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Perturbation properties of some classes of operators

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RIASSUNTO: *Sia X uno spazio di Banach complesso e $\mathcal{L}(X)$ l'algebra di di Banach di tutti gli operatori lineari limitati in X. Considerate le seguenti famiglie di operatori:*

$$\mathcal{D}(X) = \{T \in \mathcal{L}(X) : T(X) \textit{ is closed and } N(T) \subseteq \bigcap_{n=1}^{\infty} T^n(X)\},$$
$$\mathcal{S}(X) = \{T \in \mathcal{D}(X) : T \textit{ is relatively regular}\}.$$

si determinanno i punti interni di $\mathcal{D}(X)$ e $\mathcal{S}(X)$, si dimostrano inoltre alcuni teoremi di perturbazione.

ABSTRACT: *Let X be a complex Banach space and $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators on X. We consider the following classes of operators:*

$$\mathcal{D}(X) = \{T \in \mathcal{L}(X) : T(X) \textit{ is closed and } N(T) \subseteq \bigcap_{n=1}^{\infty} T^n(X)\},$$
$$\mathcal{S}(X) = \{T \in \mathcal{D}(X) : T \textit{ is relatively regular}\}.$$

We determine the interior points of $\mathcal{D}(X)$ and $\mathcal{S}(X)$ and prove some perturbation theorems.

1 – Introduction and terminology

Throughout this paper X denotes a Banach space over the complex

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field $\mathbb{C}$ and $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators on X. If $T \in \mathcal{L}(X)$ we denote by $N(T)$ the kernel of T and by $\alpha(T)$ the dimension of $N(T)$. The range of T is denoted by $T(X)$ and we define $\beta(T) = \operatorname{codim} T(X)$.

$T \in \mathcal{L}(X)$ is called relatively regular if $TST = T$ for some $S \in \mathcal{L}(X)$. $\mathcal{R}(X)$ will denote the set of all relatively regular operators.

We shall make use of the following results [1, p. 10]:

1. $T \in R(X)$ if and only if $N(T)$ and $T(X)$ are closed complemented subspaces of X.
2. If $TST = T$ for some $S \in \mathcal{L}(X)$, then TS is a projection onto $T(X)$ and $I - ST$ is a projection onto $N(T)$.

An operator T is called an Atkinson operator if $T \in \mathcal{R}(X)$ and at least one of $\alpha(T)$, $\beta(T)$ is finite. The set of Atkinson operators will be denoted by $\mathcal{A}(X)$.

We write $\mathcal{C}(X)$ for the set of operators having closed range. The class of semi-Fredholm operators is defined by

$$\mathcal{SF}(X) = \{T \in \mathcal{C}(X) : \alpha(T) < \infty \ \text{ or } \ \beta(T) < \infty\}.$$

We have $\mathcal{A}(X) \subseteq \mathcal{SF}(X)$. The index of $T \in \mathcal{SF}(X)$ is given by $\operatorname{ind}(T) = \alpha(T) - \beta(T)$.

The following result is well known (for proofs see [1] and [3]).

Theorem 1. *Let $T \in \mathcal{A}(X)$ (resp. $T \in \mathcal{SF}(X)$). Then there exists $\delta > 0$ such that*

(a) *$T - B \in \mathcal{A}(X)$ (resp. $T - B \in \mathcal{SF}(X)$), $\alpha(T-B) \leq \alpha(T)$, $\beta(T-B) \leq \beta(T)$ and $\operatorname{ind}(T-B) = \operatorname{ind}(T)$ for all $B \in \mathcal{L}(X)$ with $||B|| < \delta$;*

(b) *$\alpha(T - \lambda I)$ is a constant $\leq \alpha(T)$, $\beta(T - \lambda I)$ is a constant $\leq \beta(T)$ for $0 < |\lambda| < \delta$.*

The above theorem shows that $\mathcal{A}(X)$ and $\mathcal{SF}(X)$ are open subsets of $\mathcal{L}(X)$. Furthermore, the continuity of the index shows that the *jump* of $T \in \mathcal{SF}(X)$

$$j(T) = \begin{cases} \alpha(T) - \alpha(T - \lambda I) & (0 < |\lambda| < \delta) \quad if \quad \alpha(T) < \infty \\ \beta(T) - \beta(T - \lambda I) & (0 < |\lambda| < \delta) \quad if \quad \beta(T) < \infty \end{cases}$$

is unambigously defined.

Proposition 1. *If* $T \in \mathcal{SF}(X)$ *then* $j(T) = 0 \iff N(T) \subseteq \bigcap_{n\geq 1} T^n(X)$.

Proof. [14, Proposition 2.2]. ∎

We now list various classes of bounded linear operators which will be discussed:

$$\mathcal{SF}_0(X) = \{T \in \mathcal{SF}(X) : \alpha(T) = 0 \text{ or } \beta(T) = 0\};$$

$$\mathcal{B}(X) = \{T \in \mathcal{L}(X) : N(T) \subseteq T(X)\};$$

$$\mathcal{M}(X) = \{T \in \mathcal{L}(X) : T \text{ is left or right invertible in } \mathcal{L}(X)\};$$

$$\mathcal{D}(X) = \{T \in \mathcal{C}(X) : N(T) \subseteq \bigcap_{n\geq 1} T^n(X)\};$$

$$\mathcal{S}(X) = \{T \in \mathcal{R}(X) : N(T) \subseteq \bigcap_{n\geq 1} T^n(X)\}.$$

It is well known that $\mathcal{M}(X)$ is open. $\mathcal{SF}_0(X)$ is open by Theorem 1. An operator in $\mathcal{S}(X)$ is called an operator of Saphar type. Such operators have an important property:

> $T \in \mathcal{S}(X)$ *if and only if there is a neigbourhood* $U \subset \mathbb{C}$ *of* 0 *and a holomorphic function* $F : U \to \mathcal{L}(X)$ *such that*
>
> $$(T - \lambda I)F(\lambda)(T - \lambda I) = T - \lambda I \quad \textit{for all } \lambda \in U.$$

For a proof see [7, Théorème 2.6] or [12, Theorem 1.4].

2 – Interior points of $\mathcal{D}(X)$ and $\mathcal{S}(X)$

If $\mathcal{H}$ is a subset of $\mathcal{L}(X)$ we write $\mathrm{int}(\mathcal{H})$ for the set of interior points of $\mathcal{H}$.

Proposition 2. *If* $T \in \mathrm{int}(\mathcal{B}(X))$ *then* $N(T) = \{0\}$ *or* $T(X) = X$.

PROOF. There exists $\delta > 0$ such that

$$S \in \mathcal{B}(X) \quad \text{whenever} \quad ||T - S|| < \delta.$$

Suppose that $N(T) \neq \{0\}$ and $T(X) \neq X$. Then there are $x_0, y_0 \in X$ with $x_0 \neq 0$, $Tx_0 = 0$, $y_0 \notin T(X)$ and $||Ty_0|| = \delta/2$. Since $y_0 \notin T(X)$ and $N(T) \subseteq T(X)$, we have $y_0 \notin N(T)$. An application of the Hahn-Banach extension theorem shows the existence of a continuous linear functional f such that

$$\alpha = f(x_0) \neq 0, \quad f(y_0) = 0 \quad \text{and} \quad ||f|| = 1$$

(see [6, Satz 36.3]). Define $S \in \mathcal{L}(X)$ by

$$Sx = Tx + f(x)Ty_0 \qquad (x \in X).$$

It follows that $||Tx - Sx|| = |f(x)|\,||Ty_0|| \leq ||x||\delta/2$, thus $||T - S|| < \delta$, hence $S \in \mathcal{B}(X)$. Since $S(X) \subseteq T(X)$, we conclude that

$$N(S) \subseteq T(X).$$

Now put $z = y_0 - x_0/\alpha$. It results that

$$Sz = Ty_0 + f\left(y_0 - \frac{1}{\alpha}x_0\right)Ty_0 = Ty_0 - Ty_0 = 0.$$

This gives $z \in T(X)$, hence $y_0 = z + x_0/\alpha \in T(X) + N(T) = T(X)$ which contradicts $y_0 \notin T(X)$. □

It is shown in [10] that neither $\mathcal{D}(X)$ nor $\mathcal{S}(X)$ are open subsets of $\mathcal{L}(X)$. But the following perturbation results are valid:

Suppose $T \in \mathcal{S}(X)$ *(resp.* $T \in \mathcal{D}(X)$*),* $B \in \mathcal{L}(X)$ *and* $B\left(\bigcap_{n=1}^{\infty} T^n(X)\right) \subseteq \bigcap_{n=1}^{\infty} T^n(X)$. *If* $||B||$ *is sufficiently small then* $T - B \in \mathcal{S}(X)$ *(resp.* $T - B \in \mathcal{D}(X)$*).*

(For proofs see [1, p. 150] (resp. [10, Corollaire 3.6]).)

Therefore a natural question arises: What are the interior points of $\mathcal{S}(X)$ and $\mathcal{D}(X)$? The following result gives an answer.

Theorem 2.

(a) $\mathrm{int}(\mathcal{D}(X)) = \mathrm{int}(\mathcal{B}(X) \cap \mathcal{C}(X)) = \mathcal{SF}_0(X)$.
(b) $\mathrm{int}(\mathcal{S}(X)) = \mathrm{int}(\mathcal{B}(X) \cap \mathcal{R}(X)) = \mathcal{M}(X)$.

Proof. (a) By Theorem 1 and Proposition 1, $\mathcal{SF}_0(X) \subseteq \mathcal{D}(X)$. Since $\mathcal{SF}_0(X)$ is open and $\mathcal{SF}_0(X) \subseteq \mathcal{D}(X) \subseteq \mathcal{B}(X) \cap \mathcal{C}(X)$, we have

$$\mathcal{SF}_0(X) \subseteq \mathrm{int}(\mathcal{D}(X)) \subseteq \mathrm{int}(\mathcal{B}(X) \cap \mathcal{C}(X)).$$

If $T \in \mathrm{int}(\mathcal{B}(X) \cap \mathcal{C}(X))$ then $T \in \mathrm{int}(\mathcal{B}(X))$, thus $\alpha(T) = 0$ or $\beta(T) = 0$, by Proposition 2. Since $T(X)$ is closed, we derive $T \in \mathcal{SF}_0(X)$.

(b) Since $\mathcal{M}(X)$ is open and $\mathcal{M}(X) \subseteq \mathcal{S}(X) \subseteq \mathcal{B}(X) \cap \mathcal{R}(X)$, we have

$$\mathcal{M}(X) \subseteq \mathrm{int}(\mathcal{S}(X)) \subseteq \mathrm{int}(\mathcal{B}(X) \cap \mathcal{R}(X)).$$

Let $T \in \mathrm{int}(\mathcal{B}(X) \cap \mathcal{R}(X))$. There is $S \in \mathcal{L}(X)$ with $TST = T$. Proposition 2 shows that $(I - ST)(X) = N(T) = \{0\}$ or $TS(X) = T(X) = X$, thus $ST = I$ or $TS = I$, therefore $T \in \mathcal{M}(X)$. ∎

Remark. If X is a Hilbert space, then $\mathcal{C}(X) = \mathcal{R}(X)$ [1, p. 12], hence $\mathcal{D}(X) = \mathcal{S}(X)$. In this special case it was shown in [8, Théorème 6.5] that $\mathrm{int}(\mathcal{D}(X)) = \mathcal{M}(X)$.

Corollary 1. *If X is a Hilbert space then $\mathrm{int}(\mathcal{S}(X))$ is dense in $\mathcal{L}(X)$.*

Proof. $\mathcal{M}(X)$ is dense in $\mathcal{L}(X)$ [4, Problem 140]. Now use Theorem 2. ∎

Corollary 2.

(a) $\mathrm{int}(\{T \in \mathcal{SF}(X) : j(T) = 0\}) = \mathcal{SF}_0(X)$.
(b) $\mathrm{int}(\{T \in \mathcal{A}(X) : j(T) = 0\}) = \mathcal{M}(X)$.

Proof. (a) follows from $\mathcal{SF}_0(X) \subseteq \{T \in \mathcal{SF}(X) : j(T) = 0\} \subseteq \mathcal{D}(X)$ (Proposition 1) and from Theorem 2.

(b) follows from $\mathcal{M}(X) \subseteq \{T \in \mathcal{A}(X) : j(T) = 0\} \subseteq \mathcal{S}(X)$ and from Theorem 2. ∎

3 – The reduced minimum modulus of operators in $\mathcal{D}(X)$

By definition, the reduced minimum modulus $\gamma(T)$ of $T \in \mathcal{L}(X) \setminus \{0\}$ is given by

$$\gamma(T) = \inf \left\{ \frac{||Tx||}{d(x, N(T))} : x \in X,\ Tx \neq 0 \right\}.$$

($d(x, N(T))$ denotes the distance of x to $N(T)$.) Observe that $\gamma(T) > 0$ if and only if $T \in \mathcal{C}(X)$ [3, Theorem IV. 1.6].

Proposition 3. *Let $T \in \mathcal{L}(X)$.*

(a) *If $T \in \mathcal{D}(X)$ then $T^n \in \mathcal{D}(X)$ for all $n \in \mathbb{N}$.*
(b) *If $T \in \mathcal{D}(X)$ then $\gamma(T^{n+m}) \geq \gamma(T^n)\gamma(T^m)$ for all $n, m \in \mathbb{N}$.*
(c) *If $T \in \mathcal{R}(X)$ and $TST = T$ for some $S \in \mathcal{L}(X)$ then $||S||^{-1} \leq \gamma(T)$.*
(d) *If $T \in \mathcal{S}(X)$ and $TST = T$ for some $S \in \mathcal{L}(X)$ then $T^n S^n T^n = T^n$ for each $n \in \mathbb{N}$.*

Proof. (a) [11, Satz 6]. (b) [2, Lemma 1]. (c) [2, Lemma 4]. (d) [13, Proposition 2]. ∎

We denote by $\sigma(T)$ the spectrum of $T \in \mathcal{L}(X)$ and by $r(T) = \max\{|\lambda| : \lambda \in \sigma(T)\}$ $(= \lim_{n\to\infty} ||T^n||^{1/n})$ the spectral radius of T. $\partial\sigma(T)$ denotes the boundary of $\sigma(T)$.

Proposition 4. *Let $T \in \mathcal{L}(X)$.*

(a) *If $\mu \in \partial\sigma(T)$ then $T - \mu I \notin \mathcal{D}(X)$.*
(b) *If $T \in \mathcal{D}(X)$ then*

$$\sup_{n \geq 1} \gamma(T^n)^{1/n} \leq \min \{|\mu| : \mu \in \partial\sigma(T)\},$$

the sequence $(\gamma(T^n)^{1/n})_{n\geq 1}$ converges and

$$\lim_{n\to\infty} \gamma(T^n)^{1/n} = \sup_{n\geq 1} \gamma(T^n)^{1/n}.$$

(c) *If $T \in \mathcal{S}(X)$ and $TST = T$ for some $S \in \mathcal{L}(X)$ then*

$$r(S)^{-1} \leq \lim_{n\to\infty} \gamma(T^n)^{1/n}.$$

Proof. (a) follows from [11, Satz 2].

(b) Fix $\mu \in \partial\sigma(T)$ such that $|\mu| = \min\{|\lambda| : \lambda \in \partial\sigma(T)\}$ and suppose that $|\mu| < \gamma(T^m)^{1/m}$ for some $m \in \mathbb{N}$. Thus $|\mu^m| < \gamma(T^m)$. Since $T^m \in \mathcal{D}(X)$ (Proposition 3(a)), Théorème 2.10 in [9] gives $T^m - \mu^m I \in \mathcal{D}(X)$. [11, Satz 6] implies now that $T - \mu I \in \mathcal{D}(X)$, but this contradicts (a). Hence $\gamma(T^m)^{1/m} \leq |\mu|$ for each $m \in \mathbb{N}$.

(b) follows from [2, remarks in connection with Lemma 1].

(c) By Proposition 3(d), $T^n S^n T^n = T^n$ for all $n \in \mathbb{N}$. Part (c) of Proposition 3 implies that $||S^n||^{-1} \leq \gamma(T^n)$ for each $n \in \mathbb{N}$, hence

$$r(S)^{-1} = \lim_{n\to\infty} \frac{1}{||S^n||^{1/n}} \leq \lim_{n\to\infty} \gamma(T^n)^{1/n}.$$

□

The following theorem is another perturbation result for operators in $\mathcal{D}(X)$ which generalizes Théorème 2.10 in [9].

Theorem 3. *If $T \in \mathcal{D}(X)$, $B \in \mathcal{L}(X)$, $TB = BT$ and $r(B) < \lim_{n\to\infty} \gamma(T^n)^{1/n}$, then $T - B \in \mathcal{D}(X)$.*

Proof. Since $r(B) = \inf_{k\geq 1} ||B^k||^{1/k} < \sup_{n\geq 1} \gamma(T^n)^{1/n}$, there exists $k \in \mathbb{N}$ such that $||B^{k+1}|| < \gamma(T^{k+1})$. By Proposition 3(a), $T^{k+1} \in \mathcal{D}(X)$, thus $T^{k+1} - B^{k+1} \in \mathcal{D}(X)$, by [10, Corollaire 3.6], since $B\left(\bigcap_{n=1}^{\infty} T^n(X)\right) \subseteq \bigcap_{n=1}^{\infty} T^n(X)$. $TB = BT$ implies

$$T^{k+1} - B^{k+1} = (T - B)(T^k + T^{k-1}B + \cdots + TB^{k-1} + B^k).$$

Therefore [11, Satz 5] shows that $T - B \in \mathcal{D}(X)$. □

The next result is proved in [10, Théorème 3.7]. It is now an immediate consequence of the last theorem.

Theorem 4. *Let $T, Q \in \mathcal{L}(X)$. If Q is quasi-nilpotent and commutes with T, then*

$$T \in \mathcal{D}(X) \text{ if and only if } T - Q \in \mathcal{D}(X).$$

We close this paper with a perturbation result concerning operators in $\mathcal{S}(X)$. For the proof we need the following proposition.

PROPOSITION 5. *If $A, B \in \mathcal{L}(X)$ commute and $AB \in \mathcal{S}(X)$, then $A, B \in \mathcal{S}(X)$.*

PROOF. [5, Theorem 10]. □

THEOREM 5. *Let $T, Q \in \mathcal{L}(X)$. If Q is quasi-nilpotent and commutes with T, then*

$$T \in \mathcal{S}(X) \text{ if and only if } T - Q \in \mathcal{S}(X).$$

PROOF. It suffices to prove the implication $T \in \mathcal{S}(X) \Longrightarrow T - Q \in \mathcal{S}(X)$. Put $S \in \mathcal{L}(X)$ such that $TST = T$. By Proposition 3(a),(d), $T^n \in \mathcal{S}(X)$ and $T^n S^n T^n = T^n$ for each $n \in \mathbb{N}$. Put $S_n := S^n T^n S^n$ $(n \in \mathbb{N})$. It follows that $T^n S_n T^n = T^n$, $S_n T^n S_n = S_n$ and $||S_n||^{1/n} \leq ||S||^2 ||T||$. There exists $k \in \mathbb{N}$ such that $||Q^{k+1}||^{1/(k+1)} < (||S||^2||T||)^{-1}$, thus $||Q^{k+1}|| < ||S_{k+1}||^{-1}$. By [1, Theorem 9 in Section 5.2], $T^{k+1} - Q^{k+1} \in \mathcal{S}(X)$. $TQ = QT$ implies

$$T^{k+1} - Q^{k+1} = (T - Q)(T^k + T^{k-1}Q + \cdots + TQ^{k-1} + Q^k),$$

hence $T - Q \in \mathcal{S}(X)$, by Proposition 5. □

REFERENCES

[1] S.R. CARADUS: *Generalized Inverses and Operator Theory*, Queen's Papers in Pure and Applied Math. No 50 (1978).

[2] K.H. FÖRSTER – M.A. KAASHOEK: *The asymptotic behaviour of the reduced minimum modulus of a Fredholm operator*, Proc. Amer. Math. Soc. **49**, 123-131, (1975).

[3] S. GOLDBERG: *Unbounded linear operators*, New York (1966).

[4] P.R. HALMOS: *A Hilbert space problem book*, 2nd ed. Princeton (1980).

[5] R. HARTE: *Taylor exactness and Kaplansky's lemma*, J. Operator Theory **25**, 399-416, (1991).

[6] H. HEUSER: *Funktionalanalysis*, 2nd ed. Stuttgart (1986).

[7] M. MBEKHTA: *Généralisation de la décomposition de Kato aux opérateurs paranormaux et spectraux*, Glasgow Math. J. **29**, 159-175, (1987).

[8] M. MBEKHTA: *Résolvant généralisé et théorie spectrale*, J. Operator Theory **21**, 69-105, (1989).

[9] M. MBEKHTA – A. OUAHAB: *Opérateurs s-régulier dans une espace de Banach et théorie spectrale*, Pub. IRMA, Lille Vol. 22, N° XII (1990).

[10] M. MBEKHTA – A. OUAHAB: *Perturbations des opérateurs s-réguliers et continuité de certain sous-espaces dans le domaine quasi-Fredholm*, Pub. IRMA, Lille Vol. 24, N° X (1991).

[11] CH. SCHMOEGER: *Ein Spektralabbildungssatz*, Arch. Math. **55**, 484-489, (1990).

[12] CH. SCHMOEGER: *The punctured neighbourhood theorem in Banach algebras*, Proc. R. Ir. Acad. **91A**, No. 2, 205-218, (1991).

[13] CH. SCHMOEGER: *Relatively regular operators and a spectral mapping theorem*, J. Math. Anal. Appl. **175**, 315-320, (1993).

[14] T.T. WEST: *A Riesz-Schauder theorem for semi-Fredholm operators*, Proc. R. Ir. Acad. **87A**, No. 2, 137-146, (1987).

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