Variational Formulations for Scattering in a 3-Dimensional Acoustic Waveguide

Tilo Arens* Drossos Gintides† Armin Lechleiter‡

Revised, 9 July 2007

Abstract

Variational formulations for direct time-harmonic scattering problems in a three dimensional waveguide are formulated and analysed. We prove that the operators defined by the corresponding forms satisfy a Gårding inequality in adequately chosen spaces of test and trial functions and depend analytically on the wave number except at the modal numbers of the waveguide. It is also shown that these operators are strictly coercive if the wave number is small enough. It follows that these scattering problems are uniquely solvable except for possibly an infinite series of exceptional values of the wave number with no finite accumulation point. Furthermore, two geometric conditions for an obstacle are given, under which uniqueness of solution always holds in the case of a Dirichlet problem.

1 Introduction

The direct scattering problem in waveguides has received much attention in recent years due to increasing interest in developing new technologies for underwater acoustics. In [2], a mathematical formulation of the problem of underwater acoustics is given using the basic assumption that the acoustic behavior of an ocean with flat bottom is well described by a waveguide bounded by two planes. On one of these planes, a Dirichlet condition is assumed to hold, a Neumann condition on the other. This description serves as a model for a stratified ocean.

The problem of scattering of acoustic waves by obstacles in a waveguide with plane boundaries is considered by Gilbert and Xu in a series of papers [10–13] and in the references therein. In [22] a detailed analysis of this scattering problem is presented. It is proved that under certain geometric assumptions on a sound soft obstacle, i.e. an obstacle with a Dirichlet boundary condition, uniqueness of solution holds. Similar geometric conditions and methods of proof appear in uniqueness results for other types of boundary conditions on the two planar boundaries of the waveguide [18, 19].

For the sound hard obstacle, i.e. an obstacle with a Neumann boundary condition, uniqueness of solution is in general an open question and a geometric condition similar to that of the Dirichlet case is not available [22]. There are many papers proving the existence of non

*Institute for Algebra und Geometry, University of Karlsruhe, Karlsruhe, Germany

†Department of Mathematics, National Technical University of Athens, Athens, Greece

‡DFG Research Training Group 1294 "Analysis, Simulation and Design of Nanotechnological Processes", University of Karlsruhe, Karlsruhe, Germany

trivial waves for the corresponding homogeneous problem, called trapped modes [4, 6, 7, 16]. The solvability of the scattering problem can be proved via boundary integral equations and the limiting absorption principle [22].

The vast majority of the papers cited above consider the problem in a classical setting. Some, such as [22], consider weak solutions of the Helmholtz equation that are elements of certain Sobolev spaces, however no explicit variational formulations appear to have been analysed, yet. For the applications we have in mind, namely solving inverse problems for scattering in a waveguide by modern methods such as the Factorization method [15], well-posed variational formulations of the direct problem are a pre-requesite.

Thus, the goal of this paper is to give proper variational formulations of the waveguide problems. To this end, a finite cylindrical section of the waveguide containing the scattering obstacle is singled out. The Dirichlet-to-Neumann map for the cylindrical outer boundary, mapping Dirichlet traces of scattered fields onto their Neumann traces, is used to define a non local boundary condition. The operator is obtained via an expansion of the solution in cylindrical coordinates, a technique well established in the literature for various types of geometries (see e.g. [21]).

By establishing various properties of the Dirichlet-to-Neumann map, it is possible to show a Gårding inequality for the variational form of the scattering problem. We can thus conclude that uniqueness of solution implies existence and that, by analyticity properties of the problem, non-uniqueness may only hold for a countable sequence of wave numbers with no finite accumulation points. This result holds independently of the actual boundary condition on the scattering obstacle.

We furthermore analyse the uniqueness results obtained in [18, 19, 22] in the framework of the new variational formulation. We conclude that uniqueness of solution for the Dirichlet scattering problem always holds if the obstacle satisfies suitable geometric conditions. We give two such conditions and the corresponding proofs of unique solvability in the last section of the paper.

In the remainder of this section, we want to present some notation and basic results used throughout the paper. The direct scattering problems to be investigated are set in a three-dimensional waveguide $\mathbb{R}^2 \times (0, h)$ of height $h > 0$. As the third coordinate axis is singled out as the one orthogonal to the waveguide, we combine the first two coordinates, writing

$$x = (x_1, x_2, x_3)^\top = (\tilde{x}, x_3)^\top, \qquad x \in \mathbb{R}^3.$$

We will use this notation throughout the paper.

The upper and the lower boundary of the waveguide are denoted by

$$\Gamma^+ := \{x \in \mathbb{R}^3 : x_3 = h\} \qquad \text{and} \qquad \Gamma^- := \{x \in \mathbb{R}^3 : x_3 = 0\},$$

respectively. A bounded and impenetrable scatterer D is assumed to be compactly contained in the waveguide, and for notational reasons we place D around the x_3 axis, i.e., we assume that

$$D \cap \{x \in \mathbb{R}^3 : \tilde{x} = 0\} \neq \emptyset.$$

The part of the waveguide not occupied by $\overline{D}$ is denoted by $\Omega := \{\mathbb{R}^2 \times (0, h)\} \setminus \overline{D}$ and we always assume that Ω is connected. Let us already point out here that later on we often work in the bounded domain $\Omega_R := \{x \in \Omega : |\tilde{x}|^2 < R^2\}$, where the radius R is assumed to be large enough such that $1 + |\tilde{x}|^2 < R^2$ for all $x \in D$. This in particular implies that $\overline{D}$ is contained in the interior of Ω_R. The cylinder

$$C_R := \{x \in \Omega : |\tilde{x}|^2 = R^2\}$$

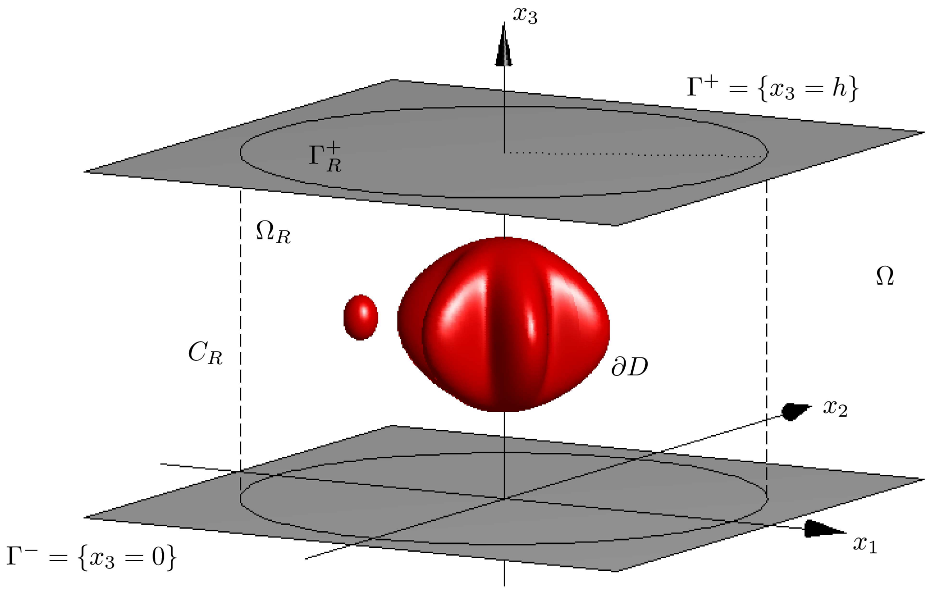


Figure 1: The obstacle D, consisting of possibly multiple disconnected components, located inside the waveguide Ω of height h intersects the x_3 axis and is contained in a cylinder of radius R around the x_3 axis.

denotes the part of the boundary of Ω_R that is contained in Ω. The two other parts of $\partial\Omega_R$, contained in the upper and lower boundary of the waveguide, are denoted as

$$\Gamma_R^+ := \{x \in \mathbb{R}^3 : |\tilde{x}| < R,\ x_3 = h\} \qquad \text{and} \qquad \Gamma_R^- := \{x \in \mathbb{R}^3 : |\tilde{x}| < R,\ x_3 = 0\},$$

respectively. We also will make use of the two-dimensional horizontal cross section

$$S_R := \{\tilde{x} \in \mathbb{R}^2 : |\tilde{x}| > R\}.$$

Figure 1 shows a diagram of the waveguide's geometry.

In the classical setting of an acoustic waveguide problem [22,23] the total field u is assumed to satisfy the Helmholtz equation

$$\Delta u + k^2 u = 0 \qquad \text{in } \Omega, \tag{1}$$

and sound soft and sound hard boundary conditions are imposed on the lower and upper boundaries of the waveguide, respectively,

$$u = 0 \quad \text{on } \Gamma^-, \qquad \text{and} \qquad \frac{\partial u}{\partial x_3} = 0 \quad \text{on } \Gamma^+. \tag{2}$$

The wave number k is often assumed to be strictly positive, however we will consider k such that $\mathrm{Re}(k) > 0$, $\mathrm{Im}(k) \geq 0$.

In the simplest case, a Dirichlet boundary condition is also imposed on the obstacle D,

$$u = 0 \qquad \text{on } \partial D. \tag{3}$$

Other possibilities which we will consider, are a Neumann condition

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial D, \tag{4}$$

or an impedance boundary condition

$$\frac{\partial u}{\partial \nu} - i\beta\, u = 0 \qquad \text{on } \partial D. \tag{5}$$

Here, and througout the paper, ν denotes the unit normal vector to ∂D directed into Ω. The quantity β termed the impedance is assumed to satisfy $\beta > 0$.

The classical direct scattering problem is then to find the scattered field $u^s = u - u^i$ subject to (1), (2) and one of the boundary conditions (3)–(5), where the given incident field u^i itself satisfies (1) and (2).

As in all scattering problems, a minimum requirement to ensure uniqueness of solution is that the scattered field additionally satisfies a radiation condition. In the case of a waveguide problem this is usually obtained by carrying out a separation of variables in $\tilde{x}$ and x_3, leading to an expansion of u^s,

$$u^s(x) = \sum_{n=1}^{\infty} \sin(\alpha_n x_3) u_n(\tilde{x}) \qquad \text{for } x \in \Omega \setminus \Omega_R, \tag{6}$$

with

$$\alpha_n := \frac{(2n-1)\,\pi}{2h}.$$

The field u^s then automatically satisfies the waveguide boundary conditions (2). The modes u_n are required to satisfy the two-dimensional Helmholtz equation

$$\Delta u_n + k_n^2 u_n = 0 \qquad \text{in } S_R \qquad \text{with } k_n := \sqrt{k^2 - \alpha_n^2}, \tag{7}$$

and the Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{1/2} \left(\frac{\partial u_n}{\partial r} - i k_n u_n \right) = 0, \qquad \text{where } r := |\tilde{x}|, \tag{8}$$

uniformly for all directions $\tilde{x}/r$.

In the case of a real wave number k, the modal wave numbers k_n are only real for a finite number of values of n, say $n \leq N$. These values correspond to modal frequencies of the waveguide and their number N is of course dependent on k. For $n > N$, the wave numbers k_n become purely imaginary, corresponding to exponentially decaying modes.

It may also be possible that $k_n = 0$ for some $n \in \mathbb{N}$ and this corresponds to an exceptional frequency. In this paper we will only state solvability results on the waveguide problem in case of absence of exceptional frequencies, i.e. we will assume throughout the arguments that

$$k \neq \alpha_n, \qquad n \in \mathbb{N}.$$

Possible exceptional frequencies of the problem impact our analysis nevertheless, as we show in detail in the sequel.

In the case of a wave number with positive imaginary part, the square root in the definition of k_n in (7) is to be understood in the sense of an analytic extension of the square root function to the complex plane. A branch cut γ can be chosen along the negative imaginary axis, such that $\sqrt{\cdot}$ is analytic in $\mathbb{C} \setminus \gamma$. It then follows that all k_n are well defined with $\arg(k_n) \in (0, \pi/2)$ and that $|k_n| \geq \rho > 0$ for all $n \in \mathbb{N}$.

2 The Dirichlet-to-Neumann map

In what is to follow, we will consider solutions to the waveguide scattering problem consisiting of (1), (2) and one of the boundary conditions (3)-(5) in a weak sense. As a technical prerequisite, we need to assume from here on that the obstacle D is a Lipschitz domain, see, e.g., [17]. Formally using Green's first identity for a solution u of the Helmholtz equation (1) we find that

$$\int_{\Omega} \left(\nabla u \nabla \overline{v} - k^2 u \overline{v}\right) \, dx = 0 \qquad \text{for all } v \in C_0^{\infty}(\Omega). \tag{9}$$

The set $C_0^{\infty}(\Omega)$ of compactly supported smooth functions in Ω is usually referred to as the set of testfunctions. The weak formulation of the direct scattering problem is now defined as follows: Find $u \in H^1_{\mathrm{loc}}(\Omega)$, the space of functions belonging to $H^1(K)$ for any compact set $K \subset \overline{\Omega}$, such that (9) holds. Moreover, u has to satisfy the waveguide boundary conditions (1) and (2) as well as one of the boundary conditions (3)-(5) on the obstacle in the trace sense. We need to impose also in this weak formulation the radiation conditions (8) on $u^s = u - u^i$. This makes indeed sense, as it follows from interior regularity results [9,17] that u^s is a smooth function in Ω.

The stated weak formulation is difficult to tackle directly. This is partially due to a lack of symmetry: The space of testfunctions differs from the solution space. It is therefore a standard procedure to proceed with a truncation of the domain by an artificial boundary, combined with the construction of a non local boundary condition on the artificial boundary, using the *Dirichlet-to-Neumann map*. This idea is of course well established, however, in contrast to the two dimensional case mapping properties of this operator are far from trivial. We carefully derive in the sequel that the Dirichlet-to-Neumann operator is bounded between suitable Sobolev spaces on the artificial boundary and afterwards, in the next section, set up a proper variational formulation. Uniqueness of solution of this formulation is investigated and established and we also show in Proposition 3.2 that each solution of this variational formulation can be extended to a solution of the weak formulation stated above and vice versa.

A key problem in setting up a variational formulation of the scattering problem in a three-dimensional waveguide is an appropriate choice of function spaces. Considering functions in $H^1(\Omega_R)$ satisfying a Dirichlet boundary condition on Γ_R^- is one possible approach. Here we choose a seemingly more complicated method via periodic extensions in the x_3 direction. The benefit is that we obtain certain expansions of the traces of the solutions on C_R for free which are directly useful in the formulation of the Dirichlet-to-Neumann map.

As a first observation, any function u continuous in $\mathbb{R}^2 \times [0,h]$ and satisfying (2) can be extended by

$$u(\tilde{x}, x_3) := \begin{cases} u(\tilde{x}, 2h - x_3), & x_3 \in (h, 2h], \\ -u(\tilde{x}, x_3 - 2h), & x_3 \in (2h, 3h], \\ -u(\tilde{x}, 4h - x_3), & x_3 \in [3h, 4h), \end{cases} \tag{10}$$

and by periodic extension with period $4h$ in the x_3 direction to a continuous function on $\mathbb{R}^3$. Taking account of the obstacle D, we define

$$M_R := \{x \in \mathbb{R}^2 \times (0, 4h) : \text{ either } x, \ (\tilde{x}, 2h - x_3), \ (\tilde{x}, x_3 - 2h) \text{ or } (\tilde{x}, 4h - x_3) \in \Omega_R\}.$$

Further denoting by $C^1_{\mathrm{per}}(M_R)$ the space of continuously differentiable functions on M_R that can be extended to $4h$-periodic continuously differentiable functions with respect to x_3 and setting

$$\|u\|_{1;M_R} := \int_{M_R} \left(|\nabla u|^2 + |u|^2\right)^{1/2} \, dx,$$

we obtain the Sobolev space $H^1_{\text{per}}(M_R)$ as the closure of $C^1_{\text{per}}(M_R)$ in norm $\|\cdot\|_{1;M_R}$. It is also a Hilbert space with the usual H^1 inner product. The set of functions in $H^1_{\text{per}}(M_R)$ satisfying (10) almost everywhere forms a closed subspace of $H^1_{\text{per}}(M_R)$.

The solution space for the variational formulation can now be defined as

$$W := \{u \in H^1(\Omega_R) : u = v|_{\Omega_R} \text{ for some } v \in H^1_{\text{per}}(M_R) \text{ and } v \text{ satisfies (10) a.e.}\}. \tag{11}$$

From the above considerations it is clear that W is the closed subspace of $H^1(\Omega_R)$ characterized by $u|_{\Gamma_R^-} = 0$ for all $u \in W$.

We now employ the theory of periodic Sobolev spaces to obtain expansion representations of the traces on C_R of functions in W. The space of traces on $\tilde{C}_R := \{x \in \partial M_R : |\tilde{x}| = R\}$ of functions in $H^1_{\text{per}}(M_R)$ is

$$H^{1/2}_{\text{per}}(\tilde{C}_R) = \Bigg\{ u(x) = \sum_{n,l\in\mathbb{Z}} U^n_l \, \exp(il\varphi) \, \exp\left(i \, \frac{2\pi n}{4h} \, x_3\right) : \\ \sum_{n,l\in\mathbb{Z}} (1 + n^2 + l^2)^{1/2} \, |U^n_l|^2 < \infty \Bigg\},$$

where we have set $x = (R\cos\varphi, R\sin\varphi, x_3)^\top$. A norm for this space is given by

$$\|u\|_{H^{1/2}_{\text{per}}(\tilde{C}_R)} := \left(\sum_{n,l\in\mathbb{Z}} (1 + n^2 + l^2)^{1/2} \, |U^n_l|^2 \right)^{1/2}.$$

Again exploiting (10), we obtain the trace space for functions in W,

$$V := \left\{ v = u|_{C_R} : u \in H^{1/2}_{\text{per}}(\tilde{C}_R), \; U^{2m}_l = 0, \; U^{2m-1}_l = -U^{-2m+1}_l, \; m \in \mathbb{N}, l \in \mathbb{Z} \right\},$$

i.e. $u|_{C_R} \in V$ for all $u \in W$ and the map $u \mapsto u|_{C_R}$ is bounded. Rather than using this definition directly, we insert the relations for the coefficients in the expansion and obtain

$$V = \Bigg\{ v(x) = \sum_{n=1}^{\infty} \sum_{l\in\mathbb{Z}} V^n_l \, \exp(il\varphi) \, \sin(\alpha_n x_3) : \\ \sum_{n=1}^{\infty} \sum_{l\in\mathbb{Z}} (1 + l^2 + \alpha_n^2)^{1/2} |V^n_l|^2 < \infty \Bigg\},$$

again using $x = (R\cos\varphi, R\sin\varphi, x_3)^\top$. This space is also a Hilbert space with the inner product

$$(u, v)_V := \sum_{n=1}^{\infty} \sum_{l\in\mathbb{Z}} (1 + l^2 + \alpha_n^2)^{1/2} \, U^n_l \, \overline{V^n_l}.$$

An element v of the dual space V' of V can be written formally as a series

$$v(x) = \sum_{n=1}^{\infty} \sum_{l\in\mathbb{Z}} V^n_l \, \exp(il\varphi) \, \sin(\alpha_n x_3), \qquad x = (R\cos\varphi, R\sin\varphi, x_3)^\top,$$

and the condition

$$\sum_{n=1}^{\infty}\sum_{l\in\mathbb{Z}}(1+l^2+\alpha_n^2)^{-1/2}|V_l^n|^2<\infty$$

assures that v is indeed a *bounded* linear form on V. The application of $v \in V'$ to $u \in V$ is then expressed as

$$\int_{C_R}\overline{u}\,v\,ds := \pi hR\sum_{n=1}^{\infty}\sum_{l\in\mathbb{Z}}\overline{U_l^n}\,V_l^n\,.$$

This definition is consistent with the usual definition of the surface integral when both u and v are such that $\overline{u}\,v$ is an integrable function on C_R.

We now turn to derive a representation of the Dirichlet-to-Neumann map on the surface C_R. A classical approach to the waveguide problem is to perform a separation of variables in cylindrical coordinates. Application of the boundary conditions (2) and the radiation conditions (8) leads to a representation of the form already encountered in the definition of V. The result is that the solution can be written as a series,

$$u(x)=\sum_{n=1}^{\infty}\sum_{l\in\mathbb{Z}}U_l^n\,H_l^{(1)}(k_nr)\,\exp(il\varphi)\,\sin(\alpha_nx_3) \tag{12}$$

for $x=(r\cos\varphi,r\sin\varphi,x_3)^\top\in\Omega_R$. Here $H_l^{(1)}$ denotes the Hankel function of the first kind and of order l. In this expression, the normal derivative on C_R can be computed,

$$\frac{\partial u}{\partial r}(x)=\sum_{n=1}^{\infty}\sum_{l\in\mathbb{Z}}k_n\,U_l^n\,H_l^{(1)'}(k_nR)\,\exp(il\varphi)\,\sin(\alpha_nx_3),$$
$$x=(R\cos\varphi,R\sin\varphi,x_3)\in C_R.$$

From this consideration, we obtain a formal definition of the Dirichlet-to-Neumann map on the space V. For $u\in V$ with Fourier coefficients U_l^n, set

$$\Lambda u(x):=\sum_{\substack{n=1\\ \alpha_n\neq k}}^{\infty}\sum_{l\in\mathbb{Z}}k_n\,U_l^n\,\frac{H_l^{(1)'}(k_nR)}{H_l^{(1)}(k_nR)}\,\exp(il\varphi)\,\sin(\alpha_nx_3). \tag{13}$$

This formal definition indeed makes sense for $k\neq 0$, $\arg(k)\in[0,\pi/2)$, as $\mathrm{Im}(k_n)\geq 0$ in this case and the modulus of the Hankel function hence strictly positive (see (A.2) in the appendix).

From the relation $H_{-l}^{(1)}(z)=(-1)^lH_l^{(1)}(z)$, we obtain that

$$\frac{H_l^{(1)'}(k_nR)}{H_l^{(1)}(k_nR)}=\frac{H_{-l}^{(1)'}(k_nR)}{H_{-l}^{(1)}(k_nR)}\,.$$

This motivates the definition of the auxiliary coefficients

$$W_l^n:=\begin{cases}U_l^n, & l=0\\ (|U_l^n|^2+|U_{-l}^n|^2)^{1/2}, & l\neq 0\end{cases}.$$

We will now address the issue of well-definedness of the operator Λ formally given in (13).

Lemma 2.1 *The Dirichlet-to-Neumann operator* Λ *defined by* (13) *is a bounded operator from* V *to* V' *for all* $k \in \mathbb{C}$ *such that* $|k| > 0$, $\arg(k) \in [0, \pi/2)$.

Proof: For $u \in V$ with Fourier coefficients U_l^n there holds

$$\begin{aligned}\|\Lambda u\|_{V'}^2 &= \pi h R \sum_{\substack{n=1\\ \alpha_n \neq k}}^{\infty} \sum_{l\in\mathbb{Z}} (1+l^2+\alpha_n^2)^{-1/2} \left| k_n\, U_l^n \, \frac{{H_l^{(1)}}'(k_n R)}{H_l^{(1)}(k_n R)} \right|^2 \\ &= \pi h R \sum_{\substack{n=1\\ \alpha_n \neq k}}^{\infty} \sum_{l=0}^{\infty} (1+l^2+\alpha_n^2)^{-1/2} \left| k_n\, W_l^n \, \frac{{H_l^{(1)}}'(k_n R)}{H_l^{(1)}(k_n R)} \right|^2 .\end{aligned}$$

From the relation (A.3) in the appendix, we obtain the estimate

$$\|\Lambda u\|_{V'}^2 \leq 2\pi h R \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} (1+l^2+\alpha_n^2)^{-1/2} |k_n\, W_l^n|^2 \left(\left| \frac{H_{l-1}^{(1)}(k_n R)}{H_l^{(1)}(k_n R)} \right|^2 + \frac{l^2}{|k_n|^2 R^2} \right)$$

The sum on the right hand side will now be split into two sums which we estimate separately. First,

$$\begin{aligned}&\sum_{\substack{n=1\\ \alpha_n \neq k}}^{\infty} \sum_{l=0}^{\infty} (1+l^2+\alpha_n^2)^{-1/2} \, |W_l^n k_n|^2 \, \frac{l^2}{|k_n|^2 R^2} \\ &\qquad \leq \frac{1}{R^2} \sum_{n=1}^{\infty} \sum_{l\in\mathbb{Z}} (1+l^2+\alpha_n^2)^{-1/2} \, l^2 \, |U_l^n|^2 \leq \frac{1}{R^2} \, \|u\|_V^2 .\end{aligned}$$

Second, as $\alpha_n = O(n)$, the estimate

$$|k_n|^2 = |\alpha_n^2 - k^2| \leq C\, \alpha_n^2$$

holds. Also note that $k_n R$ is bounded away from zero for all $n \in \{m : k \neq \alpha_m\}$. Hence Lemma A.2 and (A.10) in the appendix yield that for these terms the fraction of Hankel functions is bounded uniformly. Thus

$$\begin{aligned}&\sum_{n=1}^{\infty} \sum_{l=0}^{\infty} (1+l^2+\alpha_n^2)^{-1/2} |k_n\, W_l^n|^2 \left| \frac{H_{l-1}^{(1)}(k_n R)}{H_l^{(1)}(k_n R)} \right|^2 \\ &\qquad \leq C \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} (1+l^2+\alpha_n^2)^{-1/2} \, \alpha_n^2 \, |W_l^n|^2 \\ &\qquad\qquad \leq C \sum_{n=1}^{\infty} \sum_{l\in\mathbb{Z}} (1+l^2+\alpha_n^2)^{1/2} |U_l^n|^2 = C \, \|u\|_V^2 .\end{aligned}$$

This final estimate completes the proof. ∎

The second result addresses the issue of bounding a certain form associated with Λ from below. This estimate is valid for all wave numbers that are small enough.

Lemma 2.2 *There exists $k_0 > 0$ and $c > 0$ such that for all $k \in (0, k_0]$ the estimate*

$$-\int_{C_R} \overline{u}\, \Lambda u\, ds \geq c\, k \sum_{n=1}^{\infty} \sum_{l \in \mathbb{Z}} |U_l^n|^2 = c\, k\, \|u\|^2_{L^2(C_R)}, \qquad u \in V,$$

holds, where U_l^n denote the Fourier coefficients of $u \in V$.

Proof: We suppose that $k_0 > 0$ be so small that all k_n are purely imaginary for $0 < k \leq k_0$. For $u \in V$ with Fourier coefficients U_l^n, we start from the representation

$$-\int_{C_R} \overline{u}\, \Lambda u\, ds = -\pi h R \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} k_n \frac{H_l^{(1)'}(k_n R)}{H_l^{(1)}(k_n R)} |W_l^n|^2.$$

Using the recurrence relation (A.3) for the fraction of Hankel functions, we obtain

$$-\int_{C_R} \overline{u}\, \Lambda u\, ds = -\pi h R \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} k_n \left(\frac{H_{l-1}^{(1)}(k_n R)}{H_l^{(1)}(k_n R)} - \frac{l}{k_n R} \right) |W_l^n|^2.$$

We can then replace the fraction of Hankel functions by a fraction of modified Bessel functions (compare (A.6)) to obtain

$$-\int_{C_R} \overline{u}\, \Lambda u\, ds = \pi h R \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} \left(|k_n| \frac{K_{l-1}(|k_n| R)}{K_l(|k_n| R)} + \frac{l}{R} \right) |W_l^n|^2.$$

Using Lemma A.1 in the appendix yields

$$\begin{aligned}-\int_{C_R} \overline{u}\, \Lambda u\, ds \geq \pi h R \sum_{n=1}^{\infty} \Bigg[|k_n|\, |W_0^n|^2 + |k_n| \left(1 - \frac{1}{|k_n| R} \right) |W_1^n|^2 \\ + \sum_{l=1}^{\infty} |k_n| \left(\frac{|k_n|\, R}{|k_n|\, R + 2l} + \frac{l}{|k_n|\, R} \right) |W_l^n|^2 \Bigg].\end{aligned}$$

The binomial formula implies

$$\frac{|k_n|\, R}{|k_n|\, R + 2l} + \frac{l}{|k_n|\, R} \geq 2 \sqrt{\frac{l}{|k_n|\, R + 2l}} \geq c\, \frac{1}{\sqrt{|k_n|}}.$$

Now, observe that for $k \leq k_0$

$$|k_n|^2 = \alpha_n^2 - k^2 = k^2 \left(\frac{\alpha_n^2}{k^2} - 1 \right) \geq k^2 \left(\frac{\alpha_n^2}{k_0^2} - 1 \right) \geq c^2\, k^2,$$

and also $\sqrt{k} \geq k$ for $k \leq k_0 < 1$. Hence, we obtain

$$-\int_{C_R} \overline{u}\, \Lambda u\, ds \geq c\, k \sum_{n=1}^{\infty} \sum_{l \in \mathbb{Z}} |U_l^n|^2.$$

This is the assertion. ∎

In the case of larger wave numbers a similar bound appears not to hold. However, it is possible to give a negative bound for the real part of the form in this case.

Lemma 2.3 *Suppose $k \in \mathbb{R}_{>0}$. Then there exists $C > 0$ such that for all $u \in V$ with Fourier coefficients U_l^n, there holds*

$$\operatorname{Re}\left(-\int_{C_R} \overline{u}\,\Lambda u\,ds\right) \geq -C \sum_{n=1}^{N} \sum_{l\in\mathbb{Z}} |U_l^n|^2 \geq -C\,\|u\|^2_{L^2(C_R)},$$

where N is such that k_n is real for $n \leq N$ and imaginary for $n > N$.

Proof: Using the coefficients W_l^n and equation (A.3) from the appendix, we write

$$-\int_{C_R} \overline{u}\,\Lambda u\,ds = \pi h R \sum_{n=1}^{\infty}\sum_{l=0}^{\infty} k_n |W_l^n|^2 \left(\frac{l}{k_n R} - \frac{H^{(1)}_{l-1}(k_n R)}{H^{(1)}_{l}(k_n R)}\right).$$

We can argue as in the proof of Lemma 2.2 that all terms in this series for $n > N$ are real and positive. Hence

$$\begin{aligned}\operatorname{Re}\left(-\int_{C_R} \overline{u}\,\Lambda u\,ds\right) &\geq \pi h R\,\operatorname{Re}\left[\sum_{n=1}^{N}\sum_{l=0}^{\infty} k_n |W_l^n|^2 \left(\frac{l}{k_n R} - \frac{H^{(1)}_{l-1}(k_n R)}{H^{(1)}_{l}(k_n R)}\right)\right]\\ &\geq -\pi h \sum_{n=1}^{N}\sum_{l=0}^{\infty} k_n |W_l^n|^2 \left|\frac{H^{(1)}_{l-1}(k_n R)}{H^{(1)}_{l}(k_n R)}\right|.\end{aligned}$$

Because of the assumption that $\alpha_n \neq k$ for all $n \in \mathbb{N}$, the numbers $k_n R$, $n = 1, \ldots, N$, form a compact set bounded away from zero. Applying Lemma A.2 and (A.10) from the appendix now yields the assertion. ■

Lemma 2.4 *Let k_* be any positive number such that $k_* \neq \alpha_n$ for all $n \in \mathbb{N}$. Then there exists a neighborhood U of k_* in which the Dirichlet-to-Neumann operator depends analytically on the wave number $k \in U$.*

Proof: Let us first recall that for u, $v \in V$

$$\int_{C_R} \overline{v}\Lambda u\,dx = \pi h R \sum_{n=1}^{\infty} k_n \sum_{l\in\mathbb{Z}} \frac{{H_l^{(1)}}'(k_n R)}{H_l^{(1)}(k_n R)} U_l^n \overline{V}_l^n. \tag{14}$$

By the equivalence of weak and strong analyticity of a bounded linear operator [20, Theorem 8.12] it is sufficient to show that the latter expression depends analytically on k in some neighborhood of k_*. The number

$$k_n = \sqrt{k^2 - \frac{(2n-1)^2\pi^2}{4h^2}}$$

is again defined using the analytic extension of the square root to the complex plane with branch cut $[0, -i\infty)$ along the negative imaginary axis. Thus, k_n depends analytically on k except for certain curved branches in $\{z \in \mathbb{C}, \operatorname{Re} z > 0, \operatorname{Im} z < 0\}$ which end at the points α_n, respectively. By definition of k_* there exists an open ball U with center k_* in which we can expand k_n in an absolutely and uniformly convergent power series,

$$k_n(k) = \sum_{j=0}^{\infty} c_j^n (k - k_*)^j \qquad k \in U,\ c_j^n \in \mathbb{C}. \tag{15}$$

Next we define

$$f_n(k,u,v) := \sum_{l\in\mathbb{Z}} \frac{H_l^{(1)\prime}(k_nR)}{H_l^{(1)}(k_nR)} U_l^n V_l^n = \sum_{l\in\mathbb{Z}} \left[\frac{H_{l-1}^{(1)}(k_nR)}{H_l^{(1)}(k_nR)} - \frac{l}{k_nR}\right] U_l^n V_l^n$$

and observe that

$$k_n f_n(k,u,v) = k_n \sum_{l\in\mathbb{Z}} \frac{H_{l-1}^{(1)}(k_nR)}{H_l^{(1)}(k_nR)} U_l^n V_l^n + A_n(u,v) \tag{16}$$

can be written as an analytic part in series form and a part A_n which is a bounded bilinear form independent of k. The series in (16) converges absolutely and uniformly, as

$$\sum_{|l|\geq L} \left|\frac{H_{l-1}^{(1)}(k_nR)}{H_l^{(1)}(k_nR)}\right| |U_l^n V_l^n| \leq C \sum_{|l|\geq L} |U_l^n V_l^n| \leq \frac{C}{(1+L^2)^{1/2}} \|u\|_V \|v\|_V.$$

To show analyticity we note that each term $H_{l-1}^{(1)}(k_nR)/H_l^{(1)}(k_nR)$ is analytic in the neighborhood U of k_*. Thus, exploiting its uniform convergence, we conclude that the series in (16) is analytic in k. Therefore we can write

$$\sum_{l\in\mathbb{Z}} \frac{H_{l-1}^{(1)}(k_nR)}{H_l^{(1)}(k_nR)} U_l^n V_l^n = \sum_{j=0}^{\infty} C_j^n(u,v)(k-k_*)^j \qquad k\in U, \tag{17}$$

where the C_j^n are bounded sesquilinear forms on $V\times V$ and the series is absolutely convergent in U.

To conclude we remark that one shows as in the proof of Lemma 2.1 that

$$\sum_{n=1}^{\infty} |k_n|\,|f_n(k,u,v)| \leq C\|u\|_V\|v\|_V$$

and therefore the series in (14) is absolutely convergent. Thus, we can plug in (15) and (17) in (14) and interchange the summation, yielding

$$\int_{C_R} \overline{v}\Lambda u\,dx = \sum_{j,l=0}^{\infty} (k-k_*)^{j+l} \sum_{n=1}^{\infty} c_j^n C_l^n(u,v) + \sum_{n=1}^{\infty} A_n(u,v),$$

and the statement of the lemma follows. ∎

Corollary 2.5 *Let $0 < k < K$ such that $k,\ K \neq \alpha_n,\ n\in\mathbb{N}$. Then there exists an open connected set U including k and K such that Λ is an analytic operator function in U.*

Remark 2.6 Applying these results on analytic dependence of Λ on k makes it possible to extend the estimate of Lemma 2.3 to k such that $\mathrm{Im}(k) > 0$. It follows that there exists a function $\delta : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that, defining the set

$$U := \{k\in\mathbb{C} : \mathrm{Re}(k) > 0,\ 0 < \mathrm{Im}(k) < \delta(\mathrm{Re}(k))\},$$

the assertion of Lemma 2.3 remains valid for $k\in U$. ∎

3 Variational Formulations

We want to investigate the Dirichlet scattering problem in a variational form, using the Hilbert space W from the previous section. Taking account of the Dirichlet boundary conditions on the obstacle D leads to the closed subspace

$$W_0 := \{u \in W : u = 0 \text{ on } \partial D\}\,.$$

Suppose now we are given a weak solution u of the Dirichlet scattering problem, i.e., $u \in H^1_{\mathrm{loc}}(\Omega)$ solves (9), (2), (3) and $u^s = u - u^i$ satisfies the radiation conditions (8). We remark that $u|_{\Omega_R}$ belongs to W_0. Multiplying the equation $\Delta u + k^2 u = 0$ in Ω_R by a test function $v \in W_0$ and an application of Green's first identity yields

$$\begin{aligned} \int_{\Omega_R} \left(\nabla u \nabla \overline{v} - k^2 u \overline{v}\right) dx &= \int_{C_R} \frac{\partial u}{\partial \nu}\, \overline{v}\, ds + \int_{\Gamma_R^+ \cup \Gamma_R^-} \frac{\partial u}{\partial \nu}\, \overline{v}\, ds - \int_{\partial D} \frac{\partial u}{\partial \nu}\, \overline{v}\, ds \\ &= \int_{C_R} \frac{\partial u}{\partial \nu}\, \overline{v}\, ds, \end{aligned} \tag{18}$$

since all other boundary terms drop out either by construction of W_0 or by definition of the direct problem. Here ν denotes the unit normal to the respective surfaces pointing out of D in the case of ∂D and out of Ω_R in the case of all other surfaces. We put emphasis on the fact that the boundary integrals over C_R are to be interpreted in the sense of a dual pairing between V and V'.

The Dirichlet-to-Neumann operator, introduced in Section 2, gives us the tool to state a variational formulation of our scattering problem. Since $u^s = u - u^i$ is a radiating solution, we find that

$$\frac{\partial u}{\partial \nu} = \frac{\partial}{\partial \nu}\left(u - u^i\right) + \frac{\partial u^i}{\partial \nu} = \Lambda(u - u^i) + \frac{\partial u^i}{\partial \nu}$$

on C_R. Consequently, we have

$$\int_{\Omega_R} \left(\nabla u \nabla \overline{v} - k^2 u \overline{v}\right)\, dx - \int_{C_R} \overline{v}\, \Lambda u\, ds = \int_{C_R} \overline{v} \left(\frac{\partial u^i}{\partial \nu} - \Lambda u^i\right) ds \tag{19}$$

and we can state the following variational formulation of the scattering problem for a sound soft obstacle corresponding to (1)–(3) and the radiation condition (6)–(8), where we set

$$\mathcal{B}_{\mathrm{D}}(u, v) := \int_{\Omega_R} \left(\nabla u \nabla \overline{v} - k^2 u \overline{v}\right) dx - \int_{C_R} \overline{v} \Lambda u\, ds. \tag{20}$$

Problem 3.1 (Dirichlet Scattering Problem) *Given an incident field u^i satisfying* (1) *and* (2), *find $u \in W_0$ such that*

$$\mathcal{B}_{\mathrm{D}}(u, v) = \int_{C_R} \overline{v} \left(\frac{\partial u^i}{\partial \nu} - \Lambda u^i\right) ds \qquad \text{for all } v \in W_0.$$

Before investigating solvability of the variational problem we want to point out the equivalence between the variational and the weak formulation of the Dirichlet scattering problem, see Equation (9).

Proposition 3.2 *The restriction of any solution of the weak formulation of the Dirichlet scattering problem is a solution of the variational formulation in Problem* (3.1). *Vice versa, any solution of Problem* (3.1) *can be extended in a unique way to a solution of the weak formulation of the Dirichlet scattering problem in all of Ω.*

Proof: The first claim of the proposition is clear by construction of the variational formulation using a weak solution of the scattering problem.

Concerning the second claim, we extend a given solution $u \in W_0$ of Problem (3.1) by the following ansatz to all of Ω

$$u(x) = \sum_{n=1}^{\infty} \sum_{l \in \mathbb{Z}} U_l^n \, H_l^{(1)}(k_n \, \rho) \exp(il\varphi) \, \sin(\alpha_n x_3), \tag{21}$$

for $x = (\rho \cos \varphi, \rho \sin \varphi, x_3) \in \Omega \setminus \overline{\Omega_R}$ and U_l^n being the Fourier coefficients of $u|_{C_R}$ that we already encountered in the definition (13) of Λ. The extension u is a radiating continuation of u in the exterior of Ω_R that satisfies the waveguide boundary conditions. It remains to show that the extension fits to u at C_R, i.e., that u belongs to $H^1(U)$ for some neighborhood U of C_R in Ω and solves the Helmholtz equation weakly in the sense of (9). Therefore we observe by (12) that the series extension (21) holds in an entire neighborhood U of C_R. We remark that convergence of the series in $H^1(U)$ can be shown in the same way as we proved boundedness of the Dirichlet-to-Neumann operator. The proof is then completed by the observation that the basis functions $(H_l^{(1)}(k_n \, \rho) \exp(il\varphi) \, \sin(\alpha_n x_3))_{k,l,n}$ solve the Helmholtz equation classically and a final limit process. ■

In the subsequent arguments, we will analyse solvability of the variational problem posed in Problem (3.1). As a first step, we will show that the form satisfies a Gårding inequality, implying by the Fredholm alternative that the variational problem is solvable whenever there is at most one solution.

Lemma 3.3 *For every $s \in (1/2, 1)$, there exists a constant $C > 0$, dependent on k, such that*

$$\operatorname{Re} \mathcal{B}_{\mathrm{D}}(u,u) \geq \|u\|_W^2 - C \, \|u\|_{H^s(\Omega_R)}^2 \,, \qquad u \in W_0.$$

Proof: We rewrite

$$\mathcal{B}_{\mathrm{D}}(u,v) = \int_{\Omega_R} (\nabla u \nabla \overline{v} + u\overline{v}) \, dx - \int_{\Omega_R} (k^2+1) u\overline{v} \, dx - \int_{C_R} \overline{v} \Lambda u \, ds,$$

and immediately conclude

$$\operatorname{Re} \mathcal{B}_{\mathrm{D}}(u,u) \geq \|u\|_W^2 - (k^2+1) \, \|u\|_{L^2(\Omega_R)}^2 - \operatorname{Re} \int_{C_R} \overline{u} \Lambda u \, ds.$$

Lemma 2.3 implies

$$-\operatorname{Re} \int_{C_R} \overline{u} \Lambda u \, ds \geq -C \, \|u\|_{L^2(C_R)}^2 \geq -C \, \|u\|_{H^s(\Omega_R)}^2$$

for any $s \in (1/2, 1)$ by the trace theorem. This completes the proof. ■

Corollary 3.4 *Suppose that the variational Dirichlet problem 3.1 posseses at most one solution in W_0 for any incident field u^i. Then there exists a unique solution $u \in W_0$ for any incident field u^i.*

Proof: As W_0 is compactly imbedded in the space $\overline{W_0}^{\|\cdot\|_{H^s(\Omega_R)}} \subseteq H^s(\Omega_R)$ for any $s \in (1/2, 1)$, the operator associated with $\mathcal{B}_D(\cdot,\cdot)$ is the sum of a coercive and a compact operator. Hence the Fredholm Alternative implies solvability for any right hand side, whenever uniqueness of solution holds. ■

Theorem 3.5 *Suppose that* $\mathrm{Re}(k)$, $\mathrm{Im}(k) > 0$. *Then the Dirichlet Scattering Problem 3.1 is uniquely solvable for any incident field* u^i.

Proof: By Corollary 3.4 it suffices to show that uniqueness holds. Suppose $u \in W_0$ is a solution to the Dirichlet scattering problem for $u^i = 0$. Setting $v = u$ and multiplying the variational equation by $-\overline{k}$,

$$\int_{\Omega_R} \left(-\overline{k}\,|\nabla u|^2 + k\,|k|^2\,|u|^2\right) dx + \overline{k} \int_{C_R} \overline{u} \Lambda u \, ds = 0.$$

If we can prove that the last term on the left hand side has non negative imaginary part, then all terms have non negative imaginary part and the assertion is proved.

Now let $r > R$. Using a separation of variables ansatz in cylindrical coordinates, we can determine a solution v to the Helmholtz equation in $\Omega_r \setminus \overline{\Omega_R}$ of the form

$$v(x) = \sum_{n=1}^{\infty} \sum_{l \in \mathbb{Z}} V_l^n \, H_l^{(1)}(k_n\,\rho) \exp(il\varphi) \, \sin(\alpha_n x_3), \tag{22}$$

for $x = (\rho \cos\varphi, \rho \sin\varphi, x_3) \in \Omega_r \setminus \overline{\Omega_R}$, such that

$$v|_{C_R} = u|_{C_R} \qquad \text{and} \qquad \left.\frac{\partial v}{\partial \nu}\right|_{C_R} = \Lambda u.$$

The function v is simply the radiating continuation of u to the outside of Ω_R, i.e. it satisfies (2) and (6)–(8).

Now, using Green's first identity, we obtain

$$\begin{aligned}
\mathrm{Im}\left(\overline{k} \int_{C_R} \overline{u} \Lambda u \, ds\right) &= \mathrm{Im}\left(\overline{k} \int_{C_R} \overline{v}\, \frac{\partial v}{\partial \nu}\, ds\right) \\
&= |k|^2 \int_{\Omega_r \setminus \overline{\Omega_R}} \mathrm{Im}(k)\,|v|^2\,dx - \int_{\Omega_r \setminus \overline{\Omega_R}} \mathrm{Im}(\overline{k})\,|\nabla v|^2\,dx \\
&\qquad + \mathrm{Im}\left(\int_{C_r} \overline{k}\,\overline{v}\, \frac{\partial v}{\partial \nu}\, ds\right).
\end{aligned}$$

From (22), the last integral is seen to tend to 0 as $r \to \infty$. Hence

$$\mathrm{Im}\left(\overline{k} \int_{C_R} \overline{u} \Lambda u \, ds\right) = |k|^2 \int_{\Omega \setminus \overline{\Omega_R}} \mathrm{Im}(k)\,|v|^2\,dx - \int_{\Omega_r \setminus \overline{\Omega_R}} \mathrm{Im}(\overline{k})\,|\nabla v|^2\,dx \geq 0.$$

This completes the proof. ∎

In physical terms the last theorem is by no means surprising, as the situation treated here corresponds to energy absorption by the medium. The case of real k is much more difficult to treat and it is an open question whether uniqueness holds for all wave numbers. Here, we will establish some results detailing the nature of the set of wave numbers for which non-uniqueness may occur. The idea is to use the analyticity properties of the Dirichlet-to-Neumann map established in Lemma 2.4. To this end we will make use of a slight generalization of [14, Theorem I.5.1] proved in [3].

Theorem 3.6 *Let H denote a Hilbert space. Assume that $C \subset \mathbb{C}$ is an open connected set and that $A_\mu : H \to H$ is a compact linear operator for all $\mu \in C$ that depends analytically on μ. Then, for all $\mu \in C$ except possibly for some isolated points, the equation*

$$(I - A_\mu)\, \varphi = 0$$

has the same number of linearly independent solutions.

Applying this Theorem and also Corollary 2.5 to the situation at hand, as the Dirichlet scattering problem is always uniquely solvable for $\mathrm{Im}(k) > 0$, it follows that in a neighborhood of any real wave number k_* there exists at most a finite number of wave numbers k at which uniqueness may not hold. Indeed, otherwise the points where non-uniqueness is not guaranteed had a limit point, which contradicts the statement of Theorem 3.6. Thus we have proved the following theorem.

Theorem 3.7 *The Dirichlet Scattering Problem 3.1 is uniquely solvable for any incident field u^i except possibly for a sequence of real wave numbers $(k^{(j)})$ such that $k^{(j)} \to \infty$ $(j \to \infty)$.*

Remark 3.8 The proof of Theorem 3.6 reveals that the modal wave numbers $k = \alpha_n$ are among the sequence of wave numbers, at which non-uniqueness may hold. However, it allows for other numbers to be in this sequence. To the authors' knowledge it is an open question whether non-uniqueness may occur only at the modal wave numbers or not. ■

So far we have only discussed the case of a Dirichlet boundary condition on the obstacle D. However, the treatment of a Neumann or an impedance condition is quite similar and we briefly treat these two cases in the remainder of this section.

The Neumann condition (4) permits to discuss the sound-hard scattering problem in the space W, defined in (11). An application of Green's first identity yields as in (18)

$$\int_{\Omega_R} \left(\nabla u \nabla \overline{v} - k^2 u \overline{v}\right) dx = \int_{C_R} \frac{\partial u}{\partial \nu} \overline{v}\, ds,$$

for a weak solution $u \in H^1_{\mathrm{loc}}(\Omega)$ of the Neumann scattering problem and a testfunction $v \in W$. The boundary term on ∂D vanishes this time due to (4). Hence, for a variational formulation, we obtain again the form $\mathcal{B}_{\mathrm{D}}$ encountered for the Dirichlet problem which incorporates the radiation condition (6)–(8) for $u^s = u - u^i$ using the Dirichlet-to-Neumann operator Λ from (13).

Problem 3.9 (Neumann Scattering Problem) *Given an incident field u^i satisfying* (1) *and* (2), *find $u \in W$ such that*

$$\mathcal{B}_{\mathrm{D}}(u, v) = \int_{C_R} \overline{v} \left(\frac{\partial u^i}{\partial \nu} - \Lambda u^i \right) ds \qquad \text{for all } v \in W.$$

Because the form for the Neumann problem is the same as for the Dirichlet problem one shows the following result on existence and uniqueness of solutions by essentially copying the proof for the Dirichlet problem.

Theorem 3.10 *The Neumann Scattering Problem 3.9 is uniquely solvable for any incident field u^i except possibly for a sequence of real wave numbers $(k^{(j)})$ such that $k^{(j)} \to \infty$ $(j \to \infty)$.*

Finally, we investigate the impedance boundary condition (5) where the impedance $\beta \in L^\infty(\partial D)$ is positive. Once again, we integrate by parts using the total field $u \in H^1_{\mathrm{loc}}(\Omega)$ that satisfies the impedance scattering problem weakly and a test function $v \in W$, to find

$$\int_{\Omega_R} \left(\nabla u \nabla \overline{v} - k^2 u\overline{v}\right)\, dx = \int_{C_R} \frac{\partial u}{\partial \nu}\overline{v}\, ds - \int_{\partial D} \frac{\partial u}{\partial \nu}\overline{v}\, ds = \int_{C_R} \frac{\partial u}{\partial \nu}\overline{v}\, ds - i\int_{\partial D} \beta u \overline{v}\, ds.$$

The Dirichlet-to-Neumann operator Λ is again used as a non local boundary condition on C_R that realizes the radiation condition,

$$\int_{\Omega_R} \left(\nabla u \nabla \overline{v} - k^2 u\overline{v}\right)\, dx - \int_{C_R} \overline{v}\Lambda u\, ds + i\int_{\partial D} \beta u\overline{v}\, ds = \int_{C_R} \overline{v}\left(\frac{\partial u^i}{\partial \nu} - \Lambda u^i\right) ds. \tag{23}$$

In consequence, we use the sesquilinear form

$$\mathcal{B}_{\mathrm{I}}(u,v) = \int_{\Omega_R} \left(\nabla u \nabla \overline{v} - k^2 u\overline{v}\right)\, dx - \int_{C_R} \overline{v}\Lambda u\, ds + i\int_{C_R} \beta u\overline{v}\, ds \tag{24}$$

to state the variational formulation of the impedance scattering problem (1), (2), (5)–(8).

Problem 3.11 (Impedance Scattering Problem) *Given an incident field u^i satisfying equations* (1) *and* (2)*, find $u \in W$ such that*

$$\mathcal{B}_{\mathrm{I}}(u,v) = \int_{C_R} \overline{v}\left(\frac{\partial u^i}{\partial \nu} + \Lambda u^i\right) ds \qquad \text{for all } v \in W.$$

Proving existence and uniqueness of solutions of Problem 3.11 is much simpler than for the Dirichlet and Neumann case due to energy absorption caused by the impedance β.

Theorem 3.12 *The Impedance Scattering Problem 3.11 is uniquely solvable for any incident field u^i and all real wave numbers $k > 0$.*

Proof: By Lemma 3.3 it is easy to see that $\mathcal{B}_{\mathrm{I}}$ satisfies a Gårding inequality. Thus, if we show uniqueness of a solution of Problem 3.11 with $u^i = 0$, the Fredholm alternative theorem implies uniqueness and existence of a solution for general u^i. Hence, assume $u \in W$ solves Problem 3.11 with $u^i = 0$. It follows that

$$0 = \mathcal{B}_{\mathrm{I}}(u,u) = \int_{\Omega_R} \left(|\nabla u|^2 - k^2|u|^2\right)\, dx - \int_{C_R} \overline{u}\Lambda u\, ds + i\int_{\partial D} \beta |u|^2\, ds. \tag{25}$$

Arguing exactly as in the proof of Theorem 3.5 we see that $\mathrm{Im}\int_{C_R} \overline{u}\Lambda u\, ds = 0$ since k is real and, taking the imaginary part of (25), it follows that $\int_{\partial D} \beta|u|^2\, ds = 0$. In consequence, $u = 0$ on ∂D and the impedance boundary condition implies $\frac{\partial u}{\partial \nu} = 0$ on ∂D. Since both trace and normal trace of u vanish at Γ, Holmgren's theorem implies that $u = 0$ in Ω_R. ∎

We finally remark that one can show in analogy to Proposition 3.2 equivalence of weak and variational formulation of the Neumann and impedance scattering problems.

4 Uniqueness in Special Cases

In this section, we will investigate a solution $u \in W_0$ of the homogeneous Dirichlet problem

$$\mathcal{B}_{\mathrm{D}}(u,v) = 0 \qquad \text{for all } v \in W_0. \tag{26}$$

It is the goal to derive conditions that guarantee uniqueness of solution of the scattering problem, i.e. that u vanishes throughout Ω_R. We wish to emphasize that the conditions given here are not new. We give references below. However, what is new is that we use these conditions to prove uniqueness of solution for a variational formulation of the problem for which the Fredholm Alternative can be applied.

As a first observation, we note that using (22) and the transmission conditions on C_R listed subseqently, u can be uniquely extended to a radiating solution of the Helmholtz equation in all of Ω. For ease of notation, this extension will again be denoted by u. From [17, Theorem 4.18] we conclude that $u \in H^2_{\mathrm{loc}}(\overline{\Omega})$ if the scatterer D is of class $C^{1,1}$.

The conditions that ensure uniqueness of solution take the form of geometrical conditions on the shape of ∂D. We will give two such conditions and also show some examples demonstrating that these conditions indeed characterize different classes of scatterers. We also remind the reader at this point that the coordinate system is chosen such that there exists a point $x \in D$ such that $\tilde{x} = (x_1, x_2)^\top = 0$.

Condition 1 Suppose D of class $C^{1,1}$ is such that $\tilde{\nu} \cdot \tilde{x} \geq 0$ for all $x \in \partial D$.

Condition 2 Suppose D of class $C^{1,1}$ is such that $\nu_1 x_1 \geq 0$ for all $x \in \partial D$.

While there are some scatterers that satisfy both conditions (i.e. a sphere or a cylindrical disc parallel to the waveguide boundaries) in general they are not equivalent. Figure 2 displays some typical examples of shapes satisfying both, either or none of these conditions, respectively.

Condition 1 refers to domains which we call *cylindrically star shaped:* the intersection of the domain with any horizontal plane is star shaped. A more general version of this condition is given in [18] to cover also perturbations of the planes in an n-dimensional space. Also in [8] this condition is assumed to prove uniqueness results in waveguides.

Condition 2 states that on both sides of the plane $x_1 = 0$, the object is shaped as a graph of a function of two variables. Of course, the orientation of the x_1- and x_2-axis are arbitrary in the setting of the scattering problem investigated here. Hence the x_1-direction can be replaced by any other direction parallel to the waveguide boundaries. This condition is given in [22, p.5] as assumption A.

In the following arguments, we will assume that D satisfies Condition 1 or 2, and we will establish uniqueness of solution for the scattering problem under this assumption.

We will start with two technical lemmas. For ease of notation, we introduce n as the unit normal to $\partial\Omega$ pointing out of Ω. Hence, $n = -\nu$ on ∂D.

Lemma 4.1 *For $u \in H^2(\Omega_R)$ there holds*

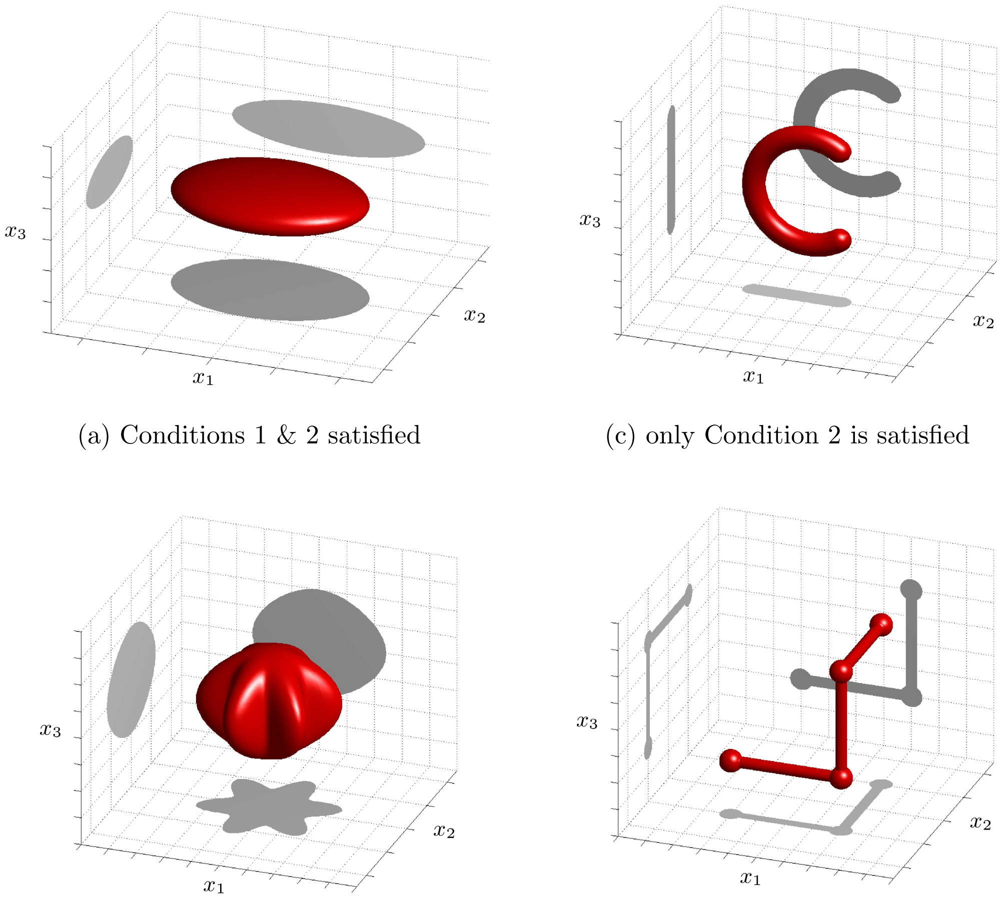
$$\int_{\partial\Omega_R} \left[\tilde{x} \cdot \left(\tilde{\nabla}\bar{u} \frac{\partial u}{\partial n} + \tilde{\nabla} u \frac{\partial \bar{u}}{\partial n} \right) - \tilde{x} \cdot \tilde{n} \left(|\nabla u|^2 - k^2 |u|^2 \right) \right] ds$$

$$= 2k^2 \int_{\Omega_R} |u|^2 \, dx - 2 \int_{\Omega_R} \left| \frac{\partial u}{\partial x_3} \right|^2 dx$$

where $\tilde{\nabla} = (\partial/\partial x_1, \partial/\partial x_2)^\top$.

Proof: Using a completeness argument, it suffices to prove the assertion for $u \in C^2(\overline{\Omega_R})$. From Green's first identity,

$$\int_{\partial\Omega_R} \phi \frac{\partial \psi}{\partial v} \, ds = \int_{\Omega_R} [\phi \, \Delta\psi + \nabla\phi \cdot \nabla\psi] \, dx, \tag{27}$$



(a) Conditions 1 & 2 satisfied

(c) only Condition 2 is satisfied

(b) Only Condition 1 is satisfied

(d) neither condition is satisfied

Figure 2: Four obstacles as examples for Conditions 1 & 2.

we first obtain, setting $\phi = |\nabla u|^2$ and $\psi = \frac{1}{2}|x|^2$, that

$$\int_{\partial\Omega_R} \tilde{x}\cdot\tilde{n}\,|\nabla u|^2\,ds = 2\int_{\Omega_R} |\nabla u|^2\,dx + \int_{\Omega_R} \tilde{x}\cdot\tilde{\nabla}\left(|\nabla u|^2\right)dx.$$

Secondly, setting $\phi = \tilde{x}\cdot\tilde{\nabla}\bar{u}$ and $\psi = u$, in (27) yields that

$$\int_{\partial\Omega_R} \tilde{x}\cdot\tilde{\nabla}\bar{u}\,\frac{\partial u}{\partial n}\,ds = \int_{\Omega_R}\left[\tilde{x}\cdot\tilde{\nabla}\bar{u}\,\Delta u + \nabla\left(\tilde{x}\cdot\tilde{\nabla}\bar{u}\right)\cdot\nabla u\right]dx,$$

and similarly for $\phi = \tilde{x} \cdot \tilde{\nabla} u$ and $\psi = \bar{u}$. So, we conclude

$$\int_{\partial \Omega_R} \left[\tilde{x} \cdot \left(\tilde{\nabla} \bar{u} \, \frac{\partial u}{\partial n} + \tilde{\nabla} u \, \frac{\partial \bar{u}}{\partial n} \right) - \tilde{x} \cdot \tilde{n} \, |\nabla u|^2 \right] ds$$

$$= \int_{\Omega_R} \left[\tilde{x} \cdot \left(\tilde{\nabla} \bar{u} \, \Delta u + \tilde{\nabla} u \, \Delta \bar{u} \right) + \nabla \left(\tilde{x} \cdot \tilde{\nabla} \bar{u} \right) \cdot \nabla u + \nabla \left(\tilde{x} \cdot \tilde{\nabla} u \right) \cdot \nabla \bar{u} \right.$$

$$\left. -2 \, |\nabla u|^2 - \tilde{x} \cdot \tilde{\nabla} \left(\left| \tilde{\nabla} u \right|^2 \right) \right] dx. \quad (28)$$

Using the identity

$$\nabla \left(\tilde{x} \cdot \tilde{\nabla} \bar{u} \right) \cdot \nabla u + \nabla \left(\tilde{x} \cdot \tilde{\nabla} u \right) \cdot \nabla \bar{u} - \tilde{x} \cdot \tilde{\nabla} \left(|\nabla u|^2 \right) = 2 \left| \tilde{\nabla} u \right|^2 ,$$

we infer that (28) can be written as

$$\int_{\partial \Omega_R} \left[\tilde{x} \cdot \left(\tilde{\nabla} \bar{u} \, \frac{\partial u}{\partial n} + \tilde{\nabla} u \, \frac{\partial \bar{u}}{\partial n} \right) - \tilde{x} \cdot \tilde{n} \, |\nabla u|^2 \right] ds$$

$$= \int_{\Omega_R} \tilde{x} \cdot \left(\tilde{\nabla} \bar{u} \, \Delta u + \tilde{\nabla} u \, \Delta \bar{u} \right) dx - 2 \int_{\Omega_R} \left(|\nabla u|^2 - \left| \tilde{\nabla} u \right|^2 \right) dx. \quad (29)$$

The right hand side of (29) can be further rewritten using the divergence theorem to obtain

$$\int_{\Omega_R} \tilde{x} \cdot \left(\tilde{\nabla} \bar{u} \, \Delta u + \tilde{\nabla} u \, \Delta \bar{u} \right) dx - 2 \int_{\Omega_R} \left| \frac{\partial u}{\partial x_3} \right|^2 dx$$

$$= -k^2 \int_{\Omega_R} \left[\nabla \cdot \left(\tilde{x} \, |u|^2 \right) - 2 \, |u|^2 \right] dx - 2 \int_{\Omega_R} \left| \frac{\partial u}{\partial x_3} \right|^2 dx$$

$$= -k^2 \int_{\partial \Omega_R} \tilde{x} \cdot \tilde{n} \, |u|^2 \, ds + 2k^2 \int_{\Omega_R} |u|^2 \, dx - 2 \int_{\Omega_R} \left| \frac{\partial u}{\partial x_3} \right|^2 dx.$$

So, (29) can be written as

$$\int_{\partial \Omega_R} \left[\tilde{x} \cdot \left(\tilde{\nabla} \bar{u} \, \frac{\partial u}{\partial n} + \tilde{\nabla} u \, \frac{\partial \bar{u}}{\partial n} \right) - \tilde{x} \cdot \tilde{n} \left(|\nabla u|^2 - k^2 \, |u|^2 \right) \right] ds$$

$$= 2k^2 \int_{\Omega_R} |u|^2 \, dx - 2 \int_{\Omega_R} \left| \frac{\partial u}{\partial x_3} \right|^2 dx.$$

This completes the proof. ∎

Lemma 4.2 *For $u \in H^2(\Omega_R)$, there holds*

$$\int_{\partial \Omega_R} \left[x_1 \left(\frac{\partial \bar{u}}{\partial x_1} \frac{\partial u}{\partial n} + \frac{\partial u}{\partial x_1} \frac{\partial \bar{u}}{\partial n} \right) - x_1 n_1 \left(|\nabla u|^2 - k^2 \, u^2 \right) \right] ds$$

$$= k^2 \int_{\Omega_R} |u|^2 \, dx - \int_{\Omega_R} |\nabla u|^2 \, dx + 2 \int_{\Omega_R} \left| \frac{\partial u}{\partial x_1} \right|^2 dx.$$

Proof: Again, it suffices to prove the assertion for $u \in C^2(\overline{\Omega_R})$. We set $\phi = |\nabla u|^2$ and $\psi = \frac{1}{2}\,|x_1|^2$ in (27) to obtain

$$\int_{\partial\Omega_R} x_1 n_1 \,|\nabla u|^2 \, ds = \int_{\Omega_R} |\nabla u|^2 \, dx + \int_{\Omega_R} x_1 e_1 \cdot \nabla \left(|\nabla u|^2\right) dx$$

where e_1 denotes the unit vector in the direction of the x_1-axis. Next, we insert $\phi = x_1\,\frac{\partial \bar{u}}{\partial x_1}$ and $\psi = u$ in Green's first identity, and so,

$$\int_{\partial\Omega_R} x_1 \frac{\partial \bar{u}}{\partial x_1} \frac{\partial u}{\partial n} \, ds = \int_{\Omega_R} x_1 \frac{\partial \bar{u}}{\partial x_1} \, \Delta u \, dx + \int_{\Omega_R} \nabla \left(x_1 \, \frac{\partial \bar{u}}{\partial x_1} \right) \cdot \nabla u \, dx$$

and similarly we deduce a relation for $\phi = x_1 \frac{\partial u}{\partial x_1}$ and $\psi = \bar{u}$. From these applications of Green's first identity we infer

$$\begin{aligned}
\int_{\partial\Omega_R} & \left[x_1 \left(\frac{\partial \bar{u}}{\partial x_1} \, \frac{\partial u}{\partial n} + \frac{\partial u}{\partial x_1} \, \frac{\partial \bar{u}}{\partial n} \right) - x_1 \, n_1 \, |\nabla u|^2 \right] ds \\
&= \int_{\Omega_R} x_1 \left(\frac{\partial \bar{u}}{\partial x_1} \, \Delta u + \frac{\partial u}{\partial x_1} \, \Delta \bar{u} \right) dx \\
&\quad + \int_{\Omega_R} \left[\nabla \left(x_1 \, \frac{\partial \bar{u}}{\partial x_1} \right) \cdot \nabla u + \nabla \left(x_1 \, \frac{\partial u}{\partial x_1} \right) \cdot \nabla \bar{u} \right] dx \\
&\quad - \int_{\Omega_R} |\nabla u|^2 \, dx - \int_{\Omega_R} x_1 \, \frac{\partial}{\partial x_1} \left(|\nabla u|^2\right) dx.
\end{aligned}$$

Using the identity

$$\left[\nabla \left(x_1 \, \frac{\partial \bar{u}}{\partial x_1} \right) \cdot \nabla u + \nabla \left(x_1 \, \frac{\partial u}{\partial x_1} \right) \cdot \nabla \bar{u} \right] - x_1 \, \frac{\partial}{\partial x_1} \left(|\nabla u|^2\right) = 2 \left| \frac{\partial u}{\partial x_1} \right|^2$$

and the divergence theorem we obtain the assertion. ■

One further ingredient for the uniqueness proof is the fact that a solution to (26) decays exponentially with distance from the obstacle. We formulate this statement in the following lemma.

Lemma 4.3 *Let w be the unique radiating extension of a solution of (26) to all of Ω. Then $w \in H^1(\Omega)$ and for some constants r, C, $c > 0$, we have*

$$|w(x)| + |\nabla w(x)| \le C \, e^{-c|x|}, \qquad |x| > r, \tag{30}$$

as well as

$$\int_{\Omega} |\nabla w|^2 \, dx = k^2 \int_{\Omega} |w|^2 \, dx. \tag{31}$$

Proof: Green's first identity together with the boundary condition $w|_{\partial D} = 0$ yields

$$0 = \int_{\Omega_R} \left(\overline{w}\Delta w - w\Delta\overline{w}\right) dx = \int_{C_R} \left(\overline{w}\Lambda w - w\overline{\Lambda w}\right) ds = 2i \, \mathrm{Im} \left(\int_{C_R} \overline{w}\Lambda w \, ds \right).$$

Recall from (14) and (A.4) that

$$- \int_{C_R} \overline{w} \Lambda w \, ds = \pi h R \sum_{n=1}^{\infty} \sum_{n \in \mathbb{Z}} \left(k_n \frac{H^{(1)}_{l-1}(k_n R)}{H^{(1)}_{l}(k_n R)} - \frac{l}{R} \right) |W_l^n|^2 \, .$$

Let us now denote by N the largest entire number such that k_n is real for $n \leq N$. It was shown in the proof of Lemma 2.2 that for $n > N$ the imaginary part of $k_n H^{(1)}_{l-1}(k_n R)/H^{(1)}_l(k_n R)$ vanishes. Hence we obtain

$$\operatorname{Im}\left(\int_{C_R} \overline{w}\Lambda w\, ds\right) = -\pi h R \sum_{n=1}^{N} k_n \sum_{l=0}^{\infty} \operatorname{Im}\left[\frac{H^{(1)}_{l-1}(k_n R)}{H^{(1)}_l(k_n R)}\right] \left|W^l_n\right|^2.$$

From the recurrence relation (A.4) it follows that

$$\operatorname{Im}\frac{H^{(1)}_{l-1}(k_n R)}{H^{(1)}_l(k_n R)} = \operatorname{Im}\frac{H^{(1)}_{l+1}(k_n R)}{H^{(1)}_l(k_n R)} = \frac{H^{(1)}_{l+1}(k_n R)H^{(2)}_l(k_n R) - H^{(2)}_{l+1}(k_n R)H^{(1)}_l(k_n R)}{2i|H^{(1)}_l(k_n R)|^2},$$

where $H^{(2)}_l = \overline{H^{(1)}_l}$. Formula 9.1.16 in [1] shows that

$$\operatorname{Im}\left[\frac{H^{(1)}_{l+1}(k_n R)}{H^{(1)}_l(k_n R)}\right] = -\frac{2}{k_n R|H^{(1)}_l(k_n R)|^2}$$

and collecting terms we arrive at

$$0 = \operatorname{Im}\left(\int_{C_R} \overline{w}\Lambda w\, ds\right) = 2\pi h \sum_{n=1}^{N}\sum_{l=0}^{\infty} \frac{\left|W^l_n\right|^2}{\left|H^{(1)}_l(k_n R)\right|^2}.$$

The last equation implies that $\left|W^l_n\right| = 0$ for $1 \leq n \leq N$ and $l \in \mathbb{N}_0$, thus, w possesses only evanescent modes. As these are exponentially decaying, (30) follows. Using (18) we infer that

$$\int_{\Omega_R} \left(|\nabla w|^2 - k^2\,|w|^2\right) dx = \int_{C_R} \frac{\partial w}{\partial \nu}\,\overline{w}\, ds \overset{(30)}{\longrightarrow} 0 \qquad \text{as } R \to \infty.$$

Hence follows (31) and the proof is complete. ∎

We are now ready to formulate the uniqueness result for the Dirichlet problem.

Theorem 4.4 *Assume that either Condition 1 or 2 holds. Then any solution of*

$$\mathcal{B}_{\mathrm{D}}(u, v) = 0 \qquad \text{for all } v \in W_0.$$

satisfies $u \equiv 0$ *in* Ω_R.

Proof: Suppose first that Condition 1 holds. We have that $\partial\Omega_R = C_R \cup \Gamma^+_R \cup \Gamma^-_R \cup \partial D$ and so the surface integral in the identity of Lemma 4.1 is written as the sum of four integrals I_{C_R}, $I_{\Gamma^+_R}$, $I_{\Gamma^-_R}$, $I_{\partial D}$ over C_R, Γ^+_R, Γ^-_R and ∂D, respectively. The integrand in all four integrals is

$$\left[\tilde{x}\cdot\left(\tilde{\nabla}\bar{u}\,\frac{\partial u}{\partial n} + \tilde{\nabla} u\,\frac{\partial \bar{u}}{\partial n}\right) - \tilde{x}\cdot\tilde{n}\left(|\nabla u|^2 - k^2\, u^2\right)\right].$$

From (30), $I_{C_R} \to 0$ $(R \to \infty)$. The integral $I_{\Gamma^+_R}$ vanishes because of the Neumann conditions and $\tilde{x}\cdot\tilde{n} = 0$ on Γ^+_R . The integral $I_{\Gamma^-_R}$ vanishes also, because on Γ^-_R Dirichlet conditions hold, so, $\tilde{\nabla} u = \tilde{n}\,(\partial u)/(\partial n)$ and again $\tilde{x}\cdot\tilde{n} = 0$.

Now letting $R \to \infty$, from (31), we conclude that

$$\int_{\partial D} \tilde{x} \cdot \tilde{n} \left| \frac{\partial u}{\partial n} \right|^2 ds = 2 \int_{\Omega} \left| \tilde{\nabla} u \right|^2 dx \geq 0.$$

Since $\tilde{x} \cdot \tilde{n} \leq 0$ for all $x \in \partial D$ we obtain that $u = 0$.

Suppose now that Condition 2 holds. Here the integral over $\partial \Omega_R$ in Lemma 4.2 is written as a sum of four integrals J_{C_R}, $J_{\Gamma_R^+}$, $J_{\Gamma_R^-}$ and $J_{\partial D}$ over C_R, Γ_R^+, Γ_R^- and ∂D respectively. The integrand in all cases is

$$\left[x_1 \left(\frac{\partial \bar{u}}{\partial x_1} \frac{\partial u}{\partial n} + \frac{\partial u}{\partial x_1} \frac{\partial \bar{u}}{\partial n} \right) - x_1 \, n_1 \left(|\nabla u|^2 - k^2 \, u^2 \right) \right].$$

From Lemma 4.3 we obtain $J_{C_R} \to 0$ as $R \to \infty$. The Neumann conditions and $x_1 n_1 = 0$ on Γ_R^+ imply that $J_{\Gamma_R^+} = 0$. Similarly, $J_{\Gamma_R^-} = 0$ because the Dirichlet conditions imply $\frac{\partial u}{\partial x_1} = n_1 \frac{\partial u}{\partial n}$. Hence, letting $R \to \infty$ and using (31) again,

$$\int_{\partial D} x_1 n_1 \left| \frac{\partial u}{\partial n} \right|^2 ds = 2 \int_{\Omega} \left| \frac{\partial u}{\partial x_1} \right|^2 dx \geq 0.$$

Since $x_1 n_1 \leq 0$ for all $x \in \partial D$, we conclude also in this case that $u = 0$. ■

Remark 4.5 In [22] a further uniqueness result is given, based on a geometric condition (Condition B of this reference) that is satisfied by star shaped obstacles. The proof given there is based on an identity similar to the one presented by us in Lemma 4.1 but without the tilde operators. We were not able to verify this identity for the waveguide. ■

A Bounds for Fractions of Bessel functions

The Hankel functions of the first kind are related to the modified Bessel functions by the relation

$$K_l(z) = \frac{\pi}{2} i^{l+1} H_l^{(1)}(iz) \qquad \text{for } l \in \mathbb{N}_0, \arg(z) \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]. \tag{A.1}$$

As the modified Bessel funktion K_l has no zeros in the right half-plane [24, p. 511] of the complex plane, it follows that the modulus of the Hankel function is stricly positive in the upper half-plane,

$$\left| H_l^{(1)}(z) \right| > 0, \qquad z \in \mathbb{C}, \quad \arg(z) \in [0, \pi), \quad l \in \mathbb{N}_0. \tag{A.2}$$

From the recurrence formula for Hankel functions

$$\frac{d}{z dz} \left(z^l H_l^{(1)}(z) \right) = z^{l-1} H_{l-1}^{(1)}(z)$$

we conclude

$$l z^{l-1} H_l^{(1)}(z) + z^l H_l^{(1)\prime}(z) = z^l H_{l-1}^{(1)}(z) \qquad \text{for } z \neq 0,$$

and therefore

$$\frac{H_l^{(1)\prime}(z)}{H_l^{(1)}(z)} = \frac{H_{l-1}^{(1)}(z)}{H_l^{(1)}(z)} - \frac{l}{z}. \tag{A.3}$$

Analogously to (A.3) the recurrence formula

$$\frac{d}{zdz}\left(z^{-l}H_l^{(1)}(z)\right) = -z^{-l-1}H_{l+1}^{(1)}(z)$$

holds and yields

$$\frac{H_l^{(1)\prime}(z)}{H_l^{(1)}(z)} = \frac{l}{z} - \frac{H_{l+1}^{(1)}(z)}{H_l^{(1)}(z)}. \tag{A.4}$$

Combining (A.3) for $l > 0$ and (A.4) for $l < 0$, bearing in mind that $H_{-l}^{(1)}(z) = (-1)^l H_l^{(1)}(z)$, we obtain

$$\frac{H_l^{(1)\prime}(z)}{H_l^{(1)}(z)} = \frac{H_{|l|-1}^{(1)}(z)}{H_{|l|}^{(1)}(z)} - \frac{|l|}{z}. \tag{A.5}$$

This formula is valid for all $l \in \mathbb{Z}$.

For purely imaginary argument it is convenient to work with the modified Bessel functions given by (A.1). We rewrite the fraction of Hankel functions as

$$\frac{H_{l-1}^{(1)}(ix)}{H_l^{(1)}(ix)} = i\frac{K_{l-1}(x)}{K_l(x)} \tag{A.6}$$

and observe that this fraction is purely imaginary if x is real, since the K_l are real functions of real arguments. From the integral representation [1, 9.6.24]

$$K_l(x) = \int_0^\infty e^{-x\cosh t}\cosh(lt)\,dt \qquad \text{for } x \in \mathbb{R}_{>0},\ l \in \mathbb{N}_0$$

we obtain immediately that

$$\frac{K_{l-1}(x)}{K_l(x)} \le 1, \qquad x \in \mathbb{R}_{>0}, \quad l \in \mathbb{N}. \tag{A.7}$$

Using the recurrence relation [1, 9.6.26]

$$K_{l-1}(x) - K_{l+1}(x) = -\frac{2l}{x}K_l(x), \qquad x \in \mathbb{R}_{>0}, \quad l \in \mathbb{N},$$

we further conclude using (A.7) that

$$\frac{K_{l+1}(x)}{K_l(x)} = \frac{K_{l-1}(x)}{K_l(x)} + \frac{2l}{x} \le \frac{x+2l}{x}, \qquad x \in \mathbb{R}_{>0}, \quad l \in \mathbb{N}.$$

Hence

$$\frac{K_l(x)}{K_{l+1}(x)} \ge \frac{x}{x+2l}, \qquad x \in \mathbb{R}_{>0}, \quad l \in \mathbb{N}. \tag{A.8}$$

Combining (A.7) and (A.8) we obtain the following Lemma.

Lemma A.1 *Suppose* $x > 0$. *Then*

$$\frac{K_{-1}(x)}{K_0(x)} \ge 1, \qquad\qquad \frac{K_0(x)}{K_1(x)} \ge 1 - \frac{2}{x},$$

and

$$1 \ge \frac{K_l(x)}{K_{l+1}(x)} \ge \frac{x}{x+2l}, \qquad l \in \mathbb{N}.$$

Proof: It remains to prove the second estimate which follows directly from (A.7) and the recurrence formula:

$$\frac{K_0(x)}{K_1(x)} = \frac{K_2(x)}{K_1(x)} - \frac{2}{x} \geq 1 - \frac{2}{x}.$$

■

With these bounds on fractions of modified Bessel functions it is also possible to obtain corresponding bounds for the modulus of fractions of Hankel functions for complex arguments. We will only require an upper bound here.

Lemma A.2 *Let* $z \in \mathbb{C}$ *such that* $|z| > 0$ *and* $\arg(z) \in (-\pi/2, \pi/2]$. *Then*

$$\left|H_l^{(1)}(z)\right| \leq \left|H_{l+1}^{(1)}(z)\right| \qquad l \in \mathbb{N}_0.$$

Proof: The proof relies on the following integral identity for the absolute value of the Bessel function $H_l^{(1)}$, which is due to the well-known Macdonald formula [24, p. 439] (see also [5]),

$$\left|H_l^{(1)}(z)\right|^2 = \frac{2}{\pi^2} \int_0^\infty e^{-\frac{t}{2} + \frac{z^2+\overline{z}^2}{2t}} K_l\left(\frac{|z|^2}{t}\right) \frac{dt}{t} \tag{A.9}$$

for $z \neq 0$ with non negative real and imaginary part. An elementary computation shows that

$$z^2 + \overline{z}^2 = z^2 + \overline{z}^2 - 2|z|^2 + 2|z|^2 = 4\,\mathrm{Im}(z)^2 + 2|z|^2,$$

and in consequence the integrand in (A.9) is real and positive. Now we exploit (A.7) to obtain the assertion of the lemma. ■

Of course, it follows from Lemma A.2 that

$$\left|\frac{H_{l-1}^{(1)}(z)}{H_l^{(1)}(z)}\right| \leq 1, \qquad l \in \mathbb{N}.$$

In the case of $l = 0$, we can also obtain an upper bound for arguments bounded away from zero from the asymptotic expansion for the Hankel functions [1, 9.2.3]. It follows that for all $\rho > 0$ there exists $C(\rho)$ such that

$$\left|\frac{H_{l-1}^{(1)}(z)}{H_l^{(1)}(z)}\right| \leq C(\rho), \qquad |z| \geq \rho, \quad \arg(z) \in \left[0, \frac{\pi}{2}\right], \quad l \in \mathbb{N}. \tag{A.10}$$

Acknowledgement

The authors would like to thank Prof. K. Kiriaki and Prof. A. Kirsch for their valuable comments. This work was supported by the Greek State Scholarship Foundation (IKY) and the German Academic Exchange Service (DAAD) through a research collaboration grant in the IKYDA Programme.

References

[1] Abramowitz M, Stegun I. *Handbook on Mathematical Functions.* Dover: Washington; 1964.

[2] D.S. Ahluwalia and J.B. Keller, Exact and asymptotic representations of the sound field in a stratified ocean. In *Wave Propagation and Underwater Acoustics*, volume 70 of *Lecture Notes in Physics*. Springer: Berlin; 1977.

[3] Arens T. *Scattering by biperiodic layered media: The integral equation approach*, in preparation.

[4] Callan M, Linton C M, Evans D V. Trapped modes in two-dimensional waveguides. *J. Fluid Mech..* 1991; **229**: 51–64.

[5] Chen Z, Liu X. An adaptive perfectly matched layer technique for time-harmonic scattering problems. *SIAM J. Numer. Anal.* 2005; **43**: 645–671.

[6] Evans D V, Levitin M, Vassiliev D. Existence theorems for trapped modes. *J. Fluid Mech.* 1994; **261**: 21–31.

[7] Evans D V, Linton C M, Ursell F. Trapped mode frequencies embedded in the continuous spectrum. *Q. J. Mech. Appl. Math.* 1993; **2**: 253–274.

[8] M. Faulhaber M. Akustische Wellen in Gebieten, die von zwei lokal gestörten parallelen Ebenen begrenzt sind. *Math. Meth. Appl. Sci.* 1982; **4**: 397–414.

[9] Gilbarg D, Trudinger N S., *Elliptic partial differential equations of second order,* 2nd edition; Springer; 1998.

[10] Gilbert R P, Xu Y. Dense sets and the projection theorem for acoustic harmonic waves in a homogeneous finite depth ocean. *Math. Meth. Appl. Sci.* 1990; **12**: 69–76.

[11] Gilbert R P, Xu Y. The propagation problem and far-field pattern in a stratified finite-depth ocean. *Math. Meth. Appl. Sci.* 1190; **12**: 199–208.

[12] Gilbert R P, Xu Y. Dense sets and the projection theorem for acoustic waves in a homogeneous finite depth ocean. *Math. Meth. in the Appl. Sci..* 1989; **12**: 67–76.

[13] Gilbert R P, Xu Y. Starting fields and far fields in ocean acoustics. *Wave Motion.* 1989; **11**: 507–524.

[14] Gohberg I C, Kreĭn M G. *Introduction to the theory of linear nonselfadjoint operators.* American Mathematical Society; 1969.

[15] Kirsch A. Characterization of the shape of a scattering obstacle using the spectral data of the far field operator. *Inverse Problems.* 1998; **14**: 1489–1512.

[16] Linton C M, McIver M, McIver P, Ratcliffe K, Zhang J. Trapped modes for off-center structures in guides. *Wave Motion.* 2002; **36**: 67–85.

[17] McLean W. *Strongly Elliptic Systems and Boundary Integral Operators.* Cambridge University Press: Cambridge; 2000.

[18] Morgenrother K, Werner P. Resonances and standing waves. *Math. Meth. Appl. Sci.* 1987; **9**: 105–126.

[19] Morgenrother K, Werner P. On the principles of limiting absorption and limiting amplitude for a class of locally perturbed waveguides. *Math. Meth. Appl. Sci.* 1988; **10**: 125–144.

[20] Mujica J. *Complex analysis in Banach spaces*. North-Holland; 1986.

[21] Nédélec J C. *Acoustic and Electromagnetic Equations, Integral Representations for Harmonic Problems.* Springer: New York; 2001.

[22] Ramm A G, Makrakis G N. Scattering by obstacles in acoustic waveguides. In *Spectral and Scattering Theory* (Ramm A G ed.). Plenum publishers; 1998: 89–110.

[23] Rozier C, Lesselier D, Angell T, Kleinman R E. Reconstruction of an impenetrable obstacle immersed in a shallow water acoustic waveguide. In *Inverse Problems of Wave Propagation and Diffraction* (Chavent G, Sabatier P C eds.). Lecture Notes in Physics. Springer: Berlin; 1997: 130–142.

[24] Watson G M. *A Treatise on the Theory of Bessel Functions.* Cambridge University Press: Cambridge; 1995.