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NEWTON REGULARIZATIONS FOR IMPEDANCE TOMOGRAPHY: CONVERGENCE BY LOCAL INJECTIVITY

ARMIN LECHLEITER† AND ANDREAS RIEDER‡

Abstract. In [*Inverse Problems* 22(2006), pp. 1967-1987] we demonstrated experimentally that the Newton-like regularization method CG-REGINN is a competitive solver for the inverse problem of the complete electrode model in 2D-electrical impedance tomography. Here we establish rigorously the observed convergence of CG-REGINN (and related schemes). To this end we prove that the underlying nonlinear operator has an injective Frechét derivative whenever the number of electrodes is sufficiently large and the discretization step size is sufficiently small. Though injectivity of the Frechét derivative is an interesting new result on its own, it is only a secondary issue here. We namely rely on it to obtain a so-called tangential cone condition in the fully discrete setting which is the main ingredient in a well-developed convergence theory for Newton-like regularization schemes.

Key words. Electrical impedance tomography, complete electrode model, tangential cone condition, nonlinear ill-posed problem, Newton regularization.

AMS subject classifications. 35R30, 65J20.

1. Introduction. In impedance imaging or electrical impedance tomography (EIT) one reconstructs the conductivity of an object by applying electric currents through the boundary of the object and recording the resulting voltages on the boundary as well, see, e.g., Borcea [2] and Cheney et al. [4] for an overview. In a practical setting currents are injected via electrodes and, usually, the same electrodes are used for voltage recording. This approach can mathematically be represented by the well-established *complete electrode model* (CEM) which we focus on here.

Let $\gamma\colon B \to [c_0, \infty[$, $c_0 > 0$, be the searched-for conductivity distribution in the simply connected Lipschitz-domain $B \subset \mathbb{R}^2$. Further, the p electrodes are denoted by $E_1, \ldots, E_p$ and are assumed to be open subsets of ∂B, the boundary of B, having positive surface measure: $|E_j| > 0$, $j = 1, \ldots, p$. Moreover, let the electrodes be connected and separated: $\overline{E}_i \cap \overline{E}_j = \emptyset$, $i \neq j$. To this electrode configuration we associate the electrode space

$$\mathcal{E}_p := \operatorname{span}\{\chi_{E_1}, \ldots, \chi_{E_p}\} \cap L^2_\diamond(\partial B) \subset L^2_\diamond(\partial B)$$

where χ_{E_i} is the indicator function of the i-th electrode and $L^2_\diamond(\partial B) = \{f \in L^2(\partial B)\colon \int_{\partial B} f \,\mathrm{d}S = 0\}$.

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The forward problem of impedance tomography under CEM now reads: Given an electrode current $I \in \mathcal{E}_p$ and a contact impedance $z > 0$, find a voltage potential $u_p \in H^1(B)$ and an electrode voltage $U \in \mathcal{E}_p$ such that

$$-\nabla\big(\gamma\nabla u_p\big) = 0 \quad \text{in } B, \tag{1.1a}$$

$$u_p + z\,\gamma\,\nu\nabla u_p = U \quad \text{on } E = \cup_{j=1}^{p} E_j, \tag{1.1b}$$

$$\frac{1}{|E_j|}\int_{E_j} \gamma\,\nu\nabla u_p \,\mathrm{d}S = I|_{E_j} \quad \text{for } j = 1, \dots p, \tag{1.1c}$$

$$\gamma\,\nu\nabla u_p = 0 \quad \text{on } \partial B \setminus \overline{E}, \tag{1.1d}$$

where ν is the outer normal on the boundary of B. The conditions $I \in \mathcal{E}_p$ and $U \in \mathcal{E}_p$ can be interpreted as conservation of charge and grounding the potential, respectively. Both restrictions are necessary to guarantee existence and uniqueness of a weak solution, see Somersalo et al. [19].

Let us briefly explain CEM: The domain B is assumed to have no electric sources or drains. Hence, the electric flux $\gamma\nabla u_p$ is divergence free which yields (1.1a). In a medical application the conductivity between skin and electrodes may be increased by dermal moisture. This effect of contact impedance is taken into account by the Robin boundary condition (1.1b). The equations in (1.1c) model the electrodes as perfect conductors: the total electric flux over an electrode agrees with the electric current on that electrode. As there is no flux over the boundary of B in-between electrodes we have the Neumann boundary condition (1.1d).

As CEM only provides finitely many independent measurements, namely $p(p-1)/2$ (see Section 4.1), one can only recover conductivities whose number of degrees of freedom is at most the number of independent measurements. From this point of view it is quite natural and meaningful to restrict the searched-for conductivities to a finite dimensional space. Here, we will work with $V_{\mathcal{T}}$, a space of piecewise polynomials over a triangulation $\mathcal{T}$ of B.

For our numerical experiments presented in [12] we discretized the elliptic equation (1.1) by a conforming finite element space S_{Υ} with respect to a subdivision Υ of B (note that $\Upsilon \neq \mathcal{T}$ in general). Thus, we can compute the finite element approximation $(u_{p,\delta}, U_\delta) \in S_{\Upsilon} \oplus \mathcal{E}_p$ to the solution $(u_p, U) \in H^1(B) \oplus \mathcal{E}_p$ of (1.1). Here, the index $\delta > 0$ denotes the discretization step size related to Υ.

After these preparations we finally formulate the inverse problem of impedance imaging in the fully-discrete setting. The corresponding forward operator is

$$F_{p,\delta} \colon V_{\mathcal{T}}^{+} \subset V_{\mathcal{T}} \to \mathcal{L}(\mathcal{E}_p), \quad F_{p,\delta}(\gamma)I = U_\delta,$$

where $V_{\mathcal{T}}^{+} = \{\sigma \in V_{\mathcal{T}} \colon \sigma \geq c_0\}$. Put differently, $F_{p,\delta}(\gamma)$ maps the applied electrode currents to the computed electrode voltages. In the EIT inverse problem we need to find $\gamma \in V_{\mathcal{T}}^{+}$ from the observed current-to-voltage mapping $\Lambda_p \in \mathcal{L}(\mathcal{E}_p)$ such that

$$F_{p,\delta}(\gamma) = \Lambda_p. \tag{1.2}$$

To our knowledge the uniqueness question remains yet to be answered: Under which assumptions on $\mathcal{E}_p$, $V_{\mathfrak{T}}$, and S_Υ is $\gamma \in V_{\mathfrak{T}}^+$ uniquely determined by Λ_p? We will contribute a 'local' answer by showing injectivity of the Frechét derivate of $F_{p,\delta}$ for $\mathcal{E}_p$ and S_Υ rich enough. Therefore, we have local uniqueness of (1.2).

However, the local uniqueness result is only a by-product of our main objective, namely, proof of convergence of the Newton-like regularization scheme CG-REGINN for solving (1.2). In [12] we reported various numerical experiments revealing CG-REGINN as a competitive solver. Regularizing effect and local convergence are guaranteed under the so-called *tangential cone condition*, see Hanke [7] and [13]. In the exploration of iterative regularization techniques for nonlinear ill-posed problems the tangential cone condition, which traces back to Scherzer [18], emerged as a minimal requirement on the nonlinearity to yield convergence and stability, see, e.g., [9, 17] for an overview and for further original references.

The tangential cone condition controls the linearization error by the nonlinear residual. For $F_{p,\delta}$ it has the following formulation as we will show: For any $\gamma \in \operatorname{int}(V_{\mathfrak{T}}^+)$ there is a ball $B_r(\gamma) \subset \operatorname{int}(V_{\mathfrak{T}}^+)$ such that, for any τ, $\sigma \in B_r(\gamma)$,*

$$\big\|F_{p,\delta}(\tau) - F_{p,\delta}(\sigma) - F'_{p,\delta}(\sigma)[\tau-\sigma]\big\|_{\mathcal{L}(\mathcal{E}_p)} \lesssim \|\tau-\sigma\|_\infty \|F_{p,\delta}(\tau) - F_{p,\delta}(\sigma)\|_{\mathcal{L}(\mathcal{E}_p)}$$

where $F'_{p,\delta}$ denotes the Fréchet derivative of $F_{p,\delta}$. Again, $\mathcal{E}_p$ and S_Υ have to be rich enough.

As we are not able to tackle injectivity of $F'_{p,\delta}$ directly, we need to take a little detour and start out from the continuum model of EIT due to Caldéron [3]. In Section 3 we verify injectivity of the Fréchet derivative of the continuum model forward operator restricted to conductivities in $V_{\mathfrak{T}}$. Here we succeed as we can rely on a very powerful tool recently introduced by Gebauer [6]: It is possible to feed currents on the boundary such that the potentials inside the object have arbitrarily high energy on some subset and arbitrarily low energy on a different one.

By a limiting process we carry over injectivity first to CEM without discretization, cf. (1.1), and then to the fully-discrete situation of $F'_{p,\delta}$ (Section 4).

We begin this paper in the next section with presenting an abstract formulation of an parameter identification problem. All three EIT models we consider here fit into our abstract framework and, therefore, we benefit from a common treatment.

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*$A \lesssim B$ indicates the existence of a generic constant m such that $A \le mB$ uniformly in all relevant parameters of the expressions A and B. The respective context will define the meaning of 'relevant parameters'.

2. Abstract framework. Let X be a real Banach space, Y be a real Hilbert space and Z^* be a subspace of Y^*, the dual of Y.

We consider a mapping $T\colon \mathcal{D}(T) \subset X \to \mathcal{L}(Z^*, Y)$ defined by

$$T(\sigma)f = u$$

where $u \in Y$ is the unique solution of the variational problem

$$a(\sigma; u, v) = \langle f, v\rangle \quad \text{for all } v \in Y.$$

Here, $\langle \cdot, \cdot \rangle$ denotes the dual pairing between Y^* and Y. The bilinear form $a(\eta; \cdot, \cdot)\colon Y \times Y \to \mathbb{R}$ is assumed to be defined for any $\eta \in X$ and to be uniformly Y-elliptic for $\eta \in \mathcal{D}(T)$ where $\mathcal{D}(T) \subset X$ has an open interior. Furthermore, we require the following representation of a,

$$a(\eta; \cdot, \cdot) = b(\eta; \cdot, \cdot) + c(\cdot, \cdot),$$

as a sum of a trilinear form b and a bilinear form c, both bounded, especially

$$|b(\eta; w, v)| \lesssim \|\eta\|_X \|w\|_Y \|v\|_Y. \tag{2.1}$$

Below, in Lemma 2.1, we show Frechét differentiability of T. To this end, we provide auxiliary estimates. By ellipticity, $\|u\|_Y^2 \lesssim a(u, u) = \langle f, u\rangle \leq \|f\|_{Y^*} \|u\|_Y$ yielding

$$\|T(\sigma)f\|_Y \lesssim \|f\|_{Y^*} \quad \text{uniformly in } \sigma \in \mathcal{D}(T). \tag{2.2}$$

Next we establish Lipschitz continuity of T. Again, by ellipticity

$$\begin{aligned}
\|T(\sigma)f - T(\gamma)f\|_Y^2 &\lesssim a(\gamma; T(\sigma)f - T(\gamma)f, T(\sigma)f - T(\gamma)f) \\
&= a(\gamma; T(\sigma)f, T(\sigma)f - T(\gamma)f) - \langle f, T(\sigma)f - T(\gamma)f\rangle \\
&= a(\gamma; T(\sigma)f, T(\sigma)f - T(\gamma)f) \\
&\qquad\qquad - a(\sigma; T(\sigma)f, T(\sigma)f - T(\gamma)f) \\
&= b(\gamma - \sigma; T(\sigma)f, T(\sigma)f - T(\gamma)f) \\
&\overset{(2.1)}{\lesssim} \|\gamma - \sigma\|_X \|T(\sigma)f\|_Y \|T(\sigma)f - T(\gamma)f\|_Y
\end{aligned}$$

which, in view of (2.2), is the Lipschitz continuity

$$\|T(\sigma)f - T(\gamma)f\|_Y \lesssim \|\gamma - \sigma\|_X \|f\|_{Y^*}. \tag{2.3}$$

LEMMA 2.1. *Let $\sigma \in \operatorname{int}(\mathcal{D}(T))$. Then, T is Frechét differentiable with $T'(\sigma) \in \mathcal{L}(X, \mathcal{L}(Z^*, Y))$ given by*

$$T'(\sigma)[h]f = u'$$

where $u' \in Y$ solves

$$a(\sigma; u', v) = -b(h; T(\sigma)f, v) \quad \textit{for all } v \in Y.$$

Proof. Clearly, the operator $T'(\sigma)$ is linear and uniformly bounded in σ:

$$\|T'(\sigma)[h]f\|_Y \lesssim \|h\|_X \|f\|_{Y^*}. \tag{2.4}$$

Now, let $h \in X$ be so small that $\sigma + h \in \mathcal{D}(T)$. With $u^+ = T(\sigma + h)f$ and $u = T(\sigma)f$ we have, by ellipticity,

$$\begin{aligned}\|u^+ - u - u'\|_Y^2 &\lesssim a(\sigma; u^+ - u - u', u^+ - u - u')\\ &= a(\sigma; u^+ - u, u^+ - u - u') - a(\sigma; u', u^+ - u - u')\\ &= a(\sigma; u^+ - u, u^+ - u - u') + b(h; u, u^+ - u - u').\end{aligned}$$

Since

$$\begin{aligned}a(\sigma; u^+ - u, u^+ - u - u') &= a(\sigma; u^+, u^+ - u - u') - a(\sigma; u, u^+ - u - u')\\ &= b(\sigma + h; u^+, u^+ - u - u') + c(u^+, u^+ - u - u')\\ &\quad - b(h; u^+, u^+ - u - u') - a(\sigma; u, u^+ - u - u')\\ &= \langle f, u^+ - u - u' \rangle\\ &\quad - b(h; u^+, u^+ - u - u') - \langle f, u^+ - u - u' \rangle\\ &= -b(h; u^+, u^+ - u - u')\end{aligned}$$

we proceed with

$$\begin{aligned}\|u^+ - u - u'\|_Y^2 &\lesssim b(h; u - u^+, u^+ - u - u')\\ &\overset{(2.1)}{\lesssim} \|h\|_X \|u^+ - u\|_Y \|u^+ - u - u'\|_Y\end{aligned}$$

resulting in

$$\|u^+ - u - u'\|_Y \lesssim \|h\|_X \|u^+ - u\|_Y \overset{(2.3)}{\lesssim} \|h\|_X^2 \|f\|_{Y^*}. \tag{2.5}$$

Hence,

$$\frac{\|T(\sigma + h)f - T(\sigma)f - u'\|_Y}{\|h\|_X} \lesssim \|h\|_X \|f\|_{Y^*} \xrightarrow{h \to 0} 0,$$

that is, $T'(\sigma)[h]f = u'$. ∎

Corollary 2.2. *Under the assumptions of Lemma* 2.1 *we have*

$$\|T(\sigma + h) - T(\sigma) - T'(\sigma)[h]\|_{\mathcal{L}(Z^*, Y)} \lesssim \big(\|\sigma\|_X + \|c\|\big) \|h\|_X \|T'(\sigma)[h]\|_{\mathcal{L}(Z^*, Y)}$$

for $\sigma \in \operatorname{int}(\mathcal{D}(T))$ *and* $\sigma + h \in \mathcal{D}(T)$.

Proof. We set $u^+ = T(\sigma+h)f$, $u = T(\sigma)f$, and $u' = T'(\sigma)[h]f$. By ellipticity,

$$\begin{aligned}\|u^+ - u\|_Y^2 &\lesssim a(\sigma+h; u^+ - u, u^+ - u)\\ &= \langle f, u^+ - u\rangle_{Y^*\times Y} - b(\sigma+h; u, u^+ - u) - c(u, u^+ - u)\\ &= -b(h; u, u^+ - u) = a(\sigma; u', u^+ - u) = b(\sigma, u', u^+ - u) + c(u', u^+ - u)\\ &\lesssim \big(\|\sigma\|_X + \|c\|\big)\|u'\|_Y\|u^+ - u\|_Y,\end{aligned}$$

that is,

$$\|u^+ - u\|_Y \lesssim \big(\|\sigma\|_X + \|c\|\big)\,\|u'\|_Y$$

which implies the assertion by the first estimate in (2.5). ◻

We close this section with the Lipschitz continuity of T'.

LEMMA 2.3. *Let* σ_1, $\sigma_2 \in \operatorname{int}\big(\mathcal{D}(T)\big)$, h_1, $h_2 \in X$, *and* f_1, $f_2 \in Z^*$. *Then,*

$$\begin{aligned}\|T'(\sigma_1)[h_1]f_1 - T'(\sigma_2)[h_2]f_2\|_Y \lesssim{}& \|\sigma_1 - \sigma_2\|_X\|h_1\|_X\|f_1\|_{Y^*}\\ &+ \|h_1 - h_2\|_X\|f_1\|_{Y^*} + \|f_1 - f_2\|_{Y^*}\|h_2\|_X.\end{aligned}$$

Proof. Set $w_{i,j,k} := T'(\sigma_i)[h_j]f_k$ for $i,j,k \in \{1,2\}$. Then,

$$\begin{aligned}\|T'(\sigma_1)[h_1]f_1 - T'(\sigma_2)[h_2]f_2\|_Y \le{}& \|w_{1,1,1} - w_{2,1,1}\|_Y + \|w_{2,1,1} - w_{2,2,1}\|_Y\\ &+ \|w_{2,2,1} - w_{2,2,2}\|_Y\end{aligned}$$

and each difference on the right will be estimated separately. We start with

$$\begin{aligned}\|w_{1,1,1} - w_{2,1,1}\|_Y^2 &\lesssim a(\sigma_1, w_{1,1,1} - w_{2,1,1}, w_{1,1,1} - w_{2,1,1})\\ &= -b(h_1, T(\sigma_1)f_1, w_{1,1,1} - w_{2,1,1}) + b(h_1, T(\sigma_2)f_1, w_{1,1,1} - w_{2,1,1})\\ &\qquad + a(\sigma_1; w_{2,1,1}, w_{1,1,1} - w_{2,1,1}) - a(\sigma_1; w_{2,1,1}, w_{1,1,1} - w_{2,1,1})\\ &\overset{(2.1)}{\lesssim} \|h_1\|_X\|T(\sigma_1)f_1 - T(\sigma_2)f_1\|_Y\|w_{1,1,1} - w_{2,1,1}\|_Y\\ &\qquad + \|\sigma_1 - \sigma_2\|_X\|w_{2,1,1}\|_Y\|w_{1,1,1} - w_{2,1,1}\|_Y.\end{aligned}$$

By (2.3) and (2.4) we obtain

$$\|w_{1,1,1} - w_{2,1,1}\|_Y \lesssim \|\sigma_1 - \sigma_2\|_X\|h_1\|_X\|f_1\|_{Y^*}.$$

Next, since

$$\begin{aligned}a(\sigma_2; w_{2,1,1} - w_{2,2,1}, v) &= -b(\sigma_2 + h_1; T(\sigma_2)f_1, v) + b(\sigma_2 + h_2; T(\sigma_2)f_1, v)\\ &= b(h_2 - h_1; T(\sigma_2)f_1, v)\\ &\overset{(2.2)}{\lesssim} \|h_2 - h_1\|_X\|f_1\|_{Y^*}\|v\|_Y\end{aligned}$$

we find

$$\|w_{2,1,1} - w_{2,2,1}\|_Y \lesssim \|h_2 - h_1\|_X \|f_1\|_{Y^*}.$$

Finally,

$$a(\sigma_2; w_{2,2,1} - w_{2,2,2}, v) = b(h_2; T(\sigma_2)(f_2 - f_1), v) \overset{(2.2)}{\lesssim} \|h_2\|_X \|f_2 - f_1\|_{Y^*} \|v\|_Y$$

gives

$$\|w_{2,2,1} - w_{2,2,2}\|_Y \lesssim \|h_2\|_X \|f_2 - f_1\|_{Y^*}$$

concluding the proof. □

3. Frechét derivative of EIT operator in the continuum model. The continuum model of EIT was introduced by Caldéron in his pioneering paper [3]: Current is applied on all of the boundary of B where also the voltages are observed.

Let $f \in L^2_\diamond(\partial B)$ be the applied current and $\gamma \in L^\infty_+(B) = \{\sigma \in L^\infty(B) : \sigma \geq c_0\}$ be the conductivity. Then, the governing equation in weak formulation is

$$\int_B \gamma \nabla u \nabla v \, \mathrm{d}x = \int_{\partial B} f v \, \mathrm{d}S \quad \text{for all } v \in H^1_\diamond(B) \tag{3.1}$$

and it has a unique solution $u \in H^1_\diamond(B) := \{v \in H^1(B) : \int_{\partial B} v \, \mathrm{d}S = 0\}$.

The inverse EIT problem in the continuum model can now be phrased as: given the *Neumann-to-Dirichlet operator*

$$\Lambda \colon f \mapsto u|_{\partial B} \tag{3.2}$$

find the conductivity γ. By classical results from the theory of partial differential equations, see, e.g., [15], Λ is known to be a bounded linear operator between $H^{-1/2}_\diamond(\partial B)$ and $H^{1/2}_\diamond(\partial B)$.

Mathematically, we have to solve an equation with the nonlinear operator F describing the forward problem, that is, we need to solve $F(\gamma) = \Lambda$ where

$$F \colon \mathcal{D}(F) \subset L^\infty(B) \to \mathcal{L}(H^{-1/2}_\diamond(\partial B), H^{1/2}_\diamond(\partial B)), \quad \gamma \mapsto \Lambda,$$

with $\mathcal{D}(F) := L^\infty_+(B)$ being a cone with vertex c_0 in the space of bounded and measurable functions. Note that $F(\gamma) = \Lambda$ is uniquely solvable, see Astala and Päivärinta [1].

Relying on the abstract framework of the former section we show Frechét differentiability of the forward operator F. To this end we introduce operator

$$T \colon \mathcal{D}(F) \subset L^\infty(B) \to \mathcal{L}(H^{-1/2}_\diamond(\partial B), H^1_\diamond(B))$$

by

$$T(\gamma) \colon f \mapsto u \quad \text{where } u \text{ is the solution of (3.1).}$$

Taking the trace of T we obtain F, more precisely we have that

$$F(\gamma) = R_{\partial B} T(\gamma)$$

with $R_{\partial B} \in \mathcal{L}\big(H^1_\diamond(B), H^{1/2}_\diamond(\partial B)\big)$ being the trace operator. Moreover, T fits into the abstract framework with $X = L^\infty(B)$, $Y = H^1_\diamond(B)$, $Z^* = H^{-1/2}_\diamond(\partial B)$, $a(\gamma; v, w) = \int_B \gamma \nabla v \nabla w \,\mathrm{d}x$, and $\langle f, v\rangle = \int_{\partial B} f v \,\mathrm{d}S$.

Therefore, by (2.3)

$$\|F(\sigma)f - F(\gamma)f\|_{H^{1/2}(\partial B)} \lesssim \|T(\sigma)f - T(\gamma)f\|_{H^1(B)} \lesssim \|\sigma - \gamma\|_\infty \|f\|_{H^{-1/2}(\partial B)}$$

implying the Lipschitz continuity of

$$F : L^\infty_+(B) \subset L^\infty(B) \to \mathcal{L}(H^t_\diamond(\partial B), H^s_\diamond(\partial B)), \quad t \geq -1/2,\ s \leq 1/2.$$

According to Lemma 2.1 F is also Frechét differentiable in $\gamma \in \operatorname{int}\big(\mathcal{D}(F)\big)$ with derivative $F'(\gamma) \in \mathcal{L}\left(L^\infty(B), \mathcal{L}(H^t_\diamond(\partial B), H^s_\diamond(\partial B))\right)$ given

$$F'(\gamma)[h]f = R_{\partial B} u' \tag{3.3}$$

where $u' \in H^1_\diamond(B)$ is the unique solution of the elliptic problem

$$\int_B \gamma \nabla u' \nabla \varphi \,\mathrm{d}x = -\int_B h \nabla u(f) \nabla \varphi \,\mathrm{d}x \quad \text{for all } \varphi \in H^1_\diamond(B). \tag{3.4}$$

In (3.4), $u(f) = T(\gamma)f \in H^1_\diamond(B)$ solves (3.1).

The following theorem is Lemma 2.3 formulated in the EIT setting.

Theorem 3.1. *Let γ_1, $\gamma_2 \in \operatorname{int}\big(\mathcal{D}(F)\big)$, h_1, $h_2 \in L^\infty(B)$, and f_1, $f_2 \in H^{-1/2}_\diamond(\partial B)$. Then,*

$$\begin{aligned}\|F'(\gamma_1)[h_1]f_1 - F'(\gamma_2)[h_2]f_2\|_{H^{1/2}(\partial B)} &\lesssim \|\gamma_1 - \gamma_2\|_\infty \|h_1\|_\infty \|f_1\|_{H^{-1/2}(\partial B)} \\ &\quad + \|h_1 - h_2\|_\infty \|f_1\|_{H^{-1/2}(\partial B)} + \|f_1 - f_2\|_{H^{-1/2}(\partial B)} \|h_2\|_\infty.\end{aligned}$$

3.1. Injectivity of F' for piecewise polynomial conductivities. From here on we restrict the conductivities to a finite dimensional space of piecewise polynomials: Embed $\overline{B}$ into a rectangular domain D which is covered by a triangulation $\mathcal{T}$. Neither is $\mathcal{T}$ assumed to be regular nor uniform, see Figure 3.1. Let $W_\mathcal{T}$ be the space of all functions defined on D which are polynomials locally on any triangle of $\mathcal{T}$ ($v \in W_\mathcal{T}$ iff $v|_\triangle$ is a polynomial for any $\triangle \in \mathcal{T}$) and set

$$V_\mathcal{T} := \{v|_B : v \in W_\mathcal{T}\} \subset L^\infty(B).$$

The following lemma states that the Fréchet derivative of $T : \mathcal{D}(F) \subset L^\infty(B) \to \mathcal{L}(L^2_\diamond(\partial B), H^1_\diamond(B))$ cannot vanish for any $\gamma \in \operatorname{int}(\mathcal{D}(F))$ in any direction $h \in V_\mathcal{T}$.

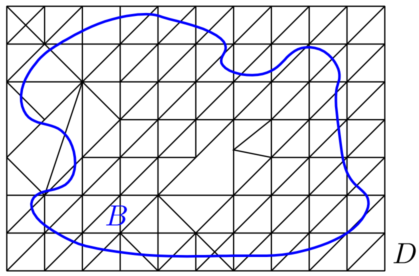

Figure 3.1: Embedding of B into the rectangular domain D which is triangulated non-regularly and non-uniformly.

Lemma 3.2. *Let $\gamma \in \operatorname{int}(\mathcal{D}(F))$. For all $h \in V_{\mathcal{T}} \setminus \{0\}$ we have*

$$\|T'(\gamma)[h]\|_{\mathcal{L}(L^2_\diamond(\partial B), H^1_\diamond(B))} \sim \|\nabla T'(\gamma)[h]\|_{\mathcal{L}(L^2_\diamond(\partial B), L^2_\diamond(B))} > 0.$$

Proof. We will rely on a result due to Gebauer [6, Theorem 2.7] which we quote here for the reader's convenience:

> *Let σ satisfy the unique continuation property (σ piecewise Lipschitz is sufficient) and let Ω_1, $\Omega_2 \subset B$ be two open sets with $\overline{\Omega_1} \cap \overline{\Omega_2} = \emptyset$. Furthermore, let $B \setminus (\overline{\Omega_1} \cup \overline{\Omega_2})$ be connected and $\overline{B} \setminus (\overline{\Omega_1} \cup \overline{\Omega_2})$ contain the relatively open set S. Then there exists a sequence of currents $\{f_n\} \subset L^2_\diamond(S)$ and corresponding potentials $\{u_n\}$, defined by the weak formulation of*
>
> $$\nabla \cdot \sigma \nabla u_n = 0, \quad \sigma \partial_\nu u_n|_{\partial B} = \begin{cases} f_n & \text{on } S, \\ 0 & \text{otherwise,} \end{cases}$$
>
> *such that*
>
> $$\lim_{n\to\infty} \int_{\Omega_1} |\nabla u_n|^2 \mathrm{d}x = \infty \text{ and } \lim_{n\to\infty} \int_{\Omega_2} |\nabla u_n|^2 \mathrm{d}x = 0.$$

We will construct Ω_1 and Ω_2 where we will distinguish two scenarios depending on $h \neq 0$:

1. h is not identically zero on the boundary ∂B. As h is locally a polynomial, there is a relatively open connected subset S of ∂B such that $\operatorname{sgn}(h)$ differs from zero and is constant in an open and connected neighborhood U of S in $\overline{B}$. In this neighborhood U we fix an open ball Ω_1 compactly contained in B. Further, we set $\Omega_2 = B \setminus \overline{U}$.

2. h is identically zero on ∂B. Here we will show that $\operatorname{supp}(h)$ is a union of triangles of $\mathcal{T}$. To this end assume that $\partial\operatorname{supp}(h)$ cuts a triangle

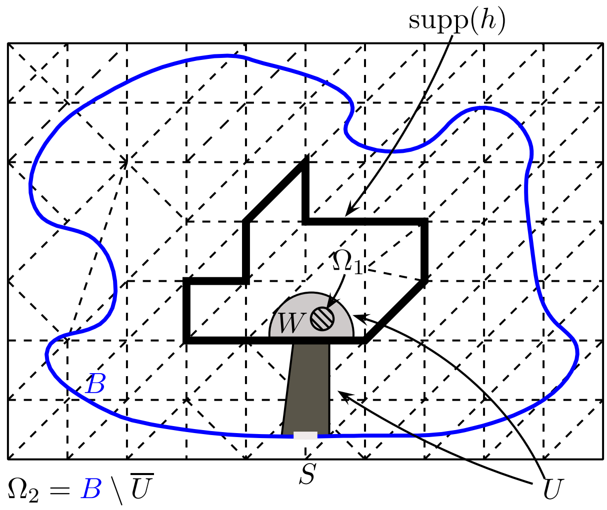

Figure 3.2: A construction of Ω_1, Ω_2, and S in the proof of Lemma 3.2. The support of $h \in V_{\mathfrak{T}}$ is the union of triangles surrounded by the bold black polygon. The sign of h is constant different from zero on the light gray semi-disk $W \subset \mathrm{supp}(h)$ and Ω_1 is the hatched disk in W. Further, U is defined as the union of W and the dark gray region connecting $\partial W \cap \partial\mathrm{supp}(h)$ with ∂B.

$\triangle$ into two pieces with positive measure. One of these pieces is in the exterior of $\mathrm{supp}(h)$ where h is identically zero. Hence, $h|_\triangle = 0$ and the intersection of a triangle with $\partial\mathrm{supp} h$ is either one edge or a node.

If h is not identically zero on $\partial\mathrm{supp}(h)$ then there is a relatively open connected subset $\widetilde{S}$ of $\partial\mathrm{supp}(h)$ such that $\mathrm{sgn}(h)$ differs from zero and is constant in an open and connected neighborhood W of $\widetilde{S}$ in $\overline{\mathrm{supp}(h)}$. We choose Ω_1 to an open ball compactly contained in W. Further, we define U to be the union of W and a neighborhood of a path connecting one point in $\widetilde{S}$ with ∂B. Observe that $\mathrm{sgn}(h) \geq 0$ on $\overline{U}$. Finally, set $\Omega_2 := B \setminus \overline{U}$ and $S := \mathrm{int}(\overline{U} \cap \partial B)$ relative to ∂B. Figure 3.2 highlights a graphical representation of the construction of Ω_1, Ω_2, and S.

If h is identically zero on $\partial\mathrm{supp}(h)$ then there is an open subset W of $\mathrm{supp}(h)$ such that $\mathrm{int}\big(\overline{W} \cap \partial\mathrm{supp}(h)\big)$ is relatively open in $\partial\mathrm{supp}(h)$ and $\mathrm{sgn}(h)$ is constant in W different from zero.
In case $\overline{W} \cap \partial B \neq \emptyset$ we set $U := W$, $S := \mathrm{int}(\overline{U} \cap \partial B)$, $\Omega_2 := B \setminus \overline{U}$, and choose Ω_1 to be an open ball compactly contained in U.
In case $\overline{W} \cap \partial B = \emptyset$ we choose Ω_1 to be an open ball compactly contained in W. Further, we define U to be the union of W and a neighborhood of a path connecting one point in $\mathrm{int}\big(\overline{W} \cap \partial\mathrm{supp}(h)\big)$ with ∂B. Observe that $\mathrm{sgn}(h) \geq 0$ on $\overline{U}$. Finally, set $\Omega_2 := B \setminus \overline{U}$ and $S := \mathrm{int}(\overline{U} \cap \partial B)$ relative to ∂B.

In all situations considered, the defined Ω_1, Ω_2 and S satisfy the hypotheses of Gebauer's theorem quoted above. Thus, there is a sequence $f_j \subset L^2_\diamond(\partial B)$ such that

$$\lim_{j\to\infty} \int_{\Omega_1} |\nabla u(f_j)|^2 \,\mathrm{d}x = \infty \quad \text{and} \quad \lim_{j\to\infty} \int_{\Omega_2} |\nabla u(f_j)|^2 \,\mathrm{d}x = 0.$$

Moreover,

$$\int_B h \,|\nabla u(f_j)|^2 \,\mathrm{d}x = \int_{\Omega_1} h \,|\nabla u(f_j)|^2 \,\mathrm{d}x + \int_{\Omega_2} h \,|\nabla u(f_j)|^2 \,\mathrm{d}x + \int_{U\setminus\Omega_1} h \,|\nabla u(f_j)|^2 \,\mathrm{d}x.$$

The sign of the integral over $U \setminus \Omega_1$ is either 0 or the sign of h in Ω_1. Therefore,

$$\int_B h \,|\nabla u(f_j)|^2 \,\mathrm{d}x \to \pm\infty \quad \text{as } j \to \infty, \tag{3.5}$$

the sign being the sign of h in Ω_1.

Assume now that

$$\|\nabla T'(\gamma)[h]\|_{\mathcal{L}(L^2_\diamond(\partial B), L^2(B))} = 0. \tag{3.6}$$

Recall: $T'(\gamma)[h]f = u'$ with $u' \in H^1_\diamond(B)$ being the solution of (3.4) and $u = u(f)$ solves (3.1). Plugging $\varphi = u(f_j)$ into (3.4) we find that

$$-\int_B h|\nabla u(f_j)|^2 \,\mathrm{d}x = \int_B \gamma \underbrace{\nabla(T'(\gamma)[h]f_j)}_{= 0 \text{ by } (3.6)} \nabla u(f_j) \,\mathrm{d}x = 0$$

contradicting (3.5), i.e., (3.6) is falsified. The stated norm equivalence is due to Poincaré's inequality, see e.g. [16]. □

Now the injectivity result for F' follows easily.

Corollary 3.3. *Under the assumptions of the former lemma we have that*

$$\min\left\{\|F'(\gamma)[h]\|_{\mathcal{L}(L^2_\diamond(\partial B))} : h \in V_{\mathfrak{T}},\ \|h\|_\infty = 1\right\} > 0.$$

Proof. Assume the claim to be false. As $V_{\mathfrak{T}}$ is finite dimensional and $F'(\gamma)$ is continuous (Theorem 3.1) there is a normalized $h \in V_{\mathfrak{T}}$ such that $F'(\gamma)[h]f = 0$ for any $f \in L^2_\diamond(\partial B)$. Further,

$$-\int_B h|\nabla u(f)|^2 \,\mathrm{d}x \overset{(3.4)}{=} \int_B \gamma\nabla(T'(\gamma)[h]f)\,\nabla u(f) \,\mathrm{d}x \overset{(3.1)}{=} \int_{\partial B} f \,\underbrace{(F'(\gamma)[h]f)}_{=0} \,\mathrm{d}S = 0$$

which cannot hold true due to (3.5). $\square$

As a by-product we obtain a tangential cone condition for the continuum model in a semi-discrete setting.

Theorem 3.4. *Let $F : V_{\mathcal{T}}^+ \subset V_{\mathcal{T}} \to \mathcal{L}\big(L^2_\diamond(\partial B)\big)$ be the EIT operator where $V_{\mathcal{T}}^+ = V_{\mathcal{T}} \cap L^\infty_+(B)$. If $\gamma \in \operatorname{int}(V_{\mathcal{T}}^+)$ then there is an open ball $B_r(\gamma) \subset \operatorname{int}(V_{\mathcal{T}}^+)$ about γ of radius $r > 0$ such that, for all τ, $\sigma \in B_r(\gamma)$,*

$$\big\|F(\tau) - F(\sigma) - F'(\sigma)[\tau - \sigma]\big\|_{\mathcal{L}(L^2_\diamond(\partial B))} \lesssim \|\tau - \sigma\|_\infty \|F(\tau) - F(\sigma)\|_{\mathcal{L}(L^2_\diamond(\partial B))}$$

where the constant depends, amongst others, on γ, r, and $V_{\mathcal{T}}$.

Proof. Setting $h = \tau - \sigma$ we find from Corollary 2.2 (recall: $c = 0$) that

$$\begin{aligned}\big\|F(\tau) - F(\sigma) - F'(\sigma)[h]\big\|_{\mathcal{L}(L^2_\diamond(\partial B))} &\leq C(B, c_0)\, \big\|T(\tau) - T(\sigma) - T'(\sigma)[h]\big\|_{\mathcal{L}(L^2_\diamond(\partial B), H^1_\diamond(B))} \\ &\leq C(B, c_0)\, \|\sigma\|_\infty \|h\|_\infty \|\nabla T'(\sigma)[h]\|_{\mathcal{L}(L^2_\diamond(\partial B), L^2_\diamond(B))} \\ &\leq C(B, c_0)\, (\|\gamma\|_\infty + \rho) \|h\|_\infty \|\nabla T'(\sigma)[h]\|_{\mathcal{L}(L^2_\diamond(\partial B), L^2_\diamond(B))}\end{aligned}$$

which holds true for all $\tau \in V_{\mathcal{T}}^+$ and all $\sigma \in B_\rho(\gamma)$ where $\rho > 0$ is such that $B_\rho(\gamma) \subset \operatorname{int}(V_{\mathcal{T}}^+)$. Tracking the constants and their dependencies is important here, therefore, we state them explicitely.

By continuity (Theorem 3.1) and injectivity (Corollary 3.3) we have that

$$\sup\left\{ \frac{\|\nabla T'(\sigma)[h]\|_{\mathcal{L}(L^2_\diamond(\partial B), L^2_\diamond(B))}}{\|F'(\sigma)[h]\|_{\mathcal{L}(L^2_\diamond(\partial B))}} : (\sigma, h) \in B_\rho(\gamma) \times V_{\mathcal{T}} \right\} < \infty. \tag{3.7}$$

Hence,

$$\big\|F(\tau) - F(\sigma) - F'(\sigma)[h]\big\|_{\mathcal{L}(L^2_\diamond(\partial B))} \leq C(B, c_0, \gamma, \rho, V_{\mathcal{T}})\, (\|\gamma\|_\infty + \rho) \|h\|_\infty \|F'(\sigma)[h]\|_{\mathcal{L}(L^2_\diamond(\partial B))}.$$

If $\|h\|_\infty < \frac{1}{C(B, c_0, \gamma, \rho, V_{\mathcal{T}})\, (\|\gamma\|_\infty + \rho)}$ then

$$\|F'(\sigma)[h]\|_{\mathcal{L}(L^2_\diamond(\partial B))} \leq \frac{\|F(\tau) - F(\sigma)\|_{\mathcal{L}(L^2_\diamond(\partial B))}}{1 - C(B, c_0, \gamma, \rho, V_{\mathcal{T}})(\|\gamma\|_\infty + \rho)\|h\|_\infty}$$

and the assertion is true for any positive r with

$$r < \min\left\{\rho,\ \frac{1}{2\, C(B, c_0, \gamma, \rho, V_{\mathcal{T}})\, (\|\gamma\|_\infty + \rho)}\right\}.$$ $\square$

Remark. *If the supremum (3.7) can be shown to be uniformly bounded in $\mathcal{T}$ then, by continuity, Theorem 3.4 implies a tangential cone condition for $F : \mathcal{D}(F) \subset L^\infty(B) \to \mathcal{L}\big(L^2_\diamond(\partial B)\big)$.*

4. Complete electrode model. For the weak formulation of (1.1) we define the bilinear form $a\colon Y_p \times Y_p \to \mathbb{R}$, $Y_p := H^1(B) \oplus \mathcal{E}_p$, by

$$a\big((v,V),(w,W)\big) := \int_B \gamma \nabla v \nabla w \, \mathrm{d}x + \frac{1}{z}\int_E (v-V)(w-W)\, \mathrm{d}S.$$

The form a is elliptic ($\gamma \geq c_0$) and continuous where the ellipticity and continuity constants depend neither on the number nor the size of the electrodes, see Hyvönen [8, Lemma 2.5, Corollary 3.6]. The weak solution $(u_p, U) \in Y_p$ of (1.1) is now given as the unique solution of the variational problem:

$$a\big((u_p,U),(w,W)\big) = \int_{\partial B} I\, W \,\mathrm{d}S \quad \text{for all } (w,W) \in Y_p. \tag{4.1}$$

The nonlinear forward operator F_p describing CEM is given by

$$F_p\colon \mathcal{D}(F) \subset L^\infty(B) \to \mathcal{L}(\mathcal{E}_p), \quad \gamma \mapsto \{I \mapsto U\},$$

that is, $F_p(\gamma)$ maps the electrode current I to the electrode voltage U of the solution of (4.1): $F_p(\gamma)I = U$.

To use the abstract framework of Section 2 we introduce $T_p\colon \mathcal{D}(F) \subset L^\infty(B) \to \mathcal{L}(\mathcal{E}_p, Y_p)$ by

$$T_p(\gamma)I = (u_p, U) \ \text{ where } (u_p,U) \text{ solves (4.1)}. \tag{4.2}$$

We emphasize that all assumptions from Section 2 are satisfied. The operators F_p and T_p are related via

$$F_p(\gamma) = R_{\mathcal{E}_p} T_p(\gamma) \tag{4.3}$$

with the restriction operator $R_{\mathcal{E}_p} \in \mathcal{L}(Y_p, \mathcal{E}_p)$, $R_{\mathcal{E}_p}(v,V) = V$. According to Lemma 2.1 the Frechét derivative $F_p'(\gamma) \in \mathcal{L}\big(L^\infty(B), \mathcal{L}(\mathcal{E}_p)\big)$, $\gamma \in \operatorname{int}\big(\mathcal{D}(F)\big)$, is given by

$$F_p'(\gamma)[h]I = R_{\mathcal{E}_p} T_p'(\gamma)[h]I = R_{\mathcal{E}_p}(u_p', U') = U'$$

where $(u_p', U') \in Y_p$ uniquely solves

$$a\big((u_p',U'),(w,W)\big) = -\int_B h \nabla u_p(I) \nabla w \,\mathrm{d}x \quad \text{for all } (w,W) \in Y_p \tag{4.4}$$

with $u_p = u_p(I)$ being the first component of the solution of (4.1) with respect to the electrode current I.

4.1. Piecewise polynomial conductivities. The symmetry of $F_p(\gamma) \in \mathcal{L}(\mathcal{E}_p)$,

$$\begin{aligned}\langle I, F_p(\gamma)J\rangle_{L^2(\partial B)} &\overset{(4.1)}{=} a\big((u_p(I), F_p(\gamma)I), (u_p(J), F_p(\gamma)J)\big)\\ &\overset{(4.1)}{=} \langle F_p(\gamma)I, J\rangle_{L^2(\partial B)},\end{aligned}$$

reveals that CEM offers only $p(p-1)/2$ independent measurements ($F_p(\gamma)$ may be represented by a symmetric matrix of order $\dim \mathcal{E}_p = p-1$). Therefore, we can only hope to recover conductivities whose number of degrees of freedom is at most the number of independent measurements.

From this point of view considering F_p in the finite-dimensional setting

$$F_p : V_{\mathfrak{J}}^+ \subset V_{\mathfrak{J}} \to \mathcal{L}(\mathcal{E}_p)$$

is reasonable ($V_{\mathfrak{J}}$ and $V_{\mathfrak{J}}^+$ are as in Section 3.1). Below in Theorem 4.3 we will validate injectivity of $F_p'(\gamma) \in \mathcal{L}\big(V_{\mathfrak{J}}, \mathcal{L}(\mathcal{E}_p)\big)$ for p sufficiently large. The proof of injectivity will be prepared by auxiliary results.

An important ingredient of our analysis are suitable estimates between solutions of CEM and the continuum model, provided by Lechleiter et al. [11]. These estimates link the injectivity result from Section 3.1 to CEM. In [11] an asymptotic analysis in the number of electrodes shows the necessary convergence properties between the CEM and the continuum model. As suitable framework, consider a sequence of forward operators $\{F_p\}_{p\in\mathbb{N}}$, where the index p corresponds to the number of electrodes used to define the forward operator F_p. To every F_p there corresponds hence a certain electrode configuration with p electrodes. The convergence formulated in (4.6) below relies on estimate (7.4) from [11] and therefore we need to adopt the geometric assumptions as specified in [11]: The Lipschitz boundary of B is assumed to be piecewise $\mathcal{C}^\infty$ where all electrodes are located, in any configuration, on the $\mathcal{C}^\infty$-patches. Let us denote by $\{E_j^p\}_{j=1}^p \subset \partial B$ the set of (connected and separated) electrodes and by $\{G_j^p\}_{j=1}^p \subset \partial B$ the set of gaps between the electrodes for the pth configuration. We require that $\sup_j |E_j^p| \to 0$ as $p \to \infty$ and $\sum_{j=1}^p |E_j^p| \to |\partial B|$, where $|E_j^p|$ and $|\partial B|$ denotes the surface measure of the electrode E_j^p and the boundary ∂B. The last condition already implies that $\sum_{j=1}^p |G_j^p| \to 0$, however, the analysis in [11] shows that we need to strengthen this assumption by

$$\lim_{p\to\infty} \sum_{j=1}^{p} \big|G_j^p\big|^\theta = 0 \tag{4.5}$$

for some exponent $\theta \in (0,1)$.[†] Now, let $P_p : L^2(\partial B) \to L^2(\partial B)$ denote the orthogonal projection onto $\mathcal{E}_p$. Then,

$$\lim_{p\to\infty} \sup \big\{ \|u(f) - u_p(P_p f)\|_{H^1(B)} \colon f \in L^2_\diamond(\partial B),\ \|f\|_{L^2(\partial B)} = 1, \\ \gamma \in \mathcal{D}(F) \big\} = 0 \tag{4.6}$$

where $u(f) = T(\gamma)f$ and $u_p(P_p f)$ are the electric potentials from (3.1) and (4.1) with respect to the boundary current f and electrode current $P_p f$,

[†]Due to the continuous embedding $H^{1/2}(\mathbb{R}) \hookrightarrow L^\theta(\mathbb{R})$ for $\theta \in [2,\infty)$ [14, Theorem 8.5] we need to replace condition (7.7) from [11], valid in three dimensions, by (4.5) for our two dimensional setting.

respectively, see Lechleiter et al. [11, Sec. 7, Remark 7.7].

Lemma 4.1. *Under the above assumptions we have that*

$$\|F'(\gamma)[h]f - F_p'(\gamma)[h]P_p f\|_{L^2(\partial B)} \lesssim \|(\mathrm{Id} - P_p)F'(\gamma)[h]f\|_{L^2(\partial B)} + \|h\|_\infty \|\nabla(u(f) - u_p(P_p f))\|_{L^2(B)}$$

for any $f \in L^2_\diamond(\partial B)$ and any $h \in L^\infty(B)$. The involved constant is uniform in $\gamma \in \mathrm{int}(\mathcal{D}(F))$ and does not depend on p or on the specific electrode configuration.

Proof. Set $u' = F'(\gamma)[h]f$. The quantity to estimate is

$$\|F'(\gamma)[h]f - F_p'(\gamma)[h]P_p f\|_{L^2(\partial B)} = \|u' - U'\|_{L^2(\partial B)} \leq \|u' - P_p u'\|_{L^2(\partial B)} + \|P_p u' - U'\|_{L^2(\partial B)}.$$

The first difference is part of the statement. Therefore, we only consider the second difference. From the Y_p-ellipticity of a we find that

$$\begin{aligned}
\|(u_p', U') - (u', P_p u')\|_{Y_p}^2 &\lesssim a\big((u_p', U') - (u', P_p u'), (u_p', U') - (u', P_p u')\big) \\
&= -\int_B h\nabla u_p \nabla(u_p' - u')\,\mathrm{d}x - a\big((u', P_p u'), (u_p', U') - (u', P_p u')\big) \\
&= -\int_B h\nabla u_p \nabla(u_p' - u')\,\mathrm{d}x - \int_B \gamma\nabla u' \nabla(u_p' - u')\,\mathrm{d}x \\
&\qquad - \frac{1}{z}\int_E \big(u' - P_p u'\big)\big(u_p' - u' - (U' - P_p u')\big)\,\mathrm{d}S \\
&\overset{(3.4)}{=} \int_B h\nabla(u - u_p)\nabla(u_p' - u')\,\mathrm{d}x \\
&\qquad - \frac{1}{z}\int_E \big(u' - P_p u'\big)\big(u_p' - u' - (U' - P_p u')\big)\,\mathrm{d}S \\
&\lesssim \|h\|_\infty \|\nabla(u - u_p)\|_{L^2} \|(u_p', U') - (u', P_p u')\|_{Y_p} \\
&\qquad + \|u' - P_p u'\|_{L^2(\partial B)} \|(u_p', U') - (u', P_p u')\|_{Y_p}.
\end{aligned}$$

Thus,

$$\|(u_p', U') - (u', P_p u')\|_{Y_p} \lesssim \|\nabla(u - u_p)\|_{L^2(B)} + \|u' - P_p u'\|_{L^2(\partial B)}, \tag{4.7}$$

yielding the stated estimate. □

Corollary 4.2. *Under the above assumptions we have that*

$$\lim_{p\to\infty} \sup\big\{\|F'(\gamma)[\cdot] - F_p'(\gamma)[\cdot]P_p\|_{\mathcal{L}(L^\infty(B), \mathcal{L}(L^2_\diamond(\partial B)))} : \gamma \in \mathrm{int}(\mathcal{D}(F))\big\} = 0.$$

Proof. In view of (4.6) and Lemma 4.1 the stated convergence is verified as soon as we have shown that

$$\lim_{p\to\infty} \sup\big\{\|(\mathrm{Id}-P_p)F'(\gamma)[h]f\|_{L^2(\partial B)} : (\gamma,h,f)\in\mathcal{M}\big\} = 0$$

where

$$\mathcal{M} := \mathrm{int}(\mathcal{D}(F))\times\{\eta\in L^\infty(B)\colon \|\eta\|_\infty\le 1\}\times\{g\in L^2_\diamond(\partial B)\colon \|g\|_{L^2(\partial B)}\le 1\}.$$

The linear operators $\{P_p\}$ converge pointwise to the identity on $L^2_\diamond(\partial B)$ and this convergence is uniform on compact subsets, see, e.g., Kress [10, Corollary 10.4].

Finally, we validate the compactness of $\mathcal{K} := \{F'(\gamma)[h]f\colon (\gamma,h,f)\in\mathcal{M}\}$ in $L^2(\partial B)$. By Theorem 3.1 the set $\mathcal{K}$ is bounded in $H^{1/2}(\partial B)$. As $H^{1/2}(\partial B)$ is compactly embedded in $L^2(\partial B)$ (Rellich's theorem, e.g., McLean [15, Theorem 3.30]), $\mathcal{K}$ is compact in $L^2(\partial B)$. ∎

THEOREM 4.3. *For p sufficiently large the Frechét derivative of*

$$F_p\colon V_{\mathcal{J}}^+\subset V_{\mathcal{J}}\to\mathcal{L}(\mathcal{E}_p)$$

is injective. More precisely: Fix $r>c_0$. Then, there is an integer $\mathbf{p}_{\mathcal{J}}=\mathbf{p}_{\mathcal{J}}(r)$ depending on r and $V_{\mathcal{J}}^+$ such that, for $p\ge\mathbf{p}_{\mathcal{J}}$,

$$\|F_p'(\gamma)[h]\|_{\mathcal{L}(\mathcal{E}_p)} \ge \frac{\Gamma(r)}{2}\,\|h\|_\infty \ \textit{for } \gamma\in\mathrm{int}(V_{\mathcal{J}}^+) \textit{ with } \|\gamma\|_\infty\le r \textit{ and } h\in V_{\mathcal{J}}.$$

Here,

$$\Gamma(r) := \min\big\{\|F'(\sigma)[v]\|_{\mathcal{L}(L^2_\diamond(\partial B))}\colon \sigma,\, v\in V_{\mathcal{J}},\ \|v\|_\infty=1,\ c_0\le\sigma\le r\big\}>0.$$

Proof. Let us first convince ourselves that $\Gamma(r)$ is well defined and positive. Indeed, the mapping $(\gamma,h)\mapsto\|F'(\gamma)[h]\|_{\mathcal{L}(L^2_\diamond(\partial B))}$ is continuous (Theorem 3.1) and non-zero (Corollary 3.3) on that compact subset of $V_{\mathcal{J}}\times V_{\mathcal{J}}$ over which the minimum is taken. Now,

$$\|F_p'(\gamma)[h]P_p\|_{\mathcal{L}(L^2_\diamond(\partial B))} \ge \|F'(\gamma)[h]\|_{\mathcal{L}(L^2_\diamond(\partial B))} - \|F'(\gamma)[h]-F_p'(\gamma)[h]P_p\|_{\mathcal{L}(L^2_\diamond(\partial B))}.$$

For $\gamma\in\mathrm{int}(V_{\mathcal{J}}^+)$ with $\|\gamma\|_\infty\le r$ we have

$$\|F'(\gamma)[h]\|_{\mathcal{L}(L^2_\diamond(\partial B))}\ge\Gamma(r)\,\|h\|_\infty.$$

Further, there is a $\mathbf{p}_{\mathcal{J}}\in\mathbb{N}$ such that

$$\sup\big\{\|F'(\sigma)[h]-F_p'(\sigma)[h]P_p\|_{\mathcal{L}(L^2_\diamond(\partial B))}\colon \sigma\in\mathrm{int}(V_{\mathcal{J}}^+)\big\} \le \frac{\Gamma(r)}{2}\,\|h\|_\infty \quad \text{for } p\ge\mathbf{p}_{\mathcal{J}}$$

which follows from Corollary 4.2. Thus,

$$\|F_p'(\gamma)[h]\|_{\mathcal{L}(\mathcal{E}_p)} \geq \|F_p'(\gamma)[h]P_p\|_{\mathcal{L}(L^2_\diamond(\partial B))} \geq \frac{\Gamma(r)}{2}\|h\|_\infty$$

and we are done. ∎

COROLLARY 4.4. *If* $\gamma \in \operatorname{int}(V_\mathfrak{T}^+)$ *then* $F_p'(\gamma) \in \mathcal{L}(V_\mathfrak{T}, \mathcal{L}(\mathcal{E}_p))$ *is injective for any* p *satisfying*

$$\|F'(\gamma)[\cdot] - F_p'(\gamma)[\cdot]P_p\|_{\mathcal{L}(V_\mathfrak{T},\mathcal{L}(L^2_\diamond(\partial B)))} < \Gamma(\|\gamma\|_\infty).$$

A necessary requirement for injectivity is $p(p-1) \geq 2\dim V_\mathfrak{T}$.

As in Section 3.1 we are now in a position to prove a tangential cone condition for F_p locally about any $\gamma \in \operatorname{int}(V_\mathfrak{T}^+)$.

THEOREM 4.5. *If* $\gamma \in \operatorname{int}(V_\mathfrak{T}^+)$ *then there is an open ball* $B_r(\gamma) \subset \operatorname{int}(V_\mathfrak{T}^+)$ *about* γ *of radius* $r > 0$ *such that, for all* τ, $\sigma \in B_r(\gamma)$,

$$\big\|F_p(\tau) - F_p(\sigma) - F_p'(\sigma)[\tau-\sigma]\big\|_{\mathcal{L}(\mathcal{E}_p)} \lesssim \|\tau-\sigma\|_\infty \|F_p(\tau) - F_p(\sigma)\|_{\mathcal{L}(\mathcal{E}_p)}$$

uniformly for all $p \geq \mathbf{p}_\mathfrak{T} = \mathbf{p}_\mathfrak{T}(r + \|\gamma\|_\infty)$*, that is, neither the involved constant nor the radius* r *depend on* p.

Proof. By Corollary 2.2 and (4.3),

$$\begin{aligned}\big\|F_p(\gamma+h) &- F_p(\gamma) - F_p'(\gamma)[h]\big\|_{\mathcal{L}(\mathcal{E}_p)} \\ &\lesssim \|T_p(\gamma+h) - T_p(\gamma) - T_p'(\gamma)[h]\big\|_{\mathcal{L}(\mathcal{E}_p,Y_p)} \\ &\lesssim \big(\|\gamma\|_\infty + z^{-1}\big)\|h\|_\infty \|T_p'(\gamma)[h]\|_{\mathcal{L}(\mathcal{E}_p,Y_p)}\end{aligned} \tag{4.8}$$

for $\gamma \in \operatorname{int}(V_\mathfrak{T}^+)$ and $\gamma + h \in V_\mathfrak{T}^+$. The constant is independent of $V_\mathfrak{T}$ and p.

Since $F_p'(\sigma)$ is injective for $\sigma \in B_\rho(\gamma) \subset \operatorname{int}(V_\mathfrak{T}^+)$ and $p \geq \mathbf{p}_\mathfrak{T}(\rho + \|\gamma\|_\infty)$ we can proceed exactly as in the proof of Theorem 3.4. It remains to show that the supremum

$$\sup\left\{\frac{\|T_p'(\sigma)[h]\|_{\mathcal{L}(\mathcal{E}_p,Y_p)}}{\|F_p'(\sigma)[h]\|_{\mathcal{L}(\mathcal{E}_p)}} : (\sigma, h) \in B_\rho(\gamma) \times V_\mathfrak{T}\right\}$$

is independent of p, compare (3.7). From Theorem 4.3 we know that $\|F_p'(\sigma)[h]\|_{\mathcal{L}(\mathcal{E}_p)} \gtrsim \|h\|_\infty$ uniformly in $\sigma \in B_\rho(\gamma)$ and uniformly in $p \geq \mathbf{p}_\mathfrak{T}$.

Because $\|T_p'(\sigma)[h]\|_{\mathcal{L}(\mathcal{E}_p,Y_p)} \lesssim \|h\|_\infty$ uniformly in p and in $\sigma \in B_\rho(\gamma)$ (Lemma 2.3), the supremum is bounded in p indeed. ∎

4.2. Finite element discretization. Let $S_\Upsilon \subset H^1(B)$ be a conforming finite element space over the subdivision Υ of B.[‡] By $\delta > 0$ denote the discretization step size associated to Υ. We require that

$$\lim_{\delta\to 0} \|w - \Pi_\delta w\|_{H^1(B)} = 0 \quad \text{for any } w \in H^1(B)$$

where $\Pi_\delta \colon H^1(B) \to S_\Upsilon$ is the orthogonal projection.

The finite element approximation $(u_{p,\delta}, U_\delta) \in Y_{p,\delta} := S_\Upsilon \oplus \mathcal{E}_p \subset Y_p$ to $(u_p, U) \in Y_p$ solves uniquely the variational problem

$$a\big((u_{p,\delta}, U_\delta), (w, W)\big) = \int_{\partial B} I\, W \,\mathrm{d}S \quad \text{for all } (w, W) \in Y_{p,\delta}, \tag{4.9}$$

compare (4.1). The corresponding EIT forward operator $F_{p,\delta}$, we consider here, is given as

$$F_{p,\delta} \colon V_{\mathcal{T}}^+ \subset V_{\mathcal{T}} \to \mathcal{L}(\mathcal{E}_p), \quad F_{p,\delta}(\gamma) I = U_\delta.$$

With the same techniques used in the previous subsection we will first verify injectivity of $F'_{p,\delta}$ and then a tangential cone condition for $F_{p,\delta}$ for $p \geq \mathbf{p}_{\mathcal{T}}$ and δ sufficiently small. Without giving details we use results of Section 2. Observe that

$$F_{p,\delta}(\gamma) = R_{\mathcal{E}_p} T_{p,\delta}(\gamma), \tag{4.10}$$

where the definition of $T_{p,\delta} \colon \mathcal{D}(F) \subset L^\infty(B) \to \mathcal{L}(\mathcal{E}_p, Y_{p,\delta})$ is obvious, compare (4.2) and (4.3).

The Frechét derivative $F'_{p,\delta}(\gamma) \in \mathcal{L}(V_{\mathcal{T}}, \mathcal{L}(\mathcal{E}_p))$, $\gamma \in \operatorname{int}(V_{\mathcal{T}}^+)$, is $F'_{p,\delta}(\gamma)[h]I = U'_\delta$ where U'_δ is the second component of $(u'_{p,\delta}, U'_\delta) \in Y_{p,\delta}$ which uniquely solves

$$a\big((u'_{p,\delta}, U'_\delta), (w, W)\big) = -\int_B h \nabla u_{p,\delta}(I) \nabla w \,\mathrm{d}x \quad \text{for all } (w, W) \in Y_{p,\delta}$$

with $u_{p,\delta}(I)$ being the first component of the solution of (4.9), compare (4.4).

Lemma 4.6. *We have that*

$$\lim_{\delta\to 0} \sup\big\{\|F'_p(\gamma) - F'_{p,\delta}(\gamma)\|_{\mathcal{L}(V_{\mathcal{T}}, \mathcal{L}(\mathcal{E}_p))} \colon \gamma \in \operatorname{int}(V_{\mathcal{T}}^+),\ \|\gamma\|_\infty \leq r,\ p \in \mathbb{N}\big\} = 0.$$

for any $r > c_0$.

Proof. All we need is finite element convergence theory. Indeed, the difference

$$\|F'_p(\gamma)[h]I - F'_{p,\delta}(\gamma)[h]I\|_{L^2(\partial B)} = \|U' - U'_\delta\|_{L^2(\partial B)}$$

[‡]Note that B is not required to be polygonal: The boundary elements of Υ may be curvilinear since no boundary conditions need to be obeyed.

will be bounded by Strang's first lemma, see, e.g., Ciarlet [5, Theorem 4.1.1]:

$$\|(u_p', U') - (u_{p,\delta}', U_\delta')\|_{Y_p} \lesssim \inf_{(w,W)\in Y_{p,\delta}} \|(u_p', U') - (w, W)\|_{Y_p} + \sup_{w\in S_\Upsilon} \frac{|\int_B h\nabla(u_p - u_{p,\delta})\nabla w \,\mathrm{d}x|}{\|w\|_{H^1(B)}}$$

uniformly in γ (and h and I). First we bound the above infimum by setting $(w, W) = (\Pi_\delta u_p', U')$:

$$\inf_{(w,W)\in Y_{p,\delta}} \|(u_p', U') - (w, W)\|_{Y_p} \le \|u_p' - \Pi_\delta u_p'\|_{H^1(B)}.$$

Now we turn to the supremum

$$\sup_{w\in S_\Upsilon} \frac{|\int_B h\nabla(u_p - u_{p,\delta})\nabla w \,\mathrm{d}x|}{\|w\|_{H^1(B)}} \le \|h\|_\infty \|u_p - u_{p,\delta}\|_{H^1(B)} \lesssim \|h\|_\infty \|u_p - \Pi_\delta u_p\|_{H^1(B)}.$$

Thus,

$$\|F_p'(\gamma)[h]I - F_{p,\delta}'(\gamma)[h]I\|_{L^2(\partial B)} \lesssim \|u_p' - \Pi_\delta u_p'\|_{H^1(B)} + \|h\|_\infty \|u_p - \Pi_\delta u_p\|_{H^1(B)} \tag{4.11}$$

and the constant does not depend on p or on γ. Both differences on the right converge to zero as $\delta \to 0$ uniformly in p and in $\gamma \in \overline{B_r(0)}$ as we show next.

First we consider $\|u_p - \Pi_\delta u_p\|_{H^1(B)}$. By the Lipschitz continuity (2.3) the mapping $\underline{V_\mathfrak{J}^+} \ni \gamma \mapsto u_p(\gamma) \in H^1(B)$ is continuous and the image of the compact ball $\overline{B_r(0)}$ is compact in $H^1(B)$ ($V_\mathfrak{J}$ is finite dimensional). Thus,

$$\lim_{\delta\to 0} \sup \left\{ \|u_p - \Pi_\delta u_p\|_{H^1(B)} : \gamma \in \overline{B_r(0)} \right\} = 0, \tag{4.12}$$

see again Kress [10, Corollary 10.4]. As $V_\mathfrak{J}^+ \ni \gamma \mapsto u_p'(\gamma) \in H^1(B)$ is also continuous (Lemma 2.3) the same line of reasoning yields

$$\lim_{\delta\to 0} \sup \left\{ \|u_p' - \Pi_\delta u_p'\|_{H^1(B)} : \gamma \in \overline{B_r(0)} \right\} = 0. \tag{4.13}$$

Both limits are even uniform in p. Assume the contrary. Then, there is an $\varepsilon > 0$ for which we can find a positive zero sequence $\{\delta_i\}_i$ and a corresponding sequence $\{p_i\}_i$ of electrode configurations such that

$$\sup \left\{ \|u_{p_i} - \Pi_{\delta_i} u_{p_i}\|_{H^1(B)} : \gamma \in \overline{B_r(0)} \right\} \ge \varepsilon, \quad \text{for all } i \in \mathbb{N}.$$

If $\{p_i\}_i$ is bounded we immediately have a contradiction. Therefore let us assume that $\{p_i\}_i$ diverges to infinity. We have

$$\|u_{p_i} - \Pi_{\delta_i} u_{p_i}\|_{H^1(B)} \le \left\|(\mathrm{Id} - \Pi_{\delta_i})u\right\|_{H^1(B)} + \left\|(\mathrm{Id} - \Pi_{\delta_i})(u_{p_i} - u)\right\|_{H^1(B)} \le \left\|(\mathrm{Id} - \Pi_{\delta_i})u\right\|_{H^1(B)} + \|u_{p_i} - u\|_{H^1(B)}.$$

The right difference on the right tends to zero uniformly in γ as $i \to \infty$, see (4.6). By the same arguments proving (4.12) the left difference on the right converges uniformly in $\gamma \in \overline{B_r(0)}$. So, we found the contradiction

$$0 < \varepsilon \le \sup\left\{\|u_{p_i} - \Pi_{\delta_i} u_{p_i}\|_{H^1(B)} : \gamma \in \overline{B_r(0)}\right\} \xrightarrow{i\to\infty} 0.$$

To establish uniformity of (4.13) in p we may argue just as above when replacing u and u_{p_i} by u' and u'_{p_i}, respectively. Now we have to take care of

$$\left\|(\mathrm{Id} - \Pi_{\delta_i})u'\right\|_{H^1(B)} \quad \text{and} \quad \|u'_{p_i} - u'\|_{H^1(B)}.$$

From above we know how to prove uniform convergence in $\gamma \in \overline{B_r(0)}$ of the left difference. The right difference we have already investigated, see (4.7) and Corollary 4.2.

Finally we have established uniformity in p of the limits (4.12) and (4.13). Recalling (4.11) we end up with

$$\lim_{\delta\to 0} \sup\left\{\|F'_p(\gamma)[h]I - F'_{p,\delta}(\gamma)[h]I\|_{L^2(\partial B)} : \gamma \in \overline{B_r(0)},\ p \in \mathbb{N}\right\} = 0.$$

As the operators act on finite-dimensional spaces, this pointwise convergence first yields

$$\lim_{\delta\to 0} \sup\left\{\|F'_p(\gamma)[h] - F'_{p,\delta}(\gamma)[h]\|_{\mathcal{L}(\mathcal{E}_p)} : \gamma \in \overline{B_r(0)},\ p \in \mathbb{N}\right\} = 0$$

for all $h \in V_{\mathfrak{T}}$ and then the result. □

Combining Lemma 4.6 with Theorem 4.3 we are able to prove injectivity.

THEOREM 4.7. *For p sufficiently large and $\delta > 0$ sufficiently small the Frechét derivative of*

$$F_{p,\delta} : V_{\mathfrak{T}}^+ \subset V_{\mathfrak{T}} \to \mathcal{L}(\mathcal{E}_p)$$

is injective. More precisely: Fix $r > c_0$. Then, there is a $\delta_{\max} = \delta_{\max}(r) > 0$ only depending on r and $V_{\mathfrak{T}}^+$ such that, for $p \ge \mathbf{p}_{\mathfrak{T}}(r)$ and $\delta \le \delta_{\max}$,

$$\|F'_{p,\delta}(\gamma)[h]\|_{\mathcal{L}(\mathcal{E}_p)} \ge \frac{\Gamma(r)}{3}\,\|h\|_\infty \qquad \text{for } \gamma \in \operatorname{int}(V_{\mathfrak{T}}^+) \text{ with } \|\gamma\|_\infty \le r \text{ and } h \in V_{\mathfrak{T}}.$$

The function Γ is defined in Theorem 4.3.

Proof. We use Theorem 4.3 and proceed exactly as in its proof:

$$\|F'_{p,\delta}(\gamma)[h]\|_{\mathcal{L}(\mathcal{E}_p)} \ge \frac{\Gamma(r)}{2}\,\|h\|_\infty - \|F'_p(\gamma)[h] - F'_{p,\delta}(\gamma)[h]\|_{\mathcal{L}(\mathcal{E}_p)}$$

for $p \ge \mathbf{p}_{\mathfrak{T}}(r)$ and $\|\gamma\|_\infty \le r$. Further, by Lemma 4.6 there is a $\delta_{\max} > 0$ such that

$$\sup\left\{\|F'_p(\gamma)[h] - F'_{p,\delta}(\gamma)[h]\|_{\mathcal{L}(\mathcal{E}_p)} : \gamma \in \overline{B_r(0)},\ p \in \mathbb{N}\right\} \le \frac{\Gamma(r)}{6}\,\|h\|_\infty$$

for all $\delta \leq \delta_{\max}$. $\square$

COROLLARY 4.8. *If* $\gamma \in \operatorname{int}(V_{\mathcal{T}}^+)$ *then* $F'_{p,\delta}(\gamma) \in \mathcal{L}(V_{\mathcal{T}}, \mathcal{L}(\mathcal{E}_p))$ *is injective for any* p *and* δ *satisfying*

$$\|F'(\gamma)[\cdot] - F'_{p,\delta}(\gamma)[\cdot]P_p\|_{\mathcal{L}(V_{\mathcal{T}},\mathcal{L}(L^2_\diamond(\partial B)))} < \Gamma(\|\gamma\|_\infty).$$

Now we are well prepared for the tangential cone condition in the fully discrete setting.

THEOREM 4.9. *If* $\gamma \in \operatorname{int}(V_{\mathcal{T}}^+)$ *then there is an open ball* $B_r(\gamma) \subset \operatorname{int}(V_{\mathcal{T}}^+)$ *about* γ *of radius* $r > 0$ *such that, for all* τ, $\sigma \in B_r(\gamma)$,

$$\big\|F_{p,\delta}(\tau) - F_{p,\delta}(\sigma) - F'_{p,\delta}(\sigma)[\tau - \sigma]\big\|_{\mathcal{L}(\mathcal{E}_p)} \lesssim \|\tau - \sigma\|_\infty \|F_{p,\delta}(\tau) - F_{p,\delta}(\sigma)\|_{\mathcal{L}(\mathcal{E}_p)}$$

uniformly for all $p \geq \mathbf{p}_{\mathcal{T}}(r + \|\gamma\|_\infty)$ *and all* $0 < \delta \leq \delta_{\max}(r + \|\gamma\|_\infty)$*, that is, neither the involved constant nor the radius* r *depend on* p *or on* δ.

Proof. Our preparatory work done in the proofs of Theorems 3.4 and 4.5 allows to be brief here. Basis of the proof are Corollary 2.2 and relation (4.10). Observe that (4.8) holds analogously, that is, F_p and T_p are replaced by $F_{p,\delta}$ and $T_{p,\delta}$, respectively. The corresponding constant is now idependent of $V_{\mathcal{T}}$, p, and δ. Arguments, already used in the proof of Theorem 4.5, confirm that

$$\sup\left\{\frac{\|T'_{p,\delta}(\sigma)[h]\|_{\mathcal{L}(\mathcal{E}_p,Y_p)}}{\|F'_{p,\delta}(\sigma)[h]\|_{\mathcal{L}(\mathcal{E}_p)}} : (\sigma, h) \in B_\rho(\gamma) \times V_{\mathcal{T}}\right\}$$

is bounded in $p \geq \mathbf{p}_{\mathcal{T}}$ and $\delta \leq \delta_{\max}$. $\square$

Remark. *In our numerical experiments* [12, Section 5] *we realized a stabilizing effect when using a much finer FE discretization for computing the Jacobian than for reconstructing the conductivity (in the notation of this paper:* Υ *much finer than* $\mathcal{T}$*). Our observation is in full agreement with the latter two theorems.*

As explained in the Introduction we finally established rigorously the reported convergence [12] of the regularizing scheme CG-REGINN. We close this paper by commenting on the convergence. To this end we shortly introduce CG-REGINN for solving the inverse EIT problem

$$F_{p,\delta}(\gamma) = \Lambda_p^\varepsilon$$

where Λ_p^ε is a perturbed version of the exact and achievable data $\Lambda_p = F_{p,\delta}(\gamma^+)$, $\gamma^+ \in \operatorname{int}(V_{\mathcal{T}}^+)$, satisfying

$$\|\Lambda_p - \Lambda_p^\varepsilon\|_{\mathcal{L}(\mathcal{E}_p)} \leq \varepsilon.$$

CG-REGINN starts out from an initial guess $\gamma_0 \in V_{\mathbb{J}}^+$ and computes a sequence $\{\gamma_n\}_n$ by

$$\gamma_{n+1} = \gamma_n + s_n.$$

The Newton correction s_n is determined as follows: Apply the conjugate gradient iteration with starting guess zero to the normal equation of the linearization

$$F'_{p,\delta}(\gamma_n)[s] = \Lambda_p^\varepsilon - F_{p,\delta}(\gamma_n)$$

and stop it as soon as the relative residual is less than a tolerance $\mu_n \in]0,1[$. That is, s_n satisfies

$$\|F'_{p,\delta}(\gamma_n)[s_n] - \Lambda_p^\varepsilon\|_{\mathcal{L}(\mathcal{E}_p)} \leq \mu_n \|\Lambda_p^\varepsilon - F_{p,\delta}(\gamma_n)\|_{\mathcal{L}(\mathcal{E}_p)}.$$

Further, CG-REGINN is stopped by a discrepancy principle: Choose $R > 0$ and pick $\gamma_{N(\varepsilon)}$ as approximate solution if

$$\|\Lambda_p^\varepsilon - F_{p,\delta}(\gamma_{N(\varepsilon)})\|_{\mathcal{L}(\mathcal{E}_p)} < R\varepsilon \leq \|\Lambda_p^\varepsilon - F_{p,\delta}(\gamma_n)\|_{\mathcal{L}(\mathcal{E}_p)}, \quad n = 0, \ldots, N(\varepsilon) - 1.$$

The results of [7, 13] in combination with Theorem 4.9 yield local convergence: Let R be sufficiently large and choose the tolerances $\{\mu_n\}$ within a certain interval. If $\gamma_0 \in B_\rho(\gamma^+)$ with ρ sufficiently small then CG-REGINN is well defined, that is, all iterates $\{\gamma_1, \ldots, \gamma_{N(\delta)}\}$ stay in $B_\rho(\gamma^+)$ and there is a number $L < 1$ such that

$$\frac{\|\Lambda_p^\varepsilon - F_{p,\delta}(\gamma_{n+1})\|_{\mathcal{L}(\mathcal{E}_p)}}{\|\Lambda_p^\varepsilon - F_{p,\delta}(\gamma_n)\|_{\mathcal{L}(\mathcal{E}_p)}} \leq L, \quad n = 0, \ldots, N(\varepsilon) - 1.$$

Moreover,

$$\lim_{\varepsilon \to 0} \gamma_{N(\varepsilon)} = \gamma^+.$$

REFERENCES

[1] K. Astala and L. Päivärinta, *Calderón's inverse conductivity problem in the plane*, Ann. Math., 163 (2006), pp. 265–99.

[2] L. Borcea, *Electrical impedance tomography*, Inverse Problems, 18 (2002), pp. R99–R136.

[3] A. Caldéron, *On an inverse boundary value problem*, in Seminar of Numerical Analysis and Its Applications to Continuum Physics, Rio de Janeiro, 1980, Soc. Brasileira de Mathemàtica, pp. 65–73. Online available from `www.ma.umist.ac.uk/bl/Calderon/`.

[4] M. Cheney, D. Isaacson, and J. Newell, *Electrical impedance tomography*, SIAM Review, 41 (1999), pp. 85–101.

[5] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, vol. 4 of Sudies in Mathematics and its Applications, North-Holland, Amsterdam, 1978.

[6] B. Gebauer, *Localized potentials in electrical impedance tomography*, Inverse Problems and Imaging, 2 (2008), pp. 251–269.

[7] M. Hanke, *Regularizing properties of a truncated Newton-CG algorithm for nonlinear inverse problems*, Numer. Funct. Anal. Optim. 18 (1997), pp. 971–993.

[8] N. Hyvönen, *Complete electrode model of electrical impedance tomography: approximation properties and characterization of inclusions*, SIAM J. Appl. Math., 64 (2004), pp. 902–931.

[9] B. Kaltenbacher, A. Neubauer, and O. Scherzer, *Iterative Regularization Methods for Nonlinear Ill-Posed Problems*, Radon Series on Computational and Applied Mathematics, de Gruyter, Berlin, 2008. To appear.

[10] R. Kress, *Linear Integral Equations*, vol. 82 of Applied Mathematical Sciences, Springer-Verlag, Berlin, 1989.

[11] A. Lechleiter, N. Hyvönen, and H. Hakula, *The factorization method applied to the complete electrode model of impedance tomography*, SIAM J. Appl. Math., 68 (2008), pp. 1097–1121.

[12] A. Lechleiter and A. Rieder, *Newton regularizations for impedance tomography: a numerical study*, Inverse Problems, 22 (2006), pp. 1967–1887.

[13] ———, *A convergence analysis of the Newton-type regularization CG-REGINN with application to impedance tomography*, Tech. Rep. 07/01, Institut für Wissenschaftliches Rechnen und Mathematische Modellbildung, Universität Karlsruhe, Germany, 2007. Available online from www.mathematik.uni-karlsruhe.de/iwrmm/seite/preprints/media/preprint%20nr.%2007-01.pdf.

[14] E. H. Lieb and M. Loss, *Analysis*, American Math. Soc., 2. ed., 2001.

[15] W. McLean, *Strongly Elliptic Systems and Boundary Integral Operators*, Cambridge University Press, 2000.

[16] J. Nečas, *Les Methodes Directes en Theorie des Equations Elliptiques*, Academia, Prague, 1967.

[17] A. Rieder, *Keine Probleme mit Inversen Problemen*, Vieweg, Wiesbaden, Germany, 2003.

[18] O. Scherzer, *The use of Morozov's discrepancy principle for Tikhonov regularization for solving nonlinear ill-posed problems*, Computing, 51 (1993), pp. 45–60.

[19] E. Somersalo, M. Cheney, and D. Isaacson, *Existence and uniqueness for electrode models for electric current computed tomography*, SIAM J. Appl. Math., 52 (1992), pp. 1023–1040.

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