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# Numerical approximation of incremental infinitesimal gradient plasticity

Patrizio Neff, Antje Sydow, and Christian Wieners

Abstract. We investigate a representative model of continuum infinitesimal gradient plasticity. The formulation is an extension of classical rate-independent infinitesimal plasticity based on the additive decomposition of the symmetric strain-tensor into elastic and plastic parts. It is assumed that dislocation processes contribute to the storage of energy in the material whereby the curl of the plastic distortion appears in the thermodynamic potential and leads to an additional nonlocal backstress tensor. The formulation is cast into a numerical framework by a saddle point approximation of the corresponding minimization problem in each incremental loading step. This allows to reformulate the (nonlocal) dissipation inequality into a point-wise flow rule and yields to a solution scheme which is a direct extension of the standard approach in classical plasticity. Our numerical results show the regularizing effects of the additional physically motivated terms.

## 1. Introduction

This article addresses the numerical analysis of a *geometrically linear isotropic gradient plasticity* model. There is an abundant literature on gradient plasticity formulations, in most cases letting the yield-stress $K_0$ depend also on some higher derivative of a scalar measure of accumulated plastic distortion [**27, 7, 10**]. Experimentally, the dependence of the yield stress on plastic gradients is well-documented (see, e. g., [**9**]), and several experimental facts testify to the length scale dependence of materials. E. g., the *Hall-Petch scaling*[1] relates the grain size in polycrystals to the yield stress in the form $K_0 = K_0^0 + \dfrac{k^+}{\sqrt{\text{grainsize}}}$.

Similarly, the *Taylor scaling* relates the yield stress $K_0$ to the dislocation density qualitatively by $K_0 = K_0^0 + k^+|\operatorname{curl}(\boldsymbol{F}_p)|$, where $\boldsymbol{F}_p$ is the incompatible intermediate configuration appearing in the multiplicative decomposition [**33**] and $\operatorname{curl}\boldsymbol{F}_p$ is a measure for the defect density.[2]

It is also known that the thinner the grains, the stiffer the material gets. As a rule one may therefore say: the smaller the sample is the stiffer it gets (while unbounded stiffness must be excluded from atomistic calculations).

From a numerical point of view the incorporation of plastic gradients may also serve the purpose of removing the mesh-sensitivity, ubiquitous in the softening case, or, more difficult to observe

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[1]There is also a reverse Hall-Petch scaling for very small grains in the nano-range.

[2]Note that we use the transpose of Gurtin's definition of the curl-operator for second order tensors, i.e., curl acts row-wise such that $\operatorname{curl} D\varphi = 0$. Formulas given in [**11**] do not obtain for this definition.

numerically, already in classical Prandtl-Reuss plasticity (shear bands and slip lines with ill-defined band width, as e.g. in the case of a plate with a hole under uniform tension [**32**]).

The incorporation of a plastic length scale, which is a natural by-product of gradient plasticity, has the potential to remove the mesh sensitivity. The presence of the internal length scale should cause the localization zones to have finite width. It makes possible the analysis of failure problems in which strain localization into shear bands occurs. However, the actual plastic length scale of a material is difficult to establish experimentally and theoretically, and it remains basically an open question how to determine the additionally appearing material constants. Moreover, it is also not entirely clear, how the shear band width depends on the characteristic length.

Models, similar in spirit to our formulation, may be found in [**22, 18**]. While gradient plasticity seems to be of high current interest [**12, 14, 13, 37, 38**] we have not been able to locate many rigorous mathematical studies of the time continuous higher gradient plasticity problem, apart for Reddy et. al. [**35**], treating a geometrically linear model of Gurtin [**13**] which is substantially different from our proposal.[3] A rigorous mathematical study of a rate-independent problem with full gradient regularization, although coming from ferroelasticity is presented in [**26, 24**].

Gurtin includes $\mathrm{curl}(\boldsymbol{F}_p)$ in the free energy and takes free variations with respect to $\boldsymbol{F}_p$, leading to a model with additional balance equations for microforces, similar, e.g., to a Cosserat or Toupin-Mindlin type model. We refer to [**30, 32**] for a model including plasticity and Cosserat effects. In modification of Gurtin's approach, in [**28, 31**] the plastic distortion $\boldsymbol{F}_p$ and thence $\mathrm{curl}(\boldsymbol{F}_p)$ is treated as an inner variable included only in the thermodynamic potential, leading to a system of evolution equations for $\boldsymbol{F}_p$ of degenerate parabolic type.

In Mielke-Müller [**25**] the time-incremental finite-strain problem is investigated. It is shown that once the update potential for one time step is established and known to be properly coercive and polyconvex as a function of $\boldsymbol{F}_e$ in the multiplicative decomposition of the deformation gradient, then, adding a regularizing term depending only on $\mathrm{curl}(\boldsymbol{F}_p)$ is indeed enough to show existence of minimizers for the new deformation and the new plastic variable.[4]

Adding a $\mathrm{curl}(\boldsymbol{F}_p)$-related term to the time-incremental problem has also been suggested in [**34**] for the description of subgrain dislocation structures. It seems therefore necessary to investigate the general structure of these class of gradient plasticity models. Our interest here resides mainly with respect to the regularizing power of these models. Focussing therefore on localization limiters we investigate a well-posed plasticity model which includes the dislocation energy density and local linear Prager kinematic hardening. In the large scale (classical) limit of vanishing plastic lengthscale our model turns into Prandtl-Reuss plasticity with linear kinematic hardening.

As far as classical rate-independent perfect elasto-plasticity is concerned we remark that global existence for the displacement has been shown only in a very weak, measure-valued sense, while the stresses could be shown to remain in $L_2(\Omega, \mathrm{Sym}(d))$. For these results we refer for example to [**3, 6, 39**]. If hardening or viscosity is added, then global classical solution are found, see, e. g., [**1, 5, 4**]. A complete theory for the classical rate-independent case remains elusive, see also

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[3] In Reddys analytical treatment of the corresponding infinitesimal model strong assumptions on the presence of the full plastic strain rate gradient (instead of the curl) are introduced in the dependence of the yield stress on the plastic gradients (isotropic hardening) in order to show existence and uniqueness. His model features a priori purely symmetric plastic strains. Such a model is called "irrotational", it does not allow for plastic spin. Local kinematic hardening is not considered.

[4] It is noteworthy that the update problem in [**25**] is a two-field minimization problem for he deformation and plastic distortion $(\varphi, \boldsymbol{F}_p)$ in the spirit of a micromorphic model [**29**] with a very special coupling between the different fields $\varphi$ and $\boldsymbol{F}_p$. Polyconvexity in $\boldsymbol{F}_e$ alone is not sufficient to obtain existence by Ball's method since $\boldsymbol{F}_e$ is not a gradient, but the term $\mathrm{curl}(\boldsymbol{F}_p)$ provides additional compactness in the spirit of Murat/Tartars method.

the remarks in [**6**]. It is therefore hoped, that including plastic gradients in the formulation will regularize the problem and lead to a well-posed model. This is what is shown in [**31**].

In the small strain gradient plasticity model proposed in [**31**] the plastic distortion $\boldsymbol{p}$ is in general not symmetric. Imposing certain form-invariance conditions (invariance of the referential, intermediate and spatial configuration under superposed rotations) on the thermodynamic potential shows that the local kinematical backstress in the linearized regime can only depend on $\mathrm{sym}(\boldsymbol{p})$, while the nonlocal backstress depends on the (in general non-symmetric) expression $\mathrm{curl}(\mathrm{curl}(\boldsymbol{p}))$.

However, assuming that plastic spin (i.e. $\mathrm{skew}(F_p^{-1}\dot{F}_p)$) generates infinite dissipation (no spin assumption, see [**23, 15**]) leads, after linearization, to a symmetric plastic distortion rate ($\dot{\boldsymbol{p}} \in \mathrm{Sym}(3)$) thereby considering only the symmetric part of the relative (elastic) stress $\boldsymbol{\alpha} = \boldsymbol{\sigma} - \mathbb{D}\boldsymbol{p} - \mathrm{curl}(\mathrm{curl}(\boldsymbol{p}))$. Taking symmetric initial values for the plastic distortion $\boldsymbol{p}$ it is thus possible to consider a gradient plasticity model in which only symmetric plastic strains occur. This is what we do here; we may rename therefore $\boldsymbol{\varepsilon}_p = \mathrm{sym}\,\boldsymbol{p}$.

Our contribution is organized as follows. In the first Section, we introduce the model step by step, where we discuss in particular the relation of flow rule and dissipation inequality for the nonlocal model. Then, in Section 3, we consider the incremental model and we derive a mixed approximation in space. It is shown that this problem corresponds to a constraint minimization problem, and we derive the characterizing KKT-system. For this KKT-system an iterative numerical scheme is developed in Section 4 by linearizing the flow rule in every step, where each iteration step itself is a nonlinear problem (solved by a nonsmooth Newton method). Finally, in Section 5 we present a numerical study which shows that the proposed scheme is a robust and efficient solution method. Moreover, the results indicate clearly the regularization properties of the nonlocal method.

## 2. A representative model for infinitesimal nonlocal plasticity

We develop a model for nonlocal plasticity in three stages: beginning with infinitesimal perfect plasticity (which is not well-posed in the primal setting), the model is first regularized by including local kinematic hardening, and finally it is extended by a nonlocal term which depends on the curl of the plastic distortion.

**2.1. Data.** Let $\Omega \subset \mathbf{R}^3$ be the reference configuration, and let $\Gamma_D \cup \Gamma_N = \partial\Omega$ be a decomposition of the boundary. The outer unit normal vector on the boundary is denoted by $\boldsymbol{n}(\boldsymbol{x})$. Let $[0,T]$ be a fixed time interval. We prescribe a displacement vector

$$\boldsymbol{u}_D \colon \Gamma_D \times [0,T] \longrightarrow \mathbf{R}^3$$

for the essential boundary conditions on $\Gamma_D$ and a load functional

$$\ell(t, \delta\boldsymbol{u}) = \int_\Omega \boldsymbol{b}(t) \cdot \delta\boldsymbol{u}\, d\boldsymbol{x} + \int_{\Gamma_N} \boldsymbol{t}_N(t) \cdot \delta\boldsymbol{u}\, d\boldsymbol{a}$$

depending on body force densities and traction force densities

$$\boldsymbol{b} \colon \Omega \times [0,T] \longrightarrow \mathbf{R}^3, \qquad \boldsymbol{t}_N \colon \Gamma_N \times [0,T] \longrightarrow \mathbf{R}^3\ .$$

In all cases below we assume that the data are sufficiently smooth.

Let $\mathrm{Sym}(d) = \{\boldsymbol{\tau} \in \mathbf{R}^{d,d} \colon \boldsymbol{\tau}^T = \boldsymbol{\tau}\}$ be the set of symmetric matrices. The elastic material properties (in the infinitesimal model) are given through the isotropic elasticity tensor $\mathbb{C}\colon \mathrm{Sym}(d) \longrightarrow \mathrm{Sym}(d)$ defined by $\mathbb{C} : \boldsymbol{\varepsilon} = 2\mu\,\boldsymbol{\varepsilon} + \lambda\,\mathrm{trace}(\boldsymbol{\varepsilon})\mathbb{1}$, depending on the Lamé constants $\lambda, \mu > 0$. On $\mathrm{Sym}(d)$, the elasticity tensor is symmetric and positive definite. For a displacement vector $\boldsymbol{u}$, the linearized strain tensor is denoted by $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \mathrm{sym}(D\boldsymbol{u})$.

We restrict ourselves to von Mises plasticity $\phi(\boldsymbol{\sigma}) = |\operatorname{dev}(\boldsymbol{\sigma})| - K_0$ depending on the yield stress $K_0 > 0$, where $\operatorname{dev}(\boldsymbol{\sigma}) = \boldsymbol{\sigma} - \frac{1}{d}\operatorname{trace}(\boldsymbol{\sigma})\mathbb{1}$ is the deviatoric part of the stress. In the following we take $d = 3$.

**2.2. Perfect plasticity revisited.** We start with the classical Prandtl-Reuss model: find displacements

$$\boldsymbol{u}\colon \overline{\Omega} \times [0,T] \longrightarrow \mathbf{R}^3 ,$$

symmetric stresses

$$\boldsymbol{\sigma}\colon \Omega \times [0,T] \longrightarrow \operatorname{Sym}(d) ,$$

symmetric plastic strains

$$\boldsymbol{\varepsilon}_p\colon \Omega \times [0,T] \longrightarrow \operatorname{Sym}(d) ,$$

and a plastic multiplier

$$\gamma\colon \Omega \times [0,T] \longrightarrow \mathbf{R},$$

satisfying the essential boundary conditions

$$\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{u}_D(\boldsymbol{x},t), \qquad (\boldsymbol{x},t) \in \Gamma_D \times [0,T] ,$$

the linear elastic constitutive relation

$$\boldsymbol{\sigma}(\boldsymbol{x},t) = \mathbb{C} : \Big(\boldsymbol{\varepsilon}(\boldsymbol{u})(\boldsymbol{x},t) - \boldsymbol{\varepsilon}_p(\boldsymbol{x},t)\Big), \qquad (\boldsymbol{x},t) \in \Omega \times [0,T],$$

the equilibrium equations

$$-\operatorname{div}\boldsymbol{\sigma}(\boldsymbol{x},t) = \boldsymbol{b}(\boldsymbol{x},t), \qquad (\boldsymbol{x},t) \in \Omega \times [0,T] , \tag{1a}$$

$$\boldsymbol{\sigma}(\boldsymbol{x},t)\boldsymbol{n}(\boldsymbol{x}) = \boldsymbol{t}_N(\boldsymbol{x},t), \qquad (\boldsymbol{x},t) \in \Gamma_N \times [0,T], \tag{1b}$$

the local flow rule

$$\dot{\boldsymbol{\varepsilon}}_p(\boldsymbol{x},t) = \gamma(\boldsymbol{x},t)\,\frac{\operatorname{dev}(\boldsymbol{\sigma}(\boldsymbol{x},t))}{|\operatorname{dev}(\boldsymbol{\sigma}(\boldsymbol{x},t))|} , \qquad (\boldsymbol{x},t) \in \Omega \times [0,T] , \tag{2}$$

and the complementary conditions

$$\gamma(\boldsymbol{x},t)\big(|\operatorname{dev}(\boldsymbol{\sigma}(\boldsymbol{x},t))| - K_0\big) = 0, \quad \gamma(\boldsymbol{x},t) \ge 0, \quad |\operatorname{dev}(\boldsymbol{\sigma}(\boldsymbol{x},t))| \le K_0 . \tag{3}$$

The flow rule and complementary condition can be reformulated in form of a dissipation inequality, and the equilibrium equation can be stated in integrated form. To this end we define the spaces $\boldsymbol{S} = L_2(\Omega, \operatorname{Sym}(d))$, $\boldsymbol{E} = \{\boldsymbol{\eta} \in \boldsymbol{S}\colon \operatorname{dev}(\boldsymbol{\eta}) = 0\}$, $\boldsymbol{K} = \{\boldsymbol{\tau} \in \boldsymbol{S}\colon |\operatorname{dev}(\boldsymbol{\tau})| \le K_0 \text{ a.e. in } \Omega\}$, and $\boldsymbol{V} = H^1(\Omega, \mathbf{R}^3)$, $\boldsymbol{V}_0 = \{\boldsymbol{v} \in \boldsymbol{V}\colon \boldsymbol{v}|_{\Gamma_D} = 0\}$, $\boldsymbol{V}(\boldsymbol{u}_D) = \{\boldsymbol{v} \in \boldsymbol{V}\colon \boldsymbol{v}|_{\Gamma_D} = \boldsymbol{u}_D\}$.

Lemma 1. *If $\boldsymbol{\sigma}(t) \in \boldsymbol{K}$, we have (the integrated dissipation inequality)*

$$\int_\Omega K_0\,|\delta\boldsymbol{\varepsilon}_p|\,d\boldsymbol{x} \ge \int_\Omega K_0\,|\dot{\boldsymbol{\varepsilon}}_p(t)|d\,\boldsymbol{x} + \int_\Omega \boldsymbol{\sigma}(t) : \big(\delta\boldsymbol{\varepsilon}_p - \dot{\boldsymbol{\varepsilon}}_p(t)\big)\,d\boldsymbol{x}\,, \qquad \delta\boldsymbol{\varepsilon}_p \in \boldsymbol{E}\,. \tag{4}$$

Proof. The flow rule (2) gives $\gamma(t) = |\dot{\boldsymbol{\varepsilon}}_p(t)|$ and thus

$$\dot{\boldsymbol{\varepsilon}}_p(t) : \boldsymbol{\sigma}(t) = \gamma\,\frac{\operatorname{dev}(\boldsymbol{\sigma}(t)) : \boldsymbol{\sigma}(t)}{|\operatorname{dev}(\boldsymbol{\sigma}(t))|} = |\dot{\boldsymbol{\varepsilon}}_p(t)|\;|\operatorname{dev}(\boldsymbol{\sigma}(t))| = K_0\;|\dot{\boldsymbol{\varepsilon}}_p(t)| ,$$

i. e., $0 = K_0\;|\dot{\boldsymbol{\varepsilon}}_p(t)| - \boldsymbol{\sigma}(t) : \dot{\boldsymbol{\varepsilon}}_p(t)$. Together with

$$K_0\;|\delta\boldsymbol{\varepsilon}_p| \ge |\operatorname{dev}(\boldsymbol{\sigma}(t))|\;|\delta\boldsymbol{\varepsilon}_p| \ge \operatorname{dev}(\boldsymbol{\sigma}(t)) : \delta\boldsymbol{\varepsilon}_p = \boldsymbol{\sigma}(t) : \delta\boldsymbol{\varepsilon}_p$$

and integrating in $\Omega$ this gives the result. □

The equilibrium equation (1) give in integrated form

$$\int_\Omega \boldsymbol{\sigma}(t) : \boldsymbol{\varepsilon}(\delta\boldsymbol{u})\, d\boldsymbol{x} = \ell(t, \delta\boldsymbol{u}), \qquad \delta\boldsymbol{u} \in \boldsymbol{V}_0$$

and adding the dissipation inequality (4) we obtain

$$\int_\Omega \boldsymbol{\sigma}(t) : \boldsymbol{\varepsilon}(\delta\boldsymbol{u})\, d\boldsymbol{x} + \int_\Omega K_0\, |\delta\boldsymbol{\varepsilon}_p|\, d\boldsymbol{x} \\ \geq \ell(t, \delta\boldsymbol{u}) + \int_\Omega K_0 |\dot{\boldsymbol{\varepsilon}}_p(t)| d\,\boldsymbol{x} + \int_\Omega \boldsymbol{\sigma}(t) : \big(\delta\boldsymbol{\varepsilon}_p - \dot{\boldsymbol{\varepsilon}}_p(t)\big)\, d\boldsymbol{x}\,.$$

Inserting the constitutive relation $\boldsymbol{\sigma}(t) = \mathbb{C} : \big(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \boldsymbol{\varepsilon}_p(t)\big)$ and testing with $\delta\boldsymbol{u} - \dot{\boldsymbol{u}}(t)$ yields

$$\int_\Omega \big(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \boldsymbol{\varepsilon}_p(t)\big) : \mathbb{C} : \boldsymbol{\varepsilon}(\delta\boldsymbol{u} - \dot{\boldsymbol{u}}(t))\, d\boldsymbol{x} + \int_\Omega K_0\, |\delta\boldsymbol{\varepsilon}_p|\, d\boldsymbol{x} \\ \geq \ell(t, \delta\boldsymbol{u} - \dot{\boldsymbol{u}}(t)) + \int_\Omega K_0 |\dot{\boldsymbol{\varepsilon}}_p(t)| d\,\boldsymbol{x} + \int_\Omega \big(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \boldsymbol{\varepsilon}_p(t)\big) : \mathbb{C} : \big(\delta\boldsymbol{\varepsilon}_p - \dot{\boldsymbol{\varepsilon}}_p(t)\big)\, d\boldsymbol{x}\ .$$

Defining the symmetric bilinear form

$$a_{\mathrm{pp}}\big((\boldsymbol{u}, \boldsymbol{\varepsilon}_p), (\delta\boldsymbol{u}, \delta\boldsymbol{\varepsilon}_p)\big) = \int_\Omega \big(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}_p\big) : \mathbb{C} : \big(\boldsymbol{\varepsilon}(\delta\boldsymbol{u}) - \delta\boldsymbol{\varepsilon}_p\big)\, d\boldsymbol{x}$$

(corresponding to perfect plasticity) and the convex, one-homogeneous integrated dissipation functional

$$j(\dot{\boldsymbol{\varepsilon}}_p) = \int_\Omega K_0\, |\dot{\boldsymbol{\varepsilon}}_p|\, d\boldsymbol{x}\,,$$

allows to rewrite the equations of quasi-static infinitesimal plasticity as variational inequality:

$$a_{\mathrm{pp}}\big((\boldsymbol{u}(t), \boldsymbol{\varepsilon}_p(t)), (\delta\boldsymbol{u}, \delta\boldsymbol{\varepsilon}_p) - (\dot{\boldsymbol{u}}(t), \dot{\boldsymbol{\varepsilon}}_p(t))\big) + j(\delta\boldsymbol{\varepsilon}_p) - j(\dot{\boldsymbol{\varepsilon}}_p(t)) \geq \ell(t, \delta\boldsymbol{u} - \dot{\boldsymbol{u}}(t))\ .$$

**2.3. Local kinematic hardening.** Since $a_{\mathrm{pp}}(\cdot,\cdot)$ is not coercive on $\boldsymbol{V}_0 \times \boldsymbol{E}$, we next study a regularized problem defining

$$a_{\mathrm{kh}}\big((\boldsymbol{u}, \boldsymbol{\varepsilon}_p), (\delta\boldsymbol{u}, \delta\boldsymbol{\varepsilon}_p)\big) = \int_\Omega \big(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}_p\big) : \mathbb{C} : \big(\boldsymbol{\varepsilon}(\delta\boldsymbol{u}) - \delta\boldsymbol{\varepsilon}_p\big)\, d\boldsymbol{x} + \int_\Omega \boldsymbol{\varepsilon}_p : \mathbb{D} : \delta\boldsymbol{\varepsilon}_p\, d\boldsymbol{x}$$

with a local kinematic hardening modulus $\mathbb{D} = \mu H_0\, \mathrm{id}$ (where $H_0 > 0$ is non-dimensional scaling factor).

For $c = 2/H_0 \geq 0$ we have $\boldsymbol{\varepsilon}_p : \mathbb{C} : \boldsymbol{\varepsilon}_p = 2\mu\, \boldsymbol{\varepsilon}_p : \boldsymbol{\varepsilon}_p = c\ \boldsymbol{\varepsilon}_p : \mathbb{D} : \boldsymbol{\varepsilon}_p$. This gives

$$(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_p) : \mathbb{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_p) + \boldsymbol{\varepsilon}_p : \mathbb{D} : \boldsymbol{\varepsilon}_p \geq \frac{1}{1+2c} \boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon} + \frac{1}{2} \boldsymbol{\varepsilon}_p : \mathbb{D} : \boldsymbol{\varepsilon}_p$$

and therefore (by Korn's inequality)

$$a_{\mathrm{kh}}\big((\boldsymbol{u}, \boldsymbol{\varepsilon}_p), (\boldsymbol{u}, \boldsymbol{\varepsilon}_p)\big) \geq C_1\ \|(\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}_p)\|^2_{L_2(\Omega)} \geq C_2\ \|(\boldsymbol{u}, \boldsymbol{\varepsilon}_p)\|^2_{\boldsymbol{V}\times\boldsymbol{E}} \tag{5}$$

with $\|\boldsymbol{v}\|_{\boldsymbol{V}} = \|\boldsymbol{v}\|_{H^1(\Omega,\mathbf{R}^3)}$ and $\|\varepsilon\|_{\boldsymbol{E}} = \|\varepsilon\|_{L_2(\Omega,\mathbf{R}^{3\times 3})}$. Now, standard theory applies [**16**, Th. 7.3]:

THEOREM 2. *A unique solution* $(\boldsymbol{u}, \boldsymbol{\varepsilon}_p) \in H^1(0,T; \boldsymbol{V} \times \boldsymbol{E})$ *with* $\boldsymbol{u}(t) \in \boldsymbol{V}(\boldsymbol{u}_D)$ *and*

$$a_{\mathrm{kh}}\big((\boldsymbol{u}(t), \boldsymbol{\varepsilon}_p(t)), (\delta\boldsymbol{u}, \delta\boldsymbol{\varepsilon}_p) - (\dot{\boldsymbol{u}}(t), \dot{\boldsymbol{\varepsilon}}_p(t))\big) + j(\delta\boldsymbol{\varepsilon}_p) - j(\dot{\boldsymbol{\varepsilon}}_p(t)) \geq \ell(t, \delta\boldsymbol{u} - \dot{\boldsymbol{u}}(t)) \tag{6}$$

*exists for all* $(\delta\boldsymbol{u}, \delta\boldsymbol{\varepsilon}_p) \in \boldsymbol{V}_0 \times \boldsymbol{E}$ *and a. a.* $t \in [0,T]$

Testing with $\delta\boldsymbol{u} = \dot{\boldsymbol{u}}(t)$ in (6) and inserting the constitutive equation for the stress yields the dissipation inequality for plasticity with hardening in the form

$$\int_\Omega K_0\,|\delta\boldsymbol{\varepsilon}_p|\,d\boldsymbol{x} \geq \int_\Omega K_0\,|\dot{\boldsymbol{\varepsilon}}_p(t)|\,d\boldsymbol{x} + \int_\Omega \big(\boldsymbol{\sigma}(t) - \mathbb{D} : \boldsymbol{\varepsilon}_p(t)\big) : (\delta\boldsymbol{\varepsilon}_p - \dot{\boldsymbol{\varepsilon}}_p(t))\,d\boldsymbol{x}\;. \tag{7}$$

Within the hardening model, $\boldsymbol{\beta} = \mathbb{D} : \boldsymbol{\varepsilon}_p$ is the local symmetric back-stress, and $\boldsymbol{\alpha} = \boldsymbol{\sigma} - \boldsymbol{\beta} = \boldsymbol{\sigma} - \mathbb{D} : \boldsymbol{\varepsilon}_p$ is the automatically symmetric relative (elastic) stress (cf. [**36**, Chap. 3.3.1]).

LEMMA 3. *If (7) holds for all $\delta\boldsymbol{\varepsilon}_p \in \boldsymbol{E}$, we have $\boldsymbol{\alpha}(t) \in \mathbf{K}$ a. e. in $\Omega$. In addition, it holds $\dot{\boldsymbol{\varepsilon}}_p(t) = \mathbf{0}$ a. e. in the elastic zone $\Omega_{\mathrm{el}}(t) = \{\boldsymbol{x} \in \Omega\colon |\operatorname{dev}\boldsymbol{\alpha}(t)| < K_0$ a. e. in some neighborhood of $\boldsymbol{x}\}$, and we have $K_0\,,\boldsymbol{\varepsilon}_p(t) = |\boldsymbol{\varepsilon}_p(t)|\operatorname{dev}(\boldsymbol{\alpha}(t))$ a. e. in $\Omega$.*

PROOF. Testing (7) with $s\,\delta\boldsymbol{\varepsilon}_p$ for $s > 0$ gives

$$s\int_\Omega K_0\,|\delta\boldsymbol{\varepsilon}_p|\,d\boldsymbol{x} \geq \int_\Omega K_0\,|\dot{\boldsymbol{\varepsilon}}_p(t)|\,d\boldsymbol{x} + s\int_\Omega \boldsymbol{\alpha}(t) : \delta\boldsymbol{\varepsilon}_p\,d\boldsymbol{x} - \int_\Omega \boldsymbol{\alpha}(t) : \dot{\boldsymbol{\varepsilon}}_p(t)\,d\boldsymbol{x}$$

and thus

$$\int_\Omega K_0\,|\delta\boldsymbol{\varepsilon}_p|\,d\boldsymbol{x} \geq \frac{1}{s}\int_\Omega K_0\,|\dot{\boldsymbol{\varepsilon}}_p(t)|\,d\boldsymbol{x} + \int_\Omega \boldsymbol{\alpha}(t) : \delta\boldsymbol{\varepsilon}_p\,d\boldsymbol{x} - \frac{1}{s}\int_\Omega \boldsymbol{\alpha}(t) : \dot{\boldsymbol{\varepsilon}}_p(t)\,d\boldsymbol{x}\;.$$

Passing to the limit $s \longrightarrow \infty$ yields

$$\int_\Omega \operatorname{dev}(\boldsymbol{\alpha}(t)) : \delta\boldsymbol{\varepsilon}_p\,d\boldsymbol{x} = \int_\Omega \boldsymbol{\alpha}(t) : \delta\boldsymbol{\varepsilon}_p\,d\boldsymbol{x} \leq \int_\Omega K_0\,|\delta\boldsymbol{\varepsilon}_p|\,d\boldsymbol{x} \tag{8}$$

for all $\delta\boldsymbol{\varepsilon}_p \in \boldsymbol{E}$. For all open subsets $\tilde\Omega \subset \Omega$ we can insert a test function $\delta\boldsymbol{\varepsilon}_p$ which is defined by $\delta\boldsymbol{\varepsilon}_p(\boldsymbol{x}) = \operatorname{dev}(\boldsymbol{\alpha}(\boldsymbol{x},t))$ for $\boldsymbol{x} \in \tilde\Omega$ and $\delta\boldsymbol{\varepsilon}_p(\boldsymbol{x}) = \mathbf{0}$ else. This results in

$$\int_{\tilde\Omega} |\operatorname{dev}(\boldsymbol{\alpha}(t))|^2\,d\boldsymbol{x} \leq \int_{\tilde\Omega} K_0\,|\operatorname{dev}(\boldsymbol{\alpha}(t))|\,d\boldsymbol{x}\,.$$

Since $\tilde\Omega \subset \Omega$ is arbitrary, this gives $|\operatorname{dev}(\boldsymbol{\alpha}(t))|^2 \leq K_0\,|\operatorname{dev}(\boldsymbol{\alpha}(t))|$ a. e. in $\Omega$, i. e., $\operatorname{dev}(\boldsymbol{\alpha}(t)) \in \boldsymbol{K}$. Now, testing (7) with $\delta\boldsymbol{\varepsilon}_p = s\,\dot{\boldsymbol{\varepsilon}}_p(t)$ on $\tilde\Omega \subset \Omega$ gives for $s > 0$

$$(s-1)\int_{\tilde\Omega} K_0\,|\dot{\boldsymbol{\varepsilon}}(t)|\,d\boldsymbol{x} \geq (s-1)\int_{\tilde\Omega} \boldsymbol{\alpha}(t) : \dot{\boldsymbol{\varepsilon}}_p(t)\,d\boldsymbol{x}\,,$$

and thus (by inserting $s = 1 \pm \frac{1}{2}$) equality

$$\int_{\tilde\Omega} K_0\,|\dot{\boldsymbol{\varepsilon}}(t)|\,d\boldsymbol{x} = \int_{\tilde\Omega} \boldsymbol{\alpha}(t) : \dot{\boldsymbol{\varepsilon}}_p(t)\,d\boldsymbol{x} = \int_{\tilde\Omega} \operatorname{dev}(\boldsymbol{\alpha}(t)) : \dot{\boldsymbol{\varepsilon}}_p(t)\,d\boldsymbol{x}\,.$$

Then, $|\operatorname{dev}(\boldsymbol{\alpha}(t))| \leq K_0$ yields $K_0\,|\dot{\boldsymbol{\varepsilon}}(t)| = \boldsymbol{\alpha}(t) : \dot{\boldsymbol{\varepsilon}}_p(t)$ a. e. in $\Omega$. In case of $|\boldsymbol{\alpha}(\boldsymbol{x},t)| < K_0$ this implies $\dot{\boldsymbol{\varepsilon}}_p(\boldsymbol{x},t) = 0$, otherwise we have equality in the Cauchy-Schwarz inequality. Thus, $s\,\operatorname{dev}(\boldsymbol{\alpha}(\boldsymbol{x},t)) = \dot{\boldsymbol{\varepsilon}}_p(\boldsymbol{x},t)$ are linearly dependent with $s = |\dot{\boldsymbol{\varepsilon}}_p(\boldsymbol{x},t)|/K_0$. □

Reformulating this lemma shows that the solution $(\boldsymbol{u}, \boldsymbol{\varepsilon}_p)$ of (6) satisfies the pointwise flow rule

$$\dot{\boldsymbol{\varepsilon}}_p = \gamma\,\frac{\operatorname{dev}(\boldsymbol{\alpha})}{|\operatorname{dev}(\boldsymbol{\alpha})|}\,,\qquad \gamma = |\dot{\boldsymbol{\varepsilon}}_p|,\qquad \boldsymbol{\sigma} = \mathbb{C} : (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}_p),\qquad \boldsymbol{\alpha} = \boldsymbol{\sigma} - \mathbb{D} : \boldsymbol{\varepsilon}_p\,, \tag{9}$$

and the complementary conditions

$$\gamma \left(|\operatorname{dev}(\boldsymbol{\alpha})| - K_0\right) = 0, \qquad \gamma \geq 0, \qquad |\operatorname{dev}(\boldsymbol{\alpha})| \leq K_0 \tag{10}$$

a. e. in $\Omega \times [0, T]$ which can also be reformulated as a subdifferential inclusion [**31**, p. 17].

Remark 4. *The equivalence of (point-wise) flow rules and (integrated) dissipation inequalities is well studied in the general framework of convex analysis* [**16**, Chap. 4], [**2**]. *Extensions of this concept are considered within the energetic plasticity approach by Mielke* [**19**]. *In general, corresponding equivalence relations hold for local models only* [**31**].

**2.4. Nonlocal dislocation based plasticity with symmetric plastic strains.** Next, we extend $a_{\mathrm{kh}}(\cdot, \cdot)$ by a further term involving nonlocal dislocation density effects to read

$$a_{\mathrm{nl}}\big((\boldsymbol{u}, \boldsymbol{\varepsilon}_p), (\delta\boldsymbol{u}, \delta\boldsymbol{\varepsilon}_p)\big) \;=\; a_{\mathrm{kh}}\big((\boldsymbol{u}, \boldsymbol{\varepsilon}_p), (\delta\boldsymbol{u}, \delta\boldsymbol{\varepsilon}_p)\big) + \int_\Omega \operatorname{curl}(\boldsymbol{\varepsilon}_p) : \mathbb{E} : \operatorname{curl}(\delta\boldsymbol{\varepsilon}_p)\, d\boldsymbol{x}$$

with $\mathbb{E} = \mu L_c^2 \,\mathrm{id}$. Here, $L_c > 0$ is the internal plastic length scale with units of length. The curl-related term represents the energy stored in the material due to deformation incompatibility. In particular, diffuse phase boundaries are energetically favored by larger values of $L_c$ while sharp phase boundaries are favored by smaller values of $L_c$.

We define the space

$$\boldsymbol{E}_{\mathrm{nl}} := \{\boldsymbol{\varepsilon}_p \in L_2(\Omega, \mathbf{R}^{d,d}) \colon \operatorname{curl}(\boldsymbol{\varepsilon}_p) \in L_2(\Omega, \mathbf{R}^{d,d}),\ \operatorname{trace}(\boldsymbol{\varepsilon}_p) = 0,\ \operatorname{skew}(\boldsymbol{\varepsilon}_p) = 0\}$$

(where the curl-operator is applied row-wise on $\boldsymbol{\varepsilon}_p$). Equipped with the norm

$$\|\boldsymbol{\varepsilon}_p\|^2_{\boldsymbol{E}_{\mathrm{nl}}} := \|\boldsymbol{\varepsilon}_p\|^2_{L_2(\Omega,\mathbf{R}^{d,d})} + \|\operatorname{curl} \boldsymbol{\varepsilon}_p\|^2_{L_2(\Omega,\mathbf{R}^{d,d})} \tag{11}$$

this is a closed subspace of $H(\mathrm{curl}, \Omega)^3$. From (5) we conclude that $a_{\mathrm{nl}}(\cdot, \cdot)$ is elliptic in $\boldsymbol{V}_0 \times \boldsymbol{E}_{\mathrm{nl}}$, so that again [**16**, Th. 7.3] applies:

Theorem 5. *A unique solution* $(\boldsymbol{u}, \boldsymbol{\varepsilon}_p) \in H^1(0, T; \boldsymbol{V} \times \boldsymbol{E}_{\mathrm{nl}})$ *with* $\boldsymbol{u}(t) \in \boldsymbol{V}(\boldsymbol{u}_D)$ *and*

$$a_{\mathrm{nl}}\big((\boldsymbol{u}(t), \boldsymbol{\varepsilon}_p(t)), (\delta\boldsymbol{u}, \delta\boldsymbol{\varepsilon}_p) - (\dot{\boldsymbol{u}}(t), \dot{\boldsymbol{\varepsilon}}_p(t))\big) + j(\delta\boldsymbol{\varepsilon}_p) - j(\dot{\boldsymbol{\varepsilon}}_p(t)) \geq \ell(t, \delta\boldsymbol{u} - \dot{\boldsymbol{u}}(t)) \tag{12}$$

*for all* $(\delta\boldsymbol{u}, \delta\boldsymbol{\varepsilon}_p) \in \boldsymbol{V}_0 \times \boldsymbol{E}_{\mathrm{nl}}$ *and a. a.* $t \in [0, T]$ *exists.*

Testing with $\delta\boldsymbol{u} = \dot{\boldsymbol{u}}(t)$ in (12) and inserting the stress $\boldsymbol{\sigma} = \mathbb{C} : (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}_p)$ yields the corresponding integrated dissipation inequality for the nonlocal plasticity model in the form

$$\begin{aligned} \int_\Omega K_0 \,|\delta\boldsymbol{\varepsilon}_p|\, d\boldsymbol{x} \;\geq\; & \int_\Omega K_0 \,|\dot{\boldsymbol{\varepsilon}}_p(t)|\, d\boldsymbol{x} + \int_\Omega \big(\boldsymbol{\sigma}(t) - \mathbb{D} : \boldsymbol{\varepsilon}_p(t)\big) : (\delta\boldsymbol{\varepsilon}_p - \dot{\boldsymbol{\varepsilon}}_p(t))\, d\boldsymbol{x} \\ & - \int_\Omega \operatorname{curl}(\boldsymbol{\varepsilon}_p(t)) : \mathbb{E} : \operatorname{curl}(\delta\boldsymbol{\varepsilon}_p - \dot{\boldsymbol{\varepsilon}}_p(t))\, d\boldsymbol{x}\,. \end{aligned} \tag{13}$$

Without additional regularity, we cannot derive a point-wise flow rule in this case (in general, the stress is not bounded in $L_\infty$). Formally, if $\operatorname{curl}(\mathbb{E} : \operatorname{curl}(\boldsymbol{\varepsilon}_p(t)))$ exists and if one includes homogeneous boundary conditions in $\boldsymbol{E}_{\mathrm{nl}}$, integration by parts yields the dissipation inequality in strong form

$$\begin{aligned} \int_\Omega K_0 \,|\delta\boldsymbol{\varepsilon}_p|\, d\boldsymbol{x} \;\geq\; & \int_\Omega K_0 \,|\dot{\boldsymbol{\varepsilon}}_p(t)|\, d\boldsymbol{x} \\ & + \int_\Omega \big(\boldsymbol{\sigma}(t) - \mathbb{D} : \boldsymbol{\varepsilon}_p(t) - \operatorname{curl}(\mathbb{E} : \operatorname{curl}(\boldsymbol{\varepsilon}_p(t)))\big) : (\delta\boldsymbol{\varepsilon}_p - \dot{\boldsymbol{\varepsilon}}_p(t))\, d\boldsymbol{x}\,. \end{aligned}$$

This corresponds to flow rule (9) and complementarity condition (10), where the relative stress has the form $\boldsymbol{\sigma} - \mathbb{D} : \boldsymbol{\varepsilon}_p - \operatorname{curl}(\mathbb{E} : \operatorname{curl}(\boldsymbol{\varepsilon}_p))$. In fact—since $\boldsymbol{\varepsilon}_p$ is symmetric—only its symmetric part enters the model. Thus, for simplicity of notation, be denote by $\boldsymbol{\alpha}$ in the following the symmetrized relative stress. This means that with the preceding provisions, a smooth solution $(\boldsymbol{u}, \boldsymbol{\varepsilon}_p)$ of (12) satisfies the pointwise flow rule

$$\dot{\boldsymbol{\varepsilon}}_p = \gamma \, \frac{\operatorname{dev}(\boldsymbol{\alpha})}{|\operatorname{dev}(\boldsymbol{\alpha})|}\,, \qquad \gamma = |\dot{\boldsymbol{\varepsilon}}_p|, \qquad \boldsymbol{\alpha} = \boldsymbol{\sigma} - \mathbb{D} : \boldsymbol{\varepsilon}_p - \operatorname{sym}(\operatorname{curl}(\mathbb{E} : \operatorname{curl}(\boldsymbol{\varepsilon}_p)))\,, \tag{14}$$

and the complementary conditions

$$\gamma\big(|\operatorname{dev}(\operatorname{sym}\boldsymbol{\alpha})| - K_0\big) = 0, \qquad \gamma \ge 0, \qquad |\operatorname{dev}(\operatorname{sym}\boldsymbol{\alpha})| \le K_0 \tag{15}$$

a. e. in $\Omega \times [0, T]$. In Theorem 8 this is addressed in more detail, where a saddle point formulation is introduced in order to provide an elegant framework which avoid extra boundary conditions and which recovers (numerically) a point-wise flow rule.

Remark 6. *This construction of a nonlocal model is only one prototype for a large class of dislocation models which can be handled in the same framework. Observe that our model can be obtained as a special non-spin case of a model in* [**31**]. *Since our main focus is the clear presentation of a numerical solution method, we do not discuss other models; see* [**31, 35, 8**] *for other proposals in this respect.*

## 3. Incremental nonlocal plasticity

Replacing the time derivative by backward difference quotients and introducing suitable finite element spaces for displacements and plastic strains directly yields the discrete analog to the nonlocal problem (12). This incremental problem is fully implicit, and the solution of the incremental problem can be characterized by a minimization problem.

Unfortunately, this approach is not compatible with standard (local) plasticity, since it requires discrete plastic strains in $H(\operatorname{curl})$. Thus, in a second step we discuss a saddle point formulation (introducing an additional variable for the approximation of $\operatorname{curl}\boldsymbol{\varepsilon}_p$), which results in a constraint minimization problem.

**3.1. A primal approach.** Let $\boldsymbol{V}^h \subset \boldsymbol{V}$ be a finite element discretization of the displacements, and let $0 = t_0 < t_1 < \cdots < t_N = T$ be a decomposition of the time interval $[0, T]$. We set $\Delta\boldsymbol{u}_D^n = \boldsymbol{u}_D(t_n) - \boldsymbol{u}_D(t_{n-1})$, $\boldsymbol{V}^h(\boldsymbol{u}_D) = \{\boldsymbol{v}^h \colon \boldsymbol{v}_h(\boldsymbol{x}) = \boldsymbol{u}_D(\boldsymbol{x})$ for all nodal points on $\Gamma_D\}$ and $\boldsymbol{V}_0^h = \boldsymbol{V}^h(\mathbf{0})$. Let $\boldsymbol{E}_{\text{nl}}^h \subset \boldsymbol{E}_{\text{nl}}$ be a finite element discretization for the plastic strains.

For fixed $t_n$ and given $(\boldsymbol{u}^{h,n-1}, \boldsymbol{\varepsilon}_p^{h,n-1}) \in \boldsymbol{V}^h \times \boldsymbol{E}_{\text{nl}}^h$ we define

$$\ell_n(\delta\boldsymbol{u}^h) = \ell(t_n, \delta\boldsymbol{u}^h), \quad \ell_{\text{nl}}^{h,n}(\delta\boldsymbol{u}^h, \delta\boldsymbol{\varepsilon}_p^h) = \ell_n(\delta\boldsymbol{u}^h) - a_{\text{nl}}\big((\boldsymbol{u}^{h,n-1}, \boldsymbol{\varepsilon}_p^{h,n-1}), (\delta\boldsymbol{u}^h, \delta\boldsymbol{\varepsilon}_p^h)\big)\,.$$

Now we obtain by [**16**, Th. 6.6] for the backward Euler discretization of (12):

Theorem 7. *A unique minimizer* $(\Delta\boldsymbol{u}^{h,n}, \Delta\boldsymbol{\varepsilon}_p^{h,n}) \in \boldsymbol{V}^h(\Delta\boldsymbol{u}_D^n) \times \boldsymbol{E}_{\text{nl}}^h$ *of*

$$J_{\text{nl}}^{h,n}(\Delta\boldsymbol{u}^h, \Delta\boldsymbol{\varepsilon}_p^h) = \frac{1}{2} a_{\text{nl}}\big((\Delta\boldsymbol{u}^h, \Delta\boldsymbol{\varepsilon}_p^h), (\Delta\boldsymbol{u}^h, \Delta\boldsymbol{\varepsilon}_p^h)\big) + j(\Delta\boldsymbol{\varepsilon}_p^h) - \ell_{\text{nl}}^{h,n}(\Delta\boldsymbol{u}^h, \Delta\boldsymbol{\varepsilon}_p^h) \tag{16}$$

*exists. Moreover, the minimizer is characterized by the discrete variational inequality*

$$\begin{aligned} a_{\text{nl}}\big((\boldsymbol{u}^{h,n-1} + \Delta\boldsymbol{u}^{h,n}, \boldsymbol{\varepsilon}_p^{h,n-1} + \Delta\boldsymbol{\varepsilon}_p^{h,n}), (\delta\boldsymbol{u}^h, \delta\boldsymbol{\varepsilon}_p^n) - (\Delta\boldsymbol{u}^{h,n}, \Delta\boldsymbol{\varepsilon}_p^{h,n})\big) \\ + j(\delta\boldsymbol{\varepsilon}_p^h) - j(\Delta\boldsymbol{\varepsilon}_p^{h,n}) \ge \ell_n(\delta\boldsymbol{u}^h - \Delta\boldsymbol{u}^{h,n})\,, \qquad (\delta\boldsymbol{u}^h, \delta\boldsymbol{\varepsilon}_p^h) \in \boldsymbol{V}_0^h \times \boldsymbol{E}_{\text{nl}}^h\,. \end{aligned} \tag{17}$$

Thus, the incremental problem is well-defined, and we set

$$(\boldsymbol{u}^{h,n}, \boldsymbol{\varepsilon}_p^{h,n}) = (\boldsymbol{u}^{h,n-1}, \boldsymbol{\varepsilon}_p^{h,n-1}) + (\Delta\boldsymbol{u}^{h,n}, \Delta\boldsymbol{\varepsilon}_p^{h,n})\,. \tag{18}$$

**3.2. A mixed formulation.** There is no need to impose or discuss explicit boundary conditions on $\boldsymbol{\varepsilon}_p$, since coercivity is automatically satisfied. This results in natural boundary conditions for the variational solution. In the dual formulation this corresponds to essential boundary conditions: for given $\boldsymbol{\varepsilon}_p \in H(\mathrm{curl},\Omega)^3$ we can define $\boldsymbol{g}_p \in H_0(\mathrm{curl},\Omega)^3 = \{\boldsymbol{g} \in H(\mathrm{curl},\Omega)^3 \colon \boldsymbol{g} \times \boldsymbol{n} = \boldsymbol{0}\}$ by

$$\int_\Omega \boldsymbol{g}_p : \mathbb{E}^{-1} : \delta\boldsymbol{g}_p \, d\boldsymbol{x} = \int_\Omega \boldsymbol{\varepsilon}_p : \mathrm{curl}(\delta\boldsymbol{g}_p) \, d\boldsymbol{x} \,, \qquad \delta\boldsymbol{g}_p \in H_0(\mathrm{curl},\Omega)^3 \,,$$

which defines $\boldsymbol{g}_p = \mathbb{E} : \mathrm{curl}(\boldsymbol{\varepsilon}_p)$ in terms of distributions.

Thus, in order to obtain a discrete flow rule for the nonlocal model, we consider a mixed formulation in $\boldsymbol{E}^h \times \boldsymbol{G}^h \subset \boldsymbol{E} \times H_0(\mathrm{curl},\Omega)^3$: we introduce a finite element function $\boldsymbol{g}_p^h \in \boldsymbol{G}^h$ approximating $\mathbb{E} : \mathrm{curl}(\varepsilon_p^h)$ in a variational (weak) sense: the pair $(\boldsymbol{g}_p^h, \boldsymbol{\varepsilon}_p^h) \in \boldsymbol{G}^h \times \boldsymbol{E}^h$ is assumed to be coupled by

$$\int_\Omega \boldsymbol{g}_p^h : \mathbb{E}^{-1} : \delta\boldsymbol{g}_p^h \, d\boldsymbol{x} = \int_\Omega \boldsymbol{\varepsilon}_p^h : \mathrm{curl}(\delta\boldsymbol{g}_p^h) \, d\boldsymbol{x} \,, \qquad \delta\boldsymbol{g}_p \in \boldsymbol{G}^h \,. \tag{19}$$

The extended bilinear form is obtained from $a_{\mathrm{nl}}(\cdot,\cdot)$ by replacing $\mathbb{E} : \mathrm{curl}(\boldsymbol{\varepsilon}_p)$ with $\boldsymbol{g}_p$, i. e.,

$$a_{\mathrm{mx}}\big((\boldsymbol{u},\boldsymbol{\varepsilon}_p,\boldsymbol{g}_p),(\delta\boldsymbol{u},\delta\boldsymbol{\varepsilon}_p,\delta\boldsymbol{g}_p)\big) \;=\; a_{\mathrm{kh}}\big((\boldsymbol{u},\boldsymbol{\varepsilon}_p),(\delta\boldsymbol{u},\delta\boldsymbol{\varepsilon}_p)\big) + \int_\Omega \boldsymbol{g}_p : \mathbb{E}^{-1} : \delta\boldsymbol{g}_p \, d\boldsymbol{x} \,.$$

The bilinear form $a_{\mathrm{mx}}(\cdot,\cdot)$ is positive definite in the discrete, extended space $\boldsymbol{V}_0^h \times \boldsymbol{E}^h \times \boldsymbol{G}^h$ and thus in the closed subspace $\boldsymbol{X}^h(\boldsymbol{0})$, where

$$\boldsymbol{X}^h(\boldsymbol{u}_D) = \{(\boldsymbol{u}^h, \boldsymbol{\varepsilon}_p^h, \boldsymbol{g}_p^h) \in \boldsymbol{V}^h(\boldsymbol{u}_D) \times \boldsymbol{E}^h \times \boldsymbol{G}^h \colon (\boldsymbol{g}_p^h, \boldsymbol{\varepsilon}_p^h) \text{ satisfies (19)}\} \,.$$

For the incremental problem we now fix the values $(\boldsymbol{u}^{h,n-1}, \boldsymbol{\varepsilon}_p^{h,n-1}, \boldsymbol{g}_p^{h,n-1}) \in \boldsymbol{X}^h(\boldsymbol{u}_D^{n-1})$ of the previous time step. Introducing

$$\ell_{\mathrm{mx}}^{h,n}(\delta\boldsymbol{u}^h, \delta\boldsymbol{\varepsilon}_p^h, \delta\boldsymbol{g}_p^h) = \ell_n(\delta\boldsymbol{u}^h) - a_{\mathrm{mx}}\big((\boldsymbol{u}^{h,n-1}, \boldsymbol{\varepsilon}_p^{h,n-1}, \boldsymbol{g}_p^{h,n-1}),(\delta\boldsymbol{u}^h, \delta\boldsymbol{\varepsilon}_p^h, \delta\boldsymbol{g}_p^h)\big) \tag{20}$$

we obtain:

THEOREM 8. *A unique minimizer* $(\Delta\boldsymbol{u}^{h,n}, \Delta\boldsymbol{\varepsilon}_p^{h,n}, \Delta\boldsymbol{g}_p^{h,n}) \in \boldsymbol{X}^h(\Delta\boldsymbol{u}_D^n)$ *of*

$$\begin{aligned} J_{\mathrm{mx}}^{h,n}(\Delta\boldsymbol{u}^h, \Delta\boldsymbol{\varepsilon}_p^h, \Delta\boldsymbol{g}_p^h) \;=\; & \frac{1}{2} a_{\mathrm{mx}}\big((\Delta\boldsymbol{u}^h, \Delta\boldsymbol{\varepsilon}_p^h, \Delta\boldsymbol{g}_p^h),(\Delta\boldsymbol{u}^h, \Delta\boldsymbol{\varepsilon}_p^h, \Delta\boldsymbol{g}_p^h)\big) \\ & + j(\Delta\boldsymbol{\varepsilon}_p^h) - \ell_{\mathrm{mx}}^{h,n}(\Delta\boldsymbol{u}^h, \Delta\boldsymbol{\varepsilon}_p^h, \Delta\boldsymbol{g}_p^h) \end{aligned}$$

*exists. Moreover,* $(\boldsymbol{u}^{h,n}, \boldsymbol{\varepsilon}_p^{h,n}, \boldsymbol{g}_p^{h,n}) = (\boldsymbol{u}^{h,n-1}, \boldsymbol{\varepsilon}_p^{h,n-1}, \boldsymbol{g}_p^{h,n-1}) + (\Delta\boldsymbol{u}^{h,n}, \Delta\boldsymbol{\varepsilon}_p^{h,n}, \Delta\boldsymbol{g}_p^{h,n})$ *is characterized by the linear variational saddle point problem*

$$\int_\Omega \boldsymbol{\sigma}^{h,n} : \mathbb{C} : \boldsymbol{\varepsilon}(\delta\boldsymbol{u}^h) \, d\boldsymbol{x} = \ell_n(\delta\boldsymbol{u}^h), \quad \delta\boldsymbol{u}^h \in \boldsymbol{V}_0^h \,, \tag{21a}$$

$$\int_\Omega \boldsymbol{g}_p^{h,n} : \mathbb{E}^{-1} : \delta\boldsymbol{g}_p^h \, d\boldsymbol{x} = \int_\Omega \boldsymbol{\varepsilon}_p^{h,n} : \mathrm{curl}(\delta\boldsymbol{g}_p^h) \, d\boldsymbol{x} \,, \qquad \delta\boldsymbol{g}_p^h \in \boldsymbol{G}^h \tag{21b}$$

*and the variational inequality*

$$\int_\Omega K_0 |\delta\boldsymbol{\varepsilon}_p^h| \, d\boldsymbol{x} \ge \int_\Omega K_0 |\Delta\boldsymbol{\varepsilon}_p^{h,n}| \, d\boldsymbol{x} + \int_\Omega \boldsymbol{\alpha}^{h,n} : (\delta\boldsymbol{\varepsilon}_p^h - \Delta\boldsymbol{\varepsilon}_p^{n,h}) \, d\boldsymbol{x} \;, \quad \delta\boldsymbol{\varepsilon}_p^h \in \boldsymbol{E}^h \,, \tag{21c}$$

*with* $\boldsymbol{\sigma}^{h,n} = \boldsymbol{\varepsilon}(\boldsymbol{u}^{h,n}) - \boldsymbol{\varepsilon}_p^{h,n}$ *and* $\boldsymbol{\alpha}^{h,n} = \boldsymbol{\sigma}^{h,n} - \mathbb{D} : \boldsymbol{\varepsilon}_p^{h,n} - \mathrm{sym}(\mathrm{curl}(\boldsymbol{g}_p^{h,n}))$.

PROOF. The proof uses standard techniques of convex analysis and is included for convenience of the reader. Since $J^{h,n}_{\rm mx}(\cdot)$ is uniformly convex, a unique minimizer $(\Delta\boldsymbol{u}^{h,n}, \Delta\boldsymbol{\varepsilon}^{h,n}_p, \Delta\boldsymbol{g}^{h,n}_p)$ in the discrete space $\boldsymbol{X}^h(\Delta\boldsymbol{u}^n_D)$ exists. Now we show (21c). Therefore, we define

$$J^{h,n}_{\rm red}(\delta\boldsymbol{\varepsilon}^h_p, \delta\boldsymbol{g}^h_p) = \frac{1}{2} a_{\rm mx}\big((\Delta\boldsymbol{u}^{h,n}, \delta\boldsymbol{\varepsilon}^h_p, \delta\boldsymbol{g}^h_p), (\Delta\boldsymbol{u}^{h,n}, \delta\boldsymbol{\varepsilon}^h_p, \delta\boldsymbol{g}^h_p)\big)\,, \qquad (\delta\boldsymbol{\varepsilon}^h_p, \delta\boldsymbol{g}^h_p) \in \boldsymbol{E}^h \times \boldsymbol{G}^h\,.$$

For any $s \in (0,1)$, the inequality

$$J^{h,n}_{\rm mx}(\Delta\boldsymbol{u}^{h,n}, \Delta\boldsymbol{\varepsilon}^{h,n}_p, \Delta\boldsymbol{g}^{h,n}_p) \le J^{h,n}_{\rm mx}\big(\Delta\boldsymbol{u}^{h,n}, \Delta\boldsymbol{\varepsilon}^{h,n}_p + s(\delta\boldsymbol{\varepsilon}^h_p - \Delta\boldsymbol{\varepsilon}^{h,n}_p), \Delta\boldsymbol{g}^{h,n}_p + s(\delta\boldsymbol{g}^h_p - \Delta\boldsymbol{g}^{h,n}_p)\big)$$

can be rewritten as

$$\begin{aligned} J^{h,n}_{\rm red}(\Delta\boldsymbol{\varepsilon}^{h,n}_p, \Delta\boldsymbol{g}^{h,n}_p) + j(\Delta\boldsymbol{\varepsilon}^{h,n}_p) \;\;\le\;\; & J^{h,n}_{\rm red}\big(\Delta\boldsymbol{\varepsilon}^{h,n}_p + s(\delta\boldsymbol{\varepsilon}^h_p - \Delta\boldsymbol{\varepsilon}^{h,n}_p), \Delta\boldsymbol{g}^{h,n}_p + s(\delta\boldsymbol{g}^h_p - \Delta\boldsymbol{g}^{h,n}_p)\big) \\ & + j\big(\Delta\boldsymbol{\varepsilon}^{h,n}_p + s(\delta\boldsymbol{\varepsilon}^h_p - \Delta\boldsymbol{\varepsilon}^{h,n}_p)\big) - s\,\ell^{h,n}_{\rm mx}(\mathbf{0}, \delta\boldsymbol{\varepsilon}^h_p - \Delta\boldsymbol{\varepsilon}^{h,n}_p)\,. \end{aligned}$$

Using the convexity of $j(\cdot)$, i. e., $j\big(\Delta\boldsymbol{\varepsilon}^{h,n}_p + s(\delta\boldsymbol{\varepsilon}^h_p - \Delta\boldsymbol{\varepsilon}^{h,n}_p)\big) = j\big((1-s)\,\Delta\boldsymbol{\varepsilon}^{h,n}_p + s\,\delta\boldsymbol{\varepsilon}^h_p)\big) \le (1-s)\,j(\Delta\boldsymbol{\varepsilon}^{h,n}_p) + s\,j(\delta\boldsymbol{\varepsilon}^h_p)$, gives

$$\begin{aligned} J^{h,n}_{\rm red}(\Delta\boldsymbol{\varepsilon}^{h,n}_p, \Delta\boldsymbol{g}^{h,n}_p) \;\;\le\;\; & J^{h,n}_{\rm red}\big(\Delta\boldsymbol{\varepsilon}^{h,n}_p + s(\delta\boldsymbol{\varepsilon}^h_p - \Delta\boldsymbol{\varepsilon}^{h,n}_p), \Delta\boldsymbol{g}^{h,n}_p + s(\Delta\boldsymbol{g}^{h,n}_p - \delta\boldsymbol{g}^h_p)\big) \\ & + s\big(j(\delta\boldsymbol{\varepsilon}^h_p) - j(\Delta\boldsymbol{\varepsilon}^{h,n}_p)\big) - s\,\ell^{h,n}_{\rm mx}(\mathbf{0}, \delta\boldsymbol{\varepsilon}^h_p - \Delta\boldsymbol{\varepsilon}^{h,n}_p) \end{aligned}$$

and thus

$$\begin{aligned} 0 \;\;\le\;\; & s\,a_{\rm mx}\big((\Delta\boldsymbol{u}^{h,n}, \Delta\boldsymbol{\varepsilon}^{h,n}_p, \Delta\boldsymbol{g}^{h,n}_p), (\mathbf{0}, \delta\boldsymbol{\varepsilon}^h_p - \Delta\boldsymbol{\varepsilon}^{n,h}_p, \delta\boldsymbol{g}^h_p - \Delta\boldsymbol{g}^{h,n}_p)\big) \\ & + s^2\,a_{\rm mx}\big((\mathbf{0}, \delta\boldsymbol{\varepsilon}^h_p - \Delta\boldsymbol{\varepsilon}^{n,h}_p, \delta\boldsymbol{g}^h_p - \Delta\boldsymbol{g}^{h,n}_p), (\mathbf{0}, \delta\boldsymbol{\varepsilon}^h_p - \Delta\boldsymbol{\varepsilon}^{n,h}_p, \delta\boldsymbol{g}^h_p - \Delta\boldsymbol{g}^{h,n}_p)\big) \\ & + s\big(j(\delta\boldsymbol{\varepsilon}^h_p) - j(\Delta\boldsymbol{\varepsilon}^{h,n}_p)\big) - s\,\ell^{h,n}_{\rm mx}(\mathbf{0}, \delta\boldsymbol{\varepsilon}^h_p - \Delta\boldsymbol{\varepsilon}^{h,n}_p)\,. \end{aligned}$$

Dividing by $s$ and then taking the limit $s \longrightarrow 0$ yields

$$\begin{aligned} 0 \;\;\le\;\; & a_{\rm mx}\big((\Delta\boldsymbol{u}^{h,n}, \Delta\boldsymbol{\varepsilon}^{h,n}_p, \Delta\boldsymbol{g}^{h,n}_p), (\mathbf{0}, \delta\boldsymbol{\varepsilon}^h_p - \Delta\boldsymbol{\varepsilon}^{n,h}_p, \delta\boldsymbol{g}^h_p - \Delta\boldsymbol{g}^{h,n}_p)\big) \\ & + j(\delta\boldsymbol{\varepsilon}^h_p) - j(\Delta\boldsymbol{\varepsilon}^{h,n}_p) - \ell^{h,n}_{\rm mx}(\mathbf{0}, \delta\boldsymbol{\varepsilon}^h_p - \Delta\boldsymbol{\varepsilon}^{h,n}_p)\,, \end{aligned}$$

i. e.,

$$\begin{aligned} j(\Delta\boldsymbol{\varepsilon}^{h,n}_p) \;\;\le\;\; & -\int_\Omega \Delta\boldsymbol{\sigma}^{h,n} : (\delta\boldsymbol{\varepsilon}^h_p - \Delta\boldsymbol{\varepsilon}^{n,h}_p)\,d\boldsymbol{x} + \int_\Omega \Delta\boldsymbol{\varepsilon}^{h,n}_p : \mathbb{D} : (\delta\boldsymbol{\varepsilon}^h_p - \Delta\boldsymbol{\varepsilon}^{n,h}_p)\,d\boldsymbol{x} \\ & + \int_\Omega \Delta\boldsymbol{g}^{h,n}_p : \mathbb{E}^{-1} : (\delta\boldsymbol{g}^h_p - \Delta\boldsymbol{g}^{h,n}_p)\,d\boldsymbol{x} + j(\delta\boldsymbol{\varepsilon}^h_p) \\ & - \int_\Omega \boldsymbol{\sigma}^{h,n-1} : (\delta\boldsymbol{\varepsilon}^h_p - \Delta\boldsymbol{\varepsilon}^{n,h}_p)\,d\boldsymbol{x} + \int_\Omega \boldsymbol{\varepsilon}^{h,n-1}_p : \mathbb{D} : (\delta\boldsymbol{\varepsilon}^h_p - \Delta\boldsymbol{\varepsilon}^{n,h}_p)\,d\boldsymbol{x} \\ & + \int_\Omega \boldsymbol{g}^{h,n-1}_p : \mathbb{E}^{-1} : (\delta\boldsymbol{g}^h_p - \Delta\boldsymbol{g}^{n,h}_p)\,d\boldsymbol{x}\;. \end{aligned}$$

The linear constraint (19)

$$\int_\Omega (\delta\boldsymbol{g}^h_p - \Delta\boldsymbol{g}^{h,n}_p) : \mathbb{E}^{-1} : \boldsymbol{g}^{h,n}_p\,d\boldsymbol{x} = \int_\Omega (\delta\boldsymbol{\varepsilon}^h_p - \Delta\boldsymbol{\varepsilon}^{h,n}_p) : \operatorname{curl}(\boldsymbol{g}^{h,n}_p)\,d\boldsymbol{x}$$

gives

$$j(\Delta\boldsymbol{\varepsilon}_p^{h,n}) \;\le\; -\int_\Omega \boldsymbol{\sigma}^{h,n} : (\delta\boldsymbol{\varepsilon}_p^h - \Delta\boldsymbol{\varepsilon}_p^{n,h})\,d\boldsymbol{x} + \int_\Omega \boldsymbol{\varepsilon}_p^{h,n} : \mathbb{D} : (\delta\boldsymbol{\varepsilon}_p^h - \Delta\boldsymbol{\varepsilon}_p^{n,h})\,d\boldsymbol{x} + \int_\Omega \operatorname{curl}(\boldsymbol{g}_p^{h,n}) : (\delta\boldsymbol{\varepsilon}_p^h - \Delta\boldsymbol{\varepsilon}_p^{h,n})\,d\boldsymbol{x} + j(\delta\boldsymbol{\varepsilon}_p^h)\;,$$

which shows the dissipation inequality (21c).
Next, we derive (21a). Since the corresponding reduced functional

$$J_{\boldsymbol{u}}^{h,n}(\Delta\boldsymbol{u}^h) = J_{\mathrm{mx}}^{h,n}(\Delta\boldsymbol{u}^h, \Delta\boldsymbol{\varepsilon}_p^{h,n}, \Delta\boldsymbol{g}_p^{h,n})\,, \quad \Delta\boldsymbol{u}^h \in \boldsymbol{V}^h(\Delta\boldsymbol{u}_D^n)$$

is quadratic with respect to $\Delta\boldsymbol{u}^h$, the minimizer $\Delta\boldsymbol{u}^{h,n}$ of $J_{\boldsymbol{u}}^{h,n}(\cdot)$ is a critical point of $J_{\boldsymbol{u}}^{h,n}(\cdot)$ and characterized by $DJ_{\boldsymbol{u}}^{h,n}(\Delta\boldsymbol{u}^{h,n})[(\delta\boldsymbol{u}^h] = 0$, i. e.,

$$a_{\mathrm{mx}}\big((\Delta\boldsymbol{u}^{h,n}, \Delta\boldsymbol{\varepsilon}_p^{h,n}, \Delta\boldsymbol{g}_p^{h,n}), (\delta\boldsymbol{u}^h, \mathbf{0}, \mathbf{0})\big) - \ell_{\mathrm{mx}}^{h,n}(\delta\boldsymbol{u}^h, \mathbf{0}, \mathbf{0}) = 0\,, \qquad \delta\boldsymbol{u}^h \in \boldsymbol{V}_0^h\;.$$

This yields for the stress increment $\Delta\boldsymbol{\sigma}^{h,n} = \mathbb{C} : \big(\boldsymbol{\varepsilon}(\Delta\boldsymbol{u}^{h,n}) - \Delta\boldsymbol{\varepsilon}_p^{h,n}\big)$

$$\int_\Omega \Delta\boldsymbol{\sigma}^{h,n} : \boldsymbol{\varepsilon}(\delta\boldsymbol{u}^h)\,d\boldsymbol{x} = \ell_{\mathrm{mx}}^{n,h}(\delta\boldsymbol{u}^h, \mathbf{0}, \mathbf{0}), \qquad \delta\boldsymbol{u}^h \in \boldsymbol{V}_0^h\,,$$

and inserting (20) gives (21a). In the same way, we obtain (21b) by minimizing

$$J_{\boldsymbol{g}}^{h,n}(\delta\boldsymbol{g}_p^h) = J_{\mathrm{mx}}^{h,n}(\Delta\boldsymbol{u}^{h,n}, \Delta\boldsymbol{\varepsilon}_p^{h,n}, \delta\boldsymbol{g}_p^h)\,, \qquad \delta\boldsymbol{g}_p^h \in \boldsymbol{G}^h$$

at given $(\Delta\boldsymbol{u}^{h,n}, \Delta\boldsymbol{\varepsilon}_p^{h,n})$. □

In the fully discrete scheme, all integrals are replaced by a quadrature formula

$$\int_\Omega \boldsymbol{v}\cdot\boldsymbol{w}\,d\boldsymbol{x} = \sum_{\boldsymbol{\xi}\in\boldsymbol{\Xi}_h} \omega_{\boldsymbol{\xi}}\,\boldsymbol{v}(\boldsymbol{\xi})\cdot\boldsymbol{w}(\boldsymbol{\xi})\,, \qquad \boldsymbol{v},\boldsymbol{w} \in \boldsymbol{V}_h\,,$$

where $\boldsymbol{\Xi}_h \subset \Omega$ are the integration points and $\omega_{\boldsymbol{\xi}}$ the corresponding quadrature weights. Then, the space $\boldsymbol{E}_h$ is identified with its values at the integration points, i. e., $\boldsymbol{E}_h = \{\boldsymbol{\varepsilon}_p \colon \boldsymbol{\Xi}_h \longrightarrow \mathrm{sl}(d)\cap\mathrm{Sym}(d)\}$. In the same way, the discrete return parameter $\gamma^{h,n}$ is approximated in $\Gamma_h = \{\gamma^h \colon \boldsymbol{\Xi}_h \longrightarrow \mathbf{R}\}$ at integration points only. This allows for a point-wise evaluation of the flow rule. We obtain (analogous to Lemma 3) independently for every integration point:

Lemma 9. *The discrete dissipation inequality*

$$K_0|\delta\boldsymbol{\varepsilon}_p^h| \ge K_0|\Delta\boldsymbol{\varepsilon}_p^{h,n}| + \boldsymbol{\alpha}^{h,n} : (\delta\boldsymbol{\varepsilon}_p^h - \Delta\boldsymbol{\varepsilon}_p^{n,h})\;, \quad \delta\boldsymbol{\varepsilon}_p^h \in \mathrm{sl}(d)\cap\mathrm{Sym}(d) \tag{22}$$

*is equivalent to the flow rule*

$$\Delta\boldsymbol{\varepsilon}_p^{h,n} = \gamma^{h,n}\,\frac{\operatorname{dev}\big(\boldsymbol{\alpha}^{h,n}\big)}{|\operatorname{dev}\big(\boldsymbol{\alpha}^{h,n}\big)|}\;, \qquad \gamma^{h,n} = |\Delta\boldsymbol{\varepsilon}_p^{h,n}| \tag{23}$$

*and the complementary conditions*

$$\gamma^{h,n}\big(|\operatorname{dev}(\boldsymbol{\alpha}^{h,n})| - K_0\big) = 0, \qquad \gamma^{h,n} \ge 0, \qquad |\operatorname{dev}(\boldsymbol{\alpha}^{h,n})| \le K_0\,. \tag{24}$$

Together, the discrete solution can be obtained by the solution of the finite element problem (21a), (21b) and—independently for every integration point $\boldsymbol{\xi} \in \boldsymbol{\Xi}_h$—by the flow rule (23) and the complementary conditions (24).

## 4. Algorithmic nonlocal plasticity

In the mixed form we can apply a projection method, where flow rule and complementary conditions are evaluated by the standard closest point projection [**36**, Chap. 3.3.2] depending on the actual trial stress, and the result is inserted into the variational problem (21a), (21b). This yields a nonlinear system for $(\boldsymbol{u}^{h,n}, \boldsymbol{g}_p^{h,n})$ which is solved with a generalized Newton method by inserting the consistent tangent modulus in both equations, linearizing the stress response in (21a) and the strain response in (21b). Note that this results in a fully coupled nonlinear formulation, since a derivative of the projection with respect to both, $\boldsymbol{\varepsilon}(\boldsymbol{u}^h)$ and $\boldsymbol{g}_p^h$, has to be provided (which makes the realization of such a method quite involved).

Thus we decided for a modified algorithmic approach. We extend the linearized projection method introduced in [**41**] to our mixed formulation of nonlocal plasticity: for a given iterate, the flow rule is linearized, so that in every step a linear variational system with linear constraints at the integration points is solved. This subproblem is—due to the constraints—nonlinear itself, but the corresponding projections on linear half spaces are easy to compute (in comparison with the fully nonlinear constraints).

In order to simplify the notation, we skip in the following the mesh parameter $h$.

**4.1. The linearized projection method.** For fixed values $(\boldsymbol{u}^{n-1}, \boldsymbol{\varepsilon}_p^{n-1}, \boldsymbol{g}_p^{n-1}) \in \boldsymbol{X}(\boldsymbol{u}_D^{n-1})$ of the preceding time step we consider the incremental problem for the computation of the approximation in step $n$ at time $t_n$. The linearized projection method iterates on the extended space $\boldsymbol{Y}^n = \boldsymbol{X}(\boldsymbol{u}_D^n) \times \Gamma$, where the linearized parameter is an independent variable.

Inserting the flow function $\phi(\boldsymbol{\alpha}) = |\operatorname{dev}(\boldsymbol{\alpha})| - K_0$ the incremental problem defined by (21a), (21b), (23), (24) is reformulated as follows: find $(\boldsymbol{u}^n, \boldsymbol{\varepsilon}_p^n, \boldsymbol{g}_p^n, \gamma^n) \in \boldsymbol{Y}^n$ with

$$\int_\Omega \big(\boldsymbol{\varepsilon}(\boldsymbol{u}^n) - \boldsymbol{\varepsilon}_p^n\big) : \mathbb{C} : \boldsymbol{\varepsilon}(\delta\boldsymbol{u})\, d\boldsymbol{x} = \ell_n(\delta\boldsymbol{u}), \quad \delta\boldsymbol{u} \in \boldsymbol{V}_0\,, \tag{25a}$$

$$\int_\Omega \boldsymbol{g}_p^n : \mathbb{E}^{-1} : \delta\boldsymbol{g}_p\, d\boldsymbol{x} = \int_\Omega \boldsymbol{\varepsilon}_p^n : \operatorname{curl}(\delta\boldsymbol{g}_p)\, d\boldsymbol{x}\,, \qquad \delta\boldsymbol{g}_p \in \boldsymbol{G}\,, \tag{25b}$$

$$\boldsymbol{\varepsilon}_p^n - \boldsymbol{\varepsilon}_p^{n-1} = \gamma^n D\phi(\boldsymbol{\alpha}^n)\,, \qquad \boldsymbol{\alpha}^n = \mathbb{C} : (\boldsymbol{\varepsilon}(\boldsymbol{u}^n) - \boldsymbol{\varepsilon}_p^n) - \mathbb{D} : \boldsymbol{\varepsilon}_p^n - \operatorname{sym}(\operatorname{curl}(\boldsymbol{g}_p^n)) \tag{25c}$$

$$\gamma^n \phi(\boldsymbol{\alpha}^n) = 0, \qquad \gamma^n \geq 0, \qquad \phi(\boldsymbol{\alpha}^n) \leq 0\,. \tag{25d}$$

This nonlinear problem (25) is solved iteratively: for an iterate $(\boldsymbol{u}^{n,k-1}, \boldsymbol{\varepsilon}_p^{n,k-1}, \boldsymbol{g}_p^{n,k-1}, \gamma^{n,k-1}) \in \boldsymbol{Y}^n$ of the previous step we define the corresponding relative elastic stress

$$\boldsymbol{\alpha}^{n,k-1} = \mathbb{C} : \big(\boldsymbol{\varepsilon}(\boldsymbol{u}^{n,k-1}) - \boldsymbol{\varepsilon}_p^{n,k-1}\big) - \mathbb{D} : \boldsymbol{\varepsilon}_p^{n,k-1} - \operatorname{sym}(\operatorname{curl}(\boldsymbol{g}_p^{n,k-1}))$$

and the linearized flow rule

$$\phi_{n,k}(\boldsymbol{\alpha}) = \phi(\boldsymbol{\alpha}^{n,k-1}) + D\phi(\boldsymbol{\alpha}^{n,k-1})(\boldsymbol{\alpha} - \boldsymbol{\alpha}^{n,k-1})\,. \tag{26}$$

The next iterate $(\boldsymbol{u}^{n,k}, \boldsymbol{\varepsilon}_p^{n,k}, \boldsymbol{g}_p^{n,k}, \gamma^{n,k}) \in \boldsymbol{Y}^n$ is determined as the solution of the linearized system

$$\int_\Omega \big(\boldsymbol{\varepsilon}(\boldsymbol{u}^{n,k}) - \boldsymbol{\varepsilon}_p^{n,k}\big) : \mathbb{C} : \boldsymbol{\varepsilon}(\delta\boldsymbol{u})\, d\boldsymbol{x} = \ell_n(\delta\boldsymbol{u}), \quad \delta\boldsymbol{u} \in \boldsymbol{V}_0\,, \tag{27a}$$

$$\int_\Omega \boldsymbol{g}_p^{n,k} : \mathbb{E}^{-1} : \delta\boldsymbol{g}_p\, d\boldsymbol{x} = \int_\Omega \boldsymbol{\varepsilon}_p^{n,k} : \operatorname{curl}(\delta\boldsymbol{g}_p)\, d\boldsymbol{x}\,, \qquad \delta\boldsymbol{g}_p \in \boldsymbol{G}\,, \tag{27b}$$

$$\boldsymbol{\varepsilon}_p^{n,k} - \boldsymbol{\varepsilon}_p^{n-1} = \gamma^{n,k} D\phi(\boldsymbol{\alpha}^{n,k-1}) + \gamma^{n,k-1} D^2\phi(\boldsymbol{\alpha}^{n,k-1}) : (\boldsymbol{\alpha}^{n,k} - \boldsymbol{\alpha}^{n,k-1})\,, \tag{27c}$$

$$\gamma^{n,k} \phi_{n,k}(\boldsymbol{\alpha}^{n,k}) = 0, \qquad \gamma^{n,k} \geq 0, \qquad \phi_{n,k}(\boldsymbol{\alpha}^{n,k}) \leq 0 \tag{27d}$$

with $\boldsymbol{\alpha}^{n,k} = \mathbb{C} : (\boldsymbol{\varepsilon}(\boldsymbol{u}^{n,k}) - \boldsymbol{\varepsilon}_p^{n,k}) - \mathbb{D} : \boldsymbol{\varepsilon}_p^{n,k} - \operatorname{sym}(\operatorname{curl}(\boldsymbol{g}_p^{n,k}))$, where the derivatives of the flow rule are

$$D\phi(\boldsymbol{\alpha}) = \frac{\operatorname{dev}(\boldsymbol{\alpha})}{|\operatorname{dev}(\boldsymbol{\alpha})|}\,, \quad D^2\phi(\boldsymbol{\alpha}) : \boldsymbol{\eta} \;=\; \frac{\operatorname{dev}(\boldsymbol{\eta})}{|\operatorname{dev}(\boldsymbol{\alpha})|} - \frac{\operatorname{dev}(\boldsymbol{\alpha}) : \operatorname{dev}(\boldsymbol{\eta})}{|\operatorname{dev}(\boldsymbol{\alpha})|^2}\,\frac{\operatorname{dev}(\boldsymbol{\alpha})}{|\operatorname{dev}(\boldsymbol{\alpha})|}\,, \quad \boldsymbol{\alpha} \neq \mathbf{0}\,.$$

For the solution of the system (27) we introduce the elastic trial stress

$$\boldsymbol{\theta}_n(\boldsymbol{u}, \boldsymbol{g}_p) = \mathbb{C} : (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}_p^{n-1}) - \mathbb{D} : \boldsymbol{\varepsilon}_p^{n-1} - \operatorname{sym}(\operatorname{curl}(\boldsymbol{g}_p))\,.$$

LEMMA 10. *The equations* (27c) *and* (27d) *have a unique solution*

$$\gamma^{n,k} \;=\; G_{n,k}(\boldsymbol{\theta}^{n,k})\,, \tag{28a}$$
$$\boldsymbol{\varepsilon}_p^{n,k} \;=\; R_{n,k}(\boldsymbol{\theta}^{n,k})\,, \tag{28b}$$

*depending only on the trial stress* $\boldsymbol{\theta}^{n,k} = \boldsymbol{\theta}_n(\boldsymbol{u}^{n,k}, \boldsymbol{g}_p^{n,k})$, *where the return parameter and the plastic strain response are defined by*

$$\begin{aligned} G_{n,k}(\boldsymbol{\theta}) &= \frac{\max\{0, \phi_{n,k}(\boldsymbol{\theta})\}}{(2+H_0)\mu} \\ R_{n,k}(\boldsymbol{\theta}) &= \boldsymbol{\varepsilon}_p^{n-1} + \frac{1}{(2+H_0)\mu}\frac{\gamma^{n,k-1}\,|\operatorname{dev}(\boldsymbol{\alpha}^{n,k-1})|}{\gamma^{n,k-1} + |\operatorname{dev}(\boldsymbol{\alpha}^{n,k-1})|} D^2\phi(\boldsymbol{\alpha}^{n,k-1}) : \boldsymbol{\theta} + G_{n,k}(\boldsymbol{\theta}) D\phi(\boldsymbol{\alpha}^{n,k-1})\,. \end{aligned}$$

PROOF. We have $(\mathbb{C}+\mathbb{D}) : (\boldsymbol{\varepsilon}_p^{n,k} - \boldsymbol{\varepsilon}_p^{n-1}) = \boldsymbol{\theta}^{n,k} - \boldsymbol{\alpha}^{n,k}$, and (27c) can be rewritten in the form

$$(\mathbb{C}+\mathbb{D})^{-1} : \big(\boldsymbol{\theta}^{n,k} - \boldsymbol{\alpha}^{n,k}\big) \;=\; \gamma^{n,k} D\phi(\boldsymbol{\alpha}^{n,k-1}) + \gamma^{n,k-1} D^2\phi(\boldsymbol{\alpha}^{n,k-1}) : (\boldsymbol{\alpha}^{n,k} - \boldsymbol{\alpha}^{n,k-1})\;.$$

This gives (using $D^2\phi(\boldsymbol{\alpha}^{n,k-1}) : \boldsymbol{\alpha}^{n,k-1} = \mathbf{0}$)

$$\Big(\operatorname{id} - (2+H_0)\mu)\gamma^{n,k-1} D^2\phi(\boldsymbol{\alpha}^{n,k-1})\Big) : \boldsymbol{\alpha}^{n,k} = \boldsymbol{\theta}^{n,k} - (2+H_0)\mu\gamma^{n,k} D\phi(\boldsymbol{\alpha}^{n,k-1})\,.$$

Inserting (27d) yields $(2+H_0)\mu\gamma^{n,k} = \max\{0, \phi_{n,k}(\boldsymbol{\theta}^{n,k})\}$ and

$$\boldsymbol{\theta}^{n,k} - \boldsymbol{\alpha}^{n,k} \;=\; \frac{\gamma^{n,k-1}\,|\operatorname{dev}(\boldsymbol{\alpha}^{n,k-1})|}{\gamma^{n,k-1} + |\operatorname{dev}(\boldsymbol{\alpha}^{n,k-1})|} D^2\phi(\boldsymbol{\alpha}^{n,k-1}) : \boldsymbol{\theta}^{n,k} + (2+H_0)\mu\gamma^{n,k} D\phi(\boldsymbol{\alpha}^{n,k-1})$$

(this can be verified by direct computation, see [**41**] for details). □

Inserting (28b) in (27a), (27b) results in a variational nonlinear problem:
find $(\boldsymbol{u}^{n,k}, \boldsymbol{g}_p^{n,k})$ such that

$$F_{n,k}(\boldsymbol{u}^{n,k}, \boldsymbol{g}_p^{n,k})[\delta\boldsymbol{u}, \delta\boldsymbol{g}_p] = \ell_n(\delta\boldsymbol{u})\,, \qquad (\delta\boldsymbol{u}, \delta\boldsymbol{g}_p) \in \boldsymbol{V}_0 \times \boldsymbol{G}\,,$$

with

$$\begin{aligned} F_{n,k}(\boldsymbol{u}, \boldsymbol{g}_p)[\delta\boldsymbol{u}, \delta\boldsymbol{g}_p] \;=\; & \int_\Omega \big(\boldsymbol{\varepsilon}(\boldsymbol{u}) - R_{n,k}(\boldsymbol{\theta}_n(\boldsymbol{u}, \boldsymbol{g}_p))\big) : \mathbb{C} : \boldsymbol{\varepsilon}(\delta\boldsymbol{u})\,d\boldsymbol{x} \\ & + \int_\Omega R_{n,k}(\boldsymbol{\theta}_n(\boldsymbol{u}, \boldsymbol{g}_p)) : \operatorname{curl}(\delta\boldsymbol{g}_p)\,d\boldsymbol{x} - \int_\Omega \boldsymbol{g}_p : \mathbb{E}^{-1} : \delta\boldsymbol{g}_p\,d\boldsymbol{x}\,. \end{aligned}$$

**4.2. The consistent tangent operator.** The plastic strain response $R_{n,k}(\cdot)$ is only nonlinear with respect to the term $\max\{0,\phi_{n,k}(\boldsymbol{\theta})\}$ and therefore a Lipschitz function. Thus, a generalized multi-valued derivative exists (see **[17]** for the definition and for properties of generalized derivatives). A consistent linearization of the plastic strain response is obtained by the specific choice

$$\begin{aligned}\mathbb{R}_{n,k}(\boldsymbol{\theta}) &= \frac{1}{(2+H_0)\mu}\Big(\frac{\gamma^{n,k-1}\,|\operatorname{dev}(\boldsymbol{\alpha}^{n,k-1})|}{\gamma^{n,k-1}+|\operatorname{dev}(\boldsymbol{\alpha}^{n,k-1})|}D^2\phi(\boldsymbol{\alpha}^{n,k-1})\\ &\qquad + \operatorname{sgn}(G_{n,k}(\boldsymbol{\theta}))\,D\phi(\boldsymbol{\alpha}^{n,k-1})\otimes D\phi(\boldsymbol{\alpha}^{n,k-1})\Big) \in \partial R_{n,k}(\boldsymbol{\theta})\end{aligned}$$

(see **[41]** for details on the subdifferential calculus for the computation of consistent tangent operators). Thus, a consistent linearization of $F_{n,k}$ is given by the symmetric bilinear form

$$\begin{aligned}&a_{n,k}(\boldsymbol{u},\boldsymbol{g}_p)[(\Delta\boldsymbol{u},\Delta\boldsymbol{g}_p),(\delta\boldsymbol{u},\delta\boldsymbol{g}_p)]\\ &\quad=\int_\Omega \boldsymbol{\varepsilon}(\Delta\boldsymbol{u}):\big(\mathbb{C}-4\mu^2\mathbb{R}_{n,k}(\boldsymbol{\theta})\big):\boldsymbol{\varepsilon}(\delta\boldsymbol{u})\,d\boldsymbol{x}+\int_\Omega 2\mu\operatorname{curl}(\Delta\boldsymbol{g}_p):\mathbb{R}_{n,k}(\boldsymbol{\theta}):\boldsymbol{\varepsilon}(\delta\boldsymbol{u})\,d\boldsymbol{x}\\ &\qquad+\int_\Omega 2\mu\,\boldsymbol{\varepsilon}(\Delta\boldsymbol{u}):\mathbb{R}_{n,k}(\boldsymbol{\theta})\big):\operatorname{curl}(\delta\boldsymbol{g}_p)\,d\boldsymbol{x}\\ &\qquad-\int_\Omega\Big(\Delta\boldsymbol{g}_p:\mathbb{E}^{-1}:\delta\boldsymbol{g}_p+\operatorname{curl}(\Delta\boldsymbol{g}_p):\mathbb{R}_{n,k}(\boldsymbol{\theta}):\operatorname{curl}(\delta\boldsymbol{g}_p)\Big)d\boldsymbol{x}\end{aligned}$$

with $\boldsymbol{\theta}=\boldsymbol{\theta}_n(\boldsymbol{u},\boldsymbol{g}_p)$. This results in the following algorithm:

S0) Start for $t_0=0$ with $\boldsymbol{u}^0=\mathbf{0}$ and $\boldsymbol{\varepsilon}_p^0=\mathbf{0}$. Set $n=1$.

S1) Choose $(\boldsymbol{u}^{n,0},\boldsymbol{\varepsilon}_p^{n,0},\boldsymbol{g}_p^{n,0},\gamma^{n,0})\in\boldsymbol{Y}^n$. Set $k=1$.

S2) Set $(\boldsymbol{u}^{n,k,0},\boldsymbol{g}_p^{n,k,0})=(\boldsymbol{u}^{n,0},\boldsymbol{g}_p^{n,0})$. Set $m=1$.

S3) Evaluate the residual $r_{n,k,m}=\ell_n(\boldsymbol{u}^{n,k,m-1})-F_{n,k}(\boldsymbol{u}^{n,k,m-1},\boldsymbol{g}_p^{n,k,m-1})$.
If $\|r_{n,k,m}\|$ is small enough, go to S5).

S4) Assemble the linearization $a_{n,k,m}=a_{n,k}(\boldsymbol{u}^{n,k,m-1},\boldsymbol{g}_p^{n,k,m-1})$.
Compute the Newton update $(\Delta\boldsymbol{u}^{n,k,m},\Delta\boldsymbol{g}_p^{n,k,m})$ by solving

$$a_{n,k,m}[(\Delta\boldsymbol{u}^{n,k,m},\Delta\boldsymbol{g}_p^{n,k,m}),(\delta\boldsymbol{u},\delta\boldsymbol{g}_p)]=r_{n,k,m}[\delta\boldsymbol{u},\delta\boldsymbol{g}_p]\,,\qquad(\delta\boldsymbol{u},\delta\boldsymbol{g}_p)\in\boldsymbol{V}_0\times\boldsymbol{G}\,,$$

choose a suitable damping parameter $\rho_{n,k,m}\in(0,1]$ and set

$$(\boldsymbol{u}^{n,k,m},\boldsymbol{g}_p^{n,k,m})=(\boldsymbol{u}^{n,k,m-1},\boldsymbol{g}_p^{n,k,m-1})+\rho_{n,k,m}(\Delta\boldsymbol{u}^{n,k,m},\Delta\boldsymbol{g}_p^{n,k,m})\,.$$

Set $m:=m+1$ and go to S3).

S5) Set $\boldsymbol{u}^{n,k}=\boldsymbol{u}^{n,k,m}$, $\boldsymbol{g}_p^{n,k}=\boldsymbol{g}_p^{n,k,m}$, $\boldsymbol{\varepsilon}_p^{n,k}=R_{n,k}(\boldsymbol{u}^{n,k},\boldsymbol{g}_p^{n,k})$, $\gamma^{n,k}=G_{n,k}(\boldsymbol{u}^{n,k},\boldsymbol{g}_p^{n,k})$.
Compute $\boldsymbol{\alpha}^{n,k}=\mathbb{C}:(\boldsymbol{\varepsilon}(\boldsymbol{u}^{n,k})-\boldsymbol{\varepsilon}_p^{n,k})-\mathbb{D}:\boldsymbol{\varepsilon}_p^{n,k}-\operatorname{sym}(\operatorname{curl}(\boldsymbol{g}_p^{n,k}))$.

S6) If $\|\boldsymbol{\varepsilon}_p^{n,k}-\boldsymbol{\varepsilon}_p^{n-1}-\gamma^{n,k}D\phi(\boldsymbol{\alpha}^{n,k})\|$ and $\max\{0,\phi(\boldsymbol{\alpha}^{n,k})\}$ are small enough,
set $\boldsymbol{u}^n=\boldsymbol{u}^{n,k}$, $\boldsymbol{\varepsilon}_p^n=\boldsymbol{\varepsilon}_p^{n,k}$, $n:=n+1$ and go to S1).

S7) $k:=k+1$ and go to S2).

In our application, the damping is realized by a simple line search: choose $\rho_{n,k,m}\in\{1,1/2,1/4,...\}$ maximal such that $\|r_{n,k,m}\|<\|r_{n,k,m-1}\|$. For sufficient small $\Delta t_n$, we have $\rho_{n,k,m}=1$ in most cases.

## 5. Numerical experiments

In this section we discuss numerical test calculations for the evaluation of the regularizing properties of the nonlocal model. The computations are realized in the parallel finite element code M++ [40]. Note that all our discretizations and all nonlinear and linear solution methods are provided in a general fashion and do not specifically depend on the studied representative model; they easily transfer to other nonlocal models without substantial changes.

**5.1. An example configuration.** For the numerical test we consider a traction problem where the load functional

$$\ell(t, \delta\boldsymbol{u}) = t \int_{\Gamma_N} \boldsymbol{t}_N \cdot \delta\boldsymbol{u}\, d\boldsymbol{a}, \qquad \boldsymbol{t}_N = (0,0,1)^T$$

depends linearly on the loading parameter $t$ (see Fig. 1 for some configuration details).

**Material parameters**
Poisson ratio $\nu = 0.29$
Young modulus $E = 206900.00$ [N/mm$^2$]
$\mu = 80193.8$ [N/mm$^2$]
$\lambda = 110743.82$ [N/mm$^2$]
yield stress $K_0 = 450.00$ [N/mm$^2$]

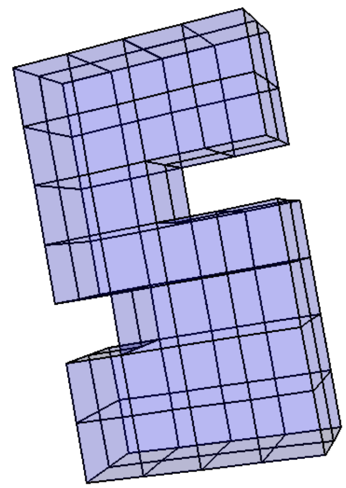

Figure 1: Initial configuration $\Omega \subset (0,4) \times (0,1) \times (0,7)$. On the bottom $\Gamma_D = (0,4) \times (0,1) \times \{0\}$ the body is fixed (homogeneous Dirichlet boundary conditions $\boldsymbol{u}_D \equiv \boldsymbol{0}$ on $\Gamma_D$), a traction force is applied on the top surface $\Gamma_N = (0,4) \times (0,1) \times \{7\}$.

**5.2. Reparametrization.** Since this model is rate independent, the parameter $t$ has no direct link to the physical time. Here we are interested in the regularization properties of the local hardening parameters $H_0$ and the nonlocal plastic length scale $L_c$, so that we will evaluate the model close to the limit load of the model of perfect plasticity. For small hardening the model is quite sensitive with respect to the case where the loading parameter $t$ is close to the limit load. This can be easily avoided by the following reparametrization.

We consider a loading cycle with $2N$ loading steps using uniform steps $s_n = n\Delta s$, $n = 1, ..., N$ with a fixed increment $\Delta s = S/N$ and $s_n = S - (n-N)\Delta s$, $n = N+1, ..., 2N$. In our tests we use $S = 0.6$ and $N = 60$. For every parameter $s_n$ we compute $t_n$ such that the external work satisfies

$$\ell(t_n, \boldsymbol{u}^n) = s_n\,. \tag{30}$$

Therefore, in every step of the nonlinear iteration we now solve the following problem: find $(\boldsymbol{u}^{n,k}, \boldsymbol{g}_p^{n,k}, t_{k,n})$ such that

$$F_{n,k}(\boldsymbol{u}^{n,k}, \boldsymbol{g}_p^{n,k})[\delta\boldsymbol{u}, \delta\boldsymbol{g}_p] = \ell(t_{n,k}, \delta\boldsymbol{u})\,, \qquad (\delta\boldsymbol{u}, \delta\boldsymbol{g}_p) \in \boldsymbol{V}_0 \times \boldsymbol{G}$$

subject to the linear constraint $\ell(t_{n,k}, \boldsymbol{u}^{n,k}) = s_n$. Thus, for the iterate $(\boldsymbol{u}^{n,k,m-1}, \boldsymbol{g}_p^{n,k,m-1}, t_{k,n,m-1})$ in S3) we compute the modified residual

$$r_{n,k,m} = \ell(t_{n,k,m-1}, \delta\boldsymbol{u}) - F_{n,k}(\boldsymbol{u}^{n,k,m-1}, \boldsymbol{g}_p^{n,k,m-1})\,,$$

and in S4) the following saddle point problem is solved: find $(\Delta \boldsymbol{u}^{n,k,m}, \Delta \boldsymbol{g}_p^{n,k,m}, \Delta t_{k,n,m})$ such that

$$\begin{aligned} a_{n,k,m}[(\Delta \boldsymbol{u}^{n,k,m}, \Delta \boldsymbol{g}_p^{n,k,m}), (\delta \boldsymbol{u}, \delta \boldsymbol{g}_p)] + \ell(\Delta t_{k,n,m}, \delta \boldsymbol{u}) &= r_{n,k,m}[\delta \boldsymbol{u}, \delta \boldsymbol{g}_p]\,, && (31a)\\ \ell(\delta t, \Delta \boldsymbol{u}^{n,k,m}) &= s_n - \ell(t_{n,k,m-1}, \boldsymbol{u}^{n,k,m-1}) && (31b) \end{aligned}$$

for all $(\delta \boldsymbol{u}, \delta \boldsymbol{g}_p) \in \boldsymbol{V}_0 \times \boldsymbol{G}$ and $\delta t \in \mathbf{R}$.

**5.3. Solution of the linearized system.** The linear system (31) has the form

$$\begin{pmatrix} K & \ell \\ \ell^T & 0 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} r \\ q \end{pmatrix}$$

with $K \in \mathbf{R}^{N_h \times N_h}$, $\ell \in \mathbf{R}^{N_h}$, where $N_h = \dim \boldsymbol{X}^h$. This bordered linear system is resolved by the following algorithm:

$$\begin{aligned} &\text{solve } Kc = \ell\,, && (32a)\\ &\text{set } t = (c^T r - q)/(c^T \ell)\,, && (32b)\\ &\text{solve } Kx = r - t\ell\,. && (32c) \end{aligned}$$

For the solution of the two linear subproblems (32a) and (32c) we use a parallel Krylov method [**42**] accelerated with a multilevel ILU preconditioner with pivoting and dropping strategy by Mayer [**20, 21**]. Note that the matrix $K = \begin{pmatrix} A & B \\ B^T & -C \end{pmatrix}$ is symmetric, invertible but indefinite ($A$ and $C$ are symmetric positive definite), so that simple cg-iterations with standard preconditioners do not apply.

**5.4. Convergence properties.** The result for a sample computation is illustrated in Fig. 3, and in Fig. 2 we illustrate the convergence with respect to the mesh size.

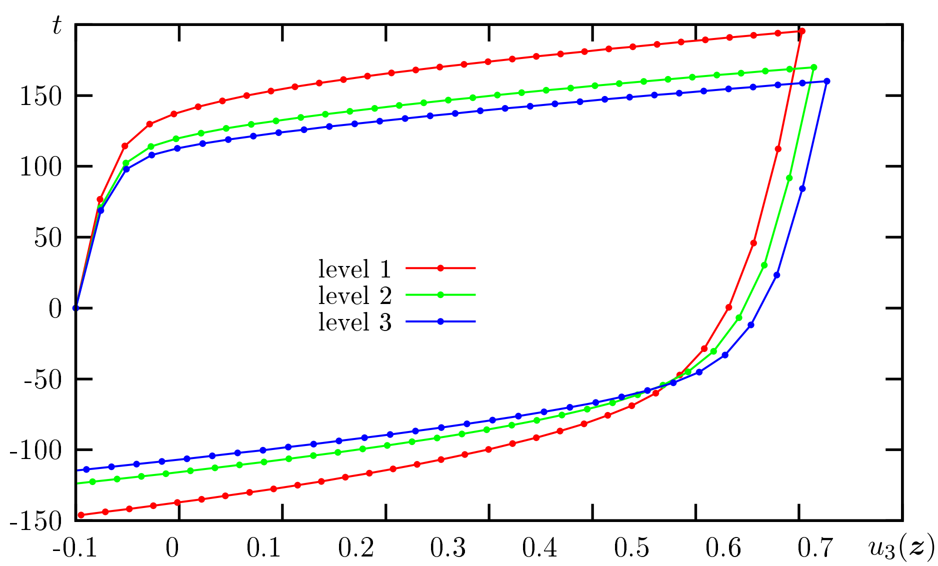


Figure 2: Mesh-convergence of the load-displacement curve for level 1,2,3 with 400, 3 200, 25 600 cells (6 904, 46 420 and 338 500 degrees of freedom) for $L_c = 0.01$ and $H_0 = 0.001$. Here, the load is proportional to $t$, and the displacement $\boldsymbol{u} = (u_1, u_2, u_3)$ is given at the point $\boldsymbol{z} = (0, 0, 7)^T$.

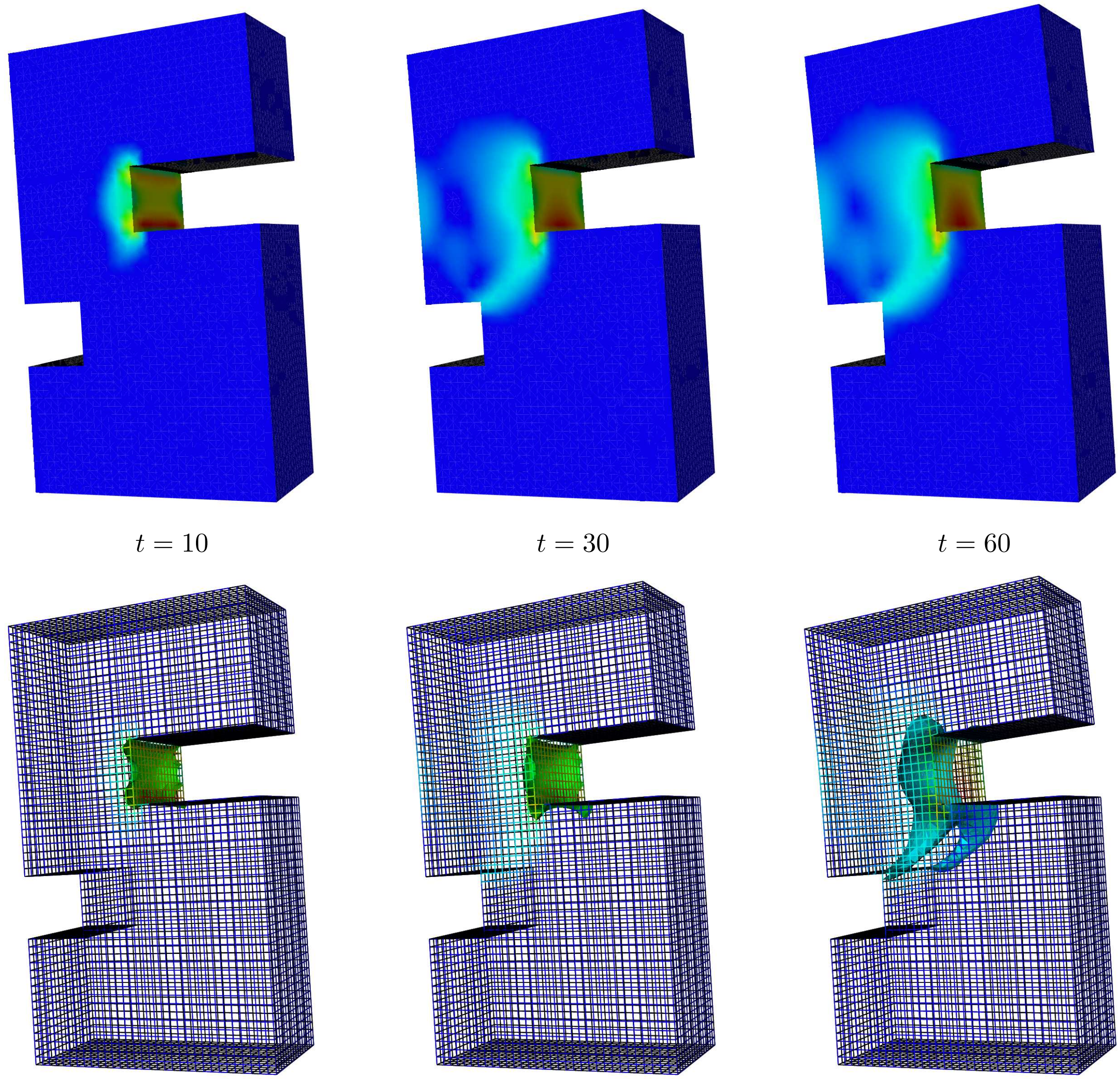


Figure 3: Distribution of the plastic strain on the boundary and on an isosurface for the nonlocal model ($L_c = 0.01$ and $H_0 = 0.001$).

Due to the reparametrization and the sufficiently small choice of the increment parameter $\Delta s$ we always observe local quadratic convergence of the nonlinear scheme; a typical convergence history is illustrated in Tab. 1.

**5.5. Evaluation of the parameter dependence.** Finally, we consider the dependence of the results on the non-dimensional hardening parameter $H_0$ and the plastic length scale $L_c$ in Fig. 4. Here, we observe clearly, that both—kinematic hardening and gradient plasticity—have a regularizing effect and can be used (for small parameters) to substitute the not well-posed model of perfect plasticity. Moreover, we observe stable convergence and approximation properties for all tested

| $k$ | $m$ | $\|r_{n,k,m}\|$ | $\max\{\phi(\boldsymbol{\sigma}^{n,k,m},0\}$ | $\|\boldsymbol{\varepsilon}_p^{n,k} - \boldsymbol{\varepsilon}_p^{n-1} - \gamma^{n,k} D\phi(\boldsymbol{\alpha}^{n,k})\|$ |
|---|---|---|---|---|
| 0 | 0 | 0.00000000 | 0.94854414 | 0.0000042244 |
| 1 | 1 | 11.53189802 | | |
| 1 | 2 | 0.11205330 | | |
| 1 | 3 | 0.00324081 | | |
| 1 | 4 | 0.00000032 | 0.12747705 | 0.0000035837 |
| 2 | 1 | 0.01711659 | | |
| 2 | 2 | 0.00003864 | | |
| 2 | 3 | $\varepsilon$ | 0.00001832 | $\varepsilon$ |
| 3 | 1 | 0.00000292 | | |
| 3 | 2 | $\varepsilon$ | $\varepsilon$ | $\varepsilon$ |

Table 1: Nonlinear convergence of the linearized projection method in loading step $n = 32$ on level 2 with 3200 cells. The iteration is started with the extrapolated solution from the previous loading step. Thus, the initial residual vanishes in the equilibrium equation. Here, we need 3 steps in the linearized projection method, where projection onto the linear half space requires 4,3, and 2 generalized Newton steps. The iteration is stopped for $\varepsilon < 10^{-9}$. Note that the algebraic error in this nonlinear test is far smaller than the approximation error of the finite element scheme.

parameters, so that the new model is now ready to use in the context of experimental data fitting or for the extension to geometrically nonlinear models. Both will be topics for further research.

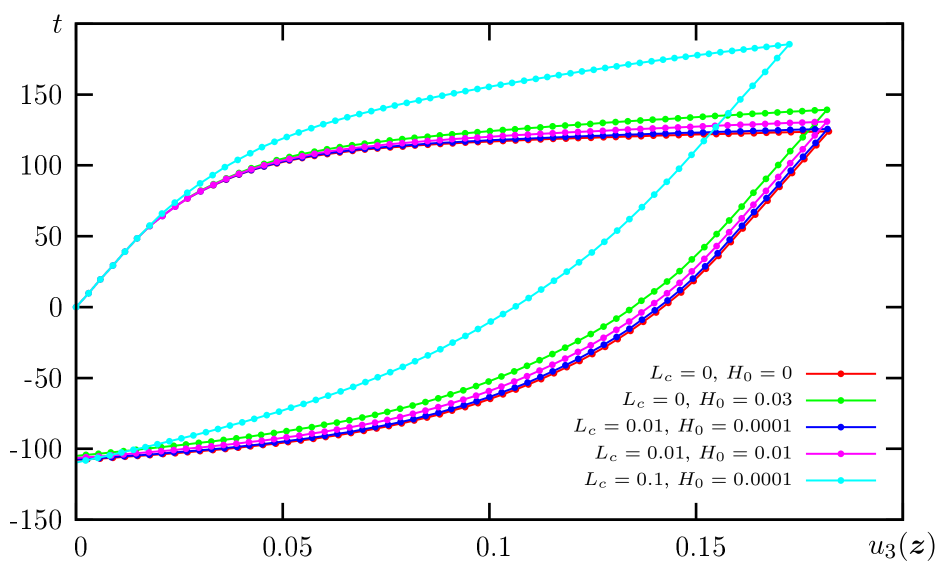


Figure 4: Load-displacement curves for the models perfect plasticity ($L_c = 0$ and $H_0 = 0$), kinematic hardening (local plasticity with $L_c = 0$), and the nonlocal model with three different parameters $(H_0, L_c)$. Here, we use 3200 cells on level 2, and again the displacement is given for the point $\boldsymbol{z} = (0, 0, 7)^T$.

Comparing with the results in Fig. 2 and Fig. 4 we observe in addition, that for coarse meshes the regularization effects caused by the FEM-discretization dominates over the regularization due to the material model; this indicates that the evaluation of regularization effects indeed require powerful numerical methods, otherwise it is not possible to separate clearly between discretization effects and material properties.

REMARK 11. *Our model does not feature plastic spin; this would required to consider non-symmetric plastic distortions* $\boldsymbol{p}$ *as well. The difficulty which one encounters for this extension is tied to the fact that coercivity is not obtained in known Hilbert spaces such as* $H(\mathrm{curl})$ *since only the symmetric part of the plastic distortion is an* $L_2(\Omega)$*-function a priori, so that a suitable analytical framework has to be developed.*

## References


[1] H. Alber, *Materials with Memory. Initial-Boundary Value Problems for Constitutive Equations with Internal Variables*, vol. 1682 of Lecture Notes in Mathematics, Springer, Berlin, 1998.

[2] J. Alberty, C. Carstensen, and D. Zarrabi, *Adaptive numerical analysis in primal elastoplasticity with hardening*, IMA Journal of Numerical Analysis, 171 (1999), pp. 175–204.

[3] G. Anzellotti and S. Luckhaus, *Dynamical evolution of elasto-perfectly plastic bodies*, Appl. Math. Optim., 15 (1987), pp. 121–140.

[4] K. Chełmiński, *Coercive approximation of viscoplasticity and plasticity*, Asymptotic Analysis, 26 (2001), pp. 105–133.

[5] ———, *Perfect plasticity as a zero relaxation limit of plasticity with isotropic hardening*, Math. Meth. Appl. Sci., 24 (2001), pp. 117–136.

[6] ———, *Global existence of weak-type solutions for models of monotone type in the theory of inelastic deformations*, Math. Meth. Appl. Sci., 25 (2002), pp. 1195–1230.

[7] R. de Borst and H. Mühlhaus, *Gradient-dependent plasticity: Formulation and algorithmic aspects*, Int. J. Num. Meth. Engrg., 35 (1992), pp. 521–539.

[8] J. Djoko, F. Ebobisse, A. McBride, and B. Reddy, *A discontinuous Galerkin formulation for classical and gradient plasticity. Part 1: Formulation and analysis*, Comp. Meth. Appl. Mech. Engrg., 196 (2007), pp. 3881–3897.

[9] N. Fleck, G. Müller, M. Ashby, and J. Hutchinson, *Strain gradient plasticity: theory and experiment*, Acta. Metall. Mater., 42 (1994), pp. 475–487.

[10] P. Gudmundson, *A unified treatment of strain gradient plasticity*, J. Mech. Phys. Solids, 52 (2004), pp. 1379–1406.

[11] M. Gurtin, *The linear theory of elasticity*, in Handbuch der Physik, S. Flügge, ed., vol. IVa/2, Springer, Heidelberg, 1972.

[12] ———, *On the plasticity of single crystals: free energy, microforces, plastic-strain gradients*, J. Mech. Phys. Solids, 48 (2000), pp. 989–1036.

[13] M. Gurtin and L. Anand, *A theory of strain-gradient plasticity for isotropic, plastically irrotational materials. Part I: Small deformations.*, J. Mech. Phys. Solids, 53 (2005), pp. 1624–1649.

[14] M. Gurtin and A. Needleman, *Boundary conditions in small-deformation, single crystal plasticity that account for the Burgers vector.*, J. Mech. Phys. Solids, 53 (2005), pp. 1–31.

[15] K. Hackl, A. Mielke, and D. Mittenhuber, *A mathematical framework for generalized standard materials in the rate-independent case.*, in Analysis and Simulation of Multifield Problems., W. Wendland, ed., vol. 12 of Lecture Notes in Applied and Computational Mechanics, Springer-Verlag, Berlin, 2003, pp. 87–100.

[16] W. Han and B. D. Reddy, *Plasticity: Mathematical Theory and Numerical Analysis*, Springer-Verlag Berlin, 1999.

[17] R. Klatte and B. Kummer, *Nonsmooth Equations in Optimization*, vol. 60 of Nonconvex Optimization and Its Applications, Kluwer, 1993.

[18] T. Liebe, A. Menzel, and P. Steinmann, *Theory and numerics of geometrically non-linear gradient plasticity*, Int. J. Engrg. Sci., 41 (2003), pp. 1603–1629.

[19] A. Mainik and A. Mielke, *Existence results for energetic models for rate-independent systems*, Calc. Var. Partial Differ. Equ., 22 (2005), pp. 73–99.

[20] J. Mayer, *A multilevel Crout ILU preconditioner with pivoting and row permutation*, Numer. Linear Algebra Appl., 14 (2007), pp. 771–789.

[21] ———, *Symmetric permutations for I-matrices to delay and avoid small pivots*, SIAM J. Sci. Comp., (2008). to appear.

[22] A. Menzel and P. Steinmann, *On the continuum formulation of higher gradient plasticity for single and polycrystals*, J. Mech. Phys. Solids, 48 (2000), pp. 1777–1796, Erratum 49, (2001), 1179–1180.

[23] A. Mielke, *Finite elastoplasticity, Lie groups and geodesics on SL(d).*, in Geometry, mechanics and dynamics. Volume in honour of the 60th birthday of J.E. Marsden., P. Newton, ed., Springer-Verlag, Berlin, 2002, pp. 61–90.

[24] ———, *A mathematical framework for generalized standard materials in the rate-independent case.*, in Multifield Problems in Solid and Fluid Mechanics., R. Helmig, A. Mielke, and B. Wohlmuth, eds., vol. 28 of Lecture Notes in Applied and Computational Mechanics, Springer-Verlag, Heidelberg, 2006, pp. 399–428.

[25] A. Mielke and S. Müller, *Lower semi-continuity and existence of minimizers in incremental finite-strain elastoplasticity*, Z. Angew. Math. u. Mech., 86 (2006), pp. 233–250.

[26] A. Mielke and A. Timofte, *An energetic material model for time-dependent ferroelectric behaviour: Existence and uniqueness.*, Math. Meth. Appl. Sci., 29 (2006), pp. 1393–1410.

[27] H. Mühlhaus and E. Aifantis, *A variational principle for gradient plasticity*, Int. J. Solids Struct., 28 (1991), pp. 845–857.

[28] P. Neff, *Remarks on invariant modelling in finite strain gradient plasticity*, to appear in: Technische Mechanik, (12/2007).

[29] ———, *Existence of minimizers for a finite-strain micromorphic elastic solid*, Preprint 2318, http://wwwbib.mathematik.tu-darmstadt.de/Math-Net/Preprints/Listen/pp04.html, Proc. Roy. Soc. Edinb. A, 136 (2006), pp. 997–1012.

[30] P. Neff and K. Chełmiński, *Infinitesimal elastic-plastic Cosserat micropolar theory. Modelling and global existence in the rate independent case.*, Preprint 2290, http://wwwbib.mathematik.tu-darmstadt.de/Math-Net/Preprints/Listen/pp03.html, Proc. Roy. Soc. Edinb. A, 135 (2005), pp. 1017–1039.

[31] P. Neff, K. Chełmiński, and H. Alber, *Notes on strain gradient plasticity. Finite strain covariant modelling and global existence in the infinitesimal rate-independent case*, Preprint 2503, http://wwwbib.mathematik.tu-darmstadt.de/Math-Net/Preprints/Listen/pp07.html, submitted to Math. Mod. Meth. Appl. Sci. (M3AS), (3/2007).

[32] P. Neff, K. Chełmiński, W. Müller, and C. Wieners, *A numerical solution method for an infinitesimal elastic-plastic Cosserat model*, Preprint 2470, http://wwwbib.mathematik.tu-darmstadt.de/Math-Net/Preprints/Listen/pp06.html, Math. Mod. Meth. Appl. Sci. (M3AS), 17 (2007), pp. 1211–1239.

[33] P. Neff and C. Wieners, *Comparison of models for finite plasticity. A numerical study*, Comput. Visual. Sci., 6 (2003), pp. 23–35.

[34] M. Ortiz, E. Repetto, and L. Stainier, *A theory of subgrain dislocation structures*, J. Mech. Phys. Solids, 48 (2000), pp. 2077–2114.

[35] B. Reddy, F. Ebobisse, and A. McBride, *Well-posedness of a model of strain gradient plasticity for plastically irrotational materials*, Int. J. Plasticity, 24 (2008), pp. 55–73.

[36] J. C. Simo and T. J. R. Hughes, *Computational Inelasticity*, Springer-Verlag Berlin, 1998.

[37] B. Svendsen, *Continuum thermodynamic models for crystal plasticity including the effects of geometrically necessary dislocations*, J. Mech. Phys. Solids, 50 (2002), pp. 1297–1329.

[38] ———, *On thermodynamic- and variational based formulations for inelastic continua with internal lengthscales*, Comp. Meth. Appl. Mech. Eng., 48-51 (2004), pp. 5429–5452.

[39] R. Temam, *A generalized Norton-Hoff model and the Prandtl-Reuss law of plasticity*, Arch. Rat. Mech. Anal., 95 (1986), pp. 137–183.

[40] C. Wieners, *Distributed point objects. A new concept for parallel finite elements*, in Domain Decomposition Methods in Science and Engineering, R. Kornhuber, R. Hoppe, J. Périaux, O. Pironneau, O. Widlund, and J. Xu, eds., vol. 40 of Lecture Notes in Computational Science and Engineering, Springer-Verlag Berlin, 2004, pp. 175–183.

[41] ———, *Nonlinear solution methods for infinitesimal perfect plasticity*, Z. Angew. Math. Mech. (ZAMM), 87 (2007), pp. 643–660.

[42] C. Wieners, M. Ammann, W. Ehlers, and T. Graf, *Parallel Krylov methods and the application to 3-d simulations of a tri-phasic porous media model in soil mechanics*, Comput. Mech., 36 (2005), pp. 409–420.

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