Cohomological Representations and Twisted Rankin-Selberg Convolutions

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Chapter 0

Introduction

Let $\mathbb{A} = \mathbb{A}(\mathbb{Q})$ be the ring of adèles of the field of rational numbers and $\mathbb{A}_f$ its finite part. Further, for an integer $n \geq 3$ let π and σ be cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ and $\mathrm{GL}_{n-1}(\mathbb{A})$, respectively. Note, that for us an automorphic representation of $\mathrm{GL}_n(\mathbb{A})$ is a topological $\mathrm{GL}_n(\mathbb{A})$-module, whose subspace of admissable vectors is an automorphic representation of the Hecke algebra $\mathscr{H}$ of $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{O}_n(\mathbb{R})$ (cf. [BJ], Section 4.6).

For such pairs (π, σ), Jacquet, Piatetski-Shapiro, and Shalika introduced an L-function $L(\pi, \sigma; s)$, that "would be" the L-function of an automorphic representation $\pi \otimes \sigma$ of $\mathrm{GL}_{n(n-1)}(\mathbb{A})$ (cf. [JPS4]). Here, the tensor product notation is in accordance to the Langlands dictionary, whenever the automorphic representation $\pi \otimes \sigma$ exists. $L(\pi, \sigma; s)$ is defined as a certain Rankin-Selberg integral, which is an entire function in s, and satisfies a functional equation of the expected type, interchanging s and $1 - s$. The family of those functions $L(\pi, \sigma; s)$ for fixed π and varying σ plays a key role in the converse theorems, whose goal is to describe an automorphic representation π analytically by its Mellin transform (cf. [CP1]).

Having such a well-working theory of complex valued L-functions already, it is only natural to try to step further. So for a fixed pair (π, σ) like before we are interested in the package of twisted L-functions $L(\pi \otimes \chi, \sigma; s)$, where χ runs through all finite Dirichlet characters. Let s_i with $i = 1, 2, \ldots$ be the critical values (cf. [Del]) of the variable s for the fixed pair (π, σ). We want to show that the function $\chi \mapsto L(\pi \otimes \chi, \sigma; s_i)$, after division by an appropriate period depending only on i and the sign of χ, takes algebraic numbers as values. Furthermore, we want to control the number fields containing those values.

In the cases $n = 2, 3$ those questions have been answered by Manin, Mazur, Swinnerton-Dyer, and Schmidt, under suitable assumptions concerning the infinity components $(\pi_\infty, \sigma_\infty)$ and the components (π_p, σ_p) at a fixed place p (cf. [Man], [MS], [Schm1]). The basic requirement on the infinity components is, that there indeed are critical values, or more specifically, that π_∞ and σ_∞ have non-trivial Lie algebra cohomology.

For general n results are known only for a certain class of cuspidal automorphic representations: In [KMS], Kazhdan, Mazur, and Schmidt discuss the case of representations, that “occur in cohomology” with constant coefficients. In this case, $s = \frac{1}{2}$ is the only critical value. The authors fix a prime p throughout the whole paper and assume, that both π and σ are unramified at p. Further they make some simplifying assumptions on the nature of the twisting Dirichlet character χ, in particular its conductor has to be a power f of p. They study the product

$$P_\infty(s) \cdot L(\pi \otimes \chi, \sigma; s),$$

where $P_\infty(s)$ is an entire function of which one knows that it can be written as an integral depending only on the Whittaker models of the infinity parts π_∞ and σ_∞. Since π and σ are cohomological, the function $P_\infty(s)$ even is uniquely determined by the number n. Kazhdan, Mazur, and Schmidt express the special value $P_\infty(\frac{1}{2}) \cdot L(\pi \otimes \chi, \sigma; \frac{1}{2})$ by relative modular symbols and show that this value is indeed an algebraic number.

In this thesis, we generalise the results of [KMS] by allowing a larger class of cohomological representations studied before by Mahnkopf (cf. [Mah]). The precise definition of this class is given in section 1.1; roughly speaking, the cohomology modules, in which our representations π and σ occur, may now have coefficients in certain finite dimensional irreducible representations $\check{M}_{\mu,\mathbb{C}}$ of $\mathrm{GL}_n(\mathbb{Q})$ resp. $\check{M}_{\nu,\mathbb{C}}$ of $\mathrm{GL}_{n-1}(\mathbb{Q})$. For the sake of coherence we show that the cohomological representations in [KMS] are indeed just a special case of ours (cf. Lemma 1.1).

The main result is Theorem 1 below, which describes $P_{\lambda,\infty}(\frac{1}{2}) \cdot L(\pi \otimes \chi, \sigma; \frac{1}{2})$ in terms of algebraic numbers and values of a pairing $\mathscr{B}_\lambda$ on cohomology. Here, $P_{\lambda,\infty}(s)$ is a modification of the entire function $P_\infty(s)$ of [KMS], depending on an at first arbitrary linear form λ on the tensor product of the coefficient systems of the cohomology modules, in which π and σ occur. Explicitly, for $n \geq 3$, under some assumptions on the infinity components of the central characters ω_π of π and ω_σ of σ, and the same assumptions on the Dirichlet character χ like in [KMS] we get

Theorem 1

$$\begin{aligned} & w_p(1) v_p(1) P_{\lambda,\infty}\left(\frac{1}{2}\right) \prod_{i=1}^{n-1} \frac{G(\chi_p^i)(1-p^{-1})}{1-p^{-i}} L(\pi \otimes \chi, \sigma; \frac{1}{2}) \\ = & \sum_u \prod_{i=1}^{n-1} \tilde{\chi}_p(u_i^i) \operatorname{vol}(K'_{u^{\varphi^{-1}}}) \mathscr{B}_\lambda((u^{-1})^{\varphi^{-1}}), \end{aligned}$$

where $u = u_p$ (with $u_\ell = 1$ for all $\ell \neq p$) is taken from a representative system for $U_n(\mathbb{Z}_p)$ modulo $U_n(\mathbb{Z}_p)^\varphi$ with $\varphi = \operatorname{diag}(f^{-1}, \ldots, f^{-n})$.

Here, w_p and v_p are the p-components of certain Whittaker functions, $\tilde{\chi}_p$ denotes the continuation of χ_p to $\mathbb{Z}_p$ by $\tilde{\chi}_p(px) = 0$ for all $x \in \mathbb{Z}_p$, and $G(\chi_p)$ denotes the Gauß sum of χ_p (cf. Chapter 2). Finally, the $K'_{u^{\varphi^{-1}}}$ are certain compact subgroups of $\mathrm{GL}_{n-1}(\hat{\mathbb{Z}})$.

We prove the theorem in Chapter 3 by constructing the pairing $\mathscr{B}_\lambda$ carefully. In Section 3.8 we show that we still have the freedom to choose $\mathscr{B}_\lambda$ in such a way that $P_{\lambda,\infty}(\frac{1}{2}) \cdot L(\pi \otimes \chi, \sigma; \frac{1}{2})$ is indeed an algebraic number. This works because $\mathscr{B}_\lambda$ (and λ) may be chosen respecting a $\mathbb{Q}$-structure, the Whittaker models of the finite parts π_f and σ_f are already defined over some number field by [Clo], and because of the one-dimensionality of certain relative Lie algebra cohomology modules proved in Corollary 1.5.

There is, however, a blemish to our result, since we cannot guarantee that $P_{\lambda,\infty}(\frac{1}{2})$ does not vanish in general. In Chapter 4 we will study this problem for $n = 3$ and will finally show

Theorem 2 *Let $n = 3$ and assume the coefficient modules $\check{M}_{\mu,\mathbb{C}}$ and $\check{M}_{\nu,\mathbb{C}}$ to be trivial. Then the period $P_{\lambda,\infty}(\frac{1}{2})$ in Theorem* 1 *does not vanish.*

This result makes use of the fact, that different sets of submodules of the relative Lie algebra cohomology can be used to prove Theorem 1. We can show that there is one such set for which $P_{\lambda,\infty}(\frac{1}{2})$ does not vanish.

Finally, in Chapter 5 we will discuss, how Theorem 2 may be generalised to cohomological representations like in Chapter 1. We will see, that in those cases it will be of interest to determine whether or not $s = \frac{1}{2}$ is a critical value.

At this place I want to use the opportunity to thank all those, whose support helped to bring this thesis into being. First and foremost, my gratitude goes to my thesis advisor, Prof. Dr. Claus-Günther Schmidt. The many fruitful discussions with him have been of great value for me. In particular, I wish to thank him for providing me with the main ideas of Section 4.5. His understanding, encouraging and personal guidance have provided a good basis for the present thesis. Also it is a pleasure for me to thank PD Dr. Stefan Kühnlein, who patiently helped me in countless details and thoroughly revised my work. His talent for translating problems into a language I could understand was vital for my progressing. Furthermore, I want to thank Prof. Laurent Clozel for supervising me during my research stay at the Université Paris-Sud. His profound knowledge in the field of automorphic representations proved very precious to me. Finally, I wish to thank Dr. Oliver Baues and Sebastian Holzmann for helpful discussions about Lie algebra theory and the members of the "Kaffeerunde" for everything, even for their odd sense of humor.

Chapter 1

Representations with non–vanishing cohomology

The first aim of this thesis is to generalise a theorem of Kazhdan, Mazur, and Schmidt, Theorem 3.5 of [KMS], concerning the algebraicity of the critical values of twisted Rankin-Selberg L-functions of pairs of automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ resp. $\mathrm{GL}_{n-1}(\mathbb{A})$. The representations they consider fulfil a special property, namely, they *occur in cohomology with constant coefficients* (cf. [loc.cit.], p. 99). In this chapter we want to introduce a larger class of representations that contains those cohomological ones. More precisely, we will study representations that occur in cohomology with coefficients in certain finite dimensional representations of $\mathrm{GL}_n(\mathbb{Q})$. In Chapter 3 we will see that the algebraicity result still holds for those new representations.

1.1 The coefficient systems

Let $n \in \mathbb{N}$ be a natural number. Throughout this thesis we write Z_n for the centre of GL_n and set $K_{n,\infty} = \mathrm{SO}_n(\mathbb{R}) Z_n(\mathbb{R})^0$, where by $Z_n(\mathbb{R})^0$ we mean the connected component of the neutral element in $Z_n(\mathbb{R})$. We will always use small letters to identify the respective Lie algebras $\mathfrak{gl}_n$, $\mathfrak{sl}_n$, $\mathfrak{so}_n$, $\mathfrak{z}_n$, $\mathfrak{k}_{n,\infty}$, ... of $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{SO}_n(\mathbb{R})$, $Z_n(\mathbb{R})$, $K_{n,\infty}$, ...

As we will prove in Section 1.4 being cohomological is a local property at infinity:

Lemma 1.1 *An irreducible cuspidal automorphic representation π occurs in cohomology in the sense of [KMS] if and only if*

$$H^{\bullet}_{\mathfrak{g}\mathrm{K}}(\mathfrak{gl}_n, K_{n,\infty}; \pi_\infty) \neq 0.$$

We want to generalise this notion. The idea is to consider irreducible cuspidal automorphic representations π that occur in cohomology modules of the kind

$H^\bullet_{\mathfrak{g}K}(\mathfrak{gl}_n, K_{n,\infty}; \pi_\infty \otimes M)$, where M is some finite dimensional rational representation of $\mathrm{GL}_n(\mathbb{R})$. The next step is to make precise how M may look like. In order to do this, we follow the work of Mahnkopf (cf. Chapter 3 of [Mah]).

Let $B_n = T_n U_n$ denote the group of upper triangular matrices in GL_n. Here, T_n resp. U_n is the standard maximal torus in GL_n resp. the unipotent radical of B_n. Let $X(T_n)$ be the set of algebraic characters $\nu : T_n \to \mathbb{G}_m$ of T_n. Then $X^+(T_n)$ resp. $X^{++}(T_n)$ denotes the set of dominant resp. dominant regular weights in $X(T_n)$. We identify $\mathbb{Z}^n$ with $X(T_n)$ by sending $\mu = (\mu_i)$ to $t \mapsto \prod_i t_i^{\mu_i}$. For a weight $\mu \in X^+(T_n)$ we denote by (ϱ_μ, M_μ) the irreducible algebraic representation of $\mathrm{GL}_n(\mathbb{Q})$ of highest weight μ. Note, that by Satz 1 of [Kra], III.1.4, such a representation always exists, and is unique up to equivalence. Since by [Clo], p. 122, the representation $\varrho_\mu : \mathrm{GL}_n(\mathbb{Q}) \to \mathrm{GL}(M_\mu)$ is defined over $\mathbb{Q}$, we may assume that M_μ is a $\mathbb{Q}$-vector space. For any extension $E/\mathbb{Q}$ we set $M_{\mu,E} := M_\mu \otimes E$.

With this new vocabulary get a more general notion of cohomological representations as follows (cf. [Mah], 3.1.1):

Definition 1.2 *The set of all irreducible cuspidal automorphic representations* π *of* $\mathrm{GL}_n(\mathbb{A})$ *satisfying*

$$H^\bullet_{\mathfrak{g}K}(\mathfrak{gl}_n, K_{n,\infty}; \pi_\infty \otimes M_{\mu,\mathbb{C}}) \neq 0 \tag{1.1}$$

for the relative Lie algebra cohomology is called $\mathrm{Coh}(\mathrm{GL}_n, \mu)$. *Here,* π_∞ *is the infinity component of* π.

Lemma 1.1 now reads: $\mathrm{Coh}(\mathrm{GL}_n, 0)$ is the set of all representations that occur in cohomology in the sense of [KMS]. In that case, $\mu = 0$ is the dominant weight and ϱ_μ is the trivial representation.

As a concluding remark in this section we should mention that as an effect of (1.1) not every dominant weight occurs as maximal weight in the coefficient system of a cohomological representation. More precisely, let us denote by $\check{\mu} \in X^+(T_n)$ the dual weight of μ, i. e. $\check{\mu}$ is the highest weight of the contragredient representation $(\check{\varrho}_\mu, \check{M}_\mu)$. As before, for any extension $E/\mathbb{Q}$ we set $\check{M}_{\mu,E} := \check{M}_\mu \otimes E$. Let further $W_{\mathrm{GL}_n} = W_{\mathrm{GL}_n}(T_n)$ be the Weyl group of GL_n, and w_{GL_n} its longest element. We view W_{GL_n} as subgroup of GL_n as usual. Then we define $X_0^+(T_n)$ resp. $X_0^{++}(T_n)$ to be the set of all dominant resp. dominant regular weights $\mu \in X(T_n)$ satisfying

$$\mu + w_{\mathrm{GL}_n}\mu = (\mathsf{wt}(\mu), \dots, \mathsf{wt}(\mu)) \tag{1.2}$$

for some $\mathsf{wt}(\mu) \in \mathbb{Z}$, where the vector on the right side should be identified with a weight in $X(T_n)$ like above. We call $\mathsf{wt}(\mu)$ the *weight* of the weight vector μ. Since $\check{\mu} = -w_{\mathrm{GL}_n}\mu$ (cf. [Kra], III.1.4), (1.2) amounts to saying that μ is self-contragredient up to twist. In particular, (1.1) implies that $\mu \in X_0^+(T_n)$ for non-empty $\mathrm{Coh}(\mathrm{GL}_n, \mu)$ by Lemme 4.9 in [Clo].

1.2 The action of the group of connected components

For later use we are interested in a submodule of $H^\bullet_{\mathfrak{g}K}(\mathfrak{gl}_n, K_{n,\infty}; \pi_\infty \otimes M_{\mu,\mathbb{C}})$ that is one-dimensional as a $\mathbb{C}$-vector space and easy to describe. In Corollary 1.5 we will see that under minor assumptions $H^\bullet_{\mathfrak{g}K}(\mathfrak{gl}_n, K_{n,\infty}; \pi_\infty \otimes M_{\mu,\mathbb{C}})^{\mathrm{O}_n(\mathbb{R})}$ is a suitable choice. In order to get there we study the action of the group $\pi_0(\mathrm{O}_n) = \mathrm{O}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$ of connected components on cohomology. The results of this section are taken from [Mah], 3.1.2.

If we put $\pi_0(\mathrm{GL}_n) = \mathrm{GL}_n(\mathbb{R})/\mathrm{GL}_n(\mathbb{R})^0$ and $\pi_0(Z_n) = Z_n/Z_n^0$ we see, that all three groups are isomorphic to the multiplicative group $\{\pm 1\}$ of order 2, and all their groups of characters are isomorphic to the group $\{\mathbf{1}, \mathrm{sgn}\}$ of (complex) characters of $\{\pm 1\}$.

We embed $\pi_0(Z_n) \hookrightarrow Z_n(\mathbb{R})$ via $\pm 1 \mapsto \pm \mathbf{1}_n$, where $\mathbf{1}_n$ is the unit matrix of $\mathrm{GL}_n(\mathbb{R})$, hence we may restrict characters ω on $Z_n(\mathbb{R})$ or on $T_n(\mathbb{R})$ to $\pi_0(Z_n)$. For simplicity, we write this $\omega|_{\pi_0}$. Moreover, the inclusion map $\mathrm{O}_n(\mathbb{R}) \hookrightarrow \mathrm{GL}_n(\mathbb{R})$ induces an isomorphism $\pi_0(\mathrm{O}_n) \cong \pi_0(\mathrm{GL}_n)$, hence we may identify $\pi_0(\mathrm{O}_n) = \pi_0(\mathrm{GL}_n)$. On the other hand, the inclusion $Z_n(\mathbb{R}) \hookrightarrow \mathrm{GL}_n(\mathbb{R})$ induces an isomorphism $\pi_0(Z_n) \cong \pi_0(\mathrm{GL}_n)$ only if n is odd. Hence, in the case n is odd we may identify $\pi_0(Z_n) = \pi_0(\mathrm{GL}_n)$ as well as their character groups. Furthermore we may embed $\pi_0(Z_n) \hookrightarrow \mathrm{GL}_n(\mathbb{R})$ via $\pm 1 \mapsto \pm \mathbf{1}_n$ and, again, restrict characters on $\mathrm{GL}_n(\mathbb{R})$ to $\pi_0(Z_n)$.

Let $\pi \in \mathrm{Coh}(\mathrm{GL}_n, \check{\mu})$ for a weight $\mu \in X_0^+(T_n)$. $\mathrm{SO}_n(\mathbb{R})$ is a normal subgroup in $\mathrm{O}_n(\mathbb{R})$, since it has index 2. We have an action of $\mathrm{O}_n(\mathbb{R})$ on

$$C^\bullet_{\mathfrak{g}K}(\mathfrak{gl}_n, K_{n,\infty}; \pi_\infty \otimes \check{M}_{\mu,\mathbb{C}}) \overset{\text{[BW], I.5.1}}{=} \left(\bigwedge^\bullet (\mathfrak{gl}_n/\mathfrak{k}_{n,\infty})^* \otimes \pi_\infty \otimes \check{M}_{\mu,\mathbb{C}} \right)^{K_{n,\infty}}$$

via the $\mathrm{GL}_n(\mathbb{R})$-module structure of the factors. Note, that the results of section I.5 of [BW] hold in this case, although $K_{n,\infty}$ is not compact. Since the action of $\mathrm{SO}_n(\mathbb{R})$ is trivial, we have even an action of the quotient $\pi_0(\mathrm{GL}_n) = \mathrm{O}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$.

We want to determine the structure of our cohomology $H^\bullet_{\mathfrak{g}K}(\mathfrak{gl}_n, K_{n,\infty}; \pi_\infty \otimes \check{M}_{\mu,\mathbb{C}})$ as a $\pi_0(\mathrm{GL}_n)$-module. We set

$$\hat{\pi}_0(\mathrm{GL}_n, \mu) = \begin{cases} \{\mathbf{1}, \mathrm{sgn}\} & \text{if } n \text{ is even ,} \\ \{\mu|_{\pi_0(Z_n)}\} = \{\mathrm{sgn}^{\mathsf{wt}(\mu)/2}\} & \text{if } n \text{ is odd.} \end{cases}$$

Using the identification $\pi_0(Z_n) = \pi_0(\mathrm{GL}_n)$ in the case n odd we may view $\hat{\pi}_0(\mathrm{GL}_n, \mu)$ as a set of characters of $\pi_0(\mathrm{GL}_n)$. We note that $\hat{\pi}_0(\mathrm{GL}_n, \mu) = \hat{\pi}_0(\mathrm{GL}_n, \check{\mu})$, since $\check{\mu} = -w_{\mathrm{GL}_n}\mu$. Lemme 3.14 in [Clo] (and its proof) then shows that $H^\bullet_{\mathfrak{g}K}(\mathfrak{gl}_n, \mathrm{O}_n(\mathbb{R})Z_n(\mathbb{R})^0; \varepsilon_\infty \otimes \pi_\infty \otimes \check{M}_{\mu,\mathbb{C}}) \neq 0$ only if $\varepsilon_\infty \in \omega_\pi|_{\pi_0}\hat{\pi}_0(\mathrm{GL}_n, \check{\mu})$, where ω_π is the *central character* of π. We set

$$m^\bullet_{\mathrm{GL}_n} := \dim_{\mathbb{C}} H^\bullet_{\mathfrak{g}K}(\mathfrak{gl}_n, \mathrm{O}_n(\mathbb{R})Z_n(\mathbb{R})^0; \varepsilon_\infty \otimes \pi_\infty \otimes \check{M}_{\mu,\mathbb{C}}),$$

where $\varepsilon_\infty \in \omega_\pi|_{\pi_0}\hat{\pi}_0(\mathrm{GL}_n, \check{\mu})$. The formula in [loc.cit.] shows that $m^{\bullet}_{\mathrm{GL}_n}$ only depends on the rank of the group GL_n. Since the restriction $\pi_\infty|_{\mathrm{SL}_n(\mathbb{R})}$ breaks into a direct sum of two irreducible representations if n is even and remains irreducible if n is odd, we thus obtain by [loc.cit.]

$$H^{\bullet}_{\mathfrak{g}\mathrm{K}}(\mathfrak{gl}_n, K_{n,\infty}; \pi_\infty \otimes \check{M}_{\mu,\mathbb{C}}) \cong m^{\bullet}_{\mathrm{GL}_n} \begin{cases} \mathrm{sgn} \oplus \mathrm{id} & \text{if } n \text{ is even}, \\ \omega_\pi|_{\pi_0}\, \mathrm{sgn}^{\mathsf{wt}(\mu)/2} & \text{if } n \text{ is odd}, \end{cases} \tag{1.3}$$

as $\pi_0(\mathrm{GL}_n)$-modules. Finally, if we set

$$b_n = \begin{cases} \frac{n^2}{4} & \text{if } n \text{ is even}, \\ \frac{n^2-1}{4} & \text{if } n \text{ is odd}, \end{cases}$$
$$t_n = \begin{cases} \frac{(n+1)^2-1}{4} - 1 & \text{if } n \text{ is even}, \\ \frac{(n+1)^2}{4} - 1 & \text{if } n \text{ is odd}, \end{cases}$$

we get from Lemme 3.14 in [Clo] that $m^{\bullet}_{\mathrm{GL}_n} \neq 0$ if and only if $b_n \leq \bullet \leq t_n$ and

$$m^{b_n}_{\mathrm{GL}_n} = m^{t_n}_{\mathrm{GL}_n} = 1.$$

1.3 A closer look on relative Lie algebra cohomology

In Chapter 3 we will need an explicit description of relative Lie algebra cohomology in order to make good use of the fact, that a cuspidal automorphic representation π lies in $\mathrm{Coh}(\mathrm{GL}_n, \check{\mu})$ for some regular weight μ. We provide such a description by the following proposition, that seems to be well known to the experts but seems not to be explicitly proved in the literature.

Proposition 1.3 *It holds*

$$H^{\bullet}_{\mathfrak{g}K}(\mathfrak{gl}_n, K_{n,\infty}; \pi_\infty \otimes \check{M}_{\mu,\mathbb{C}}) = \left(\bigwedge^{\bullet} (\mathfrak{sl}_n/\mathfrak{so}_n)^* \otimes \pi_\infty \otimes \check{M}_{\mu,\mathbb{C}} \right)^{\mathrm{SO}_n(\mathbb{R})}.$$

Proof. Since $K_{n,\infty}$ is connected, by section I.5 of [BW][1] we may write

$$H^{\bullet}_{\mathfrak{g}\mathrm{K}}(\mathfrak{gl}_n, K_{n,\infty}; \pi_\infty \otimes \check{M}_{\mu,\mathbb{C}}) = H^{\bullet}_{\mathfrak{g}\mathfrak{k}}(\mathfrak{gl}_n, \mathfrak{k}_{n,\infty}; \pi_\infty \otimes \check{M}_{\mu,\mathbb{C}}).$$

[1]Note, that since the central action is by a scalar, the $K_{n,\infty}$-invariant submodules of $\pi_\infty \otimes \check{M}_{\mu,\mathbb{C}}$ are just the same as the SO_n-invariant ones. Therefore, we may apply the results of [BW] on $K_{n,\infty}$, even if the latter is not compact. In the results we cite, the maximality of the compact subgroup is never needed.

Consider the complex

$$C^\bullet_{\mathfrak{g}\mathfrak{k}}(\mathfrak{gl}_n, \mathfrak{k}_{n,\infty}; \pi_\infty \otimes \check{M}_{\mu,\mathbb{C}}) \overset{\text{[BW], I.1.2}}{=} \mathrm{Hom}_{\mathfrak{k}_{n,\infty}}(\bigwedge^\bullet \mathfrak{gl}_n/\mathfrak{k}_{n,\infty}, \pi_\infty \otimes \check{M}_{\mu,\mathbb{C}}).$$

By Theorem I.5.3 of [loc. cit.], and since $\pi \in \mathrm{Coh}(\mathrm{GL}_n, \check{\mu})$, the central character of π_∞ equals the one of ϱ_μ, implying that $\pi_\infty \otimes \check{M}_{\mu,\mathbb{C}}$ has trivial central character. Recall that π_∞ and $M_{\check{\mu}}$ both are irreducible representations of $\mathrm{GL}_n(\mathbb{R})$. By the triviality of the central character of $\pi_\infty \otimes \check{M}_{\mu,\mathbb{C}}$ the latter uniquely corresponds to the tensor product of the irreducible representations of $\mathrm{SL}_n^\pm(\mathbb{R})$ given by restriction. We will identify the respective modules and denote them the same.

Because of $\mathfrak{k}_{n,\infty} = \mathfrak{so}_n \oplus \mathfrak{z}_n$ the vector spaces $\mathfrak{gl}_n/\mathfrak{k}_{n,\infty}$ and $\mathfrak{sl}_n/\mathfrak{so}_n$ are identical, so that we have

$$\begin{aligned} C^\bullet_{\mathfrak{g}\mathfrak{k}}(\mathfrak{gl}_n, \mathfrak{k}_{n,\infty}; \pi_\infty \otimes \check{M}_{\mu,\mathbb{C}}) &= \mathrm{Hom}_{\mathfrak{so}_n}(\bigwedge^\bullet \mathfrak{sl}_n/\mathfrak{so}_n, \pi_\infty \otimes \check{M}_{\mu,\mathbb{C}}) \\ &= C^\bullet_{\mathfrak{g}\mathfrak{k}}(\mathfrak{sl}_n, \mathfrak{so}_n; \pi_\infty \otimes \check{M}_{\mu,\mathbb{C}}), \end{aligned}$$

whence

$$H^\bullet_{\mathfrak{g}K}(\mathfrak{gl}_n, K_{n,\infty}; \pi_\infty \otimes \check{M}_{\mu,\mathbb{C}}) = H^\bullet_{\mathfrak{g}\mathfrak{k}}(\mathfrak{sl}_n, \mathfrak{so}_n; \pi_\infty \otimes \check{M}_{\mu,\mathbb{C}}).$$

Now, since $\mathrm{SO}_n(\mathbb{R})$ is connected, all that is left to show is

$$H^\bullet_{\mathfrak{g}\mathfrak{k}}(\mathfrak{sl}_n, \mathfrak{so}_n; \pi_\infty \otimes \check{M}_{\mu,\mathbb{C}}) = \mathrm{Hom}_{\mathfrak{so}_n}(\bigwedge^\bullet \mathfrak{sl}_n/\mathfrak{so}_n, \pi_\infty \otimes \check{M}_{\mu,\mathbb{C}}). \tag{1.4}$$

But this follows directly from Proposition II.3.1 of [BW], we just have to verify that we are allowed to use it. In order to do that we choose $\mathrm{SL}_n(\mathbb{R})$ as connected, reductive Lie group and $\mathrm{SO}_n(\mathbb{R})$ as its maximal compact subgroup. We have to guarantee that $d\pi_\infty(C)$ and $d\check{\varrho}_{\mu,\mathbb{C}}(C)$ are scalar operators, where C is the Casimir element of the envelopping algebra $\mathfrak{U}(\mathfrak{sl}_n)$, and $d\pi_\infty$ and $d\check{\varrho}_{\mu,\mathbb{C}}$ are the respective induced mappings on $\mathfrak{U}(\mathfrak{sl}_n)$. By Schur's Lemma (cf. [Kna2], Proposition 5.1), and since C is in the centre of $\mathfrak{U}(\mathfrak{sl}_n)$, it would suffice to show that π_∞ and $\check{M}_{\mu,\mathbb{C}}$ are irreducible $\mathrm{SL}_n(\mathbb{R})$-modules. Obviously, this does not hold in general, but since all representations are irreducible as $\mathrm{SL}_n^\pm(\mathbb{R})$-modules, we may use

Lemma 1.4 *Let $\varrho : \mathrm{SL}_n^\pm(\mathbb{R}) \to \mathrm{GL}(V)$ be an irreducible $\mathrm{SL}_n^\pm(\mathbb{R})$-module and $d\varrho$ the induced mapping on $\mathfrak{U}(\mathfrak{sl}_n)$. Then there is a scalar r such that $d\varrho(C) = r \cdot \mathrm{id}$.*

By applying the lemma on π_∞ and on $\check{M}_{\mu,\mathbb{C}}$ we may use Proposition II.3.1 of [BW] now. Since $\pi \in \mathrm{Coh}(\mathrm{GL}_n, \check{\mu})$, we get $d\pi_\infty(C) = d\check{\varrho}_{\mu,\mathbb{C}}(C)$, and therefore (1.4). This concludes the proof of Proposition 1.3. □

Proof of Lemma 1.4. If V is still irreducible as $\mathrm{SL}_n(\mathbb{R})$-module, there is nothing to show. So assume that V decomposes into a direct sum of (irreducible) $\mathrm{SL}_n(\mathbb{R})$-modules (ϱ_1, V_1) and (ϱ_2, V_2). Note, that since the index is 2 this is the only other case. If V_1 and V_2 are isomorphic, still there is nothing to show. So assume that V_1 and V_2 are not isomorphic as $\mathrm{SL}_n(\mathbb{R})$-modules. Choose $g \in \mathrm{SL}_n^{\pm}(\mathbb{R}) \smallsetminus \mathrm{SL}_n(\mathbb{R})$ and $v_1 \in V_1$ with $gv_1 \notin V_1$. Then gV_1 is not contained in V_1. Since for all $h \in \mathrm{SL}_n(\mathbb{R})$ we have

$$h(gV_1) = (gg^{-1})h(gV_1) = g(g^{-1}hg)V_1 = gV_1, \tag{1.5}$$

gV_1 is a $\mathrm{SL}_n(\mathbb{R})$-module. Note, that $\mathrm{SL}_n(\mathbb{R})$ is normal because of its index 2 in $\mathrm{SL}_n^{\pm}(\mathbb{R})$.

Since $V_1 \ncong V_2$, the only $\mathrm{SL}_n(\mathbb{R})$-submodules of V are $0, V_1, V_2$, and V, so that gV_1 is isomorphic to V_2. Then (1.5) tells us how the module structures of V_1 and V_2 are related: Clearly it is enough to proof $g^{-1}Cg = C$ to get $d\varrho_1(C) = d\varrho_2(C)$.

We may write $C = \sum_i X_i X_i^*$, where the X_i resp. the X_i^* form a basis of $\mathfrak{sl}_n$, dual to each other via the Killing form κ of $\mathfrak{sl}_n$. It holds

$$\kappa(g^{-1}X_i g, g^{-1}X_j^* g) = \kappa(X_i, X_j^*) \quad \forall g \in \mathrm{SL}_n^{\pm}(\mathbb{R}),$$

so that the basis formed by the $g^{-1}X_i g$ and the one formed by the $g^{-1}X_i^* g$ are also dual to each other. The lemma follows because of $g^{-1}Cg = \sum_i g^{-1}X_i g g^{-1} X_i^* g$ and the independence of the Casimir element of its basis. □

Now let $\pi_\infty^{(\mathrm{O}_n)}$ denote the space of $\mathrm{O}_n(\mathbb{R})$-finite vectors in the representation space of π_∞. Note, that since $\check{M}_{\mu,\mathbb{C}}$ is of finite dimension, we have $\check{M}_{\mu,\mathbb{C}}^{(\mathrm{O}_n)} = \check{M}_{\mu,\mathbb{C}}$. Let further $H^{\mathrm{O}_n(\mathbb{R})} = H_+$ and H_- denote the respective (± 1)-eigenspaces of any $\mathrm{SO}_n(\mathbb{R})$-invariant module H. We get the following corollary, which will be useful in Section 3.6.

Corollary 1.5 *(a) If n is even, then $H_{\mathfrak{g}K}^{b_n}(\mathfrak{gl}_n, K_{n,\infty}; \pi_\infty \otimes \check{M}_{\mu,\mathbb{C}})_\varepsilon$ is a one-dimensional $\mathbb{C}$-vector space for both $\varepsilon \in \{+,-\}$.*

(b) If n is odd, then $H_{\mathfrak{g}K}^{b_n}(\mathfrak{gl}_n, K_{n,\infty}; \pi_\infty \otimes \check{M}_{\mu,\mathbb{C}})_\varepsilon$ is a one-dimensional $\mathbb{C}$-vector space, if $\mathrm{sgn}(\omega_\pi(-\mathbf{1}_n)(-1)^{\mathsf{wt}(\mu)/2}) = \varepsilon$; the other one is trivial.

(c) $H_{\mathfrak{g}K}^{b_n}(\mathfrak{gl}_n, K_{n,\infty}; \pi_\infty \otimes \check{M}_{\mu,\mathbb{C}})_\pm = \left(\bigwedge^{b_n} (\mathfrak{sl}_n/\mathfrak{so}_n)^* \otimes \pi_\infty^{(\mathrm{O}_n)} \otimes \check{M}_{\mu,\mathbb{C}} \right)_\pm^{\mathrm{SO}_n(\mathbb{R})}$

Proof. By Proposition 1.3 we have

$$\begin{aligned} H_{\mathfrak{g}K}^{b_n}(\mathfrak{gl}_n, K_{n,\infty}; \pi_\infty \otimes \check{M}_{\mu,\mathbb{C}})^{\mathrm{O}_n(\mathbb{R})} &= \left(\bigwedge^{b_n} (\mathfrak{sl}_n/\mathfrak{so}_n)^* \otimes \pi_\infty \otimes \check{M}_{\mu,\mathbb{C}} \right)^{\mathrm{O}_n(\mathbb{R})} \\ &\overset{\text{[BW], I.5}}{=} \left(\bigwedge^{b_n} (\mathfrak{sl}_n/\mathfrak{so}_n)^* \otimes \pi_\infty^{(\mathrm{O}_n)} \otimes \check{M}_{\mu,\mathbb{C}} \right)^{\mathrm{O}_n(\mathbb{R})}. \end{aligned}$$

Assertion (c) follows, since $H_{\mathfrak{g}K}^{b_n}(\mathfrak{gl}_n, K_{n,\infty}; \pi_\infty \otimes \check{M}_{\mu,\mathbb{C}})$ is the direct sum of its (± 1)-eigenspaces. Assertions (a) and (b) result from (1.3). □

1.4 Proof of Lemma 1.1

Let $\pi = \pi_f \otimes \pi_\infty$ be an irreducible cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$. In the special case $\check{\mu} = 0$ of Section 1.3 we find

$$H^{b_n}_{\mathfrak{g}\mathrm{K}}(\mathfrak{gl}_n, K_{n,\infty}; \pi_\infty) = H^{b_n}_{\mathfrak{g}\mathfrak{k}}(\mathfrak{sl}_n, \mathfrak{so}_n; \pi_\infty).$$

On the one hand, assume that π occurs in (s-dimensional) cohomology with constant coefficients in the sense of [KMS], p. 99. We want to show that π lies in $\mathrm{Coh}(\mathrm{GL}_n, 0)$ by proving

$$H^{b_n}_{\mathfrak{g}\mathfrak{k}}(\mathfrak{sl}_n, \mathfrak{so}_n; \pi_\infty) \neq 0$$

with the above. By p. 122 of [KMS] we have

$$H^{\bullet}_{\mathfrak{g}\mathfrak{k}}(\mathfrak{sl}_n, \mathfrak{so}_n; \pi_\infty)^{\mathrm{O}_n(\mathbb{R})/\,\mathrm{SO}_n(\mathbb{R})} \overset{\text{[BW], I.5}}{=} H^{\bullet}_{\mathfrak{g}\mathrm{K}}(\mathfrak{sl}_n, \mathrm{O}_n; \pi_\infty^{(\mathrm{O}_n)}) \neq 0.$$

So we have found a non-trivial summand of $H^{\bullet}_{\mathfrak{g}\mathrm{K}}(\mathfrak{gl}_n, K_{n,\infty}; \pi_\infty)$, which proves the first inclusion.

On the other hand, let $\pi = \pi_f \otimes \pi_\infty$ be in $\mathrm{Coh}(\mathrm{GL}_n, 0)$. Like in Section 1.3 we see that π_∞ has trivial central character, so that we may use Corollary 1.5 to get

$$\mathbb{C} \cong H^{\bullet}_{\mathfrak{g}\mathfrak{k}}(\mathfrak{sl}_n, \mathfrak{so}_n; \pi_\infty)^{\mathrm{O}_n(\mathbb{R})/\,\mathrm{SO}_n(\mathbb{R})} \overset{\text{[BW], I.5}}{=} H^{\bullet}_{\mathfrak{g}\mathrm{K}}(\mathfrak{sl}_n, \mathrm{O}_n; \pi_\infty^{(\mathrm{O}_n)}).$$

By p. 122 of [KMS] this gives us the second inclusion.

1.5 The Langlands parameter

We denote by $\mathfrak{L}_0^+(\mathrm{GL}_n)$ the set of all pairs $(\mathsf{w}, \mathbf{l})$, where $\mathsf{w} \in \mathbb{Z}$ and $\mathbf{l} = (l_1, \ldots, l_n) \in \mathbb{Z}^n$ is a finite sequence satisfying $l_1 > \cdots > l_{\lfloor n/2 \rfloor} > 0$, $l_i + l_{n+1-i} = 0$ for all $i \in \{0, \ldots, n\}$, and the purity condition

$$\mathsf{w} + \mathbf{l} \equiv \begin{cases} 1 \bmod (2), & \text{if } n \text{ is even,} \\ 0 \bmod (2), & \text{if } n \text{ is odd,} \end{cases} \tag{1.6}$$

where we identify w with $(\mathsf{w}, \ldots, \mathsf{w})$. We note, that for n odd this immediately implies $\mathsf{w} \equiv \mathbf{l} \equiv 0 \bmod (2)$ since $l_{(n+1)/2} = 0$. If we let $\Phi_{\mathrm{GL}_n} = \Phi(\mathrm{GL}_n, T_n)$ denote the set of roots of T_n in GL_n and $\Phi^+_{\mathrm{GL}_n}$ the subset of positive roots determined by the choice of B_n, we see, that the sets $\mathfrak{L}_0^+(\mathrm{GL}_n)$ and $X_0^+(T_n)$ are in bijection:

$$\begin{array}{ccc} \mathfrak{L}_0^+(\mathrm{GL}_n) & \longleftrightarrow & X_0^+(T_n) \\ (\mathsf{w}, \mathbf{l}) & \mapsto & \mu = \dfrac{\mathsf{w} + \mathbf{l}}{2} - \varrho_n. \end{array} \tag{1.7}$$

Here,

$$\varrho_n = \frac{1}{2}\sum_{\alpha\in\Phi^+_{\mathrm{GL}_n}} \alpha = (\frac{n-1}{2}, \frac{n-3}{2}, \ldots, -\frac{n-1}{2}) \quad \in X(\mathrm{GL}_n)\otimes_{\mathbb{Z}}\mathbb{Q}$$

is the half-sum of positive roots of GL_n relative to T_n. Explicitly, we have

$$\mu = \begin{cases} \left(\frac{\mathsf{w}+l_1-(n-1)}{2}, \frac{\mathsf{w}+l_2-(n-3)}{2}, \ldots, \frac{\mathsf{w}-l_1+(n-1)}{2}\right), & \text{if } n \text{ is even,}\\ \left(\frac{\mathsf{w}+l_1-(n-1)}{2}, \frac{\mathsf{w}+l_2-(n-3)}{2}, \ldots, \frac{\mathsf{w}}{2}, \ldots, \frac{\mathsf{w}-l_1+(n-1)}{2}\right), & \text{if } n \text{ is odd.}\end{cases}$$

In the inverse direction the parameter associated with a dominant, integral weight μ reads $(\mathsf{w}, \mathbf{l})$, where $\mathsf{w} = \mu_1 + \mu_n$ is the weight of μ and $\mathbf{l} = 2(\mu + \varrho_n) - \mathsf{w}$.

To any $(\mathsf{w}, \mathbf{l}) \in \mathfrak{L}_0^+(\mathrm{GL}_n)$ we attach an induced representation of Langlands type: we write D_l for the discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ of lowest weight $l+1$ (cf. [Bum1], p. 216); we then set

$$J(\mathsf{w}, \mathbf{l}) := \begin{cases} \mathrm{Ind}_{Q(\mathbb{R})}^{\mathrm{GL}_n(\mathbb{R})}(|\cdot|_{\mathbb{R}}^{\mathsf{w}/2}\otimes D_{l_1}, \ldots, |\cdot|_{\mathbb{R}}^{\mathsf{w}/2}\otimes D_{l_{n/2}}), & \text{if } n \text{ is even,}\\ \mathrm{Ind}_{Q(\mathbb{R})}^{\mathrm{GL}_n(\mathbb{R})}(|\cdot|_{\mathbb{R}}^{\mathsf{w}/2}\otimes D_{l_1}, \ldots, |\cdot|_{\mathbb{R}}^{\mathsf{w}/2}\otimes D_{l_{(n-1)/2}}, |\cdot|_{\mathbb{R}}^{\mathsf{w}/2}), & \text{if } n \text{ is odd.}\end{cases}$$

Here, $Q \le \mathrm{GL}_n$ is the parabolic subgroup of type $(2, \ldots, 2)$ resp. $(2, \ldots, 2, 1)$.

Let $(\mathsf{w}, \mathbf{l}) \in \mathfrak{L}_0^+(\mathrm{GL}_n)$ correspond to $\mu \in X_0^+(T_n)$ as in (1.7). By (3.6) of [Mah] any $\pi \in \mathrm{Coh}(\mathrm{GL}_n, \check{\mu})$ has infinity component

$$\pi_\infty \cong \mathrm{sgn}^k \otimes J(\mathsf{w}, \mathbf{l}), \quad k \in \mathbb{Z}/2\mathbb{Z}. \tag{1.8}$$

For later use (cf. Chapter 5) we remark that to each such representation π_∞ there is a corresponding representation π_∞^{W} of the Weil group $W_{\mathbb{R}}$ of $\mathbb{R}$ via Langlands correspondence. $W_{\mathbb{R}}$ is the non-split extension of $\mathbb{C}^\times$ by $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$ given by

$$W_{\mathbb{R}} = \mathbb{C}^\times \cup j\mathbb{C}^\times,$$

where $j^2 = -1$ and $jzj^{-1} = \bar{z}$ for all $z \in \mathbb{C}^\times$. Thus any representation of $W_{\mathbb{R}}$ is determined by how elements of the form $z = re^{i\theta}$ and j act. There are exactly three types of irreducible representations, which are given explicitly in Chapter 3 of [Kna1]:

- The one-dimensional representations $(+, t)$ with $t \in \mathbb{C}$, which act via φ are given by
$$\varphi(z) = |z|^t \text{ and } \varphi(j) = +1.$$
- The one-dimensional representations $(-, t)$ with $t \in \mathbb{C}$, which act via φ are given by
$$\varphi(z) = |z|^t \text{ and } \varphi(j) = -1.$$

- The two-dimensional representations (l,t), where $l \geq 1$ is an integer and $t \in \mathbb{C}$. In those we may always choose a basis $\{u, u'\}$ such that we have

$$\varphi(re^{i\theta})u = r^{2t}e^{il\theta}u,\ \varphi(re^{i\theta})u' = r^{2t}e^{-il\theta}u',\ \varphi(j)u = u',\ \varphi(j)u' = (-1)^l u,$$

where (l,t) acts via φ.

Using this notation we have

$$\pi_\infty^{\mathrm{W}} = \begin{cases} (l_1, \frac{\mathsf{w}}{2}) \oplus (l_2, \frac{\mathsf{w}}{2}) \oplus \cdots \oplus (l_{n/2}, \frac{\mathsf{w}}{2}), & \text{if } n \text{ is even,} \\ (l_1, \frac{\mathsf{w}}{2}) \oplus (l_2, \frac{\mathsf{w}}{2}) \oplus \cdots \oplus (l_{(n-1)/2}, \frac{\mathsf{w}}{2}) \oplus (\mathrm{sgn}^k, \frac{\mathsf{w}}{2}), & \text{if } n \text{ is odd.} \end{cases}$$

We will need to determine the tensor product of two such Weil group representations. So let

$$\sigma_\infty \cong \mathrm{sgn}^{k'} \otimes J(\mathsf{w}', \mathbf{l}'), \quad k' \in \mathbb{Z}/2\mathbb{Z}$$

be a representation of $\mathrm{GL}_m(\mathbb{R})$, notation being clear from the context. Analogously, we get

$$\sigma_\infty^{\mathrm{W}} = \begin{cases} (l_1', \frac{\mathsf{w}'}{2}) \oplus (l_2', \frac{\mathsf{w}'}{2}) \oplus \cdots \oplus (l_{m/2}', \frac{\mathsf{w}'}{2}) \oplus (\mathrm{sgn}^{k'}, \frac{\mathsf{w}'}{2}), & \text{if } m \text{ is odd,} \\ (l_1', \frac{\mathsf{w}'}{2}) \oplus (l_2', \frac{\mathsf{w}'}{2}) \oplus \cdots \oplus (l_{(m-1)/2}', \frac{\mathsf{w}'}{2}), & \text{if } m \text{ is even.} \end{cases}$$

We want to calculate the tensor product of π_∞^{W} and $\sigma_\infty^{\mathrm{W}}$. Therefore we have to calculate the various tensor products of the building blocks. We distinguish three cases:

- Let σ, σ' be in $\{+,-\}$, and let t, t' be in $\mathbb{C}$. Obviously, we get

$$(\sigma, t) \otimes (\sigma', t') = \begin{cases} (+, t+t'), & \text{if } \sigma = \sigma', \\ (-, t+t'), & \text{if } \sigma \neq \sigma'. \end{cases}$$

- Let $l \geq 1$ be an integer, $\sigma \in \{+,-\}$ and t, t' in $\mathbb{C}$. Let further $\{u, u'\}$ be the special basis from the definition of (l,t) and v an arbitrary element of $(\pm, t')$. Then it is an easy calculation to show, that

$$(l,t) \otimes (\sigma, t') = (l, t+t'),$$

where an associated special basis is given by $\{u \otimes v, u' \otimes v\}$, if $\sigma = +$, and by $\{u \otimes v, -u' \otimes v\}$, if $\sigma = -$.

- Let $l, l' \geq 1$ be integers, and let t, t' be complex numbers. Let further be $\{u, u'\}$ and $\{v, v'\}$ the respective special bases of (l,t) and (l', t'). A quick calculation shows that $u \otimes v$ and $u' \otimes v'$ span a two-dimensional representation of the type $(l+l', t+t')$. Analogously,

$$\begin{array}{l}(l-l', t+t') \text{ with special basis } \{(-1)^{l'} u \otimes v', u' \otimes v\} \text{ is well-defined for } l > l', \\ (l'-l, t+t') \text{ with special basis } \{(-1)^{l} u' \otimes v, u \otimes v'\} \text{ is well-defined for } l < l'.\end{array}$$

In the case $l = l'$ the representation $(l,t) \otimes (l',t')$ is not irreducible any more, but splits into

$$\begin{array}{l}(+, t+t') \text{ spanned by } (-1)^l u \otimes v' + u' \otimes v, \\ (-, t+t') \text{ spanned by } (-1)^{l-1} u \otimes v' + u' \otimes v.\end{array}$$

We may subsume the results of this case by

$$(l,t) \otimes (l',t') = \begin{cases} (l+l', t+t') \oplus (|l-l'|, t+t'), & \text{if } l \neq l', \\ (l+l', t+t') \oplus (+, t+t') \oplus (-, t+t'), & \text{if } l = l'. \end{cases}$$

We will only be interested in the case $m = n-1$. There we get

Proposition 1.6 *The tensor product $\pi_\infty^{\mathrm{W}} \otimes \sigma_\infty^{\mathrm{W}}$ takes the value*

$$\bigoplus_{i=1}^{\frac{n}{2}} (l_i, \frac{\mathsf{w}+\mathsf{w}'}{2}) \oplus \bigoplus_{i=1}^{\frac{n}{2}} \bigoplus_{j=1}^{\frac{n}{2}-1} \left[(l_i + l'_j, \frac{\mathsf{w}+\mathsf{w}'}{2}) \oplus (|l_i - l'_j|, \frac{\mathsf{w}+\mathsf{w}'}{2}) \right], \quad \textit{if } n \textit{ is even,}$$

$$\bigoplus_{j=1}^{\frac{n-1}{2}} (l'_j, \frac{\mathsf{w}+\mathsf{w}'}{2}) \oplus \bigoplus_{i=1}^{\frac{n-1}{2}} \bigoplus_{j=1}^{\frac{n-1}{2}} \left[(l_i + l'_j, \frac{\mathsf{w}+\mathsf{w}'}{2}) \oplus (|l_i - l'_j|, \frac{\mathsf{w}+\mathsf{w}'}{2}) \right], \quad \textit{if } n \textit{ is odd,}$$

where by $(0, \frac{\mathsf{w}+\mathsf{w}'}{2})$ *we denote* $(+, \frac{\mathsf{w}+\mathsf{w}'}{2}) \oplus (-, \frac{\mathsf{w}+\mathsf{w}'}{2})$.[2]

1.6 Cohomology of locally symmetric spaces

We want to relate the results up to now to the cohomology of the orbifolds

$$S_n(K) = \mathrm{GL}_n(\mathbb{Q}) \backslash \mathrm{GL}_n(\mathbb{A}) / K K_{n,\infty},$$

where K is a compact open subgroup of $\mathrm{GL}_n(\mathbb{A}_f)$. Note, that if $\det(K) = \hat{\mathbb{Z}}^\times$ we have, by strong approximation, the isomorphism

$$\mathrm{GL}_n(\mathbb{Q}) \backslash \mathrm{GL}_n(\mathbb{A}) / K K_{n,\infty} \cong \Gamma_K \backslash \mathrm{GL}_n(\mathbb{R})^0 / \mathrm{SO}_n(\mathbb{R}),$$

[2]Note, that in [Kna1] the integer l belonging to the representation (l,t) is at least 1, so that there is no conflict of notation.

where

$$\Gamma_K := \{\gamma \in \mathrm{GL}_n(\mathbb{Q})^0 \mid \gamma_f \in K\} \subseteq \mathrm{SL}_n(\mathbb{Z}).$$

We set

$$\tilde{S}_n := \varprojlim_{K} S_n(K),$$

where K runs through all compact open subgroups of $\mathrm{GL}_n(\mathbb{A}_f)$. For any finite-dimensional representation $(\check{\varrho}_\mu, \check{M}_\mu)$ we define the locally constant sheaf $\check{\mathcal{M}}_\mu = \check{\mathcal{M}}_{\mu,K}$ on $S_n(K)$ by setting $\check{\mathcal{M}}_{\mu,K}(U)$ for any open $U \subseteq S_n(K)$ to be the set of locally constant functions $f : \mathrm{pr}^{-1}(U) \to \check{M}_\mu$ satisfying

$$\forall \gamma \in \mathrm{GL}_n(\mathbb{Q}),\ z \in \mathrm{pr}^{-1}(U) : f(\gamma z) = \check{\varrho}_\mu(\gamma)(f(z)),$$

where $\mathrm{pr} : \mathrm{GL}_n(\mathbb{A})/KK_{n,\infty} \to S_n(K)$ is the natural projection. Analogously, we get a locally constant sheaf on $\tilde{S}_n$, noted $\check{\mathcal{M}}_\mu$ as well. Similarly, for any field extension $E/\mathbb{Q}$ we denote by $\check{\mathcal{M}}_{\mu,E} = \check{\mathcal{M}}_{\mu,E,K}$ the corresponding sheaf on $S_n(K)$. Analogously to the rational case, $\mathcal{M}_{\mu,E}$ denotes as well the respective sheaf on $\tilde{S}_n$.

We then define the cohomology groups with coefficients in $\check{\mathcal{M}}_{\mu,\mathbb{C},K}$ (cf. [Clo], p. 121):

$$H^\bullet_?(\tilde{S}_n, \check{\mathcal{M}}_{\mu,\mathbb{C}}) := \varinjlim_{K} H^\bullet_?(S_n(K), \check{\mathcal{M}}_{\mu,\mathbb{C},K}), \quad ? \in \{\text{ blank, c, cusp }\}.$$

These groups are modules under the canonical action of $\mathrm{GL}_n(\mathbb{A}_f) \times \pi_0(\mathrm{GL}_n)$.

Viewed as a $\mathrm{GL}_n(\mathbb{A}_f)$-module, the cuspidal cohomology decomposes into a direct sum of $\mathrm{GL}_n(\mathbb{A}_f)$-isotypic components:

$$H^\bullet_{\mathrm{cusp}}(\tilde{S}_n, \check{\mathcal{M}}_{\mu,\mathbb{C}}) = \bigoplus_{\pi \in \mathrm{Coh}(\mathrm{GL}_n, \check{\mu})} H^\bullet_{\mathrm{cusp}}(\tilde{S}_n, \check{\mathcal{M}}_{\mu,\mathbb{C}})(\pi_f).$$

Since the actions of $\mathrm{GL}_n(\mathbb{A}_f)$ and $\pi_0(\mathrm{GL}_n)$ commute, the isotypical components are stable under the action of $\pi_0(\mathrm{GL}_n)$. Since furthermore by [Clo], Lemme 3.15 (ii) we have

$$H^\bullet_{\mathrm{cusp}}(\tilde{S}_n, \check{\mathcal{M}}_{\mu,\mathbb{C}})(\pi_f) \cong \pi_f \otimes H^\bullet_{\mathfrak{g}K}(\mathfrak{gl}_n, K_{n,\infty}; \pi_\infty \otimes \check{M}_{\mu,\mathbb{C}}) \tag{1.9}$$

and since GL_n satisfies multiplicity one by [Sha], we obtain from (1.3) and the definition of $\hat{\pi}_0(\mathrm{GL}_n, \mu)$

$$H^\bullet_{\mathrm{cusp}}(\tilde{S}_n, \check{\mathcal{M}}_{\mu,\mathbb{C}})(\pi_f) \cong \bigoplus_{\bar{\pi}_\infty \in\, \omega_\pi|_{\pi_0} \hat{\pi}_0(\mathrm{GL}_n,\mu)} m^\bullet_{\mathrm{GL}_n}(\pi_f \otimes \bar{\pi}_\infty).$$

Note, that $\omega_\pi|_{\pi_0} \in \{\mathrm{id}, \mathrm{sgn}\}$. Altogether we find a decomposition into irreducible $\mathrm{GL}_n(\mathbb{A}_f) \times \pi_0(\mathrm{GL}_n)$-modules:

$$H^\bullet_{\mathrm{cusp}}(\tilde{S}_n, \check{\mathcal{M}}_{\mu,\mathbb{C}}) \cong \bigoplus_{\pi \in \mathrm{Coh}(\mathrm{GL}_n, \check{\mu})} \ \bigoplus_{\bar{\pi}_\infty \in\, \omega_\pi|_{\pi_0} \hat{\pi}_0(\mathrm{GL}_n,\mu)} m^\bullet_{\mathrm{GL}_n}(\pi_f \otimes \bar{\pi}_\infty). \tag{1.10}$$

Since $m^{b_n} = 1$, this decomposition is multiplicity free in degree b_n.

Chapter 2

The Rankin-Selberg convolution

In this chapter we will introduce the Rankin-Selberg L-series, whose critical values we want to study in Chapter 3. Starting from the Global Birch Lemma of [KMS] we will give a first description of those values in terms of integrals over certain Whittaker functions.

We fix a non-trivial character $\tau = \bigoplus_\ell \tau_\ell : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times$, such that for all finite places ℓ the conductor of τ_ℓ equals $\mathbb{Z}_\ell$. We also denote by

$$\tau(n) := \prod_{i=1}^{n-1} \tau(n_{i,i+1})$$

the induced (generic) character of $U_n(\mathbb{A})$. For any automorphic representation $\pi = \pi_f \otimes \pi_\infty$ we write $\mathscr{W}(\pi,\tau)$ for the *Whittaker model* of π with respect to τ. This Whittaker space can be described as the restricted tensor product of the *local Whittaker spaces* defined as in [JPS2] resp. [JPS1] in the infinite resp. finite case, that is

$$\mathscr{W}(\pi,\tau) = \mathscr{W}(\pi_\infty,\tau_\infty) \otimes \bigotimes_{\ell\nmid\infty}{}' \mathscr{W}(\pi_\ell,\tau_\ell).$$

Hereby, we restrict in the following way: For any prime ℓ where π_ℓ is unramified the subspace of $\mathrm{GL}_n(\mathbb{Z}_\ell)$-fixed elements in $\mathscr{W}(\pi_\ell,\tau_\ell)$ is one-dimensional (cf. §4 in [JPS3]). Its normalised generator w_ℓ^0 is called the *new vector* of $\mathscr{W}(\pi_\ell,\tau_\ell)$, normalisation being by $w_\ell^0(\mathbf{1}_n) = 1$ (cf. (3.3) in [loc.cit.]). Now an element of $\mathscr{W}(\pi,\tau)$ is a tensor in $\bigotimes_\ell \mathscr{W}(\pi_\ell,\tau_\ell)$, where all but finitely many factors are given by the respective new vector. This is possible, since π_ℓ is unramified for all but finitely many primes ℓ.

Throughout this thesis, we will always assume that for every finite place ℓ the respective additive character τ_ℓ has exponent 0, meaning that $\mathbb{Z}_\ell$ is the biggest broken ideal of $\mathbb{Q}_\ell$ on which τ_ℓ is trivial. We may do so without loss of generality, since by [JPS1] the Whittaker models for two additive characters τ_ℓ and τ_ℓ' are isomorphic.

Now we fix a prime p, and two cuspidal automorphic representations $\pi = \pi_f \otimes \pi_\infty$ resp. $\sigma = \sigma_f \otimes \sigma_\infty$ of $\mathrm{GL}_n(\mathbb{A})$ resp. $\mathrm{GL}_{n-1}(\mathbb{A})$ both unramified at p. Following [JPS1] we now introduce the local Rankin-Selberg convolution for π and σ at some fixed prime number $\ell \neq p$: For each pair of Whittaker functions

$$(w_\ell, v_\ell) \in \mathscr{W}(\pi_\ell, \tau_\ell) \times \mathscr{W}(\sigma_\ell, \bar{\tau}_\ell)$$

the associated zeta integral

$$\psi_\ell(w_\ell, v_\ell; s) := \int\limits_{U_{n-1}(\mathbb{Q}_\ell)\backslash \mathrm{GL}_{n-1}(\mathbb{Q}_\ell)} w_\ell\begin{pmatrix} g & \\ & 1 \end{pmatrix} \cdot v_\ell(g) \cdot |\det(g)|^{s-\frac{1}{2}} dg$$

converges for $\mathrm{Re}(s)$ large enough. These zeta integrals span a fractional ideal L of the ring $\mathbb{C}[\ell^s, \ell^{-s}]$. In that way the *local L-function* $L(\pi_\ell, \sigma_\ell; s)$ is defined uniquely by fixing a polynomial $P(X) \in \mathbb{C}[X]$, such that $P(0) = 1$ and $P(\ell^{-s})^{-1}$ generates L, and by setting

$$P(\ell^{-s})^{-1} =: L(\pi_\ell, \sigma_\ell; s).$$

Obviously, we have a linear map on the tensor product $\mathscr{W}(\pi_\ell, \tau_\ell) \otimes \mathscr{W}(\sigma_\ell, \bar{\tau}_\ell)$ given by

$$\Psi_\ell : \begin{cases} \mathscr{W}(\pi_\ell, \tau_\ell) \otimes \mathscr{W}(\sigma_\ell, \bar{\tau}_\ell) \to \mathbb{C}(\ell^s), \\ w_\ell \otimes v_\ell \mapsto \Psi_\ell(w_\ell \otimes v_\ell; s) := \psi_\ell(w_\ell, v_\ell; s). \end{cases}$$

Moreover, if π_ℓ and σ_ℓ are both unramified, by §3.2 in [KMS] the zeta integral for the associated new vectors w_ℓ^0 and v_ℓ^0 represents the L-function

$$L(\pi_\ell, \sigma_\ell; s) = \Psi_\ell(w_\ell^0 \otimes v_\ell^0; s).$$

From now on we will write in short $t_\ell^0 := w_\ell^0 \otimes v_\ell^0$. Let S denote the set of primes ℓ, where π_ℓ or σ_ℓ is ramified. For any $\ell \in S$ there is a tensor $t_\ell^0 \in \mathscr{W}(\pi_\ell, \tau_\ell) \otimes \mathscr{W}(\sigma_\ell, \bar{\tau}_\ell)$ such that we have

$$L(\pi_\ell, \sigma_\ell; s) = \Psi_\ell(t_\ell^0; s).$$

Note, that for general n such a vector does not need to be pure. In the case $n = 3$ however Riedel recently showed, that there is always a choice of a pure t_ℓ^0 (cf. [Rie]).

We will now consider pairs (w, v) of global Whittaker functions on $\mathrm{GL}_n(\mathbb{A})$ and $\mathrm{GL}_{n-1}(\mathbb{A})$ given as products of local Whittaker functions $w := \prod_\ell w_\ell$ and $v := \prod_\ell v_\ell$, where we choose $w_\ell = w_\ell^0$ and $v_\ell = v_\ell^0$ for ℓ not contained in $S \cup \{p\}$. For $\ell = p$ we let w_p and v_p vary among all Whittaker functions which are right invariant under the respective *Iwahori subgroup* I_n or I_{n-1}. Here, I_n consists of those matrices in $\mathrm{GL}_n(\mathbb{Z}_p)$ which are upper triangular modulo p. For $\ell \in S$ we will choose a tensor as described above.

For any choice of $w_\infty \in \mathscr{W}_0(\pi_\infty, \tau_\infty)$ and $v_\infty \in \mathscr{W}_0(\sigma_\infty, \bar{\tau}_\infty)$, for arbitrary w_ℓ, v_ℓ for $\ell \in S$, for $(w_\ell, v_\ell) = (w_\ell^0, v_\ell^0)$ for $\ell \notin S \cup \{p\}$, and for (w_p, v_p) like in the last paragraph we get global Whittaker functions (w, v) with associated automorphic forms (ϕ, φ). Here, the 0 in the index means, that we consider the space of $\mathrm{O}_n(\mathbb{R})$-finite resp. $\mathrm{O}_{n-1}(\mathbb{R})$-finite Whittaker functions (cf. [JPS2]). Like above we set

$$\mathscr{W}_0(\pi, \tau) = \mathscr{W}_0(\pi_\infty, \tau_\infty) \otimes \bigotimes_{\ell \nmid \infty}{}' \mathscr{W}(\pi_\ell, \tau_\ell).$$

The product of all local zeta integrals then becomes a Rankin-Selberg convolution (cf. [JS])

$$\prod_\ell \psi_\ell(w_\ell, v_\ell; s) = \int_{\mathrm{GL}_{n-1}(\mathbb{Q}) \backslash \mathrm{GL}_{n-1}(\mathbb{A})} \phi\begin{pmatrix} g & \\ & 1 \end{pmatrix} \cdot \varphi(g) \cdot |\det(g)|^{s-\frac{1}{2}} dg$$

for $\mathrm{Re}(s) \gg 0$, admitting an analytic continuation to an entire function in s (cf. [CP1], Prop. 6.1). This function only depends on the pure tensor $w \otimes v$ and can be extended linearly to the algebraic tensor product of Whittaker spaces $\mathscr{W}_0(\pi, \tau) \otimes \mathscr{W}_0(\sigma, \bar{\tau})$ by sending

$$\prod_\ell w_\ell \otimes \prod_\ell v_\ell \mapsto \prod_\ell \Psi_\ell(w_\ell \otimes v_\ell; s).$$

In particular we find (up to the infinity factor) the *global L-function*

$$L(\pi, \sigma; s) := \prod_\ell L(\pi_\ell, \sigma_\ell; s)$$

in the image of this map. For each choice of the pair (w_∞, v_∞) there is an entire function $P(s)$ such that

$$P(s) \cdot L(\pi_\infty, \sigma_\infty; s) = \Psi_\ell(w_\infty \otimes v_\infty; s),$$

and therefore

$$P(s) \cdot L(\pi, \sigma; s) = \Psi_\ell(w_\infty \otimes v_\infty; s) \cdot \prod_{\ell \nmid \infty} \Psi_\ell(t_\ell^0; s).$$

Recall, that $L(\pi_\infty, \sigma_\infty; s)$ is defined by the Weil group representation as in [JPS2].

Writing each t_ℓ^0 for $\ell \in S$ as a sum of pure tensors leads to a finite sum of (global) pure tensors in $\mathscr{W}_0(\pi, \tau) \otimes \mathscr{W}_0(\sigma, \bar{\tau})$

$$\sum_j w_j \otimes v_j = (w_\infty \otimes v_\infty) \cdot \prod_{\ell \nmid \infty} t_\ell^0. \tag{2.1}$$

We fix this explicit decomposition and in what follows our formulas will depend on it. Separating finite and infinite parts we will sometimes write $w_j = w_\infty \cdot w_{j,f}$

and $v_j = v_\infty \cdot v_{j,f}$. The associated automorphic forms ϕ_j and φ_j yield the integral representation

$$P(s) \cdot L(\pi, \sigma; s) = \sum_j \int \phi_j \begin{pmatrix} g & \\ & 1 \end{pmatrix} \varphi_j(g) |\det(g)|^{s-\frac{1}{2}} dg.$$

We will in particular consider modified (w_j, v_j)'s and (ϕ_j, φ_j)'s, where at $\ell = p$ the local component (w_p^0, v_p^0) is replaced by an arbitrary pair (w_p, v_p) of Whittaker functions invariant under the respective Iwahori subgroup.

We want to consider χ-twists of π for a finite idele class character $\chi = \prod_\ell \chi_\ell$ satisfying the properties[3]

(a) $\chi_\infty = 1$,

(b) $\chi, \chi^2, \ldots, \chi^{n-1}$ have the same non-trivial conductor $f = p$-power.

Let $\tilde{\chi}_p$ denote the continuation of χ_p to $\mathbb{Z}_p$ by $\tilde{\chi}_p(px) = 0$ for all $x \in \mathbb{Z}_p$, and let further $G(\chi_p)$ denote the Gauß sum of χ_p. Then it holds

Global Birch Lemma (Kazhdan, Mazur, Schmidt) *For any choice of (w_∞, v_∞) and any (w_p, v_p) right-invariant under the respective Iwahori subgroup the corresponding triples (P, ϕ_j, φ_j) for all j satisfy*

$$\begin{aligned} & w_p(1) \cdot v_p(1) \cdot P(s) \cdot \prod_{i=1}^{n-1} \frac{G(\chi_p^i)(1-p^{-1})}{1-p^{-i}} \cdot L(\pi \otimes \chi, \sigma; s) \\ = & \sum_u \prod_{i=1}^{n-1} \tilde{\chi}_p(u_i^i) \sum_j \int_{\mathrm{GL}_{n-1}(\mathbb{Q}) \backslash \mathrm{GL}_{n-1}(\mathbb{A})} \phi_j(\begin{pmatrix} g & \\ & 1 \end{pmatrix} \varphi^{-1} u \varphi) \varphi_j(g) \chi(\det g) |\det g|^{s-\frac{1}{2}} dg, \end{aligned}$$

where $u = u_p$ (with $u_\ell = 1$ for all $\ell \neq p$) is taken from a representative system for $U_n(\mathbb{Z}_p)$ modulo $\varphi U_n(\mathbb{Z}_p) \varphi^{-1}$ with $\varphi = \mathrm{diag}(f^{-1}, \ldots, f^{-n})$.

For a proof look at page 114 of [KMS]. In view of Chapter 3 we want to reformulate the lemma such that the integrals on the right side of the formula do not involve the character χ. Let f be a non-trivial power of our fixed prime p and C_f the inverse image of the idele class group

$$\mathbb{Q}^\times \backslash \mathbb{Q}^\times \cdot \left(\mathbb{R}_{>0} \times \prod_{\ell \neq p, \infty} \mathbb{Z}_\ell^\times \times (1 + f^{2(n-1)}) \mathbb{Z}_p \right) \subset \mathbb{Q}^\times \backslash \mathbb{A}^\times$$

[3]The first assumption ensures that $P(s)$ will not change when passing from π to $\pi \otimes \chi$ since $\pi_\infty = (\pi \otimes \chi)_\infty$. The work of Schmidt and Utz indicates that the second assumption on χ may be omitted (cf. [Schm2] and [Utz]).

under the determinant map

$$\det : \mathrm{GL}_{n-1}(\mathbb{Q})\backslash \mathrm{GL}_{n-1}(\mathbb{A}) \to \mathbb{Q}^\times\backslash\mathbb{A}^\times.$$

We decompose the domain of integration into finitely many shifts of C_f as follows. Write

$$\mathrm{GL}_{n-1}(\mathbb{Q})\backslash \mathrm{GL}_{n-1}(\mathbb{A}) = \dot{\bigcup_x} C_f \,\mathrm{diag}(x, 1, \dots, 1),$$

where x runs over a representative system of $(\mathbb{Z}/f^{2(n-1)}\mathbb{Z})^\times$ in $\mathbb{Z}_p^\times$. Note, that this shift only effects the p-component.

Corollary 2.1 *Specialising to $s = \frac{1}{2}$ we get*

$$\begin{aligned} & w_p(1)\cdot v_p(1)\cdot P\left(\frac{1}{2}\right)\cdot \prod_{i=1}^{n-1}\frac{G(\chi_p^i)(1-p^{-1})}{1-p^{-i}}\cdot L(\pi\otimes\chi,\sigma;\frac{1}{2}) \\ = \; & \frac{p-1}{p} f^{2(n-1)} \sum_u \prod_{i=1}^{n-1}\tilde{\chi}_p(u_i^i)\sum_j \int_{C_f}\phi_j(\begin{pmatrix} g & \\ & 1\end{pmatrix}\varphi^{-1}u\varphi)\varphi_j(g)dg. \end{aligned}$$

Chapter 3

The algebraicity of the special L-value

From now on, let $\pi \in \mathrm{Coh}(\mathrm{GL}_n, \check{\mu})$ and $\sigma \in \mathrm{Coh}(\mathrm{GL}_{n-1}, \check{\nu})$ be two cohomological representations like we introduced them in Chapter 1. Accordingly, we have $\mu \in X_0^+(T_n)$ and $\nu \in X_0^+(T_{n-1})$. Our aim is to show that we may choose a Whittaker function $t_\infty \in \mathscr{W}_0(\pi_\infty, \tau_\infty) \otimes \mathscr{W}_0(\sigma_\infty, \bar{\tau}_\infty)$ such that for these representations the integrals on the right side of the equation in Corollary 2.1 up to a constant factor are algebraic numbers (cf. Theorem 1). The idea is to make use of the non-vanishing of cohomology for π and σ. We thus will be able to construct a pairing on cohomology having the above-mentioned integrals as values. Since both representations are already defined over the algebraic numbers, and since this pairing respects algebraicity by construction, this will prove the assertion. Note, that the results of this chapter hold, no matter what the critical values of (π, σ) are. However, we keep in mind the important case, where $s = \frac{1}{2}$ is a critical value of (π, σ), or even the only one (cf. Chapter 5 for a discussion of this point). The title of this chapter should be viewed in that light.

3.1 A map of differential forms

We will begin not by constructing a pairing on cohomology but by finding a natural pairing on differential forms instead. However, this cannot be done straight forward, since belonging to π and σ we will get differentials on different symmetric spaces. So the first thing we will have to do is to find a method that translates one type of differentials into the other. In this and the next two sections we will thus construct a chain map from the differential forms of the first type into those of the second one.

By [JPS3], Théorème (5.1) for $n \geq 3$ the global representations π and σ have finite parts π_f and σ_f with new vectors w_f resp. v_f right-invariant under some open compact subgroup $K \subseteq \mathrm{GL}_n(\hat{\mathbb{Z}})$ resp. $K' \subseteq \mathrm{GL}_{n-1}(\hat{\mathbb{Z}})$, such that the respective image under the determinant map is the full unit group $\hat{\mathbb{Z}}^\times$, i. e.

$$\det(K) = \det(K') = \hat{\mathbb{Z}}^\times.$$

We will assume $n \geq 3$ from now on. Moreover, the canonical embedding

$$j : \mathrm{GL}_{n-1} \to \mathrm{GL}_n,\ g \mapsto \begin{pmatrix} g & \\ & 1 \end{pmatrix}$$

sends K' into K, since by Théorème (4.1) of [loc.cit.] w_f is even right invariant under $j(\mathrm{GL}_{n-1}(\hat{\mathbb{Z}}))$, so we may choose K containing $j(K')$. Recall that we defined all of our additive characters τ_v to have exponent 0, what allows us to use those results.

Separating finite and infinite parts of adelic elements we write $g = (g_f, g_\infty)$ for $g \in \mathrm{GL}_n(\mathbb{A}) = \mathrm{GL}_n(\mathbb{A}_f) \times \mathrm{GL}_n(\mathbb{R})$. We put

$$\mathscr{X}_n := \mathrm{GL}_n(\mathbb{R})/\,\mathrm{O}_n(\mathbb{R}) = \mathrm{GL}_n(\mathbb{R})^0/\,\mathrm{SO}_n(\mathbb{R}) = \mathbb{R}_{>0} \times \mathscr{X}_n^1$$

with

$$\mathscr{X}_n^1 := \mathrm{SL}_n(\mathbb{R})/\,\mathrm{SO}_n(\mathbb{R}) = \mathrm{SL}_n^\pm(\mathbb{R})/\,\mathrm{O}_n(\mathbb{R}) \subseteq \mathscr{X}_n$$

and

$$\Gamma := \{\gamma \in \mathrm{GL}_n(\mathbb{Q})^0 \mid \gamma_f \in K\} \subseteq \mathrm{SL}_n(\mathbb{Z}).$$

Then by the surjectivity of the determinant map we have the bijections

$$\Gamma\backslash\mathscr{X}_n \cong \mathrm{GL}_n(\mathbb{Q})\backslash\,\mathrm{GL}_n(\mathbb{A})/K \cdot \mathrm{O}_n(\mathbb{R}) \tag{3.1}$$

and

$$\Gamma\backslash\mathscr{X}_n^1 \cong \mathrm{GL}_n(\mathbb{Q})\backslash\,\mathrm{GL}_n(\mathbb{A})/K \cdot \mathrm{SO}_n(\mathbb{R})Z_n(\mathbb{R})^0 \overset{1.6}{=} S_n(K). \tag{3.2}$$

For a proof look in [Bum1], Prop. 3.3.1.[4] The common dimension of $\mathscr{X}_n$ and $\Gamma\backslash\mathscr{X}_n$ is $d_n := \frac{n^2+n}{2}$. The same argument applies to GL_{n-1} with a discrete subgroup $\Gamma' \subseteq \mathrm{SL}_{n-1}(\mathbb{Z})$ attached to K'. The embedding $j : \mathrm{GL}_{n-1} \to \mathrm{GL}_n$ induces an embedding of symmetric spaces

$$j : \mathscr{X}_{n-1} \to \mathscr{X}_n,\ g \cdot \mathrm{O}_{n-1}(\mathbb{R}) \mapsto \begin{pmatrix} g & \\ & 1 \end{pmatrix} \cdot \mathrm{O}_n(\mathbb{R}).$$

Moreover we can create a whole family of embeddings by composing j with left translation by any element $h \in \mathrm{GL}_n(\mathbb{R})$. Set

$$j_h : \mathscr{X}_{n-1} \to \mathscr{X}_n,\ g \cdot \mathrm{O}_{n-1}(\mathbb{R}) \mapsto h \cdot \begin{pmatrix} g & \\ & 1 \end{pmatrix} \cdot \mathrm{O}_n(\mathbb{R}).$$

[4] The proof is written down only for $n = 2$ but holds for general n.

We are in particular interested in those embeddings j_h which define maps of arithmetic quotients. For any $h \in \mathrm{GL}_n(\mathbb{Q})$ let

$$\Gamma'_h := \{\gamma \in \Gamma' \mid j(\gamma) \in h^{-1}\Gamma h\}.$$

Then j_h induces a proper mapping

$$\bar{j}_h : \Gamma'_h \backslash \mathscr{X}_{n-1} \to \Gamma\backslash \mathscr{X}_n,\ \Gamma'_h g\, \mathrm{O}_{n-1}(\mathbb{R}) \mapsto \Gamma h \begin{pmatrix} g & \\ & 1 \end{pmatrix} \mathrm{O}_n(\mathbb{R}).$$

We want to compose the maps j_h with the projections p_2 into the second component of $\mathscr{X}_n = \mathbb{R}_{>0} \times \mathscr{X}_n^1$, induced by the map

$$p_2 : \mathrm{GL}_n(\mathbb{R}) \to \mathrm{SL}_n^{\pm}(\mathbb{R}),\ g \mapsto g \cdot |\det(g)|^{-1/n}.$$

Recall that the passage to quotients only effects the second component, i. e.

$$\Gamma\backslash\mathscr{X}_n = \mathbb{R}_{>0} \times \Gamma\backslash\mathscr{X}_n^1.$$

On arithmetic quotients we have the homotopy equivalence

$$\bar{p}_2 : \Gamma\backslash\mathscr{X}_n \to \Gamma\backslash\mathscr{X}_n^1.$$

Of course the same arguments apply to $n-1$ instead of n.

For each $u \in U_n(\mathbb{Q})$ the map

$$J_u := \bar{p}_2 \circ \bar{j}_u : \begin{cases} \Gamma'_u\backslash\mathscr{X}_{n-1} & \to\ \Gamma\backslash\mathscr{X}_n^1 \\ \Gamma'_u g\, \mathrm{O}_{n-1} & \mapsto\ \Gamma u \left(\begin{smallmatrix} g & \\ & 1 \end{smallmatrix}\right) \cdot |\det(g)|^{-1/n}\, \mathrm{O}_n \end{cases}$$

is proper by [KMS], p. 102. We want to keep track of the effect of these maps J_u on certain differential forms. We denote by l_u left translation by u and we decompose the map

$$p_2 \circ j_u : \mathrm{GL}_{n-1}(\mathbb{R}) \to \mathrm{SL}_n^{\pm}(\mathbb{R}),\ g \mapsto p_2(u \cdot j(g))$$

further into $p_2 \circ j_u = p_2 \circ l_u \circ j$. Since $\det(u) = 1$, the maps p_2 and l_u commute, hence we have

$$p_2 \circ j_u = l_u \circ p_2 \circ j.$$

We observe that $p_2 \circ j$ is an injective Lie group homomorphism and hence the induced map on invariant 1-forms is surjective. Specifically, letting $*$ denote dual vector space, this induced mapping

$$\delta(p_2 \circ j) : \mathfrak{sl}_n^* \to \mathfrak{gl}_{n-1}^*$$

is given by the formula

$$\delta(p_2 \circ j)(\omega)(X) := \omega(d(p_2 \circ j)(X)) \tag{3.3}$$

for $X \in \mathfrak{gl}_{n-1}$. Here $d(p_2 \circ j)$ denotes the Lie algebra homomorphism $\mathfrak{gl}_{n-1} \to \mathfrak{sl}_n$ induced by $p_2 \circ j$. Since the pullback l_u^* acts trivially on $\mathfrak{sl}_n^*$ we have

$$\delta(p_2 \circ j) = \delta(p_2 \circ j_u) = (d(p_2 \circ j_u))^*.$$

The map $\delta(p_2 \circ j)$ respects the Cartan decompositions

$$\mathfrak{sl}_n = \mathfrak{so}_n \oplus \tilde{\wp}_n \text{ and } \mathfrak{gl}_{n-1} = \mathfrak{so}_{n-1} \oplus \wp_{n-1} \left(= \mathfrak{k}_{n-1,\infty} \oplus \tilde{\wp}_{n-1}\right),$$

where $\mathfrak{so}_n$ denotes the set of skew symmetric $n \times n$ matrices and $\wp_n$ (resp. $\tilde{\wp}_n$) stands for the set of symmetric $n \times n$ matrices (resp. of trace equal to zero). In particular we have

$$\delta(p_2 \circ j)(\tilde{\wp}_n^*) = \wp_{n-1}^*.$$

We can now describe the map of differential forms

$$J_u^* : \Omega^\bullet(\Gamma \backslash \mathscr{X}_n^1, \check{\mathcal{M}}_{\mu,\mathbb{C}}) \to \Omega^\bullet(\Gamma_u' \backslash \mathscr{X}_{n-1}, \check{\mathcal{M}}_{\mu,\mathbb{C}})$$

in terms of the complex defining the Lie algebra cohomology. Note, that since $\check{M}_{\mu,\mathbb{C}}$ can be viewed as a $\mathrm{GL}_{n-1}(\mathbb{R})$-module via j, we can define a locally constant sheaf on $\tilde{S}_{n-1}$ just like in 1.6. We identify this sheaf with the one defined on $\tilde{S}_n$ and name it $\check{\mathcal{M}}_{\mu,\mathbb{C}}$ also. Since $\check{M}_{\mu,\mathbb{C}}$ can be viewed as a finite dimensional complex linear representation of $\mathrm{GL}_n(\mathbb{R})$ and therefore as one of any discrete subgroup Γ_n of $\mathrm{SL}_n^{\pm}(\mathbb{R})$, we may use Corollary VII.2.7 and VII.2.4 (5) of [BW] to get

$$\Omega^\bullet(\Gamma_n \backslash \mathscr{X}_n^1, \check{\mathcal{M}}_{\mu,\mathbb{C}}) \cong \left(\bigwedge^\bullet \tilde{\wp}_n^* \otimes C^\infty(\Gamma_n \backslash \mathrm{SL}_n^{\pm}(\mathbb{R}), \check{M}_{\mu,\mathbb{C}}) \right)^{\mathrm{O}_n(\mathbb{R})}. \tag{3.4}$$

Here, we view the sheaf $\check{\mathcal{M}}_{\mu,\mathbb{C}}$ over $S_n(K_{\Gamma_n})$ as a sheaf over the arithmetic quotient $\Gamma_n \backslash \mathscr{X}_n^1$ via (3.2).

An analogous statement holds for an arbitrary discrete subgroup Γ_{n-1} of $\mathrm{GL}_{n-1}(\mathbb{R})$, if we write $\check{\mathcal{M}}_{\mu,\mathbb{C}}$ as well for the locally constant sheaf of $\check{M}_{\mu,\mathbb{C}}$ over $\mathrm{GL}_{n-1}(\mathbb{Q}) \backslash \mathrm{GL}_{n-1}(\mathbb{A}) / K_{\Gamma_{n-1}} \mathrm{O}_{n-1}(\mathbb{R}) \overset{(3.1)}{\cong} \Gamma_{n-1} \backslash \mathscr{X}_{n-1}$ that we get like in Section 1.6:

$$\Omega^\bullet(\Gamma_{n-1} \backslash \mathscr{X}_{n-1}, \check{\mathcal{M}}_{\mu,\mathbb{C}}) \cong \left(\bigwedge^\bullet \wp_{n-1}^* \otimes C^\infty(\Gamma_{n-1} \backslash \mathrm{GL}_{n-1}(\mathbb{R}), \check{M}_{\mu,\mathbb{C}}) \right)^{\mathrm{O}_{n-1}(\mathbb{R})}. \tag{3.5}$$

We want to choose bases for $\tilde{\wp}_n^*$ and $\wp_{n-1}^*$. Thereby the dimension of $\wp_{n-1}^*$ is $d_{n-1} = \frac{n^2-n}{2}$, and the one of $\tilde{\wp}_n^*$ is $\tilde{d}_n := d_n - 1 = \frac{n^2+n}{2} - 1$. By II.7 in [Hel] for any basis $\{X_1, \ldots, X_{\tilde{d}_n}\}$ of $\tilde{\wp}_n$ there is a basis $\{\omega_1, \ldots, \omega_{\tilde{d}_n}\}$ of left-invariant differential forms in $\tilde{\wp}_n^*$ given by $\omega_i(X_j) := \delta_{ij}$ for all $1 \leq i,j \leq \tilde{d}_n$. Those forms are called *Maurer-Cartan forms.* Now we fix such a basis $\{\omega_1, \ldots, \omega_{\tilde{d}_n}\}$ of Maurer-Cartan forms such that

$$\omega_i' := \delta(p_2 \circ j)(\omega_i) \text{ for } i = 1, \ldots, d_{n-1}$$

is a basis of $\wp_{n-1}^*$ and $\omega_i' = 0$ for $i > d_{n-1}$. Then the ω_i' for $1 \leq i \leq d_{n-1}$ are Maurer-Cartan forms as well. For any set $I = \{i_1, \ldots, i_r\} \subseteq \{1, \ldots, \tilde{d}_n\}$ of pairwise disjoint elements $i_1, \ldots, i_r$ we put $\omega_I := \omega_{i_1} \wedge \ldots \wedge \omega_{i_r}$ resp. $\omega_I' := \omega_{i_1}' \wedge \ldots \wedge \omega_{i_r}'$. It holds

Lemma 3.1 *Let $r \in \mathbb{N}$. Given a differential form*

$$\eta = \sum_{|I|=r} \omega_I \phi_I \in \Omega^r(\Gamma \backslash \mathscr{X}_n^1, \check{\mathcal{M}}_{\mu,\mathbb{C}})$$

with $\phi_I \in C^\infty(\Gamma \backslash \mathrm{SL}_n^\pm(\mathbb{R}), \check{M}_{\mu,\mathbb{C}})$ we have

$$J_u^*(\eta) = \sum_{|I|=r} \omega_I'(\phi_I \circ p_2 \circ j_u) \in \Omega^r(\Gamma_u' \backslash \mathscr{X}_{n-1}, \check{\mathcal{M}}_{\mu,\mathbb{C}}).$$

Since J_u is proper we also get a map on differential forms with compact support

$$J_u^* : \Omega_c^\bullet(\Gamma \backslash \mathscr{X}_n^1, \check{\mathcal{M}}_{\mu,\mathbb{C}}) \to \Omega_c^\bullet(\Gamma_u' \backslash \mathscr{X}_{n-1}, \check{\mathcal{M}}_{\mu,\mathbb{C}}),$$

just by replacing C^∞-functions by compactly supported C^∞-functions in our description above. We will later need a version of J_u^* on differential forms with certain growth conditions (which we get just the same).

3.2 Growth conditions

The next thing is to make precise those growth conditions. Let ϕ be a function in $C^\infty(\mathrm{SL}_n^\pm(\mathbb{R}), \check{M}_{\mu,\mathbb{C}})$. With an arbitrary norm $|\cdot| : \check{M}_{\mu,\mathbb{C}} \to \mathbb{R}$ of $\check{M}_{\mu,\mathbb{C}}$ as a $\mathbb{C}$-vector space we can generalise the growth conditions described in [KMS], p. 119.

Choose a norm $|\cdot|$ of $\check{M}_{\mu,\mathbb{C}}$. The function ϕ is of *moderate growth* or *slowly increasing*, if there is a constant C and a positive integer m such that for all $g \in \mathrm{SL}_n^\pm(\mathbb{R})$ we have

$$|\phi(g)| \leq C \cdot ||g||^m,$$

where $||g|| := \mathrm{tr}({}^t g \cdot g)^{1/2}$. The function ϕ is *fast decreasing*, if for each integer m there is a constant $C = C_m$ such that this inequality holds for all g. Those concepts are well-defined (i.e. independent of the norm $|\cdot|$) since all norms on $\check{M}_{\mu,\mathbb{C}}$ are equivalent, $\check{M}_{\mu,\mathbb{C}}$ being finite dimensional as a $\mathbb{C}$-vector space.

We will denote the compactly supported C^∞-functions by C_c^∞, the fast decreasing ones by C_{fd}^∞, and the ones of moderate growth by C_{mg}^∞.

A differential form $\eta = \sum_I \omega_I \phi_I$ on $\Gamma\backslash\mathscr{X}_n^1$ is of *moderate growth* (resp. *fast decreasing*), if the ϕ_I have this property (cf. [Bor]). Following Borel we denote by $\Omega_{\mathrm{mg}}^\bullet(\Gamma\backslash\mathscr{X}_n^1, \check{\mathcal{M}}_{\mu,\mathbb{C}})$ (resp. $\Omega_{\mathrm{fd}}^\bullet(\Gamma\backslash\mathscr{X}_n^1, \check{\mathcal{M}}_{\mu,\mathbb{C}})$) the complex of forms $\eta \in \Omega^\bullet(\Gamma\backslash\mathscr{X}_n^1, \check{M}_{\mu,\mathbb{C}})$ which together with their exterior de Rham differentials $d\eta$ are of moderate growth (resp. fast decreasing).

3.3 Integration along the fibre

In this section we want to find a map from the image of J_u^* to differentials on $\Gamma'_u\backslash\mathscr{X}_{n-1}^1$ such that the composition of this map and J_u^* is a chain map of the de Rham complex. In order to do this we will *integrate along the fibre*: We consider the canonical projection

$$\pi : \Gamma'_u\backslash\mathscr{X}_{n-1} = \Gamma'_u\backslash\mathscr{X}_{n-1}^1 \times \mathbb{R}_{>0} \to \Gamma'_u\backslash\mathscr{X}_{n-1}^1$$

onto the first component and consider the push-forward π_* like in [BT], p. 37. We will show that for $n \geq 3$ the forms in

$$\Omega^\bullet(\Gamma'_u\backslash\mathscr{X}_{n-1}, \check{\mathcal{M}}_{\mu,\mathbb{C}}) = \Omega^\bullet(\Gamma'_u\backslash\mathscr{X}_{n-1}^1 \times \mathbb{R}_{>0}, \check{\mathcal{M}}_{\mu,\mathbb{C}})$$

which are in the image $J_u^*(\Omega_{\mathrm{fd}}^\bullet(\Gamma\backslash\mathscr{X}_n^1, \check{\mathcal{M}}_{\mu,\mathbb{C}}))$ can be integrated along the fibre, i. e.

Lemma 3.2 *For $n \geq 3$ the push-forward π_* is a chain map lowering the degree of forms by one, more precisely*

$$\pi_* : J_u^*(\Omega_{\mathrm{fd}}^\bullet(\Gamma\backslash\mathscr{X}_n^1, \check{\mathcal{M}}_{\mu,\mathbb{C}})) \to \Omega_{\mathrm{mg}}^{\bullet-1}(\Gamma'_u\backslash\mathscr{X}_{n-1}^1, \check{\mathcal{M}}_{\mu,\mathbb{C}}).$$

Remark *We need $n \geq 3$ only for the identifications (3.1) and (3.2). The lemma is true for $n = 2$ as well, if we view $\check{\mathcal{M}}_{\mu,\mathbb{C}}$ as a sheaf over the respective arithmetic quotients.*

Proof. Let $\eta = \sum_{|I|=\bullet} \omega_I \phi_I$ be an arbitrary differential in $\Omega_{\mathrm{fd}}^\bullet(\Gamma\backslash\mathscr{X}_n^1, \check{\mathcal{M}}_{\mu,\mathbb{C}})$. Then we have $J_u^*(\eta) = \sum_{|I|=\bullet} \omega'_I \cdot (\phi_I \circ p_2 \circ j_u)$.

We want to use the fact that ϕ_I is fast decreasing to show that for each $N > 0$ there is a constant $C_{(N)}$ independent of g such that

$$\left|\phi_I\left(u\begin{pmatrix} g & \\ & 1\end{pmatrix} \cdot |\det(g)|^{-\frac{1}{n}}\right)\right| \leq C_{(N)} \cdot \min\{|\det(g)|^{-N}, |\det(g)|^N\}. \tag{3.6}$$

By definition of fast decrease, there is a constant C_m such that $|\phi_I(g)| \leq C_m||g||^m$ for all $g \in G$. In particular, we have

$$\left|\phi_I\left(u\begin{pmatrix} g & \\ & 1\end{pmatrix}\cdot|\det(g)|^{-\frac{1}{n}}\right)\right| \leq C_m \left|\left|u\begin{pmatrix} g & \\ & 1\end{pmatrix}\cdot|\det(g)|^{-\frac{1}{n}}\right|\right|^m.$$

Now, the norm $||x|| := (\sum_{i,j} x_{ij}^2)^{1/2}$ on $\mathrm{SL}_n^{\pm}$ satisfies $||x|| \geq 1$ and

$$||x||\cdot||y^{-1}||^{-1} \leq ||x\cdot y|| \leq ||x||\cdot||y||.$$

We use this information to find an estimate

$$\left|\phi_I\left(u\begin{pmatrix} g & \\ & 1\end{pmatrix}\cdot|\det(g)|^{-\frac{1}{n}}\right)\right| \leq C_m \left|\left|\begin{pmatrix} g & \\ & 1\end{pmatrix}\cdot|\det(g)|^{-\frac{1}{n}}\right|\right|^m \begin{cases} ||u||^m & \text{for } m \geq 0,\\ ||u^{-1}||^{-m} & \text{for } m < 0.\end{cases}$$

Put $N := -\frac{m}{n(n-1)}$ and

$$C_{(N)} := C_m \cdot \begin{cases} ||u||^m & \text{for } m \geq 0,\\ ||u^{-1}||^{-m} & \text{for } m < 0.\end{cases}$$

Then we get the required inequality (3.6), because for any $m < 0$ we have

$$\left|\left|\begin{pmatrix} g & \\ & 1\end{pmatrix}\cdot|\det(g)|^{-\frac{1}{n}}\right|\right|^m = |\det(g)|^{-\frac{m}{n}}\left(1+\sum g_{ij}^2\right)^{\frac{m}{2}}.$$

Since $\sum g_{ij}^2 \geq |\det(g)|^{\frac{2}{n-1}}$, this implies for $n \geq 2$

$$\begin{aligned} ||\dots||^m \leq\ & |\det(g)|^{-\frac{m}{n}}\cdot\left(1+|\det(g)|^{\frac{2}{n-1}}\right)^{\frac{m}{2}} = \left(|\det(g)|^{-\frac{2}{n}}+|\det(g)|^{\frac{2}{n-1}-\frac{2}{n}}\right)^{\frac{m}{2}}\\ =\ & \left(|\det(g)|^{-\frac{2}{n}}+|\det(g)|^{\frac{2}{n(n-1)}}\right)^{\frac{m}{2}} \leq \min\{|\det(g)|^{-\frac{m}{n}}, |\det(g)|^{\frac{m}{n(n-1)}}\}\\ \overset{n\geq 2}{\leq}\ & \min\{|\det(g)|^{\pm N}\}.\end{aligned}$$

For the last estimate we have to distinguish the two cases $|\det(g)| \leq 1$ and $|\det(g)| > 1$.

Let t denote the global parameter of the factor $\mathbb{R}_{>0}$ in $\Gamma'_u\backslash\mathscr{X}_{n-1}$. Integration along the fiber means that for each ω'_I having the invariant differential $\frac{dt}{t} =: \omega'_{d_{n-1}}$ as a wedge factor[5] we must consider the integrals

$$\int_0^\infty \phi_I\left(u\begin{pmatrix} ht & \\ & 1\end{pmatrix}t^{\frac{1-n}{n}}\right)\frac{dt}{t} =: \check{\phi}_{I,u}(h)$$

[5] In difference to [BT] we use the multiplicative Haar measure.

for $h \in \mathrm{SL}_{n-1}^{\pm}(\mathbb{R})$. These integrals are absolutely convergent by (3.6). Moreover, the resulting functions $\check{\phi}_{I,u}$ are bounded, hence in particular, they are of moderate growth. For $\omega_{d_{n-1}} \notin I$ we set $\check{\phi}_{I,u} \equiv 0$. The same proof as for compact supports shows that integration along the fibre is a chain map lowering the degree of forms by 1, i. e.

$$\pi_* : J_u^*(\Omega_{\mathrm{fd}}^{\bullet}(\Gamma\backslash\mathscr{X}_n^1, \check{\mathcal{M}}_{\mu,\mathbb{C}})) \to \Omega_{\mathrm{mg}}^{\bullet-1}(\Gamma'_u\backslash\mathscr{X}_{n-1}^1, \check{\mathcal{M}}_{\mu,\mathbb{C}}).$$

If we write, in abuse of notation, $\omega'_{I\smallsetminus\{d_{n-1}\}}$ for the exterior product of the fitting $\omega'_i|_{\mathscr{X}_{n-1}^1}$, the image of π_* can be described by

$$\pi_* J_u^*(\eta) = \sum_{|I|=\bullet} \check{\phi}_{I,u}\omega'_{I\smallsetminus\{d_{n-1}\}}.$$

Since $\bigwedge^{\bullet}\wp_{n-1}^* \to \bigwedge^{\bullet}\tilde{\wp}_{n-1}^*$ is surjective, those restricted differentials generate $\tilde{\wp}_{n-1}^*$.

Note, that

$$d(\pi_* J_u^*(\eta)) = \pi_* J_u^*(d\eta)$$

has coefficient functions of moderate growth, since for $\eta \in \Omega_{\mathrm{fd}}^{\bullet}$ the coefficient functions of $d\eta$ are by definition also fast decreasing. So the proof of the lemma is complete. □

We thus have constructed a composed chain map

$$\Omega_{\mathrm{fd}}^{\bullet}(\Gamma\backslash\mathscr{X}_n^1, \check{\mathcal{M}}_{\mu,\mathbb{C}}) \xrightarrow{J_u^*} \mathrm{im}(J_u^*) \xrightarrow{\pi_*} \Omega_{\mathrm{mg}}^{\bullet-1}(\Gamma'_u\backslash\mathscr{X}_{n-1}^1, \check{\mathcal{M}}_{\mu,\mathbb{C}}) \tag{3.7}$$

similar to the Poincaré Lemma for forms with compact support (cf. [BT]).

Now let $\check{\mathcal{M}}_{\mu,\nu,\mathbb{C}}$ be the locally constant sheaf belonging to the tensor product $\check{M}_{\mu,\mathbb{C}} \otimes \check{M}_{\nu,\mathbb{C}}$. We want to construct a natural pairing

$$B_u : \Omega_{\mathrm{fd}}^{b_n}(\Gamma\backslash\mathscr{X}_n^1, \check{\mathcal{M}}_{\mu,\mathbb{C}}) \times \Omega_{\mathrm{fd}}^{b_{n-1}}(\Gamma'_u\backslash\mathscr{X}_{n-1}^1, \check{\mathcal{M}}_{\nu,\mathbb{C}}) \to \Omega_{\mathrm{fd}}^{\tilde{d}_{n-1}}(\Gamma'_u\backslash\mathscr{X}_{n-1}^1, \check{\mathcal{M}}_{\mu,\nu,\mathbb{C}}).$$

Note, that $b_n + b_{n-1} - 1 = \tilde{d}_{n-1} = \dim(\mathscr{X}_{n-1}^1)$. Because of (3.7) it suffices to find a pairing

$$\tilde{B}_u : \Omega_{\mathrm{mg}}^{b_n-1}(\Gamma'_u\backslash\mathscr{X}_{n-1}^1, \check{\mathcal{M}}_{\mu,\mathbb{C}}) \times \Omega_{\mathrm{fd}}^{b_{n-1}}(\Gamma'_u\backslash\mathscr{X}_{n-1}^1, \check{\mathcal{M}}_{\nu,\mathbb{C}}) \to \Omega_{\mathrm{fd}}^{\tilde{d}_{n-1}}(\Gamma'_u\backslash\mathscr{X}_{n-1}^1, \check{\mathcal{M}}_{\mu,\nu,\mathbb{C}})$$

and to set $B_u(\eta,\eta') := \tilde{B}_u(\pi_* J_u^*(\eta), \eta')$.

3.4 A pairing on the differentials

We want to construct such a natural pairing $\tilde{B}_u$ now. In order to do this we need to write down elements $\check{\eta}$ of $\Omega_{\mathrm{mg}}^{b_n-1}(\Gamma'_u\backslash\mathscr{X}_{n-1}^1, \check{\mathcal{M}}_{\mu,\mathbb{C}})$ and η' of $\Omega_{\mathrm{fd}}^{b_{n-1}}(\Gamma'_u\backslash\mathscr{X}_{n-1}^1, \check{\mathcal{M}}_{\nu,\mathbb{C}})$

quite explicitly the way suggested by (3.4). In the proof of Lemma 3.2 we saw that the ω'_i with i running from 1 to $d_{n-1} - 1 = \tilde{d}_{n-1}$ form a basis of $\tilde{\wp}^*_{n-1}$. We let $\check{I}$ and I' run through the subsets of this basis like above Lemma 3.1, and let m run through a basis of M_μ and m' through a basis of M_ν.

Then we may write $\check{\eta} = \sum_{|\check{I}|=b_n-1} \omega'_{\check{I}} \check{\phi}_{\check{I}}$ with

$$\check{\phi}_{\check{I}} = \sum_m (\check{\phi}_{\check{I},m} \otimes m) \in C^\infty_{\mathrm{mg}}(\Gamma'_u \backslash \mathrm{SL}^\pm_{n-1}(\mathbb{R}), \mathbb{C}) \otimes \check{M}_{\mu,\mathbb{C}} = C^\infty_{\mathrm{mg}}(\Gamma'_u \backslash \mathrm{SL}^\pm_{n-1}(\mathbb{R}), \check{M}_{\mu,\mathbb{C}})$$

and $\eta' = \sum_{|I'|=b_{n-1}} \omega'_{I'} \varphi_{I'}$ with

$$\varphi_{I'} = \sum_{m'} (\varphi_{I',m'} \otimes m') \in C^\infty_{\mathrm{fd}}(\Gamma'_u \backslash \mathrm{SL}^\pm_{n-1}(\mathbb{R}), \mathbb{C}) \otimes \check{M}_{\nu,\mathbb{C}} = C^\infty_{\mathrm{fd}}(\Gamma'_u \backslash \mathrm{SL}^\pm_{n-1}(\mathbb{R}), \check{M}_{\nu,\mathbb{C}}).$$

Now consider the mapping given by

$$\tilde{B}_u(\check{\eta}, \eta') := \sum (\omega'_{\check{I}} \wedge \omega'_{I'}) \sum_{m,m'} \left(\check{\phi}_{\check{I},m} \cdot \varphi_{I',m'} \otimes (m \otimes m') \right),$$

where the first sum is over all pairs of subsets $\check{I}$ and I' of $\{1, \ldots, d_{n-1}\}$ fulfilling $|\check{I}| = b_n - 1$ and $|I'| = b_{n-1}$. It is well defined, since $\tilde{B}_u(\check{\eta}, \eta')$ is invariant under $\mathrm{O}_{n-1}(\mathbb{R})$: The latter acts on $\bigwedge^{\tilde{d}_{n-1}} \tilde{\wp}^*_{n-1}$ by action on the factors. So from the invariance of $\omega'_{\check{I}}$ and $\omega'_{I'}$.[6] follows, that $\omega'_{\check{I}} \wedge \omega'_{I'}$ is also invariant under $\mathrm{O}_{n-1}(\mathbb{R})$. Using this we can show that

$$g.\tilde{B}_u(\check{\eta}, \eta') = \tilde{B}_u(g.\check{\eta}, g.\eta')$$

for all g in $\mathrm{O}_{n-1}(\mathbb{R})$. The assertion follows from the invariance of $\check{\eta}$ and η'.

A straight-forward calculation shows that $\tilde{B}_u$ is bilinear as a map of $C^\infty_{\mathrm{mg}}(\Gamma'_u \backslash \mathrm{SL}^\pm_{n-1}(\mathbb{R}), \mathbb{C})$-modules. From the definitions of moderate growth and fast decrease we get that, if η' is fast decreasing, the same is true for the image $\tilde{B}_u(\check{\eta}, \eta')$. All in all $\tilde{B}_u$ has the properties we were searching for.

We want to use this formula for $\tilde{B}_u$ to describe B_u explicitly. So if we set $\check{\eta} := \pi_* J^*_u(\eta)$ with $\eta = \sum_I \phi_I \omega_I$ like before, and if we write $\check{\phi}_{I,u} = \sum_m \check{\phi}_{I,u,m} \otimes m$, we get

$$\begin{aligned} B_u(\eta, \eta') &= \tilde{B}_u(\pi_* J^*_u(\eta), \eta') \\ &= \sum_{\substack{|I|=b_n \\ |I'|=b_{n-1}}} \varepsilon_{I,I'} \sum_{m,m'} \left(\check{\phi}_{I,u,m} \cdot \varphi_{I',m'} \otimes (m \otimes m') \right) \omega'_1 \wedge \ldots \wedge \omega'_{\tilde{d}_{n-1}}, \end{aligned} \tag{3.8}$$

where $\varepsilon_{I,I'} = \pm 1$ if $I \,\dot{\cup}\, I' = \{1, \ldots, d_{n-1}\}$ and $\varepsilon_{I,I'} = 0$ otherwise. This is true, since from the proof of Lemma 3.2 we know

$$\pi_* J^*_u(\eta) = \sum_{|I|=b_n-1} \check{\phi}_{I,u} \omega'_{I \smallsetminus \{d_{n-1}\}},$$

[6]The subgroup $\mathrm{O}_{n-1}(\mathbb{R})$ of $\mathrm{SL}^\pm_{n-1}(\mathbb{R})$ acts on $\tilde{\wp}^*_{n-1}$ by conjugation.

and since for all $h \in \mathrm{SL}^{\pm}_{n-1}(\mathbb{R})$ we have

$$\begin{aligned}
\check{\phi}_{I,u}(h) &= \int_0^\infty \phi_I \left(u \begin{pmatrix} ht & \\ & 1 \end{pmatrix} t^{\frac{1-n}{n}} \right) \frac{dt}{t} \\
&= \int_0^\infty \sum_m \phi_{I,m} \left(u \begin{pmatrix} ht & \\ & 1 \end{pmatrix} t^{\frac{1-n}{n}} \right) \otimes m \, \frac{dt}{t} \\
&= \sum_m \int_0^\infty \phi_{I,m} \left(u \begin{pmatrix} ht & \\ & 1 \end{pmatrix} t^{\frac{1-n}{n}} \right) \otimes m \, \frac{dt}{t} \\
&= \sum_m \int_0^\infty \phi_{I,m} \left(u \begin{pmatrix} ht & \\ & 1 \end{pmatrix} t^{\frac{1-n}{n}} \right) \frac{dt}{t} \otimes m \\
&= \sum_m \left(\check{\phi}_{I,u,m}(h) \otimes m \right).
\end{aligned}$$

3.5 A cohomological pairing

By Theorem 5.2 of [Bor] the inclusion $\Omega_{\mathrm{c}} \hookrightarrow \Omega_{\mathrm{fd}}$ induces isomorphisms in cohomology. In particular, each fast decreasing cohomology class can be represented by a form with compact support. This allows us to integrate over the values of B_u. By §5 of [BT][7] we get an induced pairing

$$\mathscr{B}_u : H_{\mathrm{c}}^{b_n}(\Gamma\backslash\mathscr{X}_n^1, \check{\mathcal{M}}_{\mu,\mathbb{C}}) \times H_{\mathrm{c}}^{b_{n-1}}(\Gamma'\backslash\mathscr{X}_{n-1}^1, \check{\mathcal{M}}_{\nu,\mathbb{C}}) \to \check{M}_{\mu,\mathbb{C}} \otimes \check{M}_{\nu,\mathbb{C}}$$

on cohomology. It is given by

$$\mathscr{B}_u([\eta],[\eta']) := \int_{\Gamma'_u\backslash\mathscr{X}_{n-1}^1} B_u(\eta, p_u^*(\eta')),$$

where $p_u : \Gamma'_u\backslash\mathscr{X}_{n-1}^1 \to \Gamma'\backslash\mathscr{X}_{n-1}^1$ is the natural projection. For simplicity we will write η' for $p_u^*(\eta')$.

[7] Bott and Tu work in the real case and with trivial coefficents, but the proof is the same in our situation. Note, that integration and tensoring with elements of $\check{M}_{\mu,\mathbb{C}} \otimes \check{M}_{\nu,\mathbb{C}}$ commutes.

With Section 3.4 we can describe the values of $\mathscr{B}_u$ more explicitly. Indeed we have

$$\begin{aligned}
&\int\limits_{\Gamma'_u\backslash\mathscr{X}^1_{n-1}} B_u(\eta,\eta')\\
\overset{(3.8)}{=}&\int\limits_{\Gamma'_u\backslash\mathscr{X}^1_{n-1}} \sum_{I,I'}\varepsilon_{I,I'}\sum_{m,m'}\left(\check{\phi}_{I,u,m}\cdot\varphi_{I',m'}\otimes(m\otimes m')\right)\omega'_1\wedge\ldots\wedge\omega'_{\tilde{d}_{n-1}}\\
=&\int\limits_{\Gamma'_u\backslash\mathscr{X}^1_{n-1}} \sum_{I,I'}\varepsilon_{I,I'}\sum_{m,m'}\int\limits_0^\infty \phi_{I,m}\left(u\begin{pmatrix}ht&\\&1\end{pmatrix}t^{\frac{1-n}{n}}\right)\frac{dt}{t}\cdot\varphi_{I',m'}(h)\otimes(m\otimes m')dh,
\end{aligned}$$

recognising that a left-invariant $\tilde{d}_{n-1}$-form[8] on $\mathscr{X}^1_{n-1}$ uniquely corresponds to a left-invariant measure dh on $\mathscr{X}^1_{n-1}=\mathrm{SL}_{n-1}/\mathrm{SO}_{n-1}$ induced from a Haar measure on SL_{n-1}.

But then our differentials are invariant under $\mathrm{O}_{n-1}(\mathbb{R})$ by choice[8] so that we can integrate over $\Gamma'_u\backslash\mathrm{SL}^\pm_{n-1}(\mathbb{R})$ instead of $\Gamma'_u\backslash\mathscr{X}^1_{n-1}=\Gamma'_u\backslash\mathrm{SL}^\pm_{n-1}(\mathbb{R})/\mathrm{O}_{n-1}(\mathbb{R})$. The measure dh is the push forward of a Haar measure dg of $\mathrm{GL}_{n-1}(\mathbb{R})$ under the canonical projection. Let us extend the functions $\phi_{I,m}$ and $\varphi_{I',m'}$ in such a manner that they have actions of the respective centres via the central characters ω_π resp. ω_σ of our representations π resp. σ. Then we get

$$\begin{aligned}
\int\limits_{\Gamma'_u\backslash\mathrm{GL}_{n-1}(\mathbb{R})} \sum_{I,I'}\varepsilon_{I,I'}\sum_{m,m'}\phi_{I,m}\left(u\begin{pmatrix}g&\\&1\end{pmatrix}\right)\cdot\varphi_{I',m'}(g)\cdot\omega_\pi(|\det(g)|^{-\frac{1}{n}})\\
\cdot\,\omega_\sigma(|\det(g)|^{-\frac{1}{n-1}})\otimes(m\otimes m')\,dg.
\end{aligned}$$

Now, if we assume that $\omega_\pi(r^{n-1})=\omega_\sigma(r^n)$ holds for all $r\in\mathbb{R}_{>0}$, we may therefore ignore the central parts of their arguments and get the following simplification:

$$\mathscr{B}_u([\eta],[\eta'])=\int\limits_{\Gamma'_u\backslash\mathrm{GL}_{n-1}(\mathbb{R})} \sum_{I,I'}\varepsilon_{I,I'}\sum_{m,m'}\phi_{I,m}\left(u\begin{pmatrix}g&\\&1\end{pmatrix}\right)\cdot\varphi_{I',m'}(g)\otimes(m\otimes m')\,dg.$$

3.6 The Whittaker model

Remember from Corollary 1.5, that for suitable $\varepsilon\in\{+,-\}$

$$H^{b_n}_{\mathfrak{g}\mathrm{K}}(\mathfrak{gl}_n,K_{n,\infty};\mathscr{W}_0(\pi_\infty,\tau_\infty)\otimes\check{M}_{\mu,\mathbb{C}})_\varepsilon=\left(\bigwedge^{b_n}\tilde{\wp}^*_n\otimes\mathscr{W}_0(\pi_\infty,\tau_\infty)\otimes\check{M}_{\mu,\mathbb{C}}\right)^{\mathrm{SO}_n(\mathbb{R})}_\varepsilon$$

[8] The ω'_i were defined as left-invariant differentials of $\wp_{n-1}$ in Section 3.1

is one-dimensional as a $\mathbb{C}$-vector space. We choose a generator η_∞. Using the known bases of $\tilde{\wp}_n^*$ and $\check{M}_\mu$ we can write

$$\eta_\infty = \sum_{|I|=b_n} \sum_m w_{\infty,I,m} \omega_I \otimes m$$

with Whittaker functions $w_{\infty,I,m} \in \mathscr{W}_0(\pi_\infty, \tau_\infty)$.

The Fourier transform $\mathfrak{F}(\pi) : L_0^2(\mathrm{GL}_n(\mathbb{Q})\backslash \mathrm{GL}_n(\mathbb{A})) \supset V_\pi \xrightarrow{\sim} \mathscr{W}(\pi, \tau)$ (cf. [Mah], 1.2) induces a mapping $\mathfrak{F}(\pi)^{\mathrm{coh}}$ on the spaces of $(\mathfrak{gl}_n, K_{n,\infty})$-cohomology, which commutes with the action of $\pi_0(\mathrm{O}_n) = \pi_0(\mathrm{GL}_n)$. Composing $\mathfrak{F}(\pi)^{\mathrm{coh}}$ with the injection of $\mathscr{W}(\pi_f, \tau_f)$ into Lie algebra cohomology given by

$$\begin{array}{rcl} \mathscr{W}(\pi_f, \tau_f) & \hookrightarrow & H^{b_n}_{\mathfrak{g}\mathrm{K}}(\mathfrak{gl}_n, K_{n,\infty}; \mathscr{W}(\pi, \tau) \otimes \check{M}_{\mu,\mathbb{C}})_\varepsilon \\ w_f & \mapsto & w_f \cdot \eta_\infty \end{array}$$

we get

$$\tilde{\mathfrak{F}}(\pi) = (\mathfrak{F}(\pi)^{\mathrm{coh}})^{-1} \cdot \eta_\infty : \mathscr{W}(\pi_f, \tau_f) \to H^{b_n}_{\mathfrak{g}\mathrm{K}}(\mathfrak{gl}_n, K_{n,\infty}; V_\pi \otimes \check{M}_{\mu,\mathbb{C}})_\varepsilon.$$

The image of w_f under $\tilde{\mathfrak{F}}(\pi)$ is

$$\eta := \sum_{|I|=b_n} \sum_m \phi_{I,m} \omega_I \otimes m,$$

where $\phi_{I,m}$ is the cusp form related to $w_f w_{\infty,I,m}$ by $\mathfrak{F}(\pi)$. Analogously, for a generator η'_∞ of the one-dimensional $\mathbb{C}$-vector space

$$H^{b_{n-1}}_{\mathfrak{g}\mathrm{K}}(\mathfrak{gl}_{n-1}, K_{n-1,\infty}; \mathscr{W}_0(\sigma_\infty, \bar{\tau}_\infty) \otimes \check{M}_{\nu,\mathbb{C}})_{\varepsilon'} \quad (\varepsilon' \in \{+,-\} \text{ suitable })$$

and some $w_f \in \mathscr{W}(\sigma_f, \bar{\tau}_f)$ we can construct an injection

$$\tilde{\mathfrak{F}}(\sigma) = (\mathfrak{F}(\sigma)^{\mathrm{coh}})^{-1} \cdot \eta'_\infty : \mathscr{W}(\sigma_f, \bar{\tau}_f) \hookrightarrow H^{b_{n-1}}_{\mathfrak{g}\mathrm{K}}(\mathfrak{gl}_{n-1}, K_{n-1,\infty}; V_\sigma \otimes \check{M}_{\nu,\mathbb{C}})_{\varepsilon'}$$

mapping $v_f \in \mathscr{W}(\sigma_f, \bar{\tau}_f)$ to

$$\eta' := \sum_{|I'|=b_{n-1}} \sum_{m'} \varphi_{I',m'} \omega'_{I'} \otimes m',$$

where we use $\omega'_{I'}$ the same sense as in the proof of Lemma 3.2. The rest of the notation should be clear.

Now remember from (2.1), that we may decompose an element of $\mathscr{W}_0(\pi,\tau) \otimes \mathscr{W}_0(\sigma,\bar{\tau})$ into a finite sum $\sum_j w_j \otimes v_j$ of pure tensors. Evaluating our pairing $\mathscr{B}_u$ at the corresponding η_j and η'_j we get

$$\mathscr{B}_u(\eta_j, \eta'_j) = \sum_{m,m'} \left(\sum_{I,I'} \varepsilon_{I,I'} \int\limits_{\Gamma'_u \backslash \mathrm{GL}_{n-1}(\mathbb{R})} \phi_{j,I,m}(uj(g)) \varphi_{j,I',m'}(g) dg \right) \otimes (m \otimes m'),$$

where the cusp forms $\phi_{j,I,m}$ (belonging to $w_{j,f}w_{\infty,I,m}$) and $\varphi_{j,I',m'}$ (belonging to $v_{j,f}v_{\infty,I',m'}$) are restricted to the infinity component. Note, that we are free to choose $w_{\infty,I,m}$ and $v_{\infty,I',m'}$ independent of j.

We are summing up terms exactly like those on p. 123 of [KMS], and we can apply the same arguments. In order to do this we need to introduce some notation: We denote conjugation by $\varphi := \mathrm{diag}(f^{-1}, \dots, f^{-n})$ by the superscript φ, so that if $g = (g_{ij}) \in \mathrm{GL}_n$, then $g^\varphi = \varphi g \varphi^{-1} = (f^{j-i} g_{ij})$. We will interpret $u \in U_n(\mathbb{Q})$ as an element of $U_n(\mathbb{Q}_p)$ and *not* embed $U_n(\mathbb{Q})$ diagonally into $U_n(\mathbb{A})$. From now on, we will only consider elements $u \in U_n(\mathbb{Q})$, that also lie in $U_n(\mathbb{Z}_p)^{\varphi^{-1}} \subset U_n(\mathbb{Q}_p)$. For those we write

$$K'_u := \{k \in K' \mid uj(k)u^{-1} \in K\}.$$

From [KMS] we get[9]

$$\mathscr{B}_u(\eta_j, \eta'_j) = \frac{\frac{p-1}{p} f^{2(n-1)}}{\mathrm{vol}(K'_u)} \cdot \sum_{m,m'} \left(\sum_{I,I'} \varepsilon_{I,I'} \int\limits_{C_f} \phi_{j,I,m}(j(g)u^{-1}) \varphi_{j,I',m'}(g) dg \right) \otimes (m \otimes m').$$

3.7 Main Theorem

Remember $n \geq 3$. Assume that we have

$$\omega_\pi(r^{n-1}) = \omega_\sigma(r^n) \text{ for all } r \in \mathbb{R}_{>0}$$

for the central characters ω_π of π and ω_σ of σ. Note, that these are local conditions at infinity.

We will be interested in $\mathscr{B}_u(\eta_j, \eta'_j)$ from the last section as a function of u. So if λ is an (at first) arbitrary linear form on $M_{\mu,\mathbb{C}} \otimes M_{\nu,\mathbb{C}}$, we set

$$\mathscr{B}_\lambda(u) := \lambda \circ \sum_j \left(\mathscr{B}_u(\eta_j, \eta'_j) \right).$$

Now we can express the integrals in the formula of Corollary 2.1 and thereby the value at $\frac{1}{2}$ of the Rankin-Selberg L-function in terms of the function $\mathscr{B}_\lambda(u)$. In order to do this we define $P_{I,I',m,m'}(s)$ to be the entire function belonging to the pair $(w_{\infty,I,m}, v_{\infty,I',m'})$ (cf. [KMS], 3.2) such that we have

$$\Psi(w_{\infty,I,m} \otimes v_{\infty,I',m'}; s) = P_{I,I',m,m'}(s) \cdot L(\pi_\infty, \sigma_\infty; s), \tag{3.9}$$

[9] Note, that in [KMS] the factor $\frac{p-1}{p} f^{2(n-1)}$ is actually missing in the cited formula on p. 123 and afterwards. However, this does obviously not change the statement substantially.

and

$$P_{\lambda,\infty}(s) := \sum_{I,I'} \varepsilon_{I,I'} \sum_{m,m'} \lambda(m \otimes m') P_{I,I',m,m'}(s). \tag{3.10}$$

Now let $\chi = \prod_\ell \chi_\ell$ be an idele class character satisfying

(a) $\chi_\infty = 1$,

(b) $\chi, \chi^2, \ldots, \chi^{n-1}$ have the same non-trivial conductor $f = p$-power,

like in Chapter 2. Then it holds

Theorem 1

$$\begin{aligned} & w_p(1)v_p(1)P_{\lambda,\infty}\left(\frac{1}{2}\right) \prod_{i=1}^{n-1} \frac{G(\chi_p^i)(1-p^{-1})}{1-p^{-i}} L(\pi \otimes \chi, \sigma; \frac{1}{2}) \\ = & \sum_u \prod_{i=1}^{n-1} \tilde{\chi}_p(u_i^i) \operatorname{vol}(K'_{u^{\varphi^{-1}}}) \mathscr{B}_\lambda((u^{-1})^{\varphi^{-1}}), \end{aligned}$$

where $u = u_p$ (with $u_\ell = 1$ for all $\ell \neq p$) is taken from a representative system for $U_n(\mathbb{Z}_p)$ modulo $U_n(\mathbb{Z}_p)^\varphi$ with $\varphi = \operatorname{diag}(f^{-1}, \ldots, f^{-n})$.

Proof. This is straight forward. We have

$$\begin{aligned} & w_p(1)v_p(1)P_{\lambda,\infty}(\frac{1}{2}) \prod_{i=1}^{n-1} \frac{G(\chi_p^i)(1-p^{-1})}{1-p^{-i}} L(\pi \otimes \chi, \sigma; \frac{1}{2}) \\ = & w_p(1)v_p(1) \sum_{I,I'} \varepsilon_{I,I'} \sum_{m,m'} \lambda(m \otimes m') P_{I,I',m,m'}(\frac{1}{2}) \prod_{i=1}^{n-1} \frac{G(\chi_p^i)(1-p^{-1})}{1-p^{-i}} L(\pi \otimes \chi, \sigma; \frac{1}{2}) \end{aligned}$$

from the definition of $P_{\lambda,\infty}$. Because of (3.9) we may use Corollary 2.1, the corollary of the Global Birch Lemma, in this situation. We get

$$\sum_{I,I'} \varepsilon_{I,I'} \sum_{m,m'} \lambda(m \otimes m') \left[\frac{p-1}{p} f^{2(n-1)} \sum_u \prod_{i=1}^{n-1} \tilde{\chi}_p(u_i^i) \sum_j \int_{C_f} \phi_{j,I,m}(j(g)u^{\varphi^{-1}}) \varphi_{j,I',m'}(g) dg \right],$$

where u runs through a representative system like in the statement of the theorem. A little rearrangement of terms yields

$$\frac{p-1}{p} f^{2(n-1)} \sum_u \prod_{i=1}^{n-1} \tilde{\chi}_p(u_i^i) \sum_j \sum_{m,m'} \left[\sum_{I,I'} \varepsilon_{I,I'} \int_{C_f} \phi_{j,I,m}(j(g)u^{\varphi^{-1}}) \varphi_{j,I',m'}(g) dg \right] \lambda(m \otimes m'),$$

so that we just have to put in the definition of $\mathscr{B}_\lambda$. Voilà

$$\sum_u \prod_{i=1}^{n-1} \tilde{\chi}_p(u_i^i)\,\mathrm{vol}(K'_{u^{\varphi^{-1}}})\mathscr{B}_\lambda((u^{-1})^{\varphi^{-1}}),$$

which proves the theorem. □

We do not know if $P_{\lambda,\infty}$ is non-zero at $s=\frac{1}{2}$ in general. However, in Chapter 4 we will show this in the case $n=3$ and for trivial coefficient systems.

3.8 Algebraicity

By [Sch1], Satz 1.10, the cuspidal cohomology classes restrict to zero on the border of the Borel-Serre compactification of $S_n(K)$, so that we get an injection of cuspidal cohomology into cohomology with compact support:

$$H^\bullet_{\mathrm{cusp}}(\tilde{S}_n, \check{\mathcal{M}}_{\mu,\mathbb{C}}) \hookrightarrow H^\bullet_{\mathrm{c}}(\tilde{S}_n, \check{\mathcal{M}}_{\mu,\mathbb{C}}).$$

The latter is a module under $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}) \times \mathrm{GL}_n(\mathbb{A}_f) \times \pi_0(\mathrm{GL}_n)$, where the actions of the factors commute and the (image of the) cuspidal cohomology even is defined over $\mathbb{Q}$ (cf. [Clo], Théorème 3.19). So this suggests that we try to choose the cuspidal cohomology classes $[\eta]$ and $[\eta']$ in such a way that the values of $\mathscr{B}_\lambda$ and therefore the L-values at $\frac{1}{2}$ are subject to good rationality conditions.

Let $\mathbb{Q}(\pi_f)$ denote the *field of rationality* of π_f in the notation of §3.1 in [Clo], that is the subfield of $\mathbb{C}$ fixed by the automorphisms $\alpha \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ fulfilling ${}^\alpha\pi_f \cong \pi_f$. It is a field of definition by Proposition 3.1 of [loc.cit.], and in our case in fact a number field by the Drinfel'd-Manin argument (cf. Proposition 3.16 in [loc.cit.]). For the field of rationality $\mathbb{Q}(\sigma_f)$ of σ_f the analogous statements hold.

If we denote by $F := \mathbb{Q}(\pi_f, \sigma_f)$ the smallest field that contains $\mathbb{Q}(\pi_f)$ and $\mathbb{Q}(\sigma_f)$, the global (finite) Whittaker spaces $\mathscr{W}(\pi_f, \tau_f)$ and $\mathscr{W}(\sigma_f, \bar{\tau}_f)$ carry an F-structure, whose underlying F-spaces we denote by $\mathscr{W}_F(\pi_f, \tau_f)$ resp. $\mathscr{W}_F(\sigma_f, \bar{\tau}_f)$. Now since by Corollary 1.5 the cohomology spaces

$$H^{b_n}_{\mathfrak{g}\mathrm{K}}(\mathfrak{gl}_n, K_{n,\infty}; \mathscr{W}_0(\pi_\infty, \tau_\infty) \otimes \check{M}_{\mu,\mathbb{C}})_\varepsilon$$

and

$$H^{b_{n-1}}_{\mathfrak{g}\mathrm{K}}(\mathfrak{gl}_{n-1}, K_{n-1,\infty}; \mathscr{W}_0(\sigma_\infty, \bar{\tau}_\infty) \otimes \check{M}_{\nu,\mathbb{C}})_{\varepsilon'}$$

are one-dimensional, an immediate consequence is

Proposition 3.3 *We can normalise the ∞-part η_∞ by a non-trivial scalar factor such that for any Whittaker function $w_f \in \mathscr{W}_F(\pi_f, \tau_f)$ the cohomology class $[\eta]$ attached to $w_f \cdot \eta_\infty$ is F-rational, i. e.*

$$[\eta] \in H^{b_n}_{\mathrm{cusp}}(\Gamma\backslash\mathscr{X}^1_n, \check{\mathcal{M}}_{\mu,F}) \subseteq H^{b_n}_{\mathrm{c}}(\Gamma\backslash\mathscr{X}^1_n, \check{\mathcal{M}}_{\mu,\bar{\mathbb{Q}}}).$$

An analogous normalisation of η'_∞ yields

$$[\eta'] \in H^{b_{n-1}}_{\mathrm{cusp}}(\Gamma' \backslash \mathscr{X}^1_{n-1}, \check{\mathcal{M}}_{\nu,F}) \subseteq H^{b_{n-1}}_{\mathrm{c}}(\Gamma' \backslash \mathscr{X}^1_{n-1}, \check{\mathcal{M}}_{\nu,\bar{\mathbb{Q}}})$$

with the obvious notation.

The pairings $\mathscr{B}_u$ of cohomology spaces we considered in the sections before are defined purely topologically and moreover with coefficients in an arbitrary subring of $\mathbb{C}$, in particular with coefficients in F. Furthermore we may choose the linear form λ to be induced from a linear form on the $\mathbb{Q}$-vector space M_μ or, slightly more general, from a linear form on the F-vector space $M_{\mu,F}$. By the definition of $\mathscr{B}_\lambda$ we then have

Corollary 3.4 *If the linear form λ is already defined over F, there is a choice of good local tensors t^0_ℓ of Whittaker functions for all $\ell \neq p$ such that for any "Iwahori fixed" pair*

$$(w_p, v_p) \in \mathscr{W}_F(\pi_p, \tau_p)^{I_n} \times \mathscr{W}_F(\sigma_p, \bar{\tau}_p)^{I_{n-1}}$$

the formula in Theorem 1 holds for the associated pairing $\mathscr{B}_\lambda$ with values $\mathscr{B}_\lambda(u)$ in the number field F.

Chapter 4

The non-vanishing of the fudge term

The algebraicity results of the last chapter have the one big flaw, that we can not guarantee the period $P_{\lambda,\infty}(\frac{1}{2})$ not to vanish. The second aim of this thesis is to improve this situation. So in this chapter we will study the case $n = 3$ and will show, that for trivial coefficient systems $\check{M}_{\mu,\mathbb{C}}$ and $\check{M}_{\nu,\mathbb{C}}$ we have $P_{\lambda,\infty}(\frac{1}{2}) \neq 0$ indeed (cf. Theorem 2), where we may assume λ to be the identity map. The general assumptions from the last chapter still hold. Note, that in the case studied $s = \frac{1}{2}$ is the only critical value of (π, σ) (cf. Chapter 5).

A large portion of the proof works fine with general coefficient systems, so that we will assume them to be trivial only when we need it. The idea of proof is to construct a pairing on

$$\left(\bigwedge^{2} \tilde{\wp}_3^* \otimes \mathscr{W}_0(\pi_\infty, \tau_\infty) \otimes \check{M}_{\mu,\infty}\right) \times \left(\tilde{\wp}_2^* \times \mathscr{W}_0(\sigma_\infty, \bar{\tau}_\infty) \otimes \check{M}_{\nu,\infty}\right),$$

whose image equals $P_{\lambda,\infty}(\frac{1}{2}) \cdot \mathbb{C}$ if restricted to the one-dimensional cohomology modules

$$H^2_{\mathfrak{g}K}(\mathfrak{gl}_3, K_{3,\infty}; \pi_\infty \otimes \check{M}_{\mu,\mathbb{C}})_\varepsilon \text{ and } H^1_{\mathfrak{g}K}(\mathfrak{gl}_2, K_{2,\infty}; \sigma_\infty \otimes \check{M}_{\nu,\mathbb{C}})_{\varepsilon'}.$$

This is done in Section 4.3. Here, we will need, that $s = \frac{1}{2}$ is a critical value of the pair (π, σ). It remains to show, that the restricted pairing is not trivial, thus has an image isomorphic to $\mathbb{C}$. In order to do this, we split it up into a pairing $B_{\lambda,\infty}$ on the infinite Whittaker spaces times the coefficient modules and a pairing $B_\wedge$ on the exterior powers.

In Section 4.2 we will study the $\mathrm{SO}_3(\mathbb{R})$-types resp. $\mathrm{SO}_2(\mathbb{R})$-types of $\mathscr{W}_0(\pi_\infty, \tau_\infty) \otimes \check{M}_{\mu,\mathbb{C}}$ resp. $\mathscr{W}_0(\sigma_\infty, \bar{\tau}_\infty) \otimes \check{M}_{\nu,\mathbb{C}}$ that contribute to cohomology. We will show, that those are minimal as $\mathrm{SO}_3(\mathbb{R})$-types resp. $\mathrm{SO}_2(\mathbb{R})$-types. After proving some nice properties of $B_{\lambda,\infty}$ in Section 4.4, that hold for λ like in the first paragraph, we show in Section 4.5, that under the assumption of trivial coefficient systems $B_{\lambda,\infty}$ is not trivial restricted to the cohomological types. Finally, we show in Section 4.6, that $B_\wedge \otimes B_{\lambda,\infty}$ is not trivial restricted to cohomology, which proves Theorem 2.

4.1 Notation

In this section we want to get to know the modules we will be working with in this chapter. Because of the small dimensions, everything is quite explicit.

$\mathfrak{so}_3$-modules. We may write $\mathfrak{so}_3(\mathbb{C}) := \mathfrak{so}_3 \otimes \mathbb{C} = \langle H, E_1, E_{-1}\rangle_{\mathbb{C}}$ with

$$H = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } E_{\pm 1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \pm i \\ -1 & \mp i & 0 \end{pmatrix},$$

where we have $[H, E_{\pm 1}] = \pm i E_{\pm 1}$ and $[E_1, E_{-1}] = 2iH$ for the Lie brackets. The standard torus is given by $\mathfrak{h}_3 = \langle H \rangle_{\mathbb{C}}$. We define e_1 to be the root given by $e_1(H) = 1$.

From now on, we denote $\mathscr{W}_0(\pi_\infty, \tau_\infty) \otimes \check{M}_{\mu,\mathbb{C}}$ by V. By [Mah], Proposition 6.1.3, the $\mathrm{SO}_3(\mathbb{R})$-type of V supporting cohomology in $\bigwedge^2 \tilde{\wp}_3$ is the irreducible representation of $\mathfrak{so}_3(\mathbb{C})$ with highest weight $3e_1$ with respect to the standard torus, which we denote by μ_3. It occurs with multiplicity 1. The restriction $\mu_3 \mapsto \mu_3|_{\mathfrak{so}_3}$ defines a bijection between irreducible (complex) representations of $\mathfrak{so}_3(\mathbb{C})$ and $\mathfrak{so}_3$, and we also denote by μ_3 the restriction of μ_3 to $\mathfrak{so}_3$.

Now, in general, for $k \in \mathbb{N}_0$ let $\mathscr{D}_k$ denote the irreducible $\mathfrak{so}_3$-module of highest weight ke_1. By [FH], §11.1 and §18.2, the irreducible $\mathfrak{so}_3$-modules $\mathscr{D}_k$ are given by the $2k$-th symmetric power of the standard representation of $\mathfrak{sl}_2(\mathbb{C})$ on $\mathbb{C}^2$. By Claim 11.4 of [loc.cit.] as an $\mathfrak{so}_3$-module $V_{\mu_3} \cong \mu_3$ has a basis $v_{-3}, v_{-2}, v_{-1}, v_0, v_1, v_2, v_3$ with

- $E_1 \,.\, v_a = \begin{cases} v_{a+1} & \text{if } a \neq 3 \\ 0 & \text{if } a = 3 \end{cases}$
- $E_{-1} \,.\, v_a = c_a v_{a-1}$ with

a	-3	-2	-1	0	1	2	3
c_a	0	-6	-10	-12	-12	-10	-6

By [BW], Theorem I.5.3, and since π is cohomological, we know that both the central and the infinitesimal character of V are trivial (cf. Footnote [1] in Section 1.3). Like before, we mean the central character on $K_{3,\infty}$.

Furthermore, $\tilde{\wp}_3$ is isomorphic to $\mathscr{D}_2$, so that by [FH], Claim 11.4, again it has a basis $Z_{-2}, Z_{-1}, Z_0, Z_1, Z_2$ fulfilling

- $[E_1, Z_a] = \begin{cases} Z_{a+1} & \text{if } a \neq 2 \\ 0 & \text{if } a = 2 \end{cases}$

- $[E_{-1}, Z_a] = d_a Z_{a-1}$ with

a	-2	-1	0	1	2
d_a	0	-4	-6	-6	-4

.

We normalise those basis vectors of $\tilde{\wp}_3$ by putting

$$Z_{-2} = \begin{pmatrix} 1 & -i & 0 \\ -i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; Z_{-1} = -2 \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ 1 & -i & 0 \end{pmatrix}, \; Z_0 = -4 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

$$Z_1 = 12 \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ 1 & i & 0 \end{pmatrix}, \; Z_2 = 24 \cdot \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$\mathfrak{so}_2$-modules. Analogously, we may write $\mathfrak{so}_2(\mathbb{C}) = \mathfrak{so}_2 \otimes \mathbb{C} = \mathbb{C} \cdot H$, if we identify

$$H = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ with } \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By embedding $\wp_2$ into $\tilde{\wp}_3$ via

$$X \mapsto \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} - \frac{1}{3} \operatorname{tr}(X) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

we identify $\wp_2$ with $\langle Z_{-2}, Z_0, Z_2 \rangle_{\mathbb{C}}$, and $\tilde{\wp}_2$ with $\langle Z_{-2}, Z_2 \rangle_{\mathbb{C}}$. Here, the standard torus $\mathfrak{h}_2$ is all of $\mathfrak{so}_2(\mathbb{C})$. The root that sends H to 1 will be denoted with e_1 as well.

From now on, we denote $\mathscr{W}_0(\sigma_\infty, \bar{\tau}_\infty) \otimes \check{M}_{\nu,\mathbb{C}}$ by W. Like above, the $\mathrm{SO}_2(\mathbb{R})$-types supporting cohomology in $\tilde{\wp}_2$ are μ_2 and μ_{-2}, the irreducible representations of $\mathfrak{so}_2(\mathbb{C})$ with highest weight $2e_1$ resp. $-2e_1$. Again, the $\mathfrak{so}_2$-modules obtained by restriction are denoted the same. μ_2 and μ_{-2} both occur with multiplicity 1, so that we have $W_{\mu_2} \cong \mu_2$ and $W_{\mu_{-2}} \cong \mu_{-2}$. For the same reasons as for V, the central character and the infinitesimal character of W are trivial.

For $k \in \mathbb{Z}$ let $\mathscr{D}'_k$ denote the irreducible $\mathfrak{so}_2$-module of weight ke_1.

4.2 Minimal K-types

In this section we want to study $\mu_{\pm 2}$ and μ_3. From now on, when discussing both cases simultanously we will simply talk of the *cohomological K-types* of V and W, respectively. We will show that the cohomological K-types are indeed the smallest K-types of their respective representation. The fact that thus all smaller K-types do not occur will be useful in Section 4.5.

$\mathfrak{so}_2$-modules. Let σ be in $\mathrm{Coh}(\mathrm{GL}_2, \check{\nu})$ for a dominant weight $\nu \in X_0^+(T_2)$. We are interested in the $\mathrm{SO}_2(\mathbb{R})$-types of W. By Lemma 6.1.1 of [Mah] the lowest $\mathrm{SO}_2(\mathbb{R})$-types of σ_∞ are isomorphic to $\mathscr{D}'_{\nu_1-\nu_2+2}$ resp. $\mathscr{D}'_{\nu_2-\nu_1-2}$ as $\mathfrak{so}_2$-modules. On the other hand $M_{\nu,\mathbb{C}}$ is of dimension $\nu_1 - \nu_2 + 1$ and decomposes into $\mathrm{SO}_2(\mathbb{R})$-types $\mathscr{D}'_{\nu_1-\nu_2}, \mathscr{D}'_{\nu_1-\nu_2-2}, \dots, \mathscr{D}'_{\nu_2-\nu_1}$ (cf. Theorem 2.5.4 of [Bum1]). Because of the one-dimensionality of irreducible $\mathfrak{so}_2$-modules dualisation is easy to control, so that the same is true for $\check{M}_{\nu,\mathbb{C}}$. By the Clebsch-Gordon Formula for $\mathfrak{so}_2$ we have

$$\mathscr{D}'_k \otimes \mathscr{D}'_l \cong \mathscr{D}'_{k+l}$$

for two irreducible $\mathfrak{so}_2$-modules $\mathscr{D}'_k$ and $\mathscr{D}'_l$ with highest weights k resp. l. Hence, we have to study

$$\begin{aligned} & (\mathscr{D}'_{\nu_1-\nu_2+2} \oplus \mathscr{D}'_{\nu_2-\nu_1-2}) \otimes (\mathscr{D}'_{\nu_1-\nu_2} \oplus \mathscr{D}'_{\nu_1-\nu_2-2} \oplus \cdots \oplus \mathscr{D}'_{\nu_2-\nu_1}) \\ \cong & \bigoplus_{k=2}^{2(\nu_1-\nu_2+1)} \mathscr{D}'_k \oplus \bigoplus_{l=-2(\nu_1-\nu_2+1)}^{-2} \mathscr{D}'_l. \end{aligned}$$

Note, that like in Section 1.5 we have $\nu_1 \geq \nu_2$. Altogether we see like in the proof of Lemma 6.1.1 of [Mah] that the smallest $\mathrm{SO}_2(\mathbb{R})$-types of W are isomorphic to $\mathscr{D}'_{-2}$ and $\mathscr{D}'_2$ as $\mathfrak{so}_2$-modules, and both occur with multiplicity one. Those are just μ_{-2} and μ_2.

$\mathfrak{so}_3$-modules. Let π be in $\mathrm{Coh}(\mathrm{GL}_3, \check{\mu})$ for a dominant weight $\mu \in X_0^+(T_3)$. We want to show, that the cohomological $\mathrm{SO}_3(\mathbb{R})$-type μ_3 of V is indeed the smallest $\mathrm{SO}_3(\mathbb{R})$-type. By Lemma 6.1.1 of [Mah] the lowest $\mathrm{SO}_3(\mathbb{R})$-type of π_∞ is isomorphic to $\mathscr{D}_{\mu_1-\mu_3+3}$ as an $\mathfrak{so}_3$-module. We still need to study the $\mathrm{SO}_3(\mathbb{R})$-types of $\check{M}_{\mu,\mathbb{C}}$. Therefor we split up $\check{M}_{\mu,\mathbb{C}}$ into the irreducible highest weight representation $\Gamma_{\mu_1-\mu_2,\mu_2-\mu_3}$ of $\mathrm{SL}_3(\mathbb{C})$ and the power $\det^{-\frac{w}{2}}$ of the determinant (cf. §15.5 of [FH] and (1.2)). Note, that $\check{M}_{\mu,\mathbb{C}}$ and $\Gamma_{\mu_1-\mu_2,\mu_2-\mu_3}$ have the same $\mathrm{SO}_3(\mathbb{R})$-types, and consider

Proposition 4.1 *Let $a, b \in \mathbb{N}$. Then the biggest $\mathrm{SO}_3(\mathbb{R})$-type of $\Gamma_{a,b}$ is $\mathscr{D}_{a+b}$. It occurs with multiplicity one.*

Proof. Let V_s be the standard representation of $\mathfrak{so}_3(\mathbb{C})$ on $\mathbb{C}^3$. Then $\mathfrak{so}_3(\mathbb{C})$ acts on the vector space $\mathrm{Sym}^a \check{V}_s$ of homogeneous polynomials of degree a in the variables X, Y, Z via

$$\phi_a(g)P(\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}) = \frac{d}{dt} P(e^{-tg} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix})|_{t=0}.$$

The weights are given by all expressions of the kind $(l_1 - k_1)e_1$ where we have $k_0 + k_1 + l_1 = a$ with $k_0, k_1, l_1 \in \mathbb{N}$. The belonging weight vectors are given by $(X+iY)^{k_1}(X-iY)^{l_1}Z^{k_0}$. It is just a question of thorough bookkeeping to find

$$\mathrm{Sym}^a \check{V}_s \cong \mathscr{D}_a \oplus \mathscr{D}_{a-2} \oplus \cdots \oplus \mathscr{D}_{a-2\lfloor \frac{a}{2} \rfloor}.$$

Analogously, we get

$$\mathrm{Sym}^a V_s \cong \mathscr{D}_a \oplus \mathscr{D}_{a-2} \oplus \cdots \oplus \mathscr{D}_{a-2\lfloor \frac{a}{2} \rfloor}.$$

By (13.5) of [FH] we have

$$\mathrm{Sym}^a V_s \otimes \mathrm{Sym}^b \check{V}_s = \bigoplus_{i=0}^{\min(a,b)} \Gamma_{a-i,b-i}.$$

The assertion follows by the Clebsch-Gordon formula for $\mathfrak{so}_3$ (cf. [Bou], VIII, §9)[10].
□

So the biggest $\mathrm{SO}_3(\mathbb{R})$-type of $\check{M}_{\mu,\mathbb{C}}$ is $\mathscr{D}_{\mu_1-\mu_3}$. Again by the Clebsch-Gordon formula we get that $\mathscr{D}_3$ is the smallest $\mathrm{SO}_3(\mathbb{R})$-type of V. It occurs with multiplicity one.

4.3 Splitting up the pairing

For the moment let $n \geq 3$ be arbitrary again. Assume that $s = \frac{1}{2}$ is a critical value of the pair (π, σ). In order to show that the value $P_{\lambda,\infty}(\frac{1}{2})$ from Theorem 1 does not vanish we study a pairing

$$B : \left(\bigwedge^{b_n} \tilde{\wp}_n^* \otimes \mathscr{W}_0(\pi, \tau) \otimes \check{M}_{\mu,\mathbb{C}} \right) \times \left(\bigwedge^{b_{n-1}} \tilde{\wp}_{n-1}^* \otimes \mathscr{W}_0(\sigma, \bar{\tau}) \otimes \check{M}_{\nu,\mathbb{C}} \right) \to \mathbb{C}$$

very similar to the pairing $\mathscr{B}_\lambda$ of Chapter 3 that takes the whole left side

$$w_p(1) v_p(1) P_{\lambda,\infty}\left(\frac{1}{2}\right) \prod_{i=1}^{n-1} \frac{G(\chi_p^i)(1-p^{-1})}{1-p^{-i}} L(\pi \otimes \chi, \sigma; \frac{1}{2})$$

of the formula in the theorem as a value. We construct B as a tensor product of three pairings. In this way we can split up B and study our non-vanishing problem in the factors. Those are:

[10] Note, that our $\mathfrak{so}_3$-module $\mathscr{D}_k$ relates to the $\mathfrak{sl}_2$-module $V(2k)$ in Bourbaki.

The archimedean Rankin-Selberg pairing. We define a pairing

$$B_{\lambda,\infty} : (\mathscr{W}_0(\pi_\infty, \tau_\infty) \otimes \check{M}_{\mu,\mathbb{C}}) \times (\mathscr{W}_0(\sigma_\infty, \bar{\tau}_\infty) \otimes \check{M}_{\nu,\mathbb{C}}) \to \mathbb{C}$$

on the infinite parts of the Whittaker spaces by setting

$$B_{\lambda,\infty}(\sum_m w_{m,\infty} \otimes m, \sum_{m'} v_{m',\infty} \otimes m') = \sum_{m,m'} P_{w_{m,\infty}, v_{m',\infty}}\left(\frac{1}{2}\right) \otimes \lambda(m \otimes m').$$

Here, m and m' run through bases of $\check{M}_{\mu,\mathbb{C}}$ resp. $\check{M}_{\nu,\mathbb{C}}$ like in Section 3.6, and it holds

$$\Psi(w_{m,\infty}, v_{m',\infty}; s) = P_{w_{m,\infty}, v_{m',\infty}}(s) \cdot L(\pi_\infty, \sigma_\infty; s)$$

like in Chapter 2. We will call $B_{\lambda,\infty}$ the *archimedean Rankin-Selberg pairing.*

The non-archimedean Rankin-Selberg pairing. On the finite parts of the Whittaker spaces we let

$$B_f : \mathscr{W}_0(\pi_f, \tau_f) \times \mathscr{W}_0(\sigma_f, \bar{\tau}_f) \to \mathbb{C}$$

be the pairing given by

$$B_f(w_f, v_f) = \prod_{\ell \nmid \infty} \psi(w_\ell, v_\ell; \frac{1}{2}).$$

We will call B_f the *non-archimedean Rankin-Selberg pairing.*

The pairing on the exterior powers. A problem in defining a pairing with values in $\mathbb{C}$ on the exterior powers of $\tilde{\wp}_n$ resp. $\tilde{\wp}_{n-1}$ is to make the arguments compatible. However, this is a problem we solved in Chapter 3: The differentials ω_I with $|I| = b_n$ generate $\bigwedge^{b_n} \tilde{\wp}_n^*$, and the differentials $\omega'_{I'}$ with $|I'| = b_{n-1}$ generate $\bigwedge^{b_{n-1}} \tilde{\wp}_{n-1}^*$. Thus a pairing of the sought-after type is given by

$$B_\wedge : \begin{cases} \bigwedge^{b_n} \tilde{\wp}_n^* \times \bigwedge^{b_{n-1}} \tilde{\wp}_{n-1}^* & \to \mathbb{C}, \\ (\omega_I, \omega'_{I'}) & \mapsto \varepsilon_{I,I'}. \end{cases}$$

We may now define B by putting

$$B(w, w') := \sum_{I,I'} B_{\lambda,\infty}(\sum_m w_{I,m,\infty} \otimes m, \sum_{m'} v_{I',m',\infty} \otimes m') \cdot B_f(w_{I,f}, v_{I',f}) \cdot B_\wedge(\omega_I, \omega'_{I'}),$$

where we have $w = \sum_{|I|=b_n} \sum_m w_{I,m} \omega_I \otimes m$ and $w' = \sum_{|I'|=b_{n-1}} \sum_{m'} v_{I',m'} \omega'_{I'} \otimes m'$.

The next thing now is to determine the relation between B and $\mathscr{B}_\lambda$. In order to do this we compare the special L-values $L(\pi \otimes \chi, \sigma; \frac{1}{2})$ with zeta-integrals like they occur as values of B. Let $w_{j,I,m}$ resp. $v_{j,I',m'}$ be the Whittaker functions belonging to the automorphic forms $\phi_{j,I,m}$ resp. $\varphi_{j,I',m'}$ from the proof of Theorem 1. From Chapter 2 we already know the "good tensors" t^0_ℓ for (π_ℓ, σ_ℓ) at an arbitrary prime $\ell \neq p$ fulfilling

$$t^0_\ell = \sum_j w_{j,\ell} \otimes v_{j,\ell},$$

where j runs through a finite sum independently of ℓ. Analogously, $\chi_\ell(\det) \cdot t^0_\ell$ is a "good tensor" for $(\pi_\ell \otimes \chi_\ell, \sigma_\ell)$. Like in the proof of the Global Birch Lemma (cf. [KMS]) it follows

$$L(\pi_\ell \otimes \chi_\ell, \sigma_\ell; s) = \Psi(\chi_\ell(\det) \cdot t^0_\ell; s) \text{ for } \ell \neq p, \infty.$$

At the place p we have

$$L(\pi_p \otimes \chi_p, \sigma_p; s) = 1$$

by page 113 of [KMS]. On the other hand, if we put

$$w_{j,p,\chi_p}(g) = \chi_p(\det(g)) \sum_u \prod_{i=1}^{n-1} \tilde{\chi}(u^i_i) w_{j,p}(g u^{\varphi^{-1}}),$$

where the summation over u is taken over a representative system for $U_n(\mathbb{Z}_p)$ modulo $U_n(\mathbb{Z}_p)^\varphi$, we get

$$\psi(w_{j,p,\chi_p}, v_{j,p}; s) = w_{j,p}(1) v_{j,p}(1) \prod_{i=1}^{n-1} \frac{G(\chi^i_p)(1-p^{-1})}{1-p^{-i}}$$

by Proposition 3.1 of [loc.cit.].

We still have to study $L(\pi_\infty \otimes \chi_\infty, \sigma_\infty; s)$. Note, that this is the same as $L(\pi_\infty, \sigma_\infty; s)$, since we chose χ_∞ to be trivial. The latter value can be calculated via Langlands correspondence at infinity (cf. Chapter 5). As a product of Γ-values $L(\pi_\infty \otimes \chi_\infty, \sigma_\infty; s)$ has no zeroes, and since we assumed $s = \frac{1}{2}$ to be a critical value $L(\pi_\infty \otimes \chi_\infty, \sigma_\infty; \frac{1}{2})$ is indeed a number in $\mathbb{C}^\times$. So without loss of generality we may renormalise some Whittakerfunctions and ignore the factor $L(\pi_\infty \otimes \chi_\infty, \sigma_\infty; \frac{1}{2})$.

Now that we know the respective values of the zeta integrals and the local Rankin-Selberg L-series at all places we use this information to find Whittaker functions such that $L(\pi \otimes \chi, \sigma; \frac{1}{2})$ occurs as a factor in the associated value of B. If we set

$$w_{j,f,\chi} = w_{p,\chi_p} \cdot \prod_{\ell \neq p,\infty} \chi_\ell(\det) w_{j,\ell}$$

we may define

$$w_{j,\chi} = \sum_{|I|=b_n} \sum_m w_{j,f,\chi} w_{\infty,I,m} \omega_I \otimes m \quad \text{and} \quad v_j = \sum_{|I'|=b_{n-1}} \sum_{m'} v_{j,f} v_{\infty,I',m'} \omega'_{I'} \otimes m'.$$

Compare with η and η' in Section 3.6. By Proposition 3.1 of [KMS] and (3.10) we get

$$B(w_{j,\chi}, v_j) = P_{\lambda,\infty}\left(\frac{1}{2}\right) \cdot w_p(1)v_p(1) \prod_{i=1}^{n-1} \frac{G(\chi_p^i)(1-p^{-1})}{1-p^{-i}} \cdot \prod_{\ell \neq p,\infty} \psi(\chi_\ell(\det) w_{j,\ell}, v_{j,\ell}; \frac{1}{2}).$$

All in all we have

$$\begin{aligned}\sum_j B(w_{j,\chi}, v_j) &= w_p(1)v_p(1)P_{\lambda,\infty}\left(\frac{1}{2}\right) \prod_{i=1}^{n-1} \frac{G(\chi_p^i)(1-p^{-1})}{1-p^{-i}} \prod_{\ell\neq p,\infty} \Psi(\chi_\ell(\det)\cdot t_\ell^0; \frac{1}{2}) \\ &= w_p(1)v_p(1)P_{\lambda,\infty}\left(\frac{1}{2}\right) \prod_{i=1}^{n-1} \frac{G(\chi_p^i)(1-p^{-1})}{1-p^{-i}} \cdot L(\pi\otimes\chi, \sigma; \frac{1}{2}).\end{aligned}$$

Like in the proof of the Global Birch Lemma we may express the values $B(w_{j,\chi}, v_j)$ as a sum of u-shifts, where u runs through a representative system of $U_n(\mathbb{Z}_p)$ modulo $U_n(\mathbb{Z}_p)^\varphi$.

Remark *Because of our choice of Whittaker functions the image of $B_\wedge \otimes B_{\lambda,\infty}$ restricted to the one-dimensional cohomology modules*

$$H_{\mathfrak{g}K}^{b_n}(\mathfrak{gl}_n, K_{n,\infty}; \mathscr{W}_0(\pi_\infty, \tau_\infty) \otimes \check{M}_{\mu,\mathbb{C}})^{\mathrm{O}_n(\mathbb{R})}$$

and

$$H_{\mathfrak{g}K}^{b_{n-1}}(\mathfrak{gl}_{n-1}, K_{n-1,\infty}; \mathscr{W}_0(\sigma_\infty, \bar{\tau}_\infty) \otimes \check{M}_{\nu,\mathbb{C}})^{\mathrm{O}_{n-1}(\mathbb{R})}$$

from Chapter 3 is spanned by $P_{\lambda,\infty}(\frac{1}{2})$.

To prove, that $P_{\lambda,\infty}(\frac{1}{2})$ does not vanish, it thus suffices to show that the restriction of $B_\wedge \otimes B_{\lambda,\infty}$ to cohomology is not trivial. In the case $n = 3$ and for trivial coefficient systems this will be done in the following sections.

4.4 The archimedean Rankin-Selberg pairing

In this section we want to study a special case of the archimedean Rankin-Selberg pairing $B_{\lambda,\infty}$ we introduced in the last section.[11] So from now on assume that the GL_{n-1}-module $\check{M}_{\mu,\mathbb{C}}$ contains the contragredient of $\check{M}_{\nu,\mathbb{C}}$. The manifest linear form $\lambda : \check{M}_{\mu,\mathbb{C}} \otimes \check{M}_{\nu,\mathbb{C}} \to \mathbb{C};\ m \otimes m' \mapsto m'(m)$ is defined over F as well and is compatible with the respective module structures, so that all results up to now still hold. If we talk about this special case, we will omit the subscript λ and write B_∞ for our pairing. We show the following

[11] Note, that this case contains the case of trivial coefficient systems we will be studying from Section 4.5 on.

Proposition 4.2 *The archimedean Rankin-Selberg pairing B_∞ fulfils the following properties:*

(a) It is $(\mathfrak{gl}_{n-1}, \mathrm{O}_{n-1}(\mathbb{R}))$-invariant.

(b) $B_\infty(V, W) \not\equiv 0$.

Proof. We first show that the zeta integral $\psi(w_\infty, v_\infty; \frac{1}{2})$ is $(\mathfrak{gl}_{n-1}, \mathrm{O}_{n-1}(\mathbb{R}))$-invariant as a function on the Whittaker models. Therefor we have to show its $\mathrm{O}_{n-1}(\mathbb{R})$-invariance[12], i. e.

$$\psi(\pi_\infty(h)w_\infty, \sigma_\infty(h)v_\infty; \frac{1}{2}) = \psi(w_\infty, v_\infty; \frac{1}{2})$$

for all $h \in \mathrm{O}_{n-1}(\mathbb{R})$, and its $\mathfrak{gl}_{n-1}$-equivariance, i. e.

$$\psi(d\pi_\infty(X)w_\infty, v_\infty; \frac{1}{2}) + \psi(w_\infty, d\sigma_\infty(X)v_\infty; \frac{1}{2}) = 0$$

for all $X \in \mathfrak{gl}_{n-1}$. Here, $d\pi_\infty$ and $d\sigma_\infty$ are the *infinitesimal representations* belonging to the $\mathrm{GL}_{n-1}(\mathbb{R})$-representations π_∞ resp. σ_∞, that is for all $X \in \mathfrak{gl}_{n-1}$ we have

$$d\pi_\infty(X)(w_\infty) = \frac{d}{dt}\left(\pi_\infty(\exp(tX))w_\infty\right)|_{t=0}$$

and

$$d\sigma_\infty(X)(v_\infty) = \frac{d}{dt}\left(\sigma_\infty(\exp(tX))v_\infty\right)|_{t=0}.$$

We consider the Rankin-Selberg zeta integral

$$\psi(w_\infty, v_\infty; s) = \int_{U_{n-1}(\mathbb{R})\backslash \mathrm{GL}_{n-1}(\mathbb{R})} w_\infty\begin{pmatrix} g & \\ & 1\end{pmatrix} v_\infty(g)|\det(g)|^{s-\frac{1}{2}} dg$$

on $\mathscr{W}(\pi_\infty, \tau_\infty) \times \mathscr{W}(\sigma_\infty, \bar{\tau}_\infty)$. The group $\mathrm{GL}_{n-1}(\mathbb{R})$ acts on the tensor product of Whittaker spaces via right translation, so that we have

$$\psi(\pi_\infty(h)w_\infty, \sigma_\infty(h)v_\infty; s) = \int_{U_{n-1}(\mathbb{R})\backslash \mathrm{GL}_{n-1}(\mathbb{R})} w_\infty\begin{pmatrix} gh & \\ & 1\end{pmatrix} v_\infty(gh)|\det(g)|^{s-\frac{1}{2}} dg$$

for every $h \in \mathrm{GL}_{n-1}(\mathbb{R})$. Now we change the integration variable from g to gh^{-1}. Because of the transitivity of the action on the quotient $U_{n-1}(\mathbb{R})\backslash \mathrm{GL}_{n-1}(\mathbb{R})$ we get

$$\psi(\pi_\infty(h)w_\infty, \sigma_\infty(h)v_\infty; s) = |\det(h)|^{\frac{1}{2}-s} \cdot \psi(w_\infty, v_\infty; s).$$

[12] Like in Chapter 3, we view π_∞ as a $\mathrm{GL}_{n-1}(\mathbb{R})$-module via j.

Evidently $\psi(w_\infty, v_\infty; s)$ is $\mathrm{SL}^{\pm}_{n-1}(\mathbb{R})$-invariant, so that the $\mathrm{O}_{n-1}(\mathbb{R})$-invariance of ψ on the product $\mathscr{W}_0(\pi_\infty, \tau_\infty) \times \mathscr{W}_0(\sigma_\infty, \bar{\tau}_\infty)$ of the $\mathrm{O}_{n-1}(\mathbb{R})$-finite Whittaker spaces follows.

In the special case $s = \frac{1}{2}$, the whole action of $\mathrm{GL}_{n-1}(\mathbb{R})$ is trivial, meaning

$$\psi(\pi_\infty(h)w_\infty, \sigma_\infty(h)v_\infty; \frac{1}{2}) = \psi(w_\infty, v_\infty; \frac{1}{2}) \quad \forall h \in \mathrm{GL}_{n-1}(\mathbb{R}). \tag{4.1}$$

We use this fact to prove the $\mathfrak{gl}_{n-1}$-equivariance of $\psi(w_\infty, v_\infty; \frac{1}{2})$. We have

$$\begin{aligned}
\psi(d\pi_\infty(X)w_\infty, v_\infty; \frac{1}{2}) &= \int \frac{d}{dt} w_\infty\begin{pmatrix} g\cdot\exp(tX) & \\ & 1\end{pmatrix}\Big|_{t=0} v_\infty(g)dg \\
&= \frac{d}{dt}\int w_\infty\begin{pmatrix} g\cdot\exp(tX) & \\ & 1\end{pmatrix} v_\infty(g)dg\,\Big|_{t=0} \\
&\overset{(4.1)}{=} \frac{d}{dt}\int w_\infty\begin{pmatrix} g & \\ & 1\end{pmatrix} v_\infty(g\cdot\exp(-tX))dg\,\Big|_{t=0} \\
&= \frac{d}{d(-t)}\int w_\infty\begin{pmatrix} g & \\ & 1\end{pmatrix} v_\infty(g\cdot\exp(tX))dg\,\Big|_{t=0} \\
&= -\psi(w_\infty, d\sigma_\infty(X)v_\infty; \frac{1}{2}),
\end{aligned}$$

where integration is always over $U_{n-1}(\mathbb{R})\backslash\mathrm{GL}_{n-1}(\mathbb{R})$. All in all we find that $\psi(w_\infty, v_\infty; \frac{1}{2})$ is $(\mathfrak{gl}_{n-1}, \mathrm{O}_{n-1}(\mathbb{R}))$-invariant. Since we have

$$\psi(w_\infty, v_\infty; \frac{1}{2}) = P_{w_\infty, v_\infty}\left(\frac{1}{2}\right)\cdot L(\pi_\infty, \sigma_\infty; \frac{1}{2}),$$

and $L(\pi_\infty, \sigma_\infty; \frac{1}{2})$ does not depend on the choice of our Whittaker functions, the same is true for $P_{w_\infty, v_\infty}\left(\frac{1}{2}\right)$.

Because of our choice of $\check{M}_{\mu,\mathbb{C}}$ and $\check{M}_{\nu,\mathbb{C}}$ we get for all $h \in \mathrm{GL}_{n-1}(\mathbb{R})$, for all $m \in \check{M}_{\mu,\mathbb{C}}$ and all $m' \in \check{M}_{\nu,\mathbb{C}}$ that $\varrho_\nu(h)m'\ (\varrho_\mu(\begin{pmatrix} h & \\ & 1\end{pmatrix})m) = m'(m)$ by (4.1) of [Kna2]. So in our case $\lambda(m\otimes m')$ is invariant under the action of $\mathrm{GL}_{n-1}(\mathbb{R})$, thus $\mathrm{O}_{n-1}(\mathbb{R})$-invariant. The $\mathfrak{gl}_{n-1}$-equivariance follows like above.

So up to now we identified B_∞ as the tensor product of two $(\mathfrak{gl}_{n-1}, \mathrm{O}_{n-1}(\mathbb{R}))$-invariant pairings. But then, in general, if we have two $(\mathfrak{gl}_{n-1}, \mathrm{O}_{n-1}(\mathbb{R}))$-invariant pairings

$$\langle\cdot,\cdot\rangle_W : W_1 \times W_2 \to \mathbb{C} \quad \text{and} \quad \langle\cdot,\cdot\rangle_M : M_1 \times M_2 \to \mathbb{C}$$

of $(\mathfrak{gl}_{n-1}, \mathrm{O}_{n-1}(\mathbb{R}))$-modules, the pairing

$$\langle\cdot,\cdot\rangle : (W_1\otimes M_1)\times(W_2\otimes M_2) \to \mathbb{C}$$

given by

$$\langle w_1 \otimes m_1, w_2 \otimes m_2 \rangle := \langle w_1, w_2 \rangle_W \cdot \langle m_1, m_2 \rangle_M$$

for all $w_1 \in W_1$, $w_2 \in W_2$, $m_1 \in M_1$, $m_2 \in M_2$ is $(\mathfrak{gl}_{n-1}, \mathrm{O}_{n-1}(\mathbb{R}))$-invariant as well. In what concerns the $\mathrm{O}_{n-1}(\mathbb{R})$-invariance, this is obvious. The $\mathfrak{gl}_{n-1}$-equivariance follows from

$$\begin{aligned}
\langle X(w_1 \otimes m_1), w_2 \otimes m_2 \rangle &= \langle Xw_1 \otimes m_1 + w_1 \otimes Xm_1, w_2 \otimes m_2 \rangle \\
&= \langle Xw_1 \otimes m_1, w_2 \otimes m_2 \rangle + \langle w_1 \otimes Xm_1, w_2 \otimes m_2 \rangle \\
&= \langle Xw_1, w_2 \rangle_W \cdot \langle m_1, m_2 \rangle_M + \langle w_1, w_2 \rangle_W \cdot \langle Xm_1, m_2 \rangle_M \\
&= -\langle w_1, Xw_2 \rangle_W \cdot \langle m_1, m_2 \rangle_M - \langle w_1, w_2 \rangle_W \cdot \langle m_1, Xm_2 \rangle_M \\
&\;\vdots \\
&= -\langle w_1 \otimes m_1, X(w_2 \otimes m_2) \rangle,
\end{aligned}$$

where w_1, w_2, m_1, m_2 are like above, and $X \in \mathfrak{gl}_{n-1}$ is arbitrary. Eventually, this proves (a).

To show (b) it suffices to find Whittaker functions in V and W that B_∞ does not send to zero. Recall that we fixed bases of $\check{M}_{\mu,\mathbb{C}}$ and $\check{M}_{\nu,\mathbb{C}}$ in Chapter 3. Let $m \in \check{M}_{\mu,\mathbb{C}}$ and $m' \in \check{M}_{\nu,\mathbb{C}}$ be such basis vectors fulfilling $\lambda(m \otimes m') \neq 0$. By Theorem 1.2 of [CP2] we are able to choose Whittaker functions $w_{m,\infty}$ in $\mathscr{W}_0(\pi_\infty, \tau_\infty)$ and $v_{m',\infty}$ in $\mathscr{W}_0(\sigma_\infty, \bar{\tau}_\infty)$ such that $P_{w_{m,\infty}, v_{m',\infty}}\left(\frac{1}{2}\right)$ does not vanish. Hence, $w_{m,\infty} \otimes m$ and $v_{m',\infty} \otimes m'$ are suitable choices of elements of V and W such that $B_\infty(V, W) \not\equiv 0$, so that we have showed the proposition. □

4.5 Reduction to the cohomological K-types

We return to the case $n = 3$ now. Further, we assume $\check{M}_{\mu,\mathbb{C}}$ and $\check{M}_{\nu,\mathbb{C}}$ to be trivial from now on. In that way V and W are both irreducible. The main goal is to show that B_∞ restricted to the infinity parts of the cohomology modules on which $\mathscr{B}_\lambda$ was defined in Chapter 3 does not vanish. In this section we start by showing the following

Theorem 4.3 *The pairing B_∞ is not trivial on the K-types supporting the cohomology, that is*

$$B_\infty(V_{\mu_3}, W_{\mu_2}) \not\equiv 0 \quad \text{and} \quad B_\infty(V_{\mu_3}, W_{\mu_{-2}}) \not\equiv 0.$$

Note, that the theorem holds for an arbitrary non-trivial $(\mathfrak{gl}_2, \mathrm{O}_2(\mathbb{R}))$-invariant pairing on $V \times W$ with values in $\mathbb{C}$ instead of the archimedean Rankin-Selberg pairing B_∞.

Proof. Because of the $\mathfrak{gl}_2$-equivariance of B_∞ and the irreducibility of W as $\mathfrak{gl}_2$-module we get

$$B_\infty(V, W) = B_\infty(V, W_{\mu_2}) = B_\infty(V, W_{\mu_{-2}}).$$

But then we assumed $B_\infty(V, W) \neq 0$, so that the Theorem follows from Proposition 4.4 below, which we could prove the same for μ_2 instead of μ_{-2}. □

Proposition 4.4 $B_\infty(V, W_{\mu_2}) = B_\infty(V_{\mu_3}, W_{\mu_{-2}})$.

Proof. Let $\mathfrak{U}(\mathfrak{sl}_3)$ be the universal envelopping algebra of $\mathfrak{sl}_3$, and let $p : \mathfrak{T}(\mathfrak{sl}_3) \to \mathfrak{U}(\mathfrak{sl}_3)$ be the belonging projection, where $\mathfrak{T}(\mathfrak{sl}_3)$ is the tensor algebra. Since each element of $\mathfrak{T}(\mathfrak{sl}_3)$ can be found as a representative of an element of $p(\bigoplus_{r=0}^{\infty} \bigotimes^r \tilde{\wp}_3)$, it holds

$$p(\bigoplus_{r=0}^{\infty} \bigotimes^r \tilde{\wp}_3) = \mathfrak{U}(\mathfrak{sl}_3). \tag{4.2}$$

This can be shown by proving that the left side contains a basis of $\mathfrak{sl}_3$. But then, $\tilde{\wp}_3$ is obviously contained, and we have

$$H = \frac{1}{48} i[Z_{-1}, Z_1], E_1 = \frac{1}{48}[Z_0, Z_1], E_{-1} = \frac{1}{8}[Z_{-1}, Z_0].$$

From now on we write $\tilde{\wp}_3^r$ for $\otimes^r \tilde{\wp}_3$.

A direct implication of (4.2) is $V = \sum_{r\geq 0} \tilde{\wp}_3^r \,.\, V_{\mu_3}$, where the dot denotes the action of $\mathfrak{U}(\mathfrak{sl}_3)$ on V. Thus the proof of Propostion 4.4 is reduced to showing that

$$\forall r \in \mathbb{N}_0 : B_\infty(\tilde{\wp}_3^r \,.\, V_{\mu_3}, W_{\mu_2}) \subseteq B_\infty(V_{\mu_3}, W_{\mu_2}).$$

We will do this by induction on r, where we will need the cases $r-1$ and $r-2$ in the step. Therefore we will start with $r = 1$ and $r = 2$.

Since W is irreducible as a $\mathfrak{gl}_2$-module, it is generated by a single element w_{-2} of, say, weight $-2e_1$. Like in the proof of the theorem from the $\mathfrak{gl}_2$-equivariance of B_∞ follows

$$B_\infty(V_{\mu_3}, W_{\mu_{-2}}) = B_\infty(V_{\mu_3}, w_{-2}).$$

Now consider an arbitrary $v \in V_{\mu_3}$. Since V_{μ_3} is the direct sum of its weight spaces and because of the bilinearity of B_∞ we may without loss of generality assume v to have a weight $\mathsf{wt}(v)$. But then we have

$$\mathsf{wt}(v) B_\infty(v, w_{-2}) = B_\infty(H \,.\, v, w_{-2}) = -B_\infty(v, H \,.\, w_{-2}) = 2B_\infty(v, w_{-2}),$$

so that $B_\infty(v, w_{-2}) = 0$ if $\mathsf{wt}(v) \neq 2e_1$. So all we have to study is the $2e_1$ weight space of V_{μ_3}.

The case $r = 1$. By the Clebsch-Gordan Formula for $\mathfrak{so}_3$ (cf. [Bou], VIII, §9) we have

$$\tilde{\wp}_3 \otimes V_{\mu_3} \cong \mathscr{D}_2 \otimes \mathscr{D}_3 \cong \mathscr{D}_1 \oplus \mathscr{D}_2 \oplus \mathscr{D}_3 \oplus \mathscr{D}_4 \oplus \mathscr{D}_5.$$

Since $\mathscr{D}_1$ has no $2e_1$ weight space, the $2e_1$ weight space of $\tilde{\wp}_3 \otimes V_{\mu_3}$ is four dimensional. A set of generators is given by

$$\begin{aligned}
&Z_0 \otimes v_2,\\
&Z_2 \otimes v_0,\\
&Z_2 \otimes v_0 - Z_1 \otimes v_1 + Z_0 \otimes v_2 - Z_{-1} \otimes v_3 \quad \in (\tilde{\wp}_3 \otimes V_{\mu_3})_{\mathscr{D}_2},\\
&E_{-1}(Z_2 \otimes v_1 - Z_1 \otimes v_2 + Z_0 \otimes v_3) \quad \in (\tilde{\wp}_3 \otimes V_{\mu_3})_{\mathscr{D}_3}.
\end{aligned}$$

Recall from Section 4.2 that $\mathscr{D}_3$ is the smallest $\mathrm{SO}_3(\mathbb{R})$-type in V. On the one hand, this is why $Z_2 . v_0 - Z_1 . v_1 + Z_0 . v_2 - Z_{-1} . v_3 \in (\tilde{\wp}_3 . V_{\mu_3})_{\mathscr{D}_2}$ is zero. On the other hand $E_{-1}(Z_2 . v_1 - Z_1 . v_2 + Z_0 . v_3)$ lies in the $2e_1$ weight space of $(\tilde{\wp}_3 . V_{\mu_3})_{\mathscr{D}_3}$, which is isomorphic to $\mathscr{D}_3$, since the smallest K-type always occurs with multiplicity one. But then we know that v_2 is a basis of the $2e_1$ weight space of $\mathscr{D}_3$, so that $E_{-1}(Z_2 . v_1 - Z_1 . v_2 + Z_0 . v_3)$ lies in $\mathbb{C} \cdot v_2$. Thus it holds

$$(\tilde{\wp}_3 . V_{\mu_3})_{2e_1} \subseteq \langle Z_0 . v_2,\ Z_2 . v_0 \rangle_{\mathbb{C}} + \mathbb{C} \cdot v_2.$$

From the $\mathfrak{gl}_2$-equivariance of B_∞ it follows

$$\begin{aligned}
B_\infty(Z_0 . v, w) = B_\infty(v, Z_0 . w) = 0 \quad \forall v \in V, w \in W,\\
B_\infty(Z_2 . v, w_{-2}) = B_\infty(v, Z_2 . w_{-2}) = 0 \quad \forall v \in V,
\end{aligned} \tag{4.3}$$

where we have $Z_0 . w = 0$ because of the triviality of the central character of W, and $Z_2 . w_{-2} \in (\tilde{\wp}_2 . W)_{\mathscr{D}'_0} = 0$ since $\mathscr{D}'_2$ is the minimal $\mathrm{SO}_2(\mathbb{R})$-type of W by Section 4.4. So the $\mathbb{C}$-vector space generated by $Z_0 . v_2$ and $Z_2 . v_0$ lies in the kernel of $B_\infty(\cdot, w_{-2})$, which we denote by $\mathfrak{a}$. All in all, this proves the case $r = 1$.

The case $r = 2$. A set of generators of the $2e_1$ weight space of $\tilde{\wp}_3^2 . V_{\mu_3}$ is given by

$$\{Z_i Z_j . v_k \mid -2 \le i, j \le 2,\ -3 \le k \le 3,\ i + j + k = 2\}.$$

Like in the case $r = 1$ we will show that all those generators lie in $\mathfrak{a} + \mathbb{C} \cdot v_2$. By (4.3) we already know this in the case that i or j is 0 or 2. So we only have to consider

$$\{Z_{-2}Z_1 . v_3, Z_{-1}Z_1 . v_2, Z_1 Z_{-2} . v_3, Z_1 Z_{-1} . v_2, Z_1^2 . v_0\}.$$

Since from $[Z_1, Z_{-2}] = -24E_{-1}$ and $[Z_1, Z_{-1}] = 48iH$ we have $[Z_1, Z_{-2}] . v_3 \in \mathbb{C} \cdot v_2$ and $[Z_1, Z_{-1}] . v_2 \in \mathbb{C} \cdot v_2$, it even suffices to study

$$\{Z_{-2}Z_1 . v_3, Z_1 Z_{-1} . v_2, Z_1^2 . v_0\}.$$

By the Lemma of Schur we have $\kappa(X,Y) = 6\cdot\mathrm{tr}(XY)$ for the Killing form κ on $\mathfrak{so}_3$. From this we can calculate the Casimir operator $C_3 \in \mathfrak{U}(\mathfrak{sl}_3)$ explicitly. It holds

$$C_3 = -\frac{1}{12}\left(2i\cdot H + H^2 + E_1E_{-1} - \frac{1}{48}\cdot Z_0^2 - \frac{1}{24}\cdot Z_{-2}Z_2 + \frac{1}{24}\cdot Z_{-1}Z_1\right).$$

Since the infinitesimal character of V is trivial, C_3 annihilates V, and we have

$$C_3\,.\,V_{\mu_3} = 0.$$

Because of $[Z_{-2}, Z_2] = -96i\cdot H$ and (4.3) it follows

$$Z_{-1}Z_1\,.\,V_{\mu_3} \subseteq \mathfrak{a} + \mathbb{C}\cdot v_2, \tag{4.4}$$

and even

$$Z_{-1}Z_1\,.\,V \subseteq \mathfrak{U}(\mathfrak{so}_3) + \mathfrak{a} + \mathbb{C}\cdot v_2. \tag{4.5}$$

Since $\mathscr{D}_3$ is the smallest $\mathrm{SO}_3(\mathbb{R})$-type of V, we have $(\tilde{\wp}_3\,.\,V_{\mu_3})_{\mathscr{D}_1+\mathscr{D}_2} = 0$, thus

$$\begin{aligned}&Z_2\,.\,v_{-1} - Z_1\,.\,v_0 + Z_0\,.\,v_1 - Z_{-1}\,.\,v_2 + Z_{-2}\,.\,v_3 = 0,\\ &Z_2\,.\,v_0 - Z_1\,.\,v_1 + Z_0\,.\,v_2 - Z_{-1}\,.\,v_3 = 0.\end{aligned}$$

We apply E_{-1} on the latter equation. Using $E_{-1}Z_a = [E_{-1}, Z_a] + Z_aE_{-1}$ and the known actions of E_{-1} on V_{μ_3} and $\tilde{\wp}_3$, we get

$$\begin{aligned}&Z_2\,.\,v_{-1} - Z_1\,.\,v_0 + Z_0\,.\,v_1 - Z_{-1}\,.\,v_2 + Z_{-2}\,.\,v_3 = 0,\\ &-12Z_2\,.\,v_{-1} + 8Z_1\,.\,v_0 - 4Z_0\,.\,v_1 + 4Z_{-2}\,.\,v_3 = 0.\end{aligned} \tag{4.6}$$

Now, on the one hand we have

$$\begin{aligned}0 &= (-12Z_2\,.\,v_{-1} + 8Z_1\,.\,v_0 - 4Z_0\,.\,v_1 + 4Z_{-2}\,.\,v_3)\\ &\quad -4(Z_2\,.\,v_{-1} - Z_1\,.\,v_0 + Z_0\,.\,v_1 - Z_{-1}\,.\,v_2 + Z_{-2}\,.\,v_3)\\ &= 4Z_{-1}\,.\,v_2 - 8Z_0\,.\,v_1 + 12Z_1\,.\,v_0 - 16Z_2\,.\,v_{-1}\\ &= Z_1(4Z_{-1}\,.\,v_2 - 8Z_0\,.\,v_1 + 12Z_1\,.\,v_0 - 16Z_2\,.\,v_{-1}),\end{aligned}$$

from which we get

$$Z_1^2\,.\,v_0 \in \mathfrak{a} + \mathbb{C}\cdot v_2$$

by (4.3) and (4.4). On the other hand applying Z_1 on the first equation in (4.6) we find

$$Z_1Z_{-2}\,.\,v_3 = -Z_1Z_2\,.\,v_{-1} + Z_1^2\,.\,v_0 - Z_1Z_0\,.\,v_1 + Z_1Z_{-1}\,.\,v_2,$$

which lies in $\mathfrak{a} + \mathbb{C}\cdot v_2$ by the anteceding. This shows the assertion in the case $r = 2$.

Induction. Now assume that $r > 2$ and for all $t \leq r-1$ we have

$$\tilde{\wp}_3^t \,.\, V_{\mu_3} \subseteq \mathfrak{a} + \mathbb{C} \cdot v_2.$$

In order to show the same for $t = r$ we have to consider terms like

$$Z_{k_1} \cdots Z_{k_r} \,.\, v_d \in \tilde{\wp}_3^r \,.\, V_{\mu_3},$$

where $k_1, \ldots, k_r, d$ are natural numbers whose sum is 2, since we are still studying the $2e_1$ weight space. From (4.3) we already know that

$$Z_0 \tilde{\wp}_3^{r-1} \,.\, V_{\mu_3} + Z_2 \tilde{\wp}_3^{r-1} \,.\, V_{\mu_3} \subseteq \mathfrak{a},$$

leading to

$$\tilde{\wp}_3^{r-1} Z_0 \,.\, V_{\mu_3} + \tilde{\wp}_3^{r-1} Z_2 \,.\, V_{\mu_3} \subseteq \mathfrak{a} + \mathbb{C} \cdot v_2$$

because of $[Z_k, Z_l] \in \mathfrak{so}_3$ and the induction hypothesis. Thus is suffices to study terms of the form

$$Z_1^a Z_{-1}^b Z_{-2}^c \,.\, v_d,$$

where a, b, c are natural numbers whose sum equals r, and it holds $a - b - 2c + d = 0$.

From (4.4) and the induction hypothesis we get that for all $u \in \tilde{\wp}_3^{r-2}$ and all $v \in V_{\mu_3}$ the terms $Z_{-1} Z_1 u \,.\, v$ and $Z_1 Z_{-1} u \,.\, v$ lie in $\mathfrak{a} + \mathbb{C} \cdot v_2$. Since $[Z_k, Z_l] \in \mathfrak{so}_3$, it follows that it suffices to study the case where $a = 0$ or $b = 0$. But then, the case $a = 0$ does not occur, since we would have $b + c = r \geq 3$ and $-b - 2c + d = 2$. Note, that $d \leq 3$, since $3e_1$ is the highest weight in V_{μ_3}. So we only have to study terms like

$$Z_1^a Z_{-2}^c \,.\, v_d, \tag{4.7}$$

where $a > 0$ and c are natural numbers whose sum equals r, and it holds $a - 2c + d = 0$.

We still have $(\tilde{\wp}_3 \,.\, V_{\mu_3})_{\mathscr{D}_1 + \mathscr{D}_2} = 0$, and we want to get some relations in $\tilde{\wp}_3 \,.\, V_{\mu_3}$ from that. Therefore, we study bases of the $\mathscr{D}_1$- and the $\mathscr{D}_2$-part of $\tilde{\wp}_3 \otimes V_{\mu_3}$. The $\mathscr{D}_1$-part has the highest weight vectors

$$\begin{array}{lll}
Z_2 \otimes v_{-1} - Z_1 \otimes v_0 + Z_0 \otimes v_1 - Z_{-1} \otimes v_2 + Z_{-2} \otimes v_3 & \text{for} & 1e_1, \\
E_{-1}(Z_2 \otimes v_{-1} - Z_1 \otimes v_0 + Z_0 \otimes v_1 - Z_{-1} \otimes v_2 + Z_{-2} \otimes v_3) & \text{for} & 0e_1, \\
E_{-1}^2(Z_2 \otimes v_{-1} - Z_1 \otimes v_0 + Z_0 \otimes v_1 - Z_{-1} \otimes v_2 + Z_{-2} \otimes v_3) & \text{for} & -1e_1.
\end{array}$$

The $\mathscr{D}_2$-part has the highest weight vectors

$$\begin{array}{lll}
Z_2 \otimes v_0 - Z_1 \otimes v_1 + Z_0 \otimes v_2 - Z_{-1} \otimes v_3 & \text{for} & 2e_1, \\
E_{-1}(Z_2 \otimes v_0 - Z_1 \otimes v_1 + Z_0 \otimes v_2 - Z_{-1} \otimes v_3) & \text{for} & 1e_1, \\
E_{-1}^2(Z_2 \otimes v_0 - Z_1 \otimes v_1 + Z_0 \otimes v_2 - Z_{-1} \otimes v_3) & \text{for} & 0e_1, \\
E_{-1}^3(Z_2 \otimes v_0 - Z_1 \otimes v_1 + Z_0 \otimes v_2 - Z_{-1} \otimes v_3) & \text{for} & -1e_1, \\
E_{-1}^4(Z_2 \otimes v_0 - Z_1 \otimes v_1 + Z_0 \otimes v_2 - Z_{-1} \otimes v_3) & \text{for} & -2e_1.
\end{array}$$

Like in (4.6) we get the relations

$$\begin{array}{ll} Z_2 \,.\, v_{-1} - Z_1 \,.\, v_0 + Z_0 \,.\, v_1 - Z_{-1} \,.\, v_2 + Z_{-2} \,.\, v_3 = 0 & \text{for } \ 1e_1, \\ 5Z_2 \,.\, v_{-2} - 4Z_1 \,.\, v_{-1} + 3Z_0 \,.\, v_0 - 5Z_{-1} \,.\, v_1 + Z_{-2} \,.\, v_2 = 0 & \text{for } \ 0e_1, \\ 15Z_2 \,.\, v_{-3} - 10Z_1 \,.\, v_{-2} + 6Z_0 \,.\, v_{-1} - 21Z_{-1} \,.\, v_0 - 5Z_{-2} \,.\, v_1 = 0 & \text{for } -1e_1 \end{array}$$

for the $\mathscr{D}_1$-part and

$$\begin{array}{ll} Z_2 \,.\, v_0 - Z_1 \,.\, v_1 + Z_0 \,.\, v_2 - Z_{-1} \,.\, v_3 = 0 & \text{for } \ 2e_1, \\ 3Z_2 \,.\, v_{-1} - 2Z_1 \,.\, v_0 + Z_0 \,.\, v_1 - Z_{-2} \,.\, v_3 = 0 & \text{for } \ 1e_1, \\ 5Z_2 \,.\, v_{-2} + Z_{-1} \,.\, v_1 - Z_{-2} \,.\, v_2 = 0 & \text{for } \ 0e_1, \\ 15Z_2 \,.\, v_{-3} + 10Z_1 \,.\, v_{-2} + 6Z_{-1} \,.\, v_0 - 3Z_{-2} \,.\, v_1 = 0 & \text{for } -1e_1, \\ 10Z_1 \,.\, v_{-3} + 5Z_0 \,.\, v_{-2} + 6Z_{-1} \,.\, v_{-1} - Z_{-2} \,.\, v_0 = 0 & \text{for } -2e_1 \end{array}$$

for the $\mathscr{D}_2$-part. Applying (4.7) we get for an arbitrary $u \in \tilde{\wp}_3^{r-1}$:

$$\begin{array}{lll} uZ_1 \,.\, v_0 - uZ_{-2} \,.\, v_3 & \equiv 0 \bmod (\mathfrak{a} + \mathbb{C} \cdot v_2) & \text{for } \ 1e_1, \\ -4uZ_1 \,.\, v_{-1} + uZ_{-2} \,.\, v_2 & \equiv 0 \bmod (\mathfrak{a} + \mathbb{C} \cdot v_2) & \text{for } \ 0e_1, \\ 2uZ_1 \,.\, v_{-2} + uZ_{-2} \,.\, v_1 & \equiv 0 \bmod (\mathfrak{a} + \mathbb{C} \cdot v_2) & \text{for } -1e_1 \end{array} \tag{4.8}$$

and

$$\begin{array}{lll} uZ_1 \,.\, v_1 & \equiv 0 \bmod (\mathfrak{a} + \mathbb{C} \cdot v_2) & \text{for } \ 2e_1, \\ 2uZ_1 \,.\, v_0 + uZ_{-2} \,.\, v_3 & \equiv 0 \bmod (\mathfrak{a} + \mathbb{C} \cdot v_2) & \text{for } \ 1e_1, \\ uZ_{-2} \,.\, v_2 & \equiv 0 \bmod (\mathfrak{a} + \mathbb{C} \cdot v_2) & \text{for } \ 0e_1, \\ 10uZ_1 \,.\, v_{-2} - 3uZ_{-2} \,.\, v_1 & \equiv 0 \bmod (\mathfrak{a} + \mathbb{C} \cdot v_2) & \text{for } -1e_1, \\ 10uZ_1 \,.\, v_{-3} - uZ_{-2} \,.\, v_0 & \equiv 0 \bmod (\mathfrak{a} + \mathbb{C} \cdot v_2) & \text{for } -2e_1. \end{array} \tag{4.9}$$

Note, that we still keep track of the weight of the terms in $\tilde{\wp}_3 \otimes V_{\mu_3}$ we started with. We show that every term $uZ_k \,.\, v_d$ like above vanishes modulo $\mathfrak{a} + \mathbb{C} \cdot v_2$. Therefore, we distinguish the cases where $k+d$ takes different values. From the case $r = 1$ we already know that $k + d$ only varies between -5 and 5. All in all we thus have to consider eleven cases. Five of those follow directly from the antecedent:

- **Weight** $-1e_1$. From (4.8) and (4.9) we get

  $$uZ_1 \,.\, v_{-2} \equiv uZ_{-2} \,.\, v_1 \equiv 0 \bmod (\mathfrak{a} + \mathbb{C} \cdot v_2).$$

  Again, by (4.7) we are done in this case.

- **Weight** $0e_1$. From (4.8) and (4.9) we get

  $$uZ_{-2} \,.\, v_2 \equiv uZ_1 \,.\, v_1 \equiv 0 \bmod (\mathfrak{a} + \mathbb{C} \cdot v_2).$$

  By (4.7) we are done in this case.

- **Weight** $1e_1$. From (4.8) and (4.9) we get

$$uZ_1 \,.\, v_0 \equiv uZ_{-2} \,.\, v_3 \equiv 0 \bmod (\mathfrak{a} + \mathbb{C} \cdot v_2).$$

  Again, by (4.7) we are done in this case.

- **Weight** $2e_1$. From (4.9) we get

$$uZ_1 \,.\, v_1 \equiv 0 \bmod (\mathfrak{a} + \mathbb{C} \cdot v_2).$$

  Since $k = -2$ can not happen in this case and by (4.7) we are done.

- **Weight** $5e_1$. Neither $k = -2$ nor $k = 1$ can happen in this case. By (4.7) we are done.

The rest can be reduced to those five cases as follows:

- **Weight** $3e_1$. Since $k = -2$ can not happen in this case and by (4.7) we only have to show that $uZ_1 \,.\, v_2$ lies in $\mathfrak{a} + \mathbb{C} \cdot v_2$. We know that u is of the form $Z_1^a Z_{-2}^c$ with $a + c = r - 1 \geq 2$ and $a - 2c + 1 + 2 = 2$. It follows $a = 2c - 1$ and $c \geq 1$. We use this information to study $uZ_1 \,.\, v_2$ modulo $\mathfrak{a} + \mathbb{C} \cdot v_2$:

$$uZ_1 \,.\, v_2 \equiv Z_1^{a+1} Z_{-2}^c \,.\, v_2 \equiv (Z_1^{a+1} Z_{-2}^{c-1}) Z_{-2} \,.\, v_2.$$

  Since we already studied the case of weight $0e_1$, we are done.

- **Weight** $4e_1$. Since $k = -2$ can not happen in this case and by (4.7) we only have to show that $uZ_1 \,.\, v_3$ lies in $\mathfrak{a} + \mathbb{C} \cdot v_2$. We know that u is of the form $Z_1^a Z_{-2}^c$ with $a + c = r - 1 \geq 2$ and $a - 2c + 1 + 3 = 2$. It follows $a = 2c - 2$ and $3c \geq 4$, thus $c \geq 2$. We use this information to study $uZ_1 \,.\, v_3$ modulo $\mathfrak{a} + \mathbb{C} \cdot v_2$:

$$uZ_1 \,.\, v_3 \equiv Z_1^{a+1} Z_{-2}^c \,.\, v_3 \equiv (Z_1^{a+1} Z_{-2}^{c-1}) Z_{-2} \,.\, v_3.$$

  Since we already studied the case of weight $1e_1$, we are done.

- **Weight** $-2e_1$. Because of (4.7) we only have to study $uZ_{-2} \,.\, v_0$ and $uZ_1 \,.\, v_{-3}$; because of (4.9) it even suffices to show that $uZ_{-2} \,.\, v_0$ lies in $\mathfrak{a} + \mathbb{C} \cdot v_2$. Without loss of generality let therefor u be of the form $Z_1^a Z_{-2}^c$ with $a + c = r - 1 \geq 2$ and $a - 2c - 2 + 0 = 2$. Then we have $a = 4 + 2c \geq 4$, so that we have with the case of weight $1e_1$

$$uZ_{-2} \,.\, v_0 = Z_1^a Z_{-2}^{c+1} \,.\, v_0 \equiv (Z_1^{a-1} Z_{-2}^{c+1}) Z_1 \,.\, v_0 \equiv 0 \bmod (\mathfrak{a} + \mathbb{C} \cdot v_2).$$

- **Weight** $-3e_1$. Again by (4.7), we only have to consider $uZ_{-2} \,.\, v_{-1}$. Without loss of generality let $u = Z_1^a Z_{-2}^c$ with $a + c = r - 1 \geq 2$ and $a - 2c - 2 - 1 = 2$. Then it holds $a = 5 + 2c \geq 5$. It follows with the case of weight $0e_1$

$$uZ_{-2} \,.\, v_{-1} \equiv (Z_1^{a-1} Z_{-2}^{c+1}) Z_1 \,.\, v_{-1} \equiv 0 \bmod (\mathfrak{a} + \mathbb{C} \cdot v_2).$$

- **Weight** $-4e_1$. By (4.7), the only interesting terms are of the form $uZ_{-2} \,.\, v_{-2}$. Without loss of generality let $u = Z_1^a Z_{-2}^c$ with $a + c = r - 1 \geq 2$ and $a - 2c - 2 - 2 = 2$, so that $a = 6 + 2c \geq 6$. It follows with the case of weight $-1e_1$

$$uZ_{-2} \,.\, v_{-2} = (Z_1^{a-1} Z_{-2}^{c+1}) Z_1 \,.\, v_{-2} \equiv 0 \bmod (\mathfrak{a} + \mathbb{C} \cdot v_2).$$

- **Weight** $-5e_1$. We only have to study $uZ_{-2} \,.\, v_{-3}$. For $u = Z_1^a Z_{-2}^c$ with $a + c = r - 1 \geq 2$ and $a - 2c - 2 - 3 = 2$ we have $a = 7 + 2c \geq 7$, so that it follows with the case of weight $-2e_1$

$$uZ_{-2} \,.\, v_{-3} = (Z_1^{a-1} Z_{-2}^{c+1}) Z_1 \,.\, v_{-3} \equiv 0 \bmod (\mathfrak{a} + \mathbb{C} \cdot v_2).$$

All in all, this proves Proposition 4.4. □

4.6 Reduction to cohomology

We still want to show that for $n = 3$ the value $P_{\lambda,\infty}(\frac{1}{2})$ in Theorem 1 does not vanish. Up to now we showed that B_∞ does not vanish on the cohomological K-types. Recalling the remark in Section 4.3 we want to prove, that $B_\wedge \otimes B_\infty$ restricted to cohomology is still nontrivial. Like in Section 3.6 we may write

$$\left(\bigwedge^2 \tilde{\wp}_3^* \otimes V_{\mu_3} \right)_\varepsilon^{\mathrm{SO}_3(\mathbb{R})} \qquad \text{resp.} \qquad \left(\tilde{\wp}_2^* \otimes (W_{\mu_{-2}} \oplus W_{\mu_2}) \right)_{\varepsilon'}^{\mathrm{SO}_2(\mathbb{R})}$$

for the respective cohomology spaces. This invites us to do the proof in two steps. At first we show

Proposition 4.5 $(B_\wedge \otimes B_\infty)\left(\left(\bigwedge^2 \tilde{\wp}_3^* \otimes V_{\mu_3}\right)^{\mathfrak{so}_3}, \left(\tilde{\wp}_2^* \otimes (W_{\mu_{-2}} \oplus W_{\mu_2})\right)^{\mathfrak{so}_2} \right) \not\equiv 0.$

Proof. A basis of the $\mathfrak{so}_3$-module $\bigwedge^2 \tilde{\wp}_3$ is given by

$$\begin{array}{ll}
5Z_1 \wedge Z_2, & \\
5Z_0 \wedge Z_2, & \\
3Z_{-1} \wedge Z_2 + 2Z_0 \wedge Z_1, & Z_0 \wedge Z_1 - Z_{-1} \wedge Z_2, \\
2Z_{-1} \wedge Z_1 + Z_{-2} \wedge Z_2, & Z_{-1} \wedge Z_1 - 2Z_{-2} \wedge Z_2, \\
Z_{-1} \wedge Z_0 + Z_{-2} \wedge Z_1, & 3Z_{-1} \wedge Z_0 - 2Z_{-2} \wedge Z_1, \\
Z_{-2} \wedge Z_0, & \\
Z_{-2} \wedge Z_{-1}, &
\end{array} \tag{4.10}$$

whence $\bigwedge^2 \tilde{\wp}_3$ is isomorphic to $\mathscr{D}_1 \oplus \mathscr{D}_3$ as an $\mathfrak{so}_3$-module. The same is true for its dual $\bigwedge^2 \tilde{\wp}_3^*$, since $\tilde{\wp}_3 \cong \mathscr{D}_2$ is self-contragredient as an $\mathfrak{so}_3$-module. To show the latter, let H, E_1, E_{-1} act on $\tilde{\wp}_3^*$ with the basis $\{(-1)^a Z_a^* \mid Z_a^*(Z_b) = \delta_{ab}\ \forall a, b \in \{-2,-1,0,1,2\}\}$ and compare the results with the values in Section 4.1. Note, that $(-1)^a Z_a^*$ has weight $-ae_1$. The cohomological $SO_3(\mathbb{R})$-type μ_3 of V is isomorphic to $\mathscr{D}_3$. So by the Clebsch-Gordon formula (cf. [Bou], VIII, §9) we get

$$\begin{aligned}\bigwedge^2 \tilde{\wp}_3^* \otimes V_{\mu_3} &\cong (\mathscr{D}_1 \oplus \mathscr{D}_3) \otimes \mathscr{D}_3 \\ &\cong (\mathscr{D}_2 \oplus \mathscr{D}_3 \oplus \mathscr{D}_4) \oplus (\mathscr{D}_0 \oplus \mathscr{D}_1 \oplus \mathscr{D}_2 \oplus \mathscr{D}_3 \oplus \mathscr{D}_4 \oplus \mathscr{D}_5 \oplus \mathscr{D}_6).\end{aligned}$$

The $\mathfrak{so}_3$-invariant vectors are just the $\mathscr{D}_0$-part by definition, so that it follows

$$\left(\bigwedge^2 \tilde{\wp}_3^* \otimes V_{\mu_3}\right)^{\mathfrak{so}_3} = \left(\bigwedge^2 \tilde{\wp}_3^* \otimes V_{\mu_3}\right)_{\mathscr{D}_0} \cong \mathscr{D}_0.$$

If we choose basis vectors $v'_{-3}, v'_{-2}, v'_{-1}, v'_0, v'_1, v'_2, v'_3$ of $\left(\bigwedge^2 \tilde{\wp}_3^*\right)_{\mathscr{D}_3} \cong \mathscr{D}_3$ such that the weight of each v'_a is ae_1, a generator of $\left(\bigwedge^2 \tilde{\wp}_3^* \otimes V_{\mu_3}\right)_{\mathscr{D}_0}$ is given by $\sum_{a=-3}^{3} v'_{-a} \otimes v_a$.

Now consider the two cohomological $SO_2(\mathbb{R})$-types $W_{\mu_{-2}} \cong \mathscr{D}'_{-2}$ and $W_{\mu_2} \cong \mathscr{D}'_2$. The $\mathfrak{so}_2$-types of $\tilde{\wp}_2$ are $\langle Z_2\rangle_{\mathbb{C}} \cong \mathscr{D}'_2$ and $\langle Z_{-2}\rangle_{\mathbb{C}} \cong \mathscr{D}'_{-2}$, so that we have $\tilde{\wp}_2 \cong \mathscr{D}'_{-2} \oplus \mathscr{D}'_2$. Since $\tilde{\wp}_2$ is self-contragredient with $H.Z_{-2}^* = 2iZ_{-2}^*$ and $H.Z_2^* = -2iZ_2^*$, the same is true for $\tilde{\wp}_2^*$. We get

$$\tilde{\wp}_2^* \otimes W_{\mu_{-2}} \cong (\mathscr{D}'_{-2} \oplus \mathscr{D}'_2) \otimes \mathscr{D}'_{-2} \cong \mathscr{D}'_{-4} \oplus \mathscr{D}'_0$$

and

$$\tilde{\wp}_2^* \otimes W_{\mu_2} \cong (\mathscr{D}'_{-2} \oplus \mathscr{D}'_2) \otimes \mathscr{D}'_2 \cong \mathscr{D}'_0 \oplus \mathscr{D}'_4,$$

so that in both cases it follows

$$\left(\tilde{\wp}_2^* \otimes W_{\mu_{\pm 2}}\right)^{\mathfrak{so}_2} \cong \mathscr{D}'_0.$$

Now let w_{-2} and w_2 denote basis vectors of $W_{\mu_{-2}}$ resp. W_{μ_2}, and choose a basis $\{w'_{-2}, w'_2\}$ of $\tilde{\wp}_2^*$ such that w'_{-2} has weight $-2e_1$ and w'_2 has weight $2e_1$. Then a basis of $\left(\tilde{\wp}_2^* \otimes W_{\mu_{-2}}\right)^{\mathfrak{so}_2}$ resp. $(\tilde{\wp}_2^* \otimes W_{\mu_2})^{\mathfrak{so}_2}$ is given by $w'_2 \otimes w_{-2}$ resp. $w'_{-2} \otimes w_2$.

We still have to show that the restriction of $B_\wedge \otimes B_\infty$ is not trivial. From Section 4.5 we know that $B_\infty(v_a, w_{-2})$ vanishes for $a \neq 2$. But then by Theorem 4.3 we have $0 \not\equiv B_\infty(V_{\mu_3}, W_{\mu_2}) = B_\infty(V_{\mu_3}, w_{-2})$, so that $B_\infty(v_2, w_{-2}) \neq 0$. Analogously, $B_\infty(v_{-2}, w_2) \neq 0$ and $B_\infty(v_a, w_2) = 0$ for $a \neq -2$. We showed for all $\alpha, \beta, \gamma \in \mathbb{C}$

$$\begin{aligned}&(B_\wedge \otimes B_\infty)(\alpha \cdot \sum_{a=-3}^{3} v'_{-a} \otimes v_a, \beta \cdot w'_{-2} \otimes w_2 + \gamma \cdot w'_2 \otimes w_{-2}) \\ = \;&\alpha\gamma \cdot B_\wedge(v'_{-2}, w'_2) B_\infty(v_2, w_{-2}) + \alpha\beta \cdot B_\wedge(v'_2, w'_{-2}) B_\infty(v_{-2}, w_2),\end{aligned} \tag{4.11}$$

where the values of B_∞ do not vanish. So we reduced the proof to showing

$$B_\wedge(v'_2, w'_{-2}) \neq 0 \quad \text{and} \quad B_\wedge(v'_{-2}, w'_2) \neq 0.$$

In order to do this we choose bases of $\bigwedge^2 \tilde{\wp}_3^*$ and $\tilde{\wp}_2^*$ consisting of Maurer-Cartan forms like in Section 3.1. We set

$$\omega_1 := Z_{-2}^*,\ \omega_2 := Z_2^*,\ \omega_3 := Z_0^*,\ \omega_4 := Z_{-1}^*,\ \omega_5 := Z_1^*.$$

Recalling the embedding of $\wp_2$ into $\tilde{\wp}_3$ from Section 4.1 we also put

$$\omega_1' := Z_{-2}^*,\ \omega_2' := Z_2^*,\ \omega_3' := Z_0^*,$$

where we define Z_b^* by $Z_b^*(Z_a) = \delta_{ab}$ with $a, b \in \{-2, 0, 2\}$ in analogy to the above. It follows

$$\delta(p_2 \circ j)(\omega_i) = \begin{cases} \omega_i' & \text{if } i = 1, 2, 3, \\ 0 & \text{if } i = 4, 5 \end{cases}$$

just like in Section 3.1.

Via (4.10) we can express v_{-2}', v_2' in terms of those Maurer-Cartan forms, and get

$$v_{-2}' = 5\ \omega_3 \wedge \omega_2 \text{ and } v_2' = \omega_1 \wedge \omega_3.$$

Further we may set

$$w_{-2}' := \omega_2' \text{ and } w_2' := \omega_1'.$$

So, following the definition of $\varepsilon_{I,I'}$ in Chapter 3 and taking into account that ω_3' corresponds to the differential $\frac{d}{dt}$ there, we find that $B_\wedge(v_2', w_{-2}')$ and $B_\wedge(v_{-2}', w_2')$ do not vanish. □

It remains to study the action of the groups of connected components of the respective orthogonal groups. The case of $\pi_0(\mathrm{O}_3)$ is already described by Corollary 1.5: For $\varepsilon = \operatorname{sgn}(\omega_\pi(-\mathbf{1}_n)(-1)^{\mathsf{wt}(\mu)/2}) = +$ we get

$$\left(\bigwedge^2 \tilde{\wp}_3^* \otimes V_{\mu_3}\right)_\varepsilon^{\mathrm{SO}_3(\mathbb{R})} = \left(\bigwedge^2 \tilde{\wp}_3^* \otimes V_{\mu_3}\right)^{\mathfrak{so}_3}.$$

The case of $\pi_0(\mathrm{O}_2)$ is more interesting. If we set $\delta_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ we may write

$$\mathrm{O}_2(\mathbb{R}) = \langle \delta_2 \rangle \ltimes \mathrm{SO}_2(\mathbb{R}) \quad \text{resp.} \quad \mathrm{O}_2(\mathbb{C}) = \langle \delta_2 \rangle \ltimes \mathrm{SO}_2(\mathbb{C}).$$

By [Mah], p. 624, we know how δ_2 acts on the weights of an arbitrary representation of $\mathfrak{so}_2$: Let τ be such a weight. Then we have

$$\tau^{\delta_2}(H) = \tau(\delta_2^{-1} H \delta_2) = \tau(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) = \tau(-H) = -\tau(H).$$

Thus δ_2 interchanges the two weights $-2e_1$ and $2e_1$ both in $\tilde{\wp}_2^*$ and W, whence by [loc.cit.] it also interchanges the two $\mathfrak{so}_2$-modules in $\left(\tilde{\wp}_2^* \otimes (W_{\mu_{-2}} \oplus W_{\mu_2})\right)^{\mathfrak{so}_2}$ that are isomorphic to $\mathscr{D}'_0$. Without loss of generality we may assume that the basis vectors $w'_2 \otimes w_{-2}$ and $w'_{-2} \otimes w_2$ merge under the action of δ_2. Then we get

$$\left(\tilde{\wp}_2^* \otimes (W_{\mu_{-2}} \oplus W_{\mu_2})\right)_+^{\mathrm{SO}_2(\mathbb{R})} \cong \langle w'_2 \otimes w_{-2} + w'_{-2} \otimes w_2 \rangle_{\mathbb{C}}.$$

and

$$\left(\tilde{\wp}_2^* \otimes (W_{\mu_{-2}} \oplus W_{\mu_2})\right)_-^{\mathrm{SO}_2(\mathbb{R})} \cong \langle w'_2 \otimes w_{-2} - w'_{-2} \otimes w_2 \rangle_{\mathbb{C}}.$$

But since the $(B_\wedge \otimes B_\infty)$-value in (4.11) can not be zero in the cases $\alpha = \beta = \gamma = 1$ and $\alpha = \beta = -\gamma = 1$ simultaneously by the above-mentioned, it follows that there are $\varepsilon, \varepsilon' \in \{+, -\}$, such that

$$(B_\wedge \otimes B_\infty)\Big(\Big(\bigwedge^2 \tilde{\wp}_3^* \otimes V_{\mu_3}\Big)_\varepsilon^{\mathrm{SO}_3(\mathbb{R})}, \left(\tilde{\wp}_2^* \otimes (W_{\mu_{-2}} \oplus W_{\mu_2})\right)_{\varepsilon'}^{\mathrm{SO}_2(\mathbb{R})}\Big) \not\equiv 0.$$

Recalling the remark in Section 4.3 we find

Theorem 2 *Let* $n = 3$ *and assume the coefficient modules* $\check{M}_{\mu,\mathbb{C}}$ *and* $\check{M}_{\nu,\mathbb{C}}$ *to be trivial. Then there is a choice of signs* ε *and* ε' *in Section 3.6, such that the period* $P_{\lambda,\infty}(\frac{1}{2})$ *in Theorem* 1 *does not vanish.*

Chapter 5

Conclusion and further questions

In Chapters 3 and 4 we showed under certain conditions, that the function $\chi \mapsto L(\pi \otimes \chi, \sigma; \frac{1}{2})$, after division by an appropriate period depending only on n takes algebraic numbers as values, like we hoped for. The most restricting conditions are:

- π has to be in $\mathrm{Coh}(\mathrm{GL}_3, \mu)$ for some dominant weight $\mu \in X_0^+(T_3)$,
- σ has to be in $\mathrm{Coh}(\mathrm{GL}_2, \nu)$ for some dominant weight $\nu \in X_0^+(T_2)$,
- the coefficient modules $\check{M}_{\mu,\mathbb{C}}$ and $\check{M}_{\nu,\mathbb{C}}$ are trivial.

How would one step further? While improvement at the first two points seems out of reach with the brute force methods of Chapter 4, the last point seems less persistent: We needed this assumption only in Section 4.5 and in Section 4.3[13], so that it seems possible to replace it by conditions like

- viewed as a GL_2-module, $\check{M}_{\mu,\mathbb{C}}$ has to contain the contragredient of $\check{M}_{\nu,\mathbb{C}}$,
- $s = \frac{1}{2}$ has to be a critical value of (π, σ).

As an obvious consequence the function $\chi \mapsto L(\pi \otimes \chi, \sigma; \frac{1}{2})$ from above would also depend on the coefficient systems $\check{M}_\mu$ and $\check{M}_\nu$, but still not on the character χ. Summing up, a first question would be

Question 1 *Is it possible to generalise the results of Chapter 4 to the bigger set of representations like we described it above?*

Assuming that it is, we should ask next, how restrictive those two new conditions are. While the first one seems to be natural to this kind of problems, the second one needs some discussion. In the following we will see, that in general $s = \frac{1}{2}$ is no critical value, although there are many examples, where it is. Finally we will discuss, how to interpret our results in the case of arbitrary sets of critical values.

[13]when we assumed that $s = \frac{1}{2}$ really was a critical value

5.1 Critical values

In this section we want to study the critical values of pairs (π, σ) of cohomological representations. Therefore we need to study $L(\pi_\infty \otimes \chi_\infty, \sigma_\infty; s)$.[14] Since we chose χ_∞ to be trivial, this is the same as $L(\pi_\infty, \sigma_\infty; s)$. The latter value can be calculated via Langlands correspondence at infinity as follows. Recalling the Langlands classification we may associate to π_∞ and σ_∞ representations π_∞^{W} and $\sigma_\infty^{\mathrm{W}}$ of $W_{\mathbb{R}}$ like in Section 1.5. So if we write

$$\pi_\infty \cong \mathrm{sgn}^k \otimes J(\mathsf{w}, \mathbf{l}), \quad k \in \mathbb{Z}/2\mathbb{Z}$$

and

$$\sigma_\infty \cong \mathrm{sgn}^{k'} \otimes J(\mathsf{w}', \mathbf{l}'), \quad k' \in \mathbb{Z}/2\mathbb{Z},$$

we can give a concrete formula for $L(\pi_\infty^{\mathrm{W}} \otimes \sigma_\infty^{\mathrm{W}}; s)$ by (3.6) in [Kna1] and Proposition 1.6. Taking into account that that the L-functions in [loc.cit.] are defined slightly different from those used by us, causing a shift by $\frac{1}{2}$ in the formula, we find the following expression for $L(\pi_\infty^{\mathrm{W}} \otimes \sigma_\infty^{\mathrm{W}}; s+\frac{1}{2})$:

$$\left[\prod_{i=1}^{\frac{n}{2}} 2(2\pi)^{-(s+\frac{\mathsf{w}+\mathsf{w}'+l_i}{2})} \cdot \Gamma\left(s+\frac{\mathsf{w}+\mathsf{w}'+l_i}{2}\right)\right] \cdot \left[\prod_{i=1}^{\frac{n}{2}} \prod_{\substack{j=1 \\ l'_j \neq l_i}}^{\frac{n}{2}-1} 2(2\pi)^{-(s+\frac{\mathsf{w}+\mathsf{w}'+l_i+l'_j}{2})}\right.$$
$$\cdot \Gamma\left(s+\frac{\mathsf{w}+\mathsf{w}'+l_i+l'_j}{2}\right) \cdot 2(2\pi)^{-(s+\frac{\mathsf{w}+\mathsf{w}'+|l_i-l'_j|}{2})} \cdot \Gamma\left(s+\frac{\mathsf{w}+\mathsf{w}'+|l_i-l'_j|}{2}\right)\Bigg]$$
$$\cdot\left[\prod_{i=1}^{\frac{n}{2}} \prod_{\substack{j=1 \\ l'_j = l_i}}^{\frac{n}{2}-1} 2(2\pi)^{-(s+\frac{\mathsf{w}+\mathsf{w}'}{2}+l_i)} \Gamma\left(s+\frac{\mathsf{w}+\mathsf{w}'}{2}+l_i\right) \cdot \pi^{(s+\frac{\mathsf{w}+\mathsf{w}'}{2})/2} \cdot \Gamma\left(\frac{s}{2}+\frac{\mathsf{w}+\mathsf{w}'}{4}\right)\right.$$
$$\left.\cdot\, \pi^{(s+\frac{\mathsf{w}+\mathsf{w}'}{2}+1)/2} \cdot \Gamma\left(\frac{s}{2}+\frac{\mathsf{w}+\mathsf{w}'}{4}+\frac{1}{2}\right)\right],$$

if n is even and

$$\left[\prod_{j=1}^{\frac{n-1}{2}} 2(2\pi)^{-(s+\frac{\mathsf{w}+\mathsf{w}'+l'_j}{2})} \cdot \Gamma\left(s+\frac{\mathsf{w}+\mathsf{w}'+l'_j}{2}\right)\right] \cdot \left[\prod_{i=1}^{\frac{n-1}{2}} \prod_{\substack{j=1 \\ l'_j \neq l_i}}^{\frac{n-1}{2}} 2(2\pi)^{-(s+\frac{\mathsf{w}+\mathsf{w}'+l_i+l'_j}{2})}\right.$$
$$\cdot \Gamma\left(s+\frac{\mathsf{w}+\mathsf{w}'+l_i+l'_j}{2}\right) \cdot 2(2\pi)^{-(s+\frac{\mathsf{w}+\mathsf{w}'+|l_i-l'_j|}{2})} \cdot \Gamma\left(s+\frac{\mathsf{w}+\mathsf{w}'+|l_i-l'_j|}{2}\right)\Bigg]$$
$$\cdot\left[\prod_{i=1}^{\frac{n-1}{2}} \prod_{\substack{j=1 \\ l'_j = l_i}}^{\frac{n-1}{2}} 2(2\pi)^{-(s+\frac{\mathsf{w}+\mathsf{w}'}{2}+l_i)} \Gamma\left(s+\frac{\mathsf{w}+\mathsf{w}'}{2}+l_i\right) \cdot \pi^{(s+\frac{\mathsf{w}+\mathsf{w}'}{2})/2} \cdot \Gamma\left(\frac{s}{2}+\frac{\mathsf{w}+\mathsf{w}'}{4}\right)\right.$$
$$\left.\cdot\, \pi^{(s+\frac{\mathsf{w}+\mathsf{w}'}{2}+1)/2} \cdot \Gamma\left(\frac{s}{2}+\frac{\mathsf{w}+\mathsf{w}'}{4}+\frac{1}{2}\right)\right],$$

if n is odd. By definition we have (cf. [JPS2])

$$L(\pi_\infty, \sigma_\infty; s) = L(\pi_\infty^{\mathrm{W}} \otimes \sigma_\infty^{\mathrm{W}}; s).$$

[14] We do this for general n, since the arguments are the same as in the case $n = 3$.

Now by [Del] the critical values of (π,σ) are the numbers $n+\frac{1}{2}$ for those integers n where neither $L(\pi_\infty^{\mathrm{W}}\otimes\sigma_\infty^{\mathrm{W}};s)$ nor $L((\pi_\infty^{\mathrm{W}}\otimes\sigma_\infty^{\mathrm{W}})\check{};1-s)$ have a pole. Note, that we can calculate $L((\pi_\infty^{\mathrm{W}}\otimes\sigma_\infty^{\mathrm{W}})\check{};1-s)$ easily by (1.2). One the one hand, there are quite a lot of pairs (π,σ) for which $s=\frac{1}{2}$ is indeed a critical value. For example consider pairs (π,σ) with $\mathsf{w}=-\mathsf{w}'$. In this case we get from (1.6) that $l_i\neq l'_j$ for all i and j, and the formulae above simplify a lot. Obviously, there are plenty of choices for the l_i and the l'_j giving us what we want, in particular we may set $\mathsf{w}=\mathsf{w}'=0$ and $\mathbf{l}=2\varrho_n$ and $\mathbf{l}'=2\varrho_{n-1}$, which is the case from [KMS] and from Chapter 4. On the other hand, in general $s=\frac{1}{2}$ does not need to be a critical value. Actually, quite a lot of sets of critical values seem possible. So the natural question in this situation is

Question 2 *What are in general the critical values of a pair (π,σ) of cohomological representations like in Chapters 3 and 4?*

Here, the condition, that viewed as a GL_2-module $\check{M}_{\mu,\mathbb{C}}$ has to contain the contragredient of $\check{M}_{\nu,\mathbb{C}}$, should give a relation between w and w'.

5.2 Pairs with arbitrary critical values

Obviously, our results from Chapter 3 are mainly interesting, if $s=\frac{1}{2}$ is indeed a critical value of the pair (π,σ). In Chapter 4 we even need this as an assumption. But the question remains

Question 3 *How can this work be generalised such that we can treat arbitrary*[15] *sets of critical values?*

Note, that the value $s=\frac{1}{2}$ plays a special role in this kind of problem, because of the factor $|\det(g)|^{s-\frac{1}{2}}$ in the definition of the zeta integral. In our special case we could ignore this factor, which for instance helped to prove the $\mathfrak{gl}_{n-1}$-invariance in Section 4.4. This is why simply copying the methods of this thesis to the case of another critical value should not work. However, if $s=\frac{1}{2}$ is indeed a critical value, we should be able to use the algebraicity (modulo period) of $L(\pi\otimes\chi,\sigma;\frac{1}{2})$ to show algebraicity (modulo period) of the values $L(\pi\otimes\chi,\sigma;s_i)$ for all critical values s_i. This has been done in the cases for small n that have already been studied in more detail (cf. [Man] for instance).

[15] Which sets of critical values are of interest, clearly depends on Question 2.

Symbols

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