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# ON OPERATORS OF SAPHAR TYPE

Christoph Schmoeger

**Abstract:** A bounded linear operator $T$ on a complex Banach space $X$ is called an operator of Saphar type, if $T$ is relatively regular and if its null space is contained in its generalized range $\bigcap_{n=1}^{\infty} T^n(X)$. This paper contains some characterizations of operators of Saphar type. Furthermore, for a function $f$ admissible in the analytic calculus, we obtain a necessary and sufficient condition in order that $f(T)$ is an operator of Saphar type.

## 1 – Terminology and introduction

Let $X$ denote a Banach space over the complex field $\mathbb{C}$ and let $\mathcal{L}(X)$ be the Banach algebra of all bounded linear operators on $X$. If $T \in \mathcal{L}(X)$, we denote by $N(T)$ the kernel and by $T(X)$ the range of $T$. The spectrum of $T$ is denoted by $\sigma(T)$. The resolvent set $\rho(T)$ is defined by $\rho(T) = \mathbb{C} \backslash \sigma(T)$. We write $\mathcal{H}(T)$ for the set of all complex valued functions which are analytic in some neighbourhood of $\sigma(T)$. For $f \in \mathcal{H}(T)$, the operator $f(T)$ is defined by the well known analytic calculus (see [3, §99]).

Let $T \in \mathcal{L}(X)$. Then an operator $S \in \mathcal{L}(X)$ will be called a pseudo inverse for $T$ if

$$TST = T .$$

We then say that $T$ is relatively regular. A relativity regular operator $T$ is called an operator of Saphar type if its null space $N(T)$ is contained in its generalized range

$$T^\infty(X) = \bigcap_{n=1}^{\infty} T^n(X) .$$

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We write $\mathcal{S}(X)$ for the set of all operators of Saphar type. This class of operators has been studied by P. Saphar [6] (see also [1]). Although operators in $\mathcal{S}(X)$ seem rather special, they have an important property:

**Theorem 1.** *$T$ is an operator of Saphar type if and only if there is a neighbourhood $U \subseteq \mathbb{C}$ of 0 and a holomorphic function $F\colon U \to \mathcal{L}(X)$ such that*

$$(T-\lambda I)\,F(\lambda)(T-\lambda I) = T-\lambda I \quad \text{for all } \lambda \in U\ .$$

**Proof:** [5, Théorème 2.6] or [8, Theorem 1.4]. ∎

In [6] Saphar considered the following question: if $T \in \mathcal{S}(X)$ and $A$ is an operator in $\mathcal{L}(X)$, when is $T-A$ an operator of Saphar type? In Section 2 of this paper we use the perturbation results of [6] to characterize operators of Saphar type.

Section 3 deals with Atkinson operators, i.e. relatively regular operators having at least one of $\dim N(T)$, codim $T(X)$ finite. The main results in Section 1 allow us to give a simple proof of the following well known fact: if $T$ is an Atkinson operator, then $\dim N(T-\lambda I)$ (resp. codim $(T-\lambda I)(X)$) is constant in a neighbourhood of 0 if and only if $T \in \mathcal{S}(X)$.

In Section 4 of the present paper we obtain a necessary and sufficient condition in order that $f(T) \in \mathcal{S}(X)$. Moreover, if $f(T)$ is an operator of Saphar type, we obtain a formula for a pseudo inverse for $f(T)$.

We close this section with some definitions, notations and preliminary results which we need in the sequel.

$\mathcal{R}(X)$ will denote the set of all relatively regular operators in $\mathcal{L}(X)$. Let $T \in \mathcal{L}(X)$. If $S \in \mathcal{L}(X)$ satisfies the two equations

$$TST = T \quad \text{and} \quad STS = S$$

then $S$ will be called a $g_2$-inverse for $T$. We shall make frequent use of the following results which will be quoted without further reference.

**1)** $T \in \mathcal{R}(X)$ if and only if $N(T)$ and $T(X)$ are closed complemented subspaces of $X$ (see [3, Satz 74.2]).

**2)** If $TST = T$ for some operator $S$, then $TS$ is a projection onto $T(X)$ and $I - ST$ is a projection onto $N(T)$ (see [3, p. 385]).

**3)** If $S$ is a peudo inverse for $T$, then $STS$ is a $g_2$-inverse for $T$ (simple verification).

## 2 – Perturbation properties

We begin with the following basic facts.

**Lemma 1.** *Let* $T \in \mathcal{L}(X)$.

(**a**) *If* $N(T) \subseteq T^\infty(X)$, *then* $T(T^\infty(X)) = T^\infty(X)$.

(**b**) *If* $N(T) \subseteq T^\infty(X)$ *and* $T(X)$ *is closed, then* $T^n(X)$ *is closed for each* $n \in \mathbb{N}$, *hence* $T^\infty(X)$ *is closed.*

**Proof:**

(**a**) The inclusion $T(T^\infty(X)) \subseteq T^\infty(X)$ is obvious. Let $y$ be an arbitrary element of $T^\infty(X)$. Then for every $k=1,2,...$ there exists $x_k \in X$ so that $y=T^k x_k$. If we set $z_k = x_1 - T^{k-1}x_k$ for $k \geq 1$, then $Tz_k = Tx_1 - T^k x_k = y - y = 0$, hence $z_k \in N(T) \subseteq T^{k-1}(X)$. It follows that $x_1 = z_k + T^{k-1}x_k \in T^{k-1}(X)$ for all $k \in \mathbb{N}$. Because of $y = Tx_1$ we see that $y \in T(T^\infty(X))$.

(**b**) [7, Satz 4]. ∎

**Corollary 1.** *An operator* $T$ *of Saphar type has the following properties:*

(**a**) $T^n(X)$ *is closed for each* $n \in \mathbb{N}$, $T^\infty(X)$ *is closed.*

(**b**) $T$ *maps* $T^\infty(X)$ *onto itself.*

**Lemma 2.** *Let* $T$ *be a relatively regular operator in* $\mathcal{L}(X)$, $S$ *a pseudo inverse for* $T$ *and* $A \in \mathcal{L}(X)$. *If* $\|A\| < \|S\|^{-1}$, *then*

(**a**) $N(T-A) \subseteq (I-SA)^{-1}(N(T))$.

(**b**) $T(X) \subseteq (I-AS)^{-1}((T-A)(X))$.

**Proof:**

(**a**) Since $S(T-A) = ST-SA = I-SA-(I-ST)$, we have $(I-SA)^{-1}S(T-A) = I-(I-SA)^{-1}(I-ST)$. Let $x \in N(T-A)$, then $0 = (I-SA)^{-1}S(T-A)\,x = x-(I-SA)^{-1}(I-ST)\,x$, hence $x \in (I-SA)^{-1}((I-ST)(X)) = (I-SA)^{-1}((N(T))$.

(**b**) $(I-AS)TS = TS-ASTS = T(STS)-A(STS) = (T-A)STS$, thus $TS = (I-AS)^{-1}(T-A)STS$. This gives $T(X) = (TS)(X) = [(I-AS)^{-1}(T-A)](STS)(X) \subseteq (I-AS)^{-1}((T-A)(X))$. ∎

The next theorem shows that under the hypotheses of Lemma 2 it follows that $T$ is an operator of Saphar type, if equality holds in (a) (or (b)) for *all* $A$ with $\|A\|$ sufficiently small.

**Theorem 2.** *Let $T$ be a relatively regular operator and $S \in \mathcal{L}(X)$ a pseudo inverse for $T$.*

(**a**) *If $N(T-\lambda I) = (I-\lambda S)^{-1}(N(T))$ for all $\lambda \in \mathbb{C}$ in a neighbourhood of 0, then $T \in \mathcal{S}(X)$.*

(**b**) *If $T(X) = (I-\lambda S)^{-1}((T-\lambda I)(X))$ for all $\lambda \in \mathbb{C}$ in a neighbourhood of 0, then $T \in \mathcal{S}(X)$.*

**Proof:** Put $G(\lambda) = (I-\lambda S)^1 = \sum_{n=0}^{\infty} \lambda^n S^n$ $(|\lambda| < \|S\|^{-1})$, $P = I - ST$ and $Q = TS$. Recall that $P(X) = N(T)$ and $Q(X) = T(X)$.

(**a**) Since $N(T-\lambda I) = (I-\lambda S)^{-1}(N(T))$ in a neighbourhood $U$ of 0, we have $N(T-\lambda I) = (G(\lambda)P)(X)$ for $\lambda \in U$. It follows that

$$0 = (T-\lambda I)\, G(\lambda)\, P = \sum_{n=0}^{\infty} \lambda^n\, T\, S^n\, P - \sum_{n=0}^{\infty} \lambda^{n+1}\, S^n\, P$$

$$= \underbrace{TP}_{=0} + \sum_{n=1}^{\infty} \lambda^n\,(T\, S^n\, P - S^{n-1} P) \quad \text{for all} \quad \lambda \in U\ .$$

This gives

$$TS^nP = S^{n-1}P \quad \text{for all} \quad n \in \mathbb{N}\ . \tag{1}$$

We prove by induction that for $n \ge 1$

$$P = T^n S^n P\ . \tag{2}$$

(1) shows that (2) holds for $n = 1$. Now suppose that (2) hols for some integer $n \ge 1$. This gives

$$T^{n+1}S^{n+1}P = T^n(TS^{n+1}P) = T^nS^nP = P$$

by (1). Thus (2) is proved.

Since $N(T) = P(X)$, it follows that

$$N(T) = P(X) = (T^nS^nP)(X) \subseteq T^n(X) \quad \text{for all} \quad n \ge 1\ .$$

Therefore $N(T) \subseteq T^{\infty}(X)$.

(**b**) Since $Q(X) = T(X) = (I-\lambda S)^{-1}((T-\lambda I)(X))$ in a neighbourhood $U$ of 0, we derive

$$G(\lambda)\,(T-\lambda I) = Q\, G(\lambda)\,(T-\lambda I) \quad (\lambda \in U)\ .$$

Thus

$$\begin{aligned} G(\lambda)(T-\lambda I) &= \sum_{n=0}^{\infty} \lambda^n S^n T - \sum_{n=0}^{\infty} \lambda^{n+1} S^n \\ &= T + \sum_{n=1}^{\infty} \lambda^n (S^n T - S^{n-1}) \\ &= TS\Big(T + \sum_{n=1}^{\infty} \lambda^n (S^n T - S^{n-1})\Big) \\ &= T + \sum_{n=1}^{\infty} \lambda^n (TS^{n+1} T - TS^n) \qquad (\lambda \in U) \end{aligned}$$

and so

$$S^n T - S^{n-1} = TS^{n+1} T - TS^n \qquad \text{for all} \quad n \in \mathbb{N} \ ,$$

therefore $(S^n T - S^{n-1})P = (TS^{n+1}T - TS^n)P$. Since $TP$ vanishes, we get

$$TS^n P = S^{n-1} P \qquad \text{for all} \quad n \in \mathbb{N} \ .$$

But this is equation (1). As in part (a) of this proof, it follows that $N(T) \subseteq T^\infty(X)$. ∎

Let $T \in \mathcal{R}(X)$ and let $S$ be a pseudo inverse for $T$. Define

$$\mathcal{P}_S(T) = \Big\{ A \in \mathcal{L}(X) \colon \ \|A\| < \|S\|^{-1} \text{ and } A(T^\infty(X)) \subseteq T^\infty(X) \Big\} \ .$$

**Remarks.**

**1**. The condition $A(T^\infty(X)) \subseteq T^\infty(X)$ is satisfied by any operator $A$ which commutes with $T$.

**2**. If $\|A\| < \|S\|^{-1}$ then $I - AS$ and $I - SA$ are invertible in $\mathcal{L}(X)$ and $S(I - AS)^{-1} = (I - SA)^{-1} S$.

The next result, a perturbation theorem, is due to P. Saphar [6] (see also [1, Theorem 9 in §5]).

**Theorem 3.** *Let $T$ be an operator in $\mathcal{S}(X)$ with $g_2$-inverse $S$ and suppose that $A \in \mathcal{P}_S(T)$. Then $T - A$ is an operator of Saphar type with $g_2$-inverse $S(I - AS)^{-1} = (I - SA)^{-1} S$ and*

$$N(T - A) \subseteq T^\infty(X) \subseteq (T - A)^\infty(X) \ .$$

**Corollary 2.** *Under the hypotheses of Theorem 3 the equations*

$$N(T-A) = (I-SA)^{-1}(N(T))$$

*and*

$$T(X) = (I-AS)^{-1}((T-A)(X))$$

*are valid for all* $A \in \mathcal{P}_S(T)$.

**Proof:** By Theorem 3, $I-(I-SA)^{-1}S(T-A) = (I-SA)^{-1}[I-SA-S(T-A)] = (I-SA)^{-1}(I-ST)$ is a projection onto $N(T-A)$. It follows that

$$N(T-A) = (I-SA)^{-1}((I-ST)(X)) = (I-SA)^{-1}(N(T)) \ .$$

According to Theorem 3, $(T-A)\,S(I-AS)^{-1}$ is a projection onto $(T-A)(X)$, thus $(T-A)(X) = (T-A)\,S(I-AS)^{-1}(X) = (T-A)(S(X)) = (T-A)(ST)(X) = (TST-AST)(X) = (T-AST)(X) = (I-AS)(T(X))$. This gives $T(X) = (I-AS)^{-1}((T-A)(X))$. ∎

From Theorem 2 and Corollary 2 we obtain immediately the following characterizations of operators of Saphar type.

**Theorem 4.** *Let* $T$ *be a relatively regular operator in* $\mathcal{L}(X)$. *Then the following assertions are equivalent:*

(**a**) $T \in \mathcal{S}(X)$.

(**b**) $N(T-A) = (I-SA)^{-1}(N(T))$ *whenever* $S$ *is a* $g_2$*-inverse for* $T$ *and* $A \in \mathcal{P}_S(T)$.

(**c**) $T(X) = (I-AS)^{-1}((T-A)(X))$ *whenever* $S$ *is a* $g_2$*-inverse for* $T$ *and* $A \in \mathcal{P}_S(T)$.

(**d**) *There is a pseudo inverse* $S$ *for* $T$ *such that*

$$N(T-A) = (I-SA)^{-1}(N(T)) \quad \text{for all } \ A \in \mathcal{P}_S(T) \ .$$

(**e**) *There is a pseudo inverse* $S$ *for* $T$ *such that*

$$T(X) = (I-AS)^{-1}(T-A)(X) \quad \text{for all } \ A \in \mathcal{P}_S(T) \ .$$

(**f**) *There is a pseudo inverse* $S$ *for* $T$ *such that*

$$N(T-\lambda I) = (I-\lambda S)^{-1}(N(T)) \quad \text{for all } \ |\lambda| < \|S\|^{-1} \ .$$

(**g**) *There is a pseudo inverse* $S$ *for* $T$ *such that*

$$T(X) = (I-\lambda S)^{-1}((T-\lambda I)(X)) \quad \text{for all } \ |\lambda| < \|S\|^{-1} \ .$$

## 3 – Atkinson operators

Recall that $T \in \mathcal{L}(X)$ is an Atkinson operator, if $T$ is relatively regular and at least one of the dim $N(T)$, codim $T(X)$ is finite. The set of Atkinson operators will be denoted by $\mathcal{A}(X)$.

It is well known (see [1, Theorem 5 in §5]) that $T \in \mathcal{A}(X)$ and dim $N(T) < \infty$ (resp. codim $T(X) < \infty$) if and only if $T$ is left invertible (resp. right invertible) modulo $\mathcal{K}(X)$, where $\mathcal{K}(X)$ denotes the closed ideal of compact operators on $X$. Therefore the following assertions are valid:

(**a**) $\mathcal{A}(X)$ is open in $\mathcal{L}(X)$.

(**b**) With $T$ also $T + K$ lies in $\mathcal{A}(X)$ for every $K \in \mathcal{K}(X)$.

(**c**) If $T \in \mathcal{A}(X)$ then $T^n \in \mathcal{A}(X)$ for every $n \in \mathbb{N}$.

The above results are also valid for semi-Fredholm operators, i.e., operators with closed range having at least one of $\dim N(T)$, codim $T(X)$ finite (see [3, §82]).

To each $T \in \mathcal{L}(X)$, $T \neq 0$, we can associate a number $\gamma(T)$, the minimum modulus of $T$, wich plays an important role in perturbation theory:

$$\gamma(T) = \inf\left\{\frac{\|Tx\|}{d(x, N(T))} : x \notin N(T)\right\} ,$$

where $d(x, N(T))$ is the distance of $x$ from $N(T)$. It is of central importance that $T \neq 0$ has closed range if and only if $\gamma(T) > 0$ [4, Lemma 322].

**Lemma 3.** *Suppose that $T \in \mathcal{R}(X)$ and $TST = T$. Then*

$$\|S\|^{-1} < \gamma(T) .$$

**Proof:** Let $x \in X$. Then $x - STx \in N(T)$, thus $d(x, N(T)) = d(STx, N(T)) \leq \|STx\| \leq \|S\| \, \|Tx\|$. ∎

The next theorem, the punctured neighbourhood theorem for Atkinson operators, is well-known. Part (c) of this theorem follows immediately from our results in Section 1.

**Theorem 5.** *Let $T \in \mathcal{A}(X)$ with dim $N(T) < \infty$ (resp. codim $T(X) < \infty$) and pseudo inverse $S$. Then*

(**a**) $T - A \in \mathcal{A}(X)$ *for all* $A \in \mathcal{L}(X)$ *with* $\|A\| < \|S\|^{-1}$.

(**b**) *dim* $N(T-\lambda I)$ *is a constant* $\leq$ *dim* $N(T)$ *(resp. codim* $(T-\lambda I)(X)$ *is a constant* $\leq$ *codim* $T(X)$*)* *for* $0<|\lambda|<\|S\|^{-1}$.

(**c**) *dim* $N(T-\lambda I)$ *is constant (resp. codim* $(T-\lambda I)(X)$ *is constant) for* $|\lambda|<\|S\|^{-1}$ *if and only if* $T$ *is an operator of Saphar type.*

**Proof:**

(**a**) [1, Theorem 6 in §5].

(**b**) Theorem V.1.6 and Corollary V.1.7 in [2] show that $T-\lambda I$ is a semi-Fredholm operator for $|\lambda|<\gamma(T)$ and $\dim N(T-\lambda I)$ is a constant $\leq \dim N(T)$ (resp. codim $(T-\lambda I)(X)$ is a constant $\leq$ codim $T(X)$) in the annulus $0<|\lambda|<\gamma(T)$. Since $\|S\|^{-1}<\gamma(T)$, by Lemma 3, and $T-\lambda I\in\mathcal{R}(X)$ for $|\lambda|<\|S\|^{-1}$, by (a), the proof of (b) is complete. ∎

At the beginning of this section we have seen that the set $\mathcal{A}(X)$ of all Atkinson operators is open. The following assertion is obtained from [2, Theorem V.2.6] and shows that the set $\mathcal{R}(X)$ of all relatively regular operators is in general not open.

**Lemma 4.** *Suppose that* $T\in\mathcal{L}(X)$ *has closed range and* $\dim N(T)=$ *codim* $T(X)=\infty$. *Then there exists a compact operator* $K$ *such that* $T+\lambda K$ *does not have closed range for all* $\lambda\neq 0$.

With the help of the above result we now characterize the interior points of $\mathcal{R}(X)$.

**Theorem 6.** *For an operator* $T$ *in* $\mathcal{L}(X)$ *the folllowing assertions are equivalent:*

(**a**) $T$ *is an interior point of* $\mathcal{R}(X)$.

(**b**) $T\in\mathcal{A}(X)$.

(**c**) $T+K\in\mathcal{R}(X)$ *for all* $K\in\mathcal{K}(X)$.

**Proof:**

(**a**)⇒(**b**)**:** Suppose that $T$ is an interior point of $\mathcal{R}(X)$ but $T\notin\mathcal{A}(X)$, hence $\dim N(T)=$ codim $T(X)=\infty$. Because of Lemma 4 there exists an operator $K\in\mathcal{K}(X)$ such that $T+\lambda K$ does not have closed range for all $\lambda\neq 0$, therefore $T+\lambda K\notin\mathcal{R}(X)$ for all $\lambda\neq 0$. Since $T$ is an interior point of $\mathcal{R}(X)$, it follows that $T+\lambda K\in\mathcal{R}(X)$ for $|\lambda|$ sufficiently small, a contradiction. Hence $T\in\mathcal{A}(X)$.

(**b**)⇒(**a**)**:** Clear since $\mathcal{A}(X)$ is open and $\mathcal{A}(X)\subseteq\mathcal{R}(X)$.

(**b**)$\Rightarrow$(**c**)**:** Since $T \in \mathcal{A}(X)$ implies $T + K \in \mathcal{A}(X)$ for every $K \in \mathcal{K}(X)$, we get $T + K \in \mathcal{R}(X)$ for every $K \in \mathcal{K}(X)$.

(**c**)$\Longrightarrow$(**b**)**:** We have $T + \lambda K \in \mathcal{R}(X)$ for all $\lambda \in \mathbb{C}$ and all $K \in \mathcal{K}(X)$, thus (put $\lambda = 0$) $T$ is relatively regular. Lemma 4 shows that $\dim N(T) < \infty$ or $\operatorname{codim} T(X) < \infty$. ∎

Let us write $\mathcal{C}(X)$ for the set of all operators $T \in \mathcal{L}(X)$ with $T(X)$ closed. Then, by Lemma 4, $\mathcal{C}(X)$is in general not open. If we go through the above proof and make the necessary modifications, we see that for $T \in \mathcal{L}(X)$ the following assertions are equivalent:

(**a**) $T$ is an interior point of $\mathcal{C}(X)$.

(**b**) $T$ is a semi-Fredholm operator.

(**c**) $T + K \in \mathcal{C}(X)$ for all $K \in \mathcal{K}(X)$.

## 4 – Mapping properties

We begin this section with products of relatively regular operators.

**Lemma 5.**

(**a**) *Let $T_1, T_2 \in \mathcal{R}(X)$ with pseudo inverses $S_1$ and $S_2$, respectively. If $N(T_1) \subseteq T_2(X)$, then $T_1T_2 \in \mathcal{R}(X)$ and $S_2S_1$ is a pseudo inverse for $T_1T_2$.*

(**b**) *Let $T_1, \ldots, T_m$ be relatively regular operators with pseudo inverses $S_1, \ldots, S_m$, respectively. If*

$$N(T_1 \cdots T_k) \subseteq T_{k+1}(X) \quad \text{for } k = 1, \ldots, m-1 \;, \tag{3}$$

*then $T_1 \cdots T_m$ is relatively regular and $S_1 \cdots S_m$ is a pseudo inverse for $T_1 \cdots T_m$.*

(**c**) *Let $T \in \mathcal{S}(X)$ and $TST = T$ for some $S \in \mathcal{L}(X)$. Then $T^n \in \mathcal{S}(X)$ and $T^nS^nT^n = T^n$ for all $n \in \mathbb{N}$.*

**Proof:**

(**a**) Since $(I - S_1T_1)(X) = N(T_1) \subseteq T_2(X) = (T_2S_2)(X)$, it follows that $T_2S_2(I - S_1T_1) = I - S_1T_1$, hence $T_2S_2S_1T_1 = T_2S_2 - I + S_1T_1$. Then we have

$$\begin{aligned} T_1T_2(S_2S_1)T_1T_2 &= T_1(T_2S_2 - I + S_1T_1)T_2 \\ &= T_1 \underbrace{T_2S_2T_2}_{=T_2} - T_1T_2 + \underbrace{T_1S_1T_1}_{=T_1} T_2 \\ &= T_1T_2 \;. \end{aligned}$$

(**b**) By (a) and (3), $T_1T_2(S_2S_1)T_1T_2 = T_1T_2$. Suppose that

$$T_1 \cdots T_j\,(S_j \cdots S_1)\,T_1 \cdots T_j = T_1 \cdots T_j$$

for some $j \in \{1, \ldots, m-1\}$. (3) implies that

$$N(T_1 \cdots T_j) \subseteq T_{j+1}(X)\ ,$$

consequently, by (a),

$$(T_1 \cdots T_j)\,T_{j+1}(S_{j+1}(S_j \cdots S_1))\,(T_1 \cdots T_j)\,T_{j+1} = (T_1 \cdots T_j)\,T_{j+1}\ .$$

(**c**) Since $T \in \mathcal{S}(X)$, $N(T) \subseteq T^n(X)$ for $n \geq 1$, thus $N(T^n) \subseteq T(X)$ for $n \geq 1$ [4, Lemma 511]. Now use (b). ∎

In this section, we shall consider the following question: If $T$ is an operator in $\mathcal{L}(X)$ and $f$ is a function in $\mathcal{H}(T)$, when is $f(T)$ an operator of Saphar type? Furthermore, if $f(T) \in \mathcal{S}(X)$, we shall consider the problem of finding a pseudo inverse for $f(T)$.

To this end, we need some concepts from [7] and [9]. We define

$$\rho_{rr}(T) = \Big\{\lambda \in \mathbb{C}\colon\ T - \lambda I \in \mathcal{S}(X)\Big\}$$

and

$$\rho_k(T) = \Big\{\lambda \in \mathbb{C}\colon\ \ (T-\lambda I)(X) \text{ is closed \ and } \ N(T-\lambda I) \subseteq (T-\lambda I)^\infty(X)\Big\}\ .$$

Then $\rho(T) \subseteq \rho_{rr}(T) \subseteq \rho_k(T)$. Theorem 3 in [4] shows that $\rho_k(T)$ is open. By Theorem 1, $\rho_{rr}(T)$ is open. Setting

$$\sigma_{rr}(T) = \mathbb{C}\backslash\rho_{rr}(T) \quad \text{ and } \quad \sigma_k(T) = \mathbb{C}\backslash\rho_k(T)\ ,$$

we obtain two 'essential spectra' of $T$. We have

$$\sigma_k(T) \subseteq \sigma_{rr}(T) \subseteq \sigma(T)\ .$$

We showed in [7, Satz 2] that $\partial\sigma(T) \subseteq \sigma_k(T)$, hence $\sigma_k(T) \neq \emptyset$. It was shown in [9, Theorem 3] that

$$f(\sigma_{rr}(T)) = \sigma_{rr}(f(T)) \quad \text{ for } \ T \in \mathcal{L}(X) \ \text{ and } \ f \in \mathcal{H}(T)\ .$$

**Theorem 7.** *Let $T \in \mathcal{L}(X)$, $f \in \mathcal{H}(T)$ and let $Z(f)$ denote the set of zeros of $f$ in $\sigma(T)$. Then $f(T)$ is an operator of Saphar type if and only if $Z(f) \subseteq \rho_{rr}$. In this case $Z(f)$ is finite or empty.*

**Proof:** Since $f(T)\in\mathcal{S}(X)\Longleftrightarrow 0\notin\sigma_{rr}(f(T))=f(\sigma_{rr}(T))\Longleftrightarrow Z(f)\subseteq\rho_{rr}(T)$, the first assertion is proved. If $Z(f)\subseteq\rho_{rr}(T)$, then $f$ does not vanish on $\sigma_k(T)$, since $\sigma_k(T)\subseteq\sigma_{rr}(T)$. Satz 3 in [7] shows that $f$ has at most a finite number of zeros in $\sigma(T)$. ∎

We are now going to calculate a pseudo inverse for $f(T)\in\mathcal{S}(X)$.

**Theorem 8.** *Suppose*

(**a**) $T\in\mathcal{L}(X)$ *and* $f\in\mathcal{H}(T)$ *are such that* $f(T)$ *is an operator of Saphar type,*

(**b**) $\lambda_1,\dots,\lambda_m$ *are the zeros of* $f$ *in* $\sigma(T)$ *with respective orders* $n_1,\dots,n_m$*,*

(**c**) $S_j$ *is a pseudo inverse for* $T-\lambda_j I$ $(j=1,\dots,m)$*.*

*Put*

$$S=\Big(\prod_{j=1}^{m}S_j^{n_j}\Big)h(T)^{-1}\ ,$$

*where* $h$ *is a function in* $\mathcal{H}(T)$ *such that* $f(\lambda)=(\prod_{j=1}^{m}(\lambda-\lambda_j)^{n_j})\,h(\lambda)$*. Then* $S$ *is a pseudo inverse for* $f(T)$*.*

**Proof:** Put $p(\lambda)=\prod_{j=1}^{m}(\lambda-\lambda_j)^{n_j}$. Then $f(\lambda)=p(\lambda)\,h(\lambda)$, thus

$$f(T)=p(T)\,h(T)=h(T)\,p(T)$$

and $h(T)$ is invertible in $\mathcal{L}(X)$. Use [3, Satz 80.1] to derive

$$\begin{aligned}N\Big(\prod_{j=1}^{k}(T-\lambda_jI)^{n_j}\Big)&=N\Big((T-\lambda_1I)^{n_1}\oplus\cdots\oplus N(T-\lambda_kI)^{n_k}\Big)\\&\subseteq(T-\lambda_{k+1}I)^{n_{k+1}}(X)\end{aligned}\tag{4}$$

for $k=1,\dots,m-1$. By Lemma 5(c), $(T-\lambda_jI)^{n_j}$ is relatively regular and $S_j^{n_j}$ is a pseudo inverse for $(T-\lambda_jI)^{n_j}$ $(j=1,\dots,m)$. Thus, using (4) and Lemma 5(b), we conclude that $p(T)$ is relatively regular and $B=\prod_{j=1}^{m}S_j^{n_j}$ is a pseudo inverse for $p(T)$. Therefore

$$\begin{aligned}f(T)\,Sf(T)&=f(T)\,Bh(T)^{-1}f(T)=h(T)\,p(T)\,Bh(T)^{-1}h(T)\,p(T)\\&=h(T)\,p(T)\,Bp(T)=h(T)\,p(T)=f(T)\ .\ ∎\end{aligned}$$

## REFERENCES


**[1]** Caradus, S.R. – *Generalized Inverses and Operator Theory*, Queen's Papers in Pure and Applied Math. No. 50, 1978.

**[2]** Goldberg, S. – *Unbounded linear operators*, New York, 1966.

**[3]** Heuser, H. – *Funktionalanalysis*, 2nd. ed. Stuttgart, 1986.

**[4]** Kato, T. – Perturbation theory for nullity, deficiency and other quantities of linear operators, *J. Analyse Math.*, 6 (1958), 261–322.

**[5]** Mbektha, M. – Généralisation de la décomposition de Kato aux opérateurs paranormaux et spectraux, *Glasgow Math. J.*, 29 (1987), 159–175.

**[6]** Saphar, P. – Contribution à l'étude des applications linéaires dans un espace de Banach, *Bull. Soc. Math. France*, 92 (1964), 363–384.

**[7]** Schmoeger, Ch. – Ein Spektralabbildungssatz, *Arch. Math.*, 55 (1990), 484–489.

**[8]** Schmoeger, Ch. – The punctured neighbourhood theorem in Banach algebras, *Proc. R. Ir. Acad.*, 91A(2) (1991), 205–218.

**[9]** Schmoeger, Ch. – Relatively regular operators and a spectral mapping theorem, *J. Math. Anal. Appl.* (to appear).



Christoph Schmoeger,
Mathematisches Institut I, Universität Karlsruhe,
Postfach 6980, D-7500 Karlsruhe 1 – GERMANY