http://www.mathematik.uni-karlsruhe.de/~semlv
Seminar LV, No. 28, 10 pp., *03.11.2006*

Characterization of the absolute value of complex linear functionals by functional equations

Karol Baron and Peter Volkmann

Dedicated to Dr. Jan Błaż on his 80th birthday

Introduction. By R, C we denote the spaces of real and complex numbers, respectively, and T denotes the unit circle in C, i.e.

$$T = \{\chi \mid \chi \in C,\ |\chi| = 1\}.$$

If G is an abelian group and $\varphi : G \to R$ is additive, i.e.

$$\varphi(x + y) = \varphi(x) + \varphi(y) \quad (x, y \in G),$$

then the function

$$f(x) = |\varphi(x)| \qquad (x \in G) \tag{0.1}$$

satisfies

$$\max\{f(x + y), f(x - y)\} = f(x) + f(y) \qquad (x, y \in G), \tag{0.2}$$

$$\min\{f(x + y), f(x - y)\} = |f(x) - f(y)| \qquad (x, y \in G). \tag{0.3}$$

According to [4] the functions (0.1) are characterized by (0.2) (but not by (0.3); in [3] a Pexider version of (0.2) has been studied).

Now we are looking for analogous results for the absolute value of complex linear functionals. So let V be a complex vector space, let $\varphi : V \to C$ be linear, and consider

$$f(x) = |\varphi(x)| \qquad (x \in V). \tag{0.4}$$

It is easily seen that f safisfies the functional equations

$$\sup_{\chi \in T} f(x + \chi y) = f(x) + f(y) \qquad (x, y \in V), \tag{X}$$

$$\inf_{\chi \in T} f(x + \chi y) = |f(x) - f(y)| \qquad (x, y \in V), \tag{Y}$$

where in fact the supremum in (X) is a maximum and the infimum in (Y) is a minimum. Our main result is the following:

Theorem 1. *If V is a complex vector space, then each of the functional equations* (X), (Y) *characterizes the functions $f : V \to R$ having the form* (0.4) *with some linear functional $\varphi : V \to C$.*

The proof will be given in the next sections. We start with the one-dimensional case ($V = C$): We consider equation (X) in Section 1 (cf. Proposition 1) and equation (Y) in Section 2 (cf. Proposition 2). Then, after some preparation in Section 3, we treat the general case in Section 4.

The final Section 5 is devoted to a characterization of the absolute value of complex determinants; this is similar to [5], where the real case had been treated by using the functional equation (0.2).

1. The one-dimensional case for equation (X).

Proposition 1. *The solutions $f : C \to R$ of the functional equation*

$$\sup_{\chi \in T} f(z + \chi w) = f(z) + f(w) \qquad (z, w \in C) \tag{1.1}$$

are the functions

$$f(z) = |cz| \qquad (z \in C), \tag{1.2}$$

where $c \in C$.

Proof. From the Introduction we know that the functions (1.2) solve (1.1). Now let $f : C \to R$ be a solution of (1.1). With $z = w = 0$ in (1.1) we get

$$f(0) = 0.$$

Then $z = 0$ in (1.1) gives $\sup_{\chi \in T} f(\chi w) = f(w)$, so f is constant on circles around zero, and we can write

$$f(z) = f(|z|) \qquad (z \in C). \tag{1.3}$$

For $x \geq 0$ we get from (1.1), (1.3) that $2f(x) = \sup_{\chi \in T} f(|1 + \chi| x)$, so

$$2f(x) = \sup f([0, 2x]) \qquad (x \geq 0).$$

This shows f to be increasing with respect to $x \geq 0$, i.e.

$$0 \leq p \leq q \Longrightarrow f(p) \leq f(q). \tag{1.4}$$

Now we shall prove

$$f(x + y) = f(x) + f(y) \qquad (x, y \geq 0). \tag{1.5}$$

By (1.1) we have

$$f(x) + f(y) = \sup_{\chi \in T} f(x + \chi y). \tag{1.6}$$

Since $|x+\chi y| \le x+y$, we get (cf. (1.3), (1.4))

$$f(x+\chi y) = f(|x+\chi y|) \le f(x+y) = f(x+1y) \qquad (\chi \in T),$$

therefore (1.5) follows from (1.6).
We have that $f(x)$ $(x \ge 0)$ is an increasing solution of the Cauchy equation (1.5), and by a result of Gaston Darboux (cf. e.g. [1]) we get $f(x) = cx$ (with some $c \ge 0$); finally (cf. (1.3)) $f(z) = c|z| = |cz|$ for all $z \in C$.

2. The one-dimensional case for equation (Y).

Proposition 2. *The solutions $f : C \to R$ of*

$$\inf_{\chi \in T} f(z+\chi w) = |f(z) - f(w)| \qquad (z, w \in C) \tag{2.1}$$

are the functions

$$f(z) = |cz| \qquad (z \in C), \tag{2.2}$$

where $c \in C$.

Proof. 1. Again we know that the functions (2.2) solve (2.1). Conversely, let $f : C \to R$ be a solution of (2.1). Starting as in the proof of Proposition 1, we find $f(0) = 0$ and

$$f(z) = f(|z|) \ge 0 \qquad (z \in C). \tag{2.3}$$

Equation (2.1) also implies

$$f(z) - f(w) \le |f(z) - f(w)| \le f(z-w),$$

therefore f is subadditive, i.e.

$$f(z+w) \le f(z) + f(w) \qquad (z, w \in C).$$

From this and (2.3) we get for $x \ge 0$ and $\chi \in T$ that

$$f(|1+\chi|x) = f(x+\chi x) \le f(x) + f(\chi x) = 2f(x),$$

and using $\sup\limits_{\chi \in T} f(|1+\chi|x) = \sup f([0, 2x])$ we have

$$\sup f([0, 2x]) \le 2f(x) \qquad (x \ge 0). \tag{2.4}$$

For $0 \le x \le t$ we have (cf. (2.1), (2.3))

$$|f(t) - f(x)| = \inf_{\chi \in T} f(t+\chi x) = \inf_{\chi \in T} f(|t+\chi x|) = \inf f([t-x, t+x]). \tag{2.5}$$

Observing that the interval $[t-x, t+x]$ increases with x (and the infimum decreases) leads to the following:

$$0 \le x \le y \le t \Longrightarrow |f(t) - f(y)| \le |f(t) - f(x)|. \tag{2.6}$$

2. Without loss of generality we assume

$$f(z) \not\equiv 0, \tag{2.7}$$

and we prove

$$\lim_{x \in R, x \to \infty} f(x) = \infty. \tag{2.8}$$

If (2.8) does not hold, then there are numbers $\alpha, a_1, a_2, a_3, \ldots \geq 0$ such that

$$a_1 < a_2 < a_3 < \ldots \to \infty$$

and $f(a_n) \to \alpha$ $(n \to \infty)$. With $x = a_n,\ t = a_{n+1}$ in (2.6) we obtain

$$|f(a_{n+1}) - f(y)| \leq |f(a_{n+1}) - f(a_n)| \qquad (a_n \leq y \leq a_{n+1}),$$

and from this we easily get $f(y)$ arbitrarily close to α for all sufficiently large real numbers y, i.e.

$$\lim_{x \in R, x \to \infty} f(x) = \alpha. \tag{2.9}$$

For $0 \leq y \leq x$ we have $|x + \chi y| \geq x - y$ $(\chi \in T)$, and from (2.3) we get

$$f(x + \chi y) \geq \inf f([x - y, \infty[) \quad (\chi \in T).$$

Then (2.1) implies

$$|f(x) - f(y)| \geq \inf f([x - y, \infty[) \qquad (0 \leq y \leq x).$$

With $x \to \infty$, (2.9) gives

$$|\alpha - f(y)| \geq \alpha \qquad (0 \leq y),$$

and then $y \to \infty$ gives $\alpha = 0$. So we have (2.9) with $\alpha = 0$, and then (2.4) implies $f(x) = 0$ $(x \geq 0)$. This is contradictory to (2.3), (2.7), and hence (2.8) is proven.

3. The function $f(x)$ $(x \geq 0)$ is increasing, i.e.

$$0 \leq x \leq y \Longrightarrow f(x) \leq f(y).$$

In fact, if $0 \leq x \leq y$, then because of (2.8) we can choose $t \geq y$ such that $f(t) \geq \max\{f(x), f(y)\}$, and (2.6) yields $f(x) \leq f(y)$.

Now (2.5) can be rewritten as

$$f(t) - f(x) = f(t - x) \qquad (0 \leq x \leq t),$$

which is equivalent to the Cauchy equation

$$f(x + y) = f(x) + f(y) \qquad (x, y \geq 0).$$

The end of the proof can be done as for Proposition 1.

3. A general result on the absolute value of linear functionals.

Lemma 1. *Let $f : G \to R$ be subadditive on an abelian group G, and suppose*

$$f(-x) = f(x) \qquad (x \in G).$$

Then $f(x_0) = 0$ implies

$$f(x + x_0) = f(x) \qquad (x \in G).$$

Proof. Subadditivity of f means

$$f(x + y) \leq f(x) + f(y) \qquad (x, y \in G).$$

Now, if $f(x_0) = 0$, then also $f(-x_0) = 0$, hence

$$\begin{aligned} f(x + x_0) &\leq f(x) + f(x_0) = f(x) = f(x + x_0 - x_0) \\ &\leq f(x + x_0) + f(-x_0) = f(x + x_0). \end{aligned}$$

Theorem 2. *Suppose $\Lambda = R$ or $\Lambda = C$, and let V be a vector space over Λ. A function $f : V \to R$ has the form*

$$f(x) = |\varphi(x)| \qquad (x \in V) \tag{3.1}$$

with a linear $\varphi : V \to \Lambda$ if and only if the following conditions are satisfied:

(P) $f(x + y) \leq f(x) + f(y)$ $(x, y \in V)$,

(Q) $f(\lambda x) = |\lambda| f(x)$ $(\lambda \in \Lambda, x \in V)$,

(R) *if U is a two-dimensional subspace of V, then there is $x_0 \in U, x_0 \neq 0$, such that $f(x_0) = 0$.*

Proof. If (3.1) holds with a linear $\varphi : V \to \Lambda$, then (P), (Q), (R) are easily checked. Conversely, suppose $f : V \to R$ to satisfy (P), (Q), (R). We can apply Lemma 1 to get

$$x, y \in V, f(y) = 0 \Longrightarrow f(x + y) = f(x). \tag{3.2}$$

Because of (P), (Q) the set

$$W = \{x \mid x \in V,\ f(x) = 0\}$$

is a subspace of V. From (R) it follows that $W = V$ or $\mathrm{codim}_V W = 1$. In the first case (3.1) holds with $\varphi(x) \equiv 0$. In the second case we take $x_0 \in V$ such that $f(x_0) = 1$. Then

$$V = \Lambda x_0 \oplus W$$

(direct sum), and (Q), (3.2) imply

$$x = \lambda x_0 + y,\ y \in W \Longrightarrow f(x) = |\lambda|. \tag{3.3}$$

Therefore (3.1) holds with $\varphi(x) = \lambda$, λ being taken from the decomposition of x in (3.3).

4. Application to the equations (X), (Y). Here V is a complex vector space, and we like to apply Theorem 2 (with $\Lambda = C$) to our functional equations

$$\text{(X)} \qquad \sup_{\chi \in T} f(x + \chi y) = f(x) + f(y) \qquad (x, y \in V),$$

$$\text{(Y)} \qquad \inf_{\chi \in T} f(x + \chi y) = |f(x) - f(y)| \qquad (x, y \in V).$$

If $f : V \to R$ is a solution of (X) or (Y), then (Q) holds: To see this, fix $x \in V$ and replace x by zx, y by wx in the equation, where $z, w \in C$. Then Proposition 1 or Proposition 2, respectively, yields $f(\lambda x) = |\lambda| f(x)$ $(\lambda \in C)$.

For the solutions of (X) and of (Y) also (P) holds: For equation (X) this is obvious, for equation (Y) the same reasoning as for the subadditivity of f in the beginning of the proof of Proposition 2 works.

From (P), (Q) it follows that the solutions of (X), (Y) are convex, hence continuous on finite-dimensional subspaces of V (endowed with their natural topology), and so $\sup, \inf$ in (X), (Y) are in fact $\max, \min$, respectively.

Now we like to show that the solutions of (X), (Y) also have property (R); this finishes the proof of Theorem 1. We start with equation (Y), because it is easier to handle: So let $f : V \to R$ solve (Y). To prove (R) let U be a two-dimensional subspace of V. We take linearly independent $x, y \in U$. If $f(x) = 0$ or $f(y) = 0$, we are done. If not, we may assume $f(x) = f(y)$ (cf. (Q)). Then by (Y) (the infimum being a minimum) there is a $\chi \in T$ such that $f(x + \chi y) = 0$, and we can take $x_0 = x + \chi y$.

For proving (R) for solutions of (X) we use the following simple lemma:

Lemma 2. *Let K be a compact, convex set in the euclidean plane, and suppose K to have interior points. Let Γ be a circle with maximal radius ρ contained in K. Then Γ has at least two points in common with the boundary ∂K of K.*

Proof. There is at least one point $P \in \Gamma \cap \partial K$. Suppose $\Gamma \cap \partial K = \{P\}$, and let M be the midpoint of Γ. Then it is possible to shift Γ a little bit in the direction from P to M such that the shifted circle is contained in the interior of K, and then there is a parallel circle Γ_1 in K with a radius $\rho_1 > \rho$. This is in contradiction to the maximality of ρ.

Now let $f : V \to R$ solve (X). If (R) is not true, there is a two-dimensional subspace U of V such that $x \in U \setminus \{0\}$ implies $f(x) \neq 0$. Because of (P), (Q), the function f then is a norm on U. Identifying U with C^2 and writing

$\| \cdot \| : C^2 \to [0, \infty[$ for this norm, (X) implies

$$\max_{\chi \in T} \|x + \chi y\| = \|x\| + \|y\| \qquad (x, y \in C^2). \tag{4.1}$$

For $x, y \neq 0$ we can show that the maximum is attained in exactly one point $\chi \in T$. Namely, let us suppose

$$\|x + \varepsilon_1 y\| = \|x + \varepsilon_2 y\| = \|x\| + \|y\|$$

with some $\varepsilon_1, \varepsilon_2 \in T$. Then there is $\eta \in T$ such that

$$2\|x\| + 2\|y\| = \|x + \varepsilon_1 y\| + \|x + \varepsilon_2 y\| = \|x + \varepsilon_1 y + \eta(x + \varepsilon_2 y)\| =$$
$$= \|(1 + \eta)x + (\varepsilon_1 + \eta\varepsilon_2)y\| \leq |1 + \eta| \cdot \|x\| + |\varepsilon_1 + \eta\varepsilon_2| \cdot \|y\| \leq 2\|x\| + 2\|y\|,$$

where $\leq$ is equality if and only if $\eta = 1$ and $\varepsilon_1 = \varepsilon_2$.

In C^2 we consider the set

$$S = \{(1, \lambda) \mid \lambda \in C\}.$$

The distance of two points $(1, \lambda_1),\ (1, \lambda_2)$ is

$$\|(0, \lambda_1 - \lambda_2)\| = |\lambda_1 - \lambda_2| \cdot \|(0, 1)\|,$$

and therefore S can be considered as a (real) euclidean plane. The circles Γ in S have the form

$$\Gamma = \{x + \chi y \mid \chi \in T\}, \tag{4.2}$$

where $x \in S$ and $y = (0, r)$ with some $r > 0$. We have $(1, 0) \in S$, therefore S contains at least one point of norm $\beta = \|(1, 0)\|$. If we choose $\gamma > \beta$, then the set

$$K = S \cap \{u \mid u \in C^2,\ \|u\| \leq \gamma\}$$

is a subset of the euclidean plane S having all the properties mentioned in Lemma 2. According to that lemma, there is a circle (4.2) having two points in common with $\partial K = S \cap \{u \mid u \in C^2,\ \|u\| = \gamma\}$. This means that for the function $\|x + \chi y\|$ $(\chi \in T)$ the maximal value γ is attained for two different $\chi_1, \chi_2 \in T$, in contradiction to the above discussion about the maximum (4.1).

5. The absolute value of complex determinants. For $\Lambda = R$ and $\Lambda = C$ the function

$$F(x_1, \ldots, x_n) = |\det(x_1, \ldots, x_n)| \qquad (x_1, \ldots, x_n \in \Lambda^n) \tag{5.1}$$

can be characterized as

$$F : \Lambda^n \times \ldots \times \Lambda^n \to [0, \infty[\tag{5.2}$$

such that:

(A) $F(\dots, a, \dots, b, \dots) = F(\dots, a+b, \dots, b, \dots) = F(\dots, a, \dots, a+b, \dots)$,

(B) $F(\dots, \lambda a, \dots) = |\lambda| F(\dots, a, \dots)$,

(C) $F(e_1, \dots, e_n) = 1$.

Here

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Concerning this characterization, cf. e.g. [2] for the case $\Lambda = R$; it also holds for $\Lambda = C$, the proof is not difficult. As a simple consequence we get:

Theorem 3. *Suppose $F : \Lambda^n \times \dots \times \Lambda^n \to R$, where $\Lambda = R$ or $\Lambda = C$. Then* (5.1) *holds if and only if* (B), (C) *and the following two conditions are satisfied:*

(D) $F(\dots, a, \dots, a, \dots) = 0$,

(E) $F(x_1, \dots, x_n)$ *is subadditive in each variable.*

Proof. 1. The function (5.1) fulfills the conditions (B), (C), (D), (E).

2. Conversely, suppose (B), (C), (D), (E) to hold. Because of (B), (E) we have (5.2), and it is sufficient to prove (A). So let us show

$$F(a, b, x_3, \dots, x_n) = F(a+b, b, x_3, \dots, x_n); \tag{5.3}$$

then (A) follows because (5.3) also will be true when permuting the arguments. We define $f : \Lambda^n \to [0, \infty[$ as

$$f(x) = F(x, b, x_3, \dots, x_n) \qquad (x \in \Lambda^n).$$

Because of (E), this function is subadditive, from (D) we have $f(b) = 0$, and from (B) we know $f(-x) = f(x)$. By Lemma 1 in Section 3 we get $f(a+b) = f(a)$, and this is just (5.3).

Remark 1. Theorem 3 is very similar to the well known characterization of the function

$$F(x_1, \dots, x_n) = \det(x_1, \dots, x_n) \qquad (x_1, \dots, x_n \in \Lambda^n)$$

(Λ being an arbitrary field) by the properties (C), (D) and

(B′) $F(\dots, \lambda a, \dots) = \lambda F(\dots, a, \dots)$,

(E′) $F(x_1, \dots, x_n)$ *is additive in each variable.*

Remark 2. The main result of [5] is Theorem 3 for the case $\Lambda = R$ with (E) replaced by the following condition:

(F) *The functions* $f(x) = F(x_1, \ldots, x_{k-1}, x, x_{k+1}, \ldots, x_n)$ $(x \in R^n)$ *are solutions of*

$$\max\{f(x+y), f(x-y)\} = f(x) + f(y) \qquad (x, y \in R^n). \tag{5.4}$$

This result now is a simple consequence of our Theorem 3: Every solution of (5.4) obviously is subadditive, hence (F) implies (E). It can be added that (F) works in the same way if we replace (5.4) by the functional equation

$$\min\{f(x+y), f(x-y)\} = |f(x) - f(y)| \qquad (x, y \in R^n), \tag{5.5}$$

for its solutions $f : R^n \to R$ also are subadditive. According to the Introduction, the absolute values of additive functions on R^n are solutions of (5.5), but (cf. [4]) they are not characterized by this functional equation.

Theorem 4. *For* $F : C^n \times \ldots \times C^n \to R$ *we have*

$$F(x_1, \ldots, x_n) = |\det(x_1, \ldots, x_n)|$$

if and only if (C), (D), *and the following condition are fulfilled:*

(G) *The functions* $f(x) = F(x_1, \ldots, x_{k-1}, x, x_{k+1}, \ldots, x_n)$ $(x \in C^n)$ *solve at least one of the functional equations*

$$\sup_{\chi \in T} f(x + \chi y) = f(x) + f(y) \qquad (x, y \in C^n), \tag{5.6}$$

$$\inf_{\chi \in T} f(x + \chi y) = |f(x) - f(y)| \qquad (x, y \in C^n). \tag{5.7}$$

Proof. Because of Theorem 3 it is sufficient to show that (G) implies (B), (E). This is true: the solutions of (5.6) and of (5.7) have the properties (P), (Q) $(\Lambda = C,\ V = C^n$; cf. Section 4).

Acknowledgement. This research was supported by the Mathematics Department of the Silesian University at Katowice (Iterative Functional Equations and Real Analysis Program). The paper had been written during the second author's stay at Katowice in August 2006; he gratefully acknowledges the hospitality of the Mathematics Department.

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Typescript: Marion Ewald.

Authors' addresses:
K. Baron, Instytut Matematyki, Uniwersytet Śląski, Bankowa 14, 40-007 Katowice, Poland.
P. Volkmann, Institut für Analysis, Universität, 76128 Karlsruhe, Germany.