3$ V), 0)

(23) $\square_{D3}$ = DPS(t)

Whenever, the market value of equity falls below the amount of payments due to senior debt, rational acting equityholders will abandon the project. The corresponding schedule of critical project values, $V^c(t)$, determines a lower free boundary of the problem. If project value falls below the critical value, i.e. $V < V^c(t)$, then all debt turns due immediately.

Thus, the boundary conditions for equity, loan guarantee, and senior debt during phase three are

(24a) $E3(V^c(t), t) = \max (V^c(t) - DPS(t) - DPJ(t) - BPS(t) - BPJ(t),0)$

(24b) $E3_V(\infty, t) = 1$

(24c) $E3(V, t_d) = \max (V - DPS(t_d) - DPJ(td), 0)$

(25a) $G3(V^c(t), t) = \max (\min (DPJ(t) + BPJ(t), DPS(t) + BPS(t) + DPJ(t) + BPJ(t) - V^c(t)), 0)$

(25b) $G3_V(\infty, t) = 0$

(25c) $G3(V, t_d) = \max (\min (DPJ(t_d), DPS(t_d) - V), 0)$

(26a) $D3(V^c(t) = \min (V^c(t), DPS(t) + BPS(t))$

(26b) $D3(\infty, t) = RBS(t)$

(26c) $D3(V, t_d) = \min (V, DPS(t_d))$

where RBS(t) is the value of risk-free debt with the same terms as senior debt.

In order to numerically solve for the current value of equity, E, loan guarantee, G, senior debt, D, price guarantee, S, and value of the incomplete project, W, the interactive character of the project must be considered. The interrelations between the financial components as depicted in Figure 1 determine the succession of calculations. For example, to solve for the current value of equity, E, first E_3, E_2, G_3, G_2, G_1, W_1 , and E_1 must be determined.

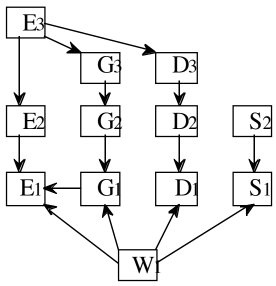


Figure 1

For hypothetical figures, the value of all relevant financial components has been calculated by an explicit finite difference approach as presented in Appendix D. Table 1 displays quarterly input figures of the example.

Table 1 Quarterly Financial Positions During Construction

t	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
IT(t)	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
IE(t)	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25	25
BPS(t)	50	102	155	209	265	323	383	445	508	573	640	710	781	854	930	1008
DPS(t)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
BPJ(t)	25	51	77	104	131	160	189	218	249	280	312	345	379	413	448	485
DPJ(t)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

During construction assume that quarterly CAPEX, IT(t), is $ 100 mill.This sum is financed to 25% by equity, 50% by senior debt, and 25% by junior debt. Since no cash flows are generated during this phase, interest payments of 12% due to senior, and of 10% due to junior debt are not paid, rather cumulate.

Table 2 Quarterly Financial Positions During Operations under Price Guarantees

t	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
BPS(t)	1008	1008	1008	1008	1008	1008	1008	1008	1008	1008	1008	1008	1008	1008	1008	1008
DPS(t)	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30
BPJ(t)	485	485	485	485	485	485	485	485	485	485	485	485	485	485	485	485
DPJ(t)	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12
CF(t)	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100

During the first operating phase, the second phase overall, goverment's price guarantee takes the form of guaranteed quarterly cash flows of $ 100 mill. which are sufficient to cover debt payments currently due, and therefore abandonment during this phase is impossible.

Table 3 Quarterly Financial Positions During Operations until Repayment of Debt

t	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
BPS(t)	1008	1008	1008	1008	1008	1008	1008	1008	1008	1008	1008	1008	1008	1008	1008	0
DPS(t)	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	1038
BPJ(t)	485	485	485	485	485	485	485	485	485	485	485	485	485	485	485	0
DPJ(t)	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	497

During the second operating phase, the third phase overall, debt payments currently due and bond principal due at t_d have to be paid from cash flows that accrue exclusively from operations.

As depicted in Figure 2 all financial components are affected by changes in project value. Equity profits most from increases in project value. However, if project value falls below $ 1166 mill., equity will abandon, which affects the value of all other financial components. If project value drops far enough, the incomplete project, senior debt, price guarantee, and loan guarantee become worthless. As project value becomes large, senior debt approaches its risk-free equivalent, and both guarantees become worthless.

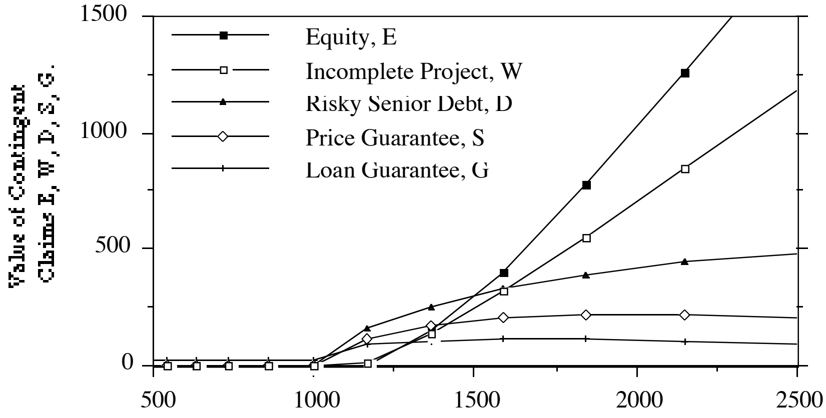


Figure 2

Extensions

With this base model changes in the contractual agreements can easily be considered. Assume, for example, that the consortium was able to pursuade the host government to guarantee for junior debt, then only a slight change in the model accounts for the new situation. Simply replace equation (3) by

(27) $\square_{E1} = - IE(t)$

However, if host government insists on completing the project it might take away the abandonment option in phase one by urging the consortium to issue a surety bond in amount of equity contribution in return for governments price guarantee. This situation can be captured by replacing (8a) by

(28) $E1(0, t) = - PVI(t)$

where PVI(t) is the present value of obligatory contribution for construction funds by equity.

Another case, that government offered the consortium to double the scale of the project at time t_d at a cost of I_D, would be easy to incorporate simply by replacing (24c)

(29) $E3(v, t_d) = \max(\max(V - t_d, 0) + V - DPS(t_d) - DPJ(t_d), 0)$

For each modification in contractual agreements the effect on the value of all financial components of the project can easily be determined. In particular, for complicated projects the advantages of this CCA approach cannot be overestimated since most other valuation techniques fail to model the situation in a satisfactory manner.

5.6 Conclusion

Real options describe the flexibility inherent to real investment projects. From project to project, the quantity and quality of real options can vary considerable. It is therefore essential to choose the model that best reflects the project's flexibility.

For real investments, three different groups of models are available, in general. These are models for operating options, investment opportunities, and project financings.

Operating options belong to already existing projects. Among operating options, that to shut down temporarily and that to abandon are the most important. They provide the firm the possibility to limit potential losses.

Investment opportunities represent real investments that have yet to be carried out. In this category of real options, growth options possess the highest relevance, since they assess the value of potential future expansion. Growth options can sucessively be contingent on each other, so that one can apply them differently according to the number of exercise moments in time. Besides growth options, there are several other applications that evaluate indirect effects on investment opportunities, such as construction time flexibility and the flexibility to meet constantly increasing demand.

Project financing reflects a more general approach to assessing flexibility and the contingent claims character of real investments. Using a set of generalized PDEs, it is not only possible to evaluate both financial and real investments simultaneously, but also their interrelations in single projects.

The general technique for evaluating real options quantitatively is Contingent Claims Analysis (CCA). Unlike traditional, static methods, such as Net Present Value (NPV) and Discounted Cash Flow (DCF), CCA explicitly recognizes flexibility as an essential source of value, and therefore can be used to quantitatively assess real investments containing flexibility, or real options.

Recent CCA applications for real investments show, that the part of project value attributable to real options can be considerable. For many real investments, both NPV and DCF techniques significantly underestimate the projects *true* value, because they entirely ignore the value of real options. Furthermore, CCA is capable providing concrete decision rules for timing the exercise of real options. It is thus possible to determine a schedule of critical conditions, under which it is optimal to shut down, abandon, or invest.

Although CCA has immense advantages, some difficulties arise when CCA is applied to real options.

In the context of real options, the difficulties arise for two main reasons. First, real investments are seldom explicitly traded assets. Second, real investments are not always held exclusively for speculative purposes.

If real investments are not traded assets, riskless trading strategies become an impossibility. As a consequence, different portfolio strategies must be employed which involve estimation procedures for risk-adjusted rates of return.

If real investments are not held for speculative mainly purposes but rather for consumption, their return can fall below the return of otherwise identical financial investments. To overcome this difference, the return, or that is the growth rate of real investment value, must be estimated.

Both estimates are crucial inputs for real option pricing models. They are also difficult to obtain. In some cases, futures markets' estimates for both rates facilitate the problem's solution.

The following summarizes the main properties of the previous models.

In the group of operating options, McDonald and Siegel (1985) present a simple approach to analytically evaluate operations when temporary shut down is possible. Although, the model adequately captures the main effects of uncertainty on investment decisions, it ignores a few important factors: Shut down and resumption are costless, fixed cost are not considered at all, and project life is independent on intensity of usage.

Dixit (1989) introduces a more advanced model to assess operations with the possibility of shutting down temporarily. Even though his model ignores project's limited life and can be solved only iteratively, it provides not only the value of the real option, but also yields a decision rule, which determines under what conditions shut down or resumption is optimal.

For abandonment options, Myers and Majd (1983) derive an approach, that can only be solved with a numerical approximation as demonstrated in appendix A. Their model is general, but does not allow the possibility that both project and salvage value are prices of non-traded assets. Furthermore, the author of this paper could not confirm their numerical examples, because important input data were omitted.

Brennan and Schwartz (1985) outline a very general model that recognizes both temporary shut down and abandonment. The solution to the model involves a complex numerical approximation procedure, which is discussed in appendix B. The model yields the value of the project and an optimal operating policy for responding to current output prices. In principle, the author of this paper could confirm the numerical examples of Brennan and Schwartz, but due to limited computational capacity the author's calculations have only an approximative character.

In the group of investment opportunities, Kemna (1987) presents a closed-form solution for a single stage growth option. Although the approach captures the main effects of uncertainty on investment opportunities, it represents a simplistic growth option, that only allows for investing a fixed amount at a predetermined date. Formally, her model resembles a European call option on a futures contract.

McDonald and Siegel (1986) develop a different closed-form solution to single stage growth options. In their model, growth options can be exercised prematurely, have random lives, and random exercise prices. Moreover, their model provides an optimal decision rule for when to exercise the growth option, that is when to invest.

Kemna (1987) treated a simple two stage growth option. Her model resembles a compound option on a futures contract and has properties similar to her single stage approach.

Pindyck (1988b) combines multi stage growth options and temporary shut down options to derive a microeconomic model for the value of entire firms. He concentrates on capacity choice, utilization, and firm value in terms of marginal irreversible investment decisions. Even though the model illustrates the main effects of uncertainty on investment decsions, it contains a few simplifications: Investments are incremental, there are no delivery lags for capacity installation, and installed capacity does not depreciate.

In the group of investment opportunities, Pindyck (1988a) develops a model for evaluating the degree of flexibility to meet constantly increasing demand with alternative production technologies. He provides an analytical solution for the value of flexibility and a decision rule for when to invest. Although the model presents a way to assess production flexibility, it focuses on a narrow aspect of the problem and treats changes in demand as deterministic.

In another model, which examines indirect effects on the value of investment opportunities, Majd and Pindyck (1987) show how to assess construction time flexibility. Many real investments require a certain construction time before completion, during which construction can be stopped and resumed. Majd and Pindyck make a few simplifying assumptions: construction can only take place at a single velocity, interruption and resumption are costless, and stopped projects incur no maintenance costs. They show, that construction time flexibility can constitute a significant part of project's value. In addition, the author of this paper could generally be confirm their numerical examples using a numerical procedure as described in appendix C.

Finally, project financing includes both financial and real options and focuses on interactive effects between both option types. The framework of Mason and Merton (1985) presents a very general approach to deriving individually tailored models, that generate numerical approximations of the value of the project's financial components. In particular, it is possible to determine the consequences of contractual agreements between the parties involved in the financing. Due to the complex structure of many project financings, it is sometimes costly to develop respective individual formal models. Appendix D shows how to compute the value of project financings.

In summary, the models to evaluate real options have quite different qualities. A few of them are very simple and limited and therefore give only a rough idea of the influence of uncertainty on real investments. Others focus on very specific properties of real investments and thus reveal only particular aspects of projects. Moreover, nearly all real option pricing models are faced with considerable difficulties concerning the estimation of risk-adjusted discount and growth rates. Although many models have defects, the advanced ones prove more accurate than currently employed approaches such as NPV and DCF techinques, nevertheless.

Appendix: Numerical Prodecures to Evaluate Real Options

All four appendices introduce a numerical approximation technique, the explicit finite difference method. The context in which it's applied, however, differs from appendix to appendix. Not only the general goal, but numerous boundary conditions make each treatment unique, although the basic concept remains the same. Each is intended to run through the computational aspect of its appropriate model. The mathematical derivation is followed by a short description of the numerical algorithm, a PASCAL program, and the ouput of the program.

Appendix A: Abandonment

This appendix shows how to solve for Myers and Majd´s (1983) approach to abandonment value by the explicit finite difference method. The function F of the abandonment option value relative to the salvage value must satisfy the following PDE.

(1) $$\frac{1}{2}\sigma^2 W^2 F_{WW} + (\delta_S - \delta_P)\, W F_W - \delta_S F + F_t = 0$$

subject to the boundary conditions

(2a) $F(W^c(t), t) = \max(1 - W, 0)$

(2b) $F_W(W^c(t), t) = 0$

(2c) $\lim_{V \to \infty} F(V, t) = 0$

(2d) $F(W, t^*) = \max(1 - W, 0)$

A logarithmic transformation simplifies further calculations.

(3) $X = \ln W$

$$F_{WW}(W, t) = (G_{XX}(X, t) - G_X(X, t))\, e^{-2X}$$

$$F_W(W, t) = G_X(X, t)\, e^{-X}$$

$$F_t(W, t) = G_t(X, t)$$

The PDE and the boundary conditions become

(4) $$\frac{1}{2}\square^2 G_{XX} + (\square_S - \square_P - \square^2/2)\, G_X - \square_S G + G_t = 0$$

(5a) $$G(X^c(t), t) = \max(1 - e^X, 0)$$

(5b) $$G_X(X^c(t), t) = 0$$

(5c) $$\lim_{X \to \infty} G(X, t) = 0$$

(5d) $$G(X, t^*) = \max(1 - e^X, 0)$$

The explicit finite difference method replaces continuous variables by discrete ones and partial differentials by finite differences in a specified way. For $G(X, t) = G(i\Delta X, j\Delta t) = G_{i,j}$, where $0 < i < m$ and $0 < j < n$, the following substitutions apply.

(6) $$G_{XX} \approx \frac{G_{i+1,\,j-1} - 2\,G_{i,\,j-1} + G_{i-1,\,j-1}}{\Delta X^2}$$

$$G_X \approx \frac{G_{i+1,\,j-1} - G_{i-1,\,j-1}}{2\,\Delta X}$$

$$G_t \approx \frac{G_{i,\,j} - G_{i,j-1}}{\Delta t}$$

The PDE is now discrete and has become a difference equation.

(7) $$G_{i,j} = \frac{1}{(1 + \square_S \Delta t)}\,[\,a\,G_{i-1,\,j-1} + b\,G_{i,\,j-1} + c\,G_{i+1,\,j-1}]$$

where

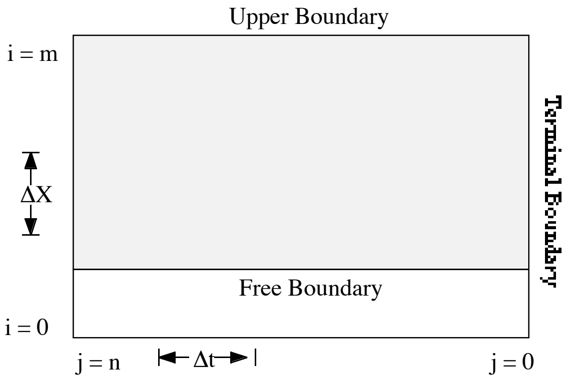
(8) $$a = \frac{\Delta t}{2\,\Delta X}\left[\frac{\square^2}{\Delta X} - (\square_S - \square_P - \square^2/2)\right]$$

$$b = 1 - \frac{\Delta t\,\square^2}{\Delta X^2}$$

$$c = \frac{\Delta t}{2\,\Delta X}\left[\frac{\square^2}{\Delta X} + (\square_S - \square_P - \square^2/2)\right]$$

The coefficients a, b, and c sum to 1, and are independent of i. In order to avoid instabilities in the calculation of G, it is necessary that a, b, and c be non-negative, which can be assured by appropriate choice of ΔX and Δt.

The solution procedure is a backward dynamic programming algorithm as demonstrated in the following picture.



Project Value-Time Space for G

First, for j = 0, the terminal boundary condition is set by using (5d). Then, for j = 1, and i = m, the upper boundary is fixed with (5c). For i = m-1, m-2, ... the values of $G_{i,j}$ are calculated with (7). Since for some i < m it may pay to abandon early, e.g. for low X, $G_{i,j}$ is the maximum between the value of immediate exercise and the value of the option, if maintained. If $G_{i,j}$ for (5a) is greater than $G_{i,j}$ for (7), the free boundary has been reached. The values between free boundary and zero are equivalent to the respective exercise values. The procedure then continues with j = j+1 until j = n.

```
program abandonment;                          {  numerical approximation for the value of abandonment           }
 uses                                         {  according to a model of s. c. myers and s. majd [1983], copyright by }
  definitions, functions, output, user;       {  christian von drathen, potosistrasse 9, 2000 hamburg 55, germany }
 var                                          {  files included: program, def.lib, f.lib, io.lib, u.lib          }
  abandon: fpt;                               {  project value-time space                                        }
begin                                         {  numerical approximation by an explicit finite difference approach }
 abandon := explicit(gp, gs, f, h, f, f, vp, vs, rho);
 print_field(abandon);                        {  output of project value-time space                              }
end.
```

```
unit definitions;                                  {  file belongs to program abandonment                              }
interface                                          {  contains global constants, variables, structures, and functions  }
 const
  max_y = 33;                                      {  number of points in y dimension                                  }
  max_x = 281;                                     {  number of  points in x dimension                                 }
  y_max = 20;                                      {  largest value in y dimension                                     }
  x_max = 70;                                      {  largest value in x dimension                                     }
  row_max = 2;                                     {  largest output value in y dimension                              }
  row_min = 0;                                     {  smallest output value in y dimesion                              }
  col_num = 8;                                     {  number of columns in output                                      }
  col_max = 70;                                    {  largest output value in x dimesion                               }
  col_min = 0;                                     {  smallest output value in x dimension                             }
  digits = 8;                                      {  number of digits in output                                       }
  decimals = 4;                                    {  number of decimals in output                                     }
  unit_y = 5;                                      {  scale factor for labels in y dimension                           }
  unit_x = 1;                                      {  scale factor for labels in x dimension                           }
  unit_xy = 5;                                     {  scale factor for values in value-time space                      }
 type
  index_y = 1..max_y;                              {  index for y dimesion in field                                    }
  index_x = 1..max_x;                              {  index for x dimesion in field                                    }
  field = array[index_y, index_x] of extended;     {  field contains stock price-time space                            }
  fpt = ^field;                                    {  pointer to field                                                 }
 var
  dy: extended;                                    {  increment in y dimension                                         }
  dx: extended;                                    {  increment in x dimension                                         }

 function max (num1, num2: extended): extended;    {  function maximum                                                 }
 function min (num1, num2: extended): extended;    {  function minimum                                                 }

implementation
 function max;
 begin
  if num1 >= num2 then
   max := num1
  else
   max := num2;
 end;
 function min;
 begin
  if num1 <= num2 then
   min := num1
  else
   min := num2;
 end;

end.
```

```
unit functions;                          {  file belongs to program abandonment                }

interface                                {  contains function with main algorithm              }
 uses
  definitions;
                                         {  parameters: }
 function explicit (function gp (i1: index_y;    {  gamma project value                        }
      j1: index_x): extended;  function gs (i2: index_y;     {  gamma slavage value           }
      j2: index_x): extended;  function l (i3: index_y;      {  lower boundary                }
      j3: index_x): extended;  function h (i4: index_y;      {  upper boundary                }
      j4: index_x): extended;  function f (i5: index_y;      {  terminal boundary             }
      j5: index_x): extended;  function ex (i6: index_y;     {  exercise value                }
      j6: index_x): extended;  function vp (i7: index_y;     {  variance project value        }
      j7: index_x): extended;  function vs (i8: index_y;     {  variance salvage value        }
      j8: index_x): extended;  function rho (i9: index_y;    {  correlation rho               }
      j9: index_x): extended): fpt;
implementation

 function explicit;
  var
   panel: fpt;                           {  project value-construction time space              }
   i: index_y;
   j: index_x;
   s2: extended;                         {  variance   }
   prob1, prob2, prob3: extended;        {  see appendix a: eq. 7 & 8                          }
   disc: extended;                       {  discount factor                                    }
   exercise: extended;                   {  exercise value                                     }
   const1, const2: extended;             {  gamma project value, gamma salvage value           }
 begin
  new(panel);                            {  initialize   }
  dy := ln(y_max * unit_y) / (max_y - 1);
  dx := x_max * unit_x / (max_x - 1);
  for i := 1 to max_y do
   for j := 1 to max_x do
    panel^[i, j] := 0;
  for i := 1 to max_y do
   panel^[i, 1] := f(i, 1);
  const1 := gp(i, j);
  const2 := gs(i, j);
  s2 := vp(i, j) - 2 * rho(i, j) * sqrt(vp(i, j) * vs(i, j)) + vs(i, j);
  disc := 1 + dx * const1;
  prob1 := dx / (2 * dy) * (s2 / dy - (const1 - const2 - s2 / 2));
  prob2 := 1 - s2 * dx / sqr(dy);
  prob3 := dx / (2 * dy) * (s2 / dy + (const1 - const2 - s2 / 2));
  if (prob1 < 0) or (prob2 < 0) or (prob3 < 0) then
   begin                                 {  stable results guaranteed ?                        }
    writeln('WARNING: Negative Probabilities in EXPLICIT');
    writeln('prob1:', prob1 : digits : decimals);
    writeln('prob2:', prob2 : digits : decimals);
    writeln('prob3:', prob3 : digits : decimals);
   end;
  for j := 2 to max_x do
   begin
    panel^[max_y, j] := h(max_y, j);     {  upper boundary                                     }
    for i := max_y - 1 downto 2 do
     begin
      panel^[i, j] := (prob1 * panel^[i - 1, j - 1] + prob2 * panel^[i, j - 1] + prob3 * panel^[i + 1, j - 1]) / disc;
      exercise := ex(i, j);
      if panel^[i, j] < exercise then    {  free boundary reached ?                            }
       panel^[i, j] := exercise;
     end;
    panel^[1, j] := l(1, j);
   end;
```

```
  explicit := panel;
 end;

end.
```

```
unit output;                              {   file belongs to program abandonment     }

interface                                 {   contains output procedures              }
 uses
  definitions, user;

 procedure print_field (z: fpt);          {   prints stock price-time space           }

implementation
 function maxi (index1, index2: integer): integer;
 begin
  if index1 >= index2 then
   maxi := index1
  else
   maxi := index2;
 end;
 function mini (index1, index2: integer): integer;
 begin
  if index1 <= index2 then
   mini := index1
  else
   mini := index2;
 end;

 procedure print_field;
  var
   x: set of index_x;
   y: set of index_y;
   i: index_y;
   j: index_x;
   n: integer;
   h: extended;

 begin
  writeln('ABANDONMENT VALUE');
  writeln('');
  writeln('(gamma p = ', gp(1, 1) : 4 : 2, ', gamma s = ', gs(1, 1) : 4 : 2, ', var p = ', vp(1, 1) : 4 : 2, ', var s = ', vs(1, 1) : 4 : 2, ',
rho = ', rho(1, 1) : 4 : 2, ' )');
  writeln('');
  x := [];
  case col_num of
   1:
    x := [max_x];
   2:
    x := [1, max_x];
   otherwise
    for n := 0 to col_num - 1 do
     x := x + [mini(maxi(round(max_x - (col_min + n * (col_max - col_min) / (col_num - 1)) * unit_x / dx), 1), max_x)];
  end;
  y := [];
  for i := 1 to max_y do
   begin
    h := exp((i - 1) * dy) / unit_y;
    if (h <= row_max) and (h >= row_min) then
     y := y + [i];
   end;
  write('G(X,t) |' : (digits + 1));
  for j := max_x downto 1 do
   if j in x then
    write(((max_x - j) * dx / unit_x) : digits : decimals);
  writeln('');
  for n := digits downto 1 do
   write('-');
```

```
  write('+');
  for j := max_x downto 1 do
   if j in x then
    for n := digits downto 1 do
     write('-');
  writeln('');
  for i := max_y downto 1 do
   if i in y then
    begin
     write((exp((i - 1) * dy) / unit_y) : digits : decimals, '|');
     for j := max_x downto 1 do
      if j in x then
       write((z^[i, j] / unit_xy) : digits : decimals);
     writeln('');
    end;
  writeln('');
 end;

end.
```

```
unit user;                          {  file belongs to program abandonment                  }

interface                           {  contains user defined functions for specific application}
 uses
  definitions;


 function gp (i: index_y;           {  gamma project value                                  }
      j: index_x): extended;
 function gs (i: index_y;           {  gamma salvage value                                  }
      j: index_x): extended;
 function l (i: index_y;            {  lower boundary                                       }
      j: index_x): extended;
 function h (i: index_y;            {  upper boundary                                       }
      j: index_x): extended;
 function f (i: index_y;            {  terminal boundary                                    }
      j: index_x): extended;
 function vp (i: index_y;           {  variance project value                               }
      j: index_x): extended;
 function vs (i: index_y;           {  variance salvage value                               }
      j: index_x): extended;
 function rho (i: index_y;          {  correlation between project and salvage value        }
      j: index_x): extended;
implementation
 function gp;
 begin
  gp := 0.06;
 end;
 function gs;
 begin
  gs := 0.07;
 end;
 function h;
 begin
  h := 0;
 end;
 function f;
 begin
  f := max(1 * unit_y - exp((i - 1) * dy), 0);
 end;
 function vp;
 begin
  vp := 0.04;
 end;
 function vs;
 begin
  vs := 0.04;
 end;
 function rho;
 begin
  rho := 0.0;
 end;
end.
```

ABANDONMENT VALUE

(gamma p = 0.06, gamma s = 0.07, var p = 0.04, var s = 0.04, rho = 0.00)

G(X,t)	0.0000	10.0000	20.0000	30.0000	40.0000	50.0000	60.0000	70.0000
2.0000	0.1796	0.1793	0.1786	0.1768	0.1715	0.1561	0.1077	0.0000
1.7319	0.2001	0.1999	0.1993	0.1977	0.1930	0.1790	0.1328	0.0000
1.4998	0.2230	0.2229	0.2224	0.2209	0.2168	0.2043	0.1616	0.0000
1.2988	0.2485	0.2484	0.2479	0.2467	0.2432	0.2322	0.1938	0.0000
1.1247	0.2769	0.2768	0.2764	0.2754	0.2724	0.2631	0.2297	0.0000
0.9739	0.3085	0.3084	0.3081	0.3073	0.3048	0.2971	0.2692	0.0261
0.8434	0.3436	0.3436	0.3433	0.3427	0.3408	0.3346	0.3125	0.1566
0.7303	0.3828	0.3827	0.3826	0.3821	0.3806	0.3760	0.3596	0.2697
0.6325	0.4264	0.4264	0.4262	0.4259	0.4249	0.4216	0.4106	0.3675
0.5477	0.4750	0.4749	0.4749	0.4747	0.4740	0.4720	0.4659	0.4523
0.4743	0.5291	0.5290	0.5290	0.5289	0.5286	0.5277	0.5257	0.5257
0.4107	0.5893	0.5893	0.5893	0.5893	0.5893	0.5893	0.5893	0.5893
0.3557	0.6443	0.6443	0.6443	0.6443	0.6443	0.6443	0.6443	0.6443
0.3080	0.6920	0.6920	0.6920	0.6920	0.6920	0.6920	0.6920	0.6920
0.2667	0.7333	0.7333	0.7333	0.7333	0.7333	0.7333	0.7333	0.7333
0.2310	0.7690	0.7690	0.7690	0.7690	0.7690	0.7690	0.7690	0.7690
0.2000	0.8000	0.8000	0.8000	0.8000	0.8000	0.8000	0.8000	0.8000

Appendix B: Temporary Shut Down and Abandonment

This appendix shows how to solve for the value of a facility that can be shut down temporarily or abandoned, as described by Brennan and Schwartz (1985). The explicit finite difference method approximates evaluations for v(S,Q) and w(S,Q) is.

The functions v(S,Q) and w(S,Q) must satisfy the following differential equations

(1a) $\frac{1}{2}\square^2 S^2 V_{SS} + (r \square \square)\, S\, V_S \square q V_Q + q(S \square a) \square\square\square (r + \square_1)\, V = 0$

(1b) $\frac{1}{2}\square^2 S^2 W_{SS} + (r \square \square)\, S\, W_S \square f \square (r + \square_0)\, W = 0$

subject to the boundary conditions

(2a) $W(S_0^*, Q) = 0$

(2b) $W_S(S_0^*, Q) = 0$

(2c) $V(S_1^*, Q) = \max\,(W(S_1^*, Q) \square k_1(Q), 0)$

(2d) $V_S(S_1^*, Q) = \square \begin{matrix} W_S(S_1^*, Q) & \text{if } V(S_1^*, Q) > 0 \\ 0 & \text{if } V(S_1^*, Q) = 0 \end{matrix}$

(2e) $W(S_2^*, Q) = V(S_2^*, Q) \square k_2(Q)$

(2f) $W_S(S_2^*, Q) = V_S(S_2^*, Q)$

(2g) $V(S, 0) = 0$

(2h) $W(S, 0) = 0$

The coefficients r, $\square$, $\square_1$, $\square_2$, $\square$, and f are constants and a is a function of Q. The free boundaries S_0^*, S_1^*, and S_2^* are implicit and must be determined jointly with the solution of v and w. Matters simplify somewhat (as in appendix A) when variables are log-transformed

(3) $X = \ln S$

$V_{SS}(S, Q) = (G_{XX}(X, Q) \square G_X(X,Q))\ e^{\square 2X}$

$V_S(S, Q) = G_X(X,Q)\ e^{\square X}$

$V_Q(S, Q) = G_Q(X, Q)$

$W_{SS}(S, Q) = (H_{XX}(X, Q) \square H_X(X,Q))\ e^{\square 2X}$

$W_S(S, Q) = H_X(X, Q)\ e^{\square X}$

After the transformation the differential equations look like:

(4a) $$\frac{1}{2}\square^2 G_{XX} + (r \square \square \square \square^2 / 2)\, G_X \square q G_Q + q(e^X \square a) \square \square \square (r + \square_1)\, G = 0$$

(4b) $$\frac{1}{2}\square^2 H_{XX} + (r \square \square \square \square^2 / 2)\, H_X \square f \square (r + \square_0)\, H = 0$$

This transformation does not affect the boundary conditions (2a) - (2h) except that each occurrence of S is replaced by e^X. (4a) is a PDE, whereas (4b) is an ordinary differential equation which can be solved analytically. Rearranging leads to:

(5) $$H_{XX} + \left[\frac{2r \square 2\square}{\square^2} \square 1\right] H_X + \frac{2(\square r \square \square_0)}{\square^2} H = \frac{2f}{\square^2}$$

The general solution to the homogeneous part of the ordinary differential equation must be determined via the characteristic equation:

(6) $$\square^2 + \left[\frac{2r \square 2\square}{\square^2} \square 1\right]\square + \frac{2(\square r \square \square_0)}{\square^2} = 0$$

Under the three conditions that 1.) $\square^2 \neq 0$, 2.) $r > 0$, and 3.) $\square_0 \geq 0$, it follows that the solution to (6) is a real number and that $\square_1 \neq \square_2$.

(7) $$\square_{1,2} = \frac{1}{\square^2}\left[\square(r \square \square \square \square^2/2) \pm \sqrt{(r \square \square \square \square^2/2)^2 \square 4(\square^2/2\,(\square r \square \square_0))}\right]$$

Thus, the homogeneous solution H^h has the form

(8) $$H^h = c_1 e^{\square_1 X} + c_2 e^{\square_2 X}$$

The particular solution H^p has the form

(9) $$H^p = \square \frac{f}{r + \square_0}$$

The solution to (4b) is the sum of (8) and (9):

(10) $$H(X) = c_1 e^{\square_1 X} + c_2 e^{\square_2 X} \square \frac{f}{r + \square_0}$$

The coefficients c_1 and c_2 must be determined jointly with the solution of G via the shared boundary conditions. With (2c) and (2e) it follows that

(11a) $$H(X_1^*, Q) = G(X_1^*, Q) + k_1(Q)$$

(11b) $$H(X_2^*, Q) = G(X_2^*, Q) \square k_2(Q)$$

Then

$$\text{(12a)}\quad c_2 = \frac{G(X_1^*, Q) + k_1(Q) + f/(r+\square_0) \square [G(X_2^*, Q) \square k_2(Q) + f/(r+\square_0)]\, e^{\square_1 (X_1^* \square X_2^*)}}{e^{\square_2 X_1^*} \square e^{\square_2 X_2^* + \square_1 (X_1^* \square X_2^*)}}$$

$$\text{(12b)}\quad c_1 = \frac{G(X_2^*, Q) \square k_2(Q) + f/(r+\square_0) \square c_2\, e^{\square_1 X_2^*}}{e^{\square_1 X_2^*}}$$

For the solution of (4a) the explicit finite difference method is employed. It replaces continuous variables by discrete ones and partial derivatives by finite differences. For $G(X, Q) = G(i\Delta X, j\Delta Q) = G_{i,j}$, where $0 < i < m$ and $0 < j < n$, the explicit method substitutes

$$\text{(13)}\quad G_{XX} \square \frac{G_{i+1,\, j\square 1} \square 2\, G_{i,\, j\square 1} + G_{i\square 1,\, j\square 1}}{\Delta X^2}$$

$$G_X \square \frac{G_{i+1,\, j\square 1} \square G_{i\square 1,\, j\square 1}}{2\, \Delta X}$$

$$G_Q \square \frac{G_{i,\, j} \square G_{i,\, j+1}}{\Delta Q}$$

A difference equation approximates the PDE

$$\text{(14)}\quad G_{i,j} = \frac{1}{(1 + \Delta Q / q(r + \square_1))}\, [\, a\, G_{i\square 1,\, j\square 1} + b\, G_{i,\, j\square 1} + c\, G_{i+1,\, j\square 1} + \square_i\,]$$

where

$$\text{(15)}\quad a = \frac{\Delta Q}{2\, q\, \Delta X} \left[\frac{\square^2}{\Delta X} \square (r \square \square \square \square^2 / 2) \right]$$

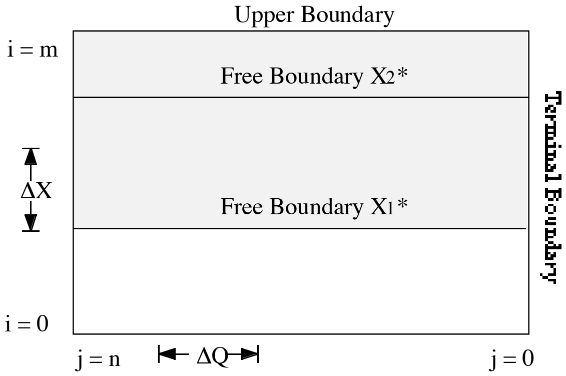
$$b = 1 \square \frac{\Delta Q\, \square^2}{q\, \Delta X^2}$$

$$c = \frac{\Delta Q}{2\, q\, \Delta X} \left[\frac{\square^2}{\Delta X} + (r \square \square \square \square^2 / 2) \right]$$

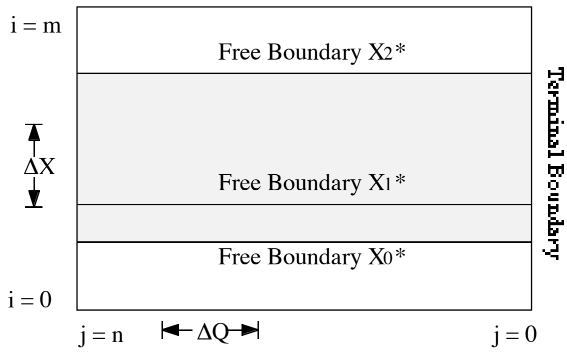
$$\square_i = \frac{\Delta Q}{q}\, [q(e^{i\, \Delta X} \square a) \square \square]$$

The coefficients a, b, and c sum to 1, are indepentent of i, and each of them ought to be non-negative in order to yield stable results in the approximation of G. Non-negativity can be achieved by appropriate choice of ΔX and ΔQ.

The solution procedure is a backward dynamic programming approach as illustrated below.



Output Price-Capacity Space for G



Output Price-Capacity Space for H

First, for j = 0, the terminal boundary for G and H is set using (2g) and (2h). Then, for j = j+1 and i = m, an artifical upper boundary in G is set. This boundary follows from the fact that for high values of X the simple NPV converges to G because operating options become neglegible. For i = m-1, m-2, ... (14) determines subsequent values of G. Since any i < m can be the free boundary, X_2*, each value i' < i is tested whether it is the free boundary, X_1*. Because X_1* and X_2* stem from the shared boundary conditions, equations (12a), (12b), and (10) are used to calculate the corresponding values for H. If boundary conditions (2a), (2b), (2d), (2f) are satisfied then X_0*, X_1*, and X_2* are found, H can be calculated for the values between X_0* and X_2*, and G rects a lower boundary at X_1*. The values of G for X < X_1* follow from (2c) and the values of H for X > X_2* from (2e). The procedure continues for j = j+ 1 until j = n.

```
program shut_down_and_abandonment;          {  numerical approximation for the value of shut down and abandonment }
 uses                                       {  according to a model of m. j. brennan and e. schwartz [1985],  author }
  definitions, functions, input_output;     {  christian von drathen, potosistrasse 9, 2000 hamburg 55, germany   }
 var                                        {  files included: program, def.lib, f.lib, io.lib                    }
  input: dpt;                               {  input data                                                         }
  factor: cpt;                              {  global variables                                                   }
  bound: bpt;                               {  free boundaries                                                    }
  open: fpt;                                {  project value-construction time space                              }
  close: fpt;                               {  project value-construction time space                              }
  i: index_y;
  j: index_x;
begin
 input := get_data(input);                  {  get input data                                                     }
 factor := init_const(input);               {  calculate basic global variables                                   }
 if factor <> nil then
  begin
   bound := init_bound(bound);              {  allocate memory for free boundaries                                }
   open := init_open(open, factor, input);  {  allocate memory for value-capacity space of opened facility        }
   close := init_close(close);              {  allocate memory for value-capacity space of closed facility        }
   for j := 2 to max_x do
    begin
     for i := max_y - 1 downto 2 do
      open := explicit(open, factor, i, j); {  numerical approximation of open by  explicit finite differences    }
     for i := max_y - 1 downto 2 do         {  estimate free boundaries                                           }
      factor := determine(factor, bound, open, input, i, j);
     for i := max_y downto 1 do
      begin                                 {  calculate remaining values in open and close                       }
       close := fill_close(close, bound, factor, i, j);
       open := fill_open(open, close, bound, factor, input, i, j);
      end;
    end;
   print_field(open, factor);               {  output of project value- capacity space for opened facility        }
   print_field(close, factor);              {  output of project value- capacity space for closed facility        }
   print_bound(bound, factor);              {  output of free  boundary vectors                                   }
  end;
end.
```

```
unit definitions;                             { file belongs to program shut down and abandonment           }
interface                                     { contains global constants, variables, structures, and  functions }
 const
  max_y = 21;                                 { number of points in y dimension                             }
  max_x = 51;                                 { number of  points in x dimension                            }
  row_max = 2.5;                              { largest output value in y dimension                         }
  row_min = 0;                                { smallest output value in y dimesion                         }
  col_num = 6;                                { number of columns in output                                 }
  col_max = 150;                              { largest output value in x dimesion                          }
  col_min = 0;                                { smallest output value in x dimension                        }
  digits = 7;                                 { number of digits in output                                  }
  decimals = 2;                               { number of decimals in output                                }
  unit_y = 10;                                { scale factor for labels in y dimension                      }
  unit_x = 1;                                 { scale factor for labels in x dimension                      }
  unit_xy = 10;                               { scale factor for values in value-time space                 }
  eps = 1;                                    { tolerance                                                   }
 type
  index_y = 1..max_y;                         { index for y dimesion in field                               }
  index_x = 1..max_x;                         { index for x dimesion in field                               }
  field = array[index_y, index_x] of extended; { field contains stock price-time space                      }
  fpt = ^field;                               { pointer to field                                            }

  data = record                               { input data                                                  }
    s_max: extended;                          { maximal spot price to be calculated                         }
    q_max: extended;                          { maximal capacity                                            }
    q: extended;                              { output                                                      }
    v: extended;                              { variance of ouput price                                     }
    r: extended;                              { interest rate                                               }
    c: extended;                              { convenience yield                                           }
    a: extended;                              { average production cost                                     }
    f: extended;                              { maintenance cost                                            }
    k1: extended;                             { cost to open                                                }
    k2: extended;                             { cost to close                                               }
    t1: extended;                             { royalties                                                   }
    t2: extended;                             { income tax                                                  }
    l0: extended;                             { tax on value of property when closed                        }
    l1: extended;                             { tax on value of property when open                          }
   end;
  dpt = ^data;                                { pointer to data                                             }

  constants = record                          { global variables                                            }
    dx: extended;                             { increment in x dimension                                    }
    dy: extended;                             { increment in y dimension                                    }
    p0: extended;                             { discount factor                                             }
    p1: extended;                             { see appendix b: eq. (15): a                                 }
    p2: extended;                             { see appendix b: eq. (15): b                                 }
    p3: extended;                             { see appendix b: eq. (15): c                                 }
    p: array[index_y] of extended;            { see appendix b: eq. (15): eta                               }
    a1: extended;                             { see appendix b: eq. (5-7)                                   }
    a2: extended;                             { see appendix b: eq. (5-7)                                   }
    a3: extended;                             { see appendix b: eq. (5-7)                                   }
    a4: extended;                             { see appendix b: eq. (5-7)                                   }
    x1: extended;                             { see appendix b: eq. (7): xi 1                               }
    x2: extended;                             { see appendix b: eq. (7): xi 2                               }
    c1: extended;                             { see appendix b: eq. (12a)                                   }
    c2: extended;                             { see appendix b: eq. (12b)                                   }
   end;
  cpt = ^constants;                           { pointer to constants                                        }

  border = record                             { free boundaries                                             }
    s0: array[index_x] of extended;           { see appendix b: eq. (2): s0                                 }
    s1: array[index_x] of extended;           { see appendix b: eq. (2): s1                                 }
    s2: array[index_x] of extended;           { see appendix b: eq. (2): s2                                 }
```

```
  end;
 bpt = ^border;                        {    pointer to border                                   }

implementation
end.
```

```
unit functions;                              {    file belongs to program shut down and abandonment          }

interface                                    {    contains local functions                                   }
 uses
  definitions;

 function init_const (input: dpt): cpt;
 function init_open (z: fpt; factor: cpt; input: dpt): fpt;
 function init_close (z: fpt): fpt;
 function init_bound (index: bpt): bpt;
 function explicit (v: fpt; factor: cpt; i: index_y; j: index_x): fpt;
 function determine (factor: cpt; bound: bpt; open: fpt; input: dpt; i: index_y; j: index_x): cpt;
 function fill_close (close: fpt; bound: bpt; factor: cpt; i: index_y; j: index_x): fpt;
 function fill_open (open: fpt; close: fpt; bound: bpt; factor: cpt; input: dpt; i: index_y; j: index_x): fpt;

implementation

 function max (a, b: extended): extended;
 begin
  if a >= b then
   max := a
  else
   max := b;
 end;

 function init_const (input: dpt): cpt;      {    sets all global constants                                  }
  var
   factor: cpt;
   i: index_y;
   s: extended;
 begin
  new(factor);
  with factor^ do
   with input^ do
    begin
     dx := q_max / (max_x - 1);
     dy := ln(s_max) / (max_y - 1);
     p0 := 1 + dx / q * (r + l1);
     p1 := dx / (2 * q * dy) * (v / dy - r + c + v / 2);
     p2 := 1 - (dx * v) / (q * sqr(dy));
     p3 := dx / (2 * q * dy) * (v / dy + r - c - v / 2);
     for i := 1 to max_y do
      begin
       s := exp((i - 1) * dy);
       p[i] := dx / q * (q * (s - a) - (t1 * q * s + max(t2 * q * (s * (1 - t1) - a), 0)));
      end;
     a1 := v / 2;
     a2 := r - c - v / 2;
     a3 := -r - l0;
     a4 := f;
     x1 := 1 / (2 * a1) * (-a2 + sqrt(sqr(a2) - 4 * a1 * a3));
     x2 := 1 / (2 * a1) * (-a2 - sqrt(sqr(a2) - 4 * a1 * a3));

     if (p1 < 0) or (p2 < 0) or (p3 < 0) then
      begin                                  {    stable results guaranteed                                  }
       writeln('ERROR: Bad Probabilities');
       writeln('p1:', p1 : 6 : 3);
       writeln('p2:', p2 : 6 : 3);
       writeln('p3:', p3 : 6 : 3);
       init_const := nil;
      end
     else
      init_const := factor;
```

```
  end;
end;

function init_open (z: fpt; factor: cpt; input: dpt): fpt;
 var                                         {    allocate memory for open and set boundaries                }
  t: extended;
  e1, e2: extended;
  f1, f2: extended;
  i: index_y;
  j: index_x;
begin
 new(z);
 for i := 1 to max_y do
  for j := 1 to max_x do
   z^[i, j] := 0;
 with factor^ do
  with input^ do
   for j := 2 to max_x do
    begin
     t := dx * (j - 1) / q;
     e1 := exp(t * ln(1 + l1 + c));
     e2 := exp(t * ln(1 + l1 + r));
     f1 := (e1 - 1) / (e1 * (l1 + c));
     f2 := (e2 - 1) / (e2 * (l1 + r));       {    set upper boundary                                         }
     z^[max_y, j] := q * (1 - t2) * ((1 - t1) * f1 * exp((max_y - 1) * dy) - a * f2);
    end;
 init_open := z;
end;

function init_close (z: fpt): fpt;           {    allocate memory for close                                  }
 var
  i: index_y;
  j: index_x;
begin
 new(z);
 for i := 1 to max_y do
  for j := 1 to max_x do
   z^[i, j] := 0;
 init_close := z;
end;

function init_bound (index: bpt): bpt;       {    allocate memory for free boundaries                        }
 var
  i: index_x;
begin
 new(index);
 with index^ do
  for i := 1 to max_x do
   begin
    s0[i] := 0;
    s1[i] := 0;
    s2[i] := 0;
   end;
 init_bound := index;
end;

function explicit (v: fpt; factor: cpt; i: index_y; j: index_x): fpt;
begin                                        {    calculate value-capacity space component by explicit finite difference  }
 with factor^ do
  v^[i, j] := max((p1 * v^[i - 1, j - 1] + p2 * v^[i, j - 1] + p3 * v^[i + 1, j - 1] + p[i]) / p0, 0);
 explicit := v;
end;
```

```
function determine (factor: cpt; bound: bpt; open: fpt; input: dpt; i: index_y; j: index_x): cpt;
 var                                    {    determine free boundaries and estimate coefficients       }
  v1, v2: extended;                     {    value of open                                             }
  sl1a, sl1b: extended;                 {    slope at price 1                                          }
  sl2a, sl2b: extended;                 {    slope at price 2                                          }
  w0: extended;                         {    value of closed when abandoned                            }
  ws0, ws1, ws2: extended;              {    slopes at free boundaries                                 }
  y0, y1, y2: extended;                 {    prices at free boundaries                                 }
  c1t, c2t: extended;                   {    coefficients see appendix b:  eq. (12a and 12b)           }
  n, m: index_y;
begin
 n := i - 1;
 with input^ do
  with factor^ do
   with bound^ do
    while (n > 1) do
     begin
      v2 := open^[i, j];
      v1 := open^[n, j];
      y2 := (i - 1) * dy;
      y1 := (n - 1) * dy;
      if (v2 >= v1 + k1 + k2) and (exp(y2) >= a) and (exp(y1 - 2 * dy) <= a) then
      begin                             {    s2 => average cost, s1 <= average cost                    }
      c2t := ((v1 + k1) - a4 / a3 - ((v2 - k2) - a4 / a3) * exp(x1 * y1) / exp(x1 * y2)) / (exp(x2 * y1) - exp(x2 * y2 + x1 * (y1 -
y2)));
      c1t := ((v2 - k2) - a4 / a3 - c2t * exp(x2 * y2)) / exp(x1 * y2);
      ws1 := c1t * x1 * exp(x1 * y1) + c2t * x2 * exp(x2 * y1);
      ws2 := c1t * x1 * exp(x1 * y2) + c2t * x2 * exp(x2 * y2);
      sl2a := (open^[i, j] - open^[i - 1, j]) / dy;
      sl2b := (open^[i + 1, j] - open^[i - 1, j]) / dy * 0.5;
      sl1a := (open^[n, j] - open^[n, j]) / dy;
      sl1b := (open^[n + 1, j] - open^[n - 1, j]) / dy * 0.5;
      if (sl2a <= ws2) and (ws2 <= sl2b) and (sl1a <= ws1) and (ws1 <= sl1b) then
      begin                             {    w'(s2) = v'(s2), w'(s1) = v'(s1)                          }
      m := n;
      y0 := (m - 1) * dy;
      w0 := a4 / a3 + c1t * exp(x1 * y0) + c2t * exp(x2 * y0);
      while (m > 1) and (w0 > eps) do
      begin                             {    w(s0) <= eps                                              }
      m := m - 1;
      y0 := (m - 1) * dy;
      w0 := a4 / a3 + c1t * exp(x1 * y0) + c2t * exp(x2 * y0);
      ws0 := x1 * c1t * exp(x1 * y0) + x2 * c2t * exp(x2 * y0);
      end;                              {    abs(w(s0)) <= eps, w'(s0) ok, w(s0) < eps                 }
      if (abs(w0) <= eps) or (abs(ws0) <= eps / dy) or ((w0 <= eps) and (ws0 >= -eps / dy)) then
      if exp(y2) - exp(y1) > s2[j] - s1[j] then
      begin                             {    boundary found                                            }
      s0[j] := exp(y0);
      s1[j] := exp(y1);
      s2[j] := exp(y2);
      c1 := c1t;
      c2 := c2t;
      end;
      end;
      end
      else
      begin
      n := 1;                           {    boundary not found                                        }
      if (v2 <= eps) and (s1[j] = 0) then
      s1[j] := exp(y2);
      end;
      if v1 = 0 then
      n := 1;
```

```
      if n > 1 then
      n := n - 1;
     end;
  determine := factor;
 end;

 function fill_close (close: fpt; bound: bpt; factor: cpt; i: index_y; j: index_x): fpt;
  var                                    {    calculate values of close by eq. (10) in appendix b                }
   y: extended;
 begin
  with bound^ do
   with factor^ do
    begin
     y := (i - 1) * dy;
     if (exp(y) >= s0[j]) and (exp(y) <= s2[j]) then
      close^[i, j] := max(a4 / a3 + c1 * exp(x1 * y) + c2 * exp(x2 * y), 0);
    end;
  fill_close := close;
 end;

 function fill_open (open: fpt; close: fpt; bound: bpt; factor: cpt; input: dpt; i: index_y; j: index_x): fpt;
  var                                    {    calculate values of open and close by eq. (2c + 2e) in appendix b     }
   y: extended;
 begin
  with input^ do
   with bound^ do
    with factor^ do
     begin
      if (s0[j] <> 0) and (s2[j] <> 0) then
       begin
       y := (i - 1) * dy;
       if exp(y) < s1[j] then
       open^[i, j] := max(close^[i, j] - k1, 0);
       if exp(y) > s2[j] then
       close^[i, j] := max(open^[i, j] - k2, 0);
       end;
     end;
  fill_open := open;
 end;

end.
```

```
unit input_output;                                { file belongs to program shut down and abandonment      }

interface                                         { contains input and output procedures                    }
 uses
  definitions;

 function get_data (input: dpt): dpt;             { simple initializaton procedure                          }
 procedure print_field (z: fpt; factor: cpt);     { prints stock price-time space                           }
 procedure print_bound (bound: bpt; factor: cpt); { prints free boundaries                                  }

implementation
 function max (a, b: integer): integer;
 begin
  if a >= b then
   max := a
  else
   max := b;
 end;

 function min (a, b: integer): integer;
 begin
  if a <= b then
   min := a
  else
   min := b;
 end;

 function get_data (input: dpt): dpt;
 begin                                            { input data from brennan and schwartz (1985) p. 148      }
  new(input);
  with input^ do
   begin
    s_max := 25;                                  { y dimension in 10 cents                                 }
    q_max := 150;                                 { q dimension in mill. lbs.                               }
    q := 10;                                      { project value  in $ 100.000                             }
    v := 0.08;
    r := 0.02;
    c := 0.01;
    a := 5;
    f := 5;
    k1 := 2;
    k2 := 2;
    t1 := 0;
    t2 := 0.5;
    l0 := 0.02;
    l1 := 0.02;
   end;
  get_data := input;
 end;

 procedure print_field (z: fpt; factor: cpt);
  var
   x: set of index_x;
   y: set of index_y;
   i: index_y;
   j: index_x;
   n: integer;
   h: extended;

 begin
  writeln('ABANDONMENT VALUE');
  writeln('');
```

```
 with factor^ do
  begin
   x := [];
   case col_num of
    1:
     x := [max_x];
    2:
     x := [1, max_x];
    otherwise
     for n := 0 to col_num - 1 do
      x := x + [min(max(round((col_min + n * (col_max - col_min) / (col_num - 1)) / dx + 1), 1), max_x)];
   end;
   y := [];
   for i := 1 to max_y do
    begin
     h := exp((i - 1) * dy) / unit_y;
     if (h <= row_max) and (h >= row_min) then
      y := y + [i];
    end;
   write('|' : (digits + 2));
   for j := max_x downto 1 do
    if j in x then
     write(((j - 1) * dx / unit_x) : digits : decimals);
   writeln('');
   for n := digits + 1 downto 1 do
    write('-');
   write('+');
   for j := max_x downto 1 do
    if j in x then
     for n := digits downto 1 do
      write('-');
   writeln('');
   for i := max_y downto 1 do
    if i in y then
     begin
      write((exp((i - 1) * dy) / unit_y) : digits : decimals, ' |');
      for j := max_x downto 1 do
      if j in x then
      write((z^[i, j] / unit_xy) : digits : decimals);
      writeln('');
     end;
   if 0 >= row_min then
    begin
     write(0.0 : digits : decimals, ' |');
     for j := max_x downto 1 do
      if j in x then
      write(0.0 : digits : decimals);
     writeln('');
    end;
   writeln('');
  end;
end;

procedure print_bound (bound: bpt; factor: cpt);
 var
  x: set of index_x;
  i: index_y;
  j: index_x;
  n: integer;
  h: extended;

begin
 with bound^ do
```

```
  with factor^ do
   begin
    x := [];
    case col_num of
     1:
      x := [max_x];
     2:
      x := [1, max_x];
     otherwise
      for n := 0 to col_num - 1 do
      x := x + [min(max(round((col_min + n * (col_max - col_min) / (col_num - 1)) / dx + 1), 1), max_x)];
    end;
    write('|' : (digits + 2));
    for j := max_x downto 1 do
     if j in x then
      write(((j - 1) * dx / unit_x) : digits : decimals);
    writeln('');
    for n := digits + 1 downto 1 do
     write('-');
    write('+');
    for j := max_x downto 1 do
     if j in x then
      for n := digits downto 1 do
      write('-');
    writeln('');
    write('S2 |' : (digits + 2));
    for j := max_x downto 1 do
     if j in x then
      write((s2[j] / unit_y) : digits : decimals);
    writeln('');
    write('S1 |' : (digits + 2));
    for j := max_x downto 1 do
     if j in x then
      write((s1[j] / unit_y) : digits : decimals);
    writeln('');
    write('S0 |' : (digits + 2));
    for j := max_x downto 1 do
     if j in x then
      write((s0[j] / unit_y) : digits : decimals);
    writeln('');
    writeln('');
   end;
 end;


end.
```

SHUT DOWN AND ABANDONMENT VALUE

v(S,Q)	150.00	120.00	90.00	60.00	30.00	0.00
2.50	121.43	100.96	78.74	54.61	28.42	0.00
2.13	99.48	82.61	64.35	44.59	23.20	0.00
1.81	80.82	67.00	52.10	36.05	18.74	0.00
1.54	64.98	53.74	41.68	28.77	14.92	0.00
1.31	51.58	42.50	32.85	22.58	11.67	0.00
1.12	40.28	33.03	25.38	17.34	8.90	0.00
0.95	30.81	25.07	19.11	12.92	6.56	0.00
0.81	22.94	18.45	13.90	9.24	4.59	0.00
0.69	16.44	13.02	9.62	6.20	2.97	0.00
0.59	11.24	8.79	6.17	3.77	1.67	0.00
0.50	7.02	5.51	3.59	1.91	0.68	0.00
0.43	4.02	3.04	1.56	0.58	0.03	0.00
0.36	1.77	1.25	0.29	0.00	0.00	0.00
0.31	0.13	0.08	0.00	0.00	0.00	0.00
0.26	0.00	0.00	0.00	0.00	0.00	0.00
0.22	0.00	0.00	0.00	0.00	0.00	0.00
0.19	0.00	0.00	0.00	0.00	0.00	0.00
0.16	0.00	0.00	0.00	0.00	0.00	0.00
0.14	0.00	0.00	0.00	0.00	0.00	0.00
0.12	0.00	0.00	0.00	0.00	0.00	0.00
0.10	0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00

w(S,Q)	150.00	120.00	90.00	60.00	30.00	0.00
2.50	121.23	100.76	78.54	54.41	0.00	0.00
2.13	99.28	82.41	64.15	44.39	0.00	0.00
1.81	80.62	66.80	51.90	35.85	0.00	0.00
1.54	64.78	53.54	41.48	28.57	0.00	0.00
1.31	51.38	42.30	32.65	22.38	0.00	0.00
1.12	40.08	32.83	25.18	17.14	0.00	0.00
0.95	30.61	24.87	18.91	12.72	0.00	0.00
0.81	22.74	18.25	13.70	9.04	0.00	0.00
0.69	16.24	12.82	9.42	6.00	0.00	0.00
0.59	11.16	8.67	6.08	3.57	0.00	0.00
0.50	7.22	5.51	3.58	1.91	0.00	0.00
0.43	4.22	3.16	1.76	0.78	0.00	0.00
0.36	1.97	1.45	0.49	0.11	0.00	0.00
0.31	0.33	0.28	0.00	0.00	0.00	0.00
0.26	0.00	0.00	0.00	0.00	0.00	0.00
0.22	0.00	0.00	0.00	0.00	0.00	0.00
0.19	0.00	0.00	0.00	0.00	0.00	0.00
0.16	0.00	0.00	0.00	0.00	0.00	0.00
0.14	0.00	0.00	0.00	0.00	0.00	0.00
0.12	0.00	0.00	0.00	0.00	0.00	0.00
0.10	0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00

S*	150.00	120.00	90.00	60.00	30.00	0.00
S2	0.69	0.69	0.69	0.59	0.00	0.00
S1	0.50	0.36	0.43	0.43	0.43	0.00
S0	0.26	0.26	0.31	0.31	0.00	0.00

Appendix C: Construction Time Flexibility

This appendix uses the expicit finite difference method to show how to approximate the value of construction time flexibilty as introduced by Majd and Pindyck (1987).

The relevant functions, F(V, K) and f(V, K), must satisfy the differential equations:

(1a) $\frac{1}{2}\sigma^2 V^2 F_{VV} + (r - \delta) V F_V - r F - k F_K - k = 0$

(1b) $\frac{1}{2}\sigma^2 V^2 f_{VV} + (r - \delta) V f_V - r f = 0$

subject to the boundary conditions

(2a) $F(V, 0) = V$

(2b) $\lim_{V \to \infty} F_V(V, K) = e^{-\delta K / k}$

(2c) $f(0, K) = 0$

Further, a free boundary, V(K)*, a shared boundary condition for F and f, is defined by

(2d) $F(V^*, K) = \frac{V^*}{\beta} F_V(V^*, K)$

The ordinary differential equation for f has an analytic solution

(3) $f(V, K) = a V^{\beta}$

where $\beta = \frac{-(r - \delta - \sigma^2/2) + \sqrt{(r - \delta - \sigma^2/2)^2 + 2 r \sigma^2}}{2}$

The coefficient a must be determined jointly with the solution for F via the shared boundary condition, (2d).

With the analytic solution for f, the problem reduces to the solution of the PDE (1) subject to boundary conditions (2a), (2b), and (2d).

In order to simplify the implementation two transformations prove helpful.

(4a) $X = \ln V$

(4b) $G(X, K) = F(V, K)\, e^{r K / k}$

$F_{SS}(V, K) = (G_{XX}(X, K) - G_X(X, K))\, e^{-2X} e^{-r K / k}$

$F_S(V, K) = G_X(X, K)\, e^{-X} e^{-r K / k}$

$F_K(V, K) = G_t(X, K)\, e^{-r K / k}$

The PDE for the upper region, X > X*, and boundary conditions become:

(5) $$\frac{1}{2}\sigma^2 G_{XX} + (r - \delta - \sigma^2/2)\,G_X - k\,G_K - k\,e^{-rK/k} = 0$$

and accordingly,

(6a) $$G(X, 0) = e^X$$

(6b) $$\lim_{X \to \infty} \left(G_X(X, K)\, e^{-X} e^{-rK/k}\right) = e^{-\delta K/k}$$

(6c) $$G(X^*, K) = G_X(X^*, K) / \beta$$

The finite difference method transforms continuous variables into discrete ones, and replaces partial derivatives by finite differences. The explicit form of this approximation method characterizes itself by a specific choice of finite differences for this substitution. Let G(X,k) = G(iΔX, jΔk) = $G_{i,j}$ where 0 < i < m and 0 < j < n. The explicit finite difference method substitutes

(7) $$G_{XX} \approx \frac{G_{i+1,\,j-1} - 2\,G_{i,\,j-1} + G_{i-1,\,j-1}}{\Delta X^2}$$

$$G_X \approx \frac{G_{i+1,\,j-1} - G_{i-1,\,j-1}}{2\,\Delta X}$$

$$G_K \approx \frac{G_{i,\,j+1} - G_{i,j}}{\Delta K}$$

The PDE then becomes a difference equation

(8) $$G_{i,j} = a\,G_{i-1,\,j-1} + b\,G_{i,\,j-1} + c\,G_{i+1,\,j-1} + \phi_{j-1}$$

where

$$(9) \qquad a = \frac{\Delta K}{2\,k\,\Delta X}\left[\frac{\square^2}{\Delta X} \square (r \square \square \square \square^2/2)\right]$$

$$b = 1 \square \frac{\Delta K\,\square^2}{k\,\Delta X^2}$$

$$c = \frac{\Delta K}{2\,k\,\Delta X}\left[\frac{\square^2}{\Delta X} + (r \square \square \square \square^2/2)\right]$$

$$\square_j = \square\,\Delta K\, e^{r\,j\,\Delta K/k}$$

The coefficients of (9) are independent of i, and a + b + c = 1. For the stability of the explicit solution it is necessary that a, b, and c be non-negative which can be achieved by appropriate choice of ΔX and ΔK.

The terminal boundary, (6a), becomes.

$$(10) \qquad G_{i,\,j} = e^{i\,\Delta X}$$

the upper boundary condition, (6b), can be expressed as

$$(11) \qquad \lim_{X \square \infty} (G_X(X, K)\, e^{\square X} e^{\square r K/k}) = e^{\square \square K/k}$$

$$\square \qquad G_X(m\,\Delta X, j\,\Delta K) = e^{m\,\Delta X} e^{r\,j\,\Delta K/k} e^{\square \square j\,\Delta K/k}$$

$$\square \qquad \frac{G_{m+1,\,j} \square G_{m \square 1,\,j}}{2\,\Delta X} = e^{m\,\Delta X + (r \square \square)\,j\,\Delta K/k}$$

$$\square \qquad G_{m+1,j} = 2\,\Delta X\, e^{m\,\Delta X + (r \square \square)\,j\,\Delta K/k} + G_{m \square 1,\,j}$$

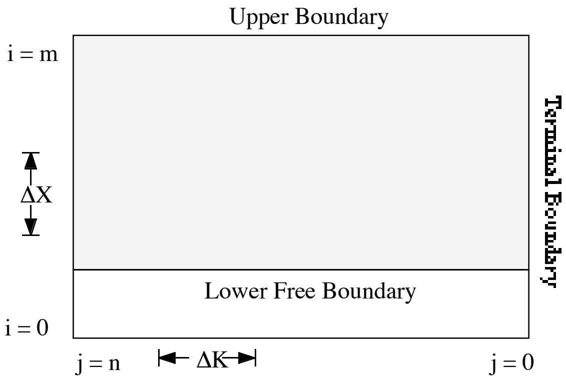
Now, $G_{m+1,j}$ can be substituted into (8) to yield the upper boundary condition

$$(12) \qquad G_{m,\,j} = (a + c)\, G_{m \square 1,\, j \square 1} + b\, G_{m,\, j \square 1} + c\; 2\,\Delta X\, e^{m\,\Delta X + (r \square \square)\,j\,\Delta K/k} + \square_{j \square 1}$$

Finally, the free boundary becomes

$$(13) \qquad G_{i^*,\,j} = G_{i^*+1,\,j} / (1 + \Delta X\, \square)$$

The solution procedure is a backward dynamic programming approach as illustrated in the following figure.



Project Value-Construction Time Space for G

First, the values of G at the terminal boundary, j = 0, are determined by using (10). Then, stepping back to j = 1, (12) is used to calculate $G_{m,j}$, and (8) yields the values for i = m-1, m-2,... . Each time equation (8) is used to calculate a value for $G_{i,j}$, (13) must be employed to check if the free boundary has been reached. Due to discretization error, however, (13) is unlikely to hold exactly, so the check allows a specified tolerance, □, within which the condition must hold:

(14) $G_{i^*,j} \, \square \, G_{i^*+1,j} / (1 + \Delta X \square) \leq \square$

The tolerance, □, is chosen arbitrarily to be (ΔX / 2). Once (14) identifies the free boundary, the coefficient, a, can be determined and equation (3) calculates the values below the boundary until the lower bound is reached. The procedure then steps back to j := j+1 and continues in this way until j = n.

```
program construction_time_flexibility;       {  numerical approximation for the value of construction time flexibility  }
 uses                                        {  according to a model of s. majd and r. s. pindyck [1987], copyright by  }
  definitions, functions, output, user;      {  christian von drathen, potosistrasse 9, 2000 hamburg 55, germany        }
 var                                         {  files included: program, def.lib, f.lib, io.lib, u.lib                  }
  space: fpt;                                {  project value-construction time space                                   }
  lower_boundary: xpt;                       {  free lower boundary vector                                              }
begin                                        {  numerical approximation by an explicit finite difference approach       }
 space := explicit(lower_boundary, f_capacity, f_delta, f_terminal, f_lower, f_interest, f_variance);
 print_field(space);                         {  output of project value-construction time space                         }
 print_bound(lower_boundary);                {  output of free lower boundary vector                                    }
end.
```

```
unit definitions;                                      {  file belongs to program construction_time_flexibility           }

interface                                              {  contains global constants, variables, structures, and functions }
 const
  max_y = 26;                                          {  number of points in y dimension                                 }
  max_x = 61;                                          {  number of  points in x dimension                                }
  y_max = 42.52;                                       {  largest value in y dimension                                    }
  x_max = 6.0;                                         {  largest value in x dimension                                    }
  row_max = 43;                                        {  largest output value in y dimension                             }
  row_min = 0;                                         {  smallest output value in y dimesion                             }
  col_num = 7;                                         {  number of columns in output                                     }
  col_max = 6;                                         {  largest output value in x dimesion                              }
  col_min = 0;                                         {  smallest output value in x dimension                            }
  digits = 7;                                          {  number of digits in output                                      }
  decimals = 2;                                        {  number of decimals in output                                    }
 type
  index_y = 1..max_y;                                  {  index for y dimesion in field                                   }
  index_x = 1..max_x;                                  {  index for x dimesion in field                                   }
  field = array[index_y, index_x] of extended;         {  field contains stock price-time space                           }
  fpt = ^field;                                        {  pointer to field                                                }
  x_array = array[index_x] of extended;                {  free boundary vector                                            }
  xpt = ^x_array;                                      {  pointer to free boundary vector                                 }
 var
  dy: extended;                                        {  increment in y dimension                                        }
  dx: extended;                                        {  increment in x dimension                                        }
  coef: extended;                                      {  see appendix c: eq. (3)                                         }
  alpha: extended;                                     {  see appendix c: eq. (3)                                         }
 function max (num1, num2: extended): extended;        {  function maximum                                                }
 function min (num1, num2: extended): extended;        {  function minimum                                                }

implementation
 function max;
 begin
  if num1 >= num2 then
   max := num1
  else
   max := num2;
 end;
 function min;
 begin
  if num1 <= num2 then
   min := num1
  else
   min := num2;
 end;

end.
```

```
unit functions;                                    {   file belongs to program construction_time_flexibility        }

interface                                          {   contains function with main algorithm                        }
 uses
  definitions;                                     {   parameters:   }
 function explicit (var bound: xpt;                {   lower free boundary                                          }
      function f_capacity (i1: index_y;            {   capacity      }
      j1: index_x): extended;   function f_delta (i2: index_y;          {   delta                               }
      j2: index_x): extended;   function f_terminal (i3: index_y;       {   terminal boundary                   }
      j3: index_x): extended;   function f_lower (i4: index_y;          {   lower boundary                      }
      j4: index_x): extended;   function f_interest (i5: index_y;       {   interest rate                       }
      j5: index_x): extended;   function f_variance (i6: index_y;       {   variance                            }
      j6: index_x): extended): fpt;

implementation
 function explicit;
  var
   panel: fpt;                                     {   project value-construction time space                        }
   i: index_y;
   j: index_x;
   cap: extended;                                  {   capacity      }
   delta: extended;                                {   delta         }
   rate: extended;                                 {   interest rate }
   s2: extended;                                   {   variance      }
   prob1, prob2, prob3: extended;                  {   see appendix c: eq. 8 & 9                                    }
   eta: extended;                                  {   see appendix c: eq. 8 & 9                                    }
   disc: extended;                                 {   discount factor    }
 begin
  new(panel);                                      { initializing         }
  new(bound);
  dy := ln(y_max) / (max_y - 1);
  dx := x_max / (max_x - 1);
  for i := 1 to max_y do
   for j := 1 to max_x do
    panel^[i, j] := 0;
  for i := 1 to max_y do                           {   terminal boundary                                            }
   panel^[i, 1] := f_terminal(i, 1);
  cap := f_capacity(1, 1);
  delta := f_delta(1, 1);
  rate := f_interest(1, 1);
  s2 := f_variance(1, 1);
  alpha := (sqrt(sqr(rate - delta - s2 / 2) + 2 * rate * s2) - (rate - delta - s2 / 2)) / s2;
  coef := 0;
  prob1 := dx / (2 * dy * cap) * (s2 / dy - rate + delta + s2 / 2);
  prob2 := 1 - s2 * dx / (sqr(dy) * cap);
  prob3 := dx / (2 * dy * cap) * (s2 / dy + rate - delta - s2 / 2);
  if (prob1 < 0) or (prob2 < 0) or (prob3 < 0) then
   begin                                           {   stable results guaranteed ?                                  }
    writeln('WARNING: Negative Probabilities in EXPLICIT');
    writeln('prob1:', prob1 : digits : decimals);
    writeln('prob2:', prob2 : digits : decimals);
    writeln('prob3:', prob3 : digits : decimals);
   end;
  for j := 2 to max_x do
   begin
    eta := -dx * exp((j - 2) * dx * rate / cap);   {   upper boundary    }
    panel^[max_y, j] := ((prob1 + prob3) * panel^[max_y - 1, j - 1] + prob2 * panel^[max_y, j - 1] + prob3 * (2 * dy *
exp((max_y - 1) * dy + (j - 1) * (rate - delta) * dx / cap)) + eta);
    for i := max_y - 1 downto 1 do
     begin
      if (coef = 0) and (i > 1) then
       begin
       panel^[i, j] := prob1 * panel^[i - 1, j - 1] + prob2 * panel^[i, j - 1] + prob3 * panel^[i + 1, j - 1] + eta;
```

```
      if panel^[i, j] - panel^[i + 1, j] / (1 + alpha * dy) <= dy / 2 then
      begin                                          {     free boundary reached                                          }
      coef := panel^[i, j] / exp(alpha * (i - 1) * dy);
      bound^[j] := exp((i - 1) * dy);
      end;
      end
     else
      panel^[i, j] := f_lower(i, 1);
    end;
   coef := 0;
  end;
 for j := 1 to max_x do
  begin                                              {     transformation     }
   disc := exp(-rate * (j - 1) * dx / cap);
   for i := 1 to max_y do
    panel^[i, j] := disc * panel^[i, j];
  end;
 explicit := panel;
end;

end.
```

```
unit output;                              {   file belongs to program construction_time_flexibility   }

interface                                 {   contains output procedures                              }
 uses
  definitions, user;

 procedure print_field (z: fpt);          {   prints stock price-time space                           }
 procedure print_bound (bound: xpt);      {   prints free boundary                                    }

implementation

 function maxi (index1, index2: integer): integer;
 begin
  if index1 >= index2 then
   maxi := index1
  else
   maxi := index2;
 end;

 function mini (index1, index2: integer): integer;
 begin
  if index1 <= index2 then
   mini := index1
  else
   mini := index2;
 end;

 procedure print_field;
  var
   x: set of index_x;
   y: set of index_y;
   i: index_y;
   j: index_x;
   n: integer;
   h: extended;

 begin
  writeln('CONSTRUCTION TIME FLEXIBILITY');
  writeln('');
  writeln('(r = ', f_interest(1, 1) : 4 : 2, ', var = ', f_variance(1, 1) : 4 : 2, ', delta = ', f_delta(1, 1) : 4 : 2, ', K = ', x_max : 4 : 2, ', k
= ', f_capacity(1, 1) : 4 : 2, ' )');
  writeln('');
  x := [];
  case col_num of
   1:
    x := [max_x];
   2:
    x := [1, max_x];
   otherwise
    for n := 0 to col_num - 1 do
     x := x + [mini(maxi(round((col_min + n * (col_max - col_min) / (col_num - 1)) / dx) + 1, 1), max_x)];
  end;
  y := [];
  for i := 1 to max_y do
   begin
    h := exp((i - 1) * dy);
    if (h <= row_max) and (h >= row_min) then
     y := y + [i];
   end;
  write('f(V,K) |' : (digits + 1));
  for j := max_x downto 1 do
   if j in x then
    write(((j - 1) * dx) : digits : decimals);
```

```
 writeln('');
 for n := digits downto 1 do
  write('-');
 write('+');
 for j := max_x downto 1 do
  if j in x then
   for n := digits downto 1 do
    write('-');
 writeln('');
 for i := max_y downto 1 do
  if i in y then
   begin
    write((exp((i - 1) * dy)) : digits : decimals, '|');
    for j := max_x downto 1 do
     if j in x then
      write((z^[i, j]) : digits : decimals);
    writeln('');
   end;

 if 0 >= row_min then
  begin
   write(0.0 : digits : decimals, '|');
   for j := max_x downto 1 do
    if j in x then
     write(0.0 : digits : decimals);
   writeln('');
  end;
 writeln('');
end;

procedure print_bound;
 var
  x: set of index_x;
  i: index_y;
  j: index_x;
  n: integer;
  h: extended;

begin
 x := [];
 case col_num of
  1:
   x := [max_x];
  2:
   x := [1, max_x];
  otherwise
   for n := 0 to col_num - 1 do
    x := x + [mini(maxi(round((col_min + n * (col_max - col_min) / (col_num - 1)) / dx + 1), 1), max_x)];
 end;
 write('V*(K) |' : (digits + 1));
 for j := max_x downto 1 do
  if j in x then
   write(((j - 1) * dx) : digits : decimals);
 writeln('');
 for n := digits downto 1 do
  write('-');
 write('+');
 for j := max_x downto 1 do
  if j in x then
   for n := digits downto 1 do
    write('-');
 writeln('');
 write('S |' : (digits + 1));
```

```
  for j := max_x downto 1 do
   if j in x then
    write((bound^[j]) : digits : decimals);
  writeln('');
  writeln('');
 end;

end.
```

```
unit user;                          {  file belongs to program construction_time_flexibility           }

interface                           {  contains user defined functions for specific application}
 uses
  definitions;

 function f_capacity (i: index_y;   {  capacity as function of project value and construction progress  }
      j: index_x): extended;
 function f_delta (i: index_y;      {  delta as function of project value and construction progress     }
      j: index_x): extended;
 function f_terminal (i: index_y;   {  terminal boundary                                                }
      j: index_x): extended;
 function f_lower (i: index_y;      {  lower region                                                     }
      j: index_x): extended;
 function f_interest (i: index_y;   {  interest rates as function of project value and construction progress }
      j: index_x): extended;
 function f_variance (i: index_y;   {  variance as function of project value and construction progress  }
      j: index_x): extended;

implementation
 function f_capacity;
 begin
  f_capacity := 1;
 end;
 function f_delta;
 begin
  f_delta := 0.06;
 end;
 function f_terminal;
 begin
  f_terminal := exp((i - 1) * dy);
 end;
 function f_lower;
 begin
  f_lower := coef * exp(alpha * (i - 1) * dy);
 end;
 function f_interest;
 begin
  f_interest := 0.02
 end;
 function f_variance;
 begin
  f_variance := 0.04;
 end;

end.
```

CONSTRUCTION TIME FLEXIBILITY

(r = 0.02, var = 0.04, delta = 0.06, K = 6.00, k = 1.00)

f(V,K)	6.00	5.00	4.00	3.00	2.00	1.00	0.00
42.52	23.92	26.65	29.52	32.53	35.69	39.01	42.52
36.60	19.82	22.30	24.89	27.61	30.47	33.46	36.60
31.50	16.28	18.54	20.90	23.37	25.96	28.67	31.50
27.11	13.23	15.30	17.46	19.72	22.07	24.54	27.11
23.34	10.60	12.51	14.49	16.57	18.73	20.98	23.34
20.09	8.34	10.11	11.94	13.85	15.85	17.92	20.09
17.29	6.39	8.04	9.74	11.52	13.37	15.29	17.29
14.88	4.72	6.25	7.85	9.51	11.23	13.02	14.88
12.81	3.29	4.72	6.22	7.78	9.39	11.07	12.81
11.02	2.11	3.41	4.82	6.29	7.81	9.39	11.02
9.49	1.28	2.28	3.62	5.01	6.45	7.94	9.49
8.17	0.78	1.39	2.58	3.91	5.28	6.70	8.17
7.03	0.48	0.85	1.70	2.96	4.27	5.63	7.03
6.05	0.29	0.52	1.03	2.14	3.40	4.71	6.05
5.21	0.18	0.31	0.63	1.44	2.66	3.91	5.21
4.48	0.11	0.19	0.38	0.88	2.01	3.23	4.48
3.86	0.07	0.12	0.23	0.53	1.46	2.64	3.86
3.32	0.04	0.07	0.14	0.33	0.98	2.14	3.32
2.86	0.02	0.04	0.09	0.20	0.60	1.70	2.86
2.46	0.01	0.03	0.05	0.12	0.36	1.33	2.46
2.12	0.01	0.02	0.03	0.07	0.22	1.00	2.12
1.82	0.01	0.01	0.02	0.04	0.14	0.72	1.82
1.57	0.00	0.01	0.01	0.03	0.08	0.44	1.57
1.35	0.00	0.00	0.01	0.02	0.05	0.27	1.35
1.16	0.00	0.00	0.00	0.01	0.03	0.16	1.16
1.00	0.00	0.00	0.00	0.01	0.02	0.10	1.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

V*(K)	6.00	5.00	4.00	3.00	2.00	1.00	0.00
S	11.02	9.49	7.03	5.21	3.32	1.82	0.00

Appendix D: Project Financing

Section 5.5 introduced the general framework of Mason and Merton (1985) to evaluate arbitrary derivative securities, W(V, t), that are contingent upon the value, V, of one arbitrary investment and calendar time, t. This appendix show how to approximate the model by the explicit finite difference method.

Any derivative security, W, must satisfy the fundamental PDE

(1) $$\frac{1}{2}\sigma^2 V^2 W_{VV} + (r - \delta)\, V W_V + W_t - r W + \square_W = 0$$

where δ is the payout ratio of the underlying investment and $\square_W$ is a claim specific pay-in / pay-out term. δ and $\square_W$ together with σ^2 and r can be functions of V and t.

The boundary conditions for the PDE are also security specific. In general, certain functions of V and t express the conditions.

(2a) $$W(0, t) = \square_1(0, t)$$

(2b) $$W(V^c(t), t) = \square_2(V^c(t), t)$$

(3a) $$\lim_{V \to \infty} W_V(V, t) = 1$$

(3b) $$\lim_{V \to \infty} W_V(V, t) = 0$$

(3b) $$\lim_{V \to \infty} W(V, t) = \square(t)$$

(4) $$W(V, t^*) = \square(V, t^*)$$

The problem is sufficiently described when the PDE is known and three boundary conditions are specified. A lower boundary (2a) or a free boundary (2b), and a upper boundary, (3a), (3b), or (3c), and a terminal boundary condition (4) must be determined.

Before implementing the scheme, it is tranformed

(5) $$X = \ln V$$

$$W_{VV}(V, t) = (G_{XX}(X, t) - G_X(X, t))\, e^{-2X}$$

$$W_V(V, t) = G_X(X, t)\, e^{-X}$$

$$W_t(V, t) = G_t(X, t)$$

The PDE and the boundary conditions become:

(6) $\frac{1}{2}\,\square^2\, G_{XX} + (r \,\square\,\square\;\square\,\square^2 / 2)\, G_X + G_t \,\square\, r\,G + \square_W = 0$

(7a) $G(X, t) = \square_1\,(e^X, t)$

(7b) $G(X^c(t), t) = \square_2\,(e^{X^c(t)}, t)$

(8a) $\lim_{X \square \infty} G_X(X, t) = 1$

(8b) $\lim_{X \square \infty} G_X(X, t) = 0$

(8b) $\lim_{X \square \infty} G\,(X, t) = \square(t)$

(9) $W(X, t^*) = \square(e^X, t^*)$

The finite difference method transforms continuous variables into discrete ones and substitutes partial derivatives by finite differences. The explicit form specifies the differences in a particular form. Defining $G_{i,j} = G(i\Delta X, j\Delta t) = G(X, t)$, where $0 < i < m$, and $0 < j < n$, the explicit finite difference method replaces

(10) $G_{XX} \,\square\, \frac{G_{i+1,\,j\square 1} \,\square\, 2\,G_{i,\,j\square 1} + G_{i\square 1,\,j\square 1}}{\Delta X^2}$

$G_X \,\square\, \frac{G_{i+1,\,j\square 1} \,\square\, G_{i\square 1,\,j\square 1}}{2\,\Delta X}$

$G_t \,\square\, \frac{G_{i,\,j} \,\square\, G_{i,j\square 1}}{\Delta t}$

The PDE then becomes a difference equation

(11) $G_{i,j} = \frac{1}{(1 + r\,\Delta t)}\,[\,a\,G_{i\square 1,\,j\square 1} + b\,G_{i,\,j\square 1} + c\,G_{i+1,\,j\square 1} + \square_W]$

where

(12) $a = \frac{\Delta t}{2\,\Delta X}\,\square\,\frac{\square^2}{\Delta X}\,\square\,(r\,\square\,\square^2/2\,\square\,\square_W)\,\square$

$b = 1\,\square\,\frac{\Delta t\,\square^2}{\Delta X^2}$

$c = \frac{\Delta t}{2\,\Delta X}\,\square\,\frac{\square^2}{\Delta X} + (r\,\square\,\square^2/2\,\square\,\square_W)\,\square$

The coefficients a, b, and c sum to 1 and are independent of i. Appropriate choice of ΔX and Δt assures that a, b, and c are non-negative so that stable results in $G_{i,j}$ are achieved.

The lower boundary (7a) and (7b), the higher boundary (7c) and the terminal boundary (9) are functions of X and t. Each occurrence of X in these functions must be substituted by its discrete equivalent, $i\,\Delta X$.

The upper boundary condition (8a) becomes

$$(13)\qquad \lim_{X \to \infty} G_X(X, K) = e^X$$

$$\Rightarrow \qquad G_X(m\,\Delta X, j\,\Delta K) = e^{m\,\Delta X}$$

$$\Rightarrow \qquad \frac{G_{m+1,\,j} - G_{m-1,\,j}}{2\,\Delta X} = e^{m\,\Delta X}$$

$$\Rightarrow \qquad G_{m+1,j} = 2\,\Delta X\, e^{m\,\Delta X} + G_{m-1,\,j}$$

This can be substituted into (11) to yield

$$(14)\qquad G_{m,\,j} = (a + c)\, G_{m-1,\,j-1} + b\, G_{m,\,j-1} + c\; 2\,\Delta X\, e^{m\,\Delta X}$$

Similarly, the upper boundary condition (8b) develops to

$$(15)\qquad \lim_{X \to \infty} G_X(X, K) = 0$$

$$\Rightarrow \qquad G_X(m\,\Delta X, j\,\Delta K) = 0$$

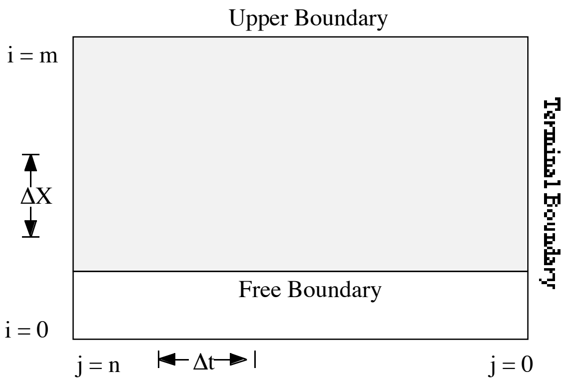
$$\Rightarrow \qquad \frac{G_{m+1,\,j} - G_{m-1,\,j}}{2\,\Delta X} = 0$$

$$\Rightarrow \qquad G_{m+1,j} = G_{m-1,\,j}$$

With the same substitution, the boundary becomes

$$(16)\qquad G_{m,\,j} = (a + c)\, G_{m-1,\,j-1} + b\, G_{m,\,j-1}$$

The solution procedure is a dynamic backward programming algorithm.



Project Value-Time Space for G

At first, for $j = 0$, the terminal boundary is set using (9). Then, for $j = 1$, the upper boundary is calculated by (8a), (8b), or (8c) respectively and (11) supplies the values for $G_{i,j}$ for $i = m-1$, $m-2$, ... When $i \Delta X < X(t)$ the lower free boundary (7b) is reached. Alternatively, if (7a) is used, at $i = 0$, the value of $G_{0,j}$ is given at the lower boundary. The procedure then returns to $j=j+1$ and repeats. Each further column is thus calculated until $j = n$.

```
program project_financing;                  {  numerical approximation for the value of a project financing          }
 uses                                       {  according to a model of s. mason and r. merton [1985], copyright by    }
  definitions, functions, output, user;     {  christian von drathen, potosistrasse 9, 2000 hamburg 55, germany       }
 var                                        {  files included: program, def.lib, f.lib, io.lib, u.lib                 }
  border, dummy: xpt;                       {  free lower boundary vector  }
  j: index_x;
begin
 get_data;                                                                        {  read input                                 }
 set_data;                                                                        {  calculate arrays                           }
 new(border);                                                                     {  allocate memory for lower boundary         }
 for j := 1 to max_x do
  border^[j] := 0;                                                                {  erase lower boundary                       }
 e3 := explicit(t2, t3, high1, p, e3b, e3l, z, e3f, e3m, r, v, border, border);   {  calculate equity in phase 3                }
 d3 := explicit(t2, t3, highc, p, d3b, d3l, d3h, d3f, z, r, v, border, dummy);    {  calculate debt in phase 3                  }
 g3 := explicit(t2, t3, high0, p, g3b, g3l, z, g3f, z, r, v, border, dummy);      {  calculate loan guarantee in phase 3        }
 for j := 1 to max_x do
  border^[j] := 0;                                                                {  erase lower boundary                       }
 e2 := explicit(t1, t2, high1, p, e2b, e2l, z, e2f, z, r, v, border, border);     {  calculate equity in phase 2                }
 d2 := explicit(t1, t2, highc, p, d2b, d2l, d2h, d2f, z, r, v, border, dummy);    {  calculate debt in phase 2                  }
 g2 := explicit(t1, t2, high0, p, g2b, g2l, z, g2f, z, r, v, border, dummy);      {  calculate loan guarantee in phase 2        }
 s2 := explicit(t1, t2, high0, p, s2b, s2l, z, z, z, r, v, border, dummy);        {  calculate cash flow guarantee in phase 2   }
 for j := 1 to max_x do
  border^[j] := 0;                                                                {  erase lower boundary                       }
 w1 := explicit(t0, t1, high1, z, w1b, z, z, w1f, w1m, r, v, border, border);     {  calculate project value in phase 1         }
 s1 := explicit(t0, t1, high0, z, z, z, z, s1f, z, r, v, border, dummy);          {  calculate cash flow guarantee in phase 1   }
 d1 := explicit(t0, t1, highc, z, d1b, d1l, d1h, d1f, z, r, v, border, dummy);    {  calculate debt in phase 1                  }
 g1 := explicit(t0, t1, high0, z, z, g1l, z, g1f, z, r, v, border, dummy);        {  calculate loan guarantee in phase 1        }
 e1 := explicit(t0, t1, high1, z, e1b, e1l, z, e1f, z, r, v, border, dummy);      {  calculate equity in phase 1                }
 print_array;                                                                     {  output data of project                     }
 print_field(e3, t2, 'e3');                 {  output of project value- time space for equity in phase 3              }
 print_field(e2, t1, 'e2');                 {  output of project value- time space for equity in phase 2}
 print_field(e1, t0, 'e1');                 {  output of project value- time space for equity in phase 1              }
 print_field(d3, t2, 'd3');                 {  output of project value- time space for debt in phase 3                }
 print_field(d2, t1, 'd2');                 {  output of project value- time space for debt in phase 2                }
 print_field(d1, t0, 'd1');                 {  output of project value- time space for debt in phase 1                }
 print_field(g3, t2, 'g3');                 {  output of project value- time space for loan guarantee in phase 3      }
 print_field(g2, t1, 'g2');                 {  output of project value- time space for loan guarantee in phase 2      }
 print_field(g1, t0, 'g1');                 {  output of project value- time space for loan guarantee in phase 1      }
 print_field(s2, t1, 's2');                 {  output of project value- time space for cash flow guarantee in phase 2 }
 print_field(s1, t0, 's1');                 {  output of project value- time space for cash flow guarantee in phase 1 }
 print_field(w1, t0, 'w1');                 {  output of project value- time space for project in phase 1             }
end.
```

```
unit definitions;                                   {  file belongs to program project financing                              }

interface                                           {  contains global constants, variables, structures, and functions        }
 const
  max_y = 31;                                       {  number of points in y dimension                                        }
  max_x = 49;                                       {  number of  points in x dimension                                       }
  max_t = 49;                                       {  number of  points in time array                                        }
  row_max = 10000.4;                                {  largest output value in y dimension                                    }
  row_min = 99;                                     {  smallest output value in y dimesion                                    }
  col_num = 9;                                      {  number of columns in output                                            }
  col_max = 16;                                     {  largest output value in x dimesion                                     }
  col_min = 0;                                      {  smallest output value in x dimension                                   }
  digits = 6;                                       {  number of digits in output                                             }
  decimals = 0;                                     {  number of decimals in output                                           }
  unit_y = 0.01;                                    {  scale factor for labels in y dimension                                 }
  unit_x = 0.25;                                    {  scale factor for labels in x dimension                                 }
  unit_xy = 0.01;                                   {  scale factor for values in value-time space                            }

 type
  index_y = 1..max_y;                               {  index for y dimesion in field                                          }
  index_x = 1..max_x;                               {  index for x dimesion in field                                          }
  index_t = 1..max_t;                               {  index for time array                                                   }

  field = array[index_y, index_x] of extended;      {  field contains stock price-time space                                  }
  fpt = ^field;                                     {  pointer to field                                                       }

  x_array = array[index_x] of extended;             {  array contains free boundary                                           }
  xpt = ^x_array;                                   {  pointer to x_array                                                     }

  t_array = array[index_t] of extended;             {  array contains investment data                                         }
  tpt = ^t_array;                                   {  pointer to t_array                                                     }

 var
  dy: extended;                                     {  increment in y dimension                                               }
  dx: extended;                                     {  increment in x dimension                                               }

 function max (num1, num2: extended): extended;
 function min (num1, num2: extended): extended;

implementation

 function max (num1, num2: extended): extended;
 begin
  if num1 >= num2 then
   max := num1
  else
   max := num2;
 end;

 function min (num1, num2: extended): extended;
 begin
  if num1 <= num2 then
   min := num1
  else
   min := num2;
 end;

end.
```

```
unit functions;                              {  file belongs to program project financing                }

interface                                    {  contains function with main algorithm                    }
 uses
  definitions;
                                             {  parameters:               }
 function explicit (ta, tb: index_t;         {  lower, higher time index       }
      function high (i1: index_y;            {  upper boundary function        }
      j1: index_x;   t1: index_t): extended;  function a (i2: index_y;     {  cash flow function              }
      j2: index_x;   t2: index_t): extended;  function b (i3: index_y;     {  payout function                 }
      j3: index_x;   t3: index_t): extended;  function l (i4: index_y;     {  lower boundary function         }
      j4: index_x;   t4: index_t): extended;  function h (i5: index_y;     {  upper boundary value            }
      j5: index_x;   t5: index_t): extended;  function f (i6: index_y;     {  terminal boundary function      }
      j6: index_x;   t6: index_t): extended;  function m (i7: index_y;     {  conditon for lower boundary     }
      j7: index_x;   t7: index_t): extended;  function r (i8: index_y;     {  riskless rate                   }
      j8: index_x;   t8: index_t): extended;  function v (i9: index_y;     {  project variance                }
      j9: index_x;   t9: index_t): extended;  border, result: xpt): fpt;   {  free boundary input, output     }
 function highc (i: index_y;                 {  constant upper boundary        }
      j: index_x;    t: index_t): extended;
 function high1 (i: index_y;                 {  marginal increase at upper boundary is 1                 }
      j: index_x;    t: index_t): extended;
 function high0 (i: index_y;                 {  marginal increase at upper boundary is 0                 }
      j: index_x;    t: index_t): extended;

implementation
 var
  panel: fpt;                                {  project value-time space pointer                         }
  ceiling: xpt;                              {  constant upper boundary        }
  prob1, prob2, prob3: extended;             {  see appendix d: (12)      }
  disc: extended;                            {  discount factor           }
  const1, const2: extended;                  {  cash flow, payout dummies }

 function explicit;
  var
   i: index_y;
   j: index_x;
   t: index_t;
   rate: extended;                           {  interest rate             }
   s2: extended;                             {  variance                  }
 begin
  new(ceiling);                              {  initialize                }
  new(panel);
  dy := ln(row_max * unit_y) / (max_y - 1);
  dx := (tb - ta) * unit_x / (max_x - 1);
  for i := 1 to max_y do
   for j := 1 to max_x do
    panel^[i, j] := 0;
  for i := 1 to max_y do
   panel^[i, 1] := f(i, 1, tb);              {  terminal boundary              }
  for j := 2 to max_x do
   begin
    t := ta + round((tb - ta) / (max_x - 1) * (max_x - j));
    ceiling^[j] := h(max_y, j, t);           {  constant higher boundary       }
    for i := max_y downto 1 do
     begin
      if j = max_x - round((t - ta) * (max_x - 1) / (tb - ta)) then
       begin                                 {  current cash flow and payout                             }
       const1 := a(i, j, t);
       const2 := b(i, j, t);
       end
      else
       begin
       const1 := 0;
```

```
    const2 := 0;
    end;
   rate := r(i, 1, t);                              {    initialization              }
   s2 := v(i, 1, t);
   disc := 1 + dx * rate;
   prob1 := dx / (2 * dy) * (s2 / dy - rate + s2 / 2 + const1);
   prob2 := 1 - s2 * dx / sqr(dy);
   prob3 := dx / (2 * dy) * (s2 / dy + rate - s2 / 2 - const1);
   if (prob1 < 0) or (prob2 < 0) or (prob3 < 0) then
    begin                                           {    stable results possible ? }
    writeln('WARNING: Negative Probabilities in EXPLICIT');
    writeln('prob1:', prob1 : digits : decimals);
    writeln('prob2:', prob2 : digits : decimals);
    writeln('prob3:', prob3 : digits : decimals);
    end;
   if i = max_y then
    panel^[i, j] := high(i, j, 1)                   {    constant upper boundary     }
   else if (i = 1) or (exp((i - 1) * dy) <= border^[j]) then
    panel^[i, j] := l(i, j, t)                      {    lower or free boundary (internal)                          }
   else
    begin                                           {    explicit finite difference calculation                     }
    panel^[i, j] := (prob1 * panel^[i - 1, j - 1] + prob2 * panel^[i, j - 1] + prob3 * panel^[i + 1, j - 1] + const2) / disc;
    if panel^[i, j] < m(1, 1, t) then               {    free boundary (external)    }
    begin
    panel^[i, j] := l(1, j, t);                     {    boundary condition (set)    }
    if border^[j] = 0 then
    result^[j] := exp((i - 1) * dy);                {    critical project value    }
    end;
    end;
  end;
 end;
 dispose(ceiling);
 explicit := panel;
end;

function highc;
begin
 highc := ceiling^[j];
end;

function high1;
begin
 high1 := ((prob1 + prob3) * panel^[i - 1, j - 1] + prob2 * panel^[i, j - 1] + prob3 * (2 * dy * exp((i - 1) * dy)) + const2) / disc;
end;

function high0;
begin
 high0 := ((prob1 + prob3) * panel^[i - 1, j - 1] + prob2 * panel^[i, j - 1] + const2) / disc;
end;

end.
```

```
unit output;                          {  file belongs to program project financing          }

interface                             {  contains output procedures                         }
 uses
  definitions;

 procedure print_field (z: fpt;       {  output of project value-time space                 }
      ta: index_t;
      name: string);

implementation

 function maxi (index1, index2: integer): integer;
 begin
  if index1 >= index2 then
   maxi := index1
  else
   maxi := index2;
 end;

 function mini (index1, index2: integer): integer;
 begin
  if index1 <= index2 then
   mini := index1
  else
   mini := index2;
 end;

 procedure print_field;
  var
   x: set of index_x;
   y: set of index_y;
   i: index_y;
   j: index_x;
   n: integer;
   h: extended;

 begin
  x := [];
  case col_num of
   1:
    x := [max_x];
   2:
    x := [1, max_x];
   otherwise
    for n := 0 to col_num - 1 do
     x := x + [mini(maxi(round((col_min + n * (col_max - col_min) / (col_num - 1)) * unit_x / dx) + 1, 1), max_x)];
  end;
  y := [];
  for i := 1 to max_y do
   begin
    h := exp((i - 1) * dy) / unit_y;
    if (h <= row_max) and (h >= row_min) then
     y := y + [i];
   end;
  write(name : digits, ' |');
  for j := max_x downto 1 do
   if j in x then
    write((((max_x - j) * dx / unit_x) + ta) : digits : decimals);
  writeln('');
  for n := digits + 1 downto 1 do
   write('-');
  write('+');
```

```
  for j := max_x downto 1 do
   if j in x then
    for n := digits downto 1 do
     write('-');
  writeln('');
  for i := max_y downto 1 do
   if i in y then
    begin
     write((exp((i - 1) * dy) / unit_y) : digits : decimals, ' |');
     for j := max_x downto 1 do
      if j in x then
       write((z^[i, j] / unit_xy) : digits : decimals);
     writeln('');
    end;
  writeln('');
 end;

end.
```

```
unit user;                      {  file belongs to program project value                        }

interface                       {  contains user defined functions for specific application}
 uses
  definitions;

 var
  t0, t1, t2, t3: index_t;
  e1, e2, e3: fpt;              {  project value-time space for equity in phase 1, 2, 3          }
  d1, d2, d3: fpt;              {  project value-time space for debt in phase 1, 2, 3            }
  g1, g2, g3: fpt;              {  project value-time space for loan guarantee in phase 1, 2, 3  }
  s1, s2: fpt;                  {  project value-time space for cash flow guarantee in phase 1, 2 }
  w1: fpt;                      {  project value-time space for project in phase 1               }
  dps: t_array;                 {  debt payment currently due to senior debt                     }
  dpj: t_array;                 {  debt payment currently due to junior debt                     }
  bps: t_array;                 {  bond principal of senior debt currently outstanding           }
  bpj: t_array;                 {  bond principal of junior debt currently outstanding}
  rbs: t_array;                 {  value of risk-free bond with same terms as senior debt        }
  cf: t_array;                  {  schedule of guaranteed cash flows during phase 2              }
  pve: t_array;                 {  present value of equity during phase 2                        }
  pvd: t_array;                 {  present value of senior debt during phase 2                   }
  pvg: t_array;                 {  present value of loan guarantee during phase 2                }
  pvs: t_array;                 {  present value of guaranteed cash flows during phase 2         }
  it: t_array;                  {  schedule of total investment funds attributed                 }
  ie: t_array;                  {  schedule of equity investment funds attributed                }

 procedure get_data;            {  input procedure                                               }
 procedure set_data;            {  procedure to calculate time series out of input               }
 procedure print_array;         {  print time series of input                                    }
 function z (i: index_y;        {  zero function                                                 }
      j: index_x;    t: index_t): extended;
 function r (i: index_y;        {  interest rate function                                        }
      j: index_x;    t: index_t): extended;
 function v (i: index_y;        {  variance function                                             }
      j: index_x;    t: index_t): extended;
 function p (i: index_y;        {  cash flow function                                            }
      j: index_x;    t: index_t): extended;
 function e3b (i: index_y;      {  payout function of equity in phase 3                          }
      j: index_x;    t: index_t): extended;
 function e3l (i: index_y;      {  lower boundary function of equity in phase 3                  }
      j: index_x;    t: index_t): extended;
 function e3f (i: index_y;      {  termian boundary function of equity in phase 3                }
      j: index_x;    t: index_t): extended;
 function e3m (i: index_y;      {  critical value function of equity in phase 3                  }
      j: index_x;    t: index_t): extended;
 function d3b (i: index_y;      {  payout function of senior debt in phase 3                     }
      j: index_x;    t: index_t): extended;
 function d3l (i: index_y;      {  lower boundary function of senior debt in phase 3             }
      j: index_x;    t: index_t): extended;
 function d3h (i: index_y;      {  constant upper boundary function of senior debt in phase 3    }
      j: index_x;    t: index_t): extended;
 function d3f (i: index_y;      {  terminal boundary function of senior debt in phase 3          }
      j: index_x;    t: index_t): extended;
 function g3b (i: index_y;      {  payout function of  loan guarantee in phase 3                 }
      j: index_x;    t: index_t): extended;
 function g3l (i: index_y;      {  lower boundary function of  loan guarantee in phase 3         }
      j: index_x;    t: index_t): extended;
 function g3f (i: index_y;      {  terminal boundary function of  loan guarantee in phase 3      }
      j: index_x;    t: index_t): extended;
 function e2b (i: index_y;      {  payout function of  equitye in phase 2                        }
      j: index_x;    t: index_t): extended;
 function e2l (i: index_y;      {  lower boundary function of  equitye in phase 2                }
      j: index_x;    t: index_t): extended;
```

```
function e2f (i: index_y;                        {  terminal boundary function of  equitye in phase 2            }
     j: index_x;    t: index_t): extended;
function d2b (i: index_y;                        {  payout function of  senior debt in phase 2                   }
     j: index_x;    t: index_t): extended;
function d2l (i: index_y;                        {  lower boundary function of  senior debt in phase 2           }
     j: index_x;    t: index_t): extended;
function d2h (i: index_y;                        {  constant upper boundary function of  senior debt in phase 2  }
     j: index_x;    t: index_t): extended;
function d2f (i: index_y;                        {  terminal boundary function of  senior debt in phase 2        }
     j: index_x;    t: index_t): extended;
function g2b (i: index_y;                        {  payout function of  loan guarantee in phase 2                }
     j: index_x;    t: index_t): extended;
function g2l (i: index_y;                        {  lower boundary function of  loan guarantee in phase 2        }
     j: index_x;    t: index_t): extended;
function g2f (i: index_y;                        {  terminal boundary function of  loan guarantee in phase 2     }
     j: index_x;    t: index_t): extended;
function s2b (i: index_y;                        {  payout function of  cash flow guarantee in phase 2           }
     j: index_x;    t: index_t): extended;
function s2l (i: index_y;                        {  lower boundary function of  cash flow guarantee in phase 2   }
     j: index_x;    t: index_t): extended;
function w1b (i: index_y;                        {  payout function of  project in phase 1                       }
     j: index_x;    t: index_t): extended;
function w1f (i: index_y;                        {  terminal boundary function of  project in phase 1            }
     j: index_x;    t: index_t): extended;
function w1m (i: index_y;                        {  critical value function of  project in phase 1               }
     j: index_x;    t: index_t): extended;
function d1b (i: index_y;                        {  payout function of  debt in phase 1                          }
     j: index_x;    t: index_t): extended;
function d1l (i: index_y;                        {  lower boundary function of  debt in phase 1                  }
     j: index_x;    t: index_t): extended;
function d1h (i: index_y;                        {  constant upper boundary function of  debt in phase 1         }
     j: index_x;    t: index_t): extended;
function d1f (i: index_y;                        {  terminal boundary function of  debt in phase 1               }
     j: index_x;    t: index_t): extended;
function g1l (i: index_y;                        {  lower boundary function of  loan guarantee in phase 1        }
     j: index_x;    t: index_t): extended;
function g1f (i: index_y;                        {  terminal boundary function of  loan guarantee in phase 1     }
     j: index_x;    t: index_t): extended;
function s1f (i: index_y;                        {  terminal boundary function of  cash flow guarantee in phase 1 }
     j: index_x;    t: index_t): extended;
function e1b (i: index_y;                        {  payout function of  equity in phase 1                        }
     j: index_x;    t: index_t): extended;
function e1l (i: index_y;                        {  lower boundary function of  equity in phase 1                }
     j: index_x;    t: index_t): extended;
function e1f (i: index_y;                        {  terminal boundary function of  equity in phase 1             }
     j: index_x;    t: index_t): extended;

implementation
 var
  rf: extended;

 procedure get_data;                             {  numerical data see: chapter 5.5 example                      }
  var
   rs: extended;
   rj: extended;
   s, j, r: extended;
   t: index_t;
 begin
  t0 := 1;
  t1 := 17;
  t2 := 33;
  t3 := 49;
  rf := 0.10;
```

```
 rs := 0.12;
 rj := 0.10;
 s := 0;
 j := 0;
 for t := t0 to t1 - 1 do
  begin
   it[t] := 100 * unit_y;
   ie[t] := it[t] / 4;
   cf[t] := 0;
   s := s * (1 + rs * unit_x) + it[t] / 2;
   j := j * (1 + rj * unit_x) + it[t] / 4;
   bps[t] := s;
   bpj[t] := j;
   dps[t] := 0;
   dpj[t] := 0;
  end;
 for t := t1 to t2 - 1 do
  begin
   it[t] := 0;
   ie[t] := 0;
   cf[t] := 100 * unit_y;
   bps[t] := s;
   bpj[t] := j;
   dps[t] := s * (rs * unit_x);
   dpj[t] := j * (rj * unit_x);
  end;
 for t := t2 to t3 do
  begin
   it[t] := 0;
   ie[t] := 0;
   cf[t] := 0;
   bps[t] := s;
   bpj[t] := j;
   dps[t] := s * (rs * unit_x);
   dpj[t] := j * (rj * unit_x);
  end;
 dps[t3] := dps[t3] + bps[t3];
 dpj[t3] := dpj[t3] + bpj[t3];
 bps[t3] := 0;
 bpj[t3] := 0;
end;
procedure set_data;
 var
  t: index_t;
begin
 for t := t0 to t3 do
  begin
   pve[t] := 0;
   pvd[t] := 0;
   pvg[t] := 0;
   pvs[t] := 0;
   rbs[t] := 0;
  end;
 pvg[t2] := bpj[t2 - 1] * (1 + rf * unit_x);
 for t := t2 - 1 downto t1 do
  begin
   pve[t] := (max(cf[t] - dps[t] - dpj[t], 0) + pve[t + 1]) / (1 + rf * unit_x);
   pvd[t] := (dps[t] + pvd[t + 1]) / (1 + rf * unit_x);
   pvg[t] := (max(dps[t] + dpj[t] - cf[t], 0) + pvg[t + 1]) / (1 + rf * unit_x);
   pvs[t] := (cf[t] + pvs[t + 1]) / (1 + rf * unit_x);
  end;
 pvg[t2] := 0;
 rbs[t3] := dps[t3];
```

```
 for t := t3 - 1 downto t0 do
  rbs[t] := (dps[t] + rbs[t + 1]) / (1 + rf * unit_x);
end;
procedure print_array;
 var
  t: index_t;
begin
 writeln('t' : digits, 'it' : digits, 'ie' : digits, 'bps' : digits, 'dps' : digits, 'bpj' : digits, 'dpj' : digits, 'cf' : digits, 'rbs' : digits, 'pve' : digits, 'pvd' : digits, 'pvg' : digits, 'pvs' : digits);
 for t := t0 to t3 do
  writeln(t : digits, it[t] / unit_y : digits : decimals, ie[t] / unit_y : digits : decimals, bps[t] / unit_y : digits : decimals, dps[t] / unit_y : digits : decimals, bpj[t] / unit_y : digits : decimals, dpj[t] / unit_y : digits : decimals, cf[t] / unit_y : digits : decimals, rbs[t] / unit_y : digits : decimals, pve[t] / unit_y : digits : decimals, pvd[t] / unit_y : digits : decimals, pvg[t] / unit_y : digits : decimals, pvs[t] / unit_y : digits : decimals);
 writeln('');
end;
function z;
begin
 z := 0;
end;
function r;
begin
 r := rf;
end;
function v;
begin
 v := 0.20;
end;
function p;
begin
 p := 0.05;
end;
function e3b;
begin
 e3b := max(min(p(i, j, t) * exp((i - 1) * dy) - dps[t], 0), p(i, j, t) * exp((i - 1) * dy) - dps[t] - dpj[t]);
end;
function e3l;
begin
 e3l := max(exp((i - 1) * dy) - dps[t] - dpj[t] - bps[t] - bpj[t], 0);
end;
function e3f;
begin
 e3f := max(exp((i - 1) * dy) - dps[t] - dpj[t], 0);
end;
function e3m;
begin
 e3m := dps[t];
end;
function d3b;
begin
 d3b := dps[t];
end;
function d3l;
begin
 d3l := min(exp((i - 1) * dy), dps[t] + bps[t]);
end;
function d3h;
begin
 d3h := rbs[t];
end;
function d3f;
begin
 d3f := min(exp((i - 1) * dy), dps[t]);
```

```
end;
function g3b;
begin
 g3b := max(min(dpj[t], dpj[t] + dps[t] - p(i, j, t) * exp((i - 1) * dy)), 0);
end;
function g3l;
begin
 g3l := max(min(dpj[t] + bpj[t], dps[t] + bps[t] + dpj[t] + bpj[t] - exp((i - 1) * dy)), 0);
end;
function g3f;
begin
 g3f := max(min(dpj[t], dps[t] - exp((i - 1) * dy)), 0);
end;
function e2b;
begin
 e2b := max(max(p(i, j, t) * exp((i - 1) * dy), cf[t]) - dps[t] - dpj[t], 0);
end;
function e2l;
begin
 e2l := pve[t];
end;
function e2f;
begin
 e2f := e3^[i, max_x];
end;
function d2b;
begin
 d2b := dps[t];
end;
function d2l;
begin
 d2l := pvd[t];
end;
function d2h;
begin
 d2h := rbs[t];
end;
function d2f;
begin
 d2f := d3^[i, max_x];
end;
function g2b;
begin
 g2b := max(dps[t] + dpj[t] - max(cf[t], p(i, j, t) * exp((i - 1) * dy)), 0);
end;
function g2l;
begin
 g2l := pvg[t];
end;
function g2f;
begin
 g2f := g3^[i, max_x];
end;
function s2b;
begin
 s2b := max(cf[t] - p(i, j, t) * exp((i - 1) * dy), 0);
end;
function s2l;
begin
 s2l := pvs[t];
end;
function w1b;
begin
```

```
 w1b := -it[t];
end;
function w1f;
begin
 w1f := exp((i - 1) * dy);
end;
function w1m;
begin
 w1m := 0;
end;
function e1b;
begin
 e1b := -(ie[t] + g1^[i, j]);
end;
function e1l;
begin
 e1l := max(w1^[i, j] - bps[t] - bpj[t] - dps[t] - dpj[t], 0);
end;
function e1f;
begin
 e1f := e2^[i, max_x];
end;
function d1b;
begin
 d1b := dps[t];
end;
function d1l;
begin
 d1l := min(w1^[i, j], dps[t] + bps[t]);
end;
function d1h;
begin
 d1h := rbs[t];
end;
function d1f;
begin
 d1f := d2^[i, max_x];
end;
function g1l;
begin
 g1l := max(min(dpj[t] + bpj[t], dpj[t] + bpj[t] + dps[t] + bps[t] - w1^[i, j]), 0);
end;
function g1f;
begin
 g1f := g2^[i, max_x];
end;
function s1f;
begin
 s1f := s2^[i, max_x];
end;


end.
```

PROJECT FINANCING

TIME SERIES

t	it	ie	bps	dps	bpj	dpj	cf	rbs	pve	pvd	pvg	pvs
1	100	25	50	0	25	0	0	762	0	0	0	0
2	100	25	102	0	51	0	0	781	0	0	0	0
3	100	25	155	0	77	0	0	801	0	0	0	0
4	100	25	209	0	104	0	0	821	0	0	0	0
5	100	25	265	0	131	0	0	841	0	0	0	0
6	100	25	323	0	160	0	0	862	0	0	0	0
7	100	25	383	0	189	0	0	884	0	0	0	0
8	100	25	445	0	218	0	0	906	0	0	0	0
9	100	25	508	0	249	0	0	929	0	0	0	0
10	100	25	573	0	280	0	0	952	0	0	0	0
11	100	25	640	0	312	0	0	976	0	0	0	0
12	100	25	710	0	345	0	0	1000	0	0	0	0
13	100	25	781	0	379	0	0	1025	0	0	0	0
14	100	25	854	0	413	0	0	1051	0	0	0	0
15	100	25	930	0	448	0	0	1077	0	0	0	0
16	100	25	1008	0	485	0	0	1104	0	0	0	0
17	0	0	1008	30	485	12	100	1132	753	395	335	1306
18	0	0	1008	30	485	12	100	1130	714	374	343	1238
19	0	0	1008	30	485	12	100	1128	674	353	351	1169
20	0	0	1008	30	485	12	100	1126	633	332	360	1098
21	0	0	1008	30	485	12	100	1124	591	310	369	1026
22	0	0	1008	30	485	12	100	1121	549	288	378	951
23	0	0	1008	30	485	12	100	1119	505	265	388	875
24	0	0	1008	30	485	12	100	1117	460	241	398	797
25	0	0	1008	30	485	12	100	1115	413	217	408	717
26	0	0	1008	30	485	12	100	1112	366	192	418	635
27	0	0	1008	30	485	12	100	1110	318	167	428	551
28	0	0	1008	30	485	12	100	1107	268	140	439	465
29	0	0	1008	30	485	12	100	1105	217	114	450	376
30	0	0	1008	30	485	12	100	1102	165	86	461	286
31	0	0	1008	30	485	12	100	1100	111	58	473	193
32	0	0	1008	30	485	12	100	1097	56	29	485	98
33	0	0	1008	30	485	12	0	1094	0	0	0	0
34	0	0	1008	30	485	12	0	1091	0	0	0	0
35	0	0	1008	30	485	12	0	1088	0	0	0	0
36	0	0	1008	30	485	12	0	1085	0	0	0	0
37	0	0	1008	30	485	12	0	1082	0	0	0	0
38	0	0	1008	30	485	12	0	1079	0	0	0	0
39	0	0	1008	30	485	12	0	1076	0	0	0	0
40	0	0	1008	30	485	12	0	1072	0	0	0	0
41	0	0	1008	30	485	12	0	1069	0	0	0	0
42	0	0	1008	30	485	12	0	1065	0	0	0	0
43	0	0	1008	30	485	12	0	1062	0	0	0	0
44	0	0	1008	30	485	12	0	1058	0	0	0	0
45	0	0	1008	30	485	12	0	1054	0	0	0	0
46	0	0	1008	30	485	12	0	1050	0	0	0	0
47	0	0	1008	30	485	12	0	1046	0	0	0	0
48	0	0	1008	30	485	12	0	1042	0	0	0	0
49	0	0	0	1038	0	497	0	1038	0	0	0	0

CONTINGENT CLAIMS

e3	33	35	37	39	41	43	45	47	49
10000	12565	12218	11842	11432	10982	10485	9925	9278	8466
8577	11020	10676	10303	9897	9452	8962	8412	7782	7043
7357	9580	9243	8879	8483	8052	7580	7060	6474	5822

6310	8248	7923	7572	7193	6783	6338	5859	5339	4775
5412	7031	6722	6389	6031	5646	5235	4801	4354	3877
4642	5936	5645	5334	5000	4644	4267	3877	3497	3107
3981	4961	4691	4403	4095	3769	3426	3073	2734	2447
3415	4103	3855	3591	3311	3015	2705	2382	2060	1880
2929	3354	3129	2891	2639	2373	2093	1797	1474	1394
2512	2708	2506	2294	2070	1834	1583	1314	1006	977
2154	2155	1977	1790	1593	1386	1167	929	652	620
1848	1686	1531	1369	1199	1021	833	631	409	313
1585	1294	1160	1022	878	728	572	407	238	50
1359	968	855	739	619	497	371	240	116	0
1166	702	607	513	415	318	219	124	45	0
1000	488	410	335	258	183	100	48	0	0
858	318	257	195	141	79	31	0	0	0
736	190	147	89	61	0	0	0	0	0
631	88	65	0	0	0	0	0	0	0
541	0	0	0	0	0	0	0	0	0
464	0	0	0	0	0	0	0	0	0
398	0	0	0	0	0	0	0	0	0
341	0	0	0	0	0	0	0	0	0
293	0	0	0	0	0	0	0	0	0
251	0	0	0	0	0	0	0	0	0
215	0	0	0	0	0	0	0	0	0
185	0	0	0	0	0	0	0	0	0
158	0	0	0	0	0	0	0	0	0
136	0	0	0	0	0	0	0	0	0
117	0	0	0	0	0	0	0	0	0
100	0	0	0	0	0	0	0	0	0

e2	17	19	21	23	25	27	29	31	33
10000	14371	14199	14020	13835	13639	13425	13177	12884	12565
8577	12809	12636	12457	12271	12076	11865	11624	11338	11020
7357	11319	11144	10963	10776	10582	10379	10158	9891	9580
6310	9908	9731	9547	9357	9163	8968	8772	8549	8248
5412	8589	8408	8219	8025	7829	7637	7463	7303	7031
4642	7374	7189	6995	6794	6590	6393	6225	6132	5936
3981	6270	6080	5880	5671	5455	5242	5055	4975	4961
3415	5279	5086	4880	4661	4431	4194	3963	3816	4103
2929	4405	4209	3999	3772	3527	3262	2975	2673	3354
2512	3647	3450	3238	3006	2750	2462	2123	1670	2708
2154	3002	2808	2596	2364	2104	1804	1437	902	2155
1848	2474	2285	2078	1850	1594	1298	938	439	1686
1585	2051	1868	1670	1453	1210	934	609	216	1294
1359	1712	1540	1354	1150	930	682	410	158	968
1166	1446	1284	1113	927	732	516	301	153	702
1000	1240	1089	933	767	596	423	266	111	488
858	1080	942	798	654	515	381	217	111	318
736	955	834	707	591	413	318	217	111	190
631	867	766	591	505	413	318	217	111	88
541	753	674	591	505	413	318	217	111	0
464	753	674	591	505	413	318	217	111	0
398	753	674	591	505	413	318	217	111	0
341	753	674	591	505	413	318	217	111	0
293	753	674	591	505	413	318	217	111	0
251	753	674	591	505	413	318	217	111	0
215	753	674	591	505	413	318	217	111	0
185	753	674	591	505	413	318	217	111	0
158	753	674	591	505	413	318	217	111	0
136	753	674	591	505	413	318	217	111	0
117	753	674	591	505	413	318	217	111	0
100	753	674	591	505	413	318	217	111	0

e1	1	3	5	7	9	11	13	15	17
10000	10888	11194	11523	11878	12262	12683	13149	13681	14371
8577	9445	9749	10075	10427	10808	11223	11681	12200	12809
7357	8153	8450	8770	9113	9483	9883	10319	10800	11319
6310	6985	7274	7582	7913	8265	8642	9045	9474	9908
5412	5923	6199	6493	6807	7139	7489	7854	8228	8589
4642	4952	5212	5490	5785	6094	6416	6744	7068	7374
3981	4062	4303	4562	4838	5126	5422	5717	6003	6270
3415	3247	3466	3706	3963	4232	4506	4777	5037	5279
2929	2506	2701	2920	3158	3411	3670	3926	4173	4405
2512	1842	2009	2204	2422	2661	2913	3165	3412	3647
2154	1262	1396	1563	1759	1985	2234	2493	2752	3002
1848	778	873	1006	1175	1384	1633	1906	2189	2474
1585	403	455	547	683	864	1107	1400	1715	2051
1359	145	155	202	297	438	659	969	1323	1712
1166	0	0	0	29	120	295	605	1001	1446
1000	0	0	0	0	0	28	305	739	1240
858	0	0	0	0	0	0	67	530	1080
736	0	0	0	0	0	0	0	364	955
631	0	0	0	0	0	0	0	239	867
541	0	0	0	0	0	0	0	149	753
464	0	0	0	0	0	0	0	92	753
398	0	0	0	0	0	0	0	59	753
341	0	0	0	0	0	0	0	44	753
293	0	0	0	0	0	0	0	38	753
251	0	0	0	0	0	0	0	36	753
215	0	0	0	0	0	0	0	30	753
185	0	0	0	0	0	0	0	0	753
158	0	0	0	0	0	0	0	0	753
136	0	0	0	0	0	0	0	0	753
117	0	0	0	0	0	0	0	0	753
100	0	0	0	0	0	0	0	0	753

d3	33	35	37	39	41	43	45	47	49
10000	1094	1088	1082	1076	1069	1062	1054	1046	1038
8577	1106	1100	1094	1088	1081	1073	1065	1056	1038
7357	1101	1096	1090	1083	1076	1068	1059	1049	1038
6310	1099	1094	1089	1082	1075	1067	1058	1048	1038
5412	1097	1093	1088	1082	1074	1066	1057	1047	1038
4642	1094	1090	1086	1081	1074	1066	1057	1047	1038
3981	1089	1087	1084	1079	1073	1066	1057	1048	1038
3415	1083	1082	1080	1077	1072	1066	1058	1049	1038
2929	1073	1074	1074	1073	1071	1066	1059	1050	1038
2512	1060	1063	1065	1067	1067	1065	1060	1052	1038
2154	1043	1047	1051	1056	1060	1063	1060	1054	1038
1848	1020	1026	1032	1040	1048	1056	1059	1055	1038
1585	991	998	1006	1016	1028	1042	1054	1055	1038
1359	955	962	971	983	999	1017	1044	1057	1038
1166	911	918	927	939	956	979	1014	1044	1038
1000	859	864	872	883	899	932	958	1000	1000
858	797	800	806	815	831	854	858	858	858
736	726	727	732	737	736	736	736	736	736
631	647	648	631	631	631	631	631	631	631
541	541	541	541	541	541	541	541	541	541
464	464	464	464	464	464	464	464	464	464
398	398	398	398	398	398	398	398	398	398
341	341	341	341	341	341	341	341	341	341
293	293	293	293	293	293	293	293	293	293
251	251	251	251	251	251	251	251	251	251
215	215	215	215	215	215	215	215	215	215
185	185	185	185	185	185	185	185	185	185

158	158	158	158	158	158	158	158	158	158
136	136	136	136	136	136	136	136	136	136
117	117	117	117	117	117	117	117	117	117
100	100	100	100	100	100	100	100	100	100

d2	17	19	21	23	25	27	29	31	33
10000	1132	1128	1124	1119	1115	1110	1105	1100	1094
8577	1104	1101	1100	1101	1103	1108	1113	1111	1106
7357	1059	1058	1059	1063	1072	1085	1101	1106	1101
6310	1017	1015	1017	1023	1036	1056	1084	1104	1099
5412	973	970	970	976	989	1013	1050	1095	1097
4642	926	920	918	920	930	952	993	1066	1094
3981	876	867	860	856	859	873	907	992	1089
3415	825	811	797	786	779	779	794	858	1083
2929	774	753	733	713	693	675	662	666	1073
2512	722	696	668	638	606	569	524	458	1060
2154	673	641	606	567	522	469	397	275	1043
1848	626	589	548	502	447	381	294	155	1020
1585	582	541	496	444	384	311	219	92	991
1359	544	500	451	395	333	258	170	79	955
1166	510	463	413	356	294	220	140	80	911
1000	481	433	382	325	264	201	134	58	859
858	457	409	357	302	250	194	114	58	797
736	436	389	343	292	217	167	114	58	726
631	425	381	310	265	217	167	114	58	647
541	395	353	310	265	217	167	114	58	541
464	395	353	310	265	217	167	114	58	464
398	395	353	310	265	217	167	114	58	398
341	395	353	310	265	217	167	114	58	341
293	395	353	310	265	217	167	114	58	293
251	395	353	310	265	217	167	114	58	251
215	395	353	310	265	217	167	114	58	215
185	395	353	310	265	217	167	114	58	185
158	395	353	310	265	217	167	114	58	158
136	395	353	310	265	217	167	114	58	136
117	395	353	310	265	217	167	114	58	117
100	395	353	310	265	217	167	114	58	100

d1	1	3	5	7	9	11	13	15	17
10000	762	801	841	884	929	976	1025	1077	1132
8577	731	768	807	848	891	937	986	1038	1104
7357	703	739	777	817	859	904	952	1004	1059
6310	673	707	743	782	822	866	912	963	1017
5412	643	676	711	747	786	828	872	921	973
4642	613	645	678	713	749	789	831	877	926
3981	583	614	645	678	712	749	789	831	876
3415	552	583	612	643	675	709	746	784	825
2929	520	551	580	609	638	670	702	737	774
2512	485	518	548	575	602	630	660	690	722
2154	444	482	515	542	567	592	618	645	673
1848	394	439	480	510	533	556	579	602	626
1585	330	388	439	477	502	522	542	563	582
1359	252	322	382	441	472	491	509	527	544
1166	163	243	306	389	441	462	479	495	510
1000	0	0	208	317	405	437	452	467	481
858	0	0	0	218	352	414	429	443	457
736	0	0	0	0	280	392	410	422	436
631	0	0	0	0	0	356	394	406	425
541	0	0	0	0	0	0	382	393	395
464	0	0	0	0	0	0	372	385	395
398	0	0	0	0	0	0	349	380	395

341	0	0	0	0	0	0	0	377	395
293	0	0	0	0	0	0	0	376	395
251	0	0	0	0	0	0	0	376	395
215	0	0	0	0	0	0	0	373	395
185	0	0	0	0	0	0	0	0	395
158	0	0	0	0	0	0	0	0	395
136	0	0	0	0	0	0	0	0	395
117	0	0	0	0	0	0	0	0	395
100	0	0	0	0	0	0	0	0	395

g3	33	35	37	39	41	43	45	47	49
10000	10	6	4	2	0	0	0	0	0
8577	11	7	4	2	1	0	0	0	0
7357	14	10	6	3	1	0	0	0	0
6310	19	15	10	6	3	1	0	0	0
5412	27	22	16	11	6	2	0	0	0
4642	37	31	25	18	11	4	1	0	0
3981	51	45	37	29	19	9	2	0	0
3415	67	61	54	44	32	19	6	0	0
2929	88	83	75	65	52	35	16	1	0
2512	112	108	102	92	79	60	34	6	0
2154	140	137	133	126	115	97	66	23	0
1848	171	171	170	166	159	144	116	63	0
1585	205	208	210	211	210	203	185	135	0
1359	242	248	253	260	265	269	273	241	0
1166	281	289	298	311	322	341	362	365	0
1000	321	333	344	360	379	416	437	497	38
858	363	376	392	407	436	465	497	497	180
736	411	422	447	457	497	497	497	497	302
631	460	466	497	497	497	497	497	497	407
541	497	497	497	497	497	497	497	497	497
464	497	497	497	497	497	497	497	497	497
398	497	497	497	497	497	497	497	497	497
341	497	497	497	497	497	497	497	497	497
293	497	497	497	497	497	497	497	497	497
251	497	497	497	497	497	497	497	497	497
215	497	497	497	497	497	497	497	497	497
185	497	497	497	497	497	497	497	497	497
158	497	497	497	497	497	497	497	497	497
136	497	497	497	497	497	497	497	497	497
117	497	497	497	497	497	497	497	497	497
100	497	497	497	497	497	497	497	497	497

g2	17	19	21	23	25	27	29	31	33
10000	30	27	25	23	21	20	17	14	10
8577	31	28	26	23	22	21	18	15	11
7357	34	31	28	26	24	22	21	18	14
6310	39	36	32	29	27	25	25	24	19
5412	47	43	39	35	31	29	29	31	27
4642	56	53	48	43	38	34	33	39	37
3981	69	65	61	55	48	41	37	44	51
3415	84	81	77	71	62	52	42	44	67
2929	102	100	96	91	82	69	51	38	88
2512	122	121	120	116	108	95	71	35	112
2154	144	146	147	145	141	129	105	49	140
1848	167	172	176	179	179	173	156	102	171
1585	191	199	207	214	221	224	222	197	205
1359	216	227	238	251	264	278	296	327	242
1166	240	254	269	287	306	331	365	418	281
1000	262	279	297	319	344	380	415	473	321
858	283	302	324	347	376	410	450	473	363

736	302	321	347	368	408	428	450	473	411
631	318	337	369	388	408	428	450	473	460
541	335	351	369	388	408	428	450	473	497
464	335	351	369	388	408	428	450	473	497
398	335	351	369	388	408	428	450	473	497
341	335	351	369	388	408	428	450	473	497
293	335	351	369	388	408	428	450	473	497
251	335	351	369	388	408	428	450	473	497
215	335	351	369	388	408	428	450	473	497
185	335	351	369	388	408	428	450	473	497
158	335	351	369	388	408	428	450	473	497
136	335	351	369	388	408	428	450	473	497
117	335	351	369	388	408	428	450	473	497
100	335	351	369	388	408	428	450	473	497

g1	1	3	5	7	9	11	13	15	17
10000	45	44	43	41	40	38	35	33	30
8577	46	45	44	42	40	38	36	34	31
7357	48	47	46	45	43	41	39	37	34
6310	51	51	50	49	48	46	44	42	39
5412	56	56	56	55	54	53	51	49	47
4642	62	62	63	63	62	62	60	59	56
3981	69	70	71	72	72	72	72	71	69
3415	77	79	81	82	83	84	85	85	84
2929	86	89	92	94	96	98	100	101	102
2512	95	100	103	107	110	114	117	119	122
2154	103	111	116	121	125	130	135	139	144
1848	109	121	129	135	141	148	154	160	167
1585	110	130	142	150	157	165	174	182	191
1359	103	135	153	165	173	183	193	204	216
1166	89	134	161	180	189	200	213	226	240
1000	25	77	163	193	206	217	231	246	262
858	25	77	131	203	222	232	247	265	283
736	25	77	131	189	239	246	262	281	302
631	25	77	131	189	249	261	274	295	318
541	25	77	131	189	249	312	284	305	335
464	25	77	131	189	249	312	292	312	335
398	25	77	131	189	249	312	301	316	335
341	25	77	131	189	249	312	379	317	335
293	25	77	131	189	249	312	379	318	335
251	25	77	131	189	249	312	379	318	335
215	25	77	131	189	249	312	379	319	335
185	25	77	131	189	249	312	379	448	335
158	25	77	131	189	249	312	379	448	335
136	25	77	131	189	249	312	379	448	335
117	25	77	131	189	249	312	379	448	335
100	25	77	131	189	249	312	379	448	335

s2	17	19	21	23	25	27	29	31	33
10000	8	5	2	1	0	0	0	0	0
8577	9	6	3	1	0	0	0	0	0
7357	13	8	5	2	1	0	0	0	0
6310	19	13	8	4	2	0	0	0	0
5412	28	20	13	8	4	1	0	0	0
4642	42	31	22	13	7	3	0	0	0
3981	62	48	35	23	13	6	1	0	0
3415	90	72	54	38	24	12	3	0	0
2929	129	106	83	61	41	23	9	0	0
2512	180	151	123	95	68	43	21	3	0
2154	247	212	177	142	107	73	41	12	0
1848	337	296	255	212	168	123	80	37	0

1585	447	400	351	301	247	190	136	72	0
1359	566	512	455	397	335	265	199	141	0
1166	688	629	563	499	427	346	257	193	0
1000	812	747	674	598	518	463	338	193	0
858	934	861	785	692	637	546	376	193	0
736	1056	968	926	807	717	551	376	193	0
631	1203	1093	1026	875	717	551	376	193	0
541	1306	1169	1026	875	717	551	376	193	0
464	1306	1169	1026	875	717	551	376	193	0
398	1306	1169	1026	875	717	551	376	193	0
341	1306	1169	1026	875	717	551	376	193	0
293	1306	1169	1026	875	717	551	376	193	0
251	1306	1169	1026	875	717	551	376	193	0
215	1306	1169	1026	875	717	551	376	193	0
185	1306	1169	1026	875	717	551	376	193	0
158	1306	1169	1026	875	717	551	376	193	0
136	1306	1169	1026	875	717	551	376	193	0
117	1306	1169	1026	875	717	551	376	193	0
100	1306	1169	1026	875	717	551	376	193	0

s1	1	3	5	7	9	11	13	15	17
10000	57	51	44	37	30	24	18	13	8
8577	59	53	46	39	32	26	20	14	9
7357	65	59	53	46	38	31	24	18	13
6310	76	70	64	56	49	41	33	26	19
5412	90	86	79	72	64	55	46	37	28
4642	109	106	100	93	85	75	64	53	42
3981	131	130	126	120	112	102	90	76	62
3415	155	159	158	153	145	136	123	108	90
2929	179	190	194	192	187	178	166	149	129
2512	202	222	235	238	236	229	219	203	180
2154	217	251	277	290	292	290	283	271	247
1848	221	270	316	345	356	359	358	353	337
1585	205	273	344	398	425	436	443	446	447
1359	169	252	342	443	495	518	535	550	566
1166	115	205	303	450	560	604	633	660	688
1000	0	0	220	407	602	689	731	773	812
858	0	0	0	300	590	770	827	885	934
736	0	0	0	0	510	835	915	990	1056
631	0	0	0	0	0	838	992	1080	1203
541	0	0	0	0	0	0	1055	1149	1306
464	0	0	0	0	0	0	1100	1196	1306
398	0	0	0	0	0	0	1078	1223	1306
341	0	0	0	0	0	0	0	1236	1306
293	0	0	0	0	0	0	0	1241	1306
251	0	0	0	0	0	0	0	1242	1306
215	0	0	0	0	0	0	0	1235	1306
185	0	0	0	0	0	0	0	0	1306
158	0	0	0	0	0	0	0	0	1306
136	0	0	0	0	0	0	0	0	1306
117	0	0	0	0	0	0	0	0	1306
100	0	0	0	0	0	0	0	0	1306

w1	1	3	5	7	9	11	13	15	17
10000	8655	8794	8940	9094	9256	9427	9606	9796	10000
8577	7237	7376	7523	7677	7838	8009	8188	8377	8577
7357	6021	6160	6306	6460	6622	6792	6971	7159	7357
6310	4977	5116	5262	5416	5578	5748	5926	6114	6310
5412	4082	4221	4367	4520	4682	4851	5029	5216	5412
4642	3314	3453	3598	3752	3913	4082	4260	4446	4642
3981	2655	2793	2939	3092	3253	3422	3599	3786	3981

3415	2090	2228	2373	2526	2687	2856	3033	3219	3415
2929	1607	1743	1888	2041	2201	2370	2547	2733	2929
2512	1194	1328	1471	1624	1784	1953	2130	2316	2512
2154	844	973	1115	1266	1427	1595	1773	1959	2154
1848	552	673	810	960	1120	1289	1466	1652	1848
1585	316	422	551	698	857	1026	1203	1389	1585
1359	136	222	335	474	632	800	977	1164	1359
1166	15	74	164	287	439	607	784	970	1166
1000	0	0	39	136	274	441	618	804	1000
858	0	0	0	25	139	298	476	662	858
736	0	0	0	0	33	177	353	540	736
631	0	0	0	0	0	76	249	435	631
541	0	0	0	0	0	0	159	345	541
464	0	0	0	0	0	0	82	268	464
398	0	0	0	0	0	0	18	202	398
341	0	0	0	0	0	0	0	146	341
293	0	0	0	0	0	0	0	97	293
251	0	0	0	0	0	0	0	55	251
215	0	0	0	0	0	0	0	19	215
185	0	0	0	0	0	0	0	0	185
158	0	0	0	0	0	0	0	0	158
136	0	0	0	0	0	0	0	0	136
117	0	0	0	0	0	0	0	0	117
100	0	0	0	0	0	0	0	0	100

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