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On operators T such that $f(T)$ is hypercyclic

by

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Abstract. A bounded linear operator A on a complex, separable, infinite-dimensional Banach space X is called hypercyclic if there is a vector $x \in X$ such that $\{x, Ax, A^2x, \ldots\}$ is dense in X. Let T be a bounded linear operator on X such that T is surjective and its generalized kernel $\bigcup_{n\ge1} N(T^n)$ is dense in X. In the present paper we show that for some admissible functions f without zeros in the spectrum of T the operator $f(T)$ is hypercyclic (Theorem 1). If f has zeros in the spectrum of T and if X is a Hilbert space then $f(T)$ is the limit of hypercyclic operators (Theorem 2).

I. Introduction. Throughout this paper X denotes a complex, separable, infinite-dimensional Banach space and $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators on X. $T \in \mathcal{L}(X)$ is called *hypercyclic* if $\{x, Tx, T^2x, \ldots\}$ is dense in X for some $x \in X$. We denote by $HC(X)$ the set of all hypercyclic operators in $\mathcal{L}(X)$. If $T \in HC(X)$, then the set of all $x \in X$ with $\{x, Tx, T^2x, \ldots\}$ dense in X is a dense G_δ-set (see [8]). It is still an open problem whether there are hypercyclic operators in every separable infinite-dimensional Banach space (if $\dim X < \infty$, $HC(X) = \emptyset$; see [20]). Hypercyclic operators were studied by several authors (cf., e.g., [1]–[5], [8], [9], [12]–[14], [16], [20], [22]).

For $X = l^p$ ($p \in [1, \infty)$) or $X = c_0$, consider the shift operator S, defined by $S((x_n)_{n=1}^\infty) = (x_{n+1})_{n=1}^\infty$. By [20], $\mu S \in HC(X)$ for $\mu \in \mathbb{C}$, $|\mu| > 1$. We shall state a theorem from which it follows that, for example, $e^S, \cos(S) \in HC(X)$. We first need some definitions:

Let $T \in \mathcal{L}(X)$. The kernel of T is denoted by $N(T)$ and its generalized kernel by $K(T) = \bigcup_{k=1}^\infty N(T^k)$. We denote by $\mathcal{H}(T)$ the set of all complex-valued functions which are analytic in a neighborhood of the spectrum $\sigma(T)$ of T. We write $\varrho(T)$ for the resolvent set $\mathbb{C} \setminus \sigma(T)$. For $f \in \mathcal{H}(T)$, the operator $f(T)$ is defined by the well known analytic calculus (see [17], [21]). We write M^- for the norm-closure of M, where M is a subset of X (or $\mathcal{L}(X)$).

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If $T \in \mathcal{L}(X)$ is surjective and $K(T)^- = X$, then we shall see that $\sigma(T)$ is connected (Proposition 3). The main results of this paper read:

THEOREM 1. *Suppose that* $T \in \mathcal{L}(X)$ *is surjective,* $K(T)^- = X$, $f \in \mathcal{H}(T)$ *is not constant,* $0 \notin f(\sigma(T))$ *and* $|f(0)| = 1$. *Then* $f(T) \in HC(X)$.

THEOREM 2. *Let* X *be a Hilbert space. Suppose that* $T \in \mathcal{L}(X)$ *is surjective,* $K(T)^- = X$, $f \in \mathcal{H}(T)$ *is not constant and* $|f(\lambda_0)| = 1$ *for some* $\lambda_0 \in \sigma(T)$. *Then* $f(T) \in HC(X)^-$.

If T satisfies the conditions of Theorem 1, it follows that e^T is hypercyclic. The shift operator S on $X = l^p$ ($p \in [1,\infty)$) or $X = c_0$ is surjective, and we have $K(S)^- = X$ (the finite sequences are dense in X). Since $\sigma(S) = \{\lambda \in \mathbb{C} : |\lambda| \le 1\}$, $\cos(S) \in HC(X)$.

The proofs of Theorems 1 and 2 will be given in Section III. For these proofs, we need some preliminary results which we collect in Section II. An operator T in $\mathcal{L}(X)$ is called *supercyclic* if the set of scalar multiples of $\{x, Tx, T^2x, \ldots\}$ is dense in X for some $x \in X$.

In Section IV we shall prove the following

THEOREM 3. *Suppose that* X *is a Hilbert space,* $T \in \mathcal{L}(X)$ *is surjective,* $K(T)^- = X$ *and* $f \in \mathcal{H}(T)$ *is not constant. Then* $f(T)$ *is in the norm-closure of the set of all supercyclic operators in* $\mathcal{L}(X)$.

A result closely related to Theorem 3 can be found in [9, Theorem 4.11]:

Suppose that X *is a Banach space and* $T \in \mathcal{L}(X)$ *is a surjective backward shift, i.e.,* $K(T)^- = X$ *and* $\dim N(T) = 1$. *If* $f \in \mathcal{H}(T)$ *is not constant, then* $f(T)$ *has a dense invariant supercyclic manifold (hence is supercyclic).*

There are operators satisfying the hypotheses of Theorem 3 which are not powers of a backward shift:

EXAMPLE. Take $X = \{f \in C[0,1] : f(0) = 0\}$ with the maximum norm, fix $\lambda \in (0,1)$ and define $T \in \mathcal{L}(X)$ by $(Tf)(t) = f(\lambda t)$ ($t \in [0,1]$).

II. Preliminary results. Our first result will be needed in the proof of Theorem 1.

PROPOSITION 1. *If* $A \in \mathcal{L}(X)$ *is invertible and if the sets*

$$D = \{x \in X : A^k x \to 0 \ (k \to \infty)\}$$

and

$$\widetilde{D} = \{x \in X : A^{-k} x \to 0 \ (k \to \infty)\}$$

are both dense subsets of X, *then* $A \in HC(X)$.

This is a special case of [9, Corollary 1.5] (see also [11, Proposition]). For a related result see [5, Proposition 2.2].

For the proofs in Section III we need some concepts from the theory of semi-Fredholm operators. The reader is referred to [17] for the definitions and properties of Fredholm operators, semi-Fredholm operators and the index $\operatorname{ind}(A)$ of a semi-Fredholm operator $A \in \mathcal{L}(X)$. The following notations will be important.

For $T \in \mathcal{L}(X)$ we define

$$\varrho_F(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is Fredholm}\},$$
$$\varrho_{s\text{-}F}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is semi-Fredholm}\},$$
$$\varrho_W(T) = \{\lambda \in \varrho_F(T) : \operatorname{ind}(\lambda I - T) = 0\}$$

and

$$\sigma_W(T) = \mathbb{C} \setminus \varrho_W(T) \quad \text{(the Weyl spectrum of } T).$$

Note that $\varrho_F(T)$, $\varrho_{s\text{-}F}(T)$ and $\varrho_W(T)$ are open subsets of $\mathbb{C}$.

PROPOSITION 2. *Suppose that* $T \in \mathcal{L}(X)$ *and* $\bigcap_{n=1}^\infty T^n(X) = \{0\}$. *Then:*

(a) $T(Y) \ne Y$ *for each closed* T*-invariant subspace* $Y \ne \{0\}$.
(b) $N(\lambda I - T) = \{0\}$ *for all* $\lambda \ne 0$.
(c) $\sigma(T)$ *is connected.*
(d) $\sigma(T) = \sigma_W(T)$.
(e) *If* T *is semi-Fredholm,* $N(T) = \{0\}$ *and if* $(\alpha_n)_{n=1}^\infty$ *is a sequence in* $\varrho_{s\text{-}F}(T) \setminus \{0\}$ *with* $\lim_{n\to\infty} \alpha_n = 0$, *then*

$$\bigcap_{n=1}^\infty (\alpha_n I - T)(X) = \{0\}.$$

Proof. (a) If $T(Y) = Y$, then $Y = T^n(Y)$ for each $n \in \mathbb{N}$. Thus $Y \subseteq \bigcap_{n=1}^\infty T^n(X) = \{0\}$, hence $Y = \{0\}$.

(b) Clear, since $N(\lambda I - T) \subseteq \bigcap_{n=1}^\infty T^n(X)$ for $\lambda \ne 0$.

(c) Suppose that $\sigma(T)$ is not connected. Therefore $\sigma(T) = \sigma \cup \tau$ with σ, τ closed, $\sigma, \tau \ne \emptyset$ and $\sigma \cap \tau = \emptyset$. We choose a function $f \in \mathcal{H}(T)$ such that $f \equiv 1$ on σ and $f \equiv 0$ on τ. Put $P = f(T)$. By [17, Satz 100.1], we have $P^2 = P$, $P(X)$ and $N(P)$ are closed, T-invariant subspaces and $\sigma(T|P(X)) = \sigma$, $\sigma(T|N(P)) = \tau$. By (a), this gives $0 \in \sigma \cap \tau$, a contradiction, since $\sigma \cap \tau = \emptyset$.

(d) We first show that $0 \in \sigma_W(T)$. To this end assume that $0 \in \varrho_W(T)$. Denote by C the connected component of $\varrho_F(T)$ with $0 \in C$. Now [17, Satz 104.6a)] and (b) show that $C \subseteq \varrho(T)$, hence T is invertible in $\mathcal{L}(X)$. This contradicts $0 \in \sigma(T)$. Therefore we have

$$0 \in \sigma_W(T). \tag{1}$$

The inclusion $\varrho(T) \subseteq \varrho_W(T)$ is clear. Let $\lambda \in \varrho_W(T)$. By (1), $\lambda \ne 0$. The definition of $\varrho_W(T)$ and (b) show that $\operatorname{ind}(\lambda I - T) = -\operatorname{codim}(\lambda I - T)(X) = 0$, thus $\lambda \notin \sigma(T)$.

(e) This is a special case of [18, Proposition 1.4]. For the convenience of the reader we shall give a proof. It is related to that employed in [19, Proposition 2.7]. Since T is semi-Fredholm, each T^n ($n \in \mathbb{N}$) is semi-Fredholm, thus $T^n(X)$ is closed for each $n \in \mathbb{N}$. Moreover, $T^{-1} : T(X) \to X$ is bounded. Let $x \in \bigcap_{n=1}^{\infty}(\alpha_n I - T)(X)$. Then there exists a sequence (x_n) in X with

$$x = (T - \alpha_n I)x_n \quad (n \in \mathbb{N}),$$

hence $Tx_n = \alpha_n x_n + x$, therefore

$$x_n = T^{-1}(Tx_n) = T^{-1}(\alpha_n x_n + x).$$

It follows that

$$\|x_n\| \le \|T^{-1}\|(\|x\| + |\alpha_n|\,\|x_n\|),$$

so

$$\|x_n\| \le \frac{\|T^{-1}\|\,\|x\|}{1 - |\alpha_n|\,\|T^{-1}\|}$$

for n sufficiently large. This shows that (x_n) is bounded. It follows from $Tx_n = \alpha_n x_n + x$ that $Tx_n \to x$ ($n \to \infty$). Since $T(X)$ is closed, we get $x \in T(X)$, hence $(x_n) \subseteq T(X)$. Since $T^2(X)$ is closed and $(Tx_n) \subseteq T^2(X)$, we derive $x \in T^2(X)$. The process continues to give $x \in \bigcap_{n=1}^{\infty} T^n(X) = \{0\}$. ■

Let X^* denote the conjugate space of the Banach space X. The adjoint of a linear operator T in $\mathcal{L}(X)$ is denoted by T^*.

PROPOSITION 3. *Suppose that $T \in \mathcal{L}(X)$ is surjective and $K(T)^- = X$. Then:*

(a) *$N(T^* - \lambda I^*) = \{0\}$ for all $\lambda \neq 0$.*
(b) *$\sigma(T) = \sigma_W(T)$ is connected.*
(c) *If (α_n) is a sequence in $\varrho_{s\text{-}F}(T) \setminus \{0\}$ with $\lim_{n\to\infty} \alpha_n = 0$, then*

$$X = \Big[\bigcup_{n=1}^{\infty} N(\alpha_n I - T)\Big]^-$$

(where $[M]$ denotes the linear span of $M \subseteq X$).

Proof. It is easy to check that

$$X = K(T)^- = {}^{\perp}\Big(\bigcap_{n=1}^{\infty}(T^*)^n(X^*)\Big),$$

thus $\bigcap_{n=1}^{\infty}(T^*)^n(X^*) = \{0\}$. Now (a) follows from Proposition 2(b). Since $\sigma(T) = \sigma(T^*)$ and $\sigma_W(T) = \sigma_W(T^*)$, (b) follows from Proposition 2(c),(d). (c) is true since $\bigcap_{n=1}^{\infty}(\alpha_n I^* - T^*)(X^*) = \{0\}$ (by Proposition 2(e)), thus

$$X = {}^{\perp}\Big(\bigcap_{n=1}^{\infty}(\alpha_n I^* - T^*)(X^*)\Big) = \Big[\bigcup_{n=1}^{\infty} N(\alpha_n I - T)\Big]^-. \ ■$$

Our final result in this section is of crucial importance for the proofs of Theorems 2 and 3.

PROPOSITION 4. *Suppose that $T \in \mathcal{L}(X)$ is surjective and $K(T)^- = X$. If $f \in \mathcal{H}(T)$ is not constant, then:*

(a) *$\operatorname{ind}(\lambda I - f(T)) \ge 0$ for all $\lambda \in \varrho_{s\text{-}F}(f(T))$,*
(b) *$\sigma_W(f(T)) = f(\sigma_W(T))$,*
(c) *$\sigma(f(T)) = \sigma_W(f(T))$ is connected.*

Proof. (a) Since $N(T^* - \lambda I^*) = \{0\}$ for all $\lambda \neq 0$ (Proposition 3(a)), T^* has the single-valued extension property, i.e., for any analytic function $\varphi : D \to X^*$, $D \subseteq \mathbb{C}$ open, with $(\lambda I^* - T^*)\varphi(\lambda) = 0$, it follows that $\varphi(\lambda) \equiv 0$. By [6, Theorem 1.5], $f(T)^* = f(T^*)$ has the single-valued extension property. Therefore $\operatorname{ind}(\lambda I - f(T)) \ge 0$ for all $\lambda \in \varrho_{s\text{-}F}(f(T))$, by [7, Corollary 12].

(b) Applying (a), we get $\operatorname{ind}(\lambda I - T) \ge 0$ for all $\lambda \in \varrho_{s\text{-}F}(T)$. Thus Theorem 3.6 in [23] shows that $f(\sigma_W(T)) = \sigma_W(f(T))$.

(c) follows from (b), Proposition 3(b) and the continuity of f. ■

Remark. In general, a spectral mapping theorem for the Weyl spectrum does not hold (see [10, p. 23] or [23, Example 3.3]). If $\operatorname{ind}(\lambda I - T) \ge 0$ for all $\lambda \in \varrho_F(T)$ or $\operatorname{ind}(\lambda I - T) \le 0$ for all $\lambda \in \varrho_F(T)$, then $f(\sigma_W(T)) = \sigma_W(f(T))$ for each $f \in \mathcal{H}(T)$ (see [23, Theorem 3.6]).

III. The proofs of Theorems 1 and 2

Proof of Theorem 1. Since f is not constant and $|f(0)| = 1$, there are sequences $(\lambda_n)_{n=1}^{\infty}$ and $(\mu_n)_{n=1}^{\infty}$ in $\varrho_{s\text{-}F}(T) \setminus \{0\}$ such that

$$\begin{aligned} &\lim_{n\to\infty} \lambda_n = 0, \quad |f(\lambda_n)| < 1 \quad \text{for all } n \in \mathbb{N}, \\ &\lim_{n\to\infty} \mu_n = 0, \quad |f(\mu_n)| > 1 \quad \text{for all } n \in \mathbb{N} \quad \text{and} \\ &\lambda_i \neq \lambda_j, \quad \mu_i \neq \mu_j \quad \text{for } i \neq j. \end{aligned} \tag{2}$$

Proposition 3(c) shows that

$$X = K(T)^- = \Big[\bigcup_{n=1}^{\infty} N(\lambda_n I - T)\Big]^- = \Big[\bigcup_{n=1}^{\infty} N(\mu_n I - T)\Big]^-. \tag{3}$$

By hypothesis, $0 \notin f(\sigma(T)) = \sigma(f(T))$, thus $f(T)$ is invertible in $\mathcal{L}(X)$. Put

$$D = \{x \in X : (f(T))^k x \to 0\}, \quad \widetilde{D} = \{x \in X : (f(T))^{-k} x \to 0\}.$$

Let $x \in N(\lambda_n I - T)$. Then $Tx = \lambda_n x$, hence

$$(f(T))^k x = (f(\lambda_n))^k x \quad \text{for each } k \in \mathbb{N}$$

(see [21, Theorem 10.33]). Since $|f(\lambda_n)| < 1$, we have $(f(T))^k x \to 0$ ($k \to \infty$), hence $x \in D$, thus $N(\lambda_n I - T) \subseteq D$. (3) now shows that

$$X = \Big[\bigcup_{n=1}^{\infty} N(\lambda_n I - T)\Big]^- = D^- .$$

In the same way we derive

$$\lim_{k\to\infty} (f(T))^{-k} x = \lim_{k\to\infty} (f(\mu_n))^{-k} x = 0 \quad \text{for } x \in N(\mu_n I - T)$$

and $X = (\widetilde{D})^-$. Now use Proposition 1 to complete the proof. ■

For the proof of Theorem 2 we need some further notations: the set of all isolated points in $\sigma(T)$ which are contained in $\varrho_{\mathrm{F}}(T)$ is denoted by $\sigma_\circ(T)$. $\partial\mathbb{D}$ denotes the boundary of the unit disc $\mathbb{D}$.

In [12], D. A. Herrero proved the following description of $HC(X)^-$, X a separable, infinite-dimensional complex Hilbert space:

(4) $A \in HC(X)^-$ if and only if

(I) $\sigma_{\mathrm{W}}(A) \cup \partial\mathbb{D}$ is connected,
(II) $\sigma_\circ(A) = \emptyset$ and
(III) $\mathrm{ind}(\lambda I - A) \geq 0$ for all $\lambda \in \varrho_{\text{s-F}}(A)$.

Proof of Theorem 2. By Proposition 4(a) and (4), it only remains to show that $\sigma_{\mathrm{W}}(f(T)) \cup \partial\mathbb{D}$ is connected and $\sigma_\circ(f(T)) = \emptyset$. We have $|f(\lambda_0)| = 1$ for some $\lambda_0 \in \sigma(T)$, hence $f(\lambda_0) \in \partial\mathbb{D}$ and $f(\lambda_0) = f(\sigma(T)) = \sigma(f(T)) = \sigma_{\mathrm{W}}(f(T))$ (Proposition 4(b)), thus $f(\lambda_0) \in \sigma_{\mathrm{W}}(f(T)) \cap \partial\mathbb{D}$. Since $\sigma_{\mathrm{W}}(f(T))$ and $\partial\mathbb{D}$ are connected, $\sigma_{\mathrm{W}}(f(T)) \cup \partial\mathbb{D}$ is connected.

Suppose that $\mu \in \sigma_\circ(f(T))$ for some $\mu \in \mathbb{C}$, thus μ is an isolated point in $\sigma(f(T))$. But $\sigma(f(T))$ is connected, hence $\sigma(f(T)) = \{\mu\}$. Since $\mu \in \varrho_{\mathrm{F}}(f(T))$, we have $\varrho_{\mathrm{F}}(f(T)) = \mathbb{C}$. This gives $\dim X < \infty$ [17, Satz 104.9], a contradiction. ■

IV. Proof of Theorem 3 and final remarks

Proof of Theorem 3. Proposition 4 shows that

(5) $\sigma(f(T)) \cup \partial(r\mathbb{D}) = \sigma_{\mathrm{W}}(f(T)) \cup \partial(r\mathbb{D})$ is connected

(for some $r \geq 0$) and

(6) $\mathrm{ind}(\lambda I - f(T)) \geq 0$ for all $\lambda \in \varrho_{\text{s-F}}(f(T))$.

As in the proof of Theorem 2 we have

(7) $\sigma_\circ(f(T)) = \emptyset$.

The combination of (5)–(7) and [12, Theorem 3.3] completes the proof. ■

We close the paper with some remarks:

1. As a consequence of Theorem 1 we get the following

PROPOSITION 5. *Let T satisfy the hypotheses of Theorem 1. Then there is $x_0 \in X$ such that the solution $C^+ : y = y(t)$, $t \geq 0$, of the linear initial value problem $y' = Ty$, $y(0) = x_0$, has X as the set of ω-limit points $\Omega(C^+)$.*

The proof follows from Theorem 1 and the fact that e^{tT}, $t \in \mathbb{R}$, is a fundamental system for $y' = Ty$.

2. It is not known if there are hypercyclic oprators on every infinite-dimensional separable Banach space. It is shown in [15] that supercyclic operators exist on every complex separable Banach space X with $\dim X \leq 1$ or $\dim X = \infty$.

3. Most questions concerning hypercyclic operators can also be studied in separable locally convex spaces. In the Fréchet space of all complex sequences hypercyclic operators can be characterized by algebraic properties of their adjoints (see [16]).

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On quasi-multipliers

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Abstract. A quasi-multipliers is a generalization of the notion of a left (right, double) multiplier. The first systematic account of the general theory of quasi-multipliers on a Banach algebra with a bounded approximate identity was given in a paper by McKennon in 1977. Further developments have been made in more recent papers by Vasudevan and Goel, Kassem and Rowlands, and Lin. In this paper we consider the quasi-multipliers of algebras not hitherto considered in the literature. In particular, we study the quasi-multipliers of A^*-algebras, of the algebra of compact operators on a Banach space, and of the Pedersen ideal of a C^*-algebra. We also consider the strict topology on the quasi-multiplier space $QM(A)$ of a Banach algebra A with a bounded approximate identity. We prove that, if $M_l(A)$ (resp. $M_r(A)$) denotes the algebra of left (right) multipliers on A, then $M_l(A)+M_r(A)$ is strictly dense in $QM(A)$, thereby generalizing a theorem due to Lin.

1. Introduction. A quasi-multiplier is a generalization of the notion of a left (right, double) multiplier, and was first introduced by Akemann and Pedersen in ([1], §4). The first systematic account of the general theory of quasi-multipliers on a Banach algebra with a bounded approximate identity was given in a paper by McKennon [14] in 1977. Further developments have been made as a result of more recent contributions by Vasudevan and Goel [21], [22], Kassem and Rowlands [11], and Lin [13]. In this paper we study the quasi-multipliers of algebras not hitherto considered in the literature.

We begin by outlining the necessary background results on quasi-multipliers and then proceed to consider the quasi-multipliers $QM(A)$ of an A^*-algebra A; in particular, we improve a result due to Vasudevan and Goel ([21], Theorem 3.4) on extending a quasi-multiplier from an A^*-algebra to its auxiliary norm completion. The result enables us to define an "auxiliary" norm on $QM(A)$ and, in the special case when $QM(A)$ is a Banach algebra, we prove that under certain conditions $QM(A)$ is an A^*-algebra. In the literature topologies other than the norm topology have been defined on the quasi-multiplier space and properties established for the resulting locally

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