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Dissertation, Universität Karlsruhe (TH)
Fakultät für Mathematik, 2006

Impressum

Universitätsverlag Karlsruhe
c/o Universitätsbibliothek
Straße am Forum 2
D-76131 Karlsruhe
www.uvka.de

Universitätsverlag Karlsruhe 2007
Print on Demand

ISBN: 978-3-86644-144-6

Path Regularity for Stochastic Differential Equations in Banach Spaces

Zur Erlangung des akademischen Grades eines

Doktors der Naturwissenschaften

von der Fakultät für Mathematik der

Universität Karlsruhe

genehmigte

Dissertation

von

Dipl.-math.-oec. Johanna Dettweiler

aus Tübingen

Tag der mündlichen Prüfung: 22. November 2006

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Universität Karlsruhe

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DIAM
Delft University of Technology

Contents

Introduction

Regularity properties of solutions of stochastic differential equations have been studied by many authors ([33, 13, 15, 8, 9, 11, 6]). However, with the exception of [33], who works in L^p-spaces with $p \geq 2$ and [4], who considers spaces of Martingale-type 2, most of the optimal regularity results so far were obtained in Hilbert spaces.

In this work we consider equations of the form

$$\begin{cases} dU(t) = AU(t)dt + BdW_H(t)\,, & t \in [0,T] \\ U_0 = \xi, & \xi \quad \mathcal{F}_0\text{-measurable} \end{cases} \tag{1}$$

where $\big(A, \mathcal{D}(A)\big)$ is the generator of a C_0-semigroup on a general separable Banach space E, B a bounded linear operator from a separable real Hilbert space H into E and $W_H(t) : H \to L^2(\Omega)$ a cylindrical Wiener process with Cameron Martin space H.

In particular this setting includes stochastic partial differential equation driven by space-time white noise:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = Lu(t,x) + \dfrac{\partial w}{\partial t}(t,x), & x \in [0,1],\ t \in [0,T], \\ u(0,x) = 0, & x \in [0,1], \\ u(t,0) = u(t,1) = 0, & t \in [0,T], \end{cases} \tag{2}$$

where L is a uniformly elliptic operator of the form

$$Lf(x) = a(x)f''(x) + b(x)f'(x) + c(x)f(x), \quad x \in [0,1],$$

with coefficients $a \in C^\varepsilon[0,1]$ for some $\varepsilon > 0$ and $b, c \in L^\infty[0,1]$. We will especially study the effect of additional properties of E, A and B on the regularity of the paths of the solution of (1).

In particular we are aiming at extending the optimal regularity results known in Hilbert spaces to the Banach space setting (precise statements are given below). To this end we often assume

- A to be the generator of an *analytic semigroup* or of a C_0*-group*, in order to exploit the smoothing effects of an analytic semigroup.

- B to be in $\gamma(H,E)$, the space of γ-radonifying operators from H into E, which take the place of Hilbert Schmidt operators in the Hilbert space case and ensure the existence of Gaussian measures on E.

- Occasionally we assume that E has finite cotype which means geometrically that E does not contain l_∞^n's uniformly for all dimensions n.

This work is based on the theory of stochastic integration in Banach spaces due to [45]. Even though the problem of stochastic integration in Banach spaces was studied by many authors (see e.g. [5, 8, 19, 18, 48]) this new approach however allows direct links to deep results of functional analysis and operator theory (see [30, 58, 34]). For instance it is well known that the very satisfactory results in Hilbert spaces are due to the Itô isometry. The approach in [45] now provides for the first time an analogon using the γ-norm of the integrand. The isometry then reads as follows (compare Section 2.1):

$$\mathbb{E}\left\|\int_0^T \Phi(t)dW_H(t)\right\|^2 = \|I_\Phi\|_\gamma^2. \tag{3}$$

where I_Φ is a integral operator connected with the integrand Φ. This isometry allows us to apply results of operator theory for analyzing stochastic integrability by studying the space of operators with finite γ-norm.

This also enables us to transfer Hilbert space results to the Banach space setting as done in Theorem 3.2.7 or Theorem 4.2.1 and to develop a self contained theory of stochastic differential equations. We illustrate this theory by discussing examples as the one quoted above. The work is organized as follows:

The **first chapter** collects some concepts from Banach space valued integration, probability theory, the notion of radonifying operators and the H^∞-functional calculus. Furthermore we will develop the principle of γ-boundedness as an analogon to the well known R-boundedness (for references see the individual sections).

Chapter two outlines the concept of stochastic integration and solutions of (1) due to [45].

In **chapter three** we show existence and regularity of solutions of (1). The main result of this chapter is Theorem 3.3.1:

Assume A to be the generator of an analytic semigroup and B to be a γ-radonifying operator. Furthermore, without loss of generality, we can assume that A has a negative growth bound. We show how time regularity interacts with space regularity: consider the 'space regularity'-parameter $\eta \geq 0$ and the 'time regularity'-parameter $\theta \geq 0$. If the two parameters then satisfy $\eta + \theta < \frac{1}{2}$, we have

1. The random variables $U(t)$ take values in $\mathcal{D}((-A)^\eta)$ almost surely and we have
$$\mathbb{E}\|U(t) - U(s)\|^2_{\mathcal{D}((-A)^\eta)} \leq C|t-s|^{2\theta}\|B\|^2_{\gamma(H,E)} \quad \forall t, s \in [0,T],$$
with a constant C independent of B;
2. The process U has a version with paths in $C^\theta\big([0,T]; \mathcal{D}((-A)^\eta)\big)$.

In Section 3.3.2 this result will be carried forward to the setting where B is allowed to be unbounded. These results allow us to analyze example 2 above.

As we will see in Theorem 3.3.1 in general we can only expect solutions in $\mathcal{D}((-A)^\eta)$, $\eta < \frac{1}{2}$, but not in $\mathcal{D}((-A)^{\frac{1}{2}})$. In Hilbert spaces this was observed in [15, Theorem 5.14]. However, in **chapter four** we use geometrical properties of E and the H^∞-calculus of the operator $-A$ to show precisely when maximal regularity, i.e. solutions with paths in $\mathcal{D}((-A)^{\frac{1}{2}})$, is possible.

Let E have finite cotype and assume that $-A$ admits γ-bounded H^∞-calculus of angle $0 < \omega_\infty^\gamma(-A) < \frac{\pi}{2}$. Then the solution U of problem (SCP) has maximal regularity in the sense that for all $t \in [0,T]$ we have $U(t) \in \mathcal{D}((-A)^{\frac{1}{2}})$ almost surely and

$$\mathbb{E}\|(-A)^{\frac{1}{2}}U(t)\|^2 \leq C\|B\|^2_{\gamma(H,E)} \tag{4}$$

for a suitable constant C independent of $T > 0$, $t \in [0,T]$, and $B \in \gamma(H,E)$.

Also a characterization of the bounded H^∞-calculus is given in Theorem 4.2.4. As the following characterization (see Theorem 4.2.4) shows, the boundedness or the H^∞-calculus is essentially a necessary condition for maximal regularity.

Let both E and E^* have finite cotype, and let $-A$ be a sectorial operator in E of angle $0 < \omega(-A) < \frac{\pi}{2}$. Then $-A$ admits a bounded H^∞-calculus if and only if

$$\begin{aligned} dU(t) &= AU(t)\,dt + x\,dW_H(t), \qquad t \geq 0,\\ U(0) &= 0, \end{aligned}$$

and

$$\begin{aligned} d\widetilde{U}(t) &= A^\odot \widetilde{U}(t)\,dt + x^\odot\,dW_H(t), \qquad t \geq 0,\\ \tilde{U}(0) &= 0, \end{aligned}$$

have maximal regularity for all $x \in E$ and $x^{\odot} \in E^{\odot} := \overline{\mathcal{D}(A^*)}$, respectively.

In **chapter five** we assume A to be the generator of a C_0-group $\mathbf{S}$. We give a characterization of the existence of solutions of (1) in terms of the boundedness of the H^∞-calculus. If E and E^* have finite cotype we show in Theorem 5.3.1 that the following are equivalent:

(a) The equations

$$\begin{aligned} dU(t) &= AU(t)\,dt + x\,dW_H(t), \qquad t \geq 0, \\ U(0) &= 0, \end{aligned}$$

and

$$\begin{aligned} d\tilde{U}(t) &= A^{\odot}\tilde{U}(t)\,dt + x^{\odot}\,dW_H(t), \qquad t \geq 0, \\ \tilde{U}(0) &= 0, \end{aligned}$$

admit a weak solution in E resp. $E^{\odot}$ for all $x \in E$ resp. for all $x^{\odot} \in E^{\odot}$.

(b) A has a H^∞-calculus on each strip

$$\mathcal{S}_\omega = \{\lambda \in \mathbb{C} : |\operatorname{Re}\lambda| < \omega\}, \quad \omega > \omega_0(A).$$

In this chapter we also consider continuity of solutions and a detailed example for nonexistence for a certain class of stochastic differential equations (see Section 5.2).

Results of this work are also published in [20, 21].

Acknowledgments

My work on this thesis was firstly supported by the 'Landesgraduiertenförderung'. In this time I could spend several months in Delft to work with Jan van Neerven. I would like to thank him sincerely for all the help he gave me in this period and later on. I would also like to thank his family and the the members of the Group of Functional Analysis of the TU Delft for their warm hospitality.

Afterward I proceeded my work as an assistant of Prof. Dr. Lutz Weis. I would like to thank him for his wide support and the inspiring discussions. I am also indebted to Peer Kunstmann for his patience in discussing minor or bigger questions.

All members of the ‘Institut für Analysis’ and especially Cornelia Kaiser I like to thank for creating a cooperative and warm atmosphere.

I thank my husband for his great tolerance and him as well as my parents for their constant support and encouragement.

1. Preliminaries

1.1. Notations

Throughout this work E denotes in general a real separable Banach space with norm $\|\cdot\|$. If F is another Banach space, $\mathcal{L}(E,F)$ denotes the space of bounded linear operators from E into F. In the case that $E = F$ we simply write $\mathcal{L}(E)$. By E^* we denote the dual space of E, i.e. $E^* = \mathcal{L}(E,\mathbb{R})$, and we write formally $\langle \cdot, \cdot \rangle$ for the duality. The dual of an operator B we denote by B^*.

For a closed and linear operator A defined on a subspace $\mathcal{D}(A) \subset E$ we write $(A, \mathcal{D}(A))$ or simple A. If $(A, \mathcal{D}(A))$ is densely defined, i.e. $\mathcal{D}(A)$ is dense in E the adjoint operator $(A^*, \mathcal{D}(A^*))$ is a linear operator from $\mathcal{D}(A^*)$ into E^* where $\mathcal{D}(A^*)$ consists of all elements $x^* \in E^*$ for which there is a $y^* \in E^*$ such that

$$\langle x^*, Ax \rangle = \langle y^*, x \rangle .$$

We then set $A^*x^* := y^*$.

The *resolvent set* $\rho(A)$ of A is defined as the set of all $\lambda \in \mathbb{C}$ for which $(\lambda I - A)$ is invertible and therefore defines a bijection from $\mathcal{D}(A)$ onto E. The set $\{(\lambda I - A)^{-1} : \lambda \in \rho(A)\}$ is referred to as *resolvent* of A. We mostly write $R(\lambda, A)$ instead of $(\lambda I - A)^{-1}$.

The complement $\sigma(A) := \mathbb{C} \setminus \rho(A)$ is called the *spectrum* of A. The *spectral bound* $s(A)$ is defined as $\sup\{\lambda \in \sigma(A)\}$.

1.2. Bochner and Pettis integration

In this section we will recall two possibilities of vector valued integration in Banach spaces.

At first we describe briefly the extension of Lebesgue integration to vector-valued functions (see e.g. [23]).

Let $(\Sigma, \mathcal{S}, \mu)$ be a σ-finite measure space and E a Banach space. Consider functions $f : \Sigma \to E$ of the form

$$f(s) = \sum_{i=1}^{n} x_i \mathbf{1}_{S_i}(s)$$

where $x_i \in E$, $S_i \in \mathcal{S}$, $i = 1, \ldots, n$ ($n \in \mathbb{N}$). Those functions are called *step functions*. Here and in the following the characteristic functional $\mathbf{1}_S$ of a subset $S \in \Sigma$ is defined as

$$\mathbf{1}_S(s) = \begin{cases} 1 & \text{if } s \in S, \\ 0 & \text{else} \end{cases}.$$

A general function $f : \Sigma \to E$ is called *strongly measurable*, if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of step functions with

$$\lim_{n\to\infty} f_n(s) = f(s)$$

μ-almost everywhere.

A function $f : \Sigma \to E$ is called *weakly measurable* if the function $\langle f, x^* \rangle(\cdot) := \langle f(\cdot), x^* \rangle$ is measurable for all $x^* \in E^*$.

The famous *Pettis measurability theorem* (see [23, Theorem 2]) proves that a function $f : \Sigma \to E$ is strongly measurable if and only if

1. the function f is weakly measurable,
2. f is almost surely separable-valued, i.e. there exists a $N \subset \mathcal{S}$ with $\mu(N) = 0$ such that $f(\Sigma \setminus N)$ is contained in a separable subspace of E.

The integral of a step function $f = \sum_{i=1}^n x_i \mathbf{1}_{S_i}$ is defined by

$$\int_\Sigma f d\mu := \sum_{i=1}^n x_i \mu(S_i).$$

A function $f : \Sigma \to E$ is called *Bochner integrable*, if there exists a sequence of step functions $(f_n)_{n\in\mathbb{N}}$ with

$$\lim_{n\to\infty} \int_\Sigma \|f_n - f\| d\mu = 0.$$

In this case the limit

$$\int_\Sigma f d\mu := \lim_{n\to\infty} \int_\Sigma f_n d\mu$$

exists and defines the Bochner integral of f.

A strongly measurable function is Bochner integrable if and only if $\int_\Sigma \|f\| d\mu$ is finite. For the integral $\int_\Sigma f \mathbf{1}_S d\mu$ we will also write $\int_S f d\mu$, $S \in \Sigma$.

The Bochner integral shares many properties of the Lebesgue integral. Of importance is the estimate

$$\left\| \int_\Sigma f d\mu \right\| \leq \int_\Sigma \|f\| d\mu\,.$$

Also the following theorem will be frequently used.

Theorem 1.2.1 *Let E, F be Banach spaces and $A : E \supset D(A) \to F$ a closed linear operator. Further let $f : \Sigma \to D(A)$ be Bochner integrable. If $A \circ f : \Sigma \to F$ is Bochner integrable then*

$$\int_\Sigma f(s)d\mu(s) \in \mathcal{D}(A) \quad \textit{and}$$
$$A \int_\Sigma f(s)d\mu(s) = \int_\Sigma A \circ f(s)d\mu(s)\,.$$

The second concept of vector valued integration is called the *Pettis integration* (see [50, 55, 23]), which is a kind of weak definition of the integral.This intrinsic definition is closely related to the definition of stochastic integration given later on. It gains importance by the coherences to Bochner integration and certain measurability conditions.

A function $f : \Sigma \to E$ is called *Pettis integrable*, if it is weakly measurable and if for all subsets $A \in \mathcal{S}$ there exists an element $F_A \in E$ such that for all $x^* \in E^*$ we have

$$\langle F_A, x^* \rangle = \int_A \langle f, x^* \rangle d\mu\,.$$

In this situation we write

$$F_A = \int_A f d\mu\,.$$

In the context of Pettis integration we will be interested in 'weak' properties of f. Let $1 \leq p \leq \infty$. A function $f : \Sigma \to E$ is called *weakly* L^p if it is weakly measurable and the function $\langle f, x^* \rangle$ belongs to $L^p(\Sigma)$ for all $x^* \in E^*$.

In the literature ([55, Chapter II.3], [23]) there are given many sufficient conditions for a function $f : \Sigma \to E$ to be Pettis integrable. From [45, Preliminaris] we adopt the following list:

- If f is Bochner integrable then it is also Pettis integrable and the integrals coincide.
- If f is strongly measurable and weakly L^p for some $p > 1$, then f is also Pettis integrable. If E is a separable Banach space then by the Pettis measurability Theorem one can replace the strong measurability by the weak measurability of f.
- Pettis integrability can be characterized by properties of the Banach space E. We have the equivalence of the following statements ([55, Proposition II.3.4]):

    1. E does not contain a subspace isomorphic to c_0.

2. For an arbitrary finite measurable space $(\Sigma, \mathcal{S}, \mu)$, each strongly measurable function $f : \Sigma \to E$ which is weakly L^1 is Pettis integrable.

- ([50, Theorem 3.4]) Let $f : \Sigma \to E$ be Pettis integrable and weakly L^p, $1 \leq p \leq \infty$ and let q satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then for $g \in L^q$ the function gf is Pettis integrable and defines a bounded operator $I_f : L^q(\Sigma) \to E$ by
$$I_f g := \int_\Sigma g f d\mu$$
where the integral on the right hand side is a Pettis integral.

1.3. Elements from probability theory

In this section we will give an excerpt from probability theory which fits in our setting. The several parts can be found in various books on probability (cf. [1, 55, 29, 40]).

In the following let E be a real locally convex vector space and E^* its topological dual. Let $\mathcal{B}(E)$ be the Borel σ-algebra, i.e. $\mathcal{B}(E) = \sigma\{B \subset E : B \text{ open }\}$ and $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ a probability space. I denotes an interval of the real line $\mathbb{R}$.

An *E-valued random element* is a $\mathcal{F}$-$\mathcal{B}(E)$-measurable mapping from Ω into E. The space of all those random elements will be denoted by $L^0(\Omega, E)$. Suppose now that E is a Banach space. For $p > 0$ let $L^p(\Omega, E)$ be the subspace of all $X \in L^0(\Omega, E)$ for which
$$\int_\Omega \|X(\omega)\|^p d\mathbb{P}(\omega) < \infty$$
holds true. Elements of $L^p(\Omega, E)$ are called *p-integrable* or, in case $p = 1$ just *integrable*.

Identifying random elements which are almost surely equal, $L^p(\Omega, E)$ becomes a Banach space for $p \geq 1$ respectively a Hilbert space if $p = 2$.

1.3.1. Filtrations

Let $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ be a complete probability space. A family of increasing sub-σ-algebras $\{\mathcal{F}_t : t \geq 0\}$ of $\mathcal{F}$: $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$, $0 \leq s < t < \infty$ is called a *filtration*. A filtration is called a *standard filtration* if

1. $\mathcal{F}_0$ contains all sets $N \in \mathcal{F}$ with $\mathbb{P}\{N\} = 0$, and

2. $\mathcal{F}_t = \bigcap\limits_{s>t} \mathcal{F}_s$ for all $t \geq 0$.

1.3.2. Processes

A family $X = \{X(t)\}_{t\in I}$ of E-valued random variables $X(t)$ defined on Ω is called a *stochastic process*. If $X(t)$ is $\mathcal{F}_t$ measurable, for any $t \in I$, then the process is called *adapted*.

X can be viewed as a mapping from $I \times \Omega$ into E. If we fix an $\omega \in \Omega$, then the function $X(\cdot, \omega)$ is called a *path* or a *trajectory* of X.

A stochastic process Y is called a *version* or a *modification* of X if

$$\mathbb{P}\{\omega \in \Omega : X(t,\omega) \neq Y(t,\omega)\} = 0 \quad \text{for all } t \in I.$$

1.3.3. Gaussian measures

Let μ be a Borel measure on E. For $x^* \in E^*$ consider the mapping $f_{x^*} : E \to \mathbb{R} : x \mapsto \langle x, x^*\rangle$. For all $x^* \in E^*$ the image measure under the mapping f_{x^*} will be denoted by $\langle \mu, x^*\rangle := \mu \circ f_{x^*}^{-1}$.

A *Gaussian measure* on E is a measure whose image measure $\langle \mu, x^*\rangle$ under each functional $x^* \in E^*$ is a Gaussian measure on $\mathbb{R}$.

1.3.4. Cylindrical measures and cylindrical processes

In this section we introduce a further σ-algebra. If $\Gamma \subset E^*$, then the smallest σ-algebra on E with respect to which all elements $x^* \in \Gamma$ are measurable will be denoted by $\tilde{\mathcal{C}}(E,\Gamma)$ or simply by $\tilde{\mathcal{C}}(E)$ if $\Gamma = E^*$. In order to introduce cylindrical measures we will recall the notion of cylinders.

Let $\{x_1^*, \dots, x_n^*\} \subset E^*$, $B \in \mathcal{B}(\mathbb{R}^n)$ and $n \in \mathbb{N}$. A set of the form

$$C_{x_1^*,\dots,x_n^*,B} = \{x \in E : (\langle x, x_1^*\rangle, \dots, \langle x, x_n^*\rangle) \in B\}$$

is called a *cylinder*.

The set of all such cylinders forms an algebra $\mathcal{C}(E)$ on E but in general not a σ-algebra. The σ-algebra $\sigma(\mathcal{C}(E))$ coincides with the σ-algebra $\tilde{\mathcal{C}}(E)$. In polish spaces and so in all separable Banach spaces the σ-algebra $\tilde{\mathcal{C}}(E)$ coincides with $\mathcal{B}(E)$ (see e.g. [55, Proposition 1.4]).

Set functions $\mathcal{C}(E) \to \mathbb{R}_+$ which are σ-additive on all σ-algebras $\tilde{\mathcal{C}}(E,\Delta)$, where Δ is a finite subset of E^* are called *cylindrical measures*. The set of all such functions will be denoted by $\mathcal{M}(E)$.

A *cylindrical process* is defined as a family of linear operators $(X_t)_{t\in[0,T]}$ from E^* into the space $L^0(\Omega)$ of real valued random variables, i.e. for all $x^* \in E^*$ and all $t \in [0,T]$, $X_t(x^*)$ is a real valued random variable.

Now $(X_t)_{t\in[0,T]}$ defines the family $(\nu_t)_{t\in[0,T]}$ of cylindrical measures on $\mathcal{C}(E)$ by

$$\nu_t\left(C_{x_1^*,\dots,x_n^*,B}\right) = \mathbb{P}\left\{(X_t(x_1^*),\dots X_t(x_n^*)) \in B\right\}\,.$$

The standard cylindrical Gaussian measure

A finitely additive set function ν on $\mathcal{C}(E)$ is called a *cylindrical Gaussian measure*, if for all $x^* \in E^*$ the measure $\langle \nu, x^*\rangle$ defined by $\langle \nu, x^*\rangle(B) = \nu(C_{x^*,B})$, $B \in \sigma(\mathbb{R})$ is a Gaussian measure on $\mathbb{R}$.

A cylindrical Gaussian measure as well as a Gaussian measure μ is uniquely determined by the corresponding characteristic functional:

$$\begin{aligned}\hat{\mu}(x^*) &= \int_E \exp(i\langle x, x^*\rangle)d\mu(x)\\ &= \exp\left(i\,M(x^*) - \frac{1}{2}\langle Qx^*, x^*\rangle\right), \quad x^* \in E^*\end{aligned}$$

where

$$M(x^*) = \int_E f_{x^*}d\mu, \quad Q(x^*) = \int_E f_{x^*}^2 d\mu - M^2(x^*), \quad x^* \in E^*.$$

Here $M : E^* \to \mathbb{R}$ is a linear functional and $Q \in \mathcal{L}(E^*, E)$ is a positive symmetric operator, where by a positive and symmetric operator $Q \in \mathcal{L}(E^*, E)$ we understand an operator for which $\langle Qx^*, x^*\rangle \geq 0$ and $\langle Qx^*, y^*\rangle = \langle Qy^*, x^*\rangle$ for all $x^*, y^* \in E^*$ holds true.

If $M \equiv 0$ we call the μ a *centered (cylindrical) Gaussian measure.*

If μ happens to be a Gaussian measure in the sense of 1.3.3 above then M is called the *mean* and Q *the (Gaussian) covariance operator.*

Later on we will discuss under which conditions such an operator Q is indeed the covariance of a Gaussian measure μ.

The cylindrical Wiener Process

Now we consider the case that $E = H$ is a real separable Hilbert space with scalar product denoted by $[\cdot,\cdot]$. As usual we will identify H^* with H by the Riesz representation theorem. Set $\Delta = \{h_1, \ldots, h_n\}$ where $h_1, \ldots, h_n$, $n \in \mathbb{N}$, are orthonormal in H. We define a mapping $g_\Delta : H \to \mathbb{R}^n$ by $g \mapsto ([g, h_1], \ldots, [g, h_n])$.

A cylindrical measure on H is called a *standard cylindrical Gaussian measure* and will be denoted by γ^H if $g_\Delta(\gamma^H) := \gamma^H \circ g_\Delta^{-1}$ is a standard Gaussian measure on $\mathbb{R}^n$, i.e. a $\mathcal{N}(0, \mathbf{I})$ distributed Gaussian measure. Here $\mathbf{I}$ denotes the n-dimensional unit matrix.

Let $(h_n)_{n\in\mathbb{N}}$ be an orthonormal basis in H and $(\beta_n(t))_{t\in\mathbb{R}}$ a sequence of independent Brownian motions on a certain probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that $(\mathcal{F}_t)$ is a standard filtration with the properties (PF)

1. The Brownian motions $(\beta_n(t))_{t\in\mathbb{R}}$, $n \in \mathbb{N}$, are adapted, and
2. $\mathcal{F}_s$ and $(\beta_n(t) - \beta_n(s))$, $n \in \mathbb{N}$ are independent for $0 \leq s < t$.

Consider now for a fixed $T > 0$ the following family of operators

$$\{W_H(t)\}_{t\in[0,T]} : H \to L^2(\mathbb{P}), \quad h \mapsto \sum_{n=1}^{\infty} \beta_n(t)[h, h_n].$$

One can easily verify that the series on the right hand side converges in $L^2(\mathbb{P})$ and that these operators are linear and bounded.

This process is called the *cylindrical Brownian motion with Cameron-Martin-space* H or shortly *the cylindrical Wiener Process* and has the following characterizing properties:

1. For all $h \in H$, $\{W_H(t)h\}_{t\in[0,T]}$ is real-valued $\{\mathcal{F}_t\}$-adapted Brownian motion;
2. For all $s, t \in [0, T]$ and $g, h \in H$ we have
$$\mathbb{E}\big(W_H(s)g \cdot W_H(t)h\big) = (s \wedge t)[g, h]_H \,.$$

In the following we will shortly call W_H $\mathcal{F}_t$-adapted.

Remark 1.3.1 If we denote the corresponding family of cylindrical measures on H by $\{\gamma_t^H\}_{t\geq 0}$ then γ_1^H equals the standard cylindrical Gaussian measure γ^H on H.

1.3.5. The Reproducing kernel Hilbert space

We have seen in Section 1.3.4 that we can define a cylindrical centered Gaussian measure μ on a separable Banach space E by a positive and symmetric operator $Q \in \mathcal{L}(E^*, E)$. The aim of this section is to find a Hilbert space $(H_Q, [\cdot, \cdot]_Q)$ continuously embedded in E by an embedding $i_Q : H \to E$ such that μ is just the image cylindrical measure of the standard cylindrical Gaussian measure γ^{H_Q} (see [2, 15]). The presentation of this subject follows [42, 43]. See also the references therein.

We define a scalar product on the range of Q, denoted by $\mathrm{Ran}(Q)$, by

$$[Qx^*, Qy^*]_Q := \langle Qx^*, y^* \rangle .$$

One can easily check that this defines indeed a scalar product. It is well defined since if either $Qx^* = 0$ or $Qy^* = 0$ then surely $[Qx^*, Q^*_y]_Q = 0$ because of the symmetry of Q. The positivity results from the positivity of Q. To check that $[\cdot, \cdot]$ is definite assume that $[Qx^*, Qx^*]_Q := \langle Qx^*, x^* \rangle = 0$, then by the Cauchy-Schwarz inequality we have for all $y^* \in E^*$

$$|\langle Qx^*, y^* \rangle| \leq \langle Qx^*, x^* \rangle^{\frac{1}{2}} \langle Qy^*, y^* \rangle^{\frac{1}{2}} = 0.$$

Therefore, $Qx^* = 0$.

Now we complete $\mathrm{Ran}(Q)$ with respect to $[\cdot, \cdot]_Q$. The resulting Hilbert space H_Q is called *the reproducing kernel Hilbert space* (briefly *RKHS*). In the next step we will show that the natural inclusion $i : \mathrm{Ran}(Q) \to E$ extends to a bounded and actually injective mapping $i_Q : H_Q \to E$.

First we consider the mapping $Q : E^* \to \mathrm{Ran}(Q) \hookrightarrow H_Q$ with respect to the norm $\| \cdot \|_Q$ induced by $[\cdot, \cdot]_Q$. This mapping is bounded since for all $x^* \in E^*$

$$\|Qx^*\|_Q^2 = \langle Qx^*, x^* \rangle \leq \|Q\| \|x^*\|^2.$$

Next we compute for $y^* \in E^*$

$$\langle Qx^*, y^* \rangle \leq \|Qx^*\|_Q \|Qy^*\|_Q \leq \|Qx^*\|_Q \|Q\|_{\mathcal{L}(E^*, H_Q)} \|y^*\|.$$

By taking the supremum over all $y^* \in E^*$ with $\|y^*\| \leq 1$ we obtain

$$\|Qx^*\|_E \leq \|Q\|_{\mathcal{L}(E^*, H_Q)} \|Qx^*\|_Q$$

which shows that the inclusion $\mathrm{Ran}(Q) \hookrightarrow E$ is bounded with respect to the scalar product $[\cdot, \cdot]_Q$ and admits a continuous extension to a mapping $i_Q : H_Q \to E$.

Set $h_{x^*} := Qx^*$ for $x^* \in E^*$. Then we have the identity

$$i_Q h_{x^*} = Qx^*$$

and we proceed by computing for all $y^* \in E^*$

$$[h_{x^*}, h_{y^*}] = \langle Qx^*, y^* \rangle = \langle i_Q h_{x^*}, y^* \rangle = [h_{x^*}, i_Q^* y^*].$$

Since the elements h_{x^*}, $x^* \in E^*$, span a dense subset of H_Q we have the identity

$$h_{y^*} = i_Q^* y^*$$

and thus
$$Qy^* = i_Q(h_{y^*}) = i_Q(i_Q^* y^*)$$
which shows that $Q = i_Q \circ i_Q^*$.

Finally assume that for an element $g \in H_Q$ we have $i_Q g = 0$. Then for all $y^* \in E^*$
$$[g, h_{y^*}]_Q = [g, i_Q^* y^*] = \langle i_Q g, y^* \rangle = 0.$$
Again due to density we find $g = 0$ which means that i_Q is injective.

From the Hahn-Banach Theorem it follows that if E is separable, then E^* is separable in the weak* topology. Since i_Q^* is weak*-to-weakly continuous it follows that H_Q is weakly separable, and hence as a Hilbert space also separable.

In the following we will work rarely with the RKHS but with arbitrary real separable Hilbert spaces. The justification follows from the following proposition. There it is stated that the RKHS enjoys a certain minimality property relative to the factorization $i_Q \circ i_Q^*$ of Q.

Proposition 1.3.2 *Let $Q \in \mathcal{L}(E^*, E)$ be positive and symmetric. Let H be a real separable Hilbert space and $T \in \mathcal{L}(H, E)$ an operator with the property $T \circ T^* = Q$. Then there exists a unique linear bounded operator $P : H \to H_Q$ such that the following diagram commutes.*
$$\begin{tikzcd} H \arrow[r, "T"] \arrow[d, "P"'] & E \arrow[d, "I"] \\ H_Q \arrow[r, "i_Q"'] & E \end{tikzcd}$$
In particular, identifying $i_Q(H_Q)$ with H_Q, we have $H_Q = Ran(T)$ as a subset of E.

Proof. Let $H_0 = (\mathrm{Ker}T)^\perp = \overline{\mathrm{Ran}(T^*)}$ and π_0 the orthogonal projection from H onto H_0. Now we define an operator $P_0 : \mathrm{Ran}(T^*) \to H_Q$ by $P_0(T^*x^*) := i_Q^* x^*$. This operator extends to an isometry from H_0 to H_Q since by $T \circ T^* = Q$ we have
$$\|T^*x^*\|_{H_0}^2 = \langle Qx^*, x^* \rangle = \|i_Q^* x^*\|_Q^2.$$
Now we define $P := P_0 \circ \pi_0$. To see that $T = i_Q \circ P$ consider an arbitrary $h \in H$. Since $H = H_0 \oplus \mathrm{Ker}T$ we may write $h = T^*x^* + g$ for suitable $x^* \in E^*$, $g \in \mathrm{Ker}T$ and compute
$$T(h) = T(T^*x^* + g) = TT^*x^* = i_Q i_Q^* x^* = i_Q P_0(T^*x^*) = i_Q P(h)$$
which shows $T = i_Q \circ P$.

For the uniqueness consider a further operator $\tilde{P}$ with $T = i_Q \circ \tilde{P}$. Then $i_Q(\tilde{P} - P) = 0$ and therefore $\tilde{P} - P = 0$ by the injectivity of i_Q. $\square$

1.4. Randonifying operators

If we deal with a cylindrical measure the question arises if we can therefrom obtain a regular measure on a possibly other space. Can we characterize those mappings which map cylindrical measures into "normal" ones? The results below are taken from [55, Chapter IV.5].

Let for the moment E be a locally convex topological vector space and E^* its topological dual. Further let μ be a cylindrical measure on $\mathcal{C}(E)$. One says then μ *admits a Radon extension*, if there exists a Radon measure $\tilde{\mu}$ which equals μ on $\mathcal{C}(E)$.

We recall that a *Radon measure* μ on E is a finite Borel measure with the additional property

$$\mu(B) = \sup\{\mu(K) : K \subset B, K \text{ compact}\}$$

for each $B \in \mathcal{B}(E)$. In other words: for every $B \in \mathcal{B}(E)$ and every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset B$, such that

$$\mu(B \setminus K_\varepsilon) < \varepsilon. \tag{$*$}$$

A measure is called *tight* if $(*)$ holds for $B = E$. Under a *uniformly tight family of measures* M we understand a family of radon measures for which $(*)$ holds for every $\mu \in M$.

If E is a separable Banach space (or more general a Polish space) then μ admits a Radon extension if and only if μ is σ-additive on $\mathcal{C}(E)$. This follows because in this case we have $\tilde{\mathcal{C}}(E) = \mathcal{B}(E)$ ([55, Theorem1.2]) and further because in those spaces we have that every Borel measure is in fact a Radon measure (see e.g. [24, Satz VIII.1.5.]).

Let E, F be both locally convex topological vector spaces. A linear and continuous mapping $T : E \to F$ is called *radonifying* for μ, if the image cylindrical measure $T(\mu)$ admits a Radon extension on F.

In the following let H be a separable Hilbert space with with ONB $(h_n)_{n\in\mathbb{N}}$ and E a separable Banach space. By γ^H we denote again the standard cylindrical Gaussian measure. A radonifying operator $T : H \to E$ will be called *γ-radonifying* if the cylindrical image measure of γ^H extends to a σ-additive measure on E. This is the main setting in this work and we will also write *radonifying* instead of *γ-radonifying*.

The next proposition ties up to considerations in (1.3.4). It is proved in [55, Chapter III and Proposition VI.3.3.].

Proposition 1.4.1 *Let E be a locally convex topological vector space and H be a real Hilbert space. Let $T \in \mathcal{L}(H, E)$. The cylindrical image measure $\mu := T(\gamma^H)$ is a centered cylindrical Gaussian measure on E whose characteristic functional is given by*

$$\begin{aligned}\hat{\mu}(x^*) &= \int_E \exp(i\langle x, x^*\rangle) d\mu(x) \\ &= \exp\left(-\frac{1}{2}\langle (T \circ T^*)x^*, x^*\rangle\right), \quad x^* \in E^*.\end{aligned}$$

The operator $T \circ T^*$ is positive and symmetric and determines a centered cylindrical Gaussian measure. The next proposition (see [43, 2.4 Proposition]) answers the question when such an operator is indeed a Gaussian covariance.

Proposition 1.4.2 *Let E be a separable real Banach space and H a separable real Hilbert space. Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of independent standard Gaussian variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For $T \in \mathcal{L}(H, E)$ the following assertions are equivalent:*

1. *$T \circ T^*$ is the covariance of a Gaussian measure μ on E.*
2. *There exists an orthonormal basis $(h_n)_{n\in\mathbb{N}}$ of H such that the Gaussian series $\sum\limits_{n=1}^{\infty} \gamma_n T h_n$ converges in $L^2(\Omega, E)$.*

In this situation we have that for every orthonormal basis $(h_n)_{n\in\mathbb{N}}$ and every $p \in [1, \infty)$ the series converges unconditionally in $L^p(\Omega, E)$ and almost surely and we have

$$\int_E \|x\|^p d\mu(x) = \mathbb{E}\left\|\sum_{n=1}^{\infty} \gamma_n T h_n\right\|^p$$

Thus a bounded operator $T \in \mathcal{L}(H, E)$ is *γ-radonifying* if it satisfies the equivalent conditions of the last theorem. For such an operator we define the *γ-norm* $\|T\|_{\gamma(H;E)}$ by

$$\|T\|_{\gamma(H;E)} := \left(\int_E \|x\|^2 d\mu(x)\right)^{\frac{1}{2}} = \left(\mathbb{E}\left\|\sum_{n=1}^{\infty} \gamma_n T h_n\right\|^2\right)^{\frac{1}{2}}.$$

If it is clear from the context which spaces E and H are meant we shall also write $\|\cdot\|_\gamma$ instead of $\|\cdot\|_{\gamma(H;E)}$. It is easy to see that $\|\cdot\|_\gamma$ defines indeed a norm. The space of all operators satisfying the equivalent conditions in the last theorem forms a linear subspace of $\mathcal{L}(H, E)$ and will be denoted by $\gamma(H, E)$. Moreover it is known (see [48, Lemma 32]) that $\gamma(H, E)$ endowed with the norm $\|\cdot\|_\gamma$ is a real Banach space.

The operator T is said to be *almost summing* if the partial sums $\sum_{n=1}^{N} \gamma_n T h_n$ are uniformly bounded in $L^2(\Omega; E)$. Every γ-radonifying operator is almost summing and we have

$$\|T\|^2_{\gamma(H,E)} = \sup_{N\geq 1} \mathbb{E}\left\|\sum_{n=1}^{N} \gamma_n T h_n\right\|^2. \tag{1.1}$$

If E does not contain a closed subspace isomorphic to c_0, then a celebrated theorem of Hoffmann-Jorgensen and Kwapień [36, Theorem 9.29] implies that every almost summing operator from H to E is γ-radonifying. For more information we refer to [3, 36, 48, 55].

For Gaussian measures we have the following useful domination result.

Proposition 1.4.3 *Let $R \in \mathcal{L}(E^*, E)$ be the covariance of a Gaussian measure ν and let $\mathcal{Q} \subset \mathcal{L}(E^*, E)$ be a family of positive symmetric operators. If for every $x^* \in E^*$ and every $Q \in \mathcal{Q}$ the estimate*

$$\langle Qx^*, x^* \rangle \leq \langle Rx^*, x^* \rangle$$

holds true, then every $Q \in \mathcal{Q}$ is a covariance of a Gaussian measure μ_Q. Further the family $M_{\mathcal{Q}} = \{\mu_Q\}$ is uniformly tight and for all $p \geq 1$ and all $\mu_Q \in M_{\mathcal{Q}}$ we have

$$\int_E \|x\|^p d\mu_Q(x) \leq \int_E \|x\|^p d\nu(x).$$

For the proof of the first part of the proposition and the tightness assertion see [3, 3.3.1. Theorem] and the proof thereafter. The last part is shown in [3, 3.3.7. Corollary].

γ-radonifying operators share an ideal property which turns out to be very important in our framework.

Proposition 1.4.4 *Consider $T \in \gamma(H, E)$, $S \in \mathcal{L}(H_1, H)$ and $U \in \mathcal{L}(E, E_1)$, where H_1, H are Hilbert spaces end E, E_1 are Banach spaces. Then the composition $U \circ T \circ S$ is again γ-radonifying and we obtain the norm estimate*

$$\|U \circ T \circ S\|_{\gamma(H_1,E_1)} \leq \|S\| \|T\|_{\gamma(H,E)} \|U\|.$$

Proof. One can easily see that $U \circ T$ is γ-radonifying and we have

$$\|U \circ T\|_{\gamma(H;E_1)} \leq \|U\| \|T\|_{\gamma(H,E)}.$$

Hence it suffices to show that $T \circ S \in \gamma(H_1, E)$ with

$$\|T \circ S\|_{\gamma(H_1,E)} \leq \|T\|_{\gamma(H,E)} \|S\|. \tag{1.2}$$

To show this let $R := T \circ T^*$ and $Q = T \circ S \circ S^* \circ T^*$. Then,

$$\langle Qx^*, x^* \rangle = \|S^* \circ T^* x^*\|_{H_1}^2 \leq \|S\|^2 \|T^* x^*\|_H^2 = \|S\|^2 \langle Rx^*, x^* \rangle.$$

Now, an easy rescaling argument together with Proposition 1.4.3 show that Q is a Gaussian covariance and (1.2) holds true. $\square$

If E is itself a Hilbert space it is much easier to decide whether $T \in \mathcal{L}(H,E)$ is γ-radonifying. By writing out the respective norms one can see that in the Hilbert space case T is γ-radonifying if and only if T is a Hilbert-Schmidt operator. We have

$$\|T\|_\gamma = \|T\|_{HS}$$

where $\|\cdot\|_{HS}$ is the Hilbert-Schmidt norm defined by

$$\|T\|_{HS}^2 := \sum_{n=1}^{\infty} \|Th_n\|^2,$$

where $(h_n)_{n\in\mathbb{N}}$ is again an arbitrary orthonormal base in H.

Definition 1.4.5 Let (S,Σ,ν) be a finite measure space. We call $\varphi : S \to E$ *weakly* L^p if the function $\langle \varphi, x^* \rangle$ is measurable and belongs to $L^p(S)$ for all $x^* \in E^*$.

We say that a function $\Phi : (0,T) \to \mathcal{L}(H,E)$ is H*-weakly* L^2 if for all $x^* \in E^*$ the map $t \mapsto \Phi^*(t)x^*$ is strongly measurable and satisfies

$$\int_0^T \|\Phi^*(t)x^*\|_H^2 \, dt < \infty \,.$$

Let $\Phi : (0,T) \to \mathcal{L}(H,E)$ be H-weakly L^2. We say that Φ *represents* an operator $T \in \mathcal{L}(L^2(0,T;H),E)$ if

$$Tf = \int_0^T \Phi(t)f(t)\, dt, \quad f \in L^2(0,T;H), \tag{1.3}$$

where the integral exist as an Pettis integral in E. In this situation we sometimes write $T = I_\Phi$. Note that this operator is the adjoint of the operator $x^* \mapsto \Phi^*(\cdot)x^*$ from E^* into $L^2(0,T;H)$. We denote by $\gamma(0,T;H,E)$ the vector space of all functions $\Phi : (0,T) \to \mathcal{L}(H,E)$ which represent a γ-radonifying operator $I_\Phi \in \mathcal{L}(L^2(0,T;H),E)$. For such a function we define

$$\|\Phi\|_{\gamma(0,T;H,E)} := \|I_\Phi\|_{\gamma(L^2(0,T;H),E)}.$$

It is easy to see that for all $\Phi \in \gamma(0,T;H,E)$ the reflected function $t \mapsto \Phi(T-t)$ belongs to $\gamma(0,T;H,E)$ with equal norm. Moreover, for all $t \in (0,T)$ the restriction $\Phi|_{(0,t)}$ belongs to $\gamma(0,t;H,E)$, and an easy application of Kahane's contraction principle gives

$$\big\|\Phi|_{(0,t)}\big\|_{\gamma(0,t;H,E)} \le \|\Phi\|_{\gamma(0,T;H,E)}. \tag{1.4}$$

The following simple lemma will be useful.

Lemma 1.4.6 *If $g \in L^2(0,T)$ and $B \in \gamma(H,E)$, then the function $gB : t \mapsto g(t)B$ belongs to $\gamma(0,T;H,E)$ and we have*

$$\|gB\|_{\gamma(0,T;H,E)} = \|g\|_{L^2(0,T)} \|B\|_{\gamma(H,E)}.$$

Proof. Let (f_n) and (h_m) be orthonormal bases for $L^2(0,T)$ and H, respectively, and note that $(f_n \otimes h_m)$ is an orthonormal basis for $L^2(0,T;H)$. Let (γ_{mn}) be a doubly indexed Gaussian sequence and define

$$\xi_m := \sum_n \gamma_{mn} \int_0^T f_n(t)g(t)\,dt.$$

The sum defining each ξ_m converges in $L^2(\Omega)$ and is $N(0,\|g\|^2)$ distributed, and the resulting i.i.d. sequence (ξ_m) is Gaussian.

Define $S : L^2(0,T;H) \to E$ by

$$Sf := \int_0^T g(t)Bf(t)\,dt, \quad f \in L^2(0,T;H).$$

Then gB represents S and we have

$$\begin{aligned}\|S\|^2_{\gamma(L^2(0,T;H),E)} &= \mathbb{E}\Big\|\sum_m \sum_n \gamma_{mn} \int_0^T f_n(t)g(t)Bh_m\,dt\Big\|^2 \\ &= \mathbb{E}\Big\|\sum_m \xi_m B h_m\Big\|^2 = \|g\|^2\,\|B\|^2_{\gamma(H,E)}.\end{aligned}$$

□

Remark 1.4.7 Observe that $SS^* = \|g\|^2_{L^2(0,T)} BB^*$. Since by assumption BB^* is a Gaussian covariance operator, the same is true for SS^* and the result follows.

For $H = \mathbb{R}$ the above definitions simplify by canonically identifying $\mathcal{L}(\mathbb{R},E)$ with E. Accordingly, a function $\varphi : (0,T) \to E$ which is weakly-L^2 is said to *represent* an operator $T \in \mathcal{L}(L^2(0,T),E)$ if

$$\langle Tf, x^*\rangle = \int_0^T \langle \varphi(t), x^*\rangle f(t)\;dt, \quad f \in L^2(0,T),\; x^* \in E^*,$$

and we write $\varphi \in \gamma(0,T;E)$ if the operator $T = I_\varphi$ is γ-radonifying. As before we define $\|\varphi\|_{\gamma(0,T;E)} := \|I_\varphi\|_{\gamma(L^2(0,T),E)}$.

1.5. Boundedness with respect to random sequences

In the recent time there is a growing interest in the so called *R-boundedness* (see e.g. [58, 30]). In the context of this work we need the related notion of γ-boundedness which is very similar to R-boundedness and many results can be shown in the same manner as in [58, Chapter 2]. To include all cases we formulate the results in an even more general way. In the following we will introduce a generalized definition of boundedness with respect to certain random sequences and see that many of the results known from the R-boundedness hold also in the general case.

Definition 1.5.1 Let (ξ_n) be any sequence of symmetric real valued random variables not necessarily independent with finite second moments, i.e. for all $n \in \mathbb{N}$ exists a constant C_n with $\mathbb{E}\|\xi_n\|^2 < C_n$. A set $\tau \subset B(X,Y)$ is called *bounded with respect to* (ξ_n) (or shorter (ξ_n)*-bounded* if there is a constant $C \in \mathbb{R}$ such that for all $T_1, \dots, T_m \in \tau$ and $x_1, \dots, x_m \in X$

$$\Big(\mathbb{E}\Big\|\sum_{n=1}^{m} \xi_n T_n x_n\Big\|_Y^2\Big)^{1/2} \leq C\Big(\mathbb{E}\Big\|\sum_{n=1}^{m} \xi_n x_n\Big\|_X^2\Big)^{1/2}. \tag{1.5}$$

The smallest constant C, for which (1.5) holds is denoted by $\Xi(\tau)$.

Examples 1.5.2 a) If $(\xi_n) = (r_n)$ is a sequence of independent Rade macher random variables (i.e. $\mathbb{P}(r_n = 1) = \mathbb{P}(r_n = -1) = 1/2$) and (1.5) holds for $\tau \subset B(X,Y)$ we say that τ is *R-bounded* and denote its bound by $R(\tau)$.

b)If $(\xi_n) = (\gamma_n)$ is a sequence of real-valued independent $\mathcal{N}(0,1)$-distributed Gaussian random variables we say accordingly γ*-bounded* with bound $\Gamma(\tau)$.

Lemma 1.5.3 *Let $\tau \subset B(X,Y)$ be a (ξ_n)-bounded collection with bound C. Then the closure $\overline{\tau}$ in the strong operator topology is also (ξ_n)-bounded, with the same (ξ_n)-bound.*

Proof. Let $m \in \mathbb{N}$ be fixed and choose $T_1, \dots, T_m \in \overline{\tau}$. Then for every T_n $n = 1, \dots, m$ and every $x \in X$ there exist operators $T_{n,k} \in \tau, k \in \mathbb{N}$ and such that

$$\|(T_n - T_{n,k})x\|_Y \to 0 \quad \text{as} \quad k \to \infty \quad \text{for all} \quad n = 1, \dots, m.$$

Hence we have for certain constants C_n, $n = 1, \dots, m$,

$$\begin{aligned}\Big\|\sum_{n=1}^{m} \xi_n T_n x_n\Big\|_{L_2(\Omega;Y)} &\leq \Big\|\sum_{n=1}^{m} \xi_n (T_n - T_{n,k}) x_n\Big\|_{L_2(\Omega;Y)} \\ &\qquad + \Big\|\sum_{n=1}^{m} \xi_n T_{n,k} x_n\Big\|_{L_2(\Omega;Y)} \\ &\leq \sum_{n=1}^{m} C_n \|(T_n - T_{n,k})x_n\|_Y + C\Big\|\sum_{n=1}^{m} \xi_n x_n\Big\|_{L_2(\Omega;X)}.\end{aligned}$$

Since the first term goes to zero as k tends to infinity, the result follows. $\square$

Lemma 1.5.4 *Let G be an index set and let $T_n(s) \in B(X,Y)$ for $n \in \mathbb{N}$ and $s \in G$. Assume that*

$$T(s) = \sum_{n=1}^{\infty} T_n(s), \qquad s \in G$$

converges in the strong operator topology of $B(X,Y)$ for all $s \in G$. Then

$$\Xi(\{T(s) : s \in G\}) \le \sum_{n=1}^{\infty} \Xi(\{T_n(s) : s \in G\}).$$

The proof proceeds like the proof of [58, 2.4. Lemma].

With these lemmata we can prove the following useful proposition:

Proposition 1.5.5 *Let $J \subset \mathbb{R}$ be an interval and $t \in J \to M(t) \in B(X,Y)$ have an integrable derivative. Then $\{M(t) : t \in J\}$ is (ξ_n)-bounded. And its bound can be estimated by*

$$\Xi(\{M(t) : t \in J\}) \le \|M(a)\| + \int_a^b \|M'(s)\| ds.$$

Proof. We proceed as in the proof of [58, 2.5. Proposition].

If $J = [a,b)$, $a < b \le \infty$, and $\sigma = \{t_0, t_1, \ldots, t\}$ is a partition of J, we set

$$M_\sigma(t) = M(a) + \sum_{j=1}^{n} \mathbf{1}_{[t_{j-1},b)}(t) \int_{t_{j-1}}^{t_j} M'(s) ds.$$

Now by observing that for $\tau = \{T\}$, $T \in B(X,Y)$, we have $\Xi(\tau) = \|T\|$ and Lemma 1.5.4 we have

$$\Xi(\{M_\sigma(t) : t \in J\}) \le \|M(a)\| + \sum_{j=1}^{n} \Xi(\{N_j(t), t \in J\})$$

where $N_j(t) := \mathbf{1}_{[t_{j-1},b)}(t) A_j$ with $A_j := \int_{t_{j-1}}^{t_j} M'(s) ds$.

Fix $j \le n$. For $m \in \mathbb{N}$ choose $s_1, \ldots, s_m \in [a,b)$ and $x_1, \ldots, x_m \in X$. Set $I_j := \{i = 1, \ldots, m : \mathbf{1}_{[t_{j-1},b)}(s_i) \ne 0\}$. Now we can estimate the (ξ_n)-bound of $\{N_j(t), t \in [a,b)\}$

$$\begin{aligned} \mathbb{E}\Big\| \sum_{i=1}^{m} \xi_i N_j(s_i) x_i \Big\|^2 &= \mathbb{E}\Big\| \sum_{i \in I} \xi_i A_j x_i \Big\|^2 \\ &\le \|A_j\|^2 \, \mathbb{E}\Big\| \sum_{i \in I} \xi_i x_i \Big\|^2 \\ &\le \|A_j\|^2 \, \mathbb{E}\Big\| \sum_{i=1}^{m} \xi_i x_i \Big\|^2. \end{aligned}$$

The last inequality followed from a version of the Kahane contraction principle (see [36, Lemma 4.6]).

With this estimate we obtain

$$\begin{aligned}\Xi(\{M_\sigma(t) : t \in J\}) &\le \|M(a)\| + \sum_{j=1}^{n} \Big\| \int_{t_{j-1}}^{t_j} M'(s)ds \Big\| \\ &\le \|M(a)\| + \int_a^b \|M'(s)\|ds.\end{aligned}$$

Since $t \mapsto M(t)$ is continuous in the uniform operator topology, we can choose a sequence of partitions σ_n, $n \in \mathbb{N}$ with $M_{\sigma_n}(t) \to M(t)$ as $n \to \infty$ for $t \in J$. Proposition 1.5.3 yields now that $\{M(t) : t \in J\}$ is (ξ_n)-bounded. The case where J is an arbitrary interval can be deduced from the preceding results. □

As an application of this proposition we state the following

Lemma 1.5.6 *If A generates an analytic C_0-semigroup S on E, then for all $k \in \mathbb{N}, \varepsilon > 0$ the family $\mathcal{T}_{k,\varepsilon} := \{t^{k+\varepsilon} A^k S(t), t \in (0,T)\}$ is R-bounded.*

Proof. By the previous lemma it suffices to check that for all $k \in \mathbb{N}$, the function $t \mapsto t^{k+\varepsilon} A^k S(t)$ has an integrable derivative on $(0,T)$, and this follows from the boundedness of $\|t^k A^k S(t)\|$. □

The following multiplier result is a straightforward generalization of a result in [30], where it is formulated for the case $H = \mathbb{R}$.

We call an operator-valued function $M : (0,T) \to \mathcal{L}(E)$ *strongly measurable* if $M : (0,T) \to E$, $Mx(t) := M(t)x$, is strongly measurable for all $x \in E$.

Lemma 1.5.7 *If $M : (0,T) \to \mathcal{L}(E)$ is strongly measurable and $\{M(t) : t \in (0,T)\}$ is γ-bounded with bound Γ, then for all $\Phi \in \gamma(0,T;H,E)$ the function $M(\cdot)\Phi(\cdot)$ belongs to $\gamma(0,T;H,E)$ and*

$$\|M(\cdot)\Phi(\cdot)\|_{\gamma(0,T;H,E)} \le \Gamma \|\Phi(\cdot)\|_{\gamma(0,T;H,E)}.$$

1.6. The H^∞-calculus

For the analysis of stochastic differential equations of the form

$$\left\{\begin{array}{rcll} dU(t) &=& AU(t)dt + BdW_H(t)\,, & t \in [0,T] \\ U_0 &=& 0 & \end{array}\right. \tag{1.6}$$

we need the functional analytic calculus of A. In this equation $\big(A, \mathcal{D}(A)\big)$ denotes the generator of a C_0-semigroup on a separable Banach space E, B a bounded linear operator from a separable real Hilbert space H into E and $W_H(t) : H \to L^2(\Omega)$ a cylindrical Wiener process with Cameron Martin space H.

In order to obtain solutions and to study their regularity we will be amongst others interested in spectral properties of the operator A. In this chapter we will encounter further properties of A such as: A is the generator of an analytic semigroup or A has a bounded H^∞-calculus (see below). To define this properties we need additional definitions.

1.6.1. Sectorial operators

In this subsection we deal with operators whose spectrum is contained in a certain subset of $\mathbb{C}$, called a sector. For those operators we can define an operator calculus which leads to the definition of fractional powers. The latter will enable us to define interpolation and extrapolation spaces of the Banach space E with respect to the operator A.

The subset of the complex plane $\Sigma_\sigma = \{z \in \mathbb{C} \setminus \{0\} : -\sigma < \arg z < \sigma\}$, $\sigma \in (0, \pi)$ is called *a sector*.

Definition 1.6.1 A closed linear operator is called *sectorial of type* σ if its spectrum is contained in the closure of the sector $\overline{\Sigma_\sigma}$ for a $\sigma \in (0, \pi)$ and additionally for each $\theta \in (\sigma, \pi)$ there exists a constant M_θ such that the resolvent estimate

$$\|\lambda R(\lambda, A)\| \leq M_\theta \qquad \text{for all } \lambda \notin \overline{\Sigma_\theta} \tag{1.7}$$

holds true.

Notation 1.6.2 By $\omega(A)$ we denote the infimum of all such σ.

We use the notation $\mathcal{S}(E)$ for the class of those sectorial operators on E that are densely defined, injective and have dense range and the notation $\mathcal{S}_\sigma(E)$ to classify those operators in $\mathcal{S}(E)$ which are of type σ. In [34, 15 Appendix] it is shown that by switching to a suitable subspace of E the properties of $A \in \mathcal{S}(E)$:

1. A is densely defined,
2. A has dense range,
3. A is injective,

hold naturally. We remark that the property of injectivity even in E itself follows from the other two (see [12]).

Now we will recall two examples of sectorial operators.

C_0-Semigroups

As an example of sectorial operators of type $\sigma \leq \frac{\pi}{2}$ can serve - under appropriate conditions as explained below - the the negative generator $-A$ of a C_0-semigroup $\mathbf{T}$.

Because the notion of semigroups is central in this work we will give a short outline. The detailed theory can be found amongst many others in [25] or [49].

A family $\mathbf{T} := (T(t))_{t\geq 0}$ of operators $T(t) \in \mathcal{L}(E)$, $t \geq 0$, is called *a semigroup of bounded linear operators* or briefly *a semigroup* if

(S1) $T(0) = I$,

(S2) $T(t+s) = T(t)T(s)$ for every $t, s \geq 0$,

where I denotes the identity operator on E.

A semigroup $\mathbf{T}$ is *a uniformly continuous semigroup* if $\lim_{t\downarrow 0} \|T(t)-I\| = 0$. A semigroup $\mathbf{T}$ is a C_0*-semigroup* if it is strongly continuous, i.e.

$$\lim_{t\downarrow 0} T(t)x = x \qquad \text{for every } x \in E,$$

which implies that it has continuous orbits $t \mapsto T_t x$.

The linear operator A defined by

$$\mathcal{D}(A) = \left\{ x \in E : \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \text{ exists} \right\}$$

and

$$Ax = \lim_{t\downarrow 0} \frac{T(t)x - x}{t}$$

is *the infinitesimal generator* (or briefly *the generator*) of the semigroup $\mathbf{T}$. $\mathcal{D}(A)$ is *the domain of* A.

For every C_0-semigroup $\mathbf{T}$ there exist constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that

$$\|T(t)\| \leq Me^{\omega t}. \tag{1.8}$$

The infimum of all $\omega \in \mathbb{R}$ for which there exists a constant $M = M(\omega) \geq 1$ such that $\|T(t)\| \leq Me^{\omega t}$ is called *the growth bound of* $\mathbf{T}$. It will be denoted by $\omega_0 = \omega_0(\mathbf{T})$.

If $\omega_0 = 0$ $\mathbf{T}$ is called *uniformly bounded.* If $\omega_0 \leq 0$ and $M = 1$ $\mathbf{T}$ is called a C_0-semigroup of contractions.

For C_0-semigroups we have the following characterization due to Hille and Yosida (contraction case), Feller, Miyadera and Phillips (general case) (see [25, 3.8 Generation Theorem]).

Proposition 1.6.3 *A linear operator A is the generator of a C_0-semigroup $\mathbf{T}$ satisfying $\|T(t)\| \leq Me^{\omega t}$, if and only if*

1. *A is closed and densely defined,*
2. *for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ one has $\lambda \in \rho(A)$ and*
$$\|R(\lambda, A)^n\| \leq \frac{M}{(\operatorname{Re}\lambda - \omega)^n} \qquad \text{for all } n \in \mathbb{N}.$$

From this Proposition it follows directly, that $-A$ is sectorial of type $\sigma \leq \frac{\pi}{2}$ if $\omega_0 \leq 0$.

Sometimes it is useful to consider the C_0-semigroup $\mathbf{S} = (S(t)) = (e^{\beta t}T(\alpha t))$, $\beta \in \mathbb{C}, \alpha > 0$. If $\beta = -\omega_0$ or $\beta < -\omega_0$ then $\mathbf{S}$ will have growth bound equal or less than zero. Moreover $\mathbf{S}$ has generator $B = \alpha A + \beta I$ with domain $\mathcal{D}(B) = \mathcal{D}(A)$, $\sigma(B) = \alpha\sigma(A) + \beta$ and $R(\lambda, B) = \frac{1}{\alpha}R(\lambda - \frac{\beta}{\alpha}, A)$ for $\lambda \in \rho(B)$.

We will call $\mathbf{S}$ the (α, β)-*rescaled semigroup*. In our context we will work with the case $\beta < -\omega_0$, $\alpha = 1$. The $(1, \beta)$-rescaled semigroup with $\beta < -\omega_0$ we will therefore call briefly *the rescaled semigroup*. The precise value of β will thereby play no role and is hence suppressed in this notation.

Analytic semigroups

(see [49, 25, 34].) The class of analytic semigroups is a certain subclass to the class $\mathcal{S}(E)$ of sectorial operators as we will see below.

Let $(A, \mathcal{D}(A))$ be a closed and densely defined operator in E. furthermore let A have its spectrum outside a sector $\Sigma_\sigma = \{\lambda \in \mathbb{C} : -\sigma < \arg\lambda < \sigma\} \setminus \{0\}$ where $\sigma \in (\frac{\pi}{2}, \pi)$. The resolvent $R(\lambda, A)$ of A satisfies the estimate

$$\|\lambda R(\lambda, A)\| \leq M_{\sigma'}, \qquad \lambda \in \Sigma_{\sigma'},$$

for $\sigma' < \sigma$.

For an A with those properties we can define a family of bounded linear operators by the contour integral

$$T(z) := \frac{1}{2\pi i}\int_\Gamma e^{\lambda z}R(\lambda, A)d\lambda, \qquad z \in \Sigma_{\sigma - \frac{\pi}{2}},$$

where $\Gamma = \partial(\Sigma_\theta \setminus \{\lambda \in \mathbb{C} : |\lambda| \leq \varepsilon\})$ for some $\theta \in (\frac{\pi}{2}, \sigma)$ and $\varepsilon > 0$. For $z, z_1, z_2 \in \Sigma_{\sigma - \frac{\pi}{2}}$ we then have the following properties:

$$\begin{aligned} \frac{d}{dz}T(z) &= AT(z), \\ T(z_1)T(z_2) &= T(z_1 + z_2). \\ \|T(z)\| &\text{ is bounded in } \Sigma_\delta \text{ for every } \delta < \sigma - \tfrac{\pi}{2} \end{aligned}$$

Set $T(0) = I$ then the family $\mathbf{T} = (T(z))_{z\in\Sigma_{\sigma-\frac{\pi}{2}}\cup\{0\}}$ is analytic and has the semigroup property. We will call $\mathbf{T}$ a *bounded analytic semigroup.* We also have that $(T(s))_{s\in\mathbb{R}_+}$ defines a C_0-semigroup.

The resolvent $R(\lambda, A)$ can be recovered from $\mathbf{T}$ by

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t} T(t) dt, \qquad \operatorname{Re}\lambda > 0.$$

A detailed description of the convergence of the contour integral and the whole theory can be found in many books (see e.g. in[25, Chapter II.4]).

The next theorem (see [25, 4.6 Theorem]) presents the connection between analytic semigroups and sectorial operators. It also gives estimates which will turn out to be very useful.

Theorem 1.6.4 *For a closed and densely defined operator A on a Banach space E the following are equivalent:*

1. *For some $\delta > 0$, A generates a bounded analytic semigroup $(T(z))_{z\in\Sigma_\delta\cup\{0\}}$.*
2. *$-A$ is sectorial of type $\sigma < \frac{\pi}{2}$.*

1.6.2. Construction of the H^∞-calculus

In this subsection we will roughly outline the H^∞-functional-calculus. A *functional calculus* in general is an algebra homeomorphism Φ from an algebra $\mathcal{F}$ of functions $f : \Lambda \to \mathbb{C}$, $\Lambda \subset \mathbb{C}$, into the space $\mathcal{L}(E)$.

Such a map is called *a bounded functional calculus with respect to a norm $\|\cdot\|_\mathcal{F}$ on $\mathcal{F}$* if there is a constant $C > 0$ with

$$\|\Phi(f)\|_{\mathcal{L}(E)} \le C\|f\|_\mathcal{F} \qquad \text{for all } f \in \mathcal{F}. \tag{1.9}$$

The aim of such a calculus is that we can first calculate with functions in $\mathcal{F}$ which is often more intuitive and then transfer the results to $\mathcal{L}(E)$ due to (1.9).

Consider now the class $\mathcal{S}_\omega$ of injective sectorial operators of type ω which have dense range. For $\sigma \in (\omega, \pi)$ let $H(\Sigma_\sigma)$ denote the algebra of holomorphic functions defined on Σ_σ and $H^\infty(\Sigma_\sigma)$ the subalgebra of $H(\Sigma_\sigma)$ consisting of all bounded functions. For the construction of the functional calculus we further need the class $H_0^\infty(\Sigma_\sigma)$ which is itself a subalgebra of $H^\infty(\Sigma_\sigma)$ consisting of those functions which satisfy an estimate

$$|f(\lambda)| \le C\frac{|\lambda|^\varepsilon}{(1+|\lambda|^2)^\varepsilon}, \quad \lambda \in \Sigma_\sigma,$$

for some $C, \varepsilon > 0$.

Characteristic for these functions is that $z \mapsto \frac{f(z)}{z}$ is integrable in zero and infinity along the contour Γ of a sector $\Sigma_{\sigma'}$, i.e. $\Gamma = \partial\Sigma_{\sigma'}$, $\omega < \sigma' < \sigma$. Since on Γ for $A \in \mathcal{S}_\omega(E)$ we have the growth estimate (1.7) , we can define a bounded operator $f(A)$ on E by the contour integral

$$f(A) := \frac{1}{2\pi i}\int_\Gamma f(\lambda)R(\lambda, A)d\lambda, \qquad f \in H_0^\infty(\Sigma_\sigma).$$

One can show that this map is independent of σ' and defines an algebra homeomorphism from $H_0^\infty(\Sigma_\sigma)$ into $\mathcal{L}(E)$ (see [34, Section II.9], [26]). So far $H_0^\infty(\Sigma_\sigma)$ is to limited too satisfy our needs.

In order to extend the functional calculus from $H_0^\infty(\Sigma_\sigma)$ to $H^\infty(\Sigma_\sigma)$ we can make use of regularizing functions. Consider $A \in \mathcal{S}_\omega(E)$ and $f \in H^\infty(\Sigma_\sigma)$. A regularizing function φ is an element of $H_0^\infty(\Sigma_\sigma)$ such that $\varphi(A)$ is injective and $\varphi f \in H_0^\infty(\Sigma_\sigma)$. Then we can define the extended functional calculus in a natural way by

$$\begin{aligned} f(A) &:= \varphi(A)^{-1}(\varphi f)(A) \\ \mathcal{D}(f(A)) &= \{x \in E : (\varphi f)(A)x \in \mathcal{D}(\varphi^{-1}(A))\}. \end{aligned}$$

For the extension to the class $H^\infty(\Sigma_\sigma)$ one could choose as regularizing function the function $\psi = z(1+z)^{-2}$. To extend the calculus to the class of polynomially bounded functions we can use suitable powers ψ^α of ψ (see [27], [34]).

We have the following properties of the H^∞-calculus: for the functions $f_0(z) = 1$, $f_1(z) = z$ we have $f_0(A) = Id_E$ and $f_1(A) = A$. Furthermore for $r_\lambda(z) = (\lambda - z)^{-1}$ we obtain $r_\lambda(A) = R(\lambda, A)$.

We also can define fractional powers A^α of A, $\alpha \in C$ via the functional calculus.

The next definition characterizes an important regularity property of A which will play an important role in analyzing regularity of the stochastic differential equations treated in this work (see [34, 9.10 Definition, 9.11 Remark]).

Definition 1.6.5 We say an operator $A \in \mathcal{S}_\omega(E)$ has *a bounded $H^\infty(\Sigma_\sigma)$-functional-calculus*, $0 \le \omega < \sigma \le \pi$, if there exists a constant $C > 0$ such that for all $f \in H^\infty(\Sigma_\sigma)$, we have $f(A) \in \mathcal{L}(E)$ and

$$\|f(A)\| \le C\|f\|_{H^\infty(\Sigma_\sigma)} \tag{1.10}$$

where $\|\cdot\|$ denotes the supremum norm on Σ_σ.

Remark 1.6.6 To assure (1.10) in the definition above it suffices to have the estimate for all f in the smaller class $H_0^\infty(\Sigma_\sigma)$.

Notation 1.6.7 By $\omega_\infty = \omega_\infty(A)$ we denote the infimum over all σ with $0 \le \omega < \sigma \le \pi$ for which (1.10) holds. For the norm $\|\cdot\|_{H^\infty(\Sigma_\sigma)}$ we will also write $\|\cdot\|_\infty$.

The γ-bounded H^∞-calculus

In Chapter 4 we will encounter an assumption on the generator A which is stronger than the bounded $H^\infty(\Sigma_\sigma)$ calculus.

For a general operator $A \in \mathcal{S}_\omega(E)$ we consider the following. If A admits not only a bounded $H^\infty(\Sigma_\sigma)$ calculus but the set

$$\{f(A) :\ \|f\|_{H^\infty(\Sigma_\sigma)} \leq 1\} \tag{1.11}$$

is γ-bounded, we say that A *has a γ-bounded $H^\infty(\Sigma_\sigma)$-calculus*. We say that A *admits a γ-bounded H^∞-calculus* if it admits a γ-bounded $H^\infty(\Sigma_\sigma)$-calculus for some $0 < \sigma < \pi$. Analogously to the case in 1.6.7 we have the following.

Notation 1.6.8 By $\omega_\infty^\gamma = \omega_\infty^\gamma(A)$ we denote the infimum over all σ with $0 \leq \omega < \sigma \leq \pi$ for which (1.10) holds and the we have that the set in (1.11) is γ-bounded.

For more details we refer to [17, 31, 30, 34].

On a Hilbert space E, negative generators of C_0-contraction semigroups, as well as negative generators given by closed sectorial forms, admit a γ-bounded H^∞-calculus. It is also known that a large class of elliptic partial differential operators on regular bounded domains in $\mathbb{R}^d$ admit a γ-bounded H^∞-calculus (see [17, 34]).

The following lemma will be used in the Section 4.2. See [21] and [16, Lemma 3.1] for a related result. We use the notation B_{l^∞} for the closed unit ball of l^∞.

Lemma 1.6.9 *Assume that $-A$ admits a γ-bounded H^∞-calculus of angle $0 < \omega_\infty(-A) < \pi$. Fix a function $f \in H_0^\infty(\Sigma_\sigma)$, where $\omega < \sigma < \pi$. Then the family*

$$F = \Big\{ \sum_{n=1}^N a_n f(-2^{-n} sA) :\ N \geq 1,\ s > 0,\ a \in B_{l^\infty} \Big\}$$

is γ-bounded, with γ-bound depending only on A and σ.

Proof. For $N \geq 1$, $s > 0$, and $a \in B_{l^\infty}$ fixed, define $f_{N,s,a} : \Sigma_\sigma \to \mathbb{C}$ by

$$f_{N,s,a}(\lambda) := \sum_{n=1}^N a_n f(2^{-n} s\lambda).$$

Since $f \in H_\varepsilon^\infty(\Sigma_\sigma)$ for some $\varepsilon > 0$,

$$|f_{N,s,a}(\lambda)| \leq \sum_{n=1}^N \Big(\frac{2^{-n}s|\lambda|}{1 + (2^{-n}s|\lambda|)^2}\Big)^\varepsilon =: M_N(s|\lambda|).$$

It is elementary to check that $\sup_{r>0,n\geq 1} M_n(r) < \infty$, and therefore the family $\big\{f_{N,s,a} :$ $N \geq 1,\ s > 0,\ a \in B_{l^\infty}\big\}$ is uniformly bounded in $H^\infty(\Sigma_\sigma)$. The result now follows from the fact that $-A$ admits a γ-bounded $H^\infty(\Sigma_\sigma)$-calculus. $\square$

2. Stochastic Pettis integration and the stochastic Cauchy problem

2.1. Construction of stochastic Integrals in Banach spaces

The notion of *stochastic Pettis integrability* as developed in [45, 46, 47] is the main tool in our considerations. We recall the definitions:

Let (S, Σ, ν) be a finite measure space. Recall (see 1.4.5) that $\varphi : S \to E$ is weakly L_p if the function $\langle \varphi, x^* \rangle$ is measurable and belongs to $L_p(S)$ for all $x^* \in E^*$. A function $\Phi : (0, T) \to \mathcal{L}(H, E)$ is H-weakly L^2 if for all $x^* \in E^*$ the map $t \mapsto \Phi^*(t)x^*$ is strongly measurable and satisfies

$$\int_0^T \|\Phi^*(t)x^*\|_H^2 \, dt < \infty \, .$$

Let $\beta = \{\beta(t)\}_{t \in [0,T]}$ be a standard Brownian motion over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, adapted to some given standard filtration $\{\mathcal{F}_t\}_{t \in [0,T]}$ which fulfills an independence condition as in (PF) 2 on page 15.

A function $\varphi : (0, T) \to E$ is called *stochastically Pettis integrable* with respect to β if it is weakly L^2 and for all measurable $A \subseteq (0, T)$ there exists a random variable $Y_A \in L^2(\Omega; E)$ such that for all $x^* \in E^*$ we have

$$\langle Y_A, x^* \rangle = \int_0^T \mathbf{1}_A(t) \langle \varphi(t), x^* \rangle \, d\beta(t) \tag{2.1}$$

almost surely. In this situation we write

$$Y_A = \int_A \varphi(t) \, d\beta(t) \, .$$

We call a function $\Phi : (0,T) \to \mathcal{L}(H,E)$ *stochastically Pettis integrable* with respect to W_H (compare 1.3.4) if it is H-weakly L^2 and for all measurable $A \subseteq (0,T)$ there exists a random variable $Y_A \in L^2(\Omega; E)$ such that for all $x^* \in E^*$ we have

$$\langle Y_A, x^* \rangle = \int_0^T \mathbf{1}_A(t)\Phi^*(t)x^* dW_H(t)$$

almost surely. In this situation we write

$$Y_A = \int_A \Phi(t)\, dW_H(t)\,.$$

One can also expand Y_A in a series as follows: Fix an orthonormal basis $(h_n)_{n=1}^\infty$ for H. Upon identifying $\mathscr{L}(\mathbb{R}, E)$ with E in the canonical way, for each $n \geq 1$, the E-valued function $\Phi(\cdot)h_n$ is stochastically integrable with respect to the cylindrical $\mathbb{R}$-Wiener process (i.e., real Brownian motion) $W_H(\cdot)h_n$ and we have the *coordinate expansion* [45, Theorem 4.2]

$$Y_A = \sum_{n=1}^\infty \int_0^T 1_A(t)\Phi(t)h_n\, dW_H(t)h_n, \tag{2.2}$$

where the series converges unconditionally in $L^2(\Omega; E)$.

The following theorem from [45, Theorem 4.2] characterizes stochastic integrability in different manners. From section 1.4 we recall the definition of $I_\Phi \in \mathcal{L}(L^2(0,T;H), E)$:

$$\langle I_\Phi f, x^* \rangle := \int_0^T [\Phi^*(t)x^*, f(t)]_H\; dt, \quad f \in L^2(0,T;H),\; x^* \in E^*, \tag{2.3}$$

where $\Phi : (0,T) \to \mathcal{L}(H,E)$ is H-weakly L^2. Further we defined there

$$\|\Phi\|_{\gamma(0,T;H,E)} := \|I_\Phi\|_{\gamma(L^2(0,T;H),E)}.$$

Theorem 2.1.1 *For an H-weakly L^2 function $\Phi : (0,T) \to \mathcal{L}(H,E)$ the following assertions are equivalent:*

1. *Φ is stochastically integrable with respect to W_H.*
2. *There exists an E-valued random variable Y and a weak*-sequentially dense linear subspace F of E^* such that for all $x^* \in F$ we have*

$$\langle Y, x^* \rangle = \int_0^T \Phi^*(t)x^* dW_H(t) \quad \textit{almost surely.}$$

3. *I_Φ maps $L^2(0,T;H)$ into E and $I_\Phi \in \mathcal{L}(L^2(0,T;H),E)$ is γ-radonifying.*

4. *There exists a Gaussian measure μ on E with covariance operator Q and a weak*-sequentially dense linear subspace F of E^* such that for all $x^* \in F$ we have*

$$\int_0^T \|\Phi^*(t)x^*\|_H^2 dt = \langle Qx^*, x^*\rangle.$$

If these equivalent conditions hold, then in (2) and (4) we may take $F = E^*$.

The measure μ is the distribution of $\int_0^T \Phi(t)dW_H(t)$ and we have the isometry

$$\mathbb{E}\left\|\int_0^T \Phi(t)dW_H(t)\right\|^2 = \|I_\Phi\|_\gamma^2. \tag{2.4}$$

2.2. The stochastic Cauchy problem SCP

Let $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ be a probability space.By the stochastic Cauchy problem we mean differential equations of the form

$$\begin{cases} dU(t) = AU(t)dt + BdW_H(t)\,, & t \in [0,T] \\ U_0 = u_0, & u_0 \quad \mathcal{F}_0\text{-measurable} \end{cases} \tag{2.5}$$

where $\big(A, \mathcal{D}(A)\big)$ is the generator of a C_0-semigroup $\mathbf{S} = S(t)$ on a separable Banach space E, B a bounded linear operator from a separable real Hilbert space H into E and $W_H(t) : H \to L^2(\Omega)$ a cylindrical Wiener process with Cameron Martin space H adapted to some given standard filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ which fulfills the independence condition in (PF) 2 on page 15.

Definition 2.2.1 An E-valued process $U_{u_0} : [0,T] \times \Omega \to E$ is called a *weak solution* if for all $x^* \in D(A^*)$ the following two conditions are satisfied:

1. Almost surely, the paths $t \mapsto U_{u_0}(t)$ are integrable, where we use the notation $U_{u_0}(t)$ instead of $U_{u_0}(t,\omega)$.

2. for all $t \in [0,T]$ we have almost surely

$$\langle U_{u_0}(t), x^*\rangle = \langle u_0, x^*\rangle + \int_0^t \langle U_{u_0}(s), A^*x^*\rangle\, ds + W_H(t)B^*x^* \tag{2.6}$$

In the sequel we will specify necessary and sufficient conditions for the existence of a unique weak solution.

2.3. The existence of solutions for the stochastic Cauchy problem

The existence of a weak solution of the stochastic Cauchy problem is closely related to finiteness of the γ-norm of a certain operator. This and other equivalences are contained in the following result from [45] and will play an important role.

Theorem 2.3.1 *The following assertions are equivalent:*

1. *The problem* (2.5) *has a weak solution* $\{U_{u_0}(t)\}_{t\in[0,T]}$*;*
2. *The operator* $R \in \mathcal{L}(E^*, E)$ *defined by*

$$Rx^* := \int_0^T S(t)BB^*S^*(t)x^*\,dt, \qquad x^* \in E^*,$$

   is a Gaussian covariance operator;

3. *The operator*

$$Vf := \int_0^T S(t)Bf(t)\,dt$$

   is γ*-radonifying from* $L^2\big((0,T);H\big)$ *into* E*.*

4. *the function* $s \mapsto S(T-s)B$ *is in* $\gamma(0,T;H,E)$

In this situation, the function $t \mapsto S(t)B$ *is stochastically Pettis integrable on* $(0,T)$ *with respect to* W_H *and for all* $t \in [0,T]$ *we have*

$$U_{u_0}(t) = S(t)u_0 + \int_0^t S(t-s)B\,dW_H(s) \tag{2.7}$$

almost surely. In particular, up to a modification the problem (2.5) *has a unique weak solution. For all* $p \in [1,\infty)$ *the paths* $t \mapsto U_{u_0}(t)$ *belong to* $L^p((0,T);E)$ *almost surely, the process* $\{U_{u_0}(t)\}_{t\in[0,T]}$ *is continuous in p-th moment.*

Remark 2.3.2 $U_{u_0}(\cdot)$ is a weak solution corresponding to the problem (2.5) with initial value u_0 if and only if $U_{u_0}(\cdot) - S(\cdot)u_0$ is a weak solution corresponding to the problem (2.5) with initial value 0. Hence we will assume without loss of generality that $u_0 = 0$. We will use the notation $U(\cdot)$ instead of $U_0(\cdot)$. If we want to stress the dependence on $\omega \in \Omega$ we will also write at full length $U(\cdot,\omega)$.

Henceforth we will thus consider the problem

$$\begin{cases} dU(t) = AU(t)dt + BdW_H(t)\,, & t \in [0,T] \\ U_0 = 0 \end{cases} \tag{SCP}$$

and write SCP instead of the ‘stochastic Cauchy problem’.

So far we considered the SCP on finite intervals $[0, T]$. That this is in fact no restriction is the content of the following Lemma. We will give the short proof (see [45, Corollary 7.2.],[15]).

Lemma 2.3.3 *The following assertions are equivalent.*

1. *The problem* (SCP) *has a weak solution* $\{U(t)\}_{t\in[0,T]}$ *for some* $T > 0$.
2. *The problem* (SCP) *has a weak solution* $\{U(t)\}_{t\in[0,T]}$ *for all* $T > 0$.

Proof. The proof makes use of Proposition 1.4.3, the domination result of Gaussian covariances. Assume that the problem (SCP) has a weak solution for some $T > 0$. We denote by $R_T \in \mathcal{L}(E^*.E)$ the covariance operator as defined in Theorem 2.3.1 (2). Let now an arbitrary $\widetilde{T} > 0$ be given. To show that the problem (SCP) has a weak solution on $[0, \widetilde{T}]$ we have to show that $R_{\widetilde{T}} \in \mathcal{L}(E^*, E)$ as in 2.3.1 (2)is a Gaussian covariance.

Chose an $N \in \mathbb{N}$ such that $\widetilde{T}/N \leq T$. We have the identity

$$R_{\widetilde{T}} = \sum_{n=0}^{N-1} S(n\widetilde{T}/N) R_{\widetilde{T}/N} S^*(n\widetilde{T}/N).$$

Since R_T is a Gaussian covariance, so is $R_{\widetilde{T}/N}$ (compare (1.4)). Denote by μ the Gaussian measure with covariance $R_{\widetilde{T}/N}$. Then the image measures $\mu_n := S(n\widetilde{T}/N)\mu$, $n = 0, ..., N-1$ have covariances $S(n\widetilde{T}/N)R_{\widetilde{T}/N}S^*(n\widetilde{T}/N)$ and their convolution $\mu = \mu_0 * \mu_1 * \ldots * \mu_{N-1}$ has covariance has covariance $R_{\widetilde{T}}$. □

3. Properties of solutions of the SCP

3.1. Existence and regularity of the solution of the SCP

As we have seen in the previous chapter the solution $U : [0, T] \times \Omega \to E$ of the SCP is a predictable E-valued process. After having assured existence we will study properties of the solution. In this context we examine for a fixed $\omega \in \Omega$ the time regularity of the paths $t \mapsto U(t, \omega) =: U(t)$, i.e. whether they are continuous or Hölder continuous. For $t \in [0, T]$ it is also interesting to determine the space regularity, i.e. whether the random variables $U(t, \cdot)$ lie in certain fractional domain spaces. This leads to the next section.

3.1.1. Sobolev towers

In many cases we can prove existence or regularity of solutions of the SCP not in E but in some interpolation or extrapolation spaces. We will outline the theory of two different kinds of such 'towers' and also treat the connection between them.

1. We will give the definition of the 'Hölder-Sobolev tower' or just 'H-Sobolev tower' where a strong Hölder-continuity in 0 of the semigroup $\mathbf{S}$ is assumed.
2. If A is the generator of a C_0-semigroup we have seen in Section 1.6 that $-A$ is a sectorial operator and we can construct fractional powers. The related domains then can be used to construct the so called 'Sobolev tower'.

Since we will only give the main definitions we recommend the detailed description of Sobolev towers given in [25].

In what follows we will assume that the given C_0-semigroup $\mathbf{S} = (S(t))$ has negative growth bound $\omega_0 < 0$. This can be obtained by the rescaling procedure (see subsection 1.6.1). This is no loss of generality because for $\beta_0, \beta_1 < -\omega_0$ the norms $\|\cdot\|_\alpha^0$ with $\|x\|_\alpha^0 := \|(\beta_0 - A)^\alpha x\|$ and $\|\cdot\|_\alpha^1$ with $\|x\|_\alpha^1 := \|(\beta_1 - A)^\alpha x\|$ (with $\alpha \in \mathbb{Z}$ in the first case and $\alpha \in \mathbb{R}$ in the second case) are in fact equivalent. The spaces X_α and E_α that we will construct depend thus on the generator A but are independent of the chosen β (c.f. [25, Chapter II.5], [34, Lemma 15.22]).

For the first case we start by defining for $k \in \mathbb{Z}$

$$\|x\|_k := \begin{cases} \|A^k x\|, & x \in \mathcal{D}(A^k), \quad k > 0 \\ \|x\|, & x \in E, \quad k = 0 \\ \|A^k x\|, & x \in E, \quad k < 0 \end{cases} \tag{3.1}$$

With these norms we obtain a scale of Banach spaces

$$E_k := \begin{cases} \big(\mathcal{D}(A^k), \|\cdot\|_k\big), & k > 0 \\ \big(E, \|\cdot\|\big), & k = 0 \\ \big(E, \|\cdot\|_k\big)^{\sim}, & k < 0. \end{cases}$$

Each E_k is densely embedded in E_{k+1} and for $k \in \mathbb{Z}$ we can easily restrict or extend the semigroup $\big(S(t)\big)$ and the generator A to E_k by

$$S_k(t) := \begin{cases} S(t)_{|E_k}, & k \geq 0 \\ \text{continuous extension of } S(t) \text{ to } E_k, & k < 0 \end{cases} \tag{3.2}$$

and

$$A_k := \begin{cases} \text{the part of } A \text{ in } E_k & k \geq 0, \\ \text{unique continuous extension of the isometry} \\ A : E_1 \to E \text{ to an isometry from } E_{k+1} \text{ onto } E_k, & k < 0, \end{cases} \tag{3.3}$$

where we have $\mathcal{D}(A_k) = \{x \in E_k \,:\, Ax \in E_k\} = E_{k+1}$, $k \in \mathbb{Z}$, and the part of A in E_k, $k \geq 0$ is defined by $A_k x := A_k$ for $x \in \mathcal{D}(A_k)$.

For $\alpha \in \mathbb{R}$ we can define a scale of Banach spaces by the following.

Definition 3.1.1 Let $\big(\mathrm{S(t)}\big)$ be a C_0-semigroup on a Banach space E and let $\alpha \in \mathbb{R}$. Write $\alpha = k + \gamma$, $k \in \mathbb{Z}$, $\gamma \in [0, 1)$. The space

$$X_\alpha := \left\{ x \in E_k \,:\, \lim_{t \downarrow 0} \left\| \frac{1}{t^\gamma}(S_k(t)x - x) \right\|_k = 0 \right\}$$

equipped with the norm

$$\|x\|_\alpha := \sup_{t>0} \left\| \frac{1}{t^\gamma}(S_k(t)x - x) \right\|_k$$

is called *the abstract Hölderspace of order* α (for $\gamma = 0$ we just obtain E_k). $\big(X_\alpha\big)_{\alpha \in \mathbb{R}}$ will be called *H-Sobolev tower* .

In case (2) the definition is even simpler.

Definition 3.1.2 Let $\big(S(t)\big)$ be a C_0-semigroup with generator A. Then for $\alpha \in \mathbb{R}$ we define

$$E_\alpha := \begin{cases} \big(\mathcal{D}((-A)^\alpha), \|\cdot\|_\alpha\big), & \alpha \geq 0 \\ \big(E, \|\cdot\|_\alpha\big)^\sim, & \alpha < 0\,, \end{cases}$$

where analogous to (3.1)

$$\|x\|_\alpha := \begin{cases} \|(-A)^\alpha x\|, & x \in \mathcal{D}((-A)^\alpha), \quad \alpha \geq 0 \\ \|(-A)^\alpha x\|, & x \in E, \quad \alpha < 0\,. \end{cases}$$

The family $\big(E_\alpha\big)_{\alpha \in \mathbb{R}}$ will be called *Sobolev tower*.

Notation 3.1.3 Each X_α (respective E_α) is densely embedded in X_β (respective E_β) for $\alpha > \beta$. The embedding $X_\alpha \hookrightarrow X_\beta$ (resp. $E_\alpha \hookrightarrow E_\beta$) is denoted by $i_{\alpha,\beta}$ and if $\alpha = 0$ just i_β. For a semigroup $\big(S(t)\big)$ with generator A on E we define $\big(S_\alpha(t)\big)$ and A_α analogously to (3.2) and (3.3).

We state here an important proposition (see [25, 5.35 Proposition]) which we will use frequently and often without further mentioning.

Proposition 3.1.4 *Let $\alpha, \beta \in (0,1)$ satisfy $\alpha + \beta \neq 1$. Then the iterated abstract Hölder space $(X_\alpha)_\beta$ coincides with the abstract Hölderspace $X_{\alpha+\beta}$.*

The relation between the two families of spaces is given by the next proposition (compare [25, 5.33 Proposition]).

Proposition 3.1.5 *Let $\alpha, \beta \in (0,1)$ such that $\alpha > \beta$.*

Then $X_\alpha \hookrightarrow E_\beta \hookrightarrow X_\beta$.

3.1.2. The stochastic Fubini theorem

This theorem is proved in full generality in [15, Chapter 4.6]. For the sake of completeness we add here a version which fits better in our setting. With $\mathscr{B}(0,T)$ we mean the Borel σ-field over the interval $(0,T)$.

Theorem 3.1.6 *Let φ be a $\mathscr{B}(0,T) \otimes \mathscr{B}(0,T)$-measurable function with values in H. Assume that $\varphi(s,\cdot) \in L^2((0,T);H)$ for almost all $s \in (0,T)$ and that*

$$\int_0^T \|\varphi(s,\cdot)\|_{L^2((0,T);H)} ds < \infty\,. \tag{3.4}$$

Then

1. *the $L^2(\Omega)$-valued function*

$$s \mapsto \int_0^T \varphi(s,t) dW_H(t)$$

   is Bochner integrable;

2. *For almost all $t \in (0,T)$ the function $s \mapsto \varphi(s,t)$ belongs to $L_1((0,T);H)$, and the H-valued function*

$$t \mapsto \int_0^T \varphi(s,t) ds$$

   is square Bochner integrable;

3. *We have*

$$\int_0^T \left(\int_0^T \varphi(s,t) dW_H(t) \right) ds = \int_0^T \left(\int_0^T \varphi(s,t)\, ds \right) dW_H(t)$$

   as elements of $L^2(\Omega)$.

For the proof we will need the following proposition.

Proposition 3.1.7 *Let $\varphi \in \mathscr{B}(0,T) \otimes \mathscr{B}(0,T)$ be given such that* (3.4) *is fulfilled. Then there exists a sequence (φ_n) of functions from $(0,1) \times (0,1)$ into H of the form*

$$\varphi_n(s,t) = \sum_{j=1}^{N_n} \sum_{k=1}^{M_n} \mathbf{1}_{(s_{j-1}^n, s_j^n) \times (t_{k-1}^n, t_k^n)}(s,t) \cdot h_{jk}^n \tag{3.5}$$

where the (s_{j-1}^n, s_j^n), $j = 1 \dots N_n$, and the (t_{k-1}^n, t_k^n), $k = 1 \dots M_n$, are disjoint subintervals of $(0,T)$ and h_{jk}^n are elements of H such that

$$\lim_{n\to\infty} \int_0^T \|\varphi(s,\cdot) - \varphi_n(s,\cdot)\|_{L^2((0,T);H)} ds = 0\,. \tag{3.6}$$

Proof. Assumption (3.4) allows us to view φ as an element of $L_1((0,T); L^2((0,T);H))$. Therefore there exists a sequence of step functions (ψ_n) with $\psi_n(s,\cdot) = \sum_{j=1}^{N_n} \mathbf{1}_{(t_{j-1}^n, t_j^n)}(s) \cdot g_j^n(\cdot)$, where $g_j^n \in L^2((0,T),H)$ such that

$$\lim_{n\to\infty} \int_0^T \|\varphi(s,\cdot) - \psi_n(s,\cdot)\|_{L^2((0,T);H)} ds = 0\,.$$

Furthermore there exists for all $n \in \mathbb{N}$ and $j = 1 \dots N_n$ a sequence of step functions $(f_m^{n,j})$ with $f_m^{n,j} = \sum_{k=1}^{M_m} \mathbf{1}_{(t_{k-1}^n, t_k^n)} \cdot h_{n,j,k}^m$ such that

$$\lim_{m\to\infty} \|g_j^n - f_m^{n,j}\|_{L^2((0,T);H)} = 0\,.$$

Consider now the double indexed sequence

$$\begin{aligned}\varphi_{n,m} &:= \sum_{j=1}^{N_n} \mathbf{1}_{(t_{j-1}^n, t_j^n)} \cdot f_m^{n,j} \\ &= \sum_{j=1}^{N_n} \sum_{k=1}^{M_m} \mathbf{1}_{(t_{j-1}^n, t_j^n)\times(t_{k-1}^m, t_k^m)} \cdot h_{n,j,k}^m\,.\end{aligned}$$

For $l \geq 1$ choose n_l with $\int_0^T \|\varphi(s,\cdot) - \psi_{n_l}(s,\cdot)\|_{L^2} ds \leq \frac{1}{l}$ and m_l with $\|g_j^{n_l} - f_{m_l}^{n_l,j}\| \leq \frac{1}{lT}$ for all $j = 1, \dots, N_{n_l}$.

Then

$$\begin{aligned}&\int_0^T \|\varphi(s,\cdot) - \varphi_{n_l,m_l}(s,\cdot)\| ds \\ &\quad\leq \int_0^T \|\varphi(s,\cdot) - \psi_{n_l}\| ds + \|\psi_{n_l}(s,\cdot) - \varphi_{n_l,m_l}(s,\cdot)\| ds \\ &\quad\leq \frac{2}{l}\,.\end{aligned}$$

The sequence $(\varphi_l) := (\varphi_{n_l,m_l})$ fulfills (3.6).

$\square$

Proof of theorem 3.1.6: First we show that the function $s \mapsto \int_0^T \varphi(s,t) dW_H(t)$ is strongly measurable. Let (φ_n) be a sequence of step functions of the form (3.5) such that (3.6) holds. For all $s \in (0,T)$ the function $f_n(s) := \varphi_n(s,\cdot)$ belongs to $L^2((0,T);H)$. Then we may assume, after passing to a pointwise a.e. convergent subsequence, that for almost all $s \in (0,T)$ we have

$$f(s) := \varphi(s,\cdot) = \lim_{n\to\infty} f_n(s) \qquad \text{in } L^2((0,T);H)\,.$$

It follows that for almost all $s \in (0,T)$ we have by the Itô isometry

$$\int_0^T \varphi(s,t) dW_H(t) = \lim_{n\to\infty} \int_0^T \big(f_n(s)\big)(t) dW_H(t) \qquad \text{in } L^2(\Omega)\,.$$

But

$$\int_0^T (f_n(s))(t)dW_H(t) = \sum_{j=1}^{N_n}\sum_{k=1}^{M_n} \mathbf{1}_{s_{j-1}^n, s_j^n}(s) \cdot \big(W_H(t_k^n) - W_H(t_{k-1}^n)\big)\, h_{j,k}^n$$

which shows that the $L^2(\Omega)$-valued function $s \mapsto \int_0^T (f_n(s))(t)dW_H(t)$ is a step function.

We have shown that $s \mapsto \int_0^T \varphi(s,t)dW_H(t)$ is a.e. the limit of a sequence of $L^2(\Omega)$-valued step functions. It follows that this function is strongly measurable. Its Bochner integrability now follows from the Itô isometry since

$$\int_0^T \left\| \int_0^T \varphi(s,t)dW_H(t)\right\|_{L^2(\Omega)} ds = \int_0^T \|f(s)\|_{L^2((0,T);H)}ds \qquad < \infty\,.$$

Now we prove (2). In the proof of proposition 3.5 we have seen that φ can be viewed as an element of $L_1((0,T); L^2((0,T);H))$. Hence $\int_0^T f(s)ds$ is also in $L^2((0,T);H)$. Since $L_1((0,T); L^2((0,T);H))$ can be embedded in $L_1((0,T); L_1((0,T);H))$ where the latter is isomorphic to $L_1((0,T)\times(0,T))$ we can use Fubinis Theorem in $L_1((0,T)\times(0,T))$ to obtain that for almost all t $\left(\int_0^T f(s)ds\right)(t) = \int_0^T \varphi(s,t)ds$. Thus we showed that $t \mapsto \int_0^T \varphi(s,t)ds$ belongs to $L^2((0,T);H)$.

To prove (3) we first note that this assertion holds for any step function φ of the form

$$\varphi = \sum_{j=1}^{N}\sum_{k=1}^{M} \mathbf{1}_{(s_{j-1},s_j)\times(t_{k-1},t_k)} \cdot h_{jk}\,. \tag{3.7}$$

Indeed, by direct computation both expressions in (3) are seen to be equal to

$$\sum_{j=1}^{N}\sum_{k=1}^{M} \big(t_k - t_{k-1}\big)\big(W_H(s_j) - W_H(s_{j-1})\big)h_{jk}.$$

Now let φ be an arbitrary function fulfilling the assumptions of theorem 3.1.6 and let (φ_n) be a sequence of step functions with $\lim_{n\to\infty}\varphi_n = \varphi$ in $L_1((0,T)\,;\,L^2((0,T);H))$. Without loss of generality we may assume that each φ_n is of the form (3.5).

The Itô isometry now leads to the following estimates:

$$\begin{aligned}
&\left\| \int_0^T \left(\int_0^T \varphi_n(s,t) - \varphi(s,t)ds\right) dW_H(t)\right\|_{L^2(\Omega)} \\
&\quad = \left\| \int_0^T (\varphi_n(s,\cdot) - \varphi(s,\cdot))ds\right\|_{L^2((0,T);H)} \\
&\quad \le \int_0^T \|\varphi_n(s,\cdot) - \varphi(s,\cdot)\|_{L^2((0,T);H)}ds
\end{aligned}$$

and

$$
\begin{aligned}
&\left\| \int_0^T \left(\int_0^T (\varphi_n(s,t) - \varphi(s,t)) dW_H(t) \right) ds \right\|_{L^2(\Omega)} \\
&\leq \int_0^T \left\| \int_0^T (\varphi(s,t) - \varphi_n(s,t)) dW_H(t) \right\|_{L^2(\Omega)} ds \\
&\qquad = \int_0^T \|\varphi(s,\cdot) - \varphi_n(s,\cdot)\|_{L^2((0,T);H)} ds \, .
\end{aligned}
$$

Combining everything we obtain

$$
\begin{aligned}
\int_0^T \left(\int_0^T \varphi(s,t)\, dW_H(s) \right) dt &= \lim_{n\to\infty} \int_0^T \left(\int_0^T \varphi_n(s,t)\, dW_H(s) \right) dt \\
&= \lim_{n\to\infty} \int_0^T \left(\int_0^T \varphi_n(s,t)\, dt \right) dW_H(s) \\
&= \int_0^T \left(\int_0^T \varphi(s,t)\, dt \right) dW_H(s),
\end{aligned}
$$

the convergence being in the sense of $L^2(\Omega)$.

□

3.2. Regularizing properties of B and $\mathbf{S}$

In this section we examine how regularity of the operator B or of the semigroup $\mathbf{S}$ leads to existence and regularity of the solution of the problem

$$
\left\{ \begin{array}{rcl} dU(t) &=& AU(t) + BdW_H(t) \\ U(0) &=& 0 \end{array} \right. \tag{SCP}
$$

or of related stochastic differential equations that we will define next.

If nothing else is stated we still assume $\mathbf{S}$ to be a C_0-semigroup in E and $B \in \mathcal{L}(H,E)$.

- *Case 1:* $\alpha < 0$. Let X_α be the αth extrapolated space of the H-Sobolev tower $\big(X_\alpha\big)$ and denote by i_α the natural inclusion.

  An *X_α-valued solution of the stochastic Cauchy problem SCP* (or an *α-extended solution*) is a weak solution of the related problem

$$
\left\{ \begin{array}{rcl} dV(t) &=& A_\alpha V(t) + B_\alpha dW_H(t) \\ V(0) &=& 0 \end{array} \right. \tag{SCP$_\alpha -$}
$$

where A_α is the generator of $\{S_\alpha(t)\}$, the extended semigroup on X_α (see 3.1.1) and $B_\alpha := i_\alpha \circ B$. Surely, every weak solution of (SCP) is also a weak solution of ($\text{SCP}_\alpha-$). The converse holds also under appropriate conditions (see Proposition 3.2.1 below).

- *Case 2:* $\alpha \geq 0$. Let X_α be the αth space of the H-Sobolev tower. Assume that $\mathrm{Ran}B \subseteq X_\alpha$. Then an *$X_\alpha$-valued Solution of the (SCP)* (or an *α-restricted solution*)is a solution of the related problem

$$\begin{cases} dV(t) &= A_\alpha V(t) + BdW_H(t) \\ V(0) &= 0 \end{cases} \tag{SCP$_\alpha$+}$$

Weak solutions of ($\text{SCP}_\alpha-$) we may consider as 'generalized' solutions of ($\text{SCP}_\alpha+$) since they take values in a larger space X_α, $\alpha < 0$, solutions of ($\text{SCP}_\alpha+$) are considered to be more 'regular' than the solutions of (SCP).

The following proposition gives an answer to the question when a weak solution in a separable Banach space F gives a weak solution in the separable Banach space E, provided there is a continuous and dense embedding $j : E \hookrightarrow F$ (compare [21, Proposition 4.3]).

Proposition 3.2.1 *Let an E-valued process U be given such that the F-valued process $U_F = j \circ U$ is a weak solution of*

$$\begin{cases} d\widetilde{U}(t) = A_F\widetilde{U}(t)\,dt + B_F\,dW_H(t) \quad t \in [0,T], \\ \widetilde{U}(0) = 0. \end{cases}, \tag{3.8}$$

where A_F is the generator of the semigroup on F extending $S(t)$ and $B_F := j \circ B$.

Then U is a weak solution of (SCP).

Proof. From the definition of weak solutions it follows for $x^* = j^*y^*$ with $y^* \in F^*$:

$$\begin{aligned} \langle U(t), x^*\rangle = \langle U_F(t), y^*\rangle &= \int_0^t B_F^* S_F^*(t-s)y^* dW_H(s) \\ &= \int_0^t B^* S^*(t-s)x^* dW_H(s) \end{aligned}$$

Form the injectivity of j we derive by using the Hahn-Banach-Theorem (see [53, 3.5 Theorem]) that $j^*(F^*)$ lies weak*-dense in E^*. Theorem 2.1.1 now proves that $s \mapsto S(t-s)B$ is stochastically Pettis integrable for all $t \in [0,T]$ and $U(t) = \int_0^t S(t-s)BdW_H(s)$ almost everywhere. Since Theorem 2.3.1(3) follows directly from Theorem 2.1.1(3) we obtain that U is a solution of (SCP) $\square$

We will also need the following result where X_α corresponds to E in the preceding proposition and E corresponds to F.

Corollary 3.2.2 *Consider the solution $U = U(t)$ of (SCP). Assume that for all $t \in [0,T]$ $U(t) \in X_\alpha$ almost surely for $\alpha > 0$. Then U is an α-restricted solution.*

For the abstract Hölderspaces (X_α) we have the following results concerning the *stochastic convolution* $V(t) = \int_0^t S(t-s)BW_H(s)ds$.

Lemma 3.2.3 *Let $\{S(t)\}$ be a C_0-semigroup and let $B : H \to E$ be γ-radonifying. Set $V(t) := \int_0^t S(t-s)BW_H(s)ds$.*

Then for $0 < \alpha < \frac{1}{2}$, almost all $\omega \in \Omega$ and all $t \in [0,T]$ we have

$$V(t) \in X_\alpha\,.$$

where X_α denotes the abstract Hölder space (see definition 3.1.1).

Before starting the proof for $\beta \in (0,1)$ we define $c^\beta([0,T];E)$ as the space of all continuous functions $f : [0,T] \to E$ for which

$$\lim_{\delta\downarrow 0} \sup_{|t-s|\le\delta} \frac{\|f(t)-f(s)\|}{|t-s|^\beta} = 0. \tag{3.9}$$

Endowed with the norm

$$\|f\|_{c^\beta([0,T];E)} := \|f\| + \sup_{t\neq s} \frac{\|f(t)-f(s)\|}{|t-s|^\beta} \tag{3.10}$$

this space is a separable Banach space. For $E = \mathbb{R}$ we simply write $c^\beta[0,T]$ and denote $c_0^\beta[0,T] = \{f \in c^\beta[0,T] : f(0) = f(T) = 0\}$.

Furthermore we will write $C^\beta([0,T];E)$ for the space of all continuous functions $f : [0,T] \to \mathbb{E}$ for which

$$\sup_{t,s\in[0,T],\ t\neq s} \frac{\|f(t)-f(s)\|}{|t-s|^\beta} < \infty. \tag{3.11}$$

With the same norm as in (3.10) it becomes again a Banach space. For $E = \mathbb{R}$ we write analoguesly $C^\beta[0,T]$.

Proof. Note that for almost all $\omega \in \Omega$, $BW_H(\cdot,\omega) \in c^\alpha([0,T];E)$ for all α with $0 < \alpha <$

$\frac{1}{2}$. Fix such an ω and set $f(t) := BW_H(t,\omega)$. For a fixed $t \in [0,T]$ we compute:

$$\begin{aligned}
&\frac{1}{u^\alpha}\big(S(u)V(t) - V(t)\big)\\
&= \frac{1}{u^\alpha}\int_0^t S(u+t-r)f(r)dr - \frac{1}{u^\alpha}\int_0^t S(t-r)f(r)dr\\
&= \frac{1}{u^\alpha}\int_{-u}^{t-u} S(t-r)f(r+u)dr - \frac{1}{u^\alpha}\int_0^t S(t-r)f(r)dr\\
&= \underbrace{\frac{1}{u^\alpha}\left(\int_{-u}^{0} S(t-r)f(r+u)dr - \int_{t-u}^{t} S(t-r)f(r)dr\right)}_{=:\,A(t,u)}\\
&\quad + \underbrace{\frac{1}{u^\alpha}\int_0^{t-u} S(t-r)[f(r+u)-f(r)]dr}_{=:\,B(t,u)}
\end{aligned}$$

Now it is immediate that $\|B(t,u)\|$ tends to 0 if u tends to 0. Furthermore we obtain

$$A(u,t) = u^{1-\beta}\Bigg(\underbrace{\frac{1}{u}\int_{-u}^{0} S(t-r)f(r+u)dr}_{\longrightarrow\, \underbrace{S(t)f(0)}_{=0}} - \underbrace{\frac{1}{u}\int_{t-u}^{t} S(t-r)f(r)dr}_{\longrightarrow\, f(t)}\Bigg)$$

which also tends to 0 if u tends to 0. Altogether we obtain

$$\lim_{u\downarrow 0}\left\|\frac{1}{u^\alpha}\big(S(u)V(t) - V(t)\big)\right\| = 0$$

and therefore $\int_0^t S(t-s)BW_H(s)ds \in X_\alpha$ almost surely for all $t \in [0,T]$. ☐

Corollary 3.2.4 *Assume B is γ-radonifying from H into X_β, $\beta \in \mathbb{R}$. Then for $V(t)$ defined as in Lemma 3.2.3 we obtain almost surely for all $t \in [0,T]$*

$$V(t) \in X_{\alpha+\beta}$$

where again $0 < \alpha < \frac{1}{2}$.

Proof. In a first step assume $\alpha + \beta \neq 1$. If we apply Lemma 3.2.3 to $\widetilde{X} := X_\beta$ then $V(t) \in \widetilde{X}_\alpha = (X_\beta)_\alpha = X_{\beta+\alpha}$ (see Proposition 3.1.4). If $\alpha + \beta = 1$ choose $\widetilde{\alpha}$ with $\alpha < \widetilde{\alpha} < \frac{1}{2}$. Now, $V(t) \in X_{\beta+\widetilde{\alpha}} \subset X_{\beta+\alpha}$. ☐

Proposition 3.2.5 *Let $\{S(t)\}$ be a C_0-semigroup and let $B : H \to X_\beta$, $0 < \beta \leq 1$, be γ-radonifying. Again let the process $\{V(t)\}$ be defined by $V(t) = \int_0^t S(t-s)BdW_H(s)$. Then for $0 < \alpha < \frac{1}{2}$ the process $\mathbf{Y}$ defined by*

$$Y(t) = i_{\alpha+\beta-1}BW_H(t) + A_{\alpha+\beta-1}\int_0^t S_\beta(t-s)BW_H(s)ds$$

is an $X_{\alpha+\beta-1}$-valued weak solution of (SCP).

Proof. We first state that both components of $Y(t)$ exist in the space $X_{\alpha+\beta-1}$: Since $\beta > \alpha+\beta-1$ it follows that $BW_H(t)$ exists as an element of $X_{\alpha+\beta-1}$. Since by Corollary 3.2.4 $\int_0^t S(t-s)BW_H(s)ds \in X_{\alpha+\beta}$ almost surely and

$$\|A_{-1}\int_0^t S(t-s)BW_H(s)ds\|_{\alpha+\beta-1} = \|\int_0^t S(t-s)BW_H(s)ds\|_{\alpha+\beta}$$

this holds also for $A_{-1}\int_0^t S(t-s)BW_H(s)ds$.

To show that $\mathbf{Y}$ is indeed a weak solution we first show that it is an $X_{\beta-1}$-valued solution. We compute for $x^* \in \mathcal{D}(A_{\beta-1})$

$$\begin{aligned}
&\int_0^t \langle i_{\alpha+\beta-1}BW_H(u) + A_{\alpha+\beta-1}\left(\int_0^u S_\beta(u-s)BW_H(s)ds\right), A^*_{\alpha+\beta-1}x^*\rangle du \\
&= \int_0^t \langle i_{\beta-1}BW_H(u) + \left(\int_0^u A_{\beta-1}S_\beta(u-s)BW_H(s)ds\right), A^*_{\beta-1}x^*\rangle du \\
&= \int_0^t \int_0^u B^* i^*_{\beta-1} S^*_\beta(u-s)A^*_{\beta-1}x^* dW_H(s)du \qquad (3.12) \\
&= \int_0^t \int_s^t B^* i^*_{\beta-1} S^*_\beta(u-s)A^*_{\beta-1}x^* du\, dW_H(s) \qquad (3.13) \\
&= \int_0^t B^* i^*_{\beta-1} S^*_\beta(t-s)x^* - B^* i^*_{\beta-1}x^* dW_H(s) \\
&= \langle x^*, \int_0^t S_{\beta-1}(t-s)i_{\beta-1}BdW_H(s) - i_{\beta-1}BW_H(t)\rangle
\end{aligned}$$

where equation (3.12) follows by the Itô formula and (3.13) by the Fubini theorem. Now, the claim follows by Proposition 3.2.2.

□

We close this considerations by stating two results, which give sufficient conditions for existence and continuity of the solution.

Corollary 3.2.6 *Let B be γ-radonifying from H into X_β for some $\beta > \frac{1}{2}$. Then there exists a solution of SCP.*

The next result generalizes [15, Theorem 5.9]. The following result was also shown in [41, Theorem 2.2] with a different approach using mainly covariance functions and a further assumption on the growth of $\mathbf{S}$. Our approach does not need further assumptions on $\mathbf{S}$ and seems to be more directly by considering the paths of the weak solution.

Theorem 3.2.7 *Let $0 < \alpha < \frac{1}{2}$. Assume that $\Phi(t) := t^{-\alpha}S(t)B : H \to E$ is stochastically Pettis integrable. Then* (SCP) *has a weak solution which has a continuous modification.*

Proof. The proof follows the proof of [15, Theorem 5.9].

For $u \le s \le t, 0 < \alpha < 1$ the following identity holds true:

$$\int_u^t (t-s)^{\alpha-1}(s-u)^{-\alpha} ds = \frac{\pi}{\sin \pi\alpha}.$$

For $0 < \alpha < 1/2$ and almost all $\omega \in \Omega$ we can therefore write for the weak solution:

$$\begin{aligned}
&\int_0^t S(t-u)BdW_H(u) \\
&= \frac{\sin \pi\alpha}{\pi} \int_0^t S(t-u) \int_u^t (t-s)^{\alpha-1}(s-u)^{-\alpha} ds B dW_H(u) \\
&= \frac{\sin \pi\alpha}{\pi} \int_0^t \int_u^t \underbrace{S(t-s)(t-s)^{\alpha-1}S(s-u)(s-u)^{-\alpha}B}_{=:\Psi_{\alpha,t}(u,s)} ds dW_H(u) \\
&= \frac{\sin \pi\alpha}{\pi} \int_0^t S(t-s)(t-s)^{\alpha-1}\left[\int_0^s S(s-u)(s-u)^{-\alpha}BdW_H(u)\right] ds.
\end{aligned}$$

The last equation followed from the stochastic Fubini theorem (see theorem 3.1.6) and the assumptions over the integrability since for each $x^* \in E^*$

$$\begin{aligned}
&\left\langle \int_0^t \left(\int_u^t \Psi_{\alpha,t}(u,s) ds\right) dW_H(u)\,,\, x^*\right\rangle = \int_0^t \int_u^t \Psi_{\alpha,t}(u,s)^* ds x^* dW_H(u) \\
&= \int_0^t \int_0^s \Psi_{\alpha,t}(u,s)^* x^* dW_H(u) ds = \int_0^t \left\langle \int_0^s \Psi_{\alpha,t}(u,s) dW_H(u)\,,\, x^*\right\rangle ds \qquad (3.14) \\
&= \left\langle \int_0^t \int_0^s \Psi_{\alpha,t}(u,s) dW_H(u) ds\,,\, x^*\right\rangle
\end{aligned}$$

where (3.14) follows since $\psi_\alpha(u,s) := \Psi_{\alpha,t}(u,s)^* x^*$ fulfills the assumptions of theorem 3.1.6.

Now we claim that

$$U(t) = \frac{\sin \pi\alpha}{\pi} \int_0^t S(t-s)(t-s)^{\alpha-1} Y(s) ds \quad \text{with}$$
$$Y(s) = \int_0^s S(s-u)(s-u)^{-\alpha} B dW_H(u)$$

is continuous for almost all $\omega \in \Omega$ (which implies that $U(t)$ is the required continuous modification).

To see this choose $p \in (1, \infty)$ with $p < \frac{1}{1-\alpha}$. Then $\varphi(t) := S(t)t^{\alpha-1}$ is integrable on $\mathbb{R}_+$. Choose q with $\frac{1}{p} + \frac{1}{q} = 1$. Let $y \in L_q([0,T], E)$ and set

$$z(t) := \frac{\sin \pi\alpha}{\pi} \int_o^t S(t-s)(t-s)^{\alpha-1} y(s) ds.$$

Then the Hölder inequality yields for a certain C only depending on α, p, T

$$\sup_{t\in[0,T]} \|z(t)\|^q \leq C \int_o^T \|y(s)\|^q ds.$$

So the linear mapping from $L_q([0,T], E)$ to $L_\infty([0,T])$, $y \mapsto \varphi * y$, is bounded. Since $z(\cdot)$ is continuous if $y(\cdot)$ is continuous and since $C([0,T], E)$ is dense in $L_q([0,T], E)$ we obtain that $z(\cdot)$ is continuous if $y(\cdot) \in L_q([0,T], E)$.

To apply this to the paths $y(\cdot) = Y(\cdot, \omega)$ for a fixed ω, note, that $Y(t)$, $t \in [0,T]$, is by assumption a Gaussian variable with $\mathbb{E}\|Y(t)\|^q \leq C_1$ for a certain constant $C_1 > 0$ (see e.g. [36, Corollary 3.2]).

Therefore we get by the Fubini theorem:

$$\mathbb{E} \int_0^T \|Y(t, \cdot)\|^q dt \leq CT$$

thus $Y(t, \cdot) \in L_q([0,T], E)$ almost surely which yields the claimed continuity. □

In the case where A generates an analytic semigroup we get existence and continuity of the weak solution (compare [21, Proposition 3.2], [15, Chapter 5]).

Theorem 3.2.8 *Consider (SCP) where B is γ-radonifying and A generates an analytic semigroup.*

Then there exists a weak solution which has a continuous modification.

Proof. We claim that $Y(t) = BW_H(t) + A\int\limits_0^t S(t-s)BW_H(s)ds$, $t \in [0,T]$ defines a weak solution. To see this we compute for $x^* \in \mathcal{D}(A^*)$

$$
\begin{aligned}
&\int_0^t \langle Y(u), A^*x^*\rangle du \\
=&\int_0^t \langle BW_H(u) + A\int_0^u S(u-s)BW_H(s)ds, A^*x^*\rangle du \\
=&\int_0^t \left\langle \int_0^u S(u-s)BdW_H(s), A^*x^* \right\rangle du && (3.15)\\
=&\int_0^t \int_0^u B^*S^*(u-s)A^*x^*dW_H(s)du \\
=&\int_0^t \int_s^t B^*S^*(t-s)A^*x^*dudW_H(s) && (3.16)\\
=&\int_0^t (B^*S^*(t-s)x^* - B^*x^*)dW_H(s) \\
=&\langle \int_0^t S(t-s)BdW_H(s), x^*\rangle - W_H(t)B^*x^*\,,
\end{aligned}
$$

which means that $\{Y(t)\}$ is a weak solution. Equation (3.15) follows by the Iô formula and (3.16) by the Fubini theorem.

For the existence of a continuous modification we remark that the paths of the stochastic convolution $t \mapsto V(t,\omega) = \int_0^t S(t-s)BW_H(s,\omega)ds$ belongs to $C([0,T], \mathcal{D}(A))$ almost surely ([Theorem 5.3.5][37]). This proves that $Y(t) = BW_H(t) + A\int\limits_0^t S(t-s)BW_H(s)ds$ is almost surely continuous. □

3.3. Space-time regularity in the analytic case

From now on we assume the generator A to be analytic with negative growth bound. The latter we can do without loss of generality (see the explanations of the beginning of Section 3.1.1). Having assured the existence of weak solutions in the last section (Theorem 3.2.8) we proceed with investigating their regularity in space and time by carefully exploiting the smoothing effect of the resolvent. This smoothing property we can use:

- to obtain space regularity for the solution $U = U(t)$ as $E\|U(t)\|_\theta^2 < \infty$ for some $\theta \in \mathbb{R}$, see Definition 3.1.2,

- to obtain time regularity for $U(t)$, i.e. $\mathbb{E}\|U(t) - U(s)\| \leq C|t - s|^{\beta}$ for a constant C, all $t, s \in [0, T]$ and some $\beta > 0$ or
- to make $(-A)^{-\delta}B$ a γ-radonifying operator, i.e. even if $B : H \to E$ is not γ-radonifying it may be so into the 'larger space' $E_{-\delta}$, $\delta > 0$.

Results of this section emanate from a joint work with Jan van Neerven and Lutz Weis (see [21]).

3.3.1. Space regularity versus time regularity in case of γ-radonifying B

The next theorem describes an interplay between the first two points. It shows how a gain in space regularity must be bought by a loss of time regularity and vice versa. The theorem generalizes regularity results for the analytic case due to Da Prato and Zabczyk [15, Section 5.4] (for Hilbert spaces E) and Brzeźniak [5] (for martingale type 2 spaces E).

Theorem 3.3.1 *Assume that A is the generator of an analytic C_0-semigroup S on E. Let $B \in \gamma(H, E)$, and let U be the weak solution of problem* (SCP). *Let $\eta \geq 0$ and $\theta \geq 0$ satisfy $\eta + \theta < \frac{1}{2}$.*

1. *The random variables $U(t)$ take values in E_η almost surely and we have*

$$\mathbb{E}\|U(t) - U(s)\|^2_{E_\eta} \leq C|t - s|^{2\theta} \|B\|^2_{\gamma(H,E)} \quad \forall t, s \in [0, T],$$

   with a constant C independent of B;

2. *The process U has a modification with paths in $C^\theta([0, T]; E_\eta)$.*

Proof. We will use the notation '$\lesssim$' for estimates involving constants which do not depend on B.

Without loss of generality we assume that $\theta > 0$. Also without loss of generality we assume A to have negative growth bound. If this assumption is not fulfilled then the multiplication of $S(t)$ with $e^{-\beta t}$ for $\beta > \omega_0$ and the resulting required estimates in the proof would obstruct the view on the main ideas of the proof.

In order to bring out the ideas of the proof we begin with a formal computation. Put

$$R(t) := (-A)^\eta S(t), \quad t > 0.$$

Then,

$$
\begin{aligned}
&\big(\mathbb{E}\|U(t+h)-U(t)\|_{E_\eta}^2\big)^{\frac12}\\
&\quad=\big(\mathbb{E}\big\|(-A)^\eta[U(t+h)-U(t)]\big\|^2\big)^{\frac12}\\
&\quad=\Big(\mathbb{E}\Big\|\int_0^{t+h}R(t+h-s)B\,dW_H(s)-\int_0^t R(t-s)B\,dW_H(s)\Big\|^2\Big)^{\frac12}\\
&\quad\le\Big(\mathbb{E}\Big\|\int_t^{t+h}R(t+h-s)B\,dW_H(s)\Big\|^2\Big)^{\frac12}\\
&\qquad\qquad+\Big(\mathbb{E}\Big\|\int_0^t R(t+h-s)B-R(t-s)B\,dW_H(s)\Big\|^2\Big)^{\frac12}\\
&\quad=\|R(\cdot)B\|_{\gamma(0,h;H,E)}+\|R(\cdot+h)B-R(\cdot)B\|_{\gamma(0,t;H,E)}\\
&\quad\le\|R(\cdot)B\|_{\gamma(0,h;H,E)}+\|R(\cdot+h)B-R(\cdot)B\|_{\gamma(0,T;H,E)}.
\end{aligned}
$$

where the final estimate follows from (1.4).

If we can show that $R(\cdot)B\in\gamma(0,T;H,E)$, then $R(\cdot)B$ is stochastically integrable with respect to W_H by (2.4) and the above computation can be justified by noting that $(-A)^\eta$ is an isomorphism from E_η onto E. Assertion (1) will follow if we can show that for small h, say for $h\in(0,1)$, we have

$$\big\|R(\cdot)B\big\|_{\gamma(0,h;H,E)}\lesssim h^\theta\|B\|_{\gamma(H,E)}$$

and

$$\big\|R(\cdot+h)B-R(\cdot)B\big\|_{\gamma(0,T;H,E)}\lesssim h^\theta\|B\|_{\gamma(H,E)}.$$

We prove these estimates in two steps.

Step 1 – Fix an arbitrary $\alpha\in[\eta+\theta,\frac12)$ and $h\in(0,1)$. We first check that the two families

$$\mathcal{T}_h:=\{s^\alpha R(s):\ s\in(0,h)\}$$

and

$$\mathcal{T}^h:=\{s^\alpha[R(s+h)-R(s)]:\ s\in(0,T)\}$$

are γ-bounded, and that for small h their γ-bounds satisfy

$$\gamma(\mathcal{T}_h)\lesssim h^\theta \tag{3.17}$$

and

$$\gamma(\mathcal{T}^h)\lesssim h^\theta. \tag{3.18}$$

To prove (3.17) we apply Proposition 1.5.5 to the function $\Psi(s):=s^\alpha R(s)$ and check that its derivative

$$\Psi'(s)=s^\alpha AR(s)+\alpha s^{\alpha-1}R(s) \tag{3.19}$$

is integrable on $(0,h)$. Using the analyticity of S we have

$$\|AR(s)\| \leq \|AR(s)\| + r\|R(s)\| \lesssim s^{-(1+\eta)} + s^{-\eta} \lesssim s^{-(1+\eta)}$$

and we can estimate the first term in (3.19) by

$$\int_0^h s^\alpha \|AR(s)\|\, ds \lesssim \int_0^h s^{\alpha-(1+\eta)}\, ds \leq \int_0^h s^{\theta-1}\, ds \lesssim h^\theta,$$

where we used that $\alpha - \eta \geq \theta$. Similarly, for the second term in (3.19) we have

$$\int_0^h s^{\alpha-1}\|R(s)\|\, ds \lesssim \int_0^h s^{(\alpha-1)-\eta}\, ds \lesssim h^\theta.$$

Together with the estimate

$$\|h^\alpha R(h)\| \lesssim h^{\alpha-\eta} \leq h^\theta$$

we see that (3.17) follows from Lemma 1.5.5.

To prove (3.18) we apply Lemma 1.5.5 to the function $\Psi(s) := s^\alpha[R(s+h) - R(s)]$ and check that its derivative

$$\Psi'(s) = s^\alpha A[R(s+h) - R(s)] + \alpha s^{\alpha-1}[R(s+h) - R(s)] \tag{3.20}$$

is integrable on $(0,T)$. For the first term in (3.20) we have

$$\begin{aligned}
\int_0^T s^\alpha \big\|A[R(s+h) - R(s)]\big\|\, ds &= \int_0^T s^\alpha \Big\| \int_s^{s+h} A^2 R(u)\, du \Big\|\, ds \\
&\lesssim \int_0^T s^\alpha \Big(\int_s^{s+h} u^{-2-\eta}\, du \Big)\, ds \\
&\lesssim h^{\alpha-\eta} \underbrace{\int_0^\infty \sigma^\alpha \big[(\sigma+1)^{-1-\eta} - \sigma^{-1-\eta}\big]\, d\sigma}_{<\infty} \lesssim h^\theta.
\end{aligned}$$

Similarly, for the second term in (3.20) we have

$$\begin{aligned}
\int_0^T s^{\alpha-1}\|R(s+h) - R(s)\|\, ds &\lesssim \int_0^T s^{\alpha-1} \Big(\int_s^{s+h} u^{-1-\eta}\, du \Big)\, ds \\
&\lesssim h^{\alpha-\eta} \underbrace{\int_0^\infty \sigma^{\alpha-1} \big[(\sigma+1)^{-\eta} - \sigma^{-\eta}\big]\, d\sigma}_{<\infty} \lesssim h^\theta.
\end{aligned}$$

Finally,

$$\begin{aligned}
T^\alpha \|R(T+h) - R(T)\| &\lesssim T^\alpha \int_T^{T+h} \|AR(s)\| \, ds \\
&\lesssim T^\alpha \int_T^{T+h} s^{-1-\eta} \, ds \\
&\lesssim T^\alpha \big[(T+h)^{-\eta} - T^{-\eta}\big] \\
&\lesssim h^{\alpha-\eta} \underbrace{\Big(\sup_{t\in\mathbb{R}_+} t^\alpha \big[(t+1)^{-\eta} - t^{-\eta}\big]\Big)}_{<\infty} \lesssim h^\theta.
\end{aligned}$$

Combination of these estimates gives (3.18).

Step 2 – We combine Step 1 with Lemma 1.5.7. Recalling that $\alpha < \frac{1}{2}$, with Lemma 1.4.6 we obtain, with $\tau_{-\alpha}(t) := t^{-\alpha}$,

$$\|R(\cdot)B\|_{\gamma(0,h;H,E)} \lesssim h^\theta \|\tau_{-\alpha}B\|_{\gamma(0,h;H,E)} \le h^\theta \|\tau_{-\alpha}\|_{L^2(0,T)} \|B\|_{\gamma(H,E)}$$

and

$$\begin{aligned}
&\|R(\cdot+h)B - R(\cdot)B\|_{\gamma(0,T;H,E)} \\
&\qquad \lesssim h^\theta \|\tau_{-\alpha}B\|_{\gamma(0,T;H,E)} \le h^\theta \|\tau_{-\alpha}\|_{L^2(0,T)} \|B\|_{\gamma(H,E)}.
\end{aligned}$$

This concludes the proof of (1).

To prove (2) we apply (1) with exponents θ' and η, where $\theta' > \theta$ is such that we still have $\theta' + \eta < \frac{1}{2}$. By the Kahane-Khinchine inequalities we have, for any $q \ge 1$,

$$\Big(\mathbb{E}\|U(t) - U(s)\|_{E_\eta}^q\Big)^{\frac{1}{q}} \lesssim \Big(\mathbb{E}\|U(t) - U(s)\|_{E_\eta}^2\Big)^{\frac{1}{2}} \lesssim |t-s|^{\theta'} \|B\|_{\gamma(H,E)}.$$

The Kolmogorov-Chentsov continuity theorem now shows that U has a modification $\tilde{U}$ which is Hölder continuous, for any exponent less than $(\theta' q - 1)/q$. Since q can be chosen arbitrarily large, it follows that the paths of $\tilde{U}$ belong to $C^\theta([0,T]; E_\eta)$ almost surely. □

Remark 3.3.2 The theorem remains true if the fractional domain spaces E_θ are replaced by (real or complex) interpolation spaces and more generally, by spaces $E(\theta)$ satisfying inclusions $(E, \mathcal{D}(A))_{\theta,1} \hookrightarrow E(\theta) \hookrightarrow (E, \mathcal{D}(A))_{\theta,\infty}$.

3.3.2. Regularity if B is unbounded

Now we will discuss the third point of page 50. In Theorem 3.3.1 we assumed $B : H \to E$ to be γ-radonifying. In certain interesting applications this assumption is not satisfied or even worse, the operator may be unbounded (see section 3.4). This situation arises for instance when a stochastic partial differential equation driven by white noise is formulated as an abstract stochastic evolution in a state space E. Typically, $E = E(\mathcal{O})$ will be a space of functions one some domain $\mathcal{O}$ in $\mathbb{R}^d$. The precise choice of $E(\mathcal{O})$ is suggested by the interpretation of the equation and the expected space regularity of its solutions. The natural choice for the Hilbert space H used to model the white noise is then $L^2(\mathcal{O})$, with $B : L^2(\mathcal{O}) \to E(\mathcal{O})$ being the identity operator. However $L^2(\mathcal{O})$ may not embed into $E(\mathcal{O})$, and if it does, the embedding may fail to be γ-radonifying.

A way out of this difficulty is to interpret the equation in a suitably chosen Banach space F. Firstly, $E \cap F$ should be dense in both E and F and contain the range of B (we think of E and F as being continuously embedded in some ambient locally convex topological vector space) and the part of A in $E \cap F$ should extend uniquely to a generator A_F of an analytic C_0-semigroup on F. Secondly, B should extend to a γ-radonifying operator B_F from H into F. The idea is now to apply Theorem 3.3.1 in F to the problem

$$\begin{cases} dU(t) = A_F U(t)\,dt + B_F\,dW_H(t) & t \in [0,T], \\ U(0) = 0. \end{cases} \tag{3.21}$$

This will show that this solution will have its paths in $C^\theta([0,T];F_\eta)$ with $\theta, \eta \geq 0$ and $\theta + \eta < \frac{1}{2}$ and if F_η embeds continuously into E the solutions take values in E and are Hölder continuous in time of exponent θ.

We proceed with a simple illustration of this ideas. A more elaborate example will be worked out in the next section.

Example 3.3.3 (Simultaneously diagonalizable case). Let A be a diagonal operator on $E = l^p$, $1 \leq p < \infty$, with real eigenvalues $-\lambda_n$ satisfying $\lambda_n \geq c$ for some $c > 0$. Fix $\alpha \in (0,1)$ and define F as the space of all real sequences $(x_n)_{n\geq 1}$ such that $(\lambda_n^{-\alpha} x_n) \in l^p$. Endowed with the norm $\|(x_n)\|_F = \|(\lambda_n^{-\alpha} x_n)\|_{l^p}$, the space F is a Banach space, and we have $E \hookrightarrow F$ with a continuous and dense embedding. Let (b_n) be a sequence of nonnegative real numbers. The diagonal operator $B : (y_n) \mapsto (b_n y_n)$ defines an element of $\gamma(l^2, F)$ if and only $B_{-\alpha} : (y_n) \mapsto (\lambda_n^{-\alpha} b_n y_n)$ defines an element of $\gamma(l^2, l^p)$. By standard square function estimates the latter happens if and only if $\sum_n \lambda_n^{-\alpha p} b_n^p < \infty$. For the special case $b_n = 1$ (the white noise case), it follows that B defines an element of $\gamma(l^2, F)$ if and only $(\lambda_n^{-\alpha}) \in l^p$. Note that this condition depends on both α and p and is likely to be fulfilled if α and/or p are large enough. Also note that for $\eta > \alpha$ we have $F_\eta \hookrightarrow E = l^p$ with continuous inclusion.

In order to discuss the Problem (3.21) we must adapt section 2.2 and section 2.3 to the setting where B is unbounded. We can do this under the condition that $(-A)^{-\delta} B : \mathcal{D}(B) \to E$ extends to a bounded operator from H into E for some $\delta \in (0, 1/2)$.

First we allow B in the formulation of the (SCP)

$$\begin{cases} dU(t) &= AU(t) + B dW_H(t) \\ U(0) &= 0 \end{cases} \tag{3.22}$$

to be unbounded and generalize the Definition 2.2.1 in a natural way:

Assume $\mathcal{D}(A^*) \subset \mathcal{D}(B^*)$. A predictable E-valued process $\{U(t)\}_{t\in[0,T]}$ is called a *weak solution* if for all $x^* \in \mathcal{D}(A^*)$ the following two conditions are satisfied:

1. Almost surely, the paths $t \to \langle U(t,u_0), A^*x^*\rangle$ are integrable;
2. for all $t \in [0,T]$ we have almost surely

$$\langle U(t,u_0), x^*\rangle = \langle u_0, x^*\rangle + \int_0^t \langle U(s,u_0), A^*x^*\rangle\, ds + W_H(t)B^*x^* \tag{3.23}$$

Under the condition that $(-A)^{-\delta}B$ extends to a bounded operator for some $\delta \in (0,1/2)$ a generalization of Theorem 2.3.1 can be obtained. From [56, Proposition 4.7] we derive the following

Proposition 3.3.4 *Assume that A is the generator of an analytic semigroup. For an E-valued process U the following assertions are equivalent:*

1. *The problem (3.22) has a weak solution U.*
2. *For all $t \in [0,T]$ the operator $S(t)B : \mathcal{D}(B) \to E$ has a continuous extension to a bounded operator $H \to E$ and for all $t \in [0,T]$ the $\mathcal{L}(H,E)$ valued process $s \mapsto S(t-s)B$ is stochastically Pettis integrable on $(0,t)$ and*

$$U(t) = \int_0^t S(t-s)BdW_H(s)$$

   almost surely.

3. *The function $s \mapsto S(T-s)B$ is in $\gamma(0,T;H,E)$*

Up to a modification the problem (3.22) *has a unique weak solution. For all $p \in [1,\infty)$ the paths $t \mapsto U(t,u_0)$ belong to $L^p((0,T);E)$ almost surely, the process $\{U(t,u_0)\}_{t\in[0,T]}$ is continuous in p-th moment.*

In Proposition 3.2.1 we saw how for a bounded B a solution of (3.21) gives a weak solution of the original problem in E. In the case where B is unbounded this happens as well under appropriate conditions.

Proposition 3.3.5 *Let $B : \mathcal{D}(B) \to E$ be a possibly unbounded linear operator. Let $j : E \hookrightarrow F$ be a continuous and dense embedding and A be the part in E of an operator A_F in F which generates an analytic semigroup in F.*

Assume that there is a constant C and a $\delta \in (0,1/2)$ such that for all $h \in \mathcal{D}(B)$,

$$\|(-A)^{-\delta}Bh\| \leq C\|h\|. \tag{3.24}$$

Suppose an E-valued process U is given.

If the F-valued process jU is a weak solution the problem (3.21), then U is a weak solution of (3.22).

Proof. The proof proceeds like the proof of Proposition 3.2.1.

Henceforth we will identify $(-A)^{-\delta}B : \mathcal{D}(B) \to E$ with its continuous extension and denote it with $(-A)^{-\delta}B : H \to E$. To apply Theorem 2.1.1 it is left to show that the function $t \mapsto S(t)B$ is H-weakly L_2. This follows easily since in the case where A generates an analytic semigroup we have $S(t) : E \to \mathcal{D}(A)$ for all $t > 0$ and $\|(-A)^{\varepsilon}S(t)\| \leq Ct^{-\varepsilon}$ for a constant C and all $\varepsilon > 0$ ([37, Proposition 2.1.1]). Let $x^* \in E^*$. For a certain constant C we obtain the estimate

$$\begin{aligned}\|B^*S^*(t)x^*\| &\leq \|S(t)B\|\|x^*\| = \|S(t)(-A)^{\delta}(-A)^{-\delta}B\|\|x^*\| \\ &\leq \|(-A)^{\delta}S(t)\|\|(-A)^{-\delta}B\|\|x^*\| \leq Ct^{-\delta}\|x^*\|\end{aligned}$$

which lies in $L_2(0,T)$ if $\delta < 1/2$. □

Now we can formulate an important corollary concerning the third point of the introductory remarks of page 50. Note that if $(-A)^{\delta}B \in \gamma(H,E)$ then equivalently $B \in \gamma(H, E_{-\delta})$.

Theorem 3.3.6 *Assume that A is the generator of an analytic C_0-semigroup S on E, let $B \in \gamma(H, E_{-\delta})$ and let U be the weak solution of problem* (3.21) *with $F := E_{-\delta}$. Let $\eta \geq 0$ and $\theta \geq 0$ satisfy $\eta + \theta < \frac{1}{2}$. Then the following assertions hold:*

1. *The random variables $U(t)$ take values in $E_{\eta-\delta}$ almost surely and we have*

$$\mathbb{E}\|U(t) - U(s)\|^2_{E_{\eta-\delta}} \leq C|t-s|^{2\theta}\|B\|^2_{\gamma(H,E_{-\delta})} \quad \forall t, s \in [0,T],$$

   with a constant C independent of B;

2. *The process U has a modification with paths in $C^{\theta}([0,T]; E_{\eta-\delta})$.*

3.4. An example

We consider the following stochastic partial differential equation driven by space-time white noise $\frac{\partial w}{\partial t}(t,x)$ (also called spatio-temporal white noise, for introduction see [57, Chapter 1]):

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = Lu(t,x) + \dfrac{\partial w}{\partial t}(t,x), & x \in [0,1],\ t \in [0,T], \\ u(0,x) = 0, & x \in [0,1], \\ u(t,0) = u(t,1) = 0, & t \in [0,T], \end{cases} \tag{3.25}$$

where L is a uniformly elliptic operator of the form

$$\begin{aligned} &Lf(x) = a(x)f''(x) + b(x)f'(x) + c(x)f(x), \quad x \in [0,1], \\ &\text{with coefficients } a(x) > 0,\ x \in [0,1],\ a \in C^{\varepsilon}[0,1] \\ &\text{for some } \varepsilon > 0 \text{ and } b, c \in L^{\infty}[0,1]. \end{aligned} \tag{3.26}$$

In what follows we let $H = L^2(0,1)$ and $E = L^p(0,1)$, where the exponent p is to be chosen later on. The realization of L in E will be henceforth denoted by A.

In the following we will examine regularity of the weak solution of (3.25) by exploiting the results of the previous sections. It will turn out that it suffices to exploit properties of the case $A = \Delta$. In the case of Laplacian with Dirichlet boundary conditions we know the ONB of eigenvectors which will allow us to compute a certain γ-norm. For an arbitrary A satisfying (3.26) we have not only $\mathcal{D}(A) = \mathcal{D}(\Delta) = W^{2,p}(0,1) \cap W_0^{1,p}(0,1)$ as shown e.g. in [37, Section 3.1.1] but also the equality of the fractional domain spaces $\mathcal{D}((-A)^{\alpha}) = \mathcal{D}((-\Delta)^{\alpha})$:

In [34, 13.13 Theorem] and in [16] it is shown that under Hölder assumption on the top-order coefficient of A there exists a $\nu \geq 0$ such that $r - A$ admits a bounded H^{∞}-calculus for all $r > \nu$ and hence lies in the class of BIP of operators with bounded imaginary powers, i.e. for fixed $r > \nu$, $(r - A)^{is} \in \mathcal{L}(E)$ for each $s \in \mathbb{R}$ and there is a constant $C > 0$ such that $\|(r - A)^{is}\| \leq C$ for $|s| \leq 1$.

In the case of $-A \in$ BIP we have (see [17, 2.5. Theorem] and the references therein) that $\mathcal{D}((-A)^{\alpha}) = [E, \mathcal{D}(A)]_{\alpha}$, $\alpha \in (0,1)$ where on the right hand side $[E, \mathcal{D}(A)]_{\alpha}$ denotes the complex interpolation space of order α (for a detailed introduction of those spaces see [38, Chapter 2]). Since $\mathcal{D}(A) = \mathcal{D}(\Delta)$ this means immediately $\mathcal{D}((-A)^{\alpha}) = \mathcal{D}((-\Delta)^{\alpha})$, $\alpha \in (0,1)$ which we will need in our considerations.

Back to the example:

In a first step we formulate the problem (3.25) as an abstract stochastic evolution equation in E of the form

$$\begin{cases} dU(t) = AU(t)\,dt + I\,dW_H(t), & t \geq 0, \\ U(0) = 0, \end{cases}$$

where W_H is an H-cylindrical Brownian motion. Here we encounter the problem described in the previous section, namely that the identity operator I is unbounded as an operator from H into E. In order to overcome this problem we shall interpret the problem in a suitable extrapolation space of E.

We fix $\delta > \frac{1}{4}$ and some $r > 0$ sufficient large such that $r - A$ is invertible and lies in BIP. Let $E_{-\delta}$ denote the extrapolation space of order δ associated with A, i.e., $E_{-\delta}$ is the completion of E with respect to the norm $\|x\|_{-\delta} := \|(r-A)^{-\delta}x\|$. Since $r - A$ is invertible, $(r-A)^{\delta}$ acts as an isomorphism from E onto $E_{-\delta}$. We will show next that the identity operator I on H extends to a bounded embedding from H into $E_{-\delta}$ which is γ-radonifying.

Let Δ_H and A_H denote the realizations in H of Δ and A with Dirichlet boundary conditions, respectively. As shown e.g. in [37, Section 3.1.1] we have

$$H_1 := \mathcal{D}(A_H) = H^{2,2} \cap H_0^{1,2} = \mathcal{D}(\Delta_H) =: H_1^{\Delta}$$

with equivalent norms. Similarly,

$$E_1 := \mathcal{D}(A) = H^{2,p} \cap H_0^{1,p} = \mathcal{D}(\Delta) =: E_1^{\Delta}$$

with equivalent norms.

The considerations at the beginning of this section mean

$$E_{1-\delta}^{\Delta} := \mathcal{D}\big((-\Delta)^{1-\delta}\big) = (E, E_1^{\Delta})_{1-\delta} = (E, E_1)_{1-\delta} = \mathcal{D}\big((r-A)^{1-\delta}\big) =: E_{1-\delta}$$

with equivalent norms.

The functions $h_n(x) := \sqrt{2}\sin(n\pi x)$, $n \geq 1$, form an orthonormal basis of eigenfunctions for Δ_H with eigenvalues $-\lambda_n$, where $\lambda_n = (n\pi)^2$. If we endow H_1^{Δ} with the equivalent Hilbert norm $\|f\|_{H_1^{\Delta}} := \|\Delta_H f\|_H$, the functions $\lambda_n^{-1}h_n$ form an orthonormal basis for H_1^{Δ} and we have

$$\begin{aligned} \mathbb{E}\Big\| \sum_{n\geq 1} \gamma_n \lambda_n^{-1} h_n \Big\|^2_{E_{1-\delta}^{\Delta}} &= \mathbb{E}\Big\| \sum_{n\geq 1} \gamma_n \lambda_n^{-1} (-\Delta)^{1-\delta} h_n \Big\|^2_E \\ &= \mathbb{E}\Big\| \sum_{n\geq 1} \gamma_n (n\pi)^{-2\delta} h_n \Big\|^2_E \overset{(*)}{\lesssim} \sum_{n\geq 1} (n\pi)^{-4\delta}, \end{aligned} \tag{3.27}$$

where $(*)$ follows from a standard square function estimate together with the fact that $\|h_n\|_E \leq \sqrt{2}$. The right hand side of (3.27) is finite since we took $\delta > \frac{1}{4}$.

It follows from (3.27) that the identity operator on $\mathcal{D}(\Delta_H)$ extends to a continuous embedding from $\mathcal{D}(\Delta_H)$ into $E^{\Delta}_{1-\delta}$ which is γ-radonifying. Denoting by $E_{-\delta}$ the extrapolation space of order δ of E associated with $A - r$, we obtain a commutative diagram

$$\begin{tikzcd} H \arrow[r, hook] \arrow[d, "(r-A_H)^{-1}"'] & E_{-\delta} \\ H_1 \arrow[d, "\simeq"'] & E_{1-\delta} \arrow[u, "(r-A)"'] \\ H_1^{\Delta} \arrow[r, hook] & E^{\Delta}_{1-\delta} \arrow[u, "\simeq"'] \end{tikzcd}$$

The inclusion $H_1^{\Delta} \hookrightarrow E^{\Delta}_{1-\delta}$ being γ-radonifying, the ideal property of γ-radonifying operators implies that the resulting embedding from H into $E^{\Delta}_{-\delta}$ in the top line of the diagram is γ-radonifying; this operator is an extension of the identity operator on H. We shall denote this embedding by $I_{-\delta}$.

We are now in a position to apply Theorem 3.3.1. Fix arbitrary real numbers α, β, θ satisfying $0 \le 2\alpha + \beta < \frac{1}{2}$, $\frac{1}{4} < \delta < \theta$, $\alpha + \theta < \frac{1}{2}$, and $\beta < 2\theta - 2\delta$. Put $\eta := \theta - \delta$. Since the extrapolated operator $A_{-\delta}$ generates an analytic C_0-semigroup in $E_{-\delta}$ we may apply Theorem 3.3.1 in the space $E_{-\delta}$ to obtain a weak solution U of the problem

$$\begin{cases} dU(t) = A_{-\delta}U(t)\,dt + I_{-\delta}\,dW_H(t)\,, & t \in [0,T], \\ U(0) = 0, \end{cases}$$

with paths in the space $C^{\alpha}\big([0,T];(E_{-\delta})_{\theta}\big) = C^{\alpha}\big([0,T];E_{\eta}\big)$. Noting that $\beta < 2\eta$ we choose p so large that $\beta + \frac{1}{p} < 2\eta$. We have

$$E_{\eta} = E^{\Delta}_{\eta} = H_0^{2\eta,p} = \{f \in H^{2\eta,p} :\ f(0) = f(1) = 0\}$$

with equivalent norms [54, Chapter 4]. By the Sobolev embedding theorem,

$$H^{2\eta,p} \hookrightarrow c^{\beta}[0,1]$$

with continuous inclusion. Putting things together we obtain a continuous inclusion

$$E_{\eta} \hookrightarrow c_0^{\beta}[0,1].$$

In particular it follows that U takes values in E. Almost surely, the trajectories of U belong to $C^{\alpha}([0,T];c_0^{\beta}[0,1])$. In particular, the trajectories of U belong to $L^1(0,T;E)$ almost surely. In view of Proposition 3.2.1 and the discussion following it, we have proved the following theorem.

Theorem 3.4.1 *Let α and β be real numbers satisfying $0 \le 2\alpha + \beta < \frac{1}{2}$. Under the above assumptions on L, the problem (3.25) admits a weak solution in $L^p(0,1)$ for all $1 \le p < \infty$, and this solution has paths in $C^{\alpha}([0,T];c_0^{\beta}[0,1])$.*

This theorem improves the result of [8, Section 6], where for $A = \Delta$ and $0 \le \beta < \frac{1}{2}$ only a solution with paths in $C([0,T]; c_0^\beta[0,1])$ was obtained. Note that the ranges of the admissible Hölder exponents are independent of the operator A.

It follows from the theorem that for all $0 \le \alpha < \frac{1}{4}$ and $0 \le \beta < \frac{1}{2}$ we have a solution in $C^\alpha([0,T]; C[0,1]) \cap C([0,T]; C^\beta[0,1])$. Taking $0 \le \alpha = \beta < \frac{1}{4}$ and recalling that $C^\alpha([0,T]; C[0,1]) \cap C([0,T]; C^\alpha[0,1]) = C^\alpha([0,T] \times [0,1])$, we obtain a solution in $C^\alpha([0,T] \times [0,1])$ for all $0 \le \alpha < \frac{1}{4}$. For $A = \Delta$ the existence of a solution in $C^\alpha([0,T] \times [0,1])$ for $0 \le \alpha < \frac{1}{4}$ was proved by Da Prato and Zabczyk by very different methods, see [14] and [15, Theorem 5.20]. This result was improved by Brzeźniak [5], who obtained Theorem 3.4.1 for $L = \Delta$ and noted without proof the possible extension to a more general class of second order elliptic operators.

The method presented here applies to general uniformly elliptic operators A. In particular it extends beyond the selfadjoint case. Also, it can be extended to operators of order $2m$ on domains in higher dimensions.

Related equations have been studied by many authors and with different methods; see for example [10, 15] and the references given there.

4. Maximal Regularity

4.1. Property (α) and other geometrical assumptions on the Banach space

Let $(r_m)_{m=1}^{\infty}$, $(r'_n)_{n=1}^{\infty}$ and $(r''_{mn})_{m,n=1}^{\infty}$ be mutually independent Rademacher sequences. Then we may assume that these sequences are living in different probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$, $(\Omega', \mathcal{F}', \mathbb{P}')$ and $(\Omega'', \mathcal{F}'', \mathbb{P}'')$ and that their common distribution is just the product probability measure $\mathbb{P} \otimes \mathbb{P}' \otimes \mathbb{P}''$ on $\big(\Omega \times \Omega' \times \Omega'',\ \mathcal{F} \otimes \mathcal{F} \otimes \mathcal{F}\big)$. The expectations relative to $\mathbb{P}$, $\mathbb{P}'$ and $\mathbb{P}''$ respectively will be denoted by $\mathbb{E}$, $\mathbb{E}'$ and $\mathbb{E}''$ respectively. The same assumption we will make, if $(\gamma_m)_{m=1}^{\infty}$, $(\gamma'_n)_{n=1}^{\infty}$ and $(\gamma''_{mn})_{m,n=1}^{\infty}$ are mutually independent Gaussian sequences.

In the following we often want to compare Gaussian and Rademacher sums or even replace one with the other. It is a well-known fact that in Banach spaces one can estimate Rademacher sums against Gaussian sums. Under a certain geometric assumption on the Banach space E one also obtain the converse estimate and thus the equivalence of the sums, see below. The required assumption is called the finite cotype of E.

A Banach space E is said to have *cotype* q, $q \in [2, \infty)$, if there exists a constant $C > 0$ such that for all sequences $(x_n)_{n=1}^{m}$, $m \in \mathbb{N}$, in E, the inequality

$$\left(\sum_{n=1}^{m} \|x_n\|^q \right)^{1/q} \le C \left(\mathbb{E} \left\| \sum_{n=1}^{m} r_n x_n \right\|^2 \right)^{1/2}$$

holds true.

Now we can formulate the following proposition. Its proof can be found amongst others in [22, Proposition 12.11 and Theorem 12.27]

Proposition 4.1.1 *For all finite sequences $(x_n)_{n=1}^{N}$ in E we have*

$$\mathbb{E} \left\| \sum_{n=1}^{N} r_n x_n \right\|^2 \le \frac{1}{2}\pi\, \mathbb{E} \left\| \sum_{n=1}^{N} \gamma_n x_n \right\|^2$$

If E has finite cotype q, there exists a constant C_q such that for all sequences $(x_n)_{n=1}^N$ in E we have

$$\mathbb{E}\left\|\sum_{n=1}^{N}\gamma_n x_n\right\|^2 \leq C_q\,\mathbb{E}\left\|\sum_{n=1}^{N} r_n x_n\right\|^2 \tag{4.1}$$

Now we want to introduce a further geometric assumption on the Banach space E which is central in this work. It will allow us important regularity results of solutions of the stochastic Cauchy problem.

A Banach space E is said to have *property* (α) if there exists a constant C, depending only on E, such that

$$\mathbb{E}\,\mathbb{E}'\left\|\sum_{m=1}^{M}\sum_{n=1}^{N}\varepsilon_{mn} r_m r'_n x_{mn}\right\|^2 \leq C^2\,\mathbb{E}\,\mathbb{E}'\left\|\sum_{m=1}^{M}\sum_{n=1}^{N} r_m r'_n x_{mn}\right\|^2$$

for all choices $\varepsilon_{mn} \in \{-1,1\}$ and $x_{mn} \in E$ ($m = 1,\ldots,M,\ n = 1,\ldots,N$). The least of all such constants is called the property (α) constant of E and will be denoted by C_α.

Pisier first introduced this geometrical assumption on E in [51]. Examples of spaces with property (α) are Hilbert spaces, L_p spaces with $1 < p < \infty$ or the space L^1/H^1.

The following considerations show why this geometrical assumption gains such importance. If we examine the finiteness of the γ-norm $\|\cdot\|_{\gamma(L^2(0,T;H),E)}$ we come across expressions of doubly indexed sums like $\sum_{m,n=1}^N \gamma''_{mn} x_{mn}$, $x_{mn} \in E$, $m,n = 1,\ldots,N$, $N \in \mathbb{N}$. For technical reasons we would like to estimate $\mathbb{E}''\left\|\sum_{m,n=1}^N \gamma''_{mn} x_{mn}\right\|^2 \leq \mathbb{E}\,\mathbb{E}'\left\|\sum_{m,n=1}^N \gamma_m\gamma'_n x_{mn}\right\|^2$. This does not hold in general but it does hold if E has property (α) as shown in the proposition below. In its formulation and in the rest of this work we will use the following notational convention:

Notation 4.1.2 We use the notation $X \sim Y$ to express the fact that there exist constants $0 < c \leq C < \infty$, depending only on the Banach space E, such that $cX \leq Y \leq CX$. Similarly we use the notation $X \lesssim Y$ to express that there exists a constant $0 < C < \infty$, such that $X \leq CY$.

As stated in the next Lemma, finite cotype of a Banach space E is a weaker assumption then property (α).

Proposition 4.1.3 *If E has property (α), then E has finite cotype.*

Proof. Recall that for $1 \leq p \leq \infty$, $n \in \mathbb{N}$, l_p^n denotes $\mathbb{R}^n$ equipped with the norm $\left|\sum_{j=1}^n |a_j|^p\right|^{1/p}$ or $\max_{j\leq n}|a_j|$ if $p = \infty$ for any $a = (a_1,\ldots,a_n) \in \mathbb{R}^n$. For $\varepsilon > 0$ we say a Banach space E *contains a subspace which is* $(1+\varepsilon)$ *isomorphic to* l_p^n if there exists $x_1, \ldots, x_n \in E$ such that for all $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ we have

$$\left(\sum_{j=1}^{n} |a_j|^p \right)^{1/p} \leq \left\| \sum_{j=1}^{n} a_j x_j \right\| \leq (1+\varepsilon) \left(\sum_{j=1}^{n} |a_j|^p \right)^{1/p} \tag{4.2}$$

or (4.2) where $\left(\sum_{j=1}^n |a_j|^p\right)^{1/p}$ is replaced by $\max_{j \leq n} |a_j|$ if $p = \infty$.

E is said to contain l_p^n*'s uniformly* if it contains subspaces $(1+\varepsilon)$-isomorphic to l_p^n for all n and all $\varepsilon > 0$.

Suppose E does not have finite cotype. By the Maurey-Pisier Theorem ([39],[22]) E contains l_n^∞ uniformly. Pisier showed in [51, Remark 2.2] that in the case that E contains l_n^∞ uniformly E fails to have property (α). $\square$

Proposition 4.1.4 *For a Banach space E, the following assertions are equivalent:*

1. *E has property (α);*
2. *For all $N \geq 1$ and all sequences $(x_{mn})_{m,n=1}^N$ in E we have*

$$\mathbb{E}'' \left\| \sum_{m,n=1}^{N} r''_{mn} x_{mn} \right\|^2 \sim \mathbb{E}\,\mathbb{E}' \left\| \sum_{m,n=1}^{N} r_m r'_n x_{mn} \right\|^2 .$$

In this situation for all $N \geq 1$ and all sequences $(x_{mn})_{m,n=1}^N$ in E we have

$$\mathbb{E}'' \left\| \sum_{m,n=1}^{N} \gamma''_{mn} x_{mn} \right\|^2 \sim \mathbb{E}\,\mathbb{E}' \left\| \sum_{m,n=1}^{N} \gamma_m \gamma'_n x_{mn} \right\|^2 .$$

Proof. From the above propositions we know that a Banach space E with property (α) automatically has finite cotype and hence Rademacher and Gaussian sums are equivalent. Thus the last assertion follows easily from the equivalence of (1) and (2). The equivalence of (1) and (2) is shown in [34, 4.11 Lemma]. $\square$

Recall the definitions of Section 1.6.2. We have the following Lemma shown in [30, 6.6. Corollary]

Lemma 4.1.5 *Let $A \in \mathcal{S}_\omega(E)$. If E has Pisier's property (α), then A admits a bounded H^∞-calculus if and only if A admits a γ-bounded H^∞-calculus and one has $\omega_\infty(A) = \omega_\infty^\gamma(A)$.*

Remark 4.1.6 As one will see later in the proofs we will only use the estimate

$$\mathbb{E}'' \left\| \sum_{m,n=1}^{N} \gamma''_{mn} x_{mn} \right\|^2 \lesssim \mathbb{E}\,\mathbb{E}' \left\| \sum_{m,n=1}^{N} \gamma_m \gamma'_n x_{mn} \right\|^2 \tag{4.3}$$

even though property (α) provides the equivalence of both sides. This weaker geometrical assumption on the Banach space is also denoted with *property* (α^+) (see [44, 21]).

Examples of Banach spaces with property (α^+) are all Hilbert spaces, all Banach lattices with finite cotype (in particular the L^p-spaces for $p \in [1, \infty)$) and the Schatten classes S_p for $p \in [1, 2]$. Banach spaces with property (α^+) have finite cotype; in particular such spaces cannot contain an isomorphic copy of c_0. We refer to [45] and there references cited there for more information.

4.2. Maximal regularity in Banach spaces with finite cotype

In this section we will sharpen Theorem 3.3.1 in the case where $-A$ admits a γ-bounded H^∞-calculus. Under this assumption we will prove maximal regularity in the sense of Theorem 4.2.1 of the weak solution. Our approach requires finite cotype of the underlying Banach space.

The main result of this section, which generalizes e.g. [15, Proposition A.19], reads as follows.

Theorem 4.2.1 *Let E have finite cotype and assume that $-A$ admits γ-bounded H^∞-calculus of angle $0 < \omega_\infty^\gamma(-A) < \frac{\pi}{2}$. Then the solution U of problem* (SCP) *has maximal regularity in the sense that for all $t \in [0, T]$ we have $U(t) \in \mathcal{D}((-A)^{\frac{1}{2}})$ almost surely and*

$$\mathbb{E}\|(-A)^{\frac{1}{2}} U(t)\|^2 \leq C \|B\|^2_{\gamma(H,E)} \tag{4.4}$$

for a suitable constant C independent of $T > 0$, $t \in [0, T]$, and $B \in \gamma(H, E)$. Moreover, $(-A)^{\frac{1}{2}} U$ is continuous in all moments, i.e., for all $1 \leq p < \infty$ we have

$$\lim_{s \to t} \mathbb{E}\|(-A)^{\frac{1}{2}}(U(t) - U(s))\|^p = 0,$$

Finally, the paths of $(-A)^{\frac{1}{2}} U$ belong to $L^2(0, T; E)$ almost surely.

Proof. Following [30] we consider the function $\psi(\lambda) := \lambda^{\frac{1}{2}} e^{-\lambda}$. We shall prove the theorem for a fixed time interval $[0, T]$ with a constant C independent of T. Fix an arbitrary

$0 < t \le T$. Our starting point is the following identity, valid for $t \in [2^{-k}T, 2^{-k+1}T)$:

$$\begin{aligned}\psi(t\lambda) &= \psi(2^{-k}T\lambda) + \int_{2^{-k}T\lambda}^{t\lambda} \psi'(s)\,ds \\ &= \psi(2^{-k}T\lambda) + \int_1^2 \mathbf{1}_{[2^{-k}sT, 2^{-k+1}T)}(t)\, 2^{-k}sT\lambda\, \psi'(2^{-k}sT\lambda)\,\frac{ds}{s}.\end{aligned}$$

In order to simplify notations a little bit, throughout the rest of the proof we take $T = 1$. It is easy to check that the constant C in (4.4) can be chosen independently of T.

By the H^∞-calculus we have $\psi(-tA) = (-tA)^{\frac{1}{2}}S(t)$. Substituting this in the above identity over k, summing over $k = 1, \dots, N$, and writing $\varphi(\lambda) := \lambda\psi'(\lambda)$, for $t \in [2^{-N}, 1)$ this gives

$$\begin{aligned}\psi(-tA) &= \sum_{k=1}^N \mathbf{1}_{[2^{-k}, 2^{-k+1})}(t)\psi(-2^{-k}A) \\ &\quad + \int_1^2 \sum_{k=1}^N \mathbf{1}_{[2^{-k}s, 2^{-k+1})}(t)\varphi(-2^{-k}sA)\frac{ds}{s}.\end{aligned}$$

Hence,

$$\begin{aligned}\Big\|\mathbf{1}_{[2^{-N},1)}(-A)^{\frac{1}{2}}S(\cdot)B\Big\|_{\gamma(0,1;H,E)} &= \big\|\mathbf{1}_{[2^{-N},1)}[\psi(-(\cdot)A)]B\big\|_{\gamma(0,1;\frac{dt}{t};H,E)} \\ &\le \Big\|\sum_{k=1}^N \mathbf{1}_{[2^{-k},2^{-k+1})}\psi(-2^{-k}A)B\Big\|_{\gamma(0,1;\frac{dt}{t};H,E)} \\ &\quad + \int_1^2 \Big\|\sum_{k=1}^N \mathbf{1}_{[2^{-k}s,2^{-k+1})}\varphi(-2^{-k}sA)B\Big\|_{\gamma(0,1;\frac{dt}{t};H,E)}\frac{ds}{s}.\end{aligned}$$

Note that the sequence $(\mathbf{1}_{[2^{-k},2^{-k+1})})_{k=1}^N$ is an orthogonal system in $L^2(0,1;\frac{dt}{t})$ with $\|\mathbf{1}_{[2^{-k},2^{-k+1})}\|_2^2 = \ln 2$. If g_n is an orthonormal basis $L^2(0,1;\frac{dt}{t})$ containing $\frac{1}{\ln 2}(\mathbf{1}_{[2^{-n},2^{-n+1})})_{n=1}^N$, $(h_j)_{j\ge1}$ is an orthonormal basis of H and $(r_{jk})_{j,k\ge1}$ is a doubly indexed Rademacher sequence on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, using (4.1) we can estimate

$$\begin{aligned}&\Big\|\sum_{k=1}^N \mathbf{1}_{[2^{-k},2^{-k+1})} \otimes \psi(-2^{-k}A)B\Big\|^2_{\gamma(0,1;\frac{dt}{t};H,E)} \\ &= \mathbb{E}\Big\|\sum_{j\ge1}\sum_{n=1}^N\sum_{k=1}^N \gamma_{jn}\int_0^T \sqrt{\ln 2}\ \frac{\mathbf{1}_{[2^{-k},2^{-k+1})}(t)}{\sqrt{\ln 2}}\ g_n\ \psi(-2^{-k}A)Bh_j\,\frac{dt}{t}\Big\|^2 \\ &= \ln 2 \cdot \mathbb{E}\Big\|\sum_{j\ge1}\sum_{k=1}^N \gamma_{jk}\psi(-2^{-k}A)Bh_j\Big\|^2 \\ &\lesssim \mathbb{E}\Big\|\sum_{j\ge1}\sum_{k=1}^N r_{jk}\psi(-2^{-k}A)Bh_j\Big\|^2.\end{aligned}$$

Let $(r'_j)_{j\geq 1}$ be a Rademacher sequence independent of $(r_{jk})_{j,k\geq 1}$. Using the same randomization argument as in the proof of Proposition 4.1.4, we estimate

$$\begin{aligned}
&\mathbb{E}\Big\| \sum_{j\geq 1}\sum_{k=1}^{N} r_{jk}\psi(-2^{-k}A)Bh_j\Big\|^2 \\
&\quad = \mathbb{E}'\,\mathbb{E}\Big\| \sum_{j\geq 1} r'_j\Big(\sum_{k=1}^{N} r_{jk}\psi(-2^{-k}A)\Big)Bh_j\Big\|^2 \\
&\quad \leq \gamma(\Psi)^2\,\mathbb{E}'\Big\| \sum_{j\geq 1} r'_j Bh_j\Big\|^2 \lesssim \gamma(\Psi)^2\|B\|^2_{\gamma(H,E)}.
\end{aligned}$$

Here $\gamma(\Psi)$ is the γ-bound of the family

$$\Psi = \Big\{ \sum_{k=1}^{N} r_{jk}(\omega)\psi(-2^{-k}A) :\ N\geq 1,\ j\geq 1,\ \omega\in\Omega\Big\},$$

which is finite by Lemma 1.6.9 since we have $\psi\in H^\infty(\Sigma_\sigma)$ for all $\sigma\in(\omega_\infty^\gamma(-A),\frac{\pi}{2})$.

It follows that the function $\sum_{k\geq 1}\mathbf{1}_{[2^{-k},2^{-k+1})}\psi(-2^{-k}A)B$ defines an almost summing operator from $L^2(0,1;\frac{dt}{t},H)$ to E (compare Section 1.4). Since E has finite cotype and therefore does not contain a copy of c_0, this operator is γ-radonifying and by (1.1) we have

$$\Big\| \sum_{k\geq 1}\mathbf{1}_{[2^{-k},2^{-k+1})}\psi(-2^{-k}A)B\Big\|_{\gamma(0,1;\frac{dt}{t};H,E)} \lesssim \gamma(\Psi)\|B\|_{\gamma(H,E)}.$$

Likewise, using that for $s\in[1,2)$ the sequence $(\mathbf{1}_{[2^{-k}s,2^{-k+1})})_{k=1}^N$ is an orthogonal system in $L^2(0,1;\frac{dt}{t})$ with $\|\mathbf{1}_{[2^{-k}s,2^{-k+1})}\|_2^2=\ln(2/s)$,

$$\begin{aligned}
&\int_1^2 \Big\| \sum_{k=1}^{N}\mathbf{1}_{[2^{-k}s,2^{-k+1})}\varphi(-2^{-k}sA)B\Big\|_{\gamma(0,1;\frac{dt}{t};H,E)}\frac{ds}{s} \\
&\quad\lesssim \int_1^2 \Big(\ln(2/s)\cdot\mathbb{E}\Big\|\sum_{k=1}^{N}\sum_{j\geq 1} r_{jk}\varphi(-2^{-k}sA)Bh_j\Big\|^2\Big)^{\frac12}\frac{ds}{s} \\
&\quad\leq \gamma(\Phi)\int_1^2 \ln(2/s)\cdot\Big(\mathbb{E}'\Big\|\sum_{j\geq 1} r'_j Bh_j\Big\|^2\Big)^{\frac12}\frac{ds}{s} \lesssim \gamma(\Phi)\|B\|_{\gamma(H,E)}.
\end{aligned}$$

Here $\gamma(\Phi)$ is the γ-bound of the family

$$\Phi = \Big\{ \sum_{k=1}^{N} r_{jk}(\omega)\varphi(-2^{-k}sA) :\ N\geq 1,\ j\geq 1,\ s\in[1,2],\ \omega\in\Omega\Big\},$$

which is finite since $\varphi \in H^\infty_{\frac{1}{2}}(\Sigma_\sigma)$. Letting $N \to \infty$ as before, with monotone convergence it follows that

$$\int_1^2 \Big\| \sum_{k\geq 1} \varphi(-2^{-k}sA)B\mathbf{1}_{[2^{-k}s,2^{-k+1})}(\cdot)\Big\|_{\gamma(0,1;\frac{dt}{t};H)E)} \frac{ds}{s} \lesssim \gamma(\Phi)^2\|B\|^2_{\gamma(H,E)}.$$

As $N \to \infty$ we also obtain that $(-A)^{\frac{1}{2}}S(\cdot)B \in \gamma(0,1;H,E)$ and

$$\|(-A)^{\frac{1}{2}}S(\cdot)B\|_{\gamma(0,1;H,E)} = \lim_{N\to\infty} \|(-A)^{\frac{1}{2}}S(\cdot)B\mathbf{1}_{[2^{-N},1)}\|_{\gamma(0,1;H,E)}.$$

Putting things together we obtain that

$$\|(-A)^{\frac{1}{2}}S(\cdot)B\|_{\gamma(0,1;H,E)} \leq C\|B\|_{\gamma(H,E)}$$

with a constant C independent of B. Therefore, for all $t \in [0,1]$ the function $(-A)^{\frac{1}{2}}S(t-\cdot)B$ is H-stochastically integrable, and (4.4) follows from

$$\begin{aligned}\mathbb{E}\|(-A)^{\frac{1}{2}}U(t)\|^2 &= \|(-A)^{\frac{1}{2}}S(t-\cdot)B\|^2_{\gamma(0,t;H,E)}\\ &\leq \|(-A)^{\frac{1}{2}}S(\cdot)B\|^2_{\gamma(0,1;H,E)} \leq C\|B\|_{\gamma(H,E)}.\end{aligned}$$

This proves (4.4). By Fubini's theorem, (4.4) implies that

$$\mathbb{E}\int_0^T \|(-A)^{\frac{1}{2}}U(t)\|^2\,dt \leq TC\|B\|^2_{\gamma(H,E)}.$$

Hence the paths of $(-A)^{\frac{1}{2}}U$ belong to $L^2(0,T;E)$ almost surely. Finally the continuity in all moments follows from [45, Theorem 6.5]. ☐

Remark 4.2.2 The theorem remains true if only $\nu - A$ admits a γ-bounded H^∞-calculus for some $\nu > 0$. To see this we apply the theorem with $-A$ replaced by $\nu - A$ to obtain maximal regularity of the solution of the problem

$$\begin{cases} dU(t) = (A-\nu)U(t)\,dt + B\,dW_H(t)\,, \quad t \in [0,T],\\ U(0) = 0.\end{cases}$$

We obtain that $(\nu - A)^{\frac{1}{2}}S_\nu(\cdot)B \in \gamma(0,T;H,E)$, where $S_\nu(t) = e^{-\nu t}S(t)$. By a standard comparison argument this implies $(\nu - A)^{\frac{1}{2}}S(\cdot)B \in \gamma(0,T;H,E)$ as well, with similar estimates.

As is well known, the deterministic Cauchy problem $y' = Ay + f$, with $-A$ sectorial of angle $0 < \omega(-A) < \frac{\pi}{2}$, has maximal L^p-regularity if and only if the set $\{tR(it,A) : t \in \mathbb{R}\backslash\{0\}\}$ is R-bounded (see [58]). The following result shows that in the stochastic setting, the strictly stronger assumption that $-A$ admits a bounded H^∞-calculus is necessary for maximal regularity and actually characterizes it in the case $H = \mathbb{R}$ (which corresponds to rank one Brownian motions). In particular this shows that in L^p-spaces there are examples of analytic generators which have maximal regularity for the deterministic Cauchy problem but not always for the stochastic one.

Remark 4.2.3 In the special case where $H = \mathbb{R}$, Lemma 1.6.9 is not needed and Theorem 4.2.1 remains valid under the weaker assumption that $-A$ admits a bounded H^∞-calculus.

We use the notation $E^\odot$ for the closed subspace of all $x^* \in E^*$ such that $\lim_{t\downarrow 0} \|S^*(t)x^* - x^*\| = 0$. As is well known (see [25, 2.6 Definition]) we have $E^\odot = \overline{\mathcal{D}(A^*)}$. The part of A^* in $E^\odot$ is denoted by $A^\odot$; it is the generator of the restriction of S^* to $E^\odot$.

Theorem 4.2.4 *Let both E and E^* have finite cotype, and let $-A$ be a sectorial operator in E of angle $0 < \omega(-A) < \frac{\pi}{2}$. Then $-A$ admits a bounded H^∞-calculus if and only if*

$$\begin{aligned} dU(t) &= AU(t)\,dt + x\,dW_H(t), \qquad t \ge 0,\\ U(0) &= 0, \end{aligned}$$

and

$$\begin{aligned} d\widetilde{U}(t) &= A^\odot \widetilde{U}(t)\,dt + x^\odot\,dW_H(t), \qquad t \ge 0,\\ \tilde{U}(0) &= 0, \end{aligned}$$

have maximal regularity in the sense of Theorem 4.2.1 for all $x \in E$ and $x^\odot \in E^\odot$, respectively.

Proof. The 'only if' part is contained in the previous theorem and the remark 4.2.3, since $-A$ admits a bounded H^∞-calculus if and only if $-A^\odot$ admits a bounded H^∞-calculus [17, 34].

For the 'if' part, for all $t \in [0, T]$ we have

$$\|(-A)^{\frac{1}{2}} S(\cdot)x\|^2_{\gamma(0,t;E)} = \|(-A)^{\frac{1}{2}} S(t-\cdot)x\|^2_{\gamma(0,t;E)} = \mathbb{E}\|(-A)^{\frac{1}{2}} U(t)\|^2 \le C\|x\|^2$$

with a constant C independent of t, T, and x. Likewise,

$$\|(-A^\odot)^{\frac{1}{2}} S^\odot(\cdot)x^\odot\|^2_{\gamma(0,t;E^\odot)} \le C\|x^\odot\|^2_{\gamma(H,E^\odot)}.$$

By Proposition A.0.5, these two estimates imply that $-A$ admits a bounded H^∞-calculus. $\square$

4.3. Maximal Regularity in Banach spaces with property(α)

We have in seen that Proposition 4.1.3 that property (α) is a stronger assumption on the Banach space E than the finite cotype. In this Section we will show that similar maximal regularity as in the last section can be obtained under weaker assumptions on the generator A. The theorems in this settings can be proven with different methods which seem to be less technical as e.g. the proof of Theorem 4.2.1. However, deep results of [30] are used again.

The following proposition is a generalization of a special case [44, Theorem 6.2].

Proposition 4.3.1 *Let F be a real Banach space with property (α). Let $B \in \gamma(H,E)$ and let Ψ be a mapping from $(0,T)$ into $\mathscr{L}(E,F)$ such that for all $x \in E$ the F-valued function $t \mapsto \Psi(t)x$ belongs to $\gamma(0,T;F)$. Then $t \mapsto \Psi(t)B$ belongs to $\gamma(0,T;H,F)$ and there is a constant c, only depending on Ψ and F, such that*

$$\|\Psi(\cdot)B\|_{\gamma(0,T;H,F)} \le c\|B\|_{\gamma(H,E)}.$$

Proof. As an easy consequence of the closed graph theorem there exists a constant C, only depending on Ψ, such that

$$\|\Psi(\cdot)x\|_{\gamma(0,T;F)} \le C\|x\|, \quad \forall x \in E. \tag{4.5}$$

Note that since F does not contain a copy of c_0, the scalarly integrable function $t \mapsto f(t)\Psi(t)Bh$ is Pettis integrable for all $f \in L^2(0,T)$ and $h \in H$ [23, Theorem II.3.7].

Let $(f_m)_{m\ge1}$ be an orthonormal basis for $L^2(0,T)$ and let $(h_n)_{n\ge1}$ be an orthonormal basis for H. Then $(f_m \otimes h_n)_{m,n\ge1}$ is a doubly indexed orthonormal basis for $L^2(0,T;H)$. Let $(\gamma'_m)_{m\ge1}$ and $(\gamma''_n)_{n\ge1}$ be mutually independent Gaussian sequences, and let $(\gamma_{mn})_{m,n\ge1}$ be a doubly indexed Gaussian sequence. Then by (4.3) and (4.5),

$$\begin{aligned}
&\|I_{\Psi(\cdot)B}\|^2_{\gamma(L_2(0,T;H),F)}\\
&\quad = \mathbb{E}\Big\|\sum_{m,n\ge1}\gamma_{mn}\int_0^T f_m(s)\Psi(s)Bh_n\,ds\Big\|^2\\
&\quad \le c_+^2\,\mathbb{E}'\,\mathbb{E}''\Big\|\sum_{m\ge1}\sum_{n\ge1}\gamma'_m\gamma''_n\int_0^T f_m(s)\Psi(s)Bh_n\,ds\Big\|^2\\
&\quad = c_+^2\,\mathbb{E}''\,\mathbb{E}'\Big\|\sum_{m\ge1}\gamma'_m\int_0^T f_m(s)\Psi(s)\Big[\sum_{n\ge1}\gamma''_nBh_n\Big]\,ds\Big\|^2\\
&\quad \le c_+^2C^2\,\mathbb{E}''\Big\|\sum_{n\ge1}\gamma''_nBh_n\Big\|^2 = c_+^2C^2\|B\|^2_{\gamma(0,T;E)}
\end{aligned}$$

where c_+ is the constant in (4.3). □

The following result leads to an analogon to Theorem 4.2.1.

Theorem 4.3.2 *Assume that B is γ-radonifying and let E have property (α). If $-A$ is a sectorial operator of angle $0 < \omega < \frac{\pi}{2}$ such that for all $x \in E$ we have*

$$(-A)^{\frac{1}{2}} S(\cdot)x \in \gamma(0, T; E), \tag{4.6}$$

then the solution U of problem (2.5) has maximal regularity on the interval $[0, T]$ in the sense that for all $t \in [0, T]$, $U(t) \in \mathcal{D}((-A)^{\frac{1}{2}})$ almost surely and

$$\mathbb{E}\|U(t)\|^2_{\mathcal{D}((-A)^{\frac{1}{2}})} \leq C\|B\|^2_{\gamma(H,E)} \tag{4.7}$$

for a suitable constant C, independent of B and T. Moreover,

$$\mathbb{E}\int_0^T \left\|(-A)^{\frac{1}{2}} U(t)\right\|^2 dt \leq TC\|B\|^2_{\gamma(H,E)}. \tag{4.8}$$

In particular, the paths of U belong to $L^2(0, T; \mathcal{D}((-A)^{\frac{1}{2}}))$, almost surely. Finally, U is continuous in all moments in $\mathcal{D}((-A)^{\frac{1}{2}})$, i.e., for all $1 \leq p < \infty$ we have

$$\lim_{s \to t} \mathbb{E}\|U(t) - U(s)\|^p_{\mathcal{D}((-A)^{\frac{1}{2}})} = 0.$$

Proof. Since $(-A)^{\frac{1}{2}}$ acts as an isomorphism from $\mathcal{D}((-A)^{\frac{1}{2}})$ onto E, condition (4.6) implies that for all for all $x \in E$ we have $S(\cdot)x \in \gamma(0, T; \mathcal{D}((-A)^{\frac{1}{2}}))$. Also, since E has property (α), $\mathcal{D}((-A)^{\frac{1}{2}})$ has property (α) as well and Proposition 4.3.1 shows that $S(\cdot)B$ belongs to $\gamma(0, T; H, \mathcal{D}((-A)^{\frac{1}{2}}))$ and

$$\|S(\cdot)B\|^2_{\gamma(0,t;H,\mathcal{D}((-A)^{\frac{1}{2}}))} \leq C\|B\|_{\gamma(H,E)}$$

with a constant C which is independent of B and T. It follows that $S(t - \cdot)B$ is H-stochastically integrable in $\mathcal{D}((-A)^{\frac{1}{2}})$, and (4.7) follows from

$$\begin{aligned}\mathbb{E}\|U(t)\|^2_{\mathcal{D}((-A)^{\frac{1}{2}})} &= \|S(t - \cdot)B\|^2_{\gamma(0,t;H,\mathcal{D}((-A)^{\frac{1}{2}}))} \\ &\leq \|S(\cdot)B\|^2_{\gamma(0,T;H,\mathcal{D}((-A)^{\frac{1}{2}}))} \leq C\|B\|_{\gamma(H,E)}.\end{aligned}$$

The estimate (4.8) follows from this by Fubini's theorem. Finally the continuity in all moments follows from [45, Theorem 6.5]. □

Under the assumption of property (α) a bounded H^∞-calculus of the operator $-A$ implies a γ-bounded H^∞-calculus (compare Lemma 4.1.5). Since (4.6) follows if $-A$ admits a bounded H^∞-calculus (see [30]) we obtain the following Theorem as a corollary to the previous result:

Theorem 4.3.3 *Let B be γ-radonifying and E have property (α). If A is the generator of an analytic semigroup with the property that $-A$ has a bounded H^∞-calculus, then the solution of the problem (2.5) has maximal regularity in the sense of Theorem 4.3.2.*

4.4. An example

Let $\mathcal{O}$ be a bounded open domain in $\mathbb{R}^d$ with C^2 boundary. Consider the problem

$$\begin{cases} du(t,x) = Lu(t,x)\,dt + \sum_{k=1}^{\infty} g_k(x)\,dw_k(t), & x \in \mathcal{O},\ t \in [0,T], \\ u(0,x) = 0, & x \in \mathcal{O}, \\ u(t,x) = 0, & x \in \partial\mathcal{O},\ t \in [0,T], \end{cases} \tag{4.9}$$

where L is a second order uniformly elliptic operator of the form

$$Lf(x) = \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} b_i(x) \frac{\partial f}{\partial x_i}(x) + c(x) f(x), \quad x \in \mathcal{O},$$

with coefficients $a_{ij} = a_{ji} \in C^{\varepsilon}(\overline{\mathcal{O}})$ for some $\varepsilon > 0$ and $b_i, c \in L^{\infty}(\mathcal{O})$ with $c \le 0$. We assume that the sequence $g = (g_k)_{k\ge1}$ belongs to $L^p(\mathcal{O}; l^2)$ for some fixed $1 < p < \infty$, and that $w = (w_k)_{k\ge1}$ is a sequence of independent standard Brownian motions. A related, time-dependent version of this equation on the full space $\mathbb{R}^d$ has been considered by Krylov [33, Chapter 5.4].

Here we will show that (4.9) has a unique solution in $L^p(\mathcal{O})$, with paths belonging to $C^{\beta}([0,T]; L^p(\mathcal{O})) \cap L^2(0,T; H_0^{1,p}(\mathcal{O}))$ for $0 \le \beta < \frac{1}{2}$.

Let $1 < p < \infty$ and take $E = L^p(\mathcal{O})$. In E we consider the realization A of L with Dirichlet boundary conditions, i.e., $\mathcal{D}(A) = H^{2,p}(\mathcal{O}) \cap H_0^{1,p}(\mathcal{O})$. Let $(e_k)_{k\ge1}$ denote the standard unit basis of l^2, and define $B \in \mathcal{L}(l^2, L^p(\mathcal{O}))$ by $Bh := \sum_{k\ge1} [h, e_k]_{l^2} g_k$ for $h \in l^2$. We can rewrite (4.9) as a linear stochastic Cauchy problem of the form

$$\begin{cases} dU(t) = AU(t)dt + B\,dW_{l^2}(t), & t \in [0,T], \\ U(0) = 0, \end{cases} \tag{4.10}$$

with H_{l^2} an l^2-cylindrical Brownian motion. The operator B is γ-radonifying since by the Fubini theorem and the Kahane-Khintchine inequalities,

$$\begin{aligned} \mathbb{E}\Big\| \sum_{k\ge1} \gamma_k\, Be_k \Big\|_{L^p}^2 \eqsim_p \mathbb{E}\Big\| \sum_{k\ge1} \gamma_k\, Be_k \Big\|_{L^p}^p &= \mathbb{E}\Big\| \sum_{k\ge1} \gamma_k\, g_k \Big\|_{L^p}^p \\ &= \int_{\mathcal{O}} \mathbb{E}\Big| \sum_{k\ge1} \gamma_k\, g_k(x) \Big|^p dx \lesssim \int_{\mathcal{O}} \Big(\sum_{k\ge1} |g_k(x)|^2 \Big)^{\frac{p}{2}} dx, \end{aligned}$$

which is finite by the assumption on g. It was shown in [16] that $\nu - A$ admits a bounded H^{∞}-calculus for $\nu > 0$ sufficiently large. This calculus is γ-bounded since $L^p(\mathcal{O})$ has property (α). It follows that the assumptions of Theorems 3.3.1 and 4.2.1

(with A replaced by $A - \nu$) are satisfied. Since A is invertible we have $\mathcal{D}((-A)^{\frac{1}{2}}) = \mathcal{D}((\nu - A)^{\frac{1}{2}}) = H_0^{1,p}(\mathcal{O})$ with equivalent norms (see [16, 52, 54]). By Theorem 4.2.1 and the remark following it we obtain a unique solution U of (4.10) with paths belonging to $C^{\beta}([0,T]; L^p(\mathcal{O})) \cap L^2(0,T; H_0^{1,p}(\mathcal{O}))$ for $0 \leq \beta < \frac{1}{2}$.

5. Regularity in case that A generates a C_0-group

5.1. Path regularity of the solution

A family of bounded linear operators $\mathbf{S} = (S(t))_{t\in\mathbb{R}}$ on a Banach space E is called *a C_0-group of bounded operators* or briefly *a C_0-group* if

1. $S(0) = Id_E$,
2. $S(t+s) = S(t)S(s)$ for all $s, t \in \mathbb{R}$,
3. $\lim_{t\to 0} S(t)x = x$ for $x \in E$.

The linear operator A defined by

$$\mathcal{D}(A) = \left\{ x \in E : \lim_{t\to 0} \frac{S(t)x - x}{t} \text{ exists} \right\}$$

and

$$Ax = \lim_{t\to} \frac{S(t)x - x}{t}$$

is *the infinitesimal generator* (or briefly *the generator*) of the group $\mathbf{S}$. $\mathcal{D}(A)$ is *the domain of A*.

Since for every $t \in \mathbb{R}$ we have that $S(t)$ is invertible with $S(t)^{-1} = S(-t)$, we will be able to show in Theorem 5.1.2 that solutions of SCP are continuous. We will make use of the following Lemma.

Lemma 5.1.1 *Let H be separable and let W_H be the cylindrical Wiener process with respect to a filtration $\{\mathcal{F}(t)\}_{t\in[0,T]}$ fulfilling (PF) of page 15. Let further $\Phi : (0,T) \to \mathcal{L}(H,E)$ be stochastically integrable with respect W_H. Then the E-valued process $Y = Y_t$ given by*

$$Y_t := \int_0^t \Phi(s)\, dW_H(s), \qquad t \in [0,T],$$

is a martingale with respect to $\{\mathcal{F}(t)\}_{t\in[0,T]}$ which has a modification with continuous trajectories.

Proof. The martingale property is evident.

In the following we use an argument from [45]. To prove the existence of a continuous modification we fix an orthonormal basis $(f_m)_{m=1}^{\infty}$ in $L^2(0,T)$ and an orthonormal basis $(h_n)_{n=1}^{\infty}$ in H.

By expanding $[h_n, \Phi^*(\cdot)x^*]_H$ with respect to the basis $(f_m)_{m=1}^{\infty}$ writing $\beta_n(s) := W_H(s)h_n$ and using the coordinate expansion (2.2), for all $x^* \in E^*$ and $t \in [0,T]$ we have

$$
\begin{aligned}
\langle Y_t, x^*\rangle &= \sum_{n=1}^{\infty} \int_0^t \langle \Phi(s)h_n, x^*\rangle \, d\beta_n(s) \\
&= \sum_{n=1}^{\infty} \int_0^t [h_n, \Phi^*(s)x^*]_H \, d\beta_n(s) \\
&= \sum_{n=1}^{\infty} \int_0^t \sum_{m=1}^{\infty} f_m(s) \int_0^T f_m(u)[h_n, \Phi^*(u)x^*]_H du \, d\beta_n(s) \\
&= \sum_{n,m=1}^{\infty} \int_0^T f_m(u)[h_n, \Phi^*(u)x^*]_H \, du \int_0^t f_m(s) \, d\beta_n(s) \\
&= \sum_{n,m=1}^{\infty} \int_0^T f_m(u)\langle \Phi(u)h_n, x^*\rangle \, du \int_0^t f_m(s) \, d\beta_n(s)
\end{aligned}
$$

with convergence in $L^2(\Omega)$; this convergence is unconditional since $(h_{\pi(n)})_{n\geq 1}$ is an orthonormal basis for every permutation π of the positive integers. The Itô-Nisio theorem [35, Theorem 2.1.1 (i)$\Leftrightarrow$(v) and Theorem 2.2.1] now implies that

$$
Y_t = \sum_{n,m=1}^{\infty} \int_0^T f_m(s)\Phi(s)h_n \, ds \int_0^t f_m(s) \, dW_H(s)h_n
$$

unconditionally in $L^2(\Omega; E)$.

For $N \geq 1$ we put

$$
Y_T^{(N)} := \sum_{m,n=1}^{N} \int_0^T f_m(s)\Phi(s)h_n \, ds \int_0^T f_m(s) \, dW_H(s)h_n,
$$

For all $t \in [0,T]$,

$$
Y_t^{(N)} := \mathbb{E}(Y^{(N)}|\mathcal{F}_t) = \sum_{m,n=1}^{N} \int_0^T f_m(s)\Phi(s)h_n \, ds \int_0^t f_m(s) \, dW_H(s)h_n.
$$

In particular, for each $N \geq 1$ the process $Y^{(N)} : t \mapsto Y_t^{(N)}$ has a version with continuous trajectories.

We claim that for each $t \in [0,T]$ we have $\lim_{N\to\infty} Y_t^{(N)} = Y_t$ in $L^2(\Omega; E)$.

Let $M = (M_t)_{t\in[0,T]}$ be an E-valued square integrable continuous martingale. From Doob's inequality (see [32, 3.8 Theorem], [15, Theorem 3.8]) we obtain

$$\Big(\mathbb{E} \sup_{0\le t\le T} \|M_t\|_E^2\Big) \le 4 \sup_{0\le t\le T}\Big(\mathbb{E}\|M_t\|_E^2\Big) = 4\Big(\mathbb{E}\|M_T\|_E^2\Big) \tag{5.1}$$

and [15, Proposition 3.9] shows that the space $\mathcal{M}_T^2(E)$ of all E-valued square integrable continuous martingales $M = (M_t)_{t\in[0,T]}$ endowed with the norm

$$\|M\|_{\mathcal{M}_T^2(E)} := \Big(\mathbb{E} \sup_{0\le t\le T} \|M_t\|_E^2\Big)^{\frac{1}{2}}$$

is a Banach space. An application of the Lemma of Borel-Cantelli as in the proof of [15, Proposition 3.9] shows that the sequence $(Y^{(N)})_{N=1}^{\infty}$ is Cauchy in $\mathcal{M}_T^2(E)$. Let $Y^{(\infty)} \in \mathcal{M}_T^2(E)$ denote its limit. Then for all $t \in [0,T]$ we have

$$Y_t^{\infty} = \lim_{N\to\infty} Y_t^{(N)} = \int_0^t \Phi(s)\, dW_H(s) = Y_t$$

in $L^2(\Omega, E)$, and therefore $Y_t^{(\infty)} = Y_t$ almost surely. Thus, $Y^{(\infty)}$ is a continuous modification of Y. □

The next theorem states that the solution of the SCP is necessarily continuous if A is a generator of a C_0-group. The poof of it follows easily by the trivial observation that the group property implies that for all $0 \le t \le T$ we have

$$\int_0^t S(t-s)B dW_H(s) = S(t-T)\int_0^t S(T-s)B dW_H(s) \tag{5.2}$$

which means that the integrand does not depend on t. By Lemma 5.1.1, the right hand side has a continuous modification on $[0,T]$.

Theorem 5.1.2 *Let A be the generator of a C_0-group $\{S(t)\}_{t\ge0}$ on a real Banach space E. Furthermore let $\{W_H(t)\}_{t\ge0}$ be a cylindrical H-Wiener process, where H is a separable real Hilbert space, and let $B : H \to E$ be a bounded operator. If $\{U(t)\}_{t\ge0}$ is a weak solution of the stochastic Cauchy problem SCP*

$$\begin{cases} dU(t) = AU(t)\,dt + B\,dW_H(t), & t \ge 0, \\ U(0) = 0, \end{cases} \tag{5.3}$$

then $\{U(t)\}_{t\ge0}$ has a modification with continuous trajectories.

Proof. Fix $T \geq 0$. Following Lemma 2.3.3 it is sufficient to show that the process $\{U(t)\}_{t\in[0,T]}$ has a continuous modification.

We know from [8, 45] that if a weak solution $\{U(t)\}_{t\geq 0}$ exists then it is unique and the $\mathcal{L}(H,E)$-valued function $s \mapsto S(t-s)B$ is stochastically integrable on $(0,t)$ for every $t \geq 0$. $\{U(t)\}_{t\geq 0}$ is given by

$$\begin{aligned} U(t) &= \int_0^t S(t-s)B\, dW_H(s) \\ &= S(t-T)\int_0^t S(T-s)B\, dW_H(s), \qquad t \in [0,T]. \end{aligned}$$

By Lemma 5.1.1, the right hand side has a continuous modification on $[0,T]$. $\square$

In the next section we combine this result with a result of Brzeźniak, Peszat and Zabczyk ([7]) on nonexistence of solutions for a certain class of stochastic differential equations.

5.2. Nonexistence of solutions - an example

The results of this section emanate from a joint work with Jan van Neerven (see [20]).

So far we met various conditions on A or E under which we obtained continuity of the solutions of the SCP. The following section grew out of an attempt to examine the situation for certain special cases, where the semigroup generated by A possesses minimal smoothing properties. To explain the main idea, let C_P denote the Banach space of periodic continuous functions $f:\mathbb{R}\to\mathbb{R}$ with period 1.

In a recent paper [7], Brzeźniak, Peszat, and Zabczyk showed that for 'most' functions $f \in C_P$, the stochastic convolution with a standard real-valued Brownian motion $\beta = \{\beta_t\}_{t\geq 0}$,

$$t \mapsto (f * \beta)_t = \int_0^t f(t-s)\, d\beta_s,$$

fails to have a modification with bounded trajectories and thus also fails to have a modification with continuous trajectories. In fact the authors showed that the set of all $f \in C_P$ for which such a modification exists is of the first category in C_P. The main ingredient is a deep regularity result for random trigonometric series [28, Theorem 8.1].

This seems to suggest an approach toward a negative solution of the continuous modification problem for a class of stochastic equations in C_P. To see why, let $A = d/d\theta$ denote the generator of the left translation group $S = \{S(t)\}_{t\geq 0}$ in C_P (compare [25, 4.14 Definition]) and consider the problem

$$\begin{cases} dU(t) = AU(t)\, dt + f\, d\beta_t, & t \geq 0, \\ U(0) = 0, \end{cases} \tag{5.4}$$

where $f \in C_P$ is a given function. If this problem has a weak solution $\{U_f(t)\}_{t\geq 0}$ in C_P (in the sense of Definition 2.2.1), then for all $t \geq 0$ we have

$$\langle U_f(t), \delta_0 \rangle = \int_0^t \langle S(t-s)f, \delta_0 \rangle \, d\beta_s = \int_0^t f(t-s) \, d\beta_s$$

almost surely, where δ_0 denotes the Dirac measure at 0.

By the Brzeźniak-Peszat-Zabczyk result, the right hand side fails to have a continuous modification for all functions f outside a set of first category in C_P. Interestingly, however, from Theorem 5.1.2 it follows that precisely for these f the above problem fails to have a weak solution. This is the content of the Theorem below. It shows that problem (5.4) actually provides an example of nonexistence and, at the same time, some evidence for a positive solution to the continuous modification problem.

Theorem 5.2.1 *For a given function $f \in C_P$, the problem* (5.4) *has a weak solution if and only if the convolution process $f * \beta$ has a modification with continuous trajectories, and in this situation the weak solution has a modification with continuous trajectories.*

Proof. Suppose E is a real Banach space and let $x \in E$ be a fixed nonzero element. Let H denote the one-dimensional subspace spanned by x, endowed with the norm $\|cx\|_H := |c|$. If $\beta = \{\beta_t\}_{t\in[0,T]}$ is a standard real-valued Brownian motion, then

$$(W_H cx)(t) := c\beta_t, \qquad c \in \mathbb{R}$$

defines a cylindrical H-Wiener process.

Using this construction we see that (5.4) is a special case of (5.3) if we take $H = \text{span}\{x\}$ and $W_t^H(cx) = c\beta_t$, and define $B_f : H \to C_P$ by $B_f(cx) := cf$. By Theorem 5.1.2 and the observations above, (5.4) fails to have a weak solution whenever the convolution of f with β fails to have a continuous modification.

Let us now assume that, conversely, the convolution process $f * \beta$ has a continuous modification. Then the convolution process $t \mapsto (f * \tilde{\beta})_t$ has a continuous modification as well, where $\tilde{\beta}_t := \beta_{1+t} - \beta_1$. Indeed, this may be deduced from [29, Lemma 3.24] or from a general comparison result for Gaussian processes [36, Theorem 12.16]. Now define, for $\theta \geq 0$,

$$X_f(\theta) := \int_0^{1+\theta} f(1+\theta-s) \, d\beta_s - \int_0^{\theta} f(\theta-s) \, d\tilde{\beta}_s, \tag{5.5}$$

where on the right hand side we take the continuous modifications, and notice that

$$X_f(\theta) = \int_0^1 f(\theta - s) \, d\beta_s$$

almost surely. Hence by the Pettis measurability theorem and the stochastic Fubini theorem, (5.5) defines a centered C_P-valued Gaussian random variable X_f, and for any finite Borel measure $\mu \in (C_P)^*$ the variance of $\langle X_f, \mu\rangle$ is given by

$$\begin{aligned}\mathbb{E}\langle X_f, \mu\rangle^2 &= \mathbb{E}\Big(\int_0^1 \int_0^1 f(\theta - s)\, d\beta_s\, d\mu(\theta)\Big)^2 \\ &= \mathbb{E}\Big(\int_0^1 \int_0^1 f(\theta - s)\, d\mu(\theta)\, d\beta_s\Big)^2 \\ &= \int_0^1 \Big(\int_0^1 f(\theta - s)\, d\mu(\theta)\Big)^2 ds \\ &= \langle Q_f \mu, \mu\rangle.\end{aligned}$$

Here, the operator $Q_f \in \mathcal{L}(C_P^*, C_P)$ is defined by

$$Q_f \mu := \int_0^1 S(t) B_f B_f^* S^*(t) \mu\, dt.$$

The existence of a global weak solution U_f now follows from Lemma 2.3.3, cf. also [8, Theorem 5.3] □

Remark 5.2.2 The solution U_f is now given by

$$U_f(t) = \int_0^t S(t-s) B_f dW_H(s) = \int_0^t f(t + \theta - s)\, d\beta_s$$

almost surely.

Remark 5.2.3 We have seen in Theorem 5.1.2 that the existence of a weak solution U to problem (5.3) implies the existence of a continuous modification of U whenever A is the generator of a C_0-group. Another situation where this is known to happen is the case where A generates an analytic C_0-semigroup on E; see Theorem 3.2.8 and [8, Proposition 4.3, Theorem 6.1] as well as [15, Lemma 5.13].

5.3. A characterization of H^∞-calculus

In Chapter 4 we developed a characterization of bounded H^∞-calculus. There we used maximal regularity of solutions of suitable SCP's (compare Theorem 4.2.4). In the case that A generates a C_0-group we can exploit results of [30] (see Appendix A) to obtain a characterization of bounded H^∞-calculus by the very existence of such weak solutions.

Theorem 5.3.1 *Let E and E^* have finite cotype. For a generator A of a C_0-group* $\mathbf{S}$ *the following are equivalent.*

(*a*) *The equations*

$$\begin{aligned} dU(t) &= AU(t)\,dt + x\,dW_H(t), \qquad t \geq 0, \\ U(0) &= 0, \end{aligned}$$

and

$$\begin{aligned} d\tilde{U}(t) &= A^{\odot}\tilde{U}(t)\,dt + x^{\odot}\,dW_H(t), \qquad t \geq 0, \\ \tilde{U}(0) &= 0, \end{aligned}$$

admit a weak solution in E *resp.* $E^{\odot}$ *for all* $x \in E$ *resp. for all* $x^{\odot} \in E^{\odot}$.

(*b*) A *has a* H^{∞}*-calculus on each strip* $\mathcal{S}_{\omega} = \{\lambda \in \mathbb{C} : |\operatorname{Re}\lambda| < \omega\}, \omega > \omega_0(A)$.

Proof. In the following we will prefer notations such as $\|s \mapsto S(s)x\|_{\gamma}^2$ instead of $\|S(\cdot)x\|_{\gamma}^2$.

$(a) \Rightarrow (b)$: With the operators $B : \mathbb{R} \to E$, $t \mapsto xt$, $x \in \mathcal{D}(A)$ and $\tilde{B} : \mathbb{R} \to E^{\odot}$, $t \mapsto x^{\odot}t$, $x^{\odot} \in E^{\odot}$, we obtain for the solution $U(t) = \int_0^t S(t-s)Bd\beta_s = \int_0^t S(t-s)xd\beta_s$,

$$\mathbb{E}\|U(t)\|^2 = \|s \mapsto S(s)x\|_{\gamma([0,t],E)}^2 < \infty.$$

If we consider the rescaled semigroup $T(t) := e^{-\omega t}S(t)$, $t \geq 0$, $\omega > \omega_0(A)$ we obtain (see [47, Proposition 4.5])

$$\|t \mapsto e^{-\omega t}S(t)x\|_{\gamma([0,\infty),E)} < \infty\,.$$

Analogous considerations in the dual case with $x^{\odot} \in E^{\odot}$ show

$$\|t \mapsto e^{-\omega t}S^*(t)x^{\odot}\|_{\gamma([0,\infty),E^*)} < \infty\,.$$

We claim that there exists a constant $K \geq 0$ such that

$$\|t \mapsto e^{-\omega t}S(t)x\|_{\gamma([0,\infty),E)} < K\|x\| \tag{5.6}$$

and

$$\|t \mapsto e^{-\omega t}S^*(t)x^{\odot}\|_{\gamma([0,\infty),E^*)} < K\|x^{\odot}\|\,. \tag{5.7}$$

This is an easy application of the Closed Graph Theorem: Let (f_n) be an ONB of $L^2(0,T)$. Assume that for a sequence (x_i), with $x_i \to x$ in E, $i \in \mathbb{N}$ we have $T(\cdot)x_j \to \Phi(\cdot)$ in $\gamma(0,\infty;E)$ for $\Phi \in \gamma(0,\infty;E)$.

Then by Fatou's Lemma,

$$\begin{aligned}
&\mathbb{E}\left\|\sum_{n=1}^{N}\gamma_n\int_0^\infty f_n(t)T(t)x\,dt\right\|^2\\
&= \mathbb{E}\left[\lim_{j\to\infty}\left\|\sum_{n=1}^{N}\gamma_n\int_0^\infty f_n(t)T(t)x_j\,dt\right\|^2\right]\\
&\le \liminf_{j\to\infty}\,\mathbb{E}\left\|\sum_{n=1}^{N}\gamma_n\int_0^\infty f_n(t)T(t)x_j\,dt\right\|^2\\
&\le \liminf_{j\to\infty}\|T(\cdot)x_j\|_\gamma^2\\
&= \|\Phi(\cdot)\|_\gamma\,.
\end{aligned}$$

Taking the supremum over $N \ge 1$, we obtain that $T(\cdot)x \in \gamma(0,\infty;E)$ and $T(\cdot)x = \Phi(\cdot)x$. Now, by the Closed Graph Theorem we obtain the desired estimate (5.6) and (5.7) by analogous arguments. Having obtained the two estimates (5.6) and (5.7) we can apply A.0.4 to infer the desired H^∞-calculus of A.

$(b) \Rightarrow (a)$: Since A generates a C_0-group, A is of γ- and R-strip type as shown in Lemma A.0.3. Further, since E has finite cotype we can apply Theorem A.0.4. This theorem states that in the present situation H^∞-calculus of A on the strips $\mathcal{S} = \{\lambda \in \mathbb{C} : |\operatorname{Re}\lambda| < \omega\}$, $\omega > \omega_0(A)$ is equivalent to the estimates

$$\begin{aligned}
\|e^{-\omega t}S(t)x\|_{\gamma([0,\infty),E)} &\le C\|x\|, && x \in \mathcal{D}(A)\\
\|e^{-\omega t}S^*(t)x^*\|_{\gamma([0,\infty),E^*)} &\le C\|x^*\|, && x^* \in \mathcal{D}(A^*)\,.
\end{aligned}$$

This is equivalent to the existence of weak solutions $U(t)$ and $\tilde{U}(t)$ on arbitrary intervals $[0,T]$, $T \ge 0$. □

A. Square function estimates - results of [30]

The ideas, concepts and results of [30] are fundamental throughout wide parts of this work. The following results are explicitly used in the previous chapters.

The following lemma could be interpreted as a Fatou lemma for the space $\gamma(H,E)$.

Lemma A.0.2 *Suppose (u_ν) is an uniformly bounded net in $\mathcal{L}(H,E)$ such that $\lim_\nu u_\nu = u$ in the strong operator topology. Then*

$$\|u\|_\gamma \leq \liminf_\nu \|u_\nu\|_\gamma.$$

In connection with square function estimates we consider operators of strip type A on a Banach space E such that $\{R(\lambda, A) : |\mathrm{Re}\lambda| > a\}$ is not only bounded, but even γ–bounded (R–bounded). Such operators are called *operators of γ-strip type* (of *R-strip type*) and $w_\gamma(A)$ (or $w_R(A)$) is the infimum over all a for which the above γ–boundedness (R–boundedness) condition holds.

Lemma A.0.3 *If A generates a C_0–group $\mathbf{S} = (S(t))$, then A is of γ– (and R–) strip type with $\omega_\gamma(A), \omega_R(A) \leq \omega_0(\mathbf{S})$.*

Theorem A.0.4 *Let A be of γ–strip type operator on a Banach space E with finite cotype. Then the following conditions are equivalent*

a) A generates a C_0–group $\mathbf{S} = (S(t))$ such that for one (all) $a > \omega(-A)$ there is a constant C with

$$\begin{aligned} \|e^{-at}S(t)x\|_{\gamma(\mathbb{R}_+,E)} &\leq C\|x\|, \quad x \in \mathcal{D}(A) \\ \|e^{-at}S(t)^*x^*\|_{\gamma(\mathbb{R}_+,X^*)} &\leq C\|x^*\|, \quad x^* \in \mathcal{D}(A^*). \end{aligned}$$

b) For one (all) a with $|a| > w_\gamma(A)$ there is a constant C, such that

$$\begin{aligned} \|R(a+i\cdot, A)x\|_{\gamma(\mathbb{R},E)} &\leq C\|x\|, \quad x \in \mathcal{D}(A) \\ \|R(a+i\cdot, A)^*x^*\|_{\gamma(\mathbb{R},E^*)} &\leq C\|x^*\|, \quad x^* \in \mathcal{D}(A^*). \end{aligned}$$

c) For one (all) a with $|a| > w_\gamma(A)$ *there is a constant* C *such that for* $x \in \mathcal{D}(A)$

$$\frac{1}{C}\|x\| \le \|R(a + i\cdot, A)x\|_{\gamma(\mathbb{R}_+,E)} \le C\|x\|.$$

d) A *has a* $H_\infty(S(b))$*–calculus for one (all)* $b > w_\gamma(A)$.

Furthermore we have $w_\gamma(A) = \omega_0(\mathbf{S}) = w_{H^\infty}(A)$.

Proposition A.0.5 *Let* $-A$ *be a sectorial operator in* E *of angle* $0 < \omega(-A) < \frac{\pi}{2}$.

1. *The operator* $-A$ *admits a bounded* H^∞*-calculus if and only if for all* $x \in E$ *and* $x^\odot \in E^\odot$ *we have* $(-A)^{\frac{1}{2}} S(\cdot)x \in \gamma(\mathbb{R}_+; E)$ *and* $(-A^\odot)^{\frac{1}{2}} S^\odot(\cdot)x^\odot \in \gamma(\mathbb{R}_+; E^\odot)$.

2. *Suppose* $0 \in \varrho(A)$ *and* $T > 0$ *is arbitrary and fixed.*

   Then $-A$ *admits a bounded* H^∞*-calculus if and only if for all* $x \in E$ *and* $x^\odot \in E^\odot$ *we have* $(-A)^{\frac{1}{2}} S(\cdot)x \in \gamma(0, T; E)$ *and* $(-A^\odot)^{\frac{1}{2}} S^\odot(\cdot)x^\odot \in \gamma(0, T; E^\odot)$.

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Index

ISBN: 978-3-86644-144-6

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