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PSEUDO-INVERSES, HOLOMORPHIC FUNCTIONS AND ATKINSON THEORY IN BANACH ALGEBRAS

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Abstract.

Let A denote a complex Banach algebra with identity e and K an inessential ideal of A. Suppose that G is a region in $\mathbf{C}$ and $f: G \to A$ is holomorphic such that $f(\lambda)$ is left [resp. right] invertible modulo K for all $\lambda \in G$. In this paper we show that the *nullity* $\operatorname{nul}(f(\lambda))$ [resp. the *defect* $\operatorname{def}(f(\lambda))$] is constant in a neighbourhood of $\lambda_0 \in G$ if and only if there exists a neighbourhood $U \subseteq G$ of λ_0 and a holomorphic function $F: U \to A/\operatorname{rad}(A)$ such that

$$(f(\lambda) + \operatorname{rad}(A))F(\lambda)(f(\lambda) + \operatorname{rad}(A)) = f(\lambda) + \operatorname{rad}(A) \quad \text{for all } \lambda \in U$$

($\operatorname{rad}(A)$ denotes the radical of A).

Introduction.

In this paper we always assume that A is a complex Banach algebra with identity $e \neq 0$.

Let K be an inessential ideal of A, $G \subseteq \mathbf{C}$ a region and $f: G \to A$ be holomorphic. In [5, Theorem 5.6] we proved the following theorem:

Suppose that $f(\lambda)$ is left [resp. right] invertible in A/K for all $\lambda \in G$. Then there exist a discrete subset M of G and a constant $c \in \mathbf{N} \cup \{0\}$ such that

$$\operatorname{nul}(f(\lambda)) = c \,[\textit{resp. } \operatorname{def}(f(\lambda)) = c] \quad \textit{for all } \lambda \in G \backslash M \textit{ and}$$

$$\operatorname{nul}(f(\mu)) > c \,[\textit{resp. } \operatorname{def}(f(\mu)) > c] \quad \textit{for } \mu \in M.$$

For the definitions of *nullity* and *defect* see Section 2 and Section 3.

The aim of this paper is to show that the following conditions for λ_0 are equivalent:

(A) $\operatorname{nul}(f(\lambda)) = \operatorname{nul}(f(\lambda_0))$ [*resp.* $\operatorname{def}(f(\lambda)) = \operatorname{def}(f(\lambda_0))$] *in a neighbourhood of* λ_0.

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(B) *There is a neighbourhood U of λ_0 and a holomorphic function $F: U \to A/\mathrm{rad}(A)$ such that*

$$(f(\lambda) + \mathrm{rad}(A))F(\lambda)(f(\lambda) + \mathrm{rad}(A)) = f(\lambda) + \mathrm{rad}(A)$$

for all $\lambda \in U$.

The first section of the present paper deals with pseudo-inverses of relatively regular elements in A. In particular, we extend some results due to S. Ivanov [2]. In Section 2 we first summarize some definitions, notations and results of Atkinson and Fredholm theory in semisimple Banach algebras. Then we investigate holomorphic function with values in a primitive complex Banach algebra. Holomorphic functions in general Banach algebras are considered in Section 3.

1. Pseudo-inverses and relatively regular elements.

1.1 Definition. For each subset M of A ($M \neq \emptyset$) the *left annihilator* and the *right annihilator* are the sets

$$L(M) = \{y \in A : yM = \{0\}\} \text{ and } R(M) = \{y \in A : My = \{0\}\},$$

respectively. If $M = \{x\}$ we simply write $L(x)$ and $R(x)$. Note that $L(xA) = L(x)$ and $R(Ax) = R(x)$, since A has an identity.

Let $x \in A$. We say that x is *relatively regular* if there exists $y \in A$ such that $xyx = x$. We call y a *pseudo-inverse* of x.

The proof of the first proposition is easy and is left to the reader.

1.2 Proposition. *Suppose that y is a pseudo-inverse of x and put $p = xy$, $q = e - yx$. Then*

$$p^2 = p,\ q^2 = q,\ pA = xA,\ qA = R(x),$$
$$A(e - q) = Ax \text{ and } A(e - p) = L(x).$$

The idea of the next proposition goes back to some results due to S. Ivanov [2].

1.3 Proposition. *Suppose that $x, y, s \in A$, $xyx = x$ and $\|y\|\,\|s\| < 1$ (then $e + ys$ and $e + sy$ are invertible). Put*

$$r(s) = (e + ys)^{-1} y.$$

(a) *$r(s) = y(e + sy)^{-1}$ and $(e + sy)^{-1} s = s(e + ys)^{-1}$.*
(b) *If $R(x) \subseteq R(s)$, then $r(s)$ is a pseudo-inverse of $x + s$.*
(c) *If $sA \subseteq xA$, then $r(s)$ is a pseudo-inverse of $x + s$.*

Proof. (a) is obvious.
(b) and (c): Define $g(s) = x + s - (x + s)r(s)(x + s)$. We have

$$
\begin{aligned}
g(s) &= (x+s)(e - r(s)(x+s)) \\
&= (x+s)[(e+ys)^{-1}(e+ys-y(x+s))] \\
&= (x+s)(e+ys)^{-1}(e-yx) \\
&= [x(e+ys)-(xy-e)s](e+ys)^{-1}(e-yx) \\
&= \underbrace{x(e-yx)}_{=0} - (xy-e)s(e+ys)^{-1}(e-yx),
\end{aligned}
$$

therefore

$$g(s) = -(xy-e)s(e+ys)^{-1}(e-yx) \tag{1.1}$$

and, by (a),

$$g(s) = -(xy-e)(e+sy)^{-1}s(e-yx). \tag{1.2}$$

If $R(x) \subseteq R(s)$, Proposition 1.2 gives $(e-yx)A = R(x) \subseteq R(s)$, hence $s(e-yx) = 0$. Use (1.2) to derive $g(s) = 0$. Now suppose that $sA \subseteq xA$. By Proposition 1.2, $sA \subseteq xA = xyA = R(xy-e)$ thus $(xy-e)s = 0$. By (1.1), this implies $g(s) = 0$.

Before we state the main result of this section, we need a simple lemma, which goes back to a theorem of Sz.-Nagy [6, p. 132]. This lemma will be useful in Section 2 again.

1.4 LEMMA. *Suppose that p and q are idempotents in A such that* $\|p-q\| < 1$. *Then*

(a) $R(p) \cap qA = R(q) \cap pA = \{0\}$;

(b) $pA = pqA, qA = qpA$.

PROOF. (a) Let $g = e + (p-q)$ and $h = e - (p-q)$. Then g and h are invertible. If $z \in R(p) \cap qA$, then $pz = 0$ and $qz = z$, thus $gz = z + pz - z = 0$, therefore $z = 0$. It follows that $R(p) \cap qA = \{0\}$. Similarly, $R(q) \cap pA = \{0\}$.

(b) Since $ph = pq$ and $hA = A$, we have $pA = phA = pqA$. Similarly, $qA = qpA$.

The following theorem is a generalization of [2, Theorem 3.5]. The proof is a modification of that in [2, p. 352].

1.5 THEOREM. *Let G be a region in* $\mathbf{C}$ *and let* $f: G \to A$ *be a holomorphic function. Suppose that* $f(\lambda_0)$ *is relatively regular* ($\lambda_0 \in G$). *Then the following statements are equivalent:*

(a) *There is a neighbourhood* $U_1 \subseteq G$ *of* λ_0 *and a holomorphic function* $u: U_1 \to A$ such that

$$f(\lambda)u(\lambda)f(\lambda) = f(\lambda) \quad \text{for all } \lambda \in U_1.$$

(b) *There is a neighbourhood* $U_2 \subseteq G$ *of* λ_0 *and a holomorphic function* $q: U_2 \to A$ *such that*

$$q(\lambda)^2 = q(\lambda) \text{ and } q(\lambda)A = R(f(\lambda)) \quad \text{for all } \lambda \in U_2.$$

(c) *There is a neighbourhood* $U_3 \subseteq G$ *of* λ_0 *and a holomorphic function* $p: U_3 \to A$ *such that*

$$p(\lambda)^2 = p(\lambda) \text{ and } p(\lambda)A = f(\lambda)A \quad \text{for all } \lambda \in U_3.$$

PROOF. (a) $\Rightarrow$ (b): Let $q(\lambda) = e - u(\lambda)f(\lambda)\,(\lambda \in U_1)$. Proposition 1.2 shows $q(\lambda)^2 = q(\lambda)$ and $q(\lambda)A = R(f(\lambda))$ for $\lambda \in U_1$.

(a) $\Rightarrow$ (c): Define $p(\lambda) = f(\lambda)u(\lambda)\,(\lambda \in U_1)$. From Proposition 1.2 we obtain $p(\lambda)A = f(\lambda)A$ and $p(\lambda)^2 = p(\lambda)$ for $\lambda \in U_1$.

(b) $\Rightarrow$ (a): Define h and s on U_2 by

$$h(\lambda) = q(\lambda)q(\lambda_0) + (e - q(\lambda))(e - q(\lambda_0)),$$
$$s(\lambda) = f(\lambda)h(\lambda) - f(\lambda_0)$$

and suppose that y is a pseudo-inverse of $f(\lambda_0)$. Since $h(\lambda_0) = e$ and $s(\lambda_0) = 0$, there is a neighbourhood $U_1 \subseteq U_2$ of λ_0 such that

(1.3) $\|q(\lambda) - q(\lambda_0)\| < 1$, $\|s(\lambda)\|\,\|y\| < 1$ and $h(\lambda)$ is invertible for $\lambda \in U_1$.

Since $h(\lambda)q(\lambda_0) = q(\lambda)q(\lambda_0)$, it follows from Lemma 1.4 that $h(\lambda)q(\lambda_0)A = q(\lambda)q(\lambda_0)A = q(\lambda)A$, hence $h(\lambda)R(f(\lambda_0)) = R(f(\lambda))$, therefore $R(f(\lambda_0)) \subseteq R(f(\lambda)h(\lambda))$. This gives

$$R(f(\lambda_0)) \subseteq R(s(\lambda)) \quad \text{for all } \lambda \in U_1.$$

From (1.3) and Proposition 1.3(b) it follows that

$$r(s(\lambda)) := (e + ys(\lambda))^{-1}y \quad (\lambda \in U_1)$$

is a pseudo-inverse of $f(\lambda_0) + s(\lambda) = f(\lambda)h(\lambda) \quad (\lambda \in U_1)$.
In other words,

$$f(\lambda)h(\lambda)r(s(\lambda))f(\lambda)h(\lambda) = f(\lambda)h(\lambda) \quad \text{for } \lambda \in U_1$$

and hence

$$f(\lambda)[h(\lambda)r(s(\lambda))]f(\lambda) = f(\lambda) \quad \text{for } \lambda \in U_1$$

(recall that $h(\lambda)$ is invertible for all $\lambda \in U_1$).

(c) $\Rightarrow$ (a): Define h on U_3 by

$$h(\lambda) = p(\lambda)p(\lambda_0) + (e - p(\lambda))(e - p(\lambda_0)).$$

Since $h(\lambda_0) = e$, there is a neighbourhood $V \subseteq U_3$ of λ_0, such that $h(\lambda)$ is invertible for all $\lambda \in V$. Next define s on V by

$$s(\lambda) = h(\lambda)^{-1} f(\lambda) - f(\lambda_0)$$

and suppose that y is a pseudo-inverse of $f(\lambda_0)$. Since $s(\lambda_0) = 0$, there exists a neighbourhood $U_1 \subseteq V$ of λ_0 such that

$$(1.4) \qquad \|p(\lambda) - p(\lambda_0)\| < 1 \text{ and } \|s(\lambda)\|\,\|y\| < 1 \quad \text{for all } \lambda \in U_1.$$

Since $h(\lambda)p(\lambda_0) = p(\lambda)p(\lambda_0)$, it follows from Lemma 1.4 that

$$(1.5) \qquad h(\lambda)p(\lambda_0)A = p(\lambda)p(\lambda_0)A = p(\lambda)A = f(\lambda)A \quad (\lambda \in U_1).$$

Combining (1.5) and $h(\lambda)p(\lambda_0)A = h(\lambda)f(\lambda_0)A$, we conclude $f(\lambda_0)A = h(\lambda)^{-1}f(\lambda)A$. This gives

$$s(\lambda)A \subseteq f(\lambda_0)A \quad \text{for all } \lambda \in U_1.$$

In view of (1.4) and Proposition 1.3(c), we find that

$$r(s(\lambda)) := (e + ys(\lambda))^{-1} y \quad (\lambda \in U_1)$$

is a pseudo-inverse of $f(\lambda_0) + s(\lambda) = h(\lambda)^{-1} f(\lambda) \quad (\lambda \in U_1)$,
therefore

$$h(\lambda)^{-1} f(\lambda) r(s(\lambda)) h(\lambda)^{-1} f(\lambda) = h(\lambda)^{-1} f(\lambda),$$

thus

$$f(\lambda)[r(s(\lambda))h(\lambda)^{-1}]f(\lambda) = f(\lambda) \quad \text{for all } \lambda \in U_1.$$

2. Atkinson theory and holomorphic functions in primitive Banach algebras.

For the convenience of the reader we summarize some definitions, notations and results of Atkinson and Fredholm theory in semisimple Banach algebras (for details see [1], [4] and [5]).

Given a left ideal L of A the *quotient* is the ideal $L:A = \{a \in A : aA \subseteq L\}$. The quotient of a maximal left ideal is called a *primitive* ideal. We denote the set of all primitive ideals by $\Pi(A)$. Note that each $P \in \Pi(A)$ is closed. The *radical* of A, denoted by $\mathrm{rad}(A)$, is the intersection of all primitive ideals of A.

A is said to be *semisimple* if $\mathrm{rad}(A) = \{0\}$. A is said to be *primitive* if $\{0\} \in \Pi(A)$ (a primitive Banach algebra is semisimple).

In a semisimple Banach algebra A the *socle* of A, denoted by $\mathrm{soc}(A)$, is defined to be the sum of all minimal right ideals (which equals the sum of all minimal left ideals [1, p. 23]) or to be $\{0\}$ if A has no minimal right ideals. *Thus* $\mathrm{soc}(A)$ *is an ideal of* A.

The following definitions are fundamental to the rest of this paper.

2.1 DEFINITION. Let A be semisimple.

(a) The ideal of *inessential elements* of A is given by

$$I(A) = \cap \{P : P \in \Pi(A) \text{ and } \operatorname{soc}(A) \subseteq P\}.$$

(b) An ideal K is called *inessential* if $K \subseteq I(A)$.

(c) The *K-left Fredholm class* and the *K-right Fredholm class* are the sets

$$\Phi_l(A, K) = \{x \in A\text{: there exists } y \in A \text{ with } yx - e \in K\}$$
$$\text{and } \Phi_r(A, K) = \{x \in A\text{: there exists } y \in A \text{ with } xy - e \in K\}.$$

(d) The set of *K-Atkinson elements* is

$$\mathscr{A}(A, K) = \Phi_l(A, K) \cup \Phi_r(A, K).$$

The following proposition contains some important properties of Atkinson elements (note that $\operatorname{soc}(A)$ is an inessential ideal of A).

2.2 PROPOSITION. *Let K be an inessential ideal of a semisimple Banach algebra A.*

(a) $\Phi_l(A, \operatorname{soc}(A)) = \Phi_l(A, I(A))$, $\Phi_r(A, \operatorname{soc}(A)) = \Phi_r(A, I(A))$.

(b) $x \in \Phi_l(A, K) \Leftrightarrow$ *there exists* $p = p^2 \in \operatorname{soc}(A) \cap K$ *such that*

$$Ax = A(e - p).$$

(c) $x \in \Phi_r(A, K) \Leftrightarrow$ *there exists* $p = p^2 \in \operatorname{soc}(A) \cap K$ *such that*

$$xA = (e - p)A.$$

(d) $x \in \mathscr{A}(A, K) \Rightarrow$ *is relatively regular.*

PROOF. (a) [1, F.1.10], [4, Lemma 2.13]. (b) and (c) [4, Proposition 2.19]. (d) [5, Proposition 2.5].

A non-zero idempotent $e_0 \in A$ is called *minimal* if $e_0 A e_0$ is a division algebra. $\operatorname{Min}(A)$ denotes the set of all minimal idempotents of A.

For the rest of this section A will be a primitive algebra such that $\operatorname{Min}(A) \neq \emptyset$.

Observe that $\operatorname{Min}(A) \neq \emptyset$ if and only if $\operatorname{soc}(A) \neq \{0\}$ [1, BA.3.1].

Fix $e_0 \in \operatorname{Min}(A)$ and denote the set of all bounded linear operators on the Banach space Ae_0 by $\mathscr{L}(Ae_0)$. We shall write

$$x \to \hat{x} : A \to \mathscr{L}(Ae_0)$$

to denote the left regular representation of A on the Banach space Ae_0, that is $\hat{x}(y) = xy$ for $y \in Ae_0$. We have

$$\hat{x}(Ae_0) = xAe_0 \text{ and}$$

$$\ker(\hat{x}) = \{y \in Ae_0 : \hat{x}(y) = 0\} = R(x) \cap Ae_0 = R(x)e_0.$$

Lemma F.2.1 in [1] shows that $\dim(xAe_0)$, $\dim R(x)e_0$ and $\dim(Ae_0/xAe_0)$ are independent of the choice of $e_0 \in \mathrm{Min}(A)$.

Therefore the *rank* of x is defined by

$$\mathrm{rank}(x) = \dim(xAe_0),$$

the *nullity* of x is defined by

$$\mathrm{nul}(x) = \dim R(x)e_0.$$

We define the *defect* of x by

$$\mathrm{def}(x) = \dim(Ae_0/xAe_0).$$

The following theorem collects some important properties of rank, nullity and defect:

2.3 THEOREM.

(a) $x = 0 \Leftrightarrow \mathrm{rank}(x) = 0$.

(b) $\mathrm{soc}(A) = \{x \in A : \mathrm{rank}(x) < \infty\}$.

(c) $x \in \Phi_l(A, I(A))$ [*resp.* $x \in \Phi_r(A, I(A))$] $\Leftrightarrow$ *x is relatively regular and* $\mathrm{nul}(x) < \infty$ [*resp.* $\mathrm{def}(x) < \infty$].

(d) *If* $Ax = A(e-p)$, $p^2 = p$ *and* $e_0 \in \mathrm{Min}(A)$, *then*

$$R(x) = pA \ \textit{and} \ \mathrm{nul}(x) = \dim R(x)e_0 = \dim(pAe_0) = \mathrm{rank}(p).$$

(e) *If* $xA = (e-q)A$, $q^2 = q$ *and* $e_0 \in \mathrm{Min}(A)$, *then*

$$Ae_0 = (e-q)Ae_0 \oplus qAe_0 = xAe_0 \oplus qAe_0 \ \textit{and} \ \mathrm{def}(x) = \dim qAe_0 = \mathrm{rank}(q).$$

(f) *If p and q are idempotents in A such that* $\|p - q\| < 1$, *then*

$$\mathrm{rank}(q) = \mathrm{rank}(p).$$

(g) *Let K be an inessential ideal of A. If* $x \in \Phi_r(A, K)$ *and* $e_0 \in \mathrm{Min}(A)$, *then*

$$\mathrm{def}(x) = \dim(e_0 L(x)).$$

PROOF. (a) and (b) [1, F.2.4]. (c) [5, Theorem 3.4]. (d) and (e) clear.

(f) Lemma 1.4 can be applied. It shows that $pqA = pA$ and $R(p) \cap qA = \{0\}$. Hence $pqAe_0 = pAe_0$ for $e_0 \in \mathrm{Min}(A)$. Define the linear operator $T: qAe_0 \to pqAe_0$ by $Ty = py$ $(y \in qAe_0)$. Since $R(p) \cap qA = \{0\}$, T is bijective. This implies $\mathrm{rank}(q) = \dim(qAe_0) = \dim(T(qAe_0)) = \dim(pqAe_0) = \dim(pAe_0) = \mathrm{rank}(p)$.

(g) By Proposition 2.2 (c), there is an idempotent $p \in \mathrm{soc}(A) \cap K$ such that $xA = (e-p)A$. By (e) and (c), $\mathrm{def}(x) = \mathrm{rank}(p) < \infty$. According to [4, Lemma 2.3], there are $e_1, \ldots, e_n \in \mathrm{Min}(A)$ such that $p = e_1 + \ldots + e_n$ and $e_i e_j = 0$ $(i \neq j)$. This gives

$$pA = e_1 A \oplus \ldots \oplus e_n A \ \text{and} \ Ap = Ae_1 \oplus \ldots \oplus Ae_n.$$

It follows that

$$pAe_0 = e_1 Ae_0 \oplus \ldots \oplus e_n Ae_0 \text{ and } e_0 Ap = e_0 Ae_1 \oplus \ldots \oplus e_0 A_n.$$

Since $\dim(e_j Ae_0) = \dim(e_0 Ae_j) = 1$ ([1, F.2.1]) and $Ap = L(x)$, we have

$$\operatorname{def}(x) = \operatorname{rank}(p) = \dim(pAe_0) = n = \dim(e_0 Ap) = \dim(e_0 L(x)).$$

If K is an inessential ideal of A, then we have $\Phi_l(A, K) \subseteq \Phi_l(A, I(A))$ and $\Phi_r(A, K) \subseteq \Phi_r(A, I(A))$. From Theorem 2.3(c) we obtain:

$$x \in \Phi_l(A, K) \Rightarrow \operatorname{nul}(x) < \infty,$$

$$x \in \Phi_r(A, K) \Rightarrow \operatorname{def}(x) < \infty.$$

For the rest of this paper G will denote a region in $\mathbf{C}$ and f a holomorphic function on G with values in A.

A subset M of G is called *discrete* if M has no accumulation points in G. Thus M is at most countable.

Before we present the main results of this section, we need the following proposition:

2.4 Proposition. *Let $p: G \to A$ a continuous function. If $p(\lambda)^2 = p(\lambda)$ in G, then* $\operatorname{rank}(p(\lambda))$ *must be constant in G.*

Proof. Take $\lambda_0 \in G$. By the continuity of p, there is $\delta(\lambda_0) > 0$ such that $\|p(\lambda) - p(\lambda_0)\| < 1$ whenever $|\lambda - \lambda_0| < \delta$. Theorem 2.3(f) yields $\operatorname{rank}(p(\lambda)) = \operatorname{rank}(p(\lambda_0))$ for $|\lambda - \lambda_0| < \delta$. In other words: to each $\lambda_0 \in G$ there exists a neighbourhood $U(\lambda_0)$ of λ_0 such that $U(\lambda_0) \subseteq G$, and $\operatorname{rank}(p(\lambda))$ is constant in $U(\lambda_0)$. Since G is a region, this implies that $\operatorname{rank}(p(\lambda))$ is constant in G.

2.5 Theorem. *Let K be an inessential ideal of A and suppose that $f(\lambda) \in \Phi_l(A, K)$ for all $\lambda \in G$. Then there exist a constant α and a discrete subset M_α of G such that*

(2.1) $\operatorname{nul}(f(\lambda)) = \alpha$ *for all* $\lambda \in G \setminus M_\alpha$ *and* $\operatorname{nul}(f(\mu)) > \alpha$ *for all* $\mu \in M_\alpha$.

For $\lambda_0 \in G$ the following conditions are equivalent:

(a) $\lambda_0 \in G \setminus M_\alpha$.

(b) *There is a neighbourhood $V \subseteq G$ of λ_0 and a holomorphic function $u: V \to A$ such that*

$$f(\lambda)u(\lambda)f(\lambda) = f(\lambda) \quad \text{for all } \lambda \in V.$$

Proof. The existence of a constant α and a discrete subset $M_\alpha \subseteq G$ such that (2.1) holds follows from [5, Theorem 5.4, Theorem 5.6].

(a) $\Rightarrow$ (b): Proposition 2.2 (b), applied to $f(\lambda_0)$, shows that there exists $p = p^2 \in \operatorname{soc}(A) \cap K$ such that

(2.2) $$Af(\lambda_0) = A(e - p).$$

Therefore

(2.3) $$R(f(\lambda_0)) = pA \text{ and } \operatorname{nul}(f(\lambda_0)) = \operatorname{rank}(p)$$

(Theorem 2.3(d)). By (2.2), we can find $y \in A$ such that

(2.4) $$yf(\lambda_0) = e - p.$$

(2.3) yields $f(\lambda_0)yf(\lambda_0) = f(\lambda_0) - f(\lambda_0)p = f(\lambda_0)$, hence y is a pseudo-inverse of $f(\lambda_0)$. Since f is continuous, there is a neighbourhood $V \subseteq G \backslash M_\alpha$ of λ_0 with

$$\|f(\lambda) - f(\lambda_0)\| < \|y\|^{-1} \quad \text{for } \lambda \in V.$$

Define the functions q and u on V by

$$q(\lambda) = (e + y(f(\lambda) - f(\lambda_0)))^{-1}p \text{ and}$$

$$u(\lambda) = (e + y(f(\lambda) - f(\lambda_0)))^{-1}y.$$

Since $e + y(f(\lambda) - f(\lambda_0)) = e + yf(\lambda) - yf(\lambda_0) = e + yf(\lambda) - (e - p) = yf(\lambda) + p$, we have $u(\lambda)f(\lambda) = (e + y(f(\lambda) - f(\lambda_0)))^{-1}yf(\lambda) = (yf(\lambda) + p)^{-1}(yf(\lambda) + p - p) = e - (yf(\lambda) + p)^{-1}p$. This gives

(2.5) $$u(\lambda)f(\lambda) = e - q(\lambda) \quad \text{for } \lambda \in V.$$

It follows that

(2.6) $$R(f(\lambda)) \subseteq q(\lambda)A \quad (\lambda \in V).$$

Now fix $e_0 \in \operatorname{Min}(A)$. By (2.6),

(2.7) $$R(f(\lambda))e_0 \subseteq q(\lambda)Ae_0 \quad (\lambda \in V).$$

Since $\operatorname{nul}(f(\lambda)) = \operatorname{nul}(f(\lambda_0))$ for all $\lambda \in V$, we derive

$$\begin{aligned}\dim R(f(\lambda))e_0 &= \operatorname{nul}(f(\lambda)) = \operatorname{nul}(f(\lambda_0)) = \operatorname{rank}(p)\\ &= \dim(pAe_0) = \dim((e + y(f(\lambda) - f(\lambda_0)))^{-1}pAe_0)\\ &= \dim(q(\lambda)Ae_0)(< \infty) \quad \text{for all } \lambda \in V.\end{aligned}$$

Hence it follows from (2.7) that $R(f(\lambda))e_0 = q(\lambda)Ae_0$, which proves that $f(\lambda)q(\lambda)Ae_0 = 0$ for $\lambda \in V$. Therefore

(2.8) $$f(\lambda)q(\lambda) = 0 \quad \text{for } \lambda \in \mathrm{V},$$

because A is primitive and $e_0 \neq 0$ (use [1, p. 29]). (2.6) and (2.8) show that

$$q(\lambda)A = R(f(\lambda)) \quad (\lambda \in V).$$

The combination of (2.8) and (2.5) gives

$$f(\lambda)u(\lambda)f(\lambda) = f(\lambda) \quad \text{for all } \lambda \in V.$$

(b) $\Rightarrow$ (a): Define the function q on V by

$$q(\lambda) = e - u(\lambda)f(\lambda).$$

Hence Proposition 1.2 implies that $q(\lambda)^2 = q(\lambda)$ and

$$q(\lambda)A = R(f(\lambda)) \quad \text{for all } \lambda \in V.$$

For fixed $e_0 \in \mathrm{Min}(A)$ this gives

$$q(\lambda)Ae_0 = R(f(\lambda))e_0$$

and therefore

$$\mathrm{rank}(q(\lambda)) = \dim(q(\lambda)Ae_0) = \dim(R(f(\lambda))e_0) = \mathrm{nul}(f(\lambda))$$

for all $\lambda \in V$. Use Proposition 2.4 to derive

$$\mathrm{nul}(f(\lambda)) = \mathrm{rank}(q(\lambda)) = \text{const.} = \mathrm{nul}(f(\lambda_0)) \quad \text{in } V.$$

Thus $\lambda_0 \in G \backslash M_\alpha$.

The second main result of this section reads as follows:

2.6 THEOREM. *Let K be an inessential ideal of A and suppose that $f(\lambda) \in \Phi_r(A, K)$ for all $\lambda \in G$. Then there exist a constant β and a discrete subset M_β of G such that*

(2.10) $\mathrm{def}(f(\lambda)) = \beta$ *for all* $\lambda \in G \backslash M_\beta$ *and* $\mathrm{def}(f(\mu)) > \beta$ *for all* $\mu \in M_\beta$.

For $\lambda_0 \in G$ the following conditions are equivalent:

(a) $\lambda_0 \in G \backslash M_\beta$.

(b) *There is a neighbourhood $V \subseteq G$ of λ_0 and a holomorphic function u: $V \to A$ such that*

$$f(\lambda)u(\lambda)f(\lambda) \quad \textit{for all } \lambda \in U.$$

PROOF. The existence of a constant β and a discrete set $M_\beta \subseteq G$ such that (2.10) holds follows from [5, Theorem 5.4, Theorem 5.6].

(a) $\Rightarrow$ (b): Proposition 2.2(c), applied to $f(\lambda_0)$, shows that there exists $q = q^2 \in \mathrm{soc}(A) \cap K$ such that

$$f(\lambda_0)A = (e - q)A. \tag{2.11}$$

Therefore $L(f(\lambda_0)) = Aq$. Theorem 2.3(e) shows

$$\mathrm{def}(f(\lambda_0)) = \mathrm{rank}(q). \tag{2.12}$$

By (2.11), we can find $y \in A$ such that $f(\lambda_0)y = e - q$, this gives (recall that $q \in L(f(\lambda_0))$)

$$f(\lambda_0)yf(\lambda_0) = f(\lambda_0),$$

hence y is a pseudo-inverse of $f(\lambda_0)$. Now choose a neighbourhood $V \subseteq G\backslash M_\beta$ of λ_0 such that

$$\|f(\lambda) - f(\lambda_0)\| < \|y\|^{-1} \quad \text{for } \lambda \in V.$$

Define the functions p and u on V by

$$p(\lambda) = q(e + (f(\lambda) - f(\lambda_0))y)^{-1} \quad \text{and}$$

$u(\lambda) = y(e + (f(\lambda) - f(\lambda_0))y)^{-1} = (e + y(f(\lambda) - f(\lambda_0)))^{-1}y$ (Proposition 1.3(a)).

Since $e + (f(\lambda) - f(\lambda_0))y = f(\lambda)y + q$, we derive

$$\begin{aligned} f(\lambda)u(\lambda) &= f(\lambda)y(f(\lambda)y + q)^{-1} = (f(\lambda)y + q - q)(f(\lambda)y + q)^{-1} \\ &= e - q(f(\lambda)y + q)^{-1}. \end{aligned}$$

Therefore

(2.13) $$f(\lambda)u(\lambda) = e - p(\lambda) \quad \text{for all } \lambda \in V.$$

It follows that

(2.14) $$L(f(\lambda)) \subseteq Ap(\lambda) \quad (\lambda \in V)$$

Fix $e_0 \in \mathrm{Min}(A)$. By (2.14),

(2.15) $$e_0L(f(\lambda)) \subseteq e_0Ap(\lambda) \quad \text{for } \lambda \in V.$$

Since $Aq = L(f(\lambda_0))$ and $\mathrm{def}(f(\lambda)) = \mathrm{def}(f(\lambda_0))$ in V, we have (use Theorem 2.3(g))

$$\begin{aligned} \dim(e_0L(f(\lambda))) &= \mathrm{def}(f(\lambda)) = \mathrm{def}(f(\lambda_0)) = \dim(e_0L(f(\lambda_0))) \\ &= \dim(e_0Aq) = \dim(e_0Aq(e + (f(\lambda) - f(\lambda_0))y)^{-1}) \\ &= \dim(e_0Ap(\lambda))(< \infty) \quad \text{for all } \lambda \in V. \end{aligned}$$

By (2.15), $e_0L(f(\lambda)) = e_0Ap(\lambda)$, which shows that $e_0Ap(\lambda)f(\lambda) = 0$ for $\lambda \in V$. Since A is primitive and $e_0 \neq 0$, we conclude (use [1, p. 29])

(2.16) $$p(\lambda)f(\lambda) = 0 \quad \text{for } \lambda \in V.$$

(2.16) and (2.14) show that

$$Ap(\lambda) = L(f(\lambda)) \quad \text{for all } \lambda \in V.$$

The combination of (2.16) and (2.13) gives

$$f(\lambda)u(\lambda)f(\lambda) = f(\lambda) \quad \text{for all } \lambda \in V.$$

(b) $\Rightarrow$ (a): Define the function p on V by

$$p(\lambda) = e - f(\lambda)u(\lambda).$$

Proposition 1.2 implies that $p(\lambda)^2 = p(\lambda)$ and

$$f(\lambda)A = (e - p(\lambda))A \quad \text{for all } \lambda \in V.$$

Use Theorem 2.3 (e) to derive

$$\operatorname{def}(f(\lambda)) = \operatorname{rank}(p(\lambda)) \quad \text{for all } \lambda \in V.$$

Proposition 2.4 asserts now that

$$\operatorname{def}(f(\lambda)) = \operatorname{rank}(p(\lambda)) = \text{const.} = \operatorname{def}(f(\lambda_0)) \quad \text{in } V.$$

Thus $\lambda_0 \in G \backslash M_\beta$.

We close this section with a corollary which we need in the next section. The proof of this corollary is implicitly contained in the proofs of the preceding theorems.

2.7 COROLLARY. *Suppose*
(a) $f(\lambda) \in \Phi_l(A, \operatorname{soc}(A))$ [*resp.* $f(\lambda) \in \Phi_r(A, \operatorname{soc}(A))$] *for all* $\lambda \in G$;
(b) $\lambda_0 \in G$, $y \in A$ *is a pseudo-inverse of* $f(\lambda_0)$ *and* $p^2 = p \in \operatorname{soc}(A)$ *such that*

$$yf(\lambda_0) = e - p \quad [\textit{resp. } f(\lambda_0)y = e - p];$$

(c) V *is a neighbourhood of* λ_0 *with*

$$\|f(\lambda) - f(\lambda_0)\| < \|y\|^{-1} \quad \textit{and}$$

$$\operatorname{nul}(f(\lambda)) = \operatorname{nul}(f(\lambda_0)) \quad [\textit{resp. } \operatorname{def}(f(\lambda)) = \operatorname{def}(f(\lambda_0))]$$

for all $\lambda \in V$;
(d) $u\colon V \to A$ *is defined by*

$$u(\lambda) = (e + y(f(\lambda) - f(\lambda_0)))^{-1}y.$$

Then

$$f(\lambda)u(\lambda)f(\lambda) = f(\lambda) \quad \textit{for all } \lambda \in V.$$

3. General Banach algebras.

In this section we assume that A is an arbitrary Banach algebra. Therefore the socle of A might not exist. But the quotient algebra $A' = A/\operatorname{rad}(A)$ is semisimple [1, BA.2.2], thus $\operatorname{soc}(A')$ exists. We write x' for the coset $x + \operatorname{rad}(A) \in A'(x \in A)$, and for $S \subseteq A$ write $S' = \{x' \colon x \in S\}$.

The *presocle* of A is defined by $\operatorname{psoc}(A) = \{x \in A \colon x' \in \operatorname{soc}(A')\}$. Observe that $\operatorname{psoc}(A)$ is an ideal of A and that $\operatorname{soc}(A) = \operatorname{psoc}(A)$ if A is semisimple.

In order to extend Atkinson and Fredholm theory we need the following important fact: If $P \in \Pi(A)$, then the quotient algebra A/P is primitive [1, BA.2.6].

The *ideal of inessential elements* of A is defined to be

$$I(A) = \cap\{P\colon P \in \Pi(A) \quad \text{and } \operatorname{psoc}(A) \subseteq P\}.$$

An ideal K of A is *essential* if $K \subseteq I(A)$. If K is an inessential ideal of A, the sets

$$\Phi_l(A. K),\ \Phi_r(A, K) \quad \text{and } \mathscr{A}(A, K)$$

are defined as in Definition 2.1.

In the following proposition we collect some properties of left and right Fredholm elements. For further information and details see [1], [4] and [5].

3.1 PROPOSITION.

(a) $\Phi_l(A, I(A)) = \Phi_l(A, \operatorname{psoc}(A))$ *and* $\Phi_r(A, I(A)) = \Phi_r(A, \operatorname{psoc}(A))$.

(b) *Let* $x \in \Phi_l(A, I(A))$ [*resp.* $x \in \Phi_r(A, I(A))$]. *Then there exist* $P_1, \ldots, P_n \in \Pi(A)$ *such that*

$$x + P \in \Phi_l(A/P, \operatorname{soc}(A/P)) \ [\textit{resp. } x + P \in \Phi_r(A/P, \operatorname{soc}(A/P))]$$

for all $P \in \Pi(A)$ *and*

$$\operatorname{nul}(x + P) = 0 \ [\textit{resp. } \operatorname{def}(x + P) = 0] \quad \textit{for all } P \in \Pi(A)\backslash\{P_1, \ldots, P_n\}.$$

PROOF. [4, Proposition 2.19, Theorem 2.22].

In view of part (b) of the preceding proposition the concepts of nullity and defect can be extended as follows:

We define the *nullity function* and *defect function* on $\mathscr{A}(A, I(A)) \times \Pi(A) \to \mathbb{N} \cup \{0, \infty\}$ by

$$\nu(x)(P) = \operatorname{nul}(x + P) \quad \text{and } \delta(x)(P) = \operatorname{def}(x + P).$$

If $x \in \mathscr{A}(A, I(A))$ we define

$$\operatorname{nul}(x) = \begin{cases} \sum_{P \in \Pi(A)} \nu(x)(P) & \text{for } x \in \Phi_l(A, I(A)), \\ \infty & \text{for } x \notin \Phi_l(A, I(A)), \end{cases}$$

$$\operatorname{def}(x) = \begin{cases} \sum_{P \in \Pi(A)} \delta(x)(P) & \text{for } x \in \Phi_r(A, I(A)), \\ \infty & \text{for } x \notin \Phi_r(A, I(A)). \end{cases}$$

3.2 REMARK. If A is primitive and $x \in \Phi_l(A, I(A))$ then we have $\nu(x)(P) = 0$ for all $\{0\} \neq P \in \Pi(A)$. Therefore $\operatorname{nul}(x) = \nu(x)(\{0\})$. Similarly $\operatorname{def}(x) = \delta(x)(\{0\})$ (see [1, p. 38], [5, Remark 4.5] for details). *So our two definitions for nullity and defect coincide.*

Notation: For the rest of this paper we write $\Phi_l(A)$ and $\Phi_r(A)$ instead of

$\Phi_l(A, I(A))$ and $\Phi_r(A, I(A))$. Recall that G denotes a region in $\mathbf{C}$ and f a holomorphic function on G with values in A.

The tools developed so far suffice to establish the main results of this paper.

3.3. THEOREM. *Let K be an essential ideal of A and suppose that $f(\lambda) \in \Phi_l(A, K)$ for all $\lambda \in G$. Then there exists a discrete subset M_α of G such that*

$$\begin{aligned} &\nu(f(\lambda)) \text{ is independent of } \lambda \text{ for } \lambda \in G \backslash M_\alpha \\ &\text{and for each } \mu \in M_\alpha \text{ there is a primitive ideal } P \text{ with} \\ &\nu(f(\mu))(P) > \nu(f(\lambda))(P) \quad (\lambda \in G \backslash M_\alpha). \end{aligned} \tag{3.1}$$

For $\lambda_0 \in G$ the following conditions are equivalent:

(a) $\text{nul}(f(\lambda)) = \text{nul}(f(\lambda_0))$ *in a neighbourhood of* λ_0.

(b) *There is a neighbourhood $U \subseteq G$ of λ_0 such that*

$$\nu(f(\lambda))(P) = \nu(f(\lambda_0))(P) \quad \text{for all } P \in \Pi(A) \quad \text{and all } \lambda \in U.$$

(c) *There is a neighbourhood $V \subseteq G$ of λ_0 and a holomorphic function $F: V \to A' = A/\text{rad}(A)$ such that*

$$(f(\lambda) + \text{rad}(A))F(\lambda)(f(\lambda) + \text{rad}(A)) = f(\lambda) + \text{rad}(A) \quad \text{for all } \lambda \in V.$$

PROOF. The existence of a discrete subset M_α such that (3.1) holds follows from [5, Theorem 5.6]. By (3.1) and Proposition 3.1(b), there exist $\delta > 0$, $P_1, \dots, P_n \in \Pi(A)$ and $\alpha_1, \dots, \alpha_n \in \mathbf{N} \cup \{0\}$ with

$$\nu(f(\lambda))(P_j) = \alpha_j \leqq \nu(f(\lambda_0))(P_j) \quad (j = 1, \dots, n; \quad 0 < |\lambda - \lambda_0| < \delta), \tag{3.2}$$

$$\nu(f(\lambda))(P) = \nu(f(\lambda_0))(P) = 0 \quad (P \in \Pi(A) \backslash \{P_1, \dots, P_n\}; \quad |\lambda - \lambda_0| < \delta), \tag{3.3}$$

$$\text{and} \quad \text{nul}(f(\lambda)) = \sum_{j=1}^{n} \alpha_j \leqq \text{nul}(f(\lambda_0)) \quad (0 < |\lambda - \lambda_0| < \delta). \tag{3.4}$$

This gives the equivalence of (a) and (b).

Since $\Phi_l(A, K) \subseteq \Phi_l(A) = \Phi_l(A, \text{psoc}(A))$ (Proposition 3.1(a)) there is $a \in A$ with $af(\lambda_0) - e \in \text{psoc(A)}$, hence $a'(f(\lambda_0))' - e' \in \text{soc}(A')$. Therefore $(f(\lambda_0))' \in \Phi_l(A', \text{soc}(A'))$. Proposition 2.2 shows the existence of $p \in A$ such that

$$A'(f(\lambda_0))' = A'(e' - p') \quad \text{and } (p')^2 = p' \in \text{soc}(A').$$

Therefore $y'(f(\lambda_0))' = e' - p'$ for some $y \in A$ and $R((f(\lambda_0))') = p'A'$ (Theorem 2.3(d)). Thus $(f(\lambda_0))'y'(f(\lambda_0))' = (f(\lambda_0))'$. It follows that

$$f(\lambda_0)yf(\lambda_0) - f(\lambda_0), \quad yf(\lambda_0) - (e - p), \quad p^2 - p \in \text{rad}(A). \tag{3.5}$$

Since $p \in \operatorname{psoc}(A)$, [1, BA.3.4] shows $p' + P' \in \operatorname{soc}(A'/P')$ for all $P \in \Pi(A)$, thus, by [1, BA.2.6],

$$p + P \in \operatorname{soc}(A/P) \quad \text{for all } P \in \Pi(A). \tag{3.6}$$

Since $\operatorname{rad}(A) = \cap\{P: P \in \Pi(A)\}$, the combination of (3.5) and (3.6) gives the following result:

$$(f(\lambda_0) + P)(y + P)(f(\lambda_0) + P) = f(\lambda_0) + P, \tag{3.7}$$

$$(y + P)(f(\lambda_0) + P) = (e - p) + P \tag{3.8}$$

$$\text{and} \quad p^2 + P = p + P \in \operatorname{soc}(A/P) \tag{3.9}$$

for each $P \in \Pi(A)$.

We show next that (b) implies (c): There is a neighbourhood $V \subseteq U$ of λ_0 such that $\|f(\lambda) - f(\lambda_0)\| < \|y\|^{-1}$ for $\lambda \in V$. It follows that

$$\|(f(\lambda) - f(\lambda_0) + P\| \le \|f(\lambda) - f(\lambda_0)\| < \|y\|^{-1} \leqq \|y + P\|^{-1} \quad (\lambda \in V, P \in \Pi(A)). \tag{3.10}$$

Thus the functions

$$u(\lambda) = (e + y(f(\lambda) - f(\lambda_0)))^{-1}y \quad \text{and}$$

$$u(\lambda) + P = [(e + y(f(\lambda) - f(\lambda_0)))^{-1}y + P]\,[y + P]$$

$(P \in \Pi(A))$ are holomorphic in V. From (3.7), (3.8), (3.9), (3.10) and Corollary 2.7 we derive

$$(f(\lambda) + P)(u(\lambda) + P)(f(\lambda) + P) = f(\lambda) + P \quad \text{for all } \lambda \in V \text{ and all } P \in \Pi(A).$$

Hence

$$f(\lambda)u(\lambda)f(\lambda) - f(\lambda) \in \cap\{P: P \in \Pi(A)\} = \operatorname{rad}(A) \quad \text{for all } \lambda \in V.$$

Now define $F: V \to A/\operatorname{rad}(A)$ by $F(\lambda) = u(\lambda) + \operatorname{rad}(A)$, and (c) follows.

To complete the proof, we show that (c) implies (b). [7, Lemma 2.1] shows that there is a neighbourhood $W \subseteq V$ of λ_0 and a holomorphic function $u: W \to A$ such that

$$u(\lambda) + \operatorname{rad}(A) = F(\lambda) \quad (\lambda \in W).$$

This gives $f(\lambda)u(\lambda)f(\lambda) - f(\lambda) \in \operatorname{rad}(A) = \cap\{P: P \in \Pi(A)\}$ for all $\lambda \in W$. Hence

$$(f(\lambda) + P)(u(\lambda) + P)(f(\lambda) + P) = f(\lambda) + P \quad \text{for all } \lambda \in W \text{ and all } P \in \Pi(A).$$

From Theorem 2.5, (3.2) and (3.3), we obtain

$$v(f(\lambda))(P) = v(f(\lambda_0))(P)$$

for all $P \in \Pi(A)$ and all $\lambda \in \{\lambda \in \mathbf{C}: |\lambda - \lambda_0| < \delta\}$.

3.4 THEOREM. *Let K be an inessential ideal of A and suppose that $f(\lambda) \in \Phi_r(A, K)$ for all $\lambda \in G$. Then there exists a discrete subset M_β of G such that*

$$\delta(f(\lambda)) \text{ is independent of } \lambda \text{ for } \lambda \in G \backslash M_\beta$$

and for each $\mu \in M_\beta$ there is a primitive ideal P with

$$\delta(f(\mu))(P) > \delta(f(\lambda))(P) \quad (\lambda \in G \backslash M_\beta).$$

For $\lambda_0 \in G$ the following conditions are equivalent:

(a) $\operatorname{def}(f(\lambda)) = \operatorname{def}(f(\lambda_0))$ *in a neighbourhood of λ_0.*

(b) *There is a neighbourhood $U \subseteq G$ of λ_0 such that*

$$\delta(f(\lambda))(P) = \delta(f(\lambda_0))(P) \quad \text{for all } P \in \Pi(A) \text{ and all } \lambda \in U.$$

(c) *There is a neighbourhood $V \subseteq G$ of λ_0 and a holomorphic function $F: V \to A' = A/\operatorname{rad}(A)$ such that*

$$(f(\lambda) + \operatorname{rad}(A))F(\lambda)(f(\lambda) + \operatorname{rad}(A)) = f(\lambda) + \operatorname{rad}(A) \quad \text{for all } \lambda \in V.$$

We omit the proof, because this theorem can be proved in the same way as Theorem 3.3.

An immediate consequence of the last theorems and Theorem 1.5 is

3.5 THEOREM. *Let A be semisimple and K be an inessential ideal of A. Suppose $f(\lambda) \in \Phi_l(A, K)$ [resp. $\Phi_r(A, K)$] for all $\lambda \in G$. For $\lambda_0 \in G$ the following conditions are equivalent:*

(a) $\operatorname{nul}(f(\lambda))$ [*resp.* $\operatorname{def}(f(\lambda))$] *is constant in a neighbourhood of λ_0.*

(b) *There is a neighbourhood $U_1 \subseteq G$ of λ_0 and a holomorphic function $u: U_1 \to A$ such that*

$$f(\lambda)u(\lambda)f(\lambda) = f(\lambda) \quad \text{for all } \lambda \in U_1.$$

(c) *There is a neighbourhood $U_2 \subseteq G$ of λ_0 and a holomorphic function $p: U_2 \to A$ such that*

$$p(\lambda)^2 = p(\lambda) \text{ and } p(\lambda)A = f(\lambda)A \quad \text{for all } \lambda \in U_2.$$

(d) *There is a neighbourhood $U_3 \subseteq G$ of λ_0 and a holomorphic function $q: U_3 \to A$ such that*

$$q(\lambda)^2 = q(\lambda) \text{ and } q(\lambda)A = R(f(\lambda)) \quad \text{for all } \lambda \in U_3.$$

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