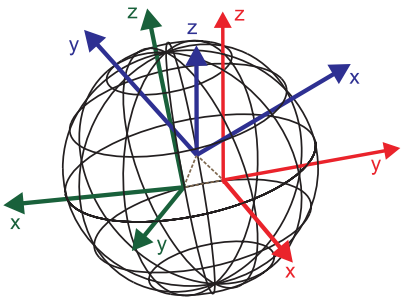
Universität Karlsruhe (TH)

Schriftenreihe des Studiengangs
Geodäsie und Geoinformatik
2007,6



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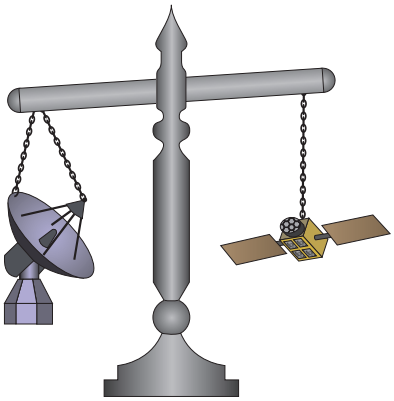

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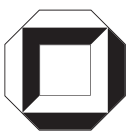

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Impressum

Universitätsverlag Karlsruhe
c/o Universitätsbibliothek
Straße am Forum 2
D-76131 Karlsruhe
www.uvka.de

Universitätsverlag Karlsruhe 2007
Print on Demand

ISSN: 1612-9733
ISBN: 978-3-86644-206-1

Contents

Summary

Terrestrial Reference Frames (TRFs) are based on heterogeneous data derived from continuous observations by different space geodetic techniques. They are computed by diverse analysis centres pursuing various strategies of mathematical modelling. Combining individual solutions to frames of superior quality (like the ITRF) usually involves homogenisation by an empirical weighting scheme. In this study, different approaches on variance component estimation have been evaluated for this purpose.

The statistically rigorous Helmert estimator has been compared with two other methods: the degree of freedom method (which is a special case of an approach known as "Förstner's method") and a simplified, approximate estimator. It turned out that the Helmert estimator is poorly beneficial due to its high requirements on computation time, since the degree of freedom method yields equivalent results. Tests have been performed, covering two elementary types of combinations:

Combining a time series of weekly SLR-solutions, all three of the investigated estimators have yielded relatively homogeneous variance components. Convergence has been achieved after a few iterations.

Three individual solutions of space geodetic techniques (GPS, SLR and VLBI) have been combined with 46 sets of local ties. Unfortunately, none of the estimators was successful in estimating 46 variance components for the local ties in addition to the three for the space geodetic solutions. Due to poor redundancy of local ties, the estimation was successful only when the weights of the local ties were fixed to a priori empirical values.

Kurzfassung

Terrestrische Referenzrahmen (TRFs) werden aus heterogenen Datensätzen abgeleitet, die aus kontinuierlichen Beobachtungen verschiedener geodätischer Raumverfahren gewonnen werden. Ihre Berechnung erfolgt durch diverse Analysezentren, die jeweils unterschiedliche Strategien der mathematischen Modellbildung verfolgen. Bei der Kombination individueller Lösungen zu Referenzrahmen übergeordneter Qualität (wie dem ITRF) erfolgt gewöhnlich eine Homogenisierung durch empirische Gewichtung der Einzellösungen. Gegenstand dieser Arbeit ist es, hierfür verschiedene Methoden der Varianzkomponentenschätzung zu evaluieren.

Die statistisch strenge Schätzung nach Helmert wurde zwei anderen Methoden vergleichend gegenübergestellt: der Schätzung nach Freiheitsgraden (einem Spezialfall eines auch als „Förstner-Methode“ bekannten Verfahrens) sowie einer vereinfachten Näherungslösung. Es ergab sich, dass der Helmert-Schätzer aufgrund hoher Rechenzeitanforderungen kaum Vorzüge gegenüber der Schätzung nach Freiheitsgraden bietet, die äquivalente Resultate liefert. Die Anwendung auf zwei elementare Kombinationstypen wurde getestet:

Bei der Kombination einer Zeitreihe aus SLR-Wochenlösungen lieferten alle drei der untersuchten Schätzer relativ homogene Varianzkomponenten. Schon nach wenigen Iterationen stellte sich Konvergenz ein.

Drei individuelle Lösungen verschiedener geodätischer Raumverfahren (GPS, SLR und VLBI) wurden gemeinsam mit 46 Datensätzen lokaler Verknüpfungsmessungen kombiniert. Leider war keiner der Schätzer in der Lage, zusätzlich zu den drei Varianzkomponenten der geodätischen Raumverfahren 46 weitere für die lokalen Messungen zu schätzen, da deren Redundanzanteile zu gering sind. Die Schätzung gelang nur, wenn die Gewichte der Verknüpfungsmessungen auf Erfahrungswerte fixiert wurden.

Résumé

Des repères de référence terrestres (TRFs) sont basés sur des données hétérogènes, déduites d'observations continues de différentes techniques de géodésie spatiale. Ils sont établis par divers centres d'analyse qui mènent des stratégies variées de modélisation mathématique. Pour combiner des solutions individuelles en vue d'obtenir des repères d'une qualité supérieure (comme l'ITRF), il faut une homogénéisation par une pondération empirique des solutions particulières. A cette fin, différentes méthodes d'estimation des composantes de la variance sont évaluées dans cette étude.

L'estimateur de Helmert, qui est statistiquement rigoureux, a été comparé à deux autres méthodes : l'estimation par le degré de liberté (qui est un cas spécial de la «méthode de Förstner») et un estimateur approximatif et simplifié. Il s'est relevé qu'en raison des besoins élevés de temps de calcul, l'estimation selon Helmert n'apporte que peu d'avantages par rapport à la méthode du degré de liberté, le dernier fournissant des résultats équivalents. L'application sur deux types élémentaires de combinaison a été testée :

Combinant une série temporelle de solutions hebdomadaires SLR, tous les trois estimateurs examinés ont produit des composantes de la variance relativement homogènes. Après très peu d'itérations, l'algorithme a convergé.

Trois solutions individuelles de différentes techniques de géodésie spatiale (GPS, SLR et VLBI) ont été combinées avec 46 jeux de rattachements locaux. Malheureusement, aucun des estimateurs n'a réussi à estimer 46 composantes de la variance pour les rattachements à part les trois pour la géodésie spatiale. A défaut de redondance suffisante des rattachements, l'estimation n'a réussi qu'en fixant les poids des rattachements à des valeurs empiriques.

1 Introduction

Providing geometrical reference frames of global dimension is a fundamental task of space geodesy. The frames are based on heterogeneous data derived from continuous observations by different techniques, covering long periods of time. Establishing a terrestrial reference frame (TRF) of superior quality means combining data from multiple individual solutions, each of them defining its own underlying frame.

Combining TRFs is and has ever been a principal field of activity of the *Laboratoire de Recherche en Géodésie* (LAREG), which is part of the French *Institut Géographique National* (IGN) and situated in the facilities of the *Ecole Nationale des Sciences Géographiques* (ENSG) at Marne la Vallée, France. Since 1995, the software CATREF (Combination and Analysis of Terrestrial Reference Frames) has been developed at the LAREG to perform combinations.

The CATREF combination algorithm implies that the individual solutions are statistically independent populations, each of them providing a full covariance matrix. But as the solutions' provenance differs in observing instruments as well as in functional and stochastic modelling, it would not be appropriate to assume a uniform level of variance. Hence, the covariance matrices are scaled by individual factors before being introduced into the combination process. The estimation of these factors (variance components) is the subject of this study.

Over the years, a multitude of approaches has been proposed in the domain of variance component estimation. Statistically rigorous estimators and approximate methods differ distinctly concerning computing time, convergence speed and adequacy of results. Up to now, two methods have been realised in CATREF software: an approximate least squares estimator (the "classical" estimator) and the degree of freedom method. Within the scope of this study, the application of other approaches is evaluated on the basis of CATREF's mathematical model. Especially the statistically rigorous Helmert estimator is focused.

As most considerations in this study are based on adjustment theory, section 2 starts up with a brief outline of the Gauß-Markov model of parameter estimation. Basic symbols and relations are introduced to be referenced later. The use of mathematical symbols is consistent as far as possible within the entire study. Where the meaning of a symbol is contextual, it is pointed out explicitly. For the ease of reading, appendix C provides an alphabetical reference of all symbols, their meaning and their page of first occurrence.

Section 3 gives a review of the approaches of variance component estimation proposed so far, selectively going into detail. It results from a bibliographic research, comprising publications since the 1970s, mainly by German geodesists. Nevertheless, the synopsis cannot be exhaustive and has been limited to relevant aspects.

The fourth section introduces the concept of TRFs. Besides some remarks on the way of establishment, their mathematical representation is described in detail, before the issue how to define the datum is discussed in particular. Finally, the combination procedure is treated, the mathematical model of which is derived.

Considering the implementation of algorithms, section 5 presents some relevant details. Considerations concerning the coding of the combination model are made to minimise memory requirements. Furthermore, the adaptivity of approaches for variance component estimation is evaluated, and (if applicable) some coding details are elucidated. A concise flowchart of the complete procedure can be found in appendix D.

Section 6 contains the results this study: Variance component estimators are tested on two authentic data sets, representing the most common applications for combination of TRFs: time

series stacking and the combination of solutions originating from different observation techniques. Diverse approaches are considered, varying in the underlying mathematical model and the chosen approximate values.

Finally, section 7 draws the conclusion of both theoretical and practical investigations. The considered estimators are evaluated with respect to strong and weak points as well as to their applicability on different types of TRF combinations. Open questions are exposed, which are probably worth for further investigations.

2 Remarks on Parameter Estimation in the Gauß-Markov Model

Parameter estimation is an indispensable, common tool in geodesy, hence no exhaustive treatment of this subject will be provided here. However, a brief summary is considered to be necessary to make the reader familiar with the notation applied in this study. Moreover, this section will serve as reference for subsequent derivations. In the following, the approach of *least squares adjustment* is presented.

A set of n differentiable functions $\mathbf{f}(\mathbf{X})$ of u parameters $\mathbf{X}$ to be estimated is developed at a point $\mathbf{X}_0$ into a Taylor series:

$$\mathbf{f}(\mathbf{X}) = \mathbf{f}(\mathbf{X}_0) + \frac{\partial \mathbf{f}}{\partial \mathbf{X}^T}(\mathbf{X} - \mathbf{X}_0) + \mathrm{O}\left\{(\mathbf{X} - \mathbf{X}_0)^2\right\}. \tag{2.1}$$

With the observations $\mathbf{L}$ of the function values $\mathbf{f}(\mathbf{X})$ and omitting terms of second order, the observation equations for adjustment in the *Gauß-Markov model* can be formulated:

$$\mathbf{l} = \mathbf{A}\mathbf{x} + \boldsymbol{\epsilon} \tag{2.2}$$

with	$\mathbf{l} = \mathbf{L} - \mathbf{f}(\mathbf{X}_0)$	$n \times 1$ vector of abridged observations
	$\mathbf{x} = \mathbf{X} - \mathbf{X}_0$	$u \times 1$ vector of abridged parameters
	$\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{X}^T} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}^T}$	$n \times u$ design matrix
	$\boldsymbol{\epsilon}$	$n \times 1$ vector of errors.

Introducing operators for the expectation ($\mathrm{E}\{\cdot\}$) and the dispersion ($\mathrm{D}\{\cdot\}$), the functional and the stochastic model are expressed as follows:

$$\mathrm{E}\{\mathbf{l}\} = \mathbf{A}\mathbf{x} = \mathbf{l} - \boldsymbol{\epsilon} \tag{2.3}$$

$$\mathrm{D}\{\mathbf{l}\} = \mathbf{C}_{ll} \tag{2.4}$$

with the $n \times n$ covariance matrix $\mathbf{C}_{ll}$ of the observations. Introducing the predicted errors $\tilde{\boldsymbol{\epsilon}}$, the $n \times 1$ vector of residuals is defined by:

$$\mathbf{v} = -\tilde{\boldsymbol{\epsilon}}\,. \tag{2.5}$$

By means of a positive definite weight matrix $\mathbf{P}$, for which usually $\mathbf{P} \propto \mathbf{C}_{ll}^{-1}$ holds, the weighted sum of squares of the residuals is minimised:

$$\mathbf{v}^\mathbf{T}\mathbf{P}\mathbf{v} \longrightarrow \min\,, \tag{2.6}$$

and the normal equation system can be derived:

$$\begin{aligned} \mathbf{A}^\mathbf{T}\mathbf{P}\mathbf{A}\hat{\mathbf{x}} &= \mathbf{A}^\mathbf{T}\mathbf{P}\mathbf{l} \\ \mathbf{N}\hat{\mathbf{x}} &= \mathbf{n}\,. \end{aligned} \tag{2.7}$$

In this context, symbols for the normal equation matrix $\mathbf{N} = \mathbf{A}^T\mathbf{P}\mathbf{A}$ and the right hand side vector $\mathbf{n} = \mathbf{A}^T\mathbf{P}\mathbf{l}$ are defined. The circumflex $\hat{}$ denoting estimated quantities, the following formulas hold in case of a regular normal equation matrix:

$$\hat{\mathbf{x}} = \mathbf{N}^{-1}\mathbf{n} \tag{2.8}$$

$$\hat{\mathbf{l}} = \mathbf{A}\hat{\mathbf{x}} \tag{2.9}$$

$$\mathbf{v} = \hat{\mathbf{l}} - \mathbf{l}\,. \tag{2.10}$$

The least squares estimator is unbiased, since $\mathrm{E}\{\hat{\mathbf{x}}\} = \mathbf{x}$, $\mathrm{E}\{\hat{\mathbf{l}}\} = \mathbf{A}\mathbf{x}$ and $\mathrm{E}\{\mathbf{v}\} = \mathbf{0}$. Assuming $\mathbf{P} = \mathbf{C}_{ll}^{-1}$ as a special case, the corresponding covariance matrices are:

$$\mathrm{D}\{\hat{\mathbf{x}}\} = \mathbf{C}_{\hat{x}\hat{x}} = \mathbf{N}^{-1} \tag{2.11}$$

$$\mathrm{D}\{\hat{\mathbf{l}}\} = \mathbf{C}_{\hat{l}\hat{l}} = \mathbf{A}\mathbf{C}_{\hat{x}\hat{x}}\mathbf{A}^T \tag{2.12}$$

$$\mathrm{D}\{\mathbf{v}\} = \mathbf{C}_{vv} = \mathbf{C}_{ll} - \mathbf{C}_{\hat{l}\hat{l}}\ . \tag{2.13}$$

Estimating the Variance of Unit Weight

For subsequent considerations it will be important to be familiar with the concept of a unitary variance factor and its estimation. The approach consists in extending the stochastic model (2.4) by scaling the covariance matrix:

$$\mathrm{D}\{\mathbf{l}\} = \mathbf{C}_{ll} = \sigma_0^2\mathbf{Q}_{ll} = \sigma_0^2\mathbf{P}^{-1}\ . \tag{2.14}$$

$\mathbf{C}_{ll}$ is decomposed into the matrix of cofactors $\mathbf{Q}_{ll}$ and a scalar factor σ_0^2, which is called the variance of unit weight or the a priori variance factor. Introducing its estimate $\hat{\sigma}_0^2$, the covariance matrices of $\hat{\mathbf{x}}$, $\hat{\mathbf{l}}$ and $\mathbf{v}$ can be decomposed analogously:

$$\mathbf{C}_{\hat{x}\hat{x}} = \hat{\sigma}_0^2\mathbf{Q}_{\hat{x}\hat{x}} \tag{2.15}$$

$$\mathbf{C}_{\hat{l}\hat{l}} = \hat{\sigma}_0^2\mathbf{Q}_{\hat{l}\hat{l}} \tag{2.16}$$

$$\mathbf{C}_{vv} = \hat{\sigma}_0^2\mathbf{Q}_{vv}\ . \tag{2.17}$$

Applying the universal formula for the expectation of an arbitrary quadratic form $\mathbf{y}^T\mathbf{M}\mathbf{y}$ (Koch, 1997, eq. (271.1)), $\mathbf{y}$ being stochastic:

$$\mathrm{E}\{\mathbf{y}^T\mathbf{M}\mathbf{y}\} = \mathrm{E}\left\{\,\mathrm{tr}\left(\mathrm{D}\{\mathbf{y}\}\,\mathbf{M}\right)\right\} + \left(\mathrm{E}\{\mathbf{y}\}\right)^T\mathbf{M}\,\mathrm{E}\{\mathbf{y}\}\ , \tag{2.18}$$

the unbiased estimate $\hat{\sigma}_0^2$ of the variance of unit weight, also called the a posteriori variance factor, is obtained by evaluating the expectation of the weighted sum of squares of the residuals:

$$\begin{aligned}
\mathrm{E}\{\mathbf{v}^T\mathbf{P}\mathbf{v}\} &= \mathrm{tr}(\mathbf{C}_{vv}\mathbf{P}) \\
&= \sigma_0^2\,\mathrm{tr}(\mathbf{Q}_{vv}\mathbf{P}) \\
&= \sigma_0^2\big[\,\mathrm{tr}(\mathbf{Q}_{ll}\mathbf{P}) - \mathrm{tr}(\mathbf{Q}_{\hat{l}\hat{l}}\mathbf{P})\big] \\
&= \sigma_0^2\cdot(n-u) \qquad (2.19)\\
\Leftrightarrow\quad \hat{\sigma}_0^2 &= \frac{\mathbf{v}^T\mathbf{P}\mathbf{v}}{n-u}\ . \qquad (2.20)
\end{aligned}$$

From (2.20) it is obvious that $\hat{\sigma}_0^2$ is unbiasedly estimable. In this context, the redundancy r is introduced, which is equivalent to the number of degrees of freedom of the adjustment problem:

$$r = n - u = \mathrm{tr}(\mathbf{Q}_{vv}\mathbf{P})\ . \tag{2.21}$$

For later considerations it will be important that the redundancy can be decomposed into redundancy numbers:

$$r_i = [\mathbf{Q}_{vv}\mathbf{P}]_{ii} \qquad \forall\, i = 1\ldots n \tag{2.22}$$

where $[\cdot]_{ii}$ extracts the i^{th} element on the main diagonal. Each redundancy number corresponds to a particular observation. Their overall sum is (cf. (2.19)):

$$\sum_{i=1}^{n} r_i = r\ . \tag{2.23}$$

3 Estimation of Variance Components

3.1 Motivation

In the linear model of parameter estimation, the a posteriori variance factor (2.20) is an estimator for the common variance level of the observations. It often serves as well as plausibility check for the assumptions made a priori – giving reason to modify them if necessary.

In the model of unit weight (2.14), it is assumed that the relations of variance levels among the observations themselves are known; they are determined by the cofactor matrix $\mathbf{Q}_{ll}$. In a way, one could consider $\mathbf{Q}_{ll}$ as given whereas σ_0^2 is free to be estimated by $\hat{\sigma}_0^2$. In practice, however, an appropriate choice of $\mathbf{Q}_{ll}$ is not always obvious.

Considering for instance a levelling network, an erroneous assumption of the uniform precision of the observed height differences (derived e. g. from calibration of the only observing instrument) would distort neither the parameters' nor the variance factor's estimates. Otherwise, in a geodetic network of observed distances and directions, one may as well assume precisions for the particular observation techniques. But in case the ratio of both variance levels is erroneous, these assumptions have a determining impact on the estimates.

Variance component estimation renders possible to estimate different levels of variance for multiple sets of observations, whereas the internal relations of each set must be fixed a priori. In the following section, the potential of this approach will be pointed out by outlining the options of variance modelling.

3.2 Variance Modelling

For the estimation of k variance components, the covariance matrix is decomposed in the following way:

$$\mathbf{C}_{ll} = \mathbf{V}_0 + \sum_{i=1}^{k} s_i \mathbf{V}_i \ . \tag{3.1}$$

$\mathbf{s} = (s_1, s_2, \ldots, s_k)^T$ is the vector of variance components, and $\mathbf{V}_i$ $(i \neq 0)$ are the $n \times n$ matrices modelling their impact. The latter must be chosen thoroughly, for there is no mechanism in the estimation process to verify their conformity to physical reality. Solely their scale factors s_i are free to be estimated. $\mathbf{V}_0$ represents a partition of the covariance matrix, the scale of which is supposed to be definitely known and where no scaling factor is to be estimated. Nevertheless, $\mathbf{V}_0 = \mathbf{0}$ holds for most applications.

Moreover, the following notations are introduced:

$$\sum_{i=0}^{k} \mathbf{V}_i = \mathbf{Q} = \mathbf{P}^{-1} \ . \tag{3.2}$$

Note that this definition of the weight matrix $\mathbf{P}$ is not in accordance with the one made in context of the variance of unit weight (2.14). However, the meaning of $\mathbf{P}$ will always clear up in the context of the respective stochastic model.

Facing the question how to establish an adequate model, two examples are given:

1. In the *model of unit weight* (2.14), there is only one variance component, namely the variance of unit weight. With $k = 1$, $\mathbf{s} = s_1 = \sigma_0^2$, $\mathbf{V}_1 = \mathbf{Q}_{ll}$ and $\mathbf{V}_0 = \mathbf{0}$, the reference to (3.1) is established.

2. When combining k solutions of Terrestrial Reference Frames (see sect. 4.4), one has k sets of n_i observations ($i = 1 \dots k$), each of them providing its own covariance matrix. Decomposing it into a factor σ_i^2 and a cofactor matrix $\mathbf{Q}_i = \mathbf{P}_i^{-1}$, the following model is obtained for the case that no correlations between the sets are assumed ($\mathbf{V}_0 = \mathbf{0}$):

$$s_i = \sigma_i^2, \quad \mathbf{V}_i = \begin{pmatrix} \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots & & \vdots \\ \mathbf{0} & \cdots & \mathbf{Q}_i & \cdots & \mathbf{0} \\ \vdots & & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix} \quad \forall\, i = 1 \dots k \tag{3.3}$$

$$\mathbf{C}_{ll} = \begin{pmatrix} \sigma_1^2\mathbf{Q}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \sigma_2^2\mathbf{Q}_2 & & \mathbf{0} \\ \vdots & & \ddots & \\ \mathbf{0} & \mathbf{0} & & \sigma_k^2\mathbf{Q}_k \end{pmatrix} . \tag{3.4}$$

Without giving examples, it ought to be mentioned that (3.1) is capable of a much more complex modelling. Estimating distance-dependent variance components for distance observations is feasible as well as the estimation of covariance components (cf. e. g. Koch, 1981; Koch, 1997, ch. 361). One deals with the latter when there are non-zero off-diagonal elements in the corresponding matrices $\mathbf{V}_i$. Anyhow, this is not subject of this study.

3.3 Historical Background

A first approach to estimate variance components was proposed by Helmert (1907). He developed a method to estimate the weights of uncorrelated sets of observations, where the observations among themselves are uncorrelated as well ((3.4) with $\mathbf{Q}_i = \mathbf{I}\ \forall\, i = 1 \dots k$; $\mathbf{I}$ denoting the identity matrix). Neglecting statistical postulates of any kind, his derivation was made in a merely heuristic manner.

Helmert's work had nearly been forgotten until Welsch (1978) reviewed it, applying matrix notation to ease the access for today's users. But it was already in the 1960s when the subject of variance component estimation was reintroduced into geodetic literature. Kubik (1967) was one of the first to renew the research. Furthermore, Grafarend (1978) mentions several papers in the domain of mathematical statistics from this epoch. Whereas the considerations had initially been restricted to variance components with or without correlations among the observations (Kubik, 1967; Kubik, 1970; Ebner, 1972; Förstner, 1979c), the model was extended to variance and covariance components later (Grafarend, 1978; Koch, 1978; Förstner, 1979a).

In numerous publications, many approaches have been made to develop a more or less optimal estimator for variance and covariance components. Considering the fact that different approaches sometimes yield the same results, the following sections intend to classify the estimators.

3.4 Estimator of Helmert Type

According to bibliographical investigations, the notion of an estimator of Helmert type was presumably introduced by Grafarend and Schaffrin (1979). Grafarend et al. (1980) give a rather general definition of its construction.

Anyway, four basically different approaches will be distinguished within this subsection, all of them yielding algebraically identical formulas for statistically rigorous estimation of variance components (Teunissen and Amiri-Simkooei, 2007). Because of their conditional equivalence to the method proposed by Helmert (1907, p. 360; Welsch, 1978), they will be referred to as *Helmert's estimator*. The general estimation formula is given to (Teunissen and Amiri-Simkooei, 2006, eq. (17) and (18)):

$$\hat{\mathbf{s}} = \mathbf{H}^{-}\mathbf{q} \tag{3.5}$$

$$\text{with} \qquad \mathbf{H} = (h_{ij}) \qquad h_{ij} = \operatorname{tr}(\mathbf{W}\mathbf{V}_i\mathbf{W}\mathbf{V}_j) \qquad i,j = 1\dots k \tag{3.6}$$

$$\mathbf{q} = (q_i) \qquad q_i = \mathbf{v}^T\mathbf{P}\mathbf{V}_i\mathbf{P}\mathbf{v} - h_{i0} \qquad i = 1\dots k \tag{3.7}$$

$$h_{i0} = \operatorname{tr}(\mathbf{W}\mathbf{V}_i\mathbf{W}\mathbf{V}_0) \qquad i = 1\dots k \tag{3.8}$$

$$\text{where} \qquad \mathbf{W} = \mathbf{P}\big(\mathbf{I} - \mathbf{A}(\mathbf{A}^T\mathbf{P}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{P}\big) = \mathbf{P}\mathbf{Q}_{vv}\mathbf{P}\ . \tag{3.9}$$

If $\mathbf{H}$ is regular, its Cayley-Inverse $\mathbf{H}^{-1}$ solves (3.5). If not, $\widehat{\mathbf{p}^T\mathbf{s}} = \mathbf{p}^T\mathbf{H}^{-}\mathbf{q}$ ($\mathbf{p} \in \mathbb{R}^k$) exists iff $\mathbf{p}$ is in the range of $\mathbf{H}$ (Koch, 1997, p. 252). The covariance matrix of the estimates is (Koch, 1997, eq. (362.11)):

$$\mathrm{D}\{\hat{\mathbf{s}}\} = 2\,\mathbf{H}^{-}\ . \tag{3.10}$$

Iteration The estimation of variance components by Helmert's method is an iterative procedure, because (3.5) is only unbiased if $\hat{\mathbf{s}} = \mathbf{1} = (1, 1, \dots, 1)^T$ (see sect. 3.4.1). To adopt this condition, the stochastic model (3.1) is refined to scale the matrices $\mathbf{V}_i$ appropriately:

$$\mathbf{C}_{ll} = \mathbf{V}_0 + \sum_{i=i}^{k} s_i\mathbf{V}_i = \mathbf{V}_0 + \sum_{i=i}^{k} s_i a_i \mathbf{T}_i\ . \tag{3.11}$$

The s_i are assumed to be 1, the $\mathbf{T}_i$ are determinately fixed and the a_i are considered to be approximate values for the variance components. They are chosen adequately for the first step of iteration and updated after each step until convergence is achieved. a_i for the $\nu+1^{\text{st}}$ iteration is computed as follows:

$$(a_i)_{\nu+1} = (a_i)_\nu \cdot (\hat{s}_i)_\nu = (a_i)_1 \prod_{j=1}^{\nu} (\hat{s}_i)_j \qquad \forall\, i = 1\dots k\ . \tag{3.12}$$

Convergence is achieved as soon as $\hat{\mathbf{s}} = \mathbf{1}$ holds within the scope of computational accuracy.

Note that the iterative approach has an impact on the statistical dispersion of the estimated variance components (Koch, 1997, eq. (363.3)). As the actual parameters of interest are factors by which the a priori covariance matrices are to be scaled, the focus is on the products $a_i\hat{s}_i$. Their variance is:

$$\hat{\mathrm{D}}\big\{\,(a_i)_\nu\,(\hat{s}_i)_\nu\,\big\} = (a_i)_\nu\,\mathrm{D}\,\big\{\,(\hat{s}_i)_\nu\,\big\} = 2\,(a_i)_\nu\,(\mathbf{H}^{-})_\nu\ . \tag{3.13}$$

Simplification The computation of the coefficients h_{ij}, q_i and h_{i0} can be simplified if the stochastic model has the following block diagonal structure:

$$\mathbf{C}_{ll} = \begin{pmatrix} \mathbf{Q}_0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \sigma_1^2\mathbf{Q}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \sigma_2^2\mathbf{Q}_2 & & \mathbf{0} \\ \vdots & \vdots & & \ddots & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & & \sigma_k^2\mathbf{Q}_k \end{pmatrix}\ . \tag{3.14}$$

Then, (3.6), (3.7) and (3.8) become:

$$h_{ij} = \delta_{ij}\big(n_i - 2\,\mathrm{tr}(\mathbf{N}^{-1}\mathbf{A}_i^T\mathbf{P}_i\mathbf{A}_i)\big) + \mathrm{tr}(\mathbf{N}^{-1}\mathbf{A}_i^T\mathbf{P}_i\mathbf{A}_i\mathbf{N}^{-1}\mathbf{A}_j^T\mathbf{P}_j\mathbf{A}_j) \qquad i,j = 1\ldots k \tag{3.15}$$

$$q_i = \mathbf{v}_i^T\mathbf{P}_i\mathbf{v}_i - h_{i0} \qquad i = 1\ldots k \tag{3.16}$$

$$h_{i0} = \mathrm{tr}(\mathbf{N}^{-1}\mathbf{A}_i^T\mathbf{P}_i\mathbf{A}_i\mathbf{N}^{-1}\mathbf{A}_0^T\mathbf{P}_0\mathbf{A}_0) \qquad i = 1\ldots k\,. \tag{3.17}$$

δ_{ij} is the Kronecker symbol, and $\mathbf{v}_i$ is that $n_i \times 1$ component of $\mathbf{v} = (\mathbf{v}_1^T, \mathbf{v}_2^T, \ldots, \mathbf{v}_k^T)^T$ that corresponds to the observations from set i. Analogously, $\mathbf{A}_i$ is the $n_i \times u$ partition of $\mathbf{A}$ corresponding to the same set.

Derivation Although the purpose of this work is application rather than derivation, four approaches will be outlined in the following: the common *BIQUE*-method, Rao's *MINQUE*-method (Rao, 1973), the *Maximum Likelihood approach* and Teunissen's *least squares variance component estimation* (LS-VCE, Teunissen and Amiri-Simkooei, 2007).

3.4.1 BIQUE

A *Best Invariant Quadratic Unbiased Estimator* (BIQUE) postulates the following properties:

- Quadratic modelling of the estimator
- Unbiasedness of the estimates
- Invariance of the estimates with respect to the parameters and their approximate values
- Minimum variance of the estimates ($\rightarrow$ best estimates)

Deriving the estimation formula, this subsection refers to the work of Koch (1978), Koch (1997) and Welsch (1984). Starting point are the functional model (2.2) and the stochastic model (3.1), whereas $\mathbf{V}_0 = \mathbf{0}$ will be assumed for the sake of simplicity. The postulated properties will be taken into account successively.

QE – Quadratic Estimator The variance of a normally distributed random vector is a quadratic function of the vector itself (Caspary and Wichmann, 1994, eq. (4.1-7)). Hence, an approach in terms of a quadratic form of the observations seems appropriate (cf. (2.18)):

$$\mathbf{p}^T\mathbf{s} = \mathrm{E}\{\mathbf{l}^T\mathbf{D}\mathbf{l}\} = \mathrm{tr}(\mathbf{D}\mathbf{C}_{ll}) + \mathbf{x}^T\mathbf{A}^T\mathbf{D}\mathbf{A}\mathbf{x}\,. \tag{3.18}$$

$\mathbf{D}$ is a symmetric $n \times n$ matrix with a priori unknown coefficients. As one can easily see by evaluating a quadratic form into a polynomial, the restriction of symmetry does not concern the generality of the approach. Neither does the estimation of a linear function $\mathbf{p}^T\mathbf{s}$ of the variance components instead of the components $\mathbf{s}$ themselves, which could be rendered possible by choosing $\mathbf{p} = \mathbf{e}_i$. $\mathbf{e}_i = (0,\ldots,0,1,0,\ldots,0)^T$ is the unit vector for the i^{th} component.

QUE – Quadratic Unbiased Estimator Unbiasedness is achieved if the variance components are independent of the parameters $\mathbf{x}$. This requires (Welsch, 1984, sect. 2.2):

$$\mathbf{A}^T\mathbf{D}\mathbf{A} = \mathbf{0}\,. \tag{3.19}$$

It follows from (3.18):

$$\mathbf{p}^T\mathbf{s} = \mathrm{tr}(\mathbf{D}\mathbf{C}_{ll})\,. \tag{3.20}$$

Comparison of coefficients yields with (3.1) and $\mathbf{p} = (p_1, p_2, \ldots, p_k)^T$:

$$p_i = \mathrm{tr}(\mathbf{D}\mathbf{V}_i) \qquad \forall\, i = 1\ldots k\,. \tag{3.21}$$

IQUE – Invariant Quadratic Unbiased Estimator Invariance of the estimates with respect to the approximate values $\mathbf{X}_0$ (cf. eq. (2.1)) is achieved by postulating (Welsch, 1984, sect. 2.3):

$$\mathbf{DA} = \mathbf{0} \; . \tag{3.22}$$

This is a more rigorous postulate than (3.19), since $\mathbf{DA} = \mathbf{0} \Rightarrow \mathbf{A}^T\mathbf{DA} = \mathbf{0}$.

BIQUE – Best Invariant Quadratic Unbiased Estimator Applying the universal formula for the variance of a quadratic form (Koch, 1997, eq. (271.2)), the variance of $\mathbf{l}^T\mathbf{Dl}$ is in consideration of (3.22):

$$\mathrm{D}\{\mathbf{l}^T\mathbf{Dl}\} = 2\,\mathrm{tr}(\mathbf{DC}_{ll}\mathbf{DC}_{ll}) \; . \tag{3.23}$$

To obtain the "best" estimates, i. e. estimates of minimum variance, (3.23) has to be minimised with respect to the unknown quantities, more precisely to $\mathbf{D}$. However, it must be taken into account that the formula contains the a priori unknown variance components s_i within $\mathbf{C}_{ll}$. To remedy this inconvenience, the iterative approach (3.11) is applied (cf. sect. 3.4), assuming the s_i to be 1 and preserving the model's consistency by introducing factors a_i to scale the matrix coefficients appropriately.

With $s_i = 1 \;\forall\, i = 1 \ldots k$, $\mathbf{C}_{ll} = \mathbf{Q}$ holds, and (3.23) is minimised by Lagrange's method, conditioned by (3.21) and (3.22) (Koch, 1997, ch. 362):

$$2\,\mathrm{tr}(\mathbf{DQDQ}) - 4\,\mathrm{tr}(\mathbf{DA}\boldsymbol{\Lambda}^T) - 4\sum_{i=1}^{k}\lambda_i\big(\,\mathrm{tr}(\mathbf{DV}_i) - p_i\big) \longrightarrow \min_{\mathbf{D}} \; . \tag{3.24}$$

The $n \times u$ matrix $\boldsymbol{\Lambda}$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_k)^T$ contain the Lagrangean multipliers. Solving (3.24), the system of equations for the BIQUE of variance components can be derived (as it is carried out in Koch (1997, pp. 249-252)). The result is (3.5) where $\mathbf{V}_0 = \mathbf{0}$.

Generally speaking, (3.5) is solely a local BIQUE, for it depends on the approximate values a_i. However, by means of numerical iteration, it becomes a uniform BIQUE.

3.4.2 MINQUE

The *Minimum Norm (Invariant) Quadratic Unbiased Estimator* (MINQUE) was initially derived by C. R. Rao in 1970. Brief descriptions can be found in Rao (1973, p. 302) and Rao and Kleffe (1988, p. 87). An extensive review with reference to geodetic applications is given by Persson (1980). This section will reproduce merely the basic idea though.

As the name implies, MINQUE is a quadratic unbiased estimator. Whether the postulate of invariance is comprised as well, depends on the specific definition. Additionally, the Euclidean norm of a certain matrix has to be minimised, being equivalent to minimising $\mathrm{tr}(\mathbf{DQDQ})$ (Persson, 1980, sect. 2.4).

Assuming normally distributed observations, the invariant MINQUE is identical to the BIQUE (Koch, 1978). (3.20) yields unbiasedness, invariance is achieved by (3.22), and minimising the term $\mathrm{tr}(\mathbf{DQDQ})$ is equivalent to minimising (3.23).

3.4.3 Maximum Likelihood Approach

The idea of *Maximum Likelihood* (ML) estimation is to maximise the probability of occurrence of the observations with respect to the parameters sought for. Practically, one has to maximise the value of a Likelihood function $\mathfrak{L}$, which means in the case of normally distributed observations:

$$\mathfrak{L}(\mathbf{l};\mathbf{x},\mathbf{s}) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det \mathbf{C}_{ll}}}\, e^{-\frac{1}{2}\mathbf{v}^T\mathbf{P}\mathbf{v}} \longrightarrow \max \ . \tag{3.25}$$

In the stochastic model of unit weight (2.14), maximisation has to be carried out with respect to $\mathbf{x}$ and $\mathbf{s} = \sigma_0^2$ (Koch, 1997, ch. 324). In the model with more than one variance component (3.1), the maximum with respect to $\mathbf{x}$ and $\mathbf{s} = (s_i)$ is sought for.

K. Kubik was the first to apply the ML-approach to the estimation of variance components in 1967, still restricting the stochastic model (3.1) to sets of uncorrelated observations ($\mathbf{Q}_i = \mathbf{I}\ \forall\, i = 1 \dots k$ in (3.4)). A conclusion of his work has been published in Kubik (1967), whereas a more detailed derivation can be found in Kubik (1970). Applicable computational algorithms are presented in Kubik (1967), Kubik (1970) and Ebner (1972).

Koch (1986) derived a ML-estimator for the general stochastic model (3.1) and obtained an equation system which is identical to the BIQUE (3.5). (Equivalence to the MINQUE follows.) Finally, Yu (1996) extended the approach to the most general model of adjustment theory, the condition adjustment with unknown parameters and constraints among the parameters.

3.4.4 Least Squares Estimation

In 1988, P. J. G. Teunissen proposed a least squares approach to variance component estimation (Teunissen and Amiri-Simkooei, 2007). Its principal idea is to transform the Gauß-Markov-Model (2.2) into the model of condition equations (cf. e. g. Koch, 1997, p. 239), using an $r \times n$ matrix $\mathbf{B}$ satisfying $\mathbf{BA} = \mathbf{0}$:

$$\mathbf{w} = \mathbf{B}\mathbf{l} \ . \tag{3.26}$$

$\mathbf{w}$ is a $r \times 1$ vector of misclosures with $\mathrm{E}\{\mathbf{w}\} = \mathbf{0}$, the covariance matrix of which is (cf. (3.1)):

$$\mathbf{C}_{ww} = \mathrm{E}\{\mathbf{w}\mathbf{w}^T\} = \mathbf{B}\mathbf{C}_{ll}\mathbf{B}^T = \mathbf{B}\mathbf{V}_0\mathbf{B}^T + \sum_{i=1}^{k} \mathbf{B}\mathbf{V}_i\mathbf{B}^T s_i \ . \tag{3.27}$$

Due to symmetry, (3.27) subsumes $\frac{r(r+1)}{2}$ individual equations of the form:

$$\mathrm{E}\{w_j w_l\} - \left[\mathbf{B}\mathbf{V}_0\mathbf{B}^T\right]_{jl} = \sum_{i=1}^{k} \left[\mathbf{B}\mathbf{V}_i\mathbf{B}^T\right]_{jl} s_i \qquad j = 1 \dots r, l = j \dots r \ , \tag{3.28}$$

which can be interpreted as a standard adjustment problem in the Gauß-Markov-Model with $\frac{r(r+1)}{2}$ observations $\left(\mathrm{E}\{w_j w_l\} - \left[\mathbf{B}\mathbf{V}_0\mathbf{B}^T\right]_{jl}\right)$ and k variance components s_i as unknown parameters to be estimated. Evaluating (3.28) with an appropriate weighting scheme by analogy to (2.8) yields (3.5).

3.5 Förstner's Method

Starting from the Helmert estimator (3.5), Förstner (1979a) derived a simplified estimator with some considerable advantages. The notion of "Förstner's method" has been given for instance by Persson (1980) and will be maintained within the scope of this study.

If the iteration process of the Helmert estimator converges, $\hat{\mathbf{s}} = \mathbf{1}$ will be finally obtained as estimate. Introducing this as an a priori assumption, (3.5) can be simplified by expanding the elements of the matrix $\mathbf{H}$ and interchanging the summation order:

$$\mathbf{q} = \mathbf{H} \cdot \hat{\mathbf{s}} \stackrel{!}{=} \mathbf{H} \cdot \mathbf{1} \tag{3.29}$$

With (3.6), (3.7) and (3.8) follows:

$$\mathbf{v}^T\mathbf{P}\mathbf{V}_i\mathbf{P}\mathbf{v} = q_i + h_{i0} \stackrel{!}{=} \sum_{j=1}^{k} h_{ij} \cdot 1 + h_{i0} = \sum_{j=0}^{k} \mathrm{tr}(\mathbf{W}\mathbf{V}_i\mathbf{W}\mathbf{V}_j) = \mathrm{tr}(\mathbf{W}\mathbf{V}_i) \; . \tag{3.30}$$

Consequently, the following relation holds:

$$1 = \frac{\mathbf{v}^T\mathbf{P}\mathbf{V}_i\mathbf{P}\mathbf{v}}{\mathrm{tr}(\mathbf{W}\mathbf{V}_i)} =: \hat{s}_{i,F} \qquad \forall\, i = 1 \ldots k \; . \tag{3.31}$$

$\hat{s}_{i,F}$ is Förstner's estimator. At the reproducing point, its estimates are identical to those of Helmert's method.

3.5.1 Estimation by Degree of Freedom

If the stochastic model is of block diagonal structure (3.14), Förstner's method can be simplified. By analogy to the simplification in section 3.4, (3.31) can be modified algebraically:

$$\hat{\sigma}^2_{i,D} = \hat{s}_{i,F} = \frac{\mathbf{v}_i^T\mathbf{P}_i\mathbf{v}_i}{n_i - \mathrm{tr}\left(\mathbf{N}^{-1}\mathbf{A}_i^T\mathbf{P}_i\mathbf{A}_i\right)} \qquad \forall\, i = 1 \ldots k \; . \tag{3.32}$$

This equation bears a striking resemblance to the estimator for the variance of unit weight (2.20). Translating (2.20) into (3.32), the numerator persists except for its decomposition by the k sets of observations, and the denominator can be identified by the redundancy number of the respective set. The latter will be shown by adapting the definition of the redundancy number (2.22) to the current stochastic model (3.14):

$$\begin{aligned} r_{i,D} &= \mathrm{tr}\left([\mathbf{Q}_{vv}\mathbf{P}]_{ii}\right) \\ &= \mathrm{tr}\left(\left[(\mathbf{Q} - \mathbf{A}(\mathbf{A}^T\mathbf{P}\mathbf{A})^{-1}\mathbf{A}^T)\mathbf{P}\right]_{ii}\right) \\ &= \mathrm{tr}\left(\mathbf{Q}_i\mathbf{P}_i\right) - \mathrm{tr}\left(\mathbf{A}_i\mathbf{N}^{-1}\mathbf{A}_i^T\mathbf{P}\right) \\ &= n_i - \mathrm{tr}\left(\mathbf{N}^{-1}\mathbf{A}_i^T\mathbf{P}_i\mathbf{A}_i\right) \; , \end{aligned} \tag{3.33}$$

where $[\cdot]_{ii}$ denotes in this context the extraction of a block matrix on the main diagonal corresponding to the i^{th} set of observations.

As redundancy is equivalent to the number of degrees of freedom, this particular case of Förstner's method is also called the estimation by degree of freedom (e. g. Sillard, 1999, p. 150).

3.5.2 Comparison to Helmert's Estimator

Compared to the estimator of Helmert type, Förstner's method features some advantages as well as some inconveniences, being concisely specified in Förstner (1979b).

First of all, the evaluation of Helmert's equation system (3.5) is notably time-consuming. For each iteration, the matrix products $\mathbf{W}\mathbf{V}_i\mathbf{W}\mathbf{V}_j$ have to be computed. Förstner's algorithm

merely involves the products $\mathbf{W}\mathbf{V}_i$. For the particular case of the estimation by degree of freedom (section 3.5.1), not even all elements of the matrix $\mathbf{W}$ must be known; only blocks on the main diagonal are relevant. Generally speaking, Förstner's method computes its results in a fraction of the time required by Helmert's estimator.

Although Förstner's estimates are identical to Helmert's in the case of convergence, there is yet no proof that convergence is achieved on the whole. Requirements for (fast) convergence are to some extent independent groups of observations and good approximate values a_i (FÖRSTNER, 1979b). In contrast to the simpler computation of Förstner's estimator within each step, its convergence speed is lower (KOCH, 1997, ch. 363), referring to the number of necessary iterations. In most practical applications, the simplicity of the computation outweighs the slowness of the convergence though.

Another inconvenience of Förstner's method is the fact that it does not provide a way to derive an estimator for the variance $\mathrm{D}\{\hat{\mathbf{s}}\}$ of the estimates (FÖRSTNER, 1979b).

For the sake of completeness it should be mentioned that Helmert's estimator may produce negative estimates, which are inadmissible for variances. However, this can normally be prevented by the choice of appropriate approximate values a_i (KOCH, 1978).

3.6 Approximate Estimators

The estimators discussed so far require a considerable amount of computing time, which can be fatal for adjustment problems of high dimensions. For this reason, some approaches have been proposed to simplify the estimation process by neglecting the postulates of statistical optimality and trying to approximate the exact estimates of Helmert type. Their use being restricted to the stochastic model of disjunctive observation groups (3.14), the starting point is always the estimation by degree of freedom (3.32).

The numerator of (3.32) consists of an n_i-dimensional quadratic form that can easily be computed. Since the number n_i of observations of a particular set is known, the matrix product $\mathbf{N}^{-1}\mathbf{A}_i^T\mathbf{P}_i\mathbf{A}_i$ is the most time-consuming element in the computation of (3.32). Hence, it is the objective of the estimators listed below to replace this term by a simpler one. Or in other words: An approximation for the redundancy number $r_{i,D}$ (see (3.33)) is sought for.

3.6.1 Helmert's Simple Estimator

A very elementary approach is to neglect the term $\mathbf{N}^{-1}\mathbf{A}_i^T\mathbf{P}_i\mathbf{A}_i$ completely by setting $r_{i,H} = n_i$. HELMERT (1907, p. 358) did so when he proposed the following estimator (which has been reviewed by WELSCH (1978) as a highly simplified alternative to his more complex estimator mentioned in section 3.3:

$$\hat{\sigma}_{i,H}^2 = \frac{\mathbf{v}_i^T\mathbf{P}_i\mathbf{v}_i}{r_{i,H}} = \frac{\mathbf{v}_i^T\mathbf{P}_i\mathbf{v}_i}{n_i} \qquad \forall\, i = 1 \ldots k\,. \tag{3.34}$$

According to simulations by WELSCH (1978), $\hat{\sigma}_{i,H}^2$ features significant biases with respect to $\hat{\sigma}_i^2$. It may serve as a rough indication at the most.

3.6.2 Kubik's Modifications

As KUBIK (1967, sect. 6) observes correctly, Helmert's simple estimator (3.34) is biased. Thus, he proposes to correct the bias simply by subtracting the overall number of unknowns u, setting

$r_{i,K} = n_i - u$. As a comparison to the estimator (3.32) implies, this correction does not yield a rigorous elimination of the estimator's bias:

$$\hat{\sigma}^2_{i,K} = \frac{\mathbf{v}_i^T \mathbf{P}_i \mathbf{v}_i}{r_{i,K}} = \frac{\mathbf{v}_i^T \mathbf{P}_i \mathbf{v}_i}{n_i - u} \qquad \forall\, i = 1 \ldots k\ . \tag{3.35}$$

3.6.3 Classical Estimator

In addition to the estimation by degree of freedom (section 3.5.1), CATREF software disposes of another, less complex estimator that is called the "classical" estimator (Altamimi et al., 2004a, eq. (13)):

$$\hat{\sigma}^2_{i,C} = \frac{\mathbf{v}_i^T \mathbf{P}_i \mathbf{v}_i}{r_{i,C}} = \frac{\mathbf{v}_i^T \mathbf{P}_i \mathbf{v}_i}{n_i - u_{i,C}} = \frac{\mathbf{v}_i^T \mathbf{P}_i \mathbf{v}_i}{n_i - \frac{n_i}{n} u} \qquad \forall\, i = 1 \ldots k\ . \tag{3.36}$$

Following the definition of redundancy in the model of unit weight (2.21), a heuristic motivation for $\hat{\sigma}^2_{i,C}$ can be given. The overall redundancy is decomposed into redundancy numbers. These numbers could be further on decomposed in the following way:

$$r = \sum_{i=1}^{k} r_{i,C} = \sum_{i=1}^{k} (n_i - u_{i,C})\ . \tag{3.37}$$

As it is not feasible to decompose the number of unknowns with respect to the sets of observations i disjointly, a way must be found to define the components $u_{i,C}$ appropriately. The approach pursued in CATREF is to distribute the overall number of unknowns among the sets i, weighted by the corresponding share of observations:

$$u_{i,C} = \frac{n_i}{n}\, u\ . \tag{3.38}$$

Due to $\sum_i n_i = n$ follows $\sum_i u_{i,C} = u$, which does not hold for Kubik's estimator (3.35) where $\sum_i u_{i,K} = k \cdot u \neq u$.

3.6.4 Persson's Estimator

Persson (1980, sect. 9.3) proposes a different approach. Introducing the number of unknowns $u_{i,P}$ which are directly concerned by the observation equations of set i, he decomposes the redundancy in the following way:

$$r_{i,P} = n_i - u_{i,P} + w_i \cdot \Delta r \tag{3.39}$$

$$w_i = \frac{\frac{n_i}{a_i}}{\sum_{i=1}^{k} \frac{n_i}{a_i}} \tag{3.40}$$

$$\Delta r = \sum_{i=1}^{k} (n_i - u_{i,P}) - r\ . \tag{3.41}$$

Due to $\sum_i u_{i,P} > u$, Persson's redundancy numbers would be significantly biased when calculated by $n_i - u_{i,P}$; their sum would not be equal to the overall redundancy. Therefore the bias Δr is

evaluated in (3.41) to be distributed among the sets of observations by means of the weighting scheme (3.40). Denoting the weight factors by w_i, the estimator itself is given by:

$$\hat{\sigma}^2_{i,P} = \frac{\mathbf{v}_i^T \mathbf{P}_i \mathbf{v}_i}{r_{i,P}} = \frac{\mathbf{v}_i^T \mathbf{P}_i \mathbf{v}_i}{n_i - u_{i,P} + w_i \cdot \Delta r} \qquad \forall\, i = 1 \dots k\ . \tag{3.42}$$

As the weight factors are dependent on the approximate values a_i, an iteration is required, updating $(a_i)_{\nu+1} = (\hat{\sigma}^2_{i,P})_\nu \ \forall\, i = 1 \dots k$ after each step ν.

Persson gives an example for application of his estimator on a geodetic network of distances and horizontal directions. He finds that the differences between these estimates and the estimates of Helmert type (there: MINQUE) never exceed 2 % at the reproducing point. Hence, considering the minor complexity of (3.42), this result is remarkable.

4 Terrestrial Reference Systems and Frames

4.1 Definitions

The notion of reference systems has its roots in classical geodetic survey. To make use of observations like directions and distances, they have to be parametrised by means of a coordinate system, the definition of which depends on the particular application. Dealing with local height networks, only one dimension is taken into account, whereas horizontal control networks necessitate the definition of a two-dimensional reference plane. Three-dimensional reference systems are considered for industrial surveying as well as in the context of space geodetic techniques like GPS. In spite of the more general meaning, this study will always refer to three-dimensional Cartesian coordinate systems with orthonormal base vectors when reference systems are concerned, restricting all considerations to the purely geometrical definition. Physical effects will not be taken into account.

Precisely, a *reference system* defines the complete conceptual definition of how a coordinate system is formed (Seeber, 2003, ch. 2.1.2), meaning principally the determination of its origin, the orientation of its axes as well as its scale. A *conventional reference system* comprises the explicit definition of all models, numerical constants and algorithms (Seeber, 2003, ch. 2.1.2).

A *terrestrial reference system (TRS)* is a spatial reference system co-rotating with the earth in its diurnal motion in space (McCarthy and Petit, 2003). Ideally, TRS satisfy the following conventions:

- The origin is close to the earth's centre of mass.
- The z-axis approximates the rotation axis of the earth, whereas the x-axis is in the meridian plane of Greenwich. The y-axis completes the triad to a right-handed system.
- The scale unit is (close to) the SI-metre.

For practical application, the sole definition of a reference system is of little use. To be able to express the observations in the system, the latter must be made accessible. This is achieved by means of a *reference frame*, which is the practical realisation of a reference system through observations (Seeber, 2003). Hence, a *terrestrial reference frame (TRF)* is the realisation of a TRS. It comprises a set of geometrically exactly defined points on or near the earth's surface with known coordinates.

However, the application of this definition presumes those points attached to a rigid earth so that the inner geometry of the frame would be time-invariant. This may be sufficiently assured for regional frames but not for those of global extent, since the earth's shape is deformed continuously by geodynamical processes like earth tides and plate tectonics (Heck, 2003). Taking this into account, TRFs can be defined by introducing linear movements of the points. These represent reality in a more adequate manner, although they cannot be more than a simple approximation of the complex actual movements.

The TRFs discussed within the scope of this study consist always of geometrically exactly defined points in 3D space with coordinates related to a certain epoch and their first time derivatives, expressed in a TRS. The points will also be referred to as *stations*. Their velocities being constant, their positions are always a function of time.

A place where several stations are gathered is called a *site*. At *co-location sites*, more than one observation technique is applied.

Lastly, *celestial reference systems (CRS)* will be briefly introduced. In contrast to TRS, they are not co-rotating with the earth but rather related to extraterrestrial objects. Their realisations are called *celestial reference frames (CRF)*.

It can be distinguished between kinematic and dynamic celestial reference systems and frames (Kovalevski and Mueller, 1989). *Kinematic CRS* are related to very distant objects (e. g. quasars). As their apparent motion is marginal or zero, the orientation of a kinematic CRF is rather stable in time and practically independent of earth motions. On the other hand, *dynamic CRS* are defined by a number of celestial bodies, the motions of which are described by a solution of a system of differential equations described in a fixed triad (Kovalevski and Mueller, 1989).

4.2 Representation of a TRF

There are two ways to represent a TRF (Altamimi et al., 2004b).

- *Explicit representation:* The TRF is given by the coordinates (x, y, z) and velocities $(\dot{x}, \dot{y}, \dot{z})$ of all comprised stations.
- *Implicit representation:* The TRF is given by the transformation parameters with respect to another TRF. In the following, it will be illustrated how to perform transformations between TRFs.

Transformation between TRFs

Given two sets of N points $\mathfrak{X}_1 = \{\boldsymbol{\xi}_1^1, \boldsymbol{\xi}_1^2, \dots \boldsymbol{\xi}_1^N\}$ and $\mathfrak{X}_2 = \{\boldsymbol{\xi}_2^1, \boldsymbol{\xi}_2^2, \dots \boldsymbol{\xi}_2^N\}$ with coordinates in the 3D-Euclidean space:

$$\boldsymbol{\xi}_1^i = \begin{pmatrix} x_1^i \\ y_1^i \\ z_1^i \end{pmatrix} \quad \text{and} \quad \boldsymbol{\xi}_2^i = \begin{pmatrix} x_2^i \\ y_2^i \\ z_2^i \end{pmatrix} \quad \forall\, i = 1 \dots N \,,$$

a similarity transforming $\mathfrak{X}_1$ into $\mathfrak{X}_2$ is:

$$\boldsymbol{\xi}_2^i = \mathbf{t} + m \cdot \mathbf{R}^* \, \boldsymbol{\xi}_1^i \qquad \forall\, i = 1 \dots N \tag{4.1}$$

with a translation vector $\mathbf{t} = (t^x, t^y, t^z)^T$, a scale factor m and an orthonormal 3×3 rotation matrix $\mathbf{R}^*$. For minor rotations and scale factors close to 1, the equation can be linearised:

$$\boldsymbol{\xi}_2^i = \boldsymbol{\xi}_1^i + \mathbf{t} + d \cdot \boldsymbol{\xi}_1^i + \mathbf{R}\, \boldsymbol{\xi}_1^i \qquad \forall\, i = 1 \dots N \tag{4.2}$$

with $d = m - 1$ and $\mathbf{R} = \mathbf{R}^* - \mathbf{I}$. The $d \cdot \mathbf{R}\, \boldsymbol{\xi}_1^i$-term is neglected, because it is a product of two small quantities. $\mathbf{R}$ comprises the differential rotation angles r^x, r^y and r^z about the respective axes:

$$\mathbf{R} = \begin{pmatrix} 0 & -r^z & r^y \\ r^z & 0 & -r^x \\ -r^y & r^x & 0 \end{pmatrix} \,.$$

This transformation holds when the coordinates are time-invariant. However, as station coordinates vary in time, the modelling of a linear movement is reasonable (cf. section 4.1). For this purpose, (4.2) is differentiated with respect to time:

$$\dot{\boldsymbol{\xi}}_2^i = \dot{\boldsymbol{\xi}}_1^i + \dot{\mathbf{t}} + \dot{d}\, \boldsymbol{\xi}_1^i + \underbrace{d \cdot \dot{\boldsymbol{\xi}}_1^i}_{\approx \mathbf{0}} + \dot{\mathbf{R}}\, \boldsymbol{\xi}_1^i + \underbrace{\mathbf{R}\, \dot{\boldsymbol{\xi}}_1^i}_{\approx \mathbf{0}} \qquad \forall\, i = 1 \dots N \,. \tag{4.3}$$

Terms of second order are neglected. Considering TRFs, d and $\mathbf{R}$ are of 10^{-5} level, and the elements of $\dot{\boldsymbol{\xi}}_1^i$ are about $10\,cm/a$, which is why their products $d{\cdot}\dot{\boldsymbol{\xi}}_1^i$ and $\mathbf{R}\,\dot{\boldsymbol{\xi}}_1^i$ have no significant impact on the resulting coordinates (Altamimi et al., 2004b).

Combining (4.2) and (4.3), the transformation between two TRFs can be formulated in one equation. With

$$\begin{aligned}\mathbf{X}_1 &= (x_1^1, y_1^1, z_1^1, \dot{x}_1^1, \dot{y}_1^1, \dot{z}_1^1, x_1^2, \ldots, \dot{z}_1^N)^T \\ \mathbf{X}_2 &= (x_2^1, y_2^1, z_2^1, \dot{x}_2^1, \dot{y}_2^1, \dot{z}_2^1, x_2^2, \ldots, \dot{z}_2^N)^T \\ \boldsymbol{\theta} &= (t^x, t^y, t^z, d, r^x, r^y, r^z, \dot{t}^x, \dot{t}^y, \dot{t}^z, \dot{d}, \dot{r}^x, \dot{r}^y, \dot{r}^z)^T\end{aligned}$$

the transformation from $\mathbf{X}_1$ to $\mathbf{X}_2$ is given by:

$$\mathbf{X}_2 = \mathbf{X}_1 + \mathbf{G}\,\boldsymbol{\theta}\ . \tag{4.4}$$

$\boldsymbol{\theta}$ is the vector of the 14 transformation parameters, and $\mathbf{G}$ is a $6N \times 14$ matrix of the following kind:

$$\mathbf{G} = \begin{pmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & 0 & x_0^i & 0 & z_0^i & -y_0^i & & & & & & & \\
0 & 1 & 0 & y_0^i & -z_0^i & 0 & x_0^i & & & & \mathbf{0} & & & \\
0 & 0 & 1 & z_0^i & y_0^i & -x_0^i & 0 & & & & & & & \\
 & & & & & & & 1 & 0 & 0 & x_0^i & 0 & z_0^i & -y_0^i \\
 & & & \approx \mathbf{0} & & & & 0 & 1 & 0 & y_0^i & -z_0^i & 0 & x_0^i \\
 & & & & & & & 0 & 0 & 1 & z_0^i & y_0^i & -x_0^i & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}\ . \tag{4.5}$$

To each station $i = 1 \ldots N$ contained in $\mathbf{X}_1$ and $\mathbf{X}_2$ correspond six lines in $\mathbf{G}$, reflecting the relations (4.2) and (4.3). x_0^i, y_0^i and z_0^i are approximate positions of station i.

4.3 Realisation of a TRF

There are two ways to realise a TRF (Altamimi et al., 2004b):

1. *Realisation by Space Geodetic Techniques.* As coordinates themselves are not observable directly, the establishment of a TRF is realised by observing them indirectly. Space geodetic techniques (cf. section 4.3.1) provide geometrical relations between the particular stations from which coordinate (and velocity) differences can be derived. Thus, a so-called *solution* is obtained. As it consists of coordinates (and maybe velocities) of the concerned stations, it defines an underlying TRF explicitly. *Analysis centres* collect observations from stations all over the world to establish solutions.

2. *Realisation by Combination.* Combining several particular TRFs yields a combined frame of superior quality that benefits from the particular features of the initial frames like observation period, network configuration or applied technique(s). The combination strategy will be illustrated in detail in section 4.4.

4.3.1 Space Geodetic Techniques

Nowadays, there are mainly five techniques to be used for the realisation of TRFs. E. g. Schön et al. (1999) classify them into kinematic and dynamic methods.

Kinematic Methods

- *Very Long Baseline Interferometry (VLBI).* A continuous signal from an extragalactic radio source is received by a set of radio telescopes. It is recorded collectively with a time tag of an atomic frequency standard. Correlating the records from all involved telescopes makes it feasible to determine the time delays of the signal's arrival at the receiving stations. As the sources are far away, it can be assumed that the signal reaches the earth as a parallel wave-front. Thus, considering the speed of light and maintaining observations of several radio sources by a network of telescopes, baseline vectors between the receiving stations can be derived from the delays.

Dynamic Methods

- *Satellite Laser Ranging (SLR).* A laser impulse, emitted from a ground station, is reflected by a satellite's reflector and re-detected by the ground station. The signal propagation delay corresponds to the distance from ground station to satellite. Observing satellites from several stations enables the estimation of their orbit parameters and yields consequently geometrical relations between the ground stations. From these, coordinate differences can be derived.

- *Lunar Laser Ranging (LLR).* By analogy with SLR, a laser impulse from a terrestrial station is sent to reflector assemblies on the moon. Returning photons of the original pulse are detected, and the delay is registered.

- *Global Positioning System (GPS).* The US-American military forces maintain 24 or more satellites in earth orbit that permanently broadcast microwave signals. These allow a receiver to determine its position relative to the satellites' reference frame. Thus, coordinate differences of the observing stations can be derived at a high level of precision.

- *Doppler Orbitography and Radiopositioning Integrated by Satellite (DORIS).* Radio signals emitted by ground beacons on the earth's surface are detected by certain satellites to evaluate the Doppler frequency shift, which is due to the relative movement. Integrating the derived relative velocities enables the derivation of distances between beacons and satellites. In contrast to the other techniques, DORIS ground stations are distributed evenly on the globe; there is no bias to the northern hemisphere. Hence, the resulting reference frames are notably stable due to the favourable configuration.

4.3.2 Datum Definition

To express a solution in a coordinate frame, a TRF must be defined. As the observation techniques provide only geometrical relations between the stations, no absolute coordinates can be derived a priori. A datum definition is necessary.

By analogy with a geodetic network, the notion of degrees of freedom exists as well in the context of TRFs. The observations by space geodetic techniques are capable of determining the shape of the frame, whereas its absolute positioning is undefined or partly defined, respectively. There are up to 14 degrees of freedom to dispose of. In terms of an implicit representation of a TRF (cf. section 4.2) they correspond to the transformation parameters $\boldsymbol{\theta}$ in (4.4).

Origin (t^x, t^y, t^z) The origin corresponds to a translation of the coordinate frame. Dynamic techniques have access to the earth's centre of mass (CoM), which is situated in a focal point of the (artificial or natural) satellites' orbital ellipses and defined as origin. For VLBI solutions, the origin is arbitrarily fixed as the CoM is inaccessible.

The earth not being a rigid body, its CoM varies with time. As most of the analysis centres do not explicitly include this effect in their models (Altamimi et al., 2002a), the CoM detected by satellites is always an average position over the whole observation period (Altamimi et al., 2004b).

Scale (d) The scale is defined by conventions, depending on physical parameters like the speed of light and the geo-gravitational constant. Furthermore, some relativistic effects have to be considered (McCarthy and Petit, 2003). Hence, all techniques are sensitive to the scale.

Orientation (r^x, r^y, r^z) The orientation of a reference frame is defined arbitrarily or conventionally, respectively. VLBI is the only technique capable to realise a conventional definition, for it has access to kinematic CRFs, the orientation of which does not vary significantly. However, it must be considered that even the orientation of a CRF has once been chosen arbitrarily.

Time evolution $(\dot{t}^x, \dot{t}^y, \dot{t}^z, \dot{d}, \dot{r}^x, \dot{r}^y, \dot{r}^z)$ The rates of the seven parameters represent another 7 degrees of freedom. As the scale is defined conventionally by physical parameters (hence observable by all techniques) and the origin is accessible by dynamic techniques, the rates of translation and scale parameters can be completely defined by observations. (For VLBI solutions the translation rate must be fixed arbitrarily.)

Since the orientation is not observable by any technique, there is no "natural" way to establish a datum for its rate. As an arbitrary definition would be rather inconvenient, some generally accepted conventions have been found for this purpose. A most common one is the use of the Tisserand system of mean axes (Altamimi et al., 2002a). To define the translation and orientation rates, one can postulate the kinetic energy to be minimal, integrated for feasibility reasons over the earth's crust. This leads to null angular momentums:

$$\iint_{\text{crust}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \, dm = \mathbf{0} \, . \tag{4.6}$$

dm is the respective infinitesimal mass element. (4.6) is known as the condition of no net rotation (NNR).

In spite of the clear definition, the approach is not easily practicable. For several reasons it is not in accordance with physical reality. Furthermore, the integration requires knowledge about the mass distribution and a continuous velocity field. Whereas the mass distribution is based on assumptions, velocities are only known for discrete points on the earth's crust or from geophysical models (e. g. NUVEL-1A, see DeMets et al., 1994).

4.3.3 Constraints Handling

Geometric observables $\mathbf{L}$ gained by space geodetic techniques establish relations $\mathbf{f}(\mathbf{X})$ between station coordinates and velocities $\mathbf{X} = (x^1, y^1, z^1, \dot{x}^1, \dot{y}^1, \dot{z}^1, x^2, \ldots, \dot{z}^N)^T$. They are linearised

as specified in section 2 at $\mathbf{X}_0 = (x_0^1, y_0^1, z_0^1, \dot{x}_0^1, \dot{y}_0^1, \dot{z}_0^1, x_0^2, \ldots, \dot{z}_0^N)^T$. $\mathbf{X}_0$ are approximate values. Application of the Gauß-Markov model (2.2) yields the system of normal equations (2.7).

Due to the configuration of the problem, the normal equation matrix $\mathbf{N}$ is singular and hence not invertible. Its rank deficiency corresponds to the datum defect, the dimension of which depends on the involved techniques. To remedy this inconvenience, the following approaches could be thought of:

1. *Fixing of Coordinates.* A certain number of coordinates are fixed to a given value (cf. e. g. SCHÖN ET AL., 1999, sect. 2.2.1). Thus, those coordinates are not estimable any more, and all other coordinates are estimated with respect to the fixed ones. Although this is the most obvious approach, its result always depends on the arbitrary choice of the coordinates to be fixed. Furthermore, the covariance matrix of the estimates is singular, because no stochastic information can be gained with respect to the fixed coordinates. Due to these inconveniences, the approach is no longer relevant for TRFs nowadays.

   The number of fixed coordinates must equal or exceed the datum defect. In case of equality, the minimum amount of supplemental information is introduced to complete the rank deficiency of the normal equation matrix. Whereas the geometry of the resulting TRF is unique, its datum depends on the arbitrary choice of the fixed coordinates. If more coordinates are fixed than necessary, not only the datum, but also the geometry is affected.

2. *Stochastic Constraints.* Pseudo-observations $\mathbf{l}_c$ of some parameters from $\mathbf{x} = \mathbf{X} - \mathbf{X}_0$ are introduced with a certain variance:

   $$D\{\mathbf{l}_c\} = \mathbf{C}_{cc} \,. \tag{4.7}$$

   Commonly, the pseudo-observations are set to zero, which is equivalent to aligning the new frame to the approximate values $\mathbf{X}_0$ taken from an old frame. This is maintained for positions and velocities of a subset of stations. Dependent on the applied variances in (4.7), ALTAMIMI ET AL. (2002b) classify stochastic constraints into three categories:

   - *tight constraints*: $\sigma \leq 10^{-10}\, m$ for positions and m/yr for velocities
   - *removable constraints*: $\sigma \approx 10^{-5}\, m$ for positions and m/yr for velocities
   - *loose constraints*: $\sigma \geq 1\, m$ for positions and $10\, cm/yr$ for velocities

3. *Pseudo-Inverse.* The pseudo-inverse (cf. e. g. ILLNER, 1985; KOCH, 1997, ch. 153; SCHÖN ET AL., 1999, sect. 2.2.3) of the normal equation matrix is used to define the so-called "inner datum". It can be obtained by extending the functional model by some condition equations that complete the rank deficiency of $\mathbf{N}$ without introducing any supplemental information. Thus, the geometry defined by the observations is not distorted, and the resulting parameters have minimum variance. However, this method yields singular covariance matrices of the estimated coordinates so that they could not be used easily for subsequent combinations.

4. *Minimum Constraints.* Any set of constraints that introduces the minimum amount of information to complete the rank deficiency of the normal equation matrix is called minimum constraints. These may be applied by fixing a number of coordinates that is equal to the datum defect (1.), constraining an according number of coordinates stochastically (2.) or choosing constraints that define the inner datum and yield the pseudo-inverse (3.).

   Initially, the use of the pseudo-inverse appears to be the most favourable of all approaches, because it does not depend on any arbitrary specification. However, the covariance matrix of the estimated coordinates is singular. SILLARD AND BOUCHER (2001) propose an

approach that likewise does not depend on any arbitrary specification and yields a regular covariance matrix in the end, the norm of which is close to minimum: Instead of constraining coordinates, a transformation to a reference TRF is modelled, the parameters of which are constrained stochastically. The according solution is not completely unique, since variances (and covariances) for the constraints have to be specified.

When minimum constraints are addressed in the following, it is referred to the approach of Sillard and Boucher (2001).

5. *Internal (Intrinsic) Constraints* (Altamimi et al., 2007). When a TRF results from stacking a time series of individual frames, the transformation parameters $\boldsymbol{\theta}$ of the individual frames with respect to the referential frame $\mathbf{X}_0$ can be constrained in a way that the time series of each particular transformation parameter has zero mean and no trend.

In today's practice, basically the strategies 2 and 4 are pursued (Altamimi et al., 2004b). They will be described in detail:

Stochastic Constraints To apply stochastic constraints, the parameter vector $\mathbf{x}$ is partitioned as follows:

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \tag{4.8}$$

with $\mathbf{x}_1$ subvector containing coordinates/velocities that are not concerned by the constraints
$\mathbf{x}_2$ subvector containing coordinates/velocities to which pseudo-observations are applied.

Involving the pseudo-observations $\mathbf{l}_c$, the Gauß-Markov model (2.2) is modified:

$$\begin{pmatrix} \mathbf{l} \\ \mathbf{l}_c \end{pmatrix} + \begin{pmatrix} \mathbf{v} \\ \mathbf{v}_c \end{pmatrix} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}}_1 \\ \hat{\mathbf{x}}_2 \end{pmatrix} . \tag{4.9}$$

The original design matrix given by $\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_2 \end{pmatrix}$ is extended for the pseudo-observations, where $\mathbf{v}_c$ are the corresponding residuals. Least squares minimisation in accordance with (2.6)

$$\begin{pmatrix} \mathbf{v}^T & \mathbf{v}_c^T \end{pmatrix} \begin{pmatrix} \mathbf{C}_{ll} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{cc} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{v} \\ \mathbf{v}_c \end{pmatrix} \longrightarrow \min \tag{4.10}$$

yields the normal equation system:

$$\begin{pmatrix} \mathbf{A}_1^T \mathbf{C}_{ll}^{-1} \mathbf{A}_1 & \mathbf{A}_1^T \mathbf{C}_{ll}^{-1} \mathbf{A}_2 \\ \mathbf{A}_2^T \mathbf{C}_{ll}^{-1} \mathbf{A}_1 & \mathbf{A}_2^T \mathbf{C}_{ll}^{-1} \mathbf{A}_2 + \mathbf{C}_{cc}^{-1} \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}}_1 \\ \hat{\mathbf{x}}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{A}_1^T \mathbf{C}_{ll}^{-1} \mathbf{l} \\ \mathbf{A}_2^T \mathbf{C}_{ll}^{-1} \mathbf{l} + \mathbf{C}_{cc}^{-1} \mathbf{l}_c \end{pmatrix} . \tag{4.11}$$

If the constraints are chosen appropriately, the normal equation matrix is regular and identical with the inverse of the covariance matrix of the parameters $\mathbf{C}_{\hat{x}\hat{x}}^{cc}$ (cf. eq. (2.11)). Obviously, it is decomposable into a part corresponding to the observations $\mathbf{l}$ and one corresponding to the constraints $\mathbf{l}_c$:

$$(\mathbf{C}_{\hat{x}\hat{x}}^{cc})^{-1} = \mathbf{N} + \mathbf{N}_c = \begin{pmatrix} \mathbf{A}_1^T \mathbf{C}_{ll}^{-1} \mathbf{A}_1 & \mathbf{A}_1^T \mathbf{C}_{ll}^{-1} \mathbf{A}_2 \\ \mathbf{A}_2^T \mathbf{C}_{ll}^{-1} \mathbf{A}_1 & \mathbf{A}_2^T \mathbf{C}_{ll}^{-1} \mathbf{A}_2 \end{pmatrix} + \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{cc}^{-1} \end{pmatrix} . \tag{4.12}$$

The objective of a regular covariance matrix of the parameters is achieved by this approach. Nevertheless, the network shape may be distorted by the arbitrarily chosen constraints. The

distortion can be minimised by applying large variances to the constraints (→ loose constraints). However, this yields an unrealistically high level of variance for the estimates, so that their dispersion can only be interpreted relatively.

When combining TRFs (cf. section 4.4), a systematic distortion of the involved TRFs' inner geometry is desirable in no case. Therefore, $\mathbf{C}_{cc}$ ($\mathbf{N}_c$, respectively) is conserved to be able to "remove" the constraints' influence in preparation of a subsequent combination (hence the appellation "removable constraints" before). The removal from the covariance matrix can be performed easily in accordance with (4.12):

$$(\mathbf{C}^{unc}_{\hat{x}\hat{x}})^{-} = \mathbf{N} = (\mathbf{C}^{cc}_{\hat{x}\hat{x}})^{-1} - \mathbf{N}_c \; . \tag{4.13}$$

Minimum Constraints It has been pointed out that the application of stochastic constraints produces certain inconveniences: Introducing supplementary information by choosing additional constraints arbitrarily always influences the result systematically. The approach of minimum constraints by Sillard and Boucher (2001) consists in completing the rank deficiency of the normal equation matrix *without* introducing supplemental information.

Similar to free network adjustment, the datum is defined by aligning the inner solution to a set of approximate coordinates (often coordinates of a network of superior order). Analogously, a TRF given explicitly by $\mathbf{X} = (x^1, y^1, z^1, \dot{x}^1, \dot{y}^1, \dot{z}^1, x^2, \ldots, \dot{z}^N)^T$ can be aligned to a datum-defining, referential TRF $\mathbf{X}_R = (x^1_R, y^1_R, z^1_R, \dot{x}^1_R, \dot{y}^1_R, \dot{z}^1_R, x^2_R, \ldots, \dot{z}^N_R)^T$.

Starting point of the derivation is the linear transformation from $\mathbf{X}_R$ to $\mathbf{X}$ (cf. (4.4)):

$$\mathbf{X} = \mathbf{X}_R + \mathbf{G}\,\boldsymbol{\theta} \; . \tag{4.14}$$

As the problem is overdetermined, inconsistency parameters $\mathbf{v}_x$ are introduced. Identifying the difference $\Delta\mathbf{X}_R = \mathbf{X} - \mathbf{X}_R$ as observations, an analogy to the Gauß-Markov model (2.2) appears with the transformation parameters $\boldsymbol{\theta}$ as unknowns:

$$\Delta\mathbf{X}_R + \mathbf{v}_x = \mathbf{G}\,\hat{\boldsymbol{\theta}} \; . \tag{4.15}$$

Minimising $\mathbf{v}_x^T\mathbf{v}_x$ yields their estimates (cf. eq. (2.8)):

$$\hat{\boldsymbol{\theta}} = (\mathbf{G}^T\mathbf{G})^{-1}\mathbf{G}^T\Delta\mathbf{X}_R = \mathbf{B}\cdot\Delta\mathbf{X}_R \tag{4.16}$$

with $\mathbf{B} = (\mathbf{G}^T\mathbf{G})^{-1}\mathbf{G}^T$. It seems obvious that the datums of $\mathbf{X}$ and $\mathbf{X}_R$ are identical if the transformation parameters' estimates $\hat{\boldsymbol{\theta}}$ vanish:

$$\hat{\boldsymbol{\theta}} = \mathbf{B}\cdot\Delta\mathbf{X}_R = \mathbf{0} \; . \tag{4.17}$$

To apply minimum constraints, these 14 equations or a subset of them, respectively, must hold. Depending on the observation technique, the rank deficiency of the normal equation matrix is generally less than 14. Thus, the matrix $\mathbf{G}$ is restricted to those columns that correspond to the datum defect of the TRF; otherwise the constraints would not be of minimum type.

Observations by dynamic techniques for instance, define an origin in a focal point of the satellites' orbital ellipses. The scale is defined by the speed of light, considering the impact of the atmospheric delay and other effects. Otherwise, no information is gained to determine the orientation parameters (and their rates) in a non-arbitrary way. Hence, $\mathbf{G}$ would consist of the 6 columns corresponding to the rotation angles and their rates (Altamimi et al., 2004b).

In practice, due to lack of available coordinates in the referential frame $\mathbf{X}_R$, the alignment can mostly be performed only with respect to a subset of stations, which both frames have in common. Thus, the matrix $\mathbf{G}$ is constituted accordingly.

The introduction of constraints into the Gauß-Markov model (2.2) proceeds by means of additional (pseudo-) observations $\mathbf{l}_{mc}$ and inconsistency parameters $\mathbf{v}_{mc}$:

$$\mathbf{l}_{mc} + \mathbf{v}_{mc} = \mathbf{B} \cdot \widehat{\Delta \mathbf{X}}_R \tag{4.18}$$

with $\mathbf{l}_{mc} = \mathbf{0}$ and $\mathrm{D}\{\mathbf{l}_{mc}\} = \mathbf{C}_{mc}$. $\mathbf{C}_{mc}$ is a diagonal matrix containing small variances for each of the transformation parameters concerned by the constraints (Altamimi et al., 2004b). Identifying finally the approximate values $\mathbf{X}_0$ in (2.1) and (4.5) by the coordinates (and velocities) of the referential frame $\mathbf{X}_R$ yields $\Delta \mathbf{X}_R \equiv \mathbf{x}$. The Gauß-Markov model (2.2) is extended:

$$\begin{pmatrix} \mathbf{l} \\ \mathbf{l}_{mc} \end{pmatrix} + \begin{pmatrix} \mathbf{v} \\ \mathbf{v}_{mc} \end{pmatrix} = \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix} \hat{\mathbf{x}} \; . \tag{4.19}$$

As no correlations are assumed between $\mathbf{l}$ and $\mathbf{l}_{mc}$, the normal equations are:

$$\left(\mathbf{N} + \mathbf{B}^T \mathbf{C}_{mc}^{-1} \mathbf{B}\right) \hat{\mathbf{x}} = \mathbf{n} \; . \tag{4.20}$$

The covariance matrix $\mathbf{C}_{\hat{x}\hat{x}}^{mc}$ of the estimated parameters is obtained by analogy to (2.11), assuming $\mathbf{P} = \mathbf{C}_{ll}^{-1}$:

$$\mathbf{C}_{\hat{x}\hat{x}}^{mc} = \left(\mathbf{N} + \mathbf{B}^T \mathbf{C}_{mc}^{-1} \mathbf{B}\right)^{-1} \; . \tag{4.21}$$

4.4 Combination of Terrestrial Reference Frames

Combining TRFs means introducing coordinates of two or more TRFs into an adjustment computation yielding a so-called combined TRF. The data from the initial TRFs are considered as observations. The combined TRF is a unique frame in which all observations can be expressed most appropriately.

The combination approach is necessitated by the amount of available observation data, which is too large to be handled within one single step of computation. The needs of computing time and memory would largely exceed the potential of today's machines. Thus, a hierarchical procedure of two or more steps seems more convenient. Firstly, TRFs are estimated from a constricted amount of input data referring to a limited observation period, a subset of stations or a certain observation technique. Then, the combined frame is derived from all TRFs resulting from the first step.

It is obvious that the combined frame is usually of superior quality, because it is based on a larger amount of observation data. Thus, its precision is enhanced with respect to the initial solutions. Furthermore, it takes advantage of the particular qualities of the input data. Either the spatial or temporal coverage is augmented, or the strong points of the combined techniques (e. g. sensitivity to datum definition parameters) are benefited from.

Strictly speaking, several solutions (cf. 4.3) may be represented in one initial TRF. However, the case of more than one solution defined in the same frame is more of theoretical interest. In practice, each particular solution defines its individual underlying reference frame.

4.4.1 The International Terrestrial Reference Frame

Considering the potential of combining TRFs, it suggests itself to establish a global frame of highest quality. Therefore, the *International Terrestrial Reference Frame* (ITRF) is being realised to serve as general reference for scientific applications.

First efforts to realise a TRF were made with the project MERIT (*Monitoring of Earth Rotation and Intercomparison of Techniques*) in 1978. It was launched by the *International Astronomical Union* (IAU) to benefit from the measures provided by space geodetic techniques that became more and more precise. The first global TRF combining data from multiple techniques was established in 1984 when the *Bureau International de l'Heure* (BIH) released its first *Terrestrial System* BTS84, followed by BTS85, BTS86 and BTS87 up to 1988 (Sillard, 1999; Altamimi et al., 2002a).

When the BIH was replaced by the *International Earth Rotation Service* (IERS) in 1988, the sequel of global reference frames was continued by the ITRF88. This first realisation of the corresponding *International Terrestrial Reference System* (ITRS) comprised the positions of 158 stations at 96 sites (Sillard, 1999). It was followed by ITRF89, ITRF90, ITRF91, ITRF92, ITRF93, ITRF94, ITRF96, ITRF97, ITRF2000 and ITRF2005. The latter was released in October 2006 and contains 608 stations located at 338 sites (Altamimi et al., 2007).

In the course of time, ITRF methodology has been refined significantly. As ITRF88 was still a reference frame of statical kind, (linear) velocities for the stations were introduced for the first time with ITRF91. Until ITRF93 (inclusively), correlations between observations had not been taken into account. ITRF96 was the first release where the covariance matrices of the individual solutions were scaled by means of variance component estimation – yielding a remarkable improvement in data quality for the combined frame (Sillard, 1999).

Until ITRF2000, the final combination step was based on various individual TRF solutions. With ITRF2005, the combination process has become more hierarchically structured. Input data are five homogeneous weekly or daily time series, one each for VLBI, SLR, GPS and two for DORIS (there are *two* independent DORIS-solutions due to logistical reasons). The time series are stacked to technique-specific long-term solutions in a first step, which are finally combined in a second step (together with local ties at co-location sites). Another novelty is the integrated estimation of earth orientation parameters with the ITRF.

With ITRF2005, the number of IERS combination centres has been augmented to three: Hitherto, combinations were performed exclusively by the IGN. Now, the *Deutsches Geodätisches Forschungsinstitut* (DGFI, Munich) and the *Natural Resources Canada* (NRCan, Ottawa) have joined to ensure competition and enable cross-validation (Altamimi et al., 2007).

4.4.2 Combination Model

The mathematical model for TRF combination, which is applied within the scope of this study, is based upon CATREF software (Altamimi et al., 2004a). Whereas CATREF is a sophisticated tool, only the very basics of its potentialities have been adopted to evaluate methods of variance component estimation.

The combination algorithm requires the following *input data:*

- k solutions $\mathbf{L}_i = (\dots x_i^p, y_i^p, z_i^p, \dot{x}_i^p, \dot{y}_i^p, \dot{z}_i^p, \dots)^T$ with referential epochs t_i^p, $p \in \mathfrak{P}_i$, and regular covariance matrices $\sigma_i^2 \mathbf{Q}_i = \sigma_i^2 \mathbf{P}_i^{-1} \; \forall \, i = 1 \dots k$, each of them defining an individual TRF. $\mathfrak{P}_i$ is the set of the indices of all points concerned by solution i.
- approximate values $\mathbf{X}_0 = (x_0^1, y_0^1, z_0^1, \dot{x}_0^1, \dot{y}_0^1, \dot{z}_0^1, x_0^2, \dots, \dot{z}_0^N)^T$ for all involved stations.
- For the application of minimum constraints to define the datum of the combined frame, a solution in a referential frame $\mathbf{X}_R = (\dots x_R^p, y_R^p, z_R^p, \dot{x}_R^p, \dot{y}_R^p, \dot{z}_R^p, \dots)^T, p \in \mathfrak{P}_R$ is needed.
- an epoch t_0 at which the combination is performed.

- k epochs t_i ($i = 1 \dots k$) at which the transformation parameters between each TRF i and the combined frame are to be estimated.
- if required, t tie-vectors $\mathbf{l}_{tie,i}$ of local surveys with covariance matrices $\mathbf{C}_{tie,i}$ $\forall\, i = 1 \dots t$ (cf. section 4.4.4).
- if required, particular variances $\sigma^2_{vel,i} \forall\, i = 1 \dots v$ for v sets of constraints on velocities (cf. section 4.4.5).

Output data:

- one solution $\mathbf{X} = (x^1, y^1, z^1, \dot{x}^1, \dot{y}^1, \dot{z}^1, x^2, \dots, \dot{z}^N)^T$ with its full covariance matrix, defining explicitly the combined frame. All estimated positions and velocities refer to t_0.
- transformation parameters $\boldsymbol{\theta}_i = (t^x_i, t^y_i, t^z_i, d_i, r^x_i, r^y_i, r^z_i, \dot{t}^x_i, \dot{t}^y_i, \dot{t}^z_i, \dot{d}_i, \dot{r}^x_i, \dot{r}^y_i, \dot{r}^z_i)^T$ $\forall\, i = 1 \dots k$ between each initial TRF i and the combined TRF, referring to t_i.

The linearised relation between each point p of an initial frame i and its coordinates in the combined frame is given by the following equations, derived from (4.2) and (4.3):

$$\begin{aligned}\begin{pmatrix} x^p_i \\ y^p_i \\ z^p_i \end{pmatrix} &= \begin{pmatrix} x^p \\ y^p \\ z^p \end{pmatrix} + (t^p_i - t_0) \begin{pmatrix} \dot{x}^p \\ \dot{y}^p \\ \dot{z}^p \end{pmatrix} + \mathbf{t}_i + d_i \begin{pmatrix} x^p \\ y^p \\ z^p \end{pmatrix} + \mathbf{R}_i \begin{pmatrix} x^p \\ y^p \\ z^p \end{pmatrix} \\ &\quad + (t^p_i - t_i) \left[\dot{\mathbf{t}}_i + \dot{d}_i \begin{pmatrix} x^p \\ y^p \\ z^p \end{pmatrix} + \dot{\mathbf{R}}_i \begin{pmatrix} x^p \\ y^p \\ z^p \end{pmatrix} \right] \qquad (4.22) \\ \begin{pmatrix} \dot{x}^p_i \\ \dot{y}^p_i \\ \dot{z}^p_i \end{pmatrix} &= \begin{pmatrix} \dot{x}^p \\ \dot{y}^p \\ \dot{z}^p \end{pmatrix} + \dot{\mathbf{t}}_i + \dot{d}_i \begin{pmatrix} x^p \\ y^p \\ z^p \end{pmatrix} + \dot{\mathbf{R}}_i \begin{pmatrix} x^p \\ y^p \\ z^p \end{pmatrix} \ . \qquad (4.23)\end{aligned}$$

In matrix notation, this yields the following observation equations:

$$\begin{pmatrix} x^p_i \\ y^p_i \\ z^p_i \\ \dot{x}^p_i \\ \dot{y}^p_i \\ \dot{z}^p_i \end{pmatrix} = \begin{pmatrix} \mathbf{I} & (t^p_i - t_0) \cdot \mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} x^p \\ y^p \\ z^p \\ \dot{x}^p \\ \dot{y}^p \\ \dot{z}^p \end{pmatrix} + \begin{pmatrix} \mathbf{A}^p & (t^p_i - t_i) \cdot \mathbf{A}^p \\ \mathbf{0} & \mathbf{A}^p \end{pmatrix} \boldsymbol{\theta}_i \qquad (4.24)$$

with

$$\mathbf{A}^p = \begin{pmatrix} 1 & 0 & 0 & x^p & 0 & z^p & -y^p \\ 0 & 1 & 0 & y^p & -z^p & 0 & x^p \\ 0 & 0 & 1 & z^p & y^p & -x^p & 0 \end{pmatrix} \ . \qquad (4.25)$$

The equations for a complete solution i are denoted as follows:

$$\mathbf{L}_i = \mathbf{A1}_i \, \mathbf{X} + \mathbf{A2}_i \, \boldsymbol{\theta}_i \qquad (4.26)$$

with

$$\mathbf{A1}_i = \begin{pmatrix} & \vdots & \vdots & \\ \dots & \mathbf{I} & (t^p_i - t_0) \cdot \mathbf{I} & \dots \\ \dots & \mathbf{0} & \mathbf{I} & \dots \\ & \vdots & \vdots & \end{pmatrix} \quad \text{and} \quad \mathbf{A2}_i = \begin{pmatrix} \vdots & \vdots \\ \mathbf{A}^p & (t^p_i - t_i) \cdot \mathbf{A}^p \\ \mathbf{0} & \mathbf{A}^p \\ \vdots & \vdots \end{pmatrix} \ .$$

Finally, the entity of n_{sol} observation equations referring to all initial solutions are established by means of:

$$\mathbf{L}_{sol} = \begin{pmatrix} \mathbf{L}_1 \\ \mathbf{L}_2 \\ \vdots \\ \mathbf{L}_k \end{pmatrix}, \quad \mathbf{A}_{sol} = \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_k \end{pmatrix} = \begin{pmatrix} \mathbf{A1}_1 & \mathbf{A2}_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{A1}_2 & \mathbf{0} & \mathbf{A2}_2 & & \mathbf{0} \\ \vdots & \vdots & & \ddots & \\ \mathbf{A1}_k & \mathbf{0} & \mathbf{0} & & \mathbf{A2}_k \end{pmatrix} \quad \text{and} \quad \boldsymbol{\vartheta} = \begin{pmatrix} \boldsymbol{\theta}_1 \\ \boldsymbol{\theta}_2 \\ \vdots \\ \boldsymbol{\theta}_k \end{pmatrix}$$

yielding:

$$\mathbf{L}_{sol} = \mathbf{A}_{sol} \begin{pmatrix} \mathbf{X} \\ \boldsymbol{\vartheta} \end{pmatrix} . \tag{4.27}$$

Taking into account that this relation is based on the linearised similarity transformation (4.2), it is inconsistent. Now, $\left(\mathbf{X}_0^T, \mathbf{0}^T\right)^T$ is chosen as reproducing point. As the transformation parameters $\boldsymbol{\vartheta}$ between similarly aligned TRFs are small quantities, their linearisation at $\mathbf{0}$ seems appropriate. Introducing $\Delta\mathbf{X} = \mathbf{X} - \mathbf{X}_0$ and the residuals $\mathbf{v}$, the Gauß-Markov model (2.2) is obtained by setting $\mathbf{A} = \mathbf{A}_{sol}$, $\mathbf{L} = \mathbf{L}_{sol}$ and $\mathbf{x} = \left(\Delta\mathbf{X}^T, \boldsymbol{\vartheta}^T\right)^T$:

$$\mathbf{l} + \mathbf{v} = \mathbf{l}_{sol} + \mathbf{v}_{sol} = \mathbf{L}_{sol} - \mathbf{A}_{sol} \begin{pmatrix} \mathbf{X}_0 \\ \mathbf{0} \end{pmatrix} + \mathbf{v}_{sol} = \mathbf{A}_{sol} \begin{pmatrix} \widehat{\Delta\mathbf{X}} \\ \hat{\boldsymbol{\vartheta}} \end{pmatrix} = \mathbf{A}\hat{\mathbf{x}} . \tag{4.28}$$

Here, $\mathbf{A}$ is made up by approximate values from $\mathbf{X}_0$.

The stochastic model is identical to (3.4) when $\sigma_i^2\mathbf{Q}_i$ are the covariance matrices of the individual solutions to be combined ($i = 1 \dots k$):

$$\mathbf{C}_{ll} = \mathbf{C}_{sol} = \begin{pmatrix} \sigma_1^2\mathbf{Q}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \sigma_2^2\mathbf{Q}_2 & & \mathbf{0} \\ \vdots & & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \sigma_k^2\mathbf{Q}_k \end{pmatrix} . \tag{4.29}$$

Taking this into account, the components of the normal equation system (2.7) are computed as follows:

$$\mathbf{N} = \begin{pmatrix} \sum_{i=1}^{k} \mathbf{A1}_i^T\mathbf{P}_i\mathbf{A1}_i & \mathbf{A1}_1^T\mathbf{P}_1\mathbf{A2}_1 & \mathbf{A1}_2^T\mathbf{P}_2\mathbf{A2}_2 & \dots & \mathbf{A1}_k^T\mathbf{P}_k\mathbf{A2}_k \\ \mathbf{A2}_1^T\mathbf{P}_1\mathbf{A1}_1 & \mathbf{A2}_1^T\mathbf{P}_1\mathbf{A2}_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{A2}_2^T\mathbf{P}_2\mathbf{A1}_2 & \mathbf{0} & \mathbf{A2}_2^T\mathbf{P}_2\mathbf{A2}_2 & & \mathbf{0} \\ \vdots & \vdots & & \ddots & \\ \mathbf{A2}_k^T\mathbf{P}_k\mathbf{A1}_k & \mathbf{0} & \mathbf{0} & & \mathbf{A2}_k^T\mathbf{P}_k\mathbf{A2}_k \end{pmatrix} , \tag{4.30}$$

$$\mathbf{n} = \begin{pmatrix} \sum_{i=1}^{k} \mathbf{A1}_i^T\mathbf{P}_i\mathbf{l}_i \\ \mathbf{A2}_1^T\mathbf{P}_1\mathbf{l}_1 \\ \mathbf{A2}_2^T\mathbf{P}_2\mathbf{l}_2 \\ \vdots \\ \mathbf{A2}_k^T\mathbf{P}_k\mathbf{l}_k \end{pmatrix} . \tag{4.31}$$

$\mathbf{l}_i$ is the $n_i \times 1$ component of $\mathbf{l}$ corresponding to the i^{th} initial solution.

4.4.3 Datum Definition

To carry out the combination, it is necessary that all initial solutions dispose of regular covariance matrices. Furthermore, it is important to avoid the distortion of the combined frame by constraints that have been applied before. Thus, it is recommended to change the datum of the initial solutions afterwards with the objective to have minimum constraints applied to each solution. ITRF combination, for instance, pursues the following strategy (Altamimi et al., 2002a):

- For solutions with *removable constraints*, the constraints are removed from the covariance matrix (cf. eq. (4.13)), and minimum constraints are added (cf. (4.21)):

$$(\mathbf{C}_{\hat{x}\hat{x}}^{mc})^{-1} = (\mathbf{C}_{\hat{x}\hat{x}}^{unc})^{-1} + \mathbf{B}^T \mathbf{C}_{mc}^{-1} \mathbf{B} \; . \tag{4.32}$$

  In a second step, the parameters themselves are modified with respect to the change of constraints:

$$\mathbf{X}^{mc} = \mathbf{C}_{\hat{x}\hat{x}}^{mc} \left((\mathbf{C}_{\hat{x}\hat{x}}^{cc})^{-1} \mathbf{X} - \mathbf{N}_c \mathbf{X}_0 \right) \; . \tag{4.33}$$

  Note that in this context $\mathbf{X}_0$ are the approximate values with which the constraints have initially been applied. They contributed to the constitution of the matrix $\mathbf{A}_2$ in (4.9).

- The parameters of *loosely constrained* solutions can be considered not to be distorted significantly by the constraints. However, the level of precision implied by the covariance matrix is highly unrealistic and to be corrected in the following way:

$$\mathbf{C}_{\hat{x}\hat{x}}^{mc} = \mathbf{C}_{\hat{x}\hat{x}}^{cc} - \mathbf{C}_{\hat{x}\hat{x}}^{cc} \mathbf{B}^T \left(\mathbf{B} \mathbf{C}_{\hat{x}\hat{x}}^{cc} \mathbf{B}^T + \mathbf{C}_{mc} \right)^{-1} \mathbf{B} \mathbf{C}_{\hat{x}\hat{x}}^{cc} \; . \tag{4.34}$$

- Solutions to which *minimum constraints* have already been applied are used as they are.

After that step, all individual solutions provide regular (or regularised) covariance matrices. However, $\mathbf{N}$ in (4.30) always has a rank deficiency of 14, which is due to the lack of datum definition for the combined frame. Basically, the approaches from section 4.3.3 can be applied as well to the combination process. Here, two of them will be discussed in detail: the fixing of parameters and the application of minimum constraints.

Fixing of Parameters By analogy to the fixing of coordinates in section 4.3.3, at least 14 of the differential transformation parameters from $\boldsymbol{\vartheta}$ are fixed to a given value, e. g. zero. These parameters cannot be selected arbitrarily though, since not every configuration is capable of remedying the rank deficiency of the normal equation matrix. Fixing e. g. 14 rotation parameters would indeed reduce the deficiency, but not completely, since the datum with respect to scale and translation of the combined frame would still be undefined.

The choice of the parameters to be fixed also depends on the desired datum. Fixing e. g. the translation parameters and its rates of all SLR-solutions to zero would equate the combined frame's origin with the averaged origin of the SLR-solutions. Although this choice may have a physical meaning, the datum definition by fixing parameters is, mathematically spoken, always arbitrary.

Without loss of generality it will now be assumed that all 14 transformation parameters of TRF k are fixed. This means that the part $\boldsymbol{\theta}_k$ is removed from the parameter vector, because the

concerned values are handled as known and do not need to be estimated any more. The design matrix $\mathbf{A}$ changes as well (cf. (4.27)):

$$\mathbf{A}_{fix} = \begin{pmatrix} \mathbf{A1}_1 & \mathbf{A2}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{A1}_2 & \mathbf{0} & \mathbf{A2}_2 & & \mathbf{0} \\ \vdots & \vdots & & \ddots & \\ \mathbf{A1}_{k-1} & \mathbf{0} & \mathbf{0} & & \mathbf{A2}_{k-1} \\ \mathbf{A1}_k & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix} . \tag{4.35}$$

The resulting normal equation matrix is regular, and the Gauß-Markov model from section 2 can be applied directly.

Minimum Constraints If minimum constraints are applied, the functional model is a synthesis of (4.28) and (4.19):

$$\mathbf{l} + \mathbf{v} = \begin{pmatrix} \mathbf{l}_{sol} \\ \mathbf{l}_{mc} \end{pmatrix} + \begin{pmatrix} \mathbf{v}_{sol} \\ \mathbf{v}_{mc} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{sol} \\ \mathbf{A}_{mc} \end{pmatrix} \hat{\mathbf{x}} = \mathbf{A}\hat{\mathbf{x}} \tag{4.36}$$

with $\mathbf{A}_{mc} = \begin{pmatrix} \mathbf{B} & \mathbf{0} \end{pmatrix}$ and $\mathbf{x}$ as defined in (4.28). The corresponding stochastic model is:

$$\mathbf{C}_{ll} = \begin{pmatrix} \mathbf{C}_{sol} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{mc} \end{pmatrix} . \tag{4.37}$$

The normal equations are (assuming $\mathbf{P} = \mathbf{C}_{ll}^{-1}$):

$$\begin{aligned} \mathbf{N}\hat{\mathbf{x}} = \mathbf{A}^{\mathbf{T}}\mathbf{P}\mathbf{A}\hat{\mathbf{x}} &= \left(\mathbf{A}_{sol}^T\mathbf{C}_{sol}^{-1}\mathbf{A}_{sol} + \mathbf{A}_{mc}^T\mathbf{C}_{mc}^{-1}\mathbf{A}_{mc}\right)\hat{\mathbf{x}} \\ &= \mathbf{A}_{sol}^T\mathbf{C}_{sol}^{-1}\mathbf{l}_{sol} + \mathbf{A}_{mc}^T\mathbf{C}_{mc}^{-1}\mathbf{l}_{mc} = \mathbf{A}^T\mathbf{P}\mathbf{l} = \mathbf{n} . \end{aligned} \tag{4.38}$$

4.4.4 Local Surveys

If the individual solutions to be combined originate from different observation techniques, identical points are required. As the observing instruments (radio telescopes, lasers, GPS antennas, ...) commonly dispose of different reference points, this requirement can only be fulfilled by means of local surveys: Baseline vectors between the stations of a site are observed and introduced into the combination. This can be realised by pursuing one of the two following strategies:

1. Introduction as observed baseline vector
2. Introduction as TRF

They will be outlined in the following:

Introduction as Observed Baseline Vector The tie vector itself is introduced as an additional set of observations. If the i^{th} tie links the stations p and q, the following observations are made:

$$\mathbf{l}_{tie,i} = \begin{pmatrix} x^q \\ y^q \\ z^q \end{pmatrix} - \begin{pmatrix} x^p \\ y^p \\ z^p \end{pmatrix} + \boldsymbol{\epsilon}_{tie,i} \tag{4.39}$$

with the corresponding errors $\boldsymbol{\epsilon}_{tie,i}$. If t tie vectors $\mathbf{l}_{tie} = (\mathbf{l}_{tie,1}^T, \mathbf{l}_{tie,2}^T, \ldots, \mathbf{l}_{tie,t}^T)^T$ are to be introduced, the functional model (4.28) is augmented by the following observation equations:

$$\mathbf{l}_{tie} + \mathbf{v}_{tie} = \mathbf{A}_{tie}\hat{\mathbf{x}} \tag{4.40}$$

$$\text{with} \quad \mathbf{A}_{tie} = \begin{pmatrix} \cdots \\ \mathbf{A}_{tie,i} \\ \cdots \end{pmatrix} = \begin{pmatrix} \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots \quad -\mathbf{I} \quad \cdots \quad \mathbf{I} \quad \cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{pmatrix} , \quad i = 1 \ldots t \, .$$

The corresponding stochastic model comprises 3×3 covariance matrices $\mathbf{C}_{tie,i}$ for all tie vectors:

$$\mathbf{C}_{tie} = \begin{pmatrix} \mathbf{C}_{tie,1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{tie,2} & & \mathbf{0} \\ \vdots & & \ddots & \\ \mathbf{0} & \mathbf{0} & & \mathbf{C}_{tie,t} \end{pmatrix} . \tag{4.41}$$

The integration into the combination model proceeds by analogy to (4.36), (4.37) and(4.38).

Introduction as TRF The whole of the local tie vectors at a co-location site establish a particular TRF that is introduced among the k initial TRF into the combination. These "tie-TRFs" comprise the positions of all stations at one site and no velocities. As their only purpose is a representation of the tie-vectors, their datum may be defined arbitrarily.

The advantage of this approach is that solutions derived from space geodetic techniques and local ties can be processed uniformly – except for one distinction: Tie-TRFs may dispose of very few observations. If there are not more than two stations at a site, a tie-TRF would comprise only 6 observations. Hence, it would not be feasible to estimate all their transformation parameters.

Indeed, only the three translations t^x, t^y and t^z are estimated. As to scale and orientation, conformity with the combined frame is assumed. In other words, the remaining parameters are "fixed" to zero, by what they theoretically could affect the datum definition of the combined frame (cf. section 4.4.3). However, as the ties are local and dispose of very short baselines, the differences in orientation and scale between the combined frame and the frames of local ties are negligible.

The transformation parameters' rates could not be estimated anyway, since tie-TRFs do not contain any information about observed velocities.

4.4.5 Constraints on Velocities

Though mathematically possible, it is not likely that stations within the same sites have different velocity vectors. To incorporate this knowledge into the combination model, the velocities of the respective stations are constrained pairwisely to be identical. If the i^{th} set of constraints concerns the stations p and q, the following observations are added:

$$\mathbf{l}_{vel,i} = \begin{pmatrix} \dot{x}^q \\ \dot{y}^q \\ \dot{z}^q \end{pmatrix} - \begin{pmatrix} \dot{x}^p \\ \dot{y}^p \\ \dot{z}^p \end{pmatrix} = \mathbf{0} \, . \tag{4.42}$$

If v velocity constraints $\mathbf{l}_{vel} = (\mathbf{l}_{vel,1}^T, \mathbf{l}_{vel,2}^T, \ldots, \mathbf{l}_{vel,v}^T)^T$ are to be introduced, the functional model (4.28) is augmented by the following observation equations:

$$\mathbf{l}_{vel} + \mathbf{v}_{vel} = \mathbf{A}_{vel}\hat{\mathbf{x}} \tag{4.43}$$

$$\text{with} \qquad \mathbf{A}_{vel} = \begin{pmatrix} \cdots \\ \mathbf{A}_{vel,i} \\ \cdots \end{pmatrix} = \begin{pmatrix} \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots \quad -\mathbf{I} \quad \cdots \quad \mathbf{I} \quad \cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{pmatrix} , \quad i = 1 \ldots v \; .$$

The corresponding stochastic model attributes a particular variance $\sigma^2_{vel,i}$ to each constraint:

$$\mathbf{C}_{vel} = \begin{pmatrix} \sigma^2_{vel,1}\mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \sigma^2_{vel,2}\mathbf{I} & & \mathbf{0} \\ \vdots & & \ddots & \\ \mathbf{0} & \mathbf{0} & & \sigma^2_{vel,v}\mathbf{I} \end{pmatrix} \; . \tag{4.44}$$

Once again, the integration into the model proceeds by analogy to (4.36), (4.37) and (4.38).

5 Implementation

The combination algorithm from section 4.4.2 has been implemented in FORTRAN 90, reading its input data from SINEX[1] format. As TRF combinations often deal with a large amount of data, some considerations have been made for economical memory management. Otherwise, delimiting the memory often is in conflict with optimising computation time. Hence, compromises had to be made under the premise of performing identical computations several times.

The following subsections explain some computational steps in detail. An overview of the sequence of steps can be found in appendix D.

5.1 Constitution of the Normal Equation Matrix

Although the normal matrix $\mathbf{N}$ is of dimension $u \times u$, its complete storage cannot be avoided. The inversion routine needs access to the whole matrix at a time. To minimise the amount of memory required beyond that, the normal matrix is constituted by computing its elements directly without storing intermediate results.

Initially, neither the datum definition nor model extensions like local ties and velocity constraints will be considered. (4.27) and (4.30) illustrate that the elements of the $u \times u$ matrix $\mathbf{N}$ are derived from the matrices $\mathbf{A} = \mathbf{A}_{sol}$ and $\mathbf{P}_i$ $(i = 1 \dots k)$. As to $\mathbf{A}$ $(n \times u)$, the matrix is not stored at all, since more than 95 % of its elements are zeros; the required coefficients are computed on demand. The weight matrices $\mathbf{P}_i$ $(n_i \times n_i)$ have to be stored, but only one at a time. More precisely, the elements of $\mathbf{N}$ are computed in the following way:

$$\mathbf{N} = \sum_{i=1}^{k} \mathbf{N}_i = \sum_{i=1}^{k} \mathbf{A}_i^T \mathbf{P}_i \mathbf{A}_i \ . \tag{5.1}$$

After $\mathbf{N}_1$ has been computed, $\mathbf{P}_1$ can be suspended from memory, and $\mathbf{P}_2$ is loaded. Then, the coefficients of $\mathbf{N}_2$ are computed and added to those of $\mathbf{N}_1$. This procedure, being repeated k times, yields the normal equation matrix $\mathbf{N}$ at its end. The computation of the vector $\mathbf{n} = \mathbf{A}^T\mathbf{P}\mathbf{l}$ pursues the same scheme.

In a second step, local ties (if not introduced as TRFs) and velocity constraints are taken into account (if necessary) by adding their contributions to the normal equation matrix:

$$\Delta\mathbf{N}_{tie} = \mathbf{A}_{tie}^T \mathbf{C}_{tie}^{-1} \mathbf{A}_{tie} = \sum_{i=1}^{t} \mathbf{A}_{tie,i}^T \mathbf{C}_{tie,i}^{-1} \mathbf{A}_{tie,i} \tag{5.2}$$

$$\Delta\mathbf{N}_{vel} = \mathbf{A}_{vel}^T \mathbf{C}_{vel}^{-1} \mathbf{A}_{vel} = \sum_{i=1}^{v} \sigma_{vel,i}^{-2} \mathbf{A}_{vel,i}^T \mathbf{A}_{vel,i} \ . \tag{5.3}$$

Finally, $\mathbf{N}$ is regularised by means of datum definition (cf. section 5.2). Subsequently, its inversion is performed by routines from the *Linear Algebra Package* (LAPACK[2]). As the matrix is replaced congruently in memory by its inverse, no supplemental storage space is needed.

[1] Software INdependent EXchange format
[2] http://www.netlib.org/lapack/

5.2 Datum Definition

The datum of the combined frame is determined by modifying the normal equation matrix $\mathbf{N}$ and the right hand side vector $\mathbf{n}$ after they have been constituted as described in section 5.1. Two methods have been realised: the fixing of parameters and the application of minimum constraints.

Fixing of Parameters When parameters are fixed, the parameter vector's number of components would be reduced by the number of fixed parameters. Thus, the normal equation matrix would change its dimensions as well. As this augments the effort of index-management in programming practice, CATREF software applies a technique of fixing (transformation) parameters (to zero) without changing the dimensions of the concerned vectors and matrices. This technique has been adopted.

Assuming again (without loss of generality) that all 14 transformation parameters of TRF k are fixed to zero (cf. section 4.4.3), $\mathbf{N}$ and $\mathbf{n}$, initially constituted pursuant to (4.30) and (4.31), are modified in the following way:

$$\begin{aligned}
\mathbf{N}' &= \begin{pmatrix} \sum_{i=1}^{k} \mathbf{A1}_i^T \mathbf{P}_i \mathbf{A1}_i & \mathbf{A1}_1^T \mathbf{P}_1 \mathbf{A2}_1 & \cdots & \mathbf{A1}_{k-1}^T \mathbf{P}_{k-1} \mathbf{A2}_{k-1} & \mathbf{0} \\ \mathbf{A2}_1^T \mathbf{P}_1 \mathbf{A1}_1 & \mathbf{A2}_1^T \mathbf{P}_1 \mathbf{A2}_1 & & \mathbf{0} & \mathbf{0} \\ \vdots & & \ddots & & \vdots \\ \mathbf{A2}_{k-1}^T \mathbf{P}_{k-1} \mathbf{A1}_{k-1} & \mathbf{0} & & \mathbf{A2}_{k-1}^T \mathbf{P}_{k-1} \mathbf{A2}_{k-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{I} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{A}_{fix}^T \mathbf{P} \mathbf{A}_{fix} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \qquad (5.4)
\end{aligned}$$

$$\mathbf{n}' = \begin{pmatrix} \sum_{i=1}^{k} \mathbf{A1}_i^T \mathbf{P}_i \mathbf{l}_i \\ \mathbf{A2}_1^T \mathbf{P}_1 \mathbf{l}_1 \\ \vdots \\ \mathbf{A2}_{k-1}^T \mathbf{P}_{k-1} \mathbf{l}_{k-1} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{fix}^T \mathbf{P} \mathbf{l} \\ \mathbf{0} \end{pmatrix} . \qquad (5.5)$$

This is equivalent to constituting the normal equation system by means of $\mathbf{A}_{fix}$ (see (4.35)) and extending it afterwards by ones on the main diagonal and zeros elsewhere. Conveniently, the procedure described above does not influence the parameters' estimates when computing $\hat{\mathbf{x}}$ by (2.8). Furthermore, the correct zero values are attributed to the fixed parameters.

However, before utilising the inverse of the normal equation matrix to estimate variance components, it must be modified a second time to assure correct results (cf. appendix D):

$$\mathbf{N}'^{-1} = \begin{pmatrix} (\mathbf{A}_{fix}^T \mathbf{P} \mathbf{A}_{fix})^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \quad \longrightarrow \quad (\mathbf{N}'^{-1})' = \begin{pmatrix} (\mathbf{A}_{fix}^T \mathbf{P} \mathbf{A}_{fix})^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} . \qquad (5.6)$$

If there are no parameters to be fixed (and the datum is defined otherwise), the modifications (5.4), (5.5) and (5.6) have no effect, and the identities $\mathbf{N}' \equiv \mathbf{N}$, $\mathbf{n}' \equiv \mathbf{n}$ and $(\mathbf{N}'^{-1})' \equiv \mathbf{N}^{-1}$ hold. Thus, no notational distinction has to be made in the following, regarding the method of datum definition.

Minimum Constraints Applying minimum constraints means modifying the normal equation matrix by adding the term $\mathbf{A}_{mc}^T \mathbf{C}_{mc}^{-1} \mathbf{A}_{mc}$ (cf. (4.38)). $\mathbf{n}$ does not need to be modified if the referential coordinates $\mathbf{X}_R$ are chosen as approximate values $\mathbf{X}_0$ for $\mathbf{X}$ (cf. section 4.3.3). Note that the positions from $\mathbf{X}_R$ have to be propagated to the epoch t_0 of the combined frame before. Otherwise, (4.17) would not hold rigorously.

If all 14 transformation parameters are concerned by the constraints, $\mathbf{C}_{mc}$ is always a 14×14 diagonal matrix of small variances, for which empirical values from CATREF software have been adopted:

$$\mathbf{C}_{mc} = 10^{-2} \cdot \mathrm{diag}\left(\sigma_t^2 \quad \sigma_t^2 \quad \sigma_t^2 \quad \sigma_d^2 \quad \sigma_r^2 \quad \sigma_r^2 \quad \sigma_r^2 \quad \sigma_{\dot{t}}^2 \quad \sigma_{\dot{t}}^2 \quad \sigma_{\dot{t}}^2 \quad \sigma_{\dot{d}}^2 \quad \sigma_{\dot{r}}^2 \quad \sigma_{\dot{r}}^2 \quad \sigma_{\dot{r}}^2\right) \tag{5.7}$$

$$\begin{aligned} \text{with} \quad \sigma_t &= 1\,mm \\ \sigma_d &= 1\,mm/R \\ \sigma_r &= 1\,mrad \cdot 1\,m/R \\ \sigma_{\dot{t}} &= 0,1\,mm/yr \\ \sigma_{\dot{d}} &= 0,1\,mm/yr/R \\ \sigma_{\dot{r}} &= 0,1\,mrad/yr \cdot 1\,m/R \\ \text{and} \quad R &= 6\,378\,km\ . \end{aligned}$$

5.3 Estimating the Variance of Unit Weight

The estimation of a global variance factor is performed additionally, based on the model of unit weight (2.14). However, contrary to the variance components, it is *not* applied as scaling factor during iteration. Of special interest is the case where $\mathbf{V}_0 = \mathbf{0}$ in (3.1). Then, any residual inconsistency of the stochastic model is absorbed by the variance components $\hat{\sigma}_i^2$, and $\hat{\sigma}_0^2 = 1$ holds as soon as convergence is achieved.

To obtain mathematically rigorous results for $\hat{\sigma}_0^2$, the computation of the overall redundancy (cf. (2.21)) of the adjustment problem should consider the method of datum definition: The overall number of observations n depends on the n_{sol} (true) observations by initial solution as well as on (pseudo-) observations relating to t local ties and v velocity constraints:

$$n = n_{sol} + 3t + 3v\ . \tag{5.8}$$

If the datum is defined by fixing f parameters, the number of parameters to be estimated decreases by this value. Thus, the overall redundancy is:

$$r = n - (u - f) \tag{5.9}$$

with the number u of *all* parameters, regardless if they are unknown or fixed. If the datum is defined by introducing c supplemental pseudo-observations applying minimum constraints, their number has to be added to the number of observations. It follows:

$$r = (n + c) - u\ . \tag{5.10}$$

It is obvious that the redundancies calculated by (5.9) and (5.10) are identical if $c = f$ holds. This shows that the redundancy of the adjustment problem is independent of the method of datum definition.

Secondly, the weighted sum of squares of the residuals will be considered, as it contributes as well to the estimation of the variance of unit weight by (2.20). If the datum is defined by fixing parameters, its computation performs as usual:

$$\mathbf{v}^T\mathbf{P}\mathbf{v} = \sum_{i=1}^{k} \mathbf{v}_i^T\mathbf{P}_i\mathbf{v}_i + \sum_{i=1}^{t} \mathbf{v}_{tie,i}^T\mathbf{C}_{tie,i}^{-1}\mathbf{v}_{tie,i} + \sum_{i=1}^{v} \sigma_{vel,i}^{-2}\mathbf{v}_{vel,i}^T\mathbf{v}_{vel,i} \; . \tag{5.11}$$

In the case of applied minimum constraints though, the residuals' weighted sum of squares is:

$$\mathbf{v}^T\mathbf{P}\mathbf{v} = \sum_{i=1}^{k} \mathbf{v}_i^T\mathbf{P}_i\mathbf{v}_i + \sum_{i=1}^{t} \mathbf{v}_{tie,i}^T\mathbf{C}_{tie,i}^{-1}\mathbf{v}_{tie,i} + \sum_{i=1}^{v} \sigma_{vel,i}^{-2}\mathbf{v}_{vel,i}^T\mathbf{v}_{vel,i} + \mathbf{v}_{mc}^T\mathbf{C}_{mc}^{-1}\mathbf{v}_{mc} \; . \tag{5.12}$$

The last term in (5.12) is numerically zero due to the configuration of the corresponding pseudo-observations. Thus, neither r nor $\mathbf{v}^T\mathbf{P}\mathbf{v}$ depend on the method of datum definition.

It can be concluded that the estimate $\hat{\sigma}_0^2$ of the variance of unit weight, as calculated by (2.20), is independent of the way the datum is defined.

5.4 Variance Component Estimation

Finally, four estimators, notably the Helmert Estimator (section 3.4), the estimation by degree of freedom (section 3.5.1), the classical estimator (section 3.6.3) and Helmert's simple estimator (section 3.6.1) have been realised. The implementation of Kubik's and Persson's methods (sections 3.6.2 and 3.6.4) have been abandoned for reasons that will be explained.

5.4.1 Helmert's Estimator

Standard Model Initially, the most simple case of a combination will be considered, where the datum is defined by fixing parameters, and neither local ties nor velocity constraints are introduced. Then, the stochastic model is (3.4), and the coefficients of the equation system (3.5) are calculated by (cf. (3.15) and (3.16)):

$$h_{ij} = \operatorname{tr}\left((\mathbf{N}'^{-1})'\mathbf{A}_i^T\mathbf{P}_i\mathbf{A}_i(\mathbf{N}'^{-1})'\mathbf{A}_j^T\mathbf{P}_j\mathbf{A}_j\right) \tag{5.13}$$

$$q_i = \mathbf{v}_i^T\mathbf{P}_i\mathbf{v}_i \; . \tag{5.14}$$

Especially the computation of h_{ij} is critical – with respect to both memory and computation time. To economise time requirements, the computation of the coefficients involves the intermediary $n_i \times n_j$ matrices $\mathbf{U}_{ij} \,\forall\, i = 1 \ldots k, j = i \ldots k$:

$$\mathbf{U}_{ij} = \mathbf{A}_i(\mathbf{N}'^{-1})'\mathbf{A}_j^T = \mathbf{U}_{ji}^T \tag{5.15}$$

$$h_{ij} = \operatorname{tr}\left(\mathbf{U}_{ij}\mathbf{P}_j\mathbf{U}_{ij}^T\mathbf{P}_i\right) \; . \tag{5.16}$$

During the computation, only four matrices must be hold in memory at a time, namely $(\mathbf{N}'^{-1})'$, $\mathbf{U}_{ij}$, $\mathbf{P}_i$ and $\mathbf{P}_j$. Nevertheless, that the computation of h_{ij} is still of considerable complexity, illustrates the following equation $\forall\, i = 1 \ldots k$, $j = i+1 \ldots k$:

$$\begin{aligned} h_{ij} = \sum_{l=1}^{n_i}\sum_{m=1}^{n_j} \Bigg[& p_{mm}q_{ll}u_{lm}^2 + \Bigg\{ \sum_{s=l+1}^{n_i} 2p_{mm}q_{sl}u_{lm}u_{sm} \\ & + \sum_{r=m+1}^{n_j} \left(2p_{mr}q_{ll}u_{lm}u_{lr} + \sum_{s=l+1}^{n_i} 2p_{mr}q_{sl}(u_{lm}u_{sr} + u_{sm}u_{lr}) \right) \Bigg\} \Bigg] \end{aligned} \tag{5.17}$$

with $\mathbf{U}_{ij} = (u_{lm})$, $\mathbf{P}_i = (p_{ls})$ and $\mathbf{P}_j = (q_{mr})$. For the computation of the diagonal elements h_{ii} of $\mathbf{H}$, it can be taken advantage of symmetry properties:

$$\begin{aligned} h_{ii} &= n_i + \sum_{l=1}^{n_i} \Bigg[-2p_{ll}u_{ll} - \sum_{s=l+1}^{n_i} 4p_{ls}u_{ls} + \sum_{m=1}^{n_i} \Bigg\{ p_{ll}p_{mm}u_{lm}^2 + \sum_{r=m+1}^{n_i} 2p_{ll}p_{mr}u_{lm}u_{lr} \\ &\quad + \sum_{s=l+1}^{n_i} \left(2p_{ls}p_{mm}u_{lm}u_{ms} + \sum_{r=m+1}^{n_i} 2p_{ls}p_{mr}(u_{lm}u_{sr} + u_{lr}u_{ms}) \right) \Bigg\} \Bigg] \end{aligned} \tag{5.18}$$

with $\mathbf{U}_{ii} = (u_{lm})$ and $\mathbf{P}_i = (p_{ls})$. Obviously, the complexity of the computation of the $\frac{k(k+1)}{2}$ different coefficients of $\mathbf{H}$ is of order $k^2 n_i^4$ (assuming $n_i = n_j \; \forall\, i, j = 1 \ldots k$). For this consideration, the constitution of $\mathbf{U}_{ij}$ has been neglected, because $\mathbf{A}$ in majority consists of zero elements. Hence, if variance components are calculated by Helmert's method, (5.17) and (5.18) determine the computing time of the entire combination.

Extended Model As the mathematical model is not always of standard kind, an example is given where some model extensions have to be considered:

- k_1 TRFs of space geodetic solutions are to be combined. For all TRFs, variance components are to be estimated.
- k_2 TRFs of local ties are considered. Their dispersion is fixed, and no variance components are to be estimated.
- $v > 0$ pairs of stations are constrained to have the same velocity.
- The datum is defined by minimum constraints.

The mathematical model is ($k_1 + k_2 = k$):

$$\begin{pmatrix} \mathbf{l}_{sol} \\ \mathbf{l}_{vel} \\ \mathbf{l}_{mc} \end{pmatrix} + \begin{pmatrix} \mathbf{v}_{sol} \\ \mathbf{v}_{vel} \\ \mathbf{v}_{mc} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{sol} \\ \mathbf{A}_{vel} \\ \mathbf{A}_{mc} \end{pmatrix} \hat{\mathbf{x}} \tag{5.19}$$

$$\mathbf{C}_{ll} = \begin{pmatrix} \sigma_1^2\mathbf{Q}_1 & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & & \vdots & \cdots & \vdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & & \sigma_{k_1}^2\mathbf{Q}_{k_1} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{Q}_{k_1+1} & & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \cdots & \vdots & & \ddots & & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & & \mathbf{Q}_k & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{C}_{vel} & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{C}_{mc} \end{pmatrix}. \tag{5.20}$$

As to the matrices $\mathbf{V}_i$, (3.3) holds $\forall\, i = 1 \ldots k_1$. The matrix $\mathbf{V}_0$ has the following form:

$$\mathbf{V}_0 = \begin{pmatrix} \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{Q}_{k_1+1} & & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \cdots & \vdots & & \ddots & & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & & \mathbf{Q}_k & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{C}_{vel} & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{C}_{mc} \end{pmatrix}. \tag{5.21}$$

The components of the equation system (3.5) are calculated as follows (cf. (3.15), (3.16) and (3.17)):

$$\mathbf{H} = (h_{ij}) \qquad i,j = 1 \dots k_1 \tag{5.22}$$

$$\mathbf{q} = (q_i) \qquad i = 1 \dots k_1 \tag{5.23}$$

$$q_i = \mathbf{v}_i^T \mathbf{P}_i \mathbf{v}_i - h_{i0} \qquad \forall\, i = 1 \dots k_1 \tag{5.24}$$

$$h_{i0} = \sum_{j=k_1+1}^{k} h_{ij} + h_{i,vel} + h_{i,mc} \qquad \forall\, i = 1 \dots k_1 \tag{5.25}$$

$$h_{i,vel} = \operatorname{tr}\left((\mathbf{N}'^{-1})' \mathbf{A}_i^T \mathbf{P}_i \mathbf{A}_i (\mathbf{N}'^{-1})' \mathbf{A}_{vel}^T \mathbf{C}_{vel}^{-1} \mathbf{A}_{vel}\right) \qquad \forall\, i = 1 \dots k_1 \tag{5.26}$$

$$h_{i,mc} = \operatorname{tr}\left((\mathbf{N}'^{-1})' \mathbf{A}_i^T \mathbf{P}_i \mathbf{A}_i (\mathbf{N}'^{-1})' \mathbf{A}_{mc}^T \mathbf{C}_{mc}^{-1} \mathbf{A}_{mc}\right) \qquad \forall\, i = 1 \dots k_1 \tag{5.27}$$

$$h_{ij} = \operatorname{tr}\left((\mathbf{N}'^{-1})' \mathbf{A}_i^T \mathbf{P}_i \mathbf{A}_i (\mathbf{N}'^{-1})' \mathbf{A}_j^T \mathbf{P}_j \mathbf{A}_j\right) \qquad \forall\, i = 1 \dots k_1, j = 1 \dots k\,. \tag{5.28}$$

Again, like in the standard model, intermediary matrices are used to economise the computation (cf. (5.15)). As to $h_{i,vel}$, these coefficients can be decomposed further on (constraint-by-constraint) so that their computation is of negligible complexity.

Results have shown that $h_{i,mc}$ is always numerically zero. This is no surprise, since the datum definition should not affect the variance estimates. Consequently, no significant error is made when these terms are neglected.

It can be concluded that the variance components estimated by Helmert's method are independent of the way the datum is defined – if no over-constraining is applied.

5.4.2 Estimation by Degree of Freedom

Standard Model The estimation of variance components by degree of freedom is much less complex. Considering the standard model where the datum is defined by fixing parameters and neither local ties nor velocity constraints are introduced, this is illustrated by the following computation formula compared to (5.17) (cf. (3.32) and (3.33)):

$$r_{i,D} = n_i - \sum_{r=1}^{u} \left(\nu_{rr} n_{rr} + \sum_{s=r+1}^{u} 2\nu_{rs} n_{rs} \right) \qquad \forall\, i = 1 \dots k \tag{5.29}$$

with $(\mathbf{N}'^{-1})' = (\nu_{rs})$ and $\mathbf{N}_i = \mathbf{A}_i^T \mathbf{P}_i \mathbf{A}_i = (n_{rs})$. $\mathbf{N}_i$ is not stored temporarily, because its coefficients appear only once in (5.29) and can be calculated on demand without inconvenience. Thus, the inverse of the normal equation matrix and $\mathbf{P}_i$ are the only matrices that need to be stored at a time. The complexity of the computation is of order ku^2, when the calculation of the elements n_{rs} is neglected because of the high proportion of zero elements in $\mathbf{A}$.

Extended Model Considering the extended model from the preceding section 5.4.1, the estimation by degree of freedom is quite simple. Variance components are estimated by means of (3.32). The distinction with respect to the standard model is that the rescaling of weights during the iteration is only performed for those data sets where variance components are to be estimated, i. e. the first k_1 TRFs corresponding to space geodetic solutions (cf. 5.20). In other words: Only the variance components of these sets are updated by the estimates from the preceding step. As to the TRFs representing local ties, constraints on velocities and minimum constraints: These datasets are always introduced with the same weights and never rescaled during iteration.

As it holds for the Helmert estimator, the datum definition by minimum constraints has no impact on the estimated variance components. This property results from the nature of minimum constraints: They subsume the minimum amount of conditions to complete the rank deficiency and eliminate the datum defect of the normal equation matrix (cf. section 4.3.3). As a consequence, there is no contribution to the redundancy of the problem and hence the respective residuals $\mathbf{v}_{mc}$ vanish – as residuals of non-redundant observations generally do.

Considering that (3.32) is based on the decomposition of the weighted sum of squares of the residuals and the redundancy (cf. section 3.5.1), to both of which minimum constraints do not contribute, they do not affect the variance component estimation by degree of freedom.

5.4.3 Classical Estimator

The the computation formula for $\hat{\sigma}^2_{i,C}$ (3.36) is applied basically in the same way as for the estimation by degree of freedom (see section 5.4.2). Nevertheless, it would be appropriate to ensure that the sum of the redundancy numbers $r_{i,C}$ equals the overall redundancy r (cf. (3.37)). For this purpose, a combination model without local surveys and without constraints on velocities is considered:

$$r = \sum_{i=1}^{k} r_{i,C} = \sum_{i=1}^{k} \left(n_i - u_{i,C} \right) . \tag{5.30}$$

If the datum is defined by fixing f parameters, the overall redundancy is given by (5.9). (3.37) can only be satisfied if the number of unknowns is decomposed in the following way (cf. (3.38)):

$$u_{i,C} = \frac{n_i}{n} \left(u - f \right) . \tag{5.31}$$

Analogously, if the datum is defined by c minimum constraints and the overall redundancy is given by (5.10), the decomposition performs as follows:

$$u_{i,C} = \frac{n_i}{n} \left(u - c \right) . \tag{5.32}$$

With $u_{i,C}$ calculated by (5.31) or (5.32), respectively, (5.30) holds because of $\sum_{i=1}^{k} n_i = n$ (cf. (5.9) and (5.10)).

5.4.4 Other Estimators

The implementation of *Helmert's simple estimator* (3.34) can easily be realised. Although this has been done, its results will not be discussed any further, since the approach is of a rather rudimentary kind. It is more of historical interest and does not satisfy today's requirements.

Considering the approaches of *Kubik* and *Persson*, it has proved that their estimators are not suitable for the combination of TRFs. If $u_{\theta,i}$ is the number of transformation parameters to be estimated for the i^{th} initial frame involved in a combination, it follows for Kubik's estimator with the total number N of points of the combined frame:

$$\begin{aligned} u = 6N + \sum_{i=1}^{k} u_{\theta,i} \quad \wedge \quad 6N \geq n_i \quad &\Rightarrow \quad r_{i,K} = n_i - 6N - \sum_{i=1}^{k} u_{\theta,i} < 0 \\ &\Rightarrow \quad \hat{\sigma}^2_{i,K} = \frac{\mathbf{v}_i^T \mathbf{P}_i \mathbf{v}_i}{r_{i,K}} < 0 . \end{aligned} \tag{5.33}$$

As to Persson's method (cf. (3.39), (3.41) and (3.42)), it follows:

$$\begin{aligned} u_{i,P} = n_i + u_{\theta,i} \quad &\Rightarrow \quad \Delta r = \sum_{i=1}^{k} (n_i - u_{i,P}) - r = -\sum_{i=1}^{k} u_{\theta,i} - r < 0 \\ &\Rightarrow \quad r_{i,P} = -u_{\theta,i} + w_i \cdot \Delta r < 0 \\ &\Rightarrow \quad \hat{\sigma}^2_{i,P} = \frac{\mathbf{v}_i^T \mathbf{P}_i \mathbf{v}_i}{r_{i,P}} < 0 \; . \end{aligned} \tag{5.34}$$

For classical problems like horizontal geodetic networks, both estimators might provide suitable approximations of variance components. Yet they fail for combinations of TRFs.

6 Numerical Tests

The implemented methods of variance component estimation have been applied to two fundamentally different kinds of combination, both of them playing an important role in the establishment of the ITRF (cf. section 4.4.1):

1. The following subsection deals with *time series stacking*. Time series are sequels of regular (e. g. daily or weekly) solutions that contain only position coordinates and no velocities. (Indeed, it would not make sense estimating velocities for a solution, the data of which span a very short period of time.) Generally, no more than one observation technique is involved in a time series.
2. The *Combination of solutions of different techniques* will be addressed in subsection 6.2. This means combining solutions derived from different observation techniques, generally containing station positions as well as velocities. As different techniques do not dispose of identical reference points at co-location sites, local surveys need to be taken into account. Furthermore, constraints on velocities are applied.

Three estimators will be considered within this section: Helmert's method, the estimation by degree of freedom and the classical estimator.

6.1 Time Series Stacking

Functional Model As it has been indicated before, solutions from time series lack of observed velocities. This has a significant impact on the functional model when intending to compute a combined frame:

- The estimation of station velocities in the combination is based exclusively on the observed positions that evolve from epoch to epoch. Thus, a station's velocity for the combined frame is only estimable if its position is observed by at least two solutions. If a station occurs in only one solution, no velocity can be derived.
- Whereas transformation parameters with respect to the combined frame can be estimated for each solution, their derivatives are not estimable. This is obvious, since the TRFs of the initial solutions are of statical kind due to lack of observed velocities.

Algebraically speaking, three modifications must be applied to the combination model (4.27) for stacking of time series:

- There are no observation equations related to observed velocities.
- The parameter vector does not contain the velocities of those stations that are observed not more than once.
- The parameter vector does not contain any time derivatives of the transformation parameters.

Data Subject of the following considerations will be a time series of 51 weekly SLR-solutions "ILRSA" from the *International Laser Ranging Service* (ILRS). The solutions have been computed by the *Agenzia Spatiale Italiana* (ASI), the primary ILRS combination centre. (That is indicated by the appended letter "A".) The temporal coverage is nearly one year (January

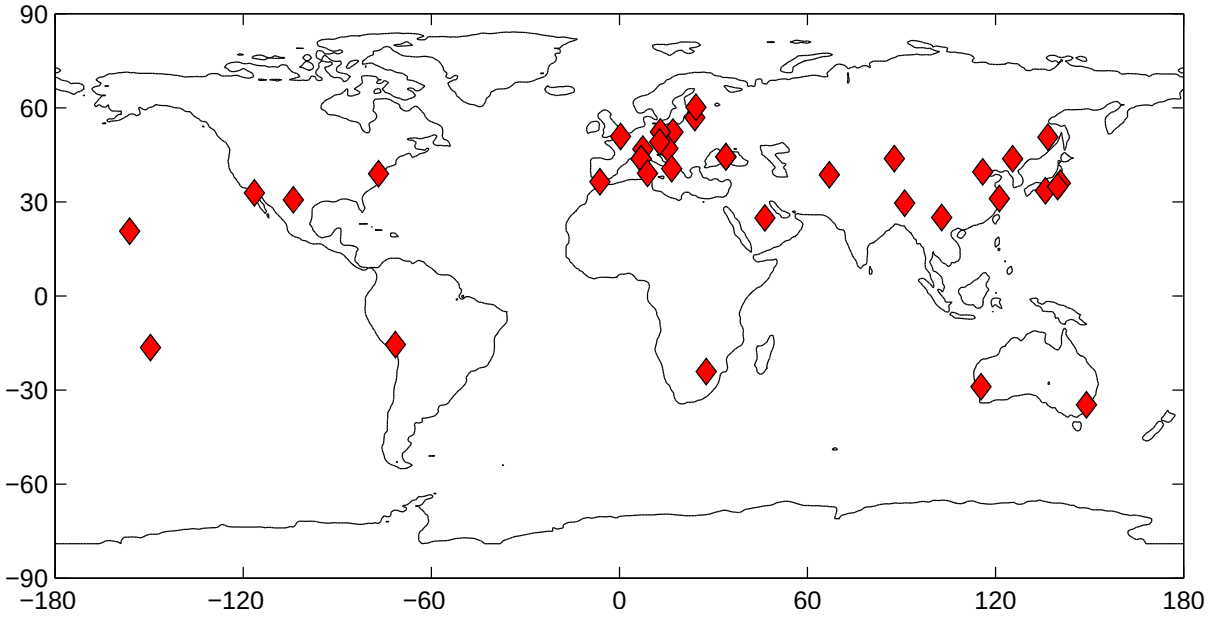

Figure 6.1: Observing stations in the ILRSA time series (2001) from weekly solutions.

to December 2001). Appendix E.1 provides an overview of the data sets in detail. Figure 6.1 visualises the geographical distribution of the comprised stations.

As the weekly solutions from ILRSA time series are loosely constrained, minimum constraints are applied by CATREF software in a preprocessing step (cf. section 4.4.3).

Datum Definition To define the datum of the combined frame, two different methods have been evaluated:

- Minimum constraints over a subset of eight stations with respect to the ITRF2000.
- Fixing of all seven transformation parameters of two arbitrarily chosen solutions.

As it has been pointed out in section 5.4, the method of datum definition does not influence the estimation of variance components. Accordingly, both variants yield identical estimates, which is why they will not be distinguished any further.

Convergence Variance components have been estimated for each of the 51 solutions and reintroduced recursively as a priori values in the subsequent iteration step. Figure 6.2 illustrates the convergence of the Helmert estimator for 10 selected solutions[1], which are representative for the whole set. The convergence behaviour of the estimation by degree of freedom and of the classical estimator do not deviate significantly; the respective plots would be similar. Numerical results can be found in table E.1.

The estimated variance of unit weight behaves as predicted in section 5.3. After the first step of iteration, its square root is 7.02. It approaches 1.0 after the second and 1.00 after the third step – converging to 1 irrespective of the applied estimator (cf. figure 6.3).

Starting with a priori values $\sqrt{a_i} = 1 \; \forall \, i = 1 \ldots k$, convergence is achieved rather quickly. Tests with values from 10^{-5} to 10^{5} have shown that their choice has no significant impact – neither on

[1] They are flagged by an asterisk in table E.1.

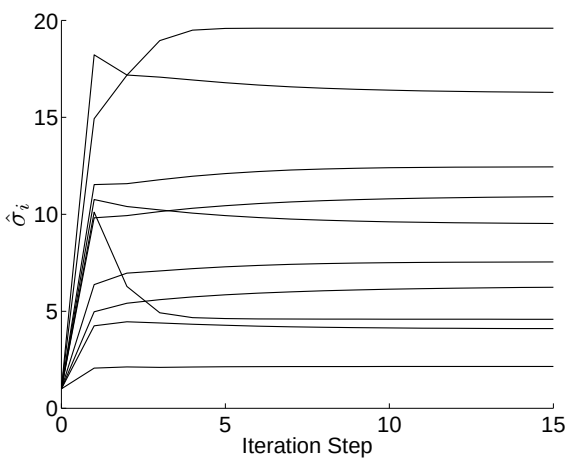

Figure 6.2: Time series stacking: Iteration of variance components by Helmert's method for 10 representatively selected solutions (cf. table E.1).

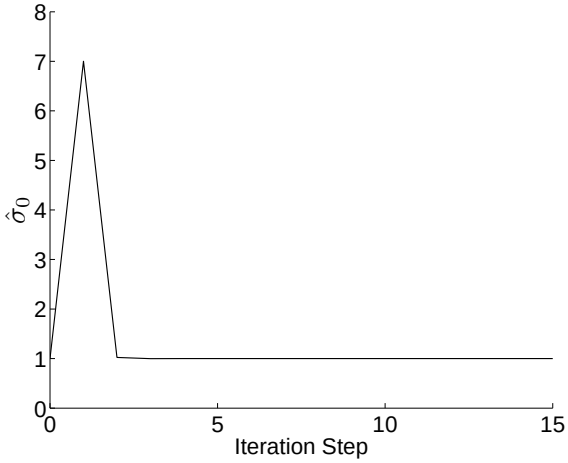

Figure 6.3: Time series stacking: Estimated variance of unit weight during iteration of variance components (applicable for all estimators).

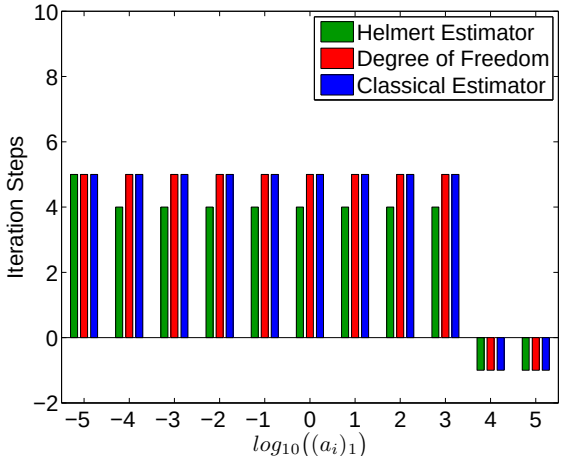

Figure 6.4: Time series stacking: Success of variance component estimation. Indicated is the number ν of necessary iteration steps after which convergence is achieved by $|(\hat{s}_i)_\nu - 1| < 10^{-1}\ \forall\, i = 1 \ldots k$, depending on the starting values $(a_i)_1$. The value -1 stands for the estimator's failure.

the convergence nor on the estimated values (cf. figure 6.4). However, if the starting values are too unrealistic, the estimation fails due to numerical instability of the normal equation matrix. Furthermore, figure 6.4 confirms that the estimation by Helmert's method always converges a bit faster – as it was stated in section 3.5.2.

Quality Evaluation The dispersion of the variance components can be estimated by (3.13) and may be an indicator for the number of necessary iteration steps. It would be desirable to specify confidence intervals that have to enclose the variance components to an adequate degree when convergence is achieved. Due to lack of information about the the applicable statistical distribution, this is not feasible rigorously. Similarly, nothing can be stated about the dispersion of the variance components' square root $\hat{\sigma}_i$.

Nevertheless, to be able to interpret the dispersion measures of the variance components at all, a sheer heuristic approach has been pursued in figure 6.6, which illustrates the convergence behaviour exemplarily for three solutions. Error bars outline an interval that is defined by the Helmert estimate plus and minus its standard deviation. The estimated standard deviations of

all variance components are given in table E.1.

Unlike the parameters' estimated dispersion, that of the variance components depends directly on the approximate values a_i and varies significantly during the first few iterations. It is biased by inexact approximate values and cannot be considered as a reliable indicator before some iterations have been made. Hence, only the confidence interval at the point of convergence should be considered. It is visualised by a grey band in the figures.

Whereas the (Helmert) estimate for the 11^{th} solution ranges within the band from the first iteration on, it takes two iterations for the 8^{th} and seven for the 33^{rd} solution (see figure 6.6). Indeed, considering all 51 solutions, the point where the respective estimate enters the heuristic confidence interval differs distinctly.

Otherwise, for combination of terrestrial reference frames the most relevant issue is the significance of the updates of estimated coordinates and velocities provided by a supplementary iteration step. This is a most natural criterion to evaluate the number of necessary iteration steps.

Figure 6.5 makes an attempt to analyse this criterion graphically. It visualises what has been named "normalised coordinate updates": the elements of the vector $\widehat{\Delta\mathbf{X}}$ of estimated updates for positions and velocities (cf. (4.28)) in the ν^{th} iteration divided by their corresponding standard deviation, the latter being evaluated at the reproducing point.

Obviously, most positions and velocities undergo no more significant updates from the fourth iteration on. Merely a small number of values still change within the range of their standard deviations. Consequently, three iterations could be considered sufficient for this particular combination problem.

Examining the values in table E.1, it proves that most estimates $\hat{\sigma}_i$ range between 2 and 5, whereas some others adopt higher values up to 19.60. This result is not expected from a time series of homogeneous data. Indeed, there are some outliers in the data that have a significant impact on the variance components. It has been verified that the homogeneity of the estimates is enhanced if the outlying stations are rejected from the respective solutions.

Comparison of Methods According to the expectations from section 3.5.2, the Helmert estimator and the estimation by degree of freedom (which is a special case of Förstner's method)

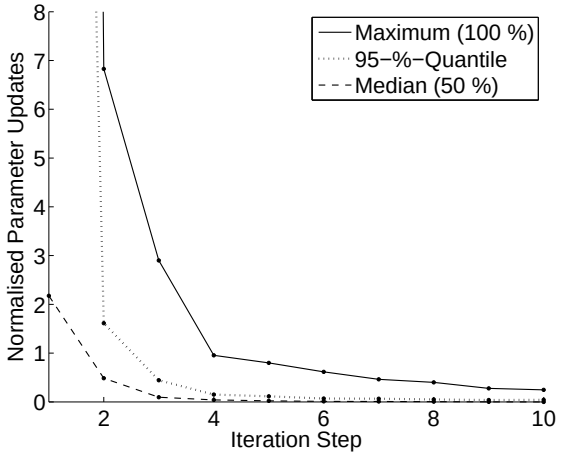

Figure 6.5: Time series stacking: Updates of coordinates and velocities in iteration step ν, given in units of the respective standard deviations.

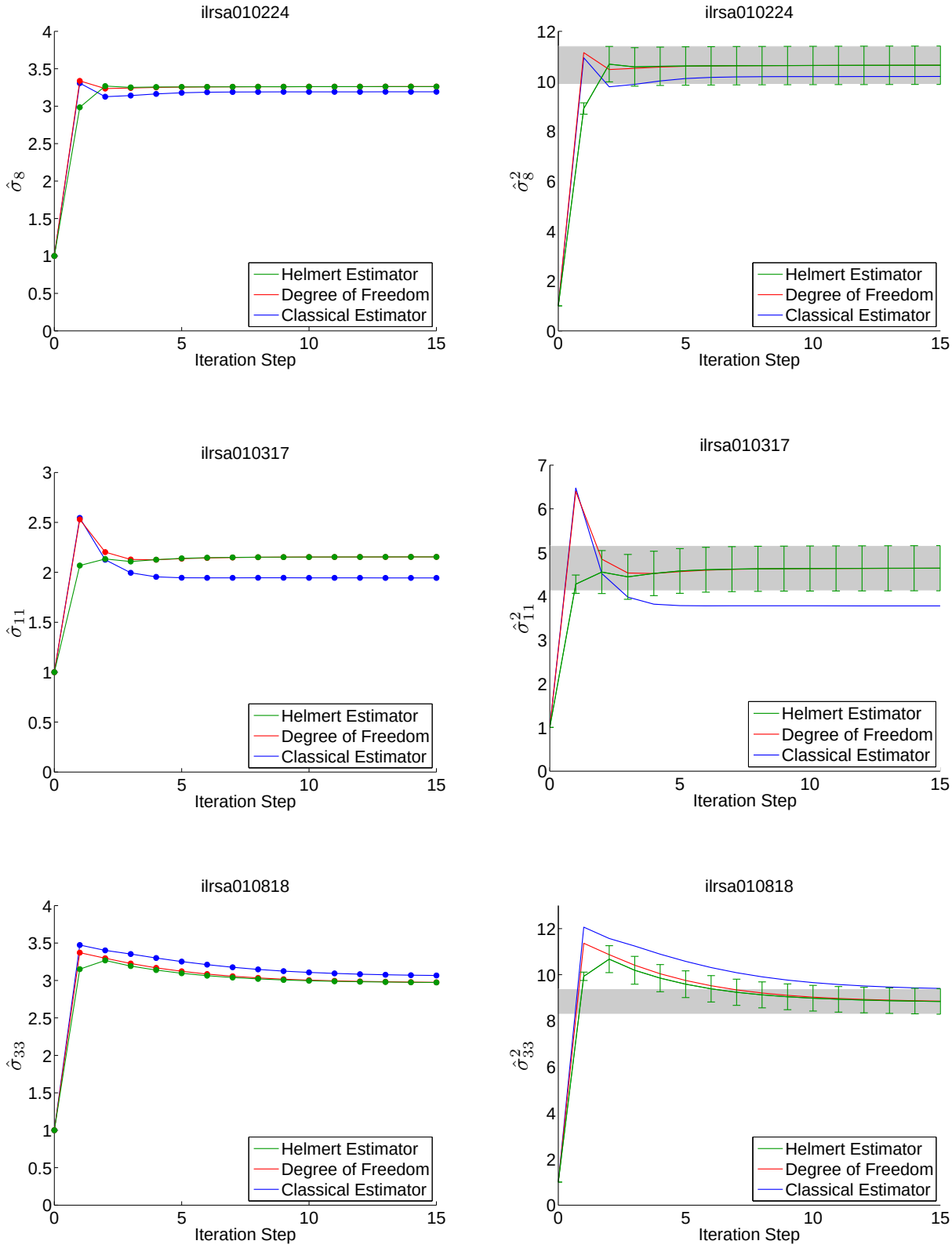

Figure 6.6: Time series stacking: Iteration of three exemplarily chosen variance components (solutions 8 (ilrsa010224), 11 (ilrsa010317) and 33 (ilrsa010818)) by three different methods, given in terms of standard deviations and variances, respectively.

yield identical results at the point of convergence. This is shown exemplarily for three initial solutions in figure 6.6.

The variance components estimated by the classical estimator deviate slightly, because the estimator is biased by construction (cf. section 3.6). It varies from solution to solution whether the estimates are inside the heuristic confidence interval or not. Anyway, it turns out that the bias is by far not significant considering the estimated positions and velocities.

Intending to evaluate the methods of variance component estimation with respect to the required computation time, figure 6.6 may be misleading. Whereas Helmert's estimator requires about 3.5 minutes for one single iteration, the other estimators manage one step in about a second (cf. table 6.1). Regarding the computation formulas (3.32) and (3.36), one would expect that the classical estimator performs notably faster than the estimation by degree of freedom. However, the difference is negligible for a small amount of data.

Table 6.1: Required computation time for one iteration step, given in minutes and seconds. The computation has been performed by an AMD Opteron 248 processor (clock frequency: 2.2 GHz).

Section		Local ties introduced	Estimator		
			Helmert	Deg. of F.	Classical
6.1	Time series stacking	–	3:27.7	0:01.12	0:01.06
6.2.1	Combination of multiple techniques	as TRF	2:23.4	0:02.33	0.02.29
6.2.3	Combination of multiple techniques	as vectors	2:19.4	0:01.30	0.01.26

It can be concluded for the ILRSA time series that the Helmert estimator is by far too time consuming, considering that the same results can be obtained by the degree of freedom method. As the latter does not differ significantly from the classical estimator in terms of computation time, the classical estimator's simplicity does not outweigh the fact that its results are biased, even though not significantly. Hence, the estimation by degree of freedom would be a most appropriate choice.

Nevertheless, the degree of freedom method does not provide any stochastic information on the estimates that can be obtained exclusively by Helmert's method. An advisable proceeding in this issue would be to iterate by degree of freedom until convergence is achieved and introduce the estimates as a priori values afterwards. Then, the Helmert estimator needs to be applied only once to yield an unbiased covariance matrix of the variance components.

6.2 Combination of Solutions of Different Techniques

As a second application, the methods of variance component estimation have been tested on three combined solutions of GPS-, SLR- and VLBI-data, being combined another time to a superior solution. The initial data sets, listed in table 6.2, are an extract of the data which the ITRF2005 is based on. Figure 6.7 visualises the geographical distribution of the 45 (exclusively co-located) sites. An overview is given by table E.3.

Since a station as a reference point is always attributed to one specific technique, none of the 101 concerned stations is observed twice. Hence, local ties are necessary to establish a link between the three initial solutions. 56 tie-vectors have been used, subsumed in 46 SINEX-files[2], one for each site. Solely for the site No. 50103 at Tidbinbilla (Australia), *two* SINEX files are on hand,

[2]available at the ITRF homepage: http://itrf.ensg.ign.fr/

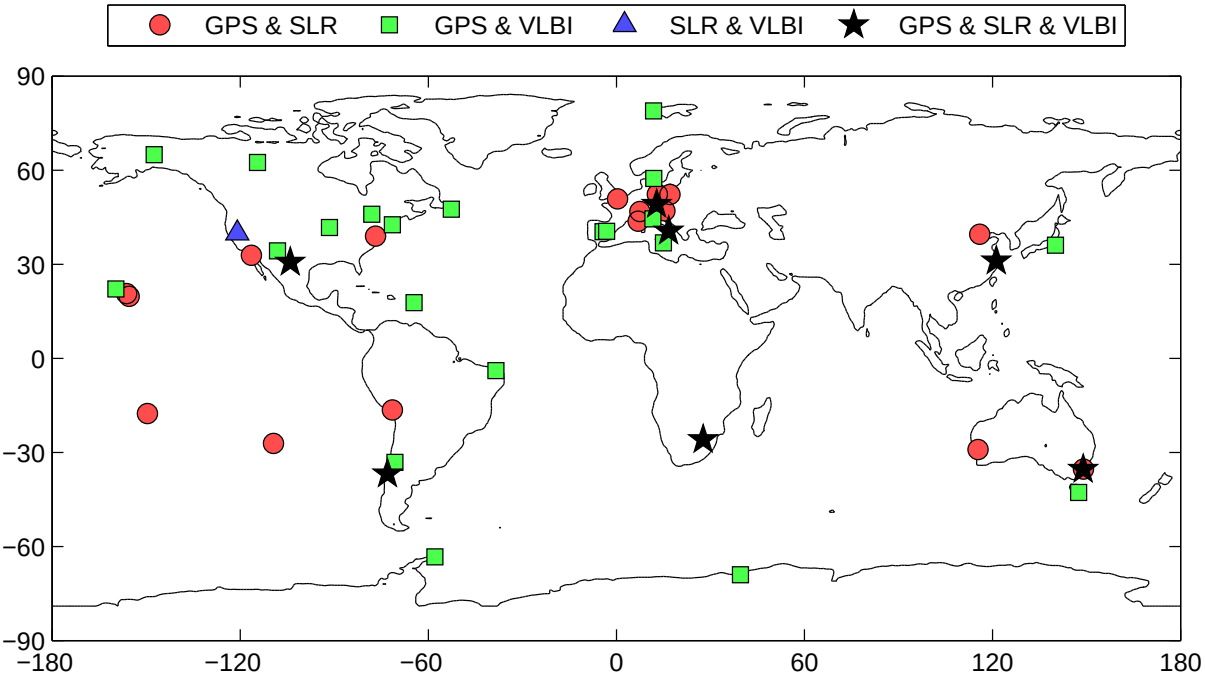

Figure 6.7: Collocated sites in the combination of solutions of different techniques.

each corresponding to one single local tie. This might be due to independent adjustments of the particular tie vectors.

Each SINEX file of local ties contains the positions of all stations within one site that are concerned by the combination. Thus, it defines explicitly a (static) TRF representing the local ties within the respective site. A covariance matrix is provided equally, whereas three types can be distinguished (cf. figure 6.8).

Ideally, the full covariance information (type C) is available, originating directly from the adjustment of local surveys. Practically, its availability is restricted merely to a subset of 15 sites. For the remainder, covariance matrices have been constituted synthetically at the LAREG, yielding type A or type B. To which degree of accuracy the resulting matrices represent physical reality, however, cannot be quantified. An overview gives table E.4.

Additionally, the velocities within a site are constrained to be identical by $56 \cdot 3$ pseudo-observations (cf. section 4.4.5). For 38 of the 56 sets of constraints, a standard deviation $\sigma_{vel,i}$ of $0.1\,mm/yr$ is assumed. As to the remainder, it is known that higher standard deviations (up to $5\,mm/yr$) yield more appropriate results. The empirical values originate from experience in ITRF-combination at the LAREG.

This time, the datum is defined by fixing all 14 transformation parameters of the GPS-solution

Table 6.2: Initial solutions for the combination involving different techniques..

Technique	Data Epoch	Stations
GPS	Jan 1996 - Jan 2006	45
SLR	Dec 1992 - Dec 2005	24
VLBI	Jul 1980 - Feb 2006	32
Total		101

$$\begin{pmatrix} X & - & - & X & - & - \\ - & X & - & - & X & - \\ - & - & X & - & - & X \\ X & - & - & X & - & - \\ - & X & - & - & X & - \\ - & - & X & - & - & X \end{pmatrix} \qquad \begin{pmatrix} X & X & X & X & - & - \\ X & X & X & - & X & - \\ X & X & X & - & - & X \\ X & - & - & X & X & X \\ - & X & - & X & X & X \\ - & - & X & X & X & X \end{pmatrix} \qquad \begin{pmatrix} X & X & X & X & X & X \\ X & X & X & X & X & X \\ X & X & X & X & X & X \\ X & X & X & X & X & X \\ X & X & X & X & X & X \\ X & X & X & X & X & X \end{pmatrix}$$

Type A Type B Type C

Figure 6.8: Types of covariance matrices for TRFs representing local tie vectors. The elements refer to the Cartesian coordinates $(x_1, y_1, z_1, x_2, y_2, z_2)$ of two stations within one site that are connected by a local tie.

to zero. Consequently, the combined solution is expressed in the same frame as the initial GPS-solution. (The respective transformation parameters are zero.)

Strictly speaking, this combination comprises 49 technically homogeneous initial solutions: three derived from space geodetic observations and 46 representing local ties. The following subsections discuss different approaches to deal with local ties in variance component estimation: In section 6.2.1 the attempt is made to estimate variance components for each individual data set, including local ties. The approach of section 6.2.2 is to fix the weights of local ties and to estimate variance components only for space geodetic solutions. Other approaches and their adequateness are discussed in section 6.2.3.

6.2.1 Estimating Variance Components for Local Ties (Approaches I and II)

In a first attempt, 49 variance components are to be estimated: $\hat{\sigma}_1^2 \dots \hat{\sigma}_3^2$ for the space geodetic solutions (GPS, SLR, VLBI) and $\hat{\sigma}_4^2 \dots \hat{\sigma}_{49}^2$ for the local ties. In the following, the convergence behaviour of this approach will be exposed for two investigations, organised by the respectively chosen a priori values a_i:

I. Initially, a priori values $a_i = 1$ are introduced for the variance components of all 49 initial solutions. Here, the Helmert estimator fails completely, yielding negative estimates (cf. table E.4, column $\mathrm{I/H_1}$).

For the estimation by degree of freedom, convergence is achieved, but rather slowly. Figure 6.9 illustrates the convergence behaviour of the GPS-, SLR- and VLBI-solutions and for 10 selected local-tie-solutions[3]. Most variance components have converged to a certain degree after about 15 iterations (which is relatively late). Figure 6.10 reveals though, that there are eight local-tie-solutions[4], for which convergence is never achieved. Their variance components fluctuate erratically about values near zero (cf. table E.4, column $\mathrm{I/D_{100}}$).

The estimated variance of unit weight approaches 1.00 after the second iteration, converging to 0.96 from the third step on (cf. figure 6.11). Its deviation from one accounts for the velocity constraints that are modelled by no variance component.

As to the classical estimator, its iteration is decidedly divergent, which is shown in figure 6.9. The variance components fluctuate arbitrarily until zero estimates render the normal equation matrix uninvertible in the 27^{th} step of iteration.

It occurs that the estimation by degree of freedom is the only method to provide relatively reasonable results – reasonable from a mathematical point of view. On the other hand, admitting

[3] They are flagged by an asterisk in the first column of table E.4.
[4] They are flagged by an asterisk in column $\mathrm{I/D_{100}}$ of table E.4.

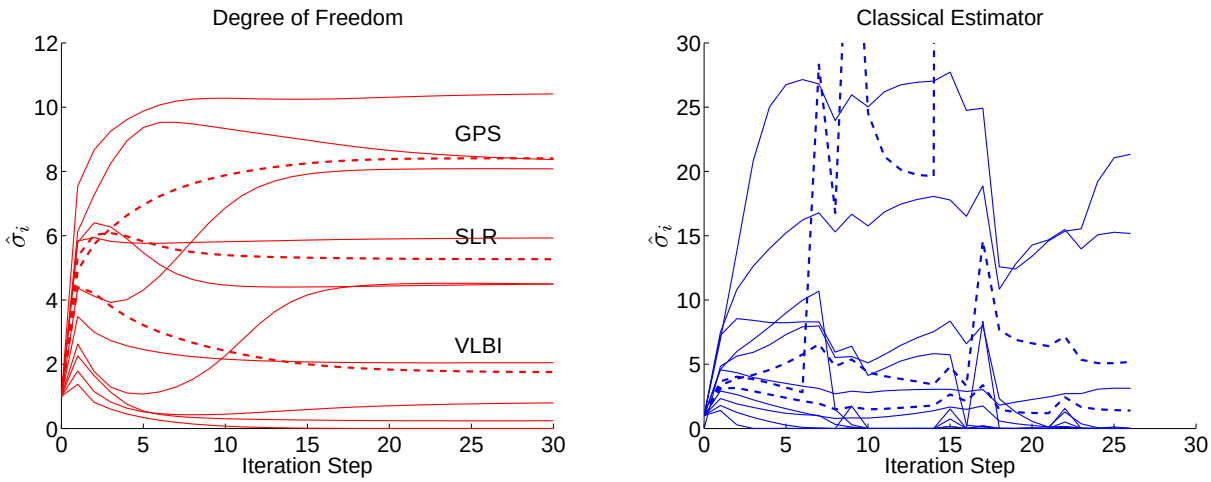

Figure 6.9: Iteration of variance components for the combination of solutions of different techniques, estimating variance components for both space geodetic solutions and local ties. Plotted are the GPS-, SLR- and VLBI-solutions (dashed) as well as 10 representatively selected local-tie-solutions (cf. table E.4).

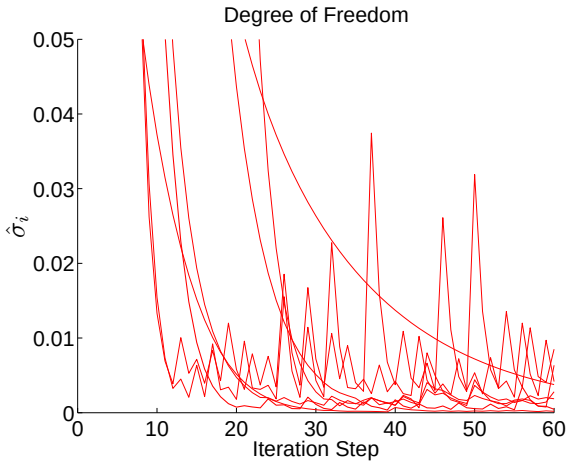

Figure 6.10: Iteration of variance components for the combination of solutions of different techniques, estimating variance components for both space geodetic solutions and local ties. Plotted are all solutions within the displayed range. (Mind the scaling of the ordinate axis!)

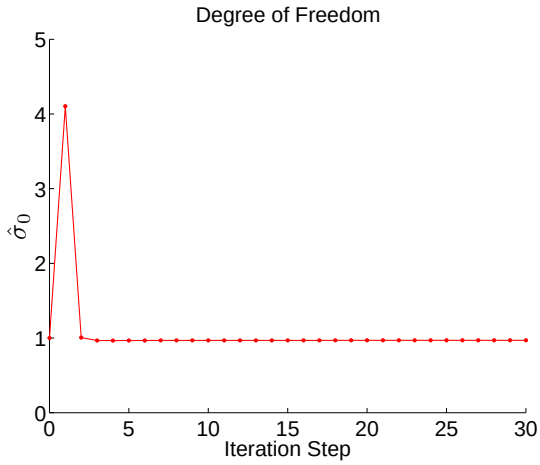

Figure 6.11: Estimated variance of unit weight for the combination of solutions of different techniques, estimating variance components for both space geodetic solutions and local ties.

variance components near zero for eight local-tie-solutions[4] is equivalent to introducing the corresponding local ties without any uncertainty. This is not in accordance with physical reality though, since the tie vectors originate from imperfect measurements.

II. In a second approach, the variance components estimated before by the degree of freedom method are introduced as a priori values for an iteration by Helmert's method. For the eight solutions that did not converge[4], an initial value of $\sqrt{(a_i)_1} = 0.001$ is assumed. Applied as usual, the Helmert estimator fails again, yielding negative estimates.

A next attempt consists in fixing the variance components of the eight problematic solutions to their initial value $a_i = 0.001^2$ as specified in section 5.4.1 ("extended model"). Thus, 41 variance components rest to be estimated, whereas eight are supposed to be definitely known. Now, Helmert's estimator succeeds instantly, reproducing the a priori values as estimates (cf. table E.4, column II/H_1). On the other hand, if the a priori values of the eight solutions in question are set to 1, it fails from the beginning.

It should be emphasised that this approach is not justified for practical purposes, being rather of theoretical interest. Firstly, it confirms once again that Helmert's method and the estimation by degree of freedom yield identical results at the reproducing point. Merely Helmert's estimator occurs to be more sensitive to starting values – and more susceptible to failure if the mathematical model is not sufficiently adequate. (Here, the degree of freedom method has been used to find suitable starting values for the Helmert estimator.)

Secondly, the estimation by Helmert's method yields statistical information about the estimates initially gained through estimation by degree of freedom (see table E.4, column II/$H_1/\hat{\sigma}_i^2$). For the space geodetic solutions (GPS, SLR, VLBI), the standard deviations of the variance components are of a reasonable order of magnitude. As to the local-tie-solutions, their standard deviations are partially about the same size as the estimates themselves. This underlines the little significance of their estimation.

Discussion One weak point is probably the little supplemental redundancy introduced by the local ties. Maximally, it may amount up to $3(N-1)$ per solution, depending on the number N of comprised stations, which is 2, 3 or 4 (see table E.4). Practically, the respective redundancy numbers are inferior, as there is also a contribution to the determination of the unknowns by the local ties.

Figure 6.12 illustrates the repartitioning of redundancy between the local-tie-solutions during the iteration. After the first step, the redundancy numbers range from 0.48 to 6.67. After some more steps however, the local ties' contribution to the overall redundancy has notably decreased, and for the eight solutions mentioned above, the contribution is practically zero. This makes the variance component estimation (3.32) unstable, and convergence cannot be achieved.

It may be conclusive that it is the poor redundancy of the local-tie-solutions that makes variance component estimation unfeasible. This is confirmed by the coincidence that all of the eight problematic solutions belong to the group of the smallest data sets, comprising only two stations. Nevertheless, rejecting these ties does not help, because then other local-tie-solutions behave in the same inconvenient way.

Another issue is the question, up to which degree the mathematical model is appropriate. An obvious deficiency is the limited availability of full covariance matrices (type C) for the local-tie-solutions. Probably, the performance of the estimation would improve if real and full covariance information would be available for all sites. This could not been confirmed though.

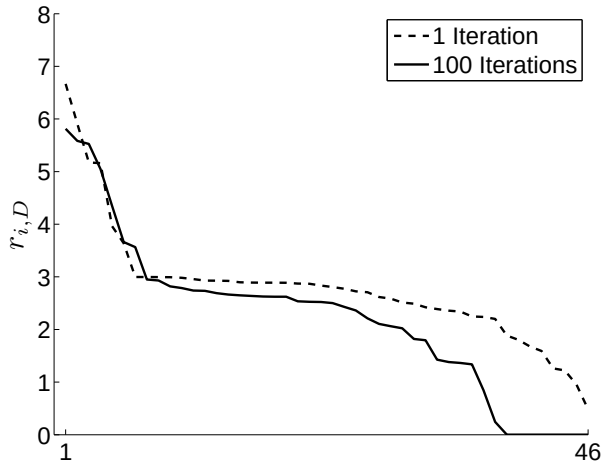

Figure 6.12: Behaviour of the redundancy numbers of the local-tie-solutions during iteration of variance components for the combination of solutions of different techniques, estimating variance components for both space geodetic solutions and local ties by the degree of freedom method. The values are sorted descendingly.

Finally, the organisation of the combination process may be questioned. In the current example, only one solution per technique is introduced into the combination, having been pre-combined before. This is the current strategy of ITRF combination. With several solutions per technique for the final combination, the overall redundancy would be enhanced, and probably better variance component estimates could be expected. The latter strategy used to be pursued for the ITRF2000 (Altamimi et al., 2002a).

6.2.2 Fixing Variance Components for Local Ties (Approach III)

As is has been pointed out, the failure of the estimations in the preceding subsection is closely related with the poor redundancy of the data sets corresponding to local ties. As this is not the case for the space geodetic solutions, the following approach consists in estimating variance components for the space geodetic solutions and fixing the weights for all the local ties to their a priori values a_i (cf. section 5.4, "extended model"). Here, empirical values have been used, originating from the long lasting experience in ITRF combination at the LAREG (cf. table E.5, column III/$\sqrt{(a_i)_1}$).

This time, both the Helmert estimator and the estimation by degree of freedom converge, yielding identical results (cf. figure 6.13 and table E.5). But in contrast to the stacking of time series (compare figure 6.2 – mind the scaling of the abscissa axis), the convergence speed is notably slower, especially for the degree of freedom method. As to the classical estimator, convergence is not achieved at all.

The slower convergence occurs as well for the estimated variance of unit weight, which is still far from one after the second iteration (cf. figure 6.13). Due to the fixed weights for local ties and constraints on velocities, it does not approach one at all, converging at 0.98.

Considering the numerical values of the variance components, it is important to understand that they are *not* an indicator for the quality the respective techniques. They are merely scaling factors, depending on the a priori cofactor matrices $\mathbf{Q}_i$. For quality evaluation, the entries of these matrices have to be taken into consideration.

After all, even if reasonable variance components could be estimated by this approach, keep in mind that they depend on the arbitrarily chosen weighting scheme for the local ties.

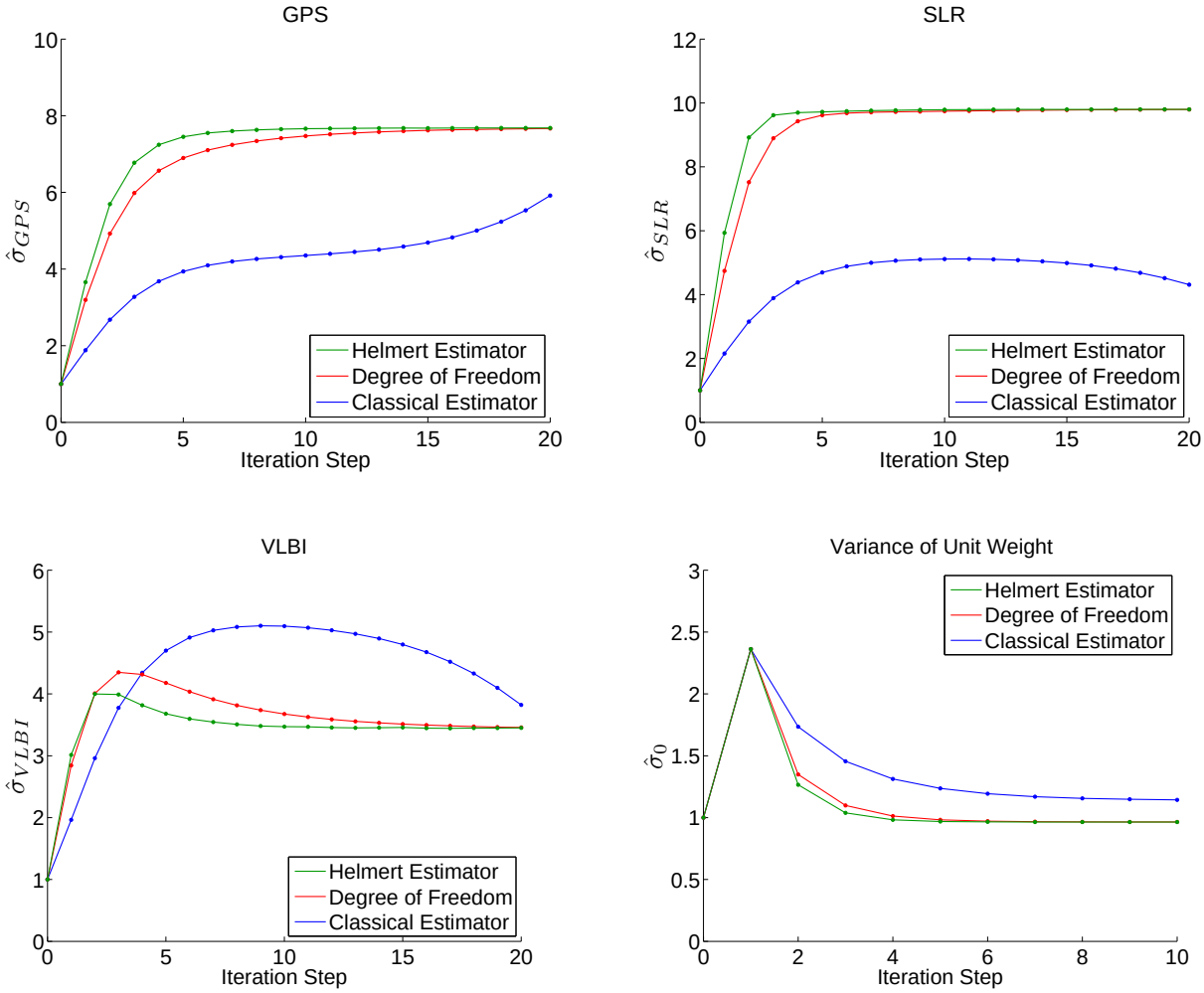

Figure 6.13: Iteration of variance components for the combination of solutions of different techniques, estimating variance components only for space geodetic solutions.

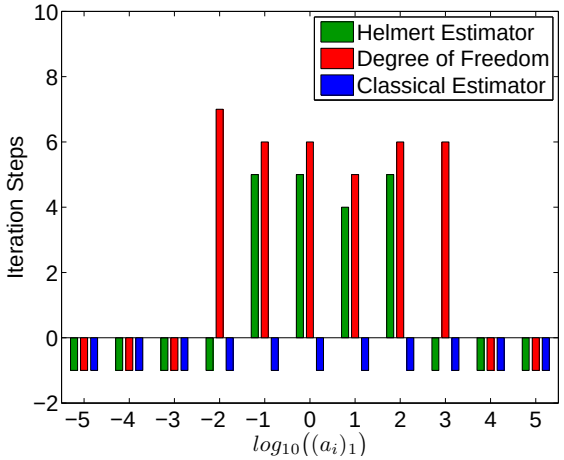

Figure 6.14: Success of variance component estimation for the combination of solutions of different techniques, estimating variance components only for space geodetic solutions. Indicated is the number ν of necessary iteration steps after which convergence is achieved by $|(\hat{s}_i)_\nu - 1| < 10^{-1}\ \forall i = 1 \dots k$, depending on the starting values $(a_i)_1$. The value -1 stands for the estimator's failure.

As figure 6.14 points out, the estimators are more sensitive to starting values this time, the Helmert Estimator more than the degree of freedom method. If the values differ too much from the final estimates at the reproducing point, the estimation fails. That may be due to a numerically not invertible equation system or – for the Helmert Estimator – negative estimates $\hat{\sigma}_i^2$.

6.2.3 Other Approaches

In order to find a strategy to combine space geodetic solutions and local ties successfully, some other approaches have been considered.

Introducing Local Ties as Vectors In all computations, the local-tie-vectors have been introduced in terms of individual TRFs to reduce implementational effort. The benefit of this approach is that local ties can be treated in exactly the same way as space geodetic solutions, because the respective data sets do not differ conceptually. However, as this is not the most natural proceeding, it has been investigated if it makes a difference for the variance component estimation if the local ties are introduced directly as observed baseline vectors (cf. section 4.4.4). It could be shown that it does not make a difference in terms of results. The gain in computation time is not crucial (see table 6.1).

Aggregation of Local Ties As it has been pointed out, redundancy of the individual data sets is an important issue for variance component estimation. Unfortunately, the datasets introducing local ties are small by construction, since measurement campaigns can never comprise reference points from more than one site at a time. Hence, one might think of aggregating these datasets to groups and estimate the variance components group-wise. Note that this proceeding is not justified from a practical point of view, since the local ties are derived from completely different sources and are, by nature, very heterogeneous. Furthermore, this approach would impose a mutual weighting of the local-tie-solutions aggregated to a group; the estimated variance components would always depend on the predefined arbitrary weighting within those groups, which cannot be desirable.

Nevertheless, an experiment has been made to perform a combination where all local ties are attributed to only one group. The internal weighting within this group has been defined by the scheme of empirical values that has already been applied in the previous section and originates from the long lasting experience in ITRF combination at the LAREG (see table E.5, column III/$\sqrt{(a_i)_1}$). Table 6.3 shows the resulting variance components.

Table 6.3: Iterated variance components for the combination of three space geodetic solutions (GPS, SLR, VLBI) with only one variance component comprising all 56 local ties. The indicated results are produced by both the Helmert estimator and the degree of freedom method after the 100th step of iteration.

Estimator	GPS	SLR	VLBI	Local Ties
$\hat{\sigma}_i$	7.02	5.98	2.13	42.82

Although stable results could be obtained, the variance components for space geodetic techniques are not in accordance with those obtained in section 6.2.2 (cf. table E.5), the same mutual weighting of local ties having been applied though. Strictly speaking, both approaches are not optimal, because both are based on arbitrarily fixed weights.

7 Conclusions

Initially, an overview of existing variance component estimators has been given, comprising six different approaches: the statistically rigorous estimator of Helmert type, Förstner's simplified method and four approximate estimators.

In a next step, the estimators have been applied to the combination of TRFs. Adapting the functional model, it turned out that two of the approximate methods are not applicable. Another one, Helmert's simple estimator, has been abandoned due to its oversimplified approach.

Finally, the three remaining estimators, i. e. the Helmert estimator, the estimation by degree of freedom (which is a special case of Förstner's method) and the "classical" estimator, have been investigated. As Helmert's method is by far the most complex one and the others had been implemented before in CATREF software, emphasis has been placed on Helmert's approach.

Two methods of datum definition have been taken into consideration: the fixing of parameters and the application of minimum constraints. It has turned out that the variance component estimates do not depend on the way the datum is defined.

A primary objective of this study was to evaluate if TRF-combinations could benefit from the Helmert estimator and its advantages. After all, Helmert's method reveals a clear distinction with regard to other estimators: It provides stochastic information about the estimates that may help specifying the number of necessary iteration steps. Unfortunately, the interpretation of the estimated standard deviations for the variance components is a delicate matter, since nothing is known (yet) about the applicable statistical distribution. This relativises the benefit of the dispersion estimates.

On the other hand, the Helmert estimator is by far the most complex one with respect to computation time, which increases quadratically with the number and quartically with the dimension of initial solutions. When it finishes a first iteration, other estimators have already converged for a long time. Furthermore, it is more sensitive to starting values and fails if they are too inaccurate. A mathematical model that is not in accordance with the data may provoke failure as well, manifesting itself mostly in negative estimates.

Nevertheless, it has been confirmed that the estimation by degree of freedom always yields the same final estimates as Helmert's method. However, it does not provide stochastic information about the estimates. As it is less susceptible and performs notably faster, it is preferable for practical purposes. If stochastic information about the estimates is required, it would be advisable to seek convergence by the degree of freedom method and to apply Helmert's estimator afterwards with the resulting estimates as reliable starting values.

The classical estimator may be a considerable alternative if the conformity of model and data is good. This is the case for the combination of homogeneous data sets as in time series. The estimates are slightly biased though, but not necessarily significantly. It is advisable to evaluate from case to case if the gain in computation time outweighs the bias.

For the combination of solutions of different techniques, none of the considered estimators performs well. The classical estimator fails, and the others do not converge rigorously. Merely if the weighting scheme of the local ties is fixed, the estimation succeeds. The data originating from many differently performed local surveys is of a relatively heterogeneous kind, though. Thus, the provided statistical information is likely not qualified to establish a weighting scheme unalteredly.

One of the main reasons for the failure of variance component estimation for the combination of solutions of different techniques is probably an insufficient redundancy of the adjustment

problem, affecting the reliability of the estimates. As it is not possible to enhance the redundancy of local tie solutions, it might be imaginable that a combination of more than one solution per observation technique at a time would yield more appropriate results. This is principally a matter of combination strategy – organising the hierarchy of successive steps, starting from the primary observable and yielding a combined global frame at its end.

Another issue is the lacking adequacy of the mathematical model. Hitherto, the stochastic information submitted with local surveys is incomplete for the most part. The availability of full covariance matrices for all local ties (if available one day) might be an important enhancement to the conformity between mathematical model and reality, probably improving the performance of variance component estimation.

Finally, none of the methods discussed can be considered adequate to estimate variance components for the combination of solutions of different techniques. Hence, the most appropriate choice is yet to specify the weighting factors based on experience, eventually inspired but not determined by more or less rigorous estimators. Hence, there is still some work to be done in this field.

Acknowledgements

The major part of this work has been carried out at the Laboratoire de Recherche en Géodésie (LAREG) and supported by the Institut Géographique National (IGN, France).

The first author would like to thank the whole LAREG staff, especially Xavier Collilieux, for cooperation and support during his stay from April to September 2006.

Additional support has been granted by the *German Academic Exchange Service* (DAAD) within the programme "Kurzfristige Studienaufenthalte für Abschlußarbeiten von Studierenden".

A References

Altamimi Z., P. Sillard, C. Boucher (2002a): *ITRF2000: A new release of the International Terrestrial Reference Frame for earth science applications.* Journal of Geophysical Research 107, B10, 2214, doi 10.1029/2001JB000561.

Altamimi Z., C. Boucher, P. Sillard (2002b): *New trends for the realization of the International Terrestrial Reference System.* Advanced Space Research 30(2):175-184.

Altamimi Z., C. Boucher, P. Sillard (2004a): *CATREF Software – Combination and Analysis of Terrestrial Reference Frames.* Technical manual, IGN, France.

Altamimi Z., P. Sillard, C. Boucher (2004b): *ITRF2000: From Theory to Implementation.* In: F. Sansò, ed.: V Hotine-Marussi Symposium on Mathematical Geodesy, 157-163, Springer, Berlin, Heidelberg.

Altamimi Z., X. Collilieux, J. Legrand, B. Garayt, C. Boucher (2007): *ITRF2005: A new release of the International Terrestrial Reference Frame based on time series of station positions and Earth Orientation Parameters.* Journal of Geophysical Research 112, B09401, doi 10.1029/2007JB004949.

Caspary W., K. Wichmann (1994): *Lineare Modelle.* Oldenbourg, Munich.

DeMets C., R. G. Gordon, D. F. Argus, S. Stein (1994): *Effect of recent revisions to the geomagnetic reversal timescale on estimates of current plate motions.* Geophysical Research Letters 21(20):2191-2194.

Ebner H. (1972): *A posteriori Varianzschätzungen für die Koordinaten unabhängiger Modelle.* ZfV 97(4):166-172.

Förstner W. (1979a): *Ein Verfahren zur Schätzung von Varianz- und Kovarianzkomponenten.* AVN 86(11-12):446-453.

Förstner W. (1979b): *Eine Optimierung der Schätzung von Varianz- und Kovarianzkomponenten.* Seminar on variance component estimation, Institute of Theoretical Geodesy, University FAF Munich, June 15th.

Förstner W. (1979c): *Konvergenzbeschleunigung bei der a posteriori Varianzschätzung.* ZfV 104(4):149-156.

Grafarend E. (1978): *Schätzung von Varianz und Kovarianz der Beobachtungen in geodätischen Ausgleichungsmodellen.* AVN 85(2):41-49.

Grafarend E., B. Schaffrin (1979): *Variance-Covariance-Component Estimation of Helmert Type.* Surveying and Mapping 39(3):225-234.

Grafarend E., A. Kleusberg, B. Schaffrin (1980): *An Introduction to the Variance Covariance Component Estimation of Helmert Type.* ZfV 105(4):161-180.

Heck B. (2003): *Rechenverfahren und Auswertemodelle in der Landesvermessung.* 3rd edition, Wichmann, Heidelberg.

Helmert F. R. (1907): *Die Ausgleichung nach der Methode der kleinsten Quadrate.* 2nd edition, Teubner, Leipzig, Berlin.

Illner I. (1985): *Datumsfestlegung in freien Netzen.* PhD thesis, DGK C 309, BADW, Munich.

KOCH K. R. (1978): *Schätzung von Varianzkomponenten.* AVN 85(7):264-269.

KOCH K. R. (1981): *Varianz- und Kovarianzkomponentenschätzung für Streckenmessungen auf Eichlinien.* AVN 88(4):125-132.

KOCH K. R. (1986): *Maximum Likelihood Estimate of Variance Components.* Bulletin Géodésique 60:329-338.

KOCH K. R. (1997): *Parameterschätzung und Hypothesentests in linearen Modellen.* 3rd edition, Dümmler, Bonn.

KOVALEVSKI J., I. I. MUELLER (1989): *Introduction.* In: J. KOVALEVSKI, I. I. MUELLER, B. KOLACZEK, eds.: *Reference frames in astronomy and geophysics*, Kluwer Academic Publishers, Dordrecht, Boston, London.

KUBIK K. (1967): *Schätzung der Gewichte der Fehlergleichungen beim Ausgleichungsproblem nach vermittelnden Beobachtungen.* ZfV 92(5):173-178.

KUBIK K. (1970): *The estimation of the weights of measured quantities within the method of least squares.* Bulletin Géodésique 95(1):21-40.

MCCARTHY D. D., G. PETIT, eds. (2003): *IERS Conventions (2003).* Verlag des Bundesamtes für Kartographie und Geodäsie, Frankfurt am Main.

PERSSON C. G. (1980): *MINQUE and related estimators for variance components in linear models.* Royal Institute of Technology, Stockholm.

RAO C. R. (1973): *Linear statistical inference and its applications.* 2nd edition, Wiley, New York.

RAO C. R., J. KLEFFE (1988): *Estimation of variance components and applications.* Elsevier Science, Amsterdam.

SCHÖN S., H. KUTTERER, M. MAYER, B. HECK (1999): *On the Datum Quality of a continental ITRF96-based Reference Network in Antarctica.* Internal report, Geodetic Institute, University of Karlsruhe.

SEEBER G. (2003): *Satellite Geodesy.* 2nd edition, DeGruyter, Berlin, New York.

SILLARD P. (1999): *Modélistation des systèmes de référence terrestres.* PhD thesis, Observatoire de Paris.

SILLARD P., C. BOUCHER (2001): *A review of algebraic constraints in terrestrial reference frame datum definition.* Journal of Geodesy 75(2-3):63-73.

TEUNISSEN P. J. G., A. R. AMIRI-SIMKOOEI (2006): *Variance component estimation by the method of least-squares.* VI Hotine-Marussi Symposium of Theoretical and Computational Geodesy, Wuhan University, China, May 29th - June 2nd.

TEUNISSEN P. J. G., A. R. AMIRI-SIMKOOEI (2007): *Least-squares variance component estimation.* Journal of Geodesy, published online, doi 10.1007/s00190-007-0157-x.

WELSCH W. (1978): *A posteriori Varianzschätzung nach Helmert.* AVN 85(2):55-63.

WELSCH W. (1984): *Grundlagen, Gebrauchsformeln und Anwendungsbeispiele der Schätzung von Varianz- und Kovarianzkomponenten.* Mensuration, Photogrammétrie, Génie rural 82(9):296-301.

YU Z. C. (1996): *A universal formula of maximum likelihood estimation of variance-covariance components.* Journal of Geodesy 70(4):233-240.

B List of Acronyms

ASI	Agenzia Spaziale Italiana
AVN	Allgemeine Vermessungsnachrichten
BADW	Bayrische Akademie der Wissenschaften
BIH	Bureau International de l'Heure
BTS	BIH Terrestrial System
BIQUE	Best Invariant Quadratic Unbiased Estimator
CATREF	Combination and Analysis of Terrestrial Reference Frames
CoM	Centre of Mass
CRF	Celestial Reference Frame
CRS	Celestial Reference System
DAAD	Deutscher Akademischer Austauschdienst
DGFI	Deutsches Geodätisches Forschungsinstitut
DGK	Deutsche Geodätische Kommission
DOMES	Directory of MERIT Sites
DORIS	Doppler Orbitography and Radiopositioning integrated by Satellite
ENSG	Ecole Nationale des Sciences Géographiques
FAF	Federal Armed Forces
FORTRAN	Formula Translator
GPS	Global Positioning System
IAU	International Astronomical Union
IERS	International Earth Rotation and Reference Systems Service
IGN	Institut Géographique National
ILRS	International Laser Ranging Service
IQUE	Invariant Quadratic Unbiased Estimator
ITRF	International Terrestrial Reference Frame
LAREG	Laboratoire de Recherche en Géodésie
LAPACK	Linear Algebra Package
LLR	Lunar Laser Ranging
LS-VCE	Least Squares Variance Component Estimation
MERIT	Monitoring of Earth Rotation and Intercomparison of Techniques
MINQUE	Minimum Norm (Invariant) Quadratic Estimator
ML	Maximum Likelihood
NNR	No Net Rotation
NRCan	National Resources Canada
PhD	Doctor of Philosophy
QE	Quadratic Estimator
QUE	Quadratic Unbiased Estimator
SI	Système International d'unités
SINEX	Software Independent Exchange Format
SLR	Satellite Laser Ranging
TH	Technische Hochschule
TRF	Terrestrial Reference Frame
TRS	Terrestrial Reference System
VCE	Variance Component Estimation
VLBI	Very Long Baseline Interferometry
ZfV	Zeitschrift für Vermessungswesen

C List of Symbols

General conventions

Matrices are denoted by capital, bold characters.
(Column-) vectors are referenced by (mostly) lower case, bold characters.
Non-bold characters stand for scalars.

$\square^T$	Transpose of a matrix or vector
$\square^{-1}$	Cayley inverse of a square matrix
$\square^-$	Generalised inverse of a square matrix
$\hat{\square}$	Estimated value
$\dot{\square}$	Derivative with respect to time
$\square'$	Modification due to fixing of parameters

Operators

The following list subsumes some operators used in this study. The numbers related to each operator indicate the pages where they are defined or appear for the first time, respectively.

$\mathrm{D}\{\cdot\}$	Dispersion	3
$\mathrm{E}\{\cdot\}$	Expectation	3
$\mathrm{O}\{\cdot\}$	Landau symbol	3
$\det(\cdot)$	Determinant	10
$\mathrm{tr}(\cdot)$	Trace of a square matrix	4
$\partial(\cdot)$	Partial differentiation	3

Specific symbols

The following list subsumes most symbols used in this study. The numbers related to each symbol indicate the pages where they are defined or appear for the first time, respectively.

$\mathbf{0}$	Matrix or vector of zeros	4
$\mathbf{1}$	Matrix or vector of ones	7
$\mathbf{A}$	$n \times u$ design matrix	3
$\mathbf{A}_i$	$n_i \times u$ component of $\mathbf{A}$ corresponding to the i^{th} set of observations; in context of stochastic constraints: partition of $\mathbf{A}$ corresponding to $\mathbf{x}_i$	8, 21
$\mathbf{A}_{fix}$	$n \times (u-f)$ design matrix of the combination model when parameters are fixed	28
$\mathbf{A}_{mc}$	$c \times u$ partition of the design matrix corresponding to minimum constraints	28
$\mathbf{A}^p$	3×7 matrix defined in (4.25)	25
$\mathbf{A}_{sol}$	$n_{sol} \times u$ partition of the design matrix corresponding to all (true) observations from initial solutions	26
$\mathbf{A}_{tie}$	$3t \times u$ partition of the design matrix corresponding to all local-ties	29
$\mathbf{A}_{tie,i}$	$3 \times u$ partition of the design matrix corresponding to the i^{th} local-tie-vector	29
$\mathbf{A}_{vel}$	$3v \times u$ partition of the design matrix corresponding to all velocity constraints	29
$\mathbf{A}_{vel,i}$	$3 \times u$ partition of the design matrix corresponding to the i^{th} set of velocity constraints	30

$\mathbf{A1}_i$	$n_i \times 6N$ partition of the design matrix comprising the derivatives of the observations from the i^{th} solution with respect to station positions and velocities	25
$\mathbf{A2}_i$	$n_i \times u_{\theta,i}$ partition of the design matrix comprising the derivatives of the observations from the i^{th} solution with respect to the transformation parameters for the i^{th} frame	25
$\mathbf{B}$	Coefficient matrix in the model of condition equations *or* $14 \times 6N$ design matrix of minimum constraints, respectively	10, 22
$\mathbf{C}_{cc}$	Covariance matrix of stochastic constraints	20
$\mathbf{C}_{ll}$	$n \times n$ covariance matrix of the observations $\mathbf{L}$ (or $\mathbf{l}$, respectively)	3
$\mathbf{C}_{mc}$	Covariance matrix of minimum constraints	23, 33
$\mathbf{C}_{sol}$	$n_{sol} \times n_{sol}$ covariance matrix corresponding to all (true) observations from initial solutions	26
$\mathbf{C}_{tie}$	$3t \times 3t$ covariance matrix of all local ties	29
$\mathbf{C}_{tie,i}$	3×3 covariance matrix of the i^{th} local-tie-vector	25
$\mathbf{C}_{vel}$	$3v \times 3v$ covariance matrix of all velocity constraints	30
$\mathbf{C}_{vv}$	$n \times n$ covariance matrix of the residuals $\mathbf{v}$	4
$\mathbf{C}_{ww}$	$r \times r$ covariance matrix of the misclosures $\mathbf{w}$	10
$\mathbf{C}_{\hat{x}\hat{x}}$	$u \times u$ covariance matrix of the estimated parameters $\hat{\mathbf{x}}$	4
$\mathbf{C}^{cc}_{\hat{x}\hat{x}}$	Covariance matrix of the stochastically constrained parameters	21
$\mathbf{C}^{mc}_{\hat{x}\hat{x}}$	Covariance matrix of the minimally constrained parameters	23
$\mathbf{C}^{unc}_{\hat{x}\hat{x}}$	Covariance matrix of the stochastically constrained parameters after removal of the constraints' contribution	22
$\mathbf{D}$	$n \times n$ symmetric matrix with a priori unknown coefficients	8
$\mathbf{G}$	$6N \times 14$ matrix defined in eq. (4.5)	17
$\mathbf{H}$	$k \times k$ coefficient matrix in the equation system for estimating variance components by Helmert's method	7
$\mathbf{I}$	Identity matrix	6
$\mathbf{L}$	$n \times 1$ vector of observations	3
$\mathbf{L}_i$	$n_i \times 1$ component of $\mathbf{L}$ corresponding to the i^{th} set of observations	24
$\mathbf{L}_{sol}$	$n_{sol} \times 1$ vector of all (true) observations corresponding to the initial solutions	26
$\mathbf{N}$	$u \times u$ normal equation matrix	3
N	Total number of points concerned by a set of coordinates or involved in a combination, respectively	16, 24
$\mathbf{N}_c$	$u \times u$ contribution of the stochastic constraints to $(\mathbf{C}^{cc}_{\hat{x}\hat{x}})^{-1}$	21
$\mathbf{N}_i$	$u \times u$ contribution of the i^{th} solution to the normal equation matrix	31
$\mathbf{P}$	$n \times n$ weight matrix, the definition of which is contextual	3, 4, 5
$\mathbf{P}_i$	$n_i \times n_i$ weight matrix corresponding to the i^{th} set of observations	6
$\mathbf{Q}$	$n \times n$ cumulated cofactor matrix of the observations in the stochastic model of variance component estimation	5
$\mathbf{Q}_i$	$n_i \times n_i$ cofactor matrix of the i^{th} set of observations	6
$\mathbf{Q}_{ll}$	$n \times n$ cofactor matrix of the observations $\mathbf{L}$ (or $\mathbf{l}$, respectively)	4
$\mathbf{Q}_{vv}$	$n \times n$ cofactor matrix of the residuals $\mathbf{v}$	4
$\mathbf{Q}_{\hat{x}\hat{x}}$	$u \times u$ cofactor matrix of the estimated parameters $\hat{\mathbf{x}}$	4
$\mathbf{R}$	Linearised 3×3 rotation matrix	16
$\mathbf{R}^*$	3×3 rotation matrix	16
$\mathbf{T}_i$	$n \times n$ matrix, proportionally related with $\mathbf{V}_i$ (cf. eq. (3.11))	7
$\mathbf{U}_{ij}$	$n_i \times n_j$ matrix defined in eq. (5.15)	34
$\mathbf{V}_i$	$n \times n$ additive component of $\mathbf{C}_{ll}$ modelling the impact of the i^{th} variance component	5
$\mathbf{V}_0$	$n \times n$ additive component of $\mathbf{C}_{ll}$ that is not concerned by any variance component	5

$\mathbf{W}$	$n \times n$ matrix defined in eq. (3.9)	7
$\mathbf{X}$	$u \times 1$ vector of unknown parameters; in context of TRFs: $6N \times 1$ vector of coordinates and velocities of all points	3, 19
$\mathbf{X}_R$	$6N \times 1$ vector of coordinates and velocities of points defined in the datum-defining, referential TRF	22
$\mathbf{X}_i$	$6N \times 1$ vector of positions and velocities of points defined in TRF i	17
$\mathbf{X}_0$	$u \times 1$ (or $6N \times 1$) vector of approximate values for $\mathbf{X}$	3, 20
$\mathbf{X}^{mc}$	$u \times 1$ vector of minimally constrained parameters	27
a_i	Approximate (a priori) value for the variance component s_i	7
c	Number of pseudo-observations introduced for application of minimum constraints	33
d	Differential scale factor	16
e	Euler's number	10
$\mathbf{e}_i$	Unit vector nullifying all elements except for the i^{th} one	8
$\mathbf{f}$	Multidimensional algebraic function	3
f	Number of fixed parameters	33
h_{i0}	Corrective coefficient in the equation system for estimating variance components by Helmert's method	7
h_{ij}	ij-element of $\mathbf{H}$	7
$h_{i,mc}$	Additive component of h_{i0} accounting for minimum constraints	36
$h_{i,vel}$	Additive component of h_{i0} accounting for constraints on velocities	36
k	Number of variance components; number of initial solutions involved in a combination of TRFs	5, 24
$\mathbf{l}$	$n \times 1$ vector of abridged observations	3
$\mathbf{l}_c$	Vector of pseudo-observations for stochastic constraints	20
$\mathbf{l}_i$	$n_i \times 1$ component of $\mathbf{l}$ corresponding to the i^{th} set of observations	26
$\mathbf{l}_{mc}$	14×1 vector of pseudo-observations for application of minimum constraints	23
$\mathbf{l}_{sol}$	$n_{sol} \times 1$ vector of all linearised (true) observations corresponding to the initial solutions	26
$\mathbf{l}_{tie}$	$3t \times 1$ vector of all local-ties	29
$\mathbf{l}_{tie,i}$	i^{th} local-tie-vector (3×1)	25
$\mathbf{l}_{vel}$	$3v \times 1$ vector of pseudo-observations for all velocity constraints	29
$\mathbf{l}_{vel,i}$	3×1 vector of pseudo-observations for the i^{th} set of velocity constraints	29
m	Scale factor	16
$\mathbf{n}$	$u \times 1$ right hand side vector in the normal equation system of the Gauß-Markov model	3
n	Overall number of observations	3
n_i	Dimension of the i^{th} set of observations	6
n_{sol}	Number of (true) observations from initial solutions	26
$\mathbf{p}$	$k \times 1$ vector defining a linear function	7
p_i	i^{th} element of $\mathbf{p}$	8
$\mathbf{q}$	$k \times 1$ right hand side vector in the equation system for estimating variance components by Helmert's method	7
q_i	i^{th} element of $\mathbf{q}$	7
r	Overall redundancy; number of degrees of freedom	4
r_i	Redundancy number of observation i	4
$r_{i,C}$	Approximation of $r_{i,D}$ by means of the classical estimator	13
$r_{i,D}$	Redundancy number of the i^{th} set of observations	11
$r_{i,H}$	Approximation of $r_{i,D}$ by means of Helmert's simple estimator	12
$r_{i,K}$	Approximation of $r_{i,D}$ by Kubik's method	13

$r_{i,P}$	Approximation of $r_{i,D}$ by Persson's method	13
r^x	Differential rotation angle for rotation about the x-axis	16
r^y	Differential rotation angle for rotation about the y-axis	16
r^z	Differential rotation angle for rotation about the z-axis	16
$\mathbf{s}$	$k \times 1$ vector of variance components	5
s_i	i^{th} variance component	5
$\hat{s}_{i,F}$	Förstner's estimator of the i^{th} variance component	11
$\mathbf{t}$	3×1 vector of translations	16
t	Number of local-tie-vectors	25
t_i	Epoch at which the transformation parameters between TRF i and the combined frame are to be estimated	25
t_i^p	Referential epoch for the position of point p in solution i	24
t_0	Epoch at which a combination is performed	24
t^x	Translation in x	16
t^y	Translation in y	16
t^z	Translation in z	16
u	Overall number of (unknown or fixed) parameters	3
$u_{i,C}$	Proportion of unknowns corresponding to the i^{th} set of observations in the model of the classical estimator	13
$u_{i,K}$	Proportion of unknowns corresponding to the i^{th} set of observations in the model of Kubik's estimator	13
$u_{i,P}$	Proportion of unknowns corresponding to the i^{th} set of observations in the model of Persson's estimator	13
$u_{\theta,i}$	Number of transformation parameters to be estimated for the i^{th} initial frame involved in a combination	37
$\mathbf{v}$	$n \times 1$ Vector of residuals	3
v	Number of sets of constrained velocities	29
$\mathbf{v}_c$	Residuals corresponding to $\mathbf{l}_c$	21
$\mathbf{v}_i$	$n_i \times 1$ component of $\mathbf{v}$, corresponding to the i^{th} set of observations	8
$\mathbf{v}_{mc}$	14×1 vector of residuals corresponding to $\mathbf{l}_{mc}$	23
$\mathbf{v}_{sol}$	$n_{sol} \times 1$ vector of residuals corresponding to all (true) observations from initial solutions	26
$\mathbf{v}_{tie}$	$3t \times 1$ vector of residuals of all local ties	29
$\mathbf{v}_{vel}$	$3v \times 1$ vector of residuals of all velocity constraints	29
$\mathbf{v}_x$	$6N \times 1$ vector of residuals in the model (4.15)	22
$\mathbf{w}$	$r \times 1$ vector of misclosures in the model of condition equations	10
$\mathbf{x}$	$u \times 1$ vector of the abridged parameters	3
$\mathbf{x}_i$	i^{th} partition of $\mathbf{x}$	21
x_j^i	x-coordinate of the i^{th} point in set j	16
y_j^i	y-coordinate of the i^{th} point in set j	16
z_j^i	z-coordinate of the i^{th} point in set j	16
$\Delta\mathbf{N}_{tie}$	Contribution of the local ties to the normal equation matrix	31
$\Delta\mathbf{N}_{vel}$	Contribution of the velocity constraints to the normal equation matrix	31
$\Delta\mathbf{X}$	Difference $\mathbf{X} - \mathbf{X}_0$	26
$\Delta\mathbf{X}_R$	Difference $\mathbf{X} - \mathbf{X}_R$	22
$\mathbf{\Lambda}$	$n \times u$ matrix of Lagrangian multipliers	9
δ_{ij}	Kronecker symbol. $\delta_{ij} = 1$ if $i = j$. $\delta_{ij} = 0$ otherwise.	8
$\boldsymbol{\epsilon}$	$n \times 1$ vector of errors	3
$\tilde{\boldsymbol{\epsilon}}$	$n \times 1$ vector of predicted errors	3

$\boldsymbol{\epsilon}_{tie,i}$	3×1 vector of errors corresponding to the i^{th} local-tie-vector	28
$\boldsymbol{\theta}$	14×1 vector of transformation parameters	17
$\boldsymbol{\theta}_i$	Transformation parameters between the i^{th} initial TRF and the combined frame, referring to t_i	25
$\boldsymbol{\vartheta}$	Transformation parameters between all initial TRFs and the combined frame (concatenation of $\boldsymbol{\theta}_i \, \forall \, i = 1 \ldots k$)	26
$\boldsymbol{\lambda}$	$k \times 1$ vector of Lagrangian multipliers	9
ν	Sequential number of the current step of iteration	7
$\boldsymbol{\xi}_j^i$	3×1 coordinate vector of the i^{th} point in set j	16
σ	Standard deviation	20
σ_0^2	A priori variance factor; variance of unit weight	4
$\hat{\sigma}_0^2$	A posteriori variance factor; estimated variance of unit weight	4
σ_i^2	Variance component of the i^{th} set of observations	6
$\hat{\sigma}_{i,D}^2$	Estimate of σ_i^2 by degree of freedom	11
$\hat{\sigma}_{i,C}^2$	Approximation of $\hat{\sigma}_i^2$ by means of the classical estimator	13
$\hat{\sigma}_{i,H}^2$	Approximation of $\hat{\sigma}_i^2$ by means of Helmert's simple estimator	12
$\hat{\sigma}_{i,K}^2$	Approximation of $\hat{\sigma}_i^2$ by Kubik's method	13
$\hat{\sigma}_{i,P}^2$	Approximation of $\hat{\sigma}_i^2$ by Persson's method	14
$\sigma_{vel,i}^2$	Variance of the i^{th} set of velocity constraints	25

D Flowchart of the Implementation

Loops	Memory	Computation		Equations
$\forall i = 1 \dots k$				
	load $\mathbf{P}_i$			
		$\mathbf{N} \rightarrow \mathbf{N} + \mathbf{A}_i^T \mathbf{P}_i \mathbf{A}_i$		(4.30), (5.1)
		$\mathbf{n} \rightarrow \mathbf{n} + \mathbf{A}_i^T \mathbf{P}_i \mathbf{l}_i$		(4.31)
	suspend $\mathbf{P}_i$			
		$\mathbf{N} \rightarrow \mathbf{N} + \mathbf{A}_{tie}^T \mathbf{C}_{tie}^{-1} \mathbf{A}_{tie}$	[1]	(5.2)
		$\mathbf{N} \rightarrow \mathbf{N} + \mathbf{A}_{vel}^T \mathbf{C}_{vel}^{-1} \mathbf{A}_{vel}$	[2]	(5.3)
		$\mathbf{N} \rightarrow \mathbf{N} + \mathbf{A}_{mc}^T \mathbf{C}_{mc}^{-1} \mathbf{A}_{mc}$	[3]	(4.38)
		$\mathbf{N} \rightarrow \mathbf{N}'$	[4]	(5.4)
		$\mathbf{n} \rightarrow \mathbf{n}'$	[4]	(5.5)
		$\mathbf{N}' \rightarrow \mathbf{N}'^{-1}$		
		$\hat{\mathbf{x}} = \mathbf{N}'^{-1} \mathbf{n}'$		(2.8)
		$\mathbf{N}'^{-1} \rightarrow (\mathbf{N}'^{-1})'$	[4]	(5.6)
$\forall i = 1 \dots k$				
		$\mathbf{v}_i = \mathbf{A}_i \hat{\mathbf{x}} - \mathbf{l}_i$		(2.9), (2.10)
		$\mathbf{U}_{ii} = \mathbf{A}_i (\mathbf{N}'^{-1})' \mathbf{A}_i^T$		(5.15)
	load $\mathbf{P}_i$			
		$\mathbf{v}_i^T \mathbf{P}_i \mathbf{v}_i$		
		$h_{ii} = \mathrm{tr}(\mathbf{U}_{ii} \mathbf{P}_i \mathbf{U}_{ii} \mathbf{P}_i)$		(5.16), (5.18)
	suspend $\mathbf{U}_{ii}$			
$\forall j = i+1 \dots k$				
		$\mathbf{U}_{ij} = \mathbf{A}_i (\mathbf{N}'^{-1})' \mathbf{A}_j^T$		(5.15)
	load $\mathbf{P}_j$			
		$h_{ij} = \mathrm{tr}(\mathbf{U}_{ij} \mathbf{P}_j \mathbf{U}_{ij}^T \mathbf{P}_i)$		(5.16), (5.17)
	suspend $\mathbf{P}_j, \mathbf{U}_{ij}$			
		$h_{i0} = h_{i0}(\mathbf{P}_i)$		(3.17)
	suspend $\mathbf{P}_i$			
		$\hat{\sigma}_0^2 = \frac{\sum_{i=1}^k \mathbf{v}_i^T \mathbf{P}_i \mathbf{v}_i}{r}$		(2.20), (2.21), (5.9) - (5.11)
		$\hat{\mathbf{s}} = \mathbf{H}^{-1} \mathbf{q}$		(3.5)
$\forall i = 1 \dots k$		$\hat{\sigma}_{i,D}^2 = \frac{\mathbf{v}_i^T \mathbf{P}_i \mathbf{v}_i}{r_{i,D}}$		(3.32), (5.29)
$\forall i = 1 \dots k$		$\hat{\sigma}_{i,C}^2 = \frac{\mathbf{v}_i^T \mathbf{P}_i \mathbf{v}_i}{r_{i,C}}$		(3.36), (5.30) - (5.32)
$\forall i = 1 \dots k$		$\hat{\sigma}_{i,H}^2 = \frac{\mathbf{v}_i^T \mathbf{P}_i \mathbf{v}_i}{n_i}$		(3.34)

[1] if local ties are introduced as observed baseline vectors
[2] if constraints on velocities are applied
[3] if minimum constraints are applied
[4] If no parameters are fixed, this transformation has no effect (cf. section 5.2).

E Data

E.1 Time Series Stacking

Table E.1 provides an overview of the ILRSA solutions that have been used for the combination in section 6.1. The column headers have the following meaning:

Sol. Name: Unique identifier of the solution, made up of "ilrsa" and a referential date. Asterisks designate the representative subset of solutions considered in figure 6.2.

Data Epoch: Period of days in which the primary observations have been made in 2001.

Pt.: Number of points (stations) comprised by the solution.

$H_1 \,|\, \hat{\sigma}_i$: Variance components estimated by Helmert's method after 1 iteration, given in terms of standard deviations.

$D_1 \,|\, \hat{\sigma}_i$: Variance components estimated by degree of freedom after 1 iteration, given in terms of standard deviations.

$C_1 \,|\, \hat{\sigma}_i$: Variance components estimated by the classical estimator after 1 iteration, given in terms of standard deviations.

$H_{20}/D_{20} \,|\, \hat{\sigma}_i$: Variance components estimated by Helmert's method after 20 iterations, given in terms of standard deviations. The estimation by degree of freedom yields an identical result at this point of convergence.

$C_{20} \,|\, \hat{\sigma}_i$: Variance components estimated by the classical estimator after 20 iterations, given in terms of standard deviations.

$H_{20} \,|\, \hat{\sigma}_i^2$: Variance components estimated by Helmert's method after 20 iterations, this time given in terms of *variances*. The variance's respective standard deviation $\sqrt{\mathrm{D}\{\hat{\sigma}_i^2)\}}$ is provided equally.

Table E.1: ILRSA time series of weekly SLR-Solutions from 2001 and iterated variance components.

Sol. Name	Data Epoch	Pt.	H_1	D_1	C_1	H_{20}/D_{20}	C_{20}	H_{20}
			$\hat{\sigma}_i$	$\hat{\sigma}_i$	$\hat{\sigma}_i$	$\hat{\sigma}_i$	$\hat{\sigma}_i$	$\hat{\sigma}_i^2$
ilrsa010106	31 Dec[a] - 7 Jan	18	4.83	5.12	4.93	5.72	5.67	32.66 ± 1.27
ilrsa010113	7 Jan - 14 Jan	21	3.32	4.25	4.03	4.36	3.95	19.03 ± 0.97
ilrsa010120*	14 Jan - 21 Jan	21	14.93	14.26	13.43	19.60	19.70	384.02 ± 3.84
ilrsa010127*	21 Jan - 28 Jan	18	10.11	10.01	9.82	4.59	4.46	21.03 ± 1.04
ilrsa010203	28 Jan - 4 Feb	19	5.56	5.71	5.60	5.58	5.65	31.09 ± 1.18
ilrsa010210	4 Feb - 11 Feb	21	2.86	3.27	3.27	2.91	2.75	8.45 ± 0.64
ilrsa010217	11 Feb - 18 Feb	21	3.88	4.21	4.15	3.91	3.84	15.27 ± 0.82
ilrsa010224	18 Feb - 25 Feb	17	2.98	3.34	3.31	3.26	3.19	10.64 ± 0.77
ilrsa010303	25 Feb - 4 Mar	19	6.53	6.57	6.57	6.53	6.70	42.68 ± 1.34
ilrsa010310	4 Mar - 11 Mar	20	3.19	3.39	3.43	3.17	3.18	10.05 ± 0.66
ilrsa010317*	11 Mar - 18 Mar	19	2.07	2.53	2.54	2.16	1.94	4.64 ± 0.52
ilrsa010324	18 Mar - 25 Mar	20	4.68	4.74	4.78	4.88	4.99	23.82 ± 0.98
ilrsa010331	25 Mar - 1 Apr	21	5.07	5.15	5.20	4.34	4.45	18.79 ± 0.86
ilrsa010407	1 Apr - 8 Apr	20	3.95	4.10	4.15	3.54	3.61	12.52 ± 0.73
ilrsa010414	8 Apr - 15 Apr	18	4.95	5.05	5.07	4.30	4.38	18.49 ± 0.93
ilrsa010421	15 Apr - 22 Apr	16	5.07	5.16	5.14	4.66	4.69	21.70 ± 1.06
ilrsa010428	22 Apr - 29 Apr	20	3.12	3.41	3.41	3.69	3.62	13.62 ± 0.77
ilrsa010505	29 Apr - 6 May	18	2.13	2.43	2.42	3.04	2.86	9.21 ± 0.68
ilrsa010512*	6 May - 13 May	20	4.98	5.11	5.16	6.27	6.35	39.35 ± 1.25
ilrsa010519	13 May - 20 May	22	5.49	5.69	5.70	7.36	7.36	54.12 ± 1.41
ilrsa010526	20 May - 27 May	23	7.75	7.80	7.98	8.81	8.97	77.53 ± 1.61
ilrsa010602	27 May - 3 Jun	23	12.38	12.12	12.04	13.96	14.23	195.00 ± 2.55
ilrsa010609	2 Jun - 10 Jun	19	10.83	10.77	10.62	12.22	12.04	149.29 ± 2.53
ilrsa010616*	10 Jun - 17 Jun	22	9.82	9.78	10.05	10.94	11.26	119.72 ± 2.02
ilrsa010623	17 Jun - 24 Jun	19	1.70	2.09	2.13	1.95	1.91	3.79 ± 0.43
ilrsa010630	23 Jun - 1 Jul	21	3.21	3.62	3.67	3.76	3.84	14.11 ± 0.75
ilrsa010707	1 Jul - 8 Jul	25	4.39	4.50	4.67	4.18	4.38	17.43 ± 0.73
ilrsa010714*	8 Jul - 15 Jul	25	18.22	17.66	17.77	16.26	17.10	264.24 ± 2.81
ilrsa010721	15 Jul - 22 Jul	23	9.83	9.80	10.08	9.12	9.56	83.14 ± 1.65
ilrsa010728*	22 Jul - 29 Jul	24	10.76	10.69	10.74	9.50	9.74	90.32 ± 1.72
ilrsa010804	29 Jul - 5 Aug	21	11.07	10.98	11.20	9.80	10.21	95.95 ± 1.86
ilrsa010811	5 Aug - 12 Aug	22	8.27	8.27	8.41	7.23	7.54	52.32 ± 1.35
ilrsa010818	11 Aug - 19 Aug	24	3.15	3.37	3.47	2.97	3.06	8.80 ± 0.55
ilrsa010825*	19 Aug - 26 Aug	28	4.25	4.40	4.54	4.10	4.30	16.79 ± 0.69
ilrsa010901	26 Aug - 2 Sep	25	5.46	5.58	5.67	4.26	4.39	18.15 ± 0.77
ilrsa010908	2 Sep - 9 Sep	22	3.62	3.85	3.93	3.23	3.30	10.41 ± 0.63
ilrsa010915	9 Sep - 16 Sep	26	3.19	3.51	3.51	2.46	2.33	6.05 ± 0.49
ilrsa010922	16 Sep - 23 Sep	26	2.97	3.37	3.42	2.38	2.20	5.66 ± 0.48
ilrsa010929	23 Sep - 30 Sep	26	3.41	3.77	3.81	2.89	2.77	8.36 ± 0.57
ilrsa011013	7 Oct - 14 Oct	23	5.90	6.06	5.97	6.04	6.14	36.48 ± 1.15
ilrsa011020	14 Oct - 21 Oct	23	9.01	8.87	8.95	8.92	9.21	79.64 ± 1.63
ilrsa011027	21 Oct - 28 Oct	25	5.04	5.19	5.10	5.94	5.94	35.32 ± 1.10
ilrsa011103	28 Oct - 4 Nov	22	3.05	3.44	3.41	3.76	3.63	14.11 ± 0.82
ilrsa011110	4 Nov - 11 Nov	22	7.10	7.01	6.73	7.71	7.64	59.50 ± 1.52
ilrsa011117	11 Nov - 18 Nov	24	4.93	5.05	5.07	5.65	5.75	31.91 ± 1.05
ilrsa011124*	18 Nov - 25 Nov	21	6.37	6.38	6.20	7.55	7.57	56.96 ± 1.51
ilrsa011201	25 Nov - 2 Dec	17	5.22	5.34	5.21	5.63	5.54	31.73 ± 1.28
ilrsa011208	2 Dec - 9 Dec	18	3.14	3.56	3.48	3.43	3.18	11.78 ± 0.84
ilrsa011215*	9 Dec - 16 Dec	21	11.53	11.17	11.04	12.45	12.84	154.96 ± 2.39
ilrsa011222	16 Dec - 23 Dec	17	4.24	4.72	4.43	4.98	4.63	24.79 ± 1.22
ilrsa011229	23 Dec - 30 Dec	16	5.20	5.26	5.22	4.59	4.47	21.06 ± 1.06
Total / $\hat{\sigma}_0$	31 Dec[a] - 30 Dec	37	7.02	7.02	7.02	1.00	1.00	

[a] December 2000

Table E.2: Stations comprised by the ILRSA time series in 2001. The last column indicates the number of solutions that comprise the respective station.

Code	DOMES	Site Name	Country	$\sim \lambda$	$\sim \varphi$	# Sol.
7080	40442M006	Fort Davis	USA	255.98°	30.68°	51
7090	50107M001	Yarragadee	Australia	115.35°	-29.05°	51
7840	13212S001	Herstmonceux	UK	0.34°	50.87°	51
7105	40451M105	Washington	USA	283.17°	39.02°	50
7501	30302M003	Hartebeeshoek	South Africa	27.69°	-25.89°	49
7237	21611S001	Changchun	China	125.44°	43.79°	48
7835	10002S001	Grasse	France	6.92°	43.75°	48
7845	10002S002	Grasse	France	6.92°	43.75°	48
7839	11001S002	Graz	Austria	15.49°	47.07°	47
7110	40497M001	Monument Peak	USA	243.58°	32.89°	46
7849	50119S001	Mount Stromlo	Australia	149.01°	-35.32°	45
7832	20101S001	Riyadh	Saudi Arabia	46.40°	24.91°	44
7838	21726S001	Simosato	Japan	135.94°	33.58°	42
7210	40445M001	Haleakala	USA	203.74°	20.71°	41
7810	14001S007	Zimmerwald	Switzerland	7.47°	46.88°	39
7836	14106S009	Potsdam	Germany	13.06°	52.38°	39
7811	12205S001	Borowiec	Poland	17.07°	52.28°	38
7837	21605S001	Shanghai	China	121.19°	31.10°	36
7403	42202M003	Arequipa	Peru	288.51°	-16.47°	32
1884	12302S002	Riga	Latvia	24.06°	56.95°	30
8834	14201S018	Wettzell	Germany	12.88°	49.14°	27
7249	21601S004	Beijing	China	115.89°	39.61°	26
7820	21609S002	Kunming	China	102.80°	25.03°	22
7806	10503S014	Metsahovi	Finland	24.39°	60.22°	19
1893	12337S006	Katzively	Ukraina	33.97°	44.39°	17
7339	21740M001	Tateyama	Japan	139.85°	34.94°	15
7941	12734S008	Matera	Italy	16.70°	40.65°	15
1873	12337S003	Simeis	Ukraina	33.99°	44.41°	14
7824	13402S007	San Fernando	Spain	353.79°	36.47°	14
7356	21613M003	Lhasa	China	91.04°	29.63°	13
7124	92201M007	Tahiti	French Polynesia	210.39°	-17.58°	8
7355	21612M002	Urumqi	China	87.71°	43.81°	5
1864	12340S002	Maidanak	Uzbekistan	66.94°	38.68°	4
1868	12341S001	Komsomolsk-na-Amure	Russia	136.74°	50.69°	2
7335	21701M002	Kashima	Japan	140.66°	35.96°	2
1863	12340S001	Maidanak	Uzbekistan	66.94°	38.69°	1
7548	12725S013	Cagliari	Italy	8.97°	39.14°	1

E.2 Combination of Solutions of Different Techniques

Table E.3: Sites involved in the combination of solutions of different techniques. The columns *GPS*, *SLR* and *VLBI* indicate the number of hosted stations per technique.

Site	GPS	SLR	VLBI	Name	Country	$\sim \lambda$	$\sim \varphi$
10002	1	2		Grasse	France	6.92°	43.75°
10317	1		1	Ny-Alesund	Norway	11.87°	78.93°
10402	1		1	Onsala	Sweden	11.93°	57.40°
11001	1	1		Graz	Austria	15.49°	47.07°
12205	1	1		Borowiec	Poland	17.07°	52.28°
12711	1		1	Medicina	Italy	11.65°	44.52°
12717	1		1	Noto	Italy	14.99°	36.88°
12734	1	1	1	Matera	Italy	16.70°	40.65°
13212	1	1		Herstmonceux	UK	0.34°	50.87°
13407	1		1	Madrid	Spain	355.75°	40.43°
13420	1		1	Yebes	Spain	356.91°	40.52°
14001	1	1		Zimmerwald	Switzerland	7.47°	46.88°
14106	1	1		Potsdam	Germany	13.07°	52.38°
14201	1	1	2	Wettzell	Germany	12.88°	49.14°
21601	1	1		Beijing	China	115.89°	39.61°
21605	1	1	1	Shanghai	China	121.20°	31.10°
21730	1		1	Tsukuba	Japan	140.09°	36.11°
30302	2	1	1	Hartebeeshoek	South Africa	27.69°	-25.89°
40101	1		1	St. John's	Canada	307.32°	47.60°
40104	1		1	Algonquin	Canada	281.93°	45.96°
40127	1		1	Yellowknife	Canada	245.52°	62.48°
40408	1		1	Fairbanks	USA	212.50°	64.98°
40424	1		2	Kauai	USA	200.34°	22.13°
40433		1	1	Quincy	USA	239.06°	39.98°
40440	1		1	Westford	USA	288.51°	42.61°
40442	1	1	1	Fort Davis	USA	255.99°	30.68°
40445	1	1		Haleakala	USA	203.74°	20.71°
40451	1	1		Washington	USA	283.17°	39.02°
40456	1		1	Pie Town	USA	251.88°	34.30°
40465	1		1	North Liberty	USA	268.43°	41.77°
40477	1		1	Mauna Kea	USA	204.54°	19.80°
40497	1	1		Monument Peak	USA	243.58°	32.89°
41602	1		1	Fortaleza	Brazil	321.57°	-3.88°
41703	1	1		Easter Islands	Chile	250.62°	-27.15°
41705	1		1	Santiago	Chile	289.33°	-33.15°
41719	1	1	1	Concepcion	Chile	286.97°	-36.84°
42202	1	1		Arequipa	Peru	288.51°	-16.47°
43201	1		1	Sainte Croix	USA	295.42°	17.76°
50103	1	1	1	Tidbinbilla	Australia	148.98°	-35.40°
50107	1	1		Yarragadee	Australia	115.35°	-29.05°
50116	1		1	Hobart	Australia	147.44°	-42.80°
50119	1	1		Mount Stromlo	Australia	149.01°	-35.32°
66006	1		1	Syowa	Antarctica	39.58°	-69.01°
66008	1		1	O'Higgins	Antarctica	302.10°	-63.32°
92201	1	1		Tahiti	French Polynesia	210.39°	-17.58°
	45	24	32				

Table E.4: Iterated variance components for the combination of solutions of different techniques, estimating variance components for both space geodetic solutions and local ties (approaches I and II).

			I			II		
				H_1	D_{100}		H_1	
Solution	$\mathbf{Q}_i$	Pt.	$\sqrt{(a_i)_1}$	$\hat{\sigma}_i$	$\hat{\sigma}_i$	$\sqrt{(a_i)_1}$	$\hat{\sigma}_i$	$\hat{\sigma}_i^2$
GPS*	C	45	1.00	5.29	8.39	8.39	8.39	70.28 ± 1.36
SLR*	C	24	1.00	4.58	5.26	5.26	5.26	27.62 ± 2.03
VLBI*	C	32	1.00	4.63	1.75	1.75	1.75	3.10 ± 0.74
10002*	C	3	1.00	6.21	4.53	4.53	4.53	20.56 ± 4.00
10317*	B	2	1.00	0.98	0.00*	0.001	(0.001)	(0.00 ± 0.00)
10402	A	2	1.00	2.10	1.43	1.43	1.43	2.05 ± 1.26
11001	A	2	1.00	0.76	0.82	0.82	0.82	0.67 ± 1.35
12205	A	2	1.00	4.04	2.76	2.76	2.76	7.65 ± 2.65
12711*	C	2	1.00	4.26	8.06	8.06	8.06	64.83 ± 7.43
12717	C	2	1.00	—	0.00*	0.001	(0.001)	(0.00 ± 0.00)
12734	C[a]	3	1.00	3.35	4.88	4.88	4.88	23.85 ± 3.72
13212	A	2	1.00	1.96	1.96	1.96	1.96	3.82 ± 1.82
13407	A	2	1.00	8.24	8.20	8.20	8.20	67.27 ± 7.55
13420	A	2	1.00	—	1.17	1.17	1.17	1.36 ± 1.78
14001*	A	2	1.00	1.98	0.26	0.26	0.26	0.07 ± 2.19
14106	A	2	1.00	0.09	0.00*	0.001	(0.001)	(0.00 ± 0.00)
14201	C	4	1.00	2.73	3.21	3.21	3.21	10.32 ± 2.15
21601	A	2	1.00	4.51	4.86	4.86	4.86	23.61 ± 4.69
21605	C	3	1.00	12.17	25.80	25.80	25.80	665.28 ± 15.94
21730	A	2	1.00	—	0.77	0.77	0.77	0.59 ± 1.00
30302	C	4	1.00	5.27	3.95	3.95	3.95	15.62 ± 2.73
40101	B	2	1.00	—	0.69	0.69	0.69	0.47 ± 1.14
40104	A	2	1.00	0.50	0.27	0.27	0.27	0.07 ± 0.30
40127	A	2	1.00	—	2.15	2.15	2.15	4.63 ± 1.95
40408*	B	2	1.00	5.86	5.95	5.95	5.95	35.38 ± 5.31
40424	C	3	1.00	2.92	1.20	1.20	1.20	1.45 ± 1.01
40433	A	2	1.00	4.57	4.72	4.72	4.72	22.31 ± 4.63
40440	A	2	1.00	0.49	0.00*	0.001	(0.001)	(0.00 ± 0.00)
40442	A	3	1.00	0.35	1.88	1.88	1.88	3.53 ± 1.17
40445	A	2	1.00	4.01	4.80	4.80	4.80	23.05 ± 4.39
40451*	A	2	1.00	3.39	2.05	2.05	2.05	4.23 ± 1.91
40456	A	2	1.00	2.45	3.65	3.65	3.65	13.29 ± 3.05
40465	A	2	1.00	1.62	1.30	1.30	1.30	1.69 ± 1.24
40477*	A	2	1.00	1.49	0.83	0.83	0.83	0.70 ± 0.97
40497	A	2	1.00	1.01	2.13	2.13	2.13	4.53 ± 2.17
41602	A	2	1.00	0.68	0.53	0.53	0.53	0.29 ± 0.49
41703	B	2	1.00	2.46	0.00*	0.001	(0.001)	(0.00 ± 0.00)
41705	A	2	1.00	2.61	2.13	2.13	2.13	4.55 ± 2.46
41719*	C	3	1.00	5.95	8.25	8.25	8.25	67.92 ± 8.70
42202	C	2	1.00	3.35	0.00*	0.001	(0.001)	(0.00 ± 0.00)
43201	A	2	1.00	1.53	0.74	0.74	0.74	0.55 ± 0.94
50103/1[b]	C	2	1.00	16.21	14.14	14.14	14.14	199.85 ± 11.74
50103/2[c]	A	2	1.00	1.85	0.38	0.38	0.38	0.15 ± 1.06
50107	C	2	1.00	25.96	25.37	25.37	25.37	643.44 ± 22.67
50116	C	2	1.00	3.52	0.00*	0.001	(0.001)	(0.00 ± 0.00)
50119*	C	2	1.00	4.46	10.47	10.47	10.47	109.55 ± 11.49
66006*	A	2	1.00	—	4.41	4.41	4.41	19.25 ± 7.19
66008	A	2	1.00	0.77	0.70	0.70	0.70	0.49 ± 0.61
92201	C	2	1.00	2.24	0.00*	0.001	(0.001)	(0.00 ± 0.00)
$\hat{\sigma}_0$				4.08	0.96		0.96	

[a] GPS is not correlated with the other techniques.
[b] Tie between GPS and SLR
[c] Tie between GPS and VLBI

Table E.5: Iterated variance components for the combination of solutions of different techniques, estimating variance components only for space geodetic solutions (approach III).

	III		
		H_{20}/D_{20}	H_{20}
Solution	$\sqrt{(a_i)_1}$	$\hat{\sigma}_i$	$\hat{\sigma}_i^2$
GPS	1.00	7.67	58.90 ± 1.47
SLR	1.00	9.80	96.03 ± 3.03
VLBI	1.00	3.45	11.94 ± 1.09
10002	5.00	(5.00)	(25.00 ± 0.00)
10317	1.00	(1.00)	(1.00 ± 0.00)
10402	1.00	(1.00)	(1.00 ± 0.00)
11001	1.00	(1.00)	(1.00 ± 0.00)
12205	5.00	(5.00)	(25.00 ± 0.00)
12711	3.00	(3.00)	(9.00 ± 0.00)
12717	3.00	(3.00)	(9.00 ± 0.00)
12734	3.00	(3.00)	(9.00 ± 0.00)
13212	1.00	(1.00)	(1.00 ± 0.00)
13407	3.80	(3.80)	(14.44 ± 0.00)
13420	1.00	(1.00)	(1.00 ± 0.00)
14001	3.00	(3.00)	(9.00 ± 0.00)
14106	1.00	(1.00)	(1.00 ± 0.00)
14201	200.00	(200.00)	(40000.00 ± 0.00)
21601	3.00	(3.00)	(9.00 ± 0.00)
21605	5.00	(5.00)	(25.00 ± 0.00)
21730	1.00	(1.00)	(1.00 ± 0.00)
30302	3.00	(3.00)	(9.00 ± 0.00)
40101	3.00	(3.00)	(9.00 ± 0.00)
40104	1.00	(1.00)	(1.00 ± 0.00)
40127	1.00	(1.00)	(1.00 ± 0.00)
40408	3.70	(3.70)	(13.69 ± 0.00)
40424	5.00	(5.00)	(25.00 ± 0.00)
40433	1.00	(1.00)	(1.00 ± 0.00)
40440	2.00	(2.00)	(4.00 ± 0.00)
40442	1.00	(1.00)	(1.00 ± 0.00)
40445	1.00	(1.00)	(1.00 ± 0.00)
40451	1.00	(1.00)	(1.00 ± 0.00)
40456	3.00	(3.00)	(9.00 ± 0.00)
40465	1.00	(1.00)	(1.00 ± 0.00)
40477	1.00	(1.00)	(1.00 ± 0.00)
40497	1.00	(1.00)	(1.00 ± 0.00)
41602	1.00	(1.00)	(1.00 ± 0.00)
41703	12.00	(12.00)	(144.00 ± 0.00)
41705	1.00	(1.00)	(1.00 ± 0.00)
41719	150.00	(150.00)	(22500.00 ± 0.00)
42202	3.00	(3.00)	(9.00 ± 0.00)
43201	1.00	(1.00)	(1.00 ± 0.00)
50103/1	10.00	(10.00)	(100.00 ± 0.00)
50103/2	1.00	(1.00)	(1.00 ± 0.00)
50107	20.00	(20.00)	(400.00 ± 0.00)
50116	3.00	(3.00)	(9.00 ± 0.00)
50119	2.00	(2.00)	(4.00 ± 0.00)
66006	30.00	(30.00)	(900.00 ± 0.00)
66008	2.00	(2.00)	(4.00 ± 0.00)
92201	1.00	(1.00)	(1.00 ± 0.00)
$\hat{\sigma}_0$		0.96	

Tables E.4 and E.5 list the initial solutions related to the combination in section 6.2.1 and 6.2.2, respectively. The column headers have the following meaning:

Solution: Initial solution, labelled by observation technique or site code, respectively. Three solutions derived from space geodetic observations are followed by 46 local-tie-solutions. In table E.4, asterisks designate the 10 representative solutions that have been selected to be plotted in figure 6.9.

$\mathbf{Q}_i$: Type of the covariance matrix as specified in figure 6.8.

Pt.: Number of points (stations) comprised by the solution.

I, II: Approaches in section 6.2.1, where the variance components for the local-tie-solutions are estimated.

III: Approach in section 6.2.2, where the variance components for the local-tie-solutions have been fixed to a priori values.

$\sqrt{(a_i)_1}$: A priori values for the variance components (starting values for the iteration), given in terms of standard deviations.

H_ν: Variance components estimated by Helmert's method after ν iterations, given in terms of standard deviations or variances, respectively. Missing values indicate that the estimation has failed and yielded negative estimates for the variances. If a variance component has been fixed (as described in section 5.4.1, extended model), the respective value is bracketed.

D_ν: Variance components estimated by degree of freedom after ν iterations, given in terms of standard deviations. For the values flagged by an asterisk, convergence is not achieved (cf. figure 6.10).

ISSN: 1612-9733
ISBN: 978-3-86644-206-1

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