ON OPERATORS OF SAPHAR TYPE

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Abstract: A bounded linear operator $T$ on a complex Banach space $X$ is called an operator of Saphar type, if $T$ is relatively regular and if its null space is contained in its generalized range $\bigcap_{n=1}^{\infty} T^n(X)$. This paper contains some characterizations of operators of Saphar type. Furthermore, for a function $f$ admissible in the analytic calculus, we obtain a necessary and sufficient condition in order that $f(T)$ is an operator of Saphar type.

1 – Terminology and introduction

Let $X$ denote a Banach space over the complex field $\mathbb{C}$ and let $\mathcal{L}(X)$ be the Banach algebra of all bounded linear operators on $X$. If $T \in \mathcal{L}(X)$, we denote by $N(T)$ the kernel and by $T(X)$ the range of $T$. The spectrum of $T$ is denoted by $\sigma(T)$. The resolvent set $\rho(T)$ is defined by $\rho(T) = \mathbb{C} \setminus \sigma(T)$. We write $\mathcal{H}(T)$ for the set of all complex valued functions which are analytic in some neighbourhood of $\sigma(T)$. For $f \in \mathcal{H}(T)$, the operator $f(T)$ is defined by the well known analytic calculus (see [3, §99]).

Let $T \in \mathcal{L}(X)$. Then an operator $S \in \mathcal{L}(X)$ will be called a pseudo inverse for $T$ if

$$TST = T.$$ 

We then say that $T$ is relatively regular. A relatively regular operator $T$ is called an operator of Saphar type if its null space $N(T)$ is contained in its generalized range

$$T^{\infty}(X) = \bigcap_{n=1}^{\infty} T^n(X).$$
We write $\mathcal{S}(X)$ for the set of all operators of Saphar type. This class of operators has been studied by P. Saphar [6] (see also [1]). Although operators in $\mathcal{S}(X)$ seem rather special, they have an important property:

**Theorem 1.** $T$ is an operator of Saphar type if and only if there is a neighbourhood $U \subseteq \mathcal{C}$ of $0$ and a holomorphic function $F: U \to \mathcal{L}(X)$ such that

$$(T - \lambda I)F(\lambda)(T - \lambda I) = T - \lambda I \quad \text{for all } \lambda \in U.$$ 

**Proof:** [5, Théorème 2.6] or [8, Theorem 1.4].

In [6] Saphar considered the following question: if $T \in \mathcal{S}(X)$ and $A$ is an operator in $\mathcal{L}(X)$, when is $T - A$ an operator of Saphar type? In Section 2 of this paper we use the perturbation results of [6] to characterize operators of Saphar type.

Section 3 deals with Atkinson operators, i.e. relatively regular operators having at least one of $\dim N(T)$, $\text{codim } T(X)$ finite. The main results in Section 1 allow us to give a simple proof of the following well known fact: if $T$ is an Atkinson operator, then $\dim N(T - \lambda I)$ (resp. $\text{codim } (T - \lambda I)(X)$) is constant in a neighbourhood of $0$ if and only if $T \in \mathcal{S}(X)$.

In Section 4 of the present paper we obtain a necessary and sufficient condition in order that $f(T) \in \mathcal{S}(X)$. Moreover, if $f(T)$ is an operator of Saphar type, we obtain a formula for a pseudo inverse for $f(T)$.

We close this section with some definitions, notations and preliminary results which we need in the sequel.

$\mathcal{R}(X)$ will denote the set of all relatively regular operators in $\mathcal{L}(X)$. Let $T \in \mathcal{L}(X)$. If $S \in \mathcal{L}(X)$ satisfies the two equations

$$TST = T \quad \text{and} \quad STS = S$$

then $S$ will be called a $g_2$-inverse for $T$. We shall make frequent use of the following results which will be quoted without further reference.

1) $T \in \mathcal{R}(X)$ if and only if $N(T)$ and $T(X)$ are closed complemented subspaces of $X$ (see [3, Satz 74.2]).

2) If $TST = T$ for some operator $S$, then $TS$ is a projection onto $T(X)$ and $I - ST$ is a projection onto $N(T)$ (see [3, p. 385]).

3) If $S$ is a pseudo inverse for $T$, then $STS$ is a $g_2$-inverse for $T$ (simple verification).
2 – Perturbation properties

We begin with the following basic facts.

**Lemma 1.** Let $T \in \mathcal{L}(X)$.

(a) If $N(T) \subseteq T^\infty(X)$, then $T(T^\infty(X)) = T^\infty(X)$.

(b) If $N(T) \subseteq T^\infty(X)$ and $T(X)$ is closed, then $T^n(X)$ is closed for each $n \in \mathbb{N}$, hence $T^\infty(X)$ is closed.

**Proof:**

(a) The inclusion $T(T^\infty(X)) \subseteq T^\infty(X)$ is obvious. Let $y$ be an arbitrary element of $T^\infty(X)$. Then for every $k = 1, 2, \ldots$ there exists $x_k \in X$ so that $y = T^k x_k$. If we set $z_k = x_1 - T^{k-1} x_k$ for $k \geq 1$, then $T z_k = T x_1 - T^k x_k = y - y = 0$, hence $z_k \in N(T) \subseteq T^{k-1}(X)$. It follows that $x_1 = z_k + T^{k-1} x_k \in T^{k-1}(X)$ for all $k \in \mathbb{N}$. Because of $y = T x_1$ we see that $y \in T(T^\infty(X))$.

(b) [7, Satz 4]. 

**Corollary 1.** An operator $T$ of Saphar type has the following properties:

(a) $T^n(X)$ is closed for each $n \in \mathbb{N}$, $T^\infty(X)$ is closed.

(b) $T$ maps $T^\infty(X)$ onto itself.

**Lemma 2.** Let $T$ be a relatively regular operator in $\mathcal{L}(X)$, $S$ a pseudo inverse for $T$ and $A \in \mathcal{L}(X)$. If $\|A\| < \|S\|^{-1}$, then

(a) $N(T-A) \subseteq (I-SA)^{-1}(N(T))$.

(b) $T(X) \subseteq (I-AS)^{-1}((T-A)(X))$.

**Proof:**

(a) Since $S(T-A) = ST-SA = I-SA-(I-ST)$, we have $(I-SA)^{-1}S(T-A) = I - (I-SA)^{-1}(I-ST)$. Let $x \in N(T-A)$, then $0 = (I-SA)^{-1}S(T-A)x = x-(I-SA)^{-1}(I-ST)x$, hence $x \in (I-SA)^{-1}((I-ST)(X)) = (I-SA)^{-1}((N(T))$.

(b) $(I-AS)TS = TS - ASTS = T(STS) - A(STS) = (T-A)STS$, thus $TS = (I-AS)^{-1}(T-A)STS$. This gives $T(X) = (TS)(X) = [(I-AS)^{-1}(T-A)](STS)(X) \subseteq (I-AS)^{-1}((T-A)(X))$. 

The next theorem shows that under the hypotheses of Lemma 2 it follows that $T$ is an operator of Saphar type, if equality holds in (a) (or (b)) for all $A$ with $\|A\|$ sufficiently small.
Theorem 2. Let $T$ be a relatively regular operator and $S \in \mathcal{L}(X)$ a pseudo inverse for $T$.

(a) If $N(T - \lambda I) = (I - \lambda S)^{-1}(N(T))$ for all $\lambda \in \mathbb{C}$ in a neighbourhood of 0, then $T \in S(X)$.

(b) If $T(X) = (I - \lambda S)^{-1}((T - \lambda I)(X))$ for all $\lambda \in \mathbb{C}$ in a neighbourhood of 0, then $T \in S(X)$.

Proof: Put $G(\lambda) = (I - \lambda S)^{-1} = \sum_{n=0}^{\infty} \lambda^n S^n$ $(|\lambda| < \|S\|^{-1})$, $P = I - ST$ and $Q = TS$. Recall that $P(X) = N(T)$ and $Q(X) = T(X)$.

(a) Since $N(T - \lambda I) = (I - \lambda S)^{-1}(N(T))$ in a neighbourhood $U$ of 0, we have $N(T - \lambda I) = (G(\lambda)P)(X)$ for $\lambda \in U$. It follows that

$$0 = (T - \lambda I)G(\lambda)P = \sum_{n=0}^{\infty} \lambda^n T^n S^n P - \sum_{n=0}^{\infty} \lambda^{n+1} S^n P = T P + \sum_{n=1}^{\infty} \lambda^n (T S^n P - S^{n-1} P) \quad \text{for all } \lambda \in U.$$

This gives

$$(1) \quad T S^n P = S^{n-1} P \quad \text{for all } n \in \mathbb{N}. \quad (2) \quad P = T^n S^n P.$$

We prove by induction that for $n \geq 1$

$$(2) \quad P = T^n S^n P.$$

(1) shows that (2) holds for $n = 1$. Now suppose that (2) holds for some integer $n \geq 1$. This gives

$$T^{n+1} S^{n+1} P = T^n (T S^{n+1} P) = T^n S^n P = P$$

by (1). Thus (2) is proved.

Since $N(T) = P(X)$, it follows that

$$N(T) = P(X) = (T^n S^n P)(X) \subseteq T^n(X) \quad \text{for all } n \geq 1.$$

Therefore $N(T) \subseteq T^\infty(X)$.

(b) Since $Q(X) = T(X) = (I - \lambda S)^{-1}((T - \lambda I)(X))$ in a neighbourhood $U$ of 0, we derive

$$G(\lambda) (T - \lambda I) = Q G(\lambda) (T - \lambda I) \quad (\lambda \in U).$$
Thus
\[ G(\lambda)(T - \lambda I) = \sum_{n=0}^{\infty} \lambda^n S^n T - \sum_{n=0}^{\infty} \lambda^{n+1} S^n \]
\[ = T + \sum_{n=1}^{\infty} \lambda^n (S^n T - S^{n+1}) \]
\[ = TS\left(T + \sum_{n=1}^{\infty} \lambda^n (S^n T - S^{n+1})\right) \]
\[ = T + \sum_{n=1}^{\infty} \lambda^n (TS^{n+1} T - TS^n) \quad (\lambda \in U) \]

and so
\[ S^n T - S^{n-1} = TS^{n+1} T - TS^n \quad \text{for all } n \in \mathbb{N}, \]

therefore \((S^n T - S^{n-1})P = (TS^{n+1} T - TS^n)P\). Since \(TP\) vanishes, we get
\[ TS^n P = S^{n-1} P \quad \text{for all } n \in \mathbb{N}. \]

But this is equation (1). As in part (a) of this proof, it follows that \(N(T) \subseteq T^\infty(X). \]

Let \(T \in \mathcal{R}(X)\) and let \(S\) be a pseudo inverse for \(T\). Define
\[ \mathcal{P}_S(T) = \{ A \in \mathcal{L}(X) : \|A\| < \|S\|^{-1} \text{ and } A(T^\infty(X)) \subseteq T^\infty(X)\} . \]

Remarks.

1. The condition \(A(T^\infty(X)) \subseteq T^\infty(X)\) is satisfied by any operator \(A\) which commutes with \(T\).

2. If \(\|A\| < \|S\|^{-1}\) then \(I - AS\) and \(I - SA\) are invertible in \(\mathcal{L}(X)\) and \(S(I - AS)^{-1} = (I - SA)^{-1} S\).

The next result, a perturbation theorem, is due to P. Saphar [6] (see also [1, Theorem 9 in §5]).

**Theorem 3.** Let \(T\) be an operator in \(\mathcal{S}(X)\) with \(g_2\)-inverse \(S\) and suppose that \(A \in \mathcal{P}_S(T)\). Then \(T - A\) is an operator of Saphar type with \(g_2\)-inverse \(S(I - AS)^{-1} = (I - SA)^{-1} S\) and
\[ N(T - A) \subseteq T^\infty(X) \subseteq (T - A)^\infty(X) . \]
Corollary 2. Under the hypotheses of Theorem 3 the equations

\[ N(T - A) = (I - SA)^{-1}(N(T)) \]

and

\[ T(X) = (I - AS)^{-1}((T - A)(X)) \]

are valid for all \( A \in P_S(T) \).

Proof: By Theorem 3, \((I - (I - SA)^{-1}S(T - A) = (I - SA)^{-1}(I - SA - S(T - A)) = (I - SA)^{-1}(I - ST)\) is a projection onto \(N(T - A)\). It follows that

\[ N(T - A) = (I - SA)^{-1}((I - ST)(X)) = (I - SA)^{-1}(N(T)) . \]

According to Theorem 3, \((T - A)S(I - AS)^{-1}\) is a projection onto \((T - A)(X)\), thus


This gives \(T(X) = (I - AS)^{-1}((T - A)(X))\). \(\blacksquare\)

From Theorem 2 and Corollary 2 we obtain immediately the following characterizations of operators of Saphar type.

Theorem 4. Let \( T \) be a relatively regular operator in \( L(X) \). Then the following assertions are equivalent:

(a) \( T \in S(X) \).

(b) \( N(T - A) = (I - SA)^{-1}(N(T)) \) whenever \( S \) is a \( g_2 \)-inverse for \( T \) and \( A \in P_S(T) \).

(c) \( T(X) = (I - AS)^{-1}((T - A)(X)) \) whenever \( S \) is a \( g_2 \)-inverse for \( T \) and \( A \in P_S(T) \).

(d) There is a pseudo inverse \( S \) for \( T \) such that

\[ N(T - A) = (I - SA)^{-1}(N(T)) \text{ for all } A \in P_S(T) . \]

(e) There is a pseudo inverse \( S \) for \( T \) such that

\[ T(X) = (I - AS)^{-1}(T - A)(X) \text{ for all } A \in P_S(T) . \]

(f) There is a pseudo inverse \( S \) for \( T \) such that

\[ N(T - \lambda I) = (I - \lambda S)^{-1}(N(T)) \text{ for all } |\lambda| < \|S\|^{-1} . \]

(g) There is a pseudo inverse \( S \) for \( T \) such that

\[ T(X) = (I - \lambda S)^{-1}((T - \lambda I)(X)) \text{ for all } |\lambda| < \|S\|^{-1} . \]
3 – Atkinson operators

Recall that \( T \in \mathcal{L}(X) \) is an Atkinson operator, if \( T \) is relatively regular and at least one of the \( \dim N(T) \), \( \operatorname{codim} T(X) \) is finite. The set of Atkinson operators will be denoted by \( A(X) \).

It is well known (see [1, Theorem 5 in §5]) that \( T \in \mathcal{A}(X) \) and \( \dim N(T) < \infty \) (resp. \( \operatorname{codim} T(X) < \infty \)) if and only if \( T \) is left invertible (resp. right invertible) modulo \( \mathcal{K}(X) \), where \( \mathcal{K}(X) \) denotes the closed ideal of compact operators on \( X \). Therefore the following assertions are valid:

(a) \( A(X) \) is open in \( \mathcal{L}(X) \).

(b) With \( T \) also \( T + K \) lies in \( \mathcal{A}(X) \) for every \( K \in \mathcal{K}(X) \).

(c) If \( T \in \mathcal{A}(X) \) then \( T^n \in \mathcal{A}(X) \) for every \( n \in \mathbb{N} \).

The above results are also valid for semi-Fredholm operators, i.e., operators with closed range having at least one of \( \dim N(T) \), \( \operatorname{codim} T(X) \) finite (see [3, §82]).

To each \( T \in \mathcal{L}(X), T \neq 0 \), we can associate a number \( \gamma(T) \), the minimum modulus of \( T \), which plays an important role in perturbation theory:

\[
\gamma(T) = \inf \left\{ \frac{\|Tx\|}{d(x, N(T))} : x \notin N(T) \right\},
\]

where \( d(x, N(T)) \) is the distance of \( x \) from \( N(T) \). It is of central importance that \( T \neq 0 \) has closed range if and only if \( \gamma(T) > 0 \) [4, Lemma 322].

**Lemma 3.** Suppose that \( T \in \mathcal{R}(X) \) and \( TST = T \). Then

\[
\|S\|^{-1} < \gamma(T).
\]

**Proof:** Let \( x \in X \). Then \( x - STx \in N(T) \), thus \( d(x, N(T)) = d(STx, N(T)) \leq \|STx\| \leq \|S\| \|Tx\| \). \( \blacksquare \)

The next theorem, the punctured neighbourhood theorem for Atkinson operators, is well-known. Part (c) of this theorem follows immediately from our results in Section 1.

**Theorem 5.** Let \( T \in \mathcal{A}(X) \) with \( \dim N(T) < \infty \) (resp. \( \operatorname{codim} T(X) < \infty \)) and pseudo inverse \( S \). Then

(a) \( T - A \in \mathcal{A}(X) \) for all \( A \in \mathcal{L}(X) \) with \( \|A\| < \|S\|^{-1} \).
(b) \( \dim N(T - \lambda I) \) is a constant \( \leq \dim N(T) \) (resp. \( \text{codim } (T - \lambda I)(X) \) is a constant \( \leq \text{codim } T(X) \)) for \( 0 < |\lambda| < \|S\|^{-1} \).

(c) \( \dim N(T - \lambda I) \) is constant (resp. \( \text{codim } (T - \lambda I)(X) \) is constant) for \( |\lambda| < \|S\|^{-1} \) if and only if \( T \) is an operator of Saphar type.

**Proof:**

(a) [1, Theorem 6 in §5].

(b) Theorem V.1.6 and Corollary V.1.7 in [2] show that \( T - \lambda I \) is a semi-Fredholm operator for \( |\lambda| < \gamma(T) \) and \( \dim N(T - \lambda I) \) is a constant \( \leq \dim N(T) \) (resp. \( \text{codim } (T - \lambda I)(X) \) is a constant \( \leq \text{codim } T(X) \)) in the annulus \( 0 < |\lambda| < \gamma(T) \). Since \( \|S\|^{-1} < \gamma(T) \), by Lemma 3, and \( T - \lambda I \in \mathcal{R}(X) \) for \( |\lambda| < \|S\|^{-1} \), by (a), the proof of (b) is complete.

At the beginning of this section we have seen that the set \( \mathcal{A}(X) \) of all Atkinson operators is open. The following assertion is obtained from [2, Theorem V.2.6] and shows that the set \( \mathcal{R}(X) \) of all relatively regular operators is in general not open.

**Lemma 4.** Suppose that \( T \in \mathcal{L}(X) \) has closed range and \( \dim N(T) = \text{codim } T(X) = \infty \). Then there exists a compact operator \( K \) such that \( T + \lambda K \) does not have closed range for all \( \lambda \neq 0 \).

With the help of the above result we now characterize the interior points of \( \mathcal{R}(X) \).

**Theorem 6.** For an operator \( T \) in \( \mathcal{L}(X) \) the following assertions are equivalent:

(a) \( T \) is an interior point of \( \mathcal{R}(X) \).

(b) \( T \in \mathcal{A}(X) \).

(c) \( T + K \in \mathcal{R}(X) \) for all \( K \in \mathcal{K}(X) \).

**Proof:**

(a)\(\Rightarrow\)(b): Suppose that \( T \) is an interior point of \( \mathcal{R}(X) \) but \( T \notin \mathcal{A}(X) \), hence \( \dim N(T) = \text{codim } T(X) = \infty \). Because of Lemma 4 there exists an operator \( K \in \mathcal{K}(X) \) such that \( T + \lambda K \) does not have closed range for all \( \lambda \neq 0 \), therefore \( T + \lambda K \notin \mathcal{R}(X) \) for all \( \lambda \neq 0 \). Since \( T \) is an interior point of \( \mathcal{R}(X) \), it follows that \( T + \lambda K \in \mathcal{R}(X) \) for \( |\lambda| \) sufficiently small, a contradiction. Hence \( T \in \mathcal{A}(X) \).

(b)\(\Rightarrow\)(a): Clear since \( \mathcal{A}(X) \) is open and \( \mathcal{A}(X) \subseteq \mathcal{R}(X) \).
(b)⇒(c): Since $T \in \mathcal{A}(X)$ implies $T + K \in \mathcal{A}(X)$ for every $K \in \mathcal{K}(X)$, we get $T + K \in \mathcal{R}(X)$ for every $K \in \mathcal{K}(X)$.

(c)⇒(b): We have $T + \lambda K \in \mathcal{R}(X)$ for all $\lambda \in \mathbb{C}$ and all $K \in \mathcal{K}(X)$, thus (put $\lambda = 0$) $T$ is relatively regular. Lemma 4 shows that $\dim N(T) < \infty$ or $\text{codim } T(X) < \infty$.

Let us write $\mathcal{C}(X)$ for the set of all operators $T \in \mathcal{L}(X)$ with $T(X)$ closed. Then, by Lemma 4, $\mathcal{C}(X)$ is in general not open. If we go through the above proof and make the necessary modifications, we see that for $T \in \mathcal{L}(X)$ the following assertions are equivalent:

(a) $T$ is an interior point of $\mathcal{C}(X)$.
(b) $T$ is a semi-Fredholm operator.
(c) $T + K \in \mathcal{C}(X)$ for all $K \in \mathcal{K}(X)$.

4 – Mapping properties

We begin this section with products of relatively regular operators.

Lemma 5.

(a) Let $T_1, T_2 \in \mathcal{R}(X)$ with pseudo inverses $S_1$ and $S_2$, respectively. If $N(T_1) \subseteq T_2(X)$, then $T_1 T_2 \in \mathcal{R}(X)$ and $S_2 S_1$ is a pseudo inverse for $T_1 T_2$.

(b) Let $T_1, \ldots, T_m$ be relatively regular operators with pseudo inverses $S_1, \ldots, S_m$, respectively. If

$$N(T_1 \cdots T_k) \subseteq T_{k+1}(X) \quad \text{for } k = 1, \ldots, m - 1,$$

then $T_1 \cdots T_m$ is relatively regular and $S_1 \cdots S_m$ is a pseudo inverse for $T_1 \cdots T_m$.

(c) Let $T \in \mathcal{S}(X)$ and $T S T = T$ for some $S \in \mathcal{L}(X)$. Then $T^n \in \mathcal{S}(X)$ and $T^n S^n T^n = T^n$ for all $n \in \mathbb{N}$.

Proof:

(a) Since $(I - S_1 T_1)(X) = N(T_1) \subseteq T_2(X) = (T_2 S_2)(X)$, it follows that $T_2 S_2(I - S_1 T_1) = I - S_1 T_1$, hence $T_2 S_2 S_1 T_1 = T_2 S_2 - I + S_1 T_1$. Then we have

$$T_1 T_2 (S_2 S_1) T_1 T_2 = T_1 (T_2 S_2 - I + S_1 T_1) T_2$$
$$= T_1 \underbrace{T_2 S_2 T_1}_T - T_1 T_2 + \underbrace{T_1 S_1 T_1 T_2}_T$$
$$= T_1 T_2.$$
(b) By (a) and (3), \( T_1T_2(S_2S_1)T_1T_2 = T_1T_2 \). Suppose that
\[
T_1 \cdots T_j (S_j \cdots S_1) T_1 \cdots T_j = T_1 \cdots T_j
\]
for some \( j \in \{1, \ldots, m-1\} \). (3) implies that
\[
N(T_1 \cdots T_j) \subseteq T_{j+1}(X) ,
\]
consequently, by (a),
\[
(T_1 \cdots T_j) T_{j+1}(S_{j+1}(S_j \cdots S_1)) (T_1 \cdots T_j) T_{j+1} = (T_1 \cdots T_j) T_{j+1} .
\]

(c) Since \( T \in \mathcal{S}(X) \), \( N(T) \subseteq T^n(X) \) for \( n \geq 1 \), thus \( N(T^n) \subseteq T(X) \) for \( n \geq 1 \) [4, Lemma 511]. Now use (b). ■

In this section, we shall consider the following question: If \( T \) is an operator in \( \mathcal{L}(X) \) and \( f \) is a function in \( \mathcal{H}(T) \), when is \( f(T) \) an operator of Saphar type? Furthermore, if \( f(T) \in \mathcal{S}(X) \), we shall consider the problem of finding a pseudo inverse for \( f(T) \).

To this end, we need some concepts from [7] and [9]. We define
\[
\rho_{rr}(T) = \left\{ \lambda \in \mathbb{C} : T - \lambda I \in \mathcal{S}(X) \right\}
\]
and
\[
\rho_k(T) = \left\{ \lambda \in \mathbb{C} : (T - \lambda I)(X) \text{ is closed and } N(T - \lambda I) \subseteq (T - \lambda I)\infty(X) \right\} .
\]
Then \( \rho(T) \subseteq \rho_{rr}(T) \subseteq \rho_k(T) \). Theorem 3 in [4] shows that \( \rho_k(T) \) is open. By Theorem 1, \( \rho_{rr}(T) \) is open. Setting
\[
\sigma_{rr}(T) = \mathbb{C} \setminus \rho_{rr}(T) \quad \text{and} \quad \sigma_k(T) = \mathbb{C} \setminus \rho_k(T) ,
\]
we obtain two ‘essential spectra’ of \( T \). We have
\[
\sigma_k(T) \subseteq \sigma_{rr}(T) \subseteq \sigma(T) .
\]
We showed in [7, Satz 2] that \( \partial \sigma(T) \subseteq \sigma_k(T) \), hence \( \sigma_k(T) \neq \emptyset \). It was shown in [9, Theorem 3] that
\[
f(\sigma_{rr}(T)) = \sigma_{rr}(f(T)) \quad \text{for } T \in \mathcal{L}(X) \text{ and } f \in \mathcal{H}(T) .
\]

**Theorem 7.** Let \( T \in \mathcal{L}(X) \), \( f \in \mathcal{H}(T) \) and let \( Z(f) \) denote the set of zeros of \( f \) in \( \sigma(T) \). Then \( f(T) \) is an operator of Saphar type if and only if \( Z(f) \subseteq \rho_{rr} \). In this case \( Z(f) \) is finite or empty.
Proof: Since \( f(T) \in \mathcal{S}(X) \iff \sigma_{rr}(f(T)) = f(\sigma_{rr}(T)) \iff Z(f) \subseteq \rho_{rr}(T), \) the first assertion is proved. If \( Z(f) \subseteq \rho_{rr}(T), \) then \( f \) does not vanish on \( \sigma_k(T), \) since \( \sigma_k(T) \subseteq \sigma_{rr}(T). \) Satz 3 in [7] shows that \( f \) has at most a finite number of zeros in \( \sigma(T). \) \( \blacksquare \)

We are now going to calculate a pseudo inverse for \( f(T) \in \mathcal{S}(X). \)

Theorem 8. Suppose

(a) \( T \in \mathcal{L}(X) \) and \( f \in \mathcal{H}(T) \) are such that \( f(T) \) is an operator of Saphar type,

(b) \( \lambda_1, \ldots, \lambda_m \) are the zeros of \( f \) in \( \sigma(T) \) with respective orders \( n_1, \ldots, n_m, \)

(c) \( S_j \) is a pseudo inverse for \( T - \lambda_j I \) \( (j = 1, \ldots, m). \)

Put

\[
S = \left( \prod_{j=1}^{m} S_j^{n_j} \right) h(T)^{-1},
\]

where \( h \) is a function in \( \mathcal{H}(T) \) such that \( f(\lambda) = \left( \prod_{j=1}^{m} (\lambda - \lambda_j)^{n_j} \right) h(\lambda). \) Then \( S \) is a pseudo inverse for \( f(T). \)

Proof: Put \( p(\lambda) = \prod_{j=1}^{m} (\lambda - \lambda_j)^{n_j}. \) Then \( f(\lambda) = p(\lambda) h(\lambda), \) thus

\[
f(T) = p(T) h(T) = h(T) p(T)
\]

and \( h(T) \) is invertible in \( \mathcal{L}(X). \) Use [3, Satz 80.1] to derive

\[
N \left( \prod_{j=1}^{k} (T - \lambda_j I)^{n_j} \right) = N \left( (T - \lambda_1 I)^{n_1} \oplus \cdots \oplus (T - \lambda_k I)^{n_k} \right)
\]

\[
\subseteq (T - \lambda_{k+1} I)^{n_{k+1}}(X)
\]

for \( k = 1, \ldots, m - 1. \) By Lemma 5(c), \( (T - \lambda_j I)^{n_j} \) is relatively regular and \( S_j^{n_j} \) is a pseudo inverse for \( (T - \lambda_j I)^{n_j} \) \( (j = 1, \ldots, m). \) Thus, using (4) and Lemma 5(b), we conclude that \( p(T) \) is relatively regular and \( B = \prod_{j=1}^{m} S_j^{n_j} \) is a pseudo inverse for \( p(T). \) Therefore

\[
f(T) S f(T) = f(T) B h(T)^{-1} f(T) = h(T) p(T) B h(T)^{-1} h(T) p(T)
\]

\[
= h(T) p(T) B p(T) = h(T) p(T) = f(T) \ . \ \blacksquare
\]
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