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A functional equation with two unknown functions

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1. Introduction. Throughout this paper I denotes a non-degenerate interval in $\mathbb{R}$, i.e., I is a convex subset of $\mathbb{R}$ with non-empty interior. We determine all continuous, strictly increasing $\varphi, \psi : I \to \mathbb{R}$ such that

$$(1.1) \qquad (\varphi+\psi)^{-1}(\varphi(x)+\psi(y)) + (\varphi+\psi)^{-1}(\varphi(y)+\psi(x)) = x+y \ (x, y \in I);$$

this will be done in the next paragraph. The third paragraph contains some background information concerning equation (1.1). For the moment we only like to mention that a more general equation (with four unknown functions) had been solved by Baják and Páles [2], but under stronger regularity conditions. In the last paragraph we give an application to the functional equation

$$(1.2) \qquad F(A_{\varphi,\psi}(x,y), A_{\psi,\varphi}(x,y)) = F(x,y) \qquad (x, y \in I),$$

where (generally) $A_{\varphi,\psi}(x,y) = (\varphi+\psi)^{-1}(\varphi(x)+\psi(y))$.

2. Solution of (1.1).

Theorem 1. *Let $\varphi, \psi : I \to \mathbb{R}$ be continuous and strictly increasing. Then* (1.1) *holds if and only if there are $a, b \in \mathbb{R}$, $a > 0$, such that*

$$(2.1) \qquad \varphi(x) + \psi(x) = ax + b \qquad (x \in I).$$

Proof. Let (1.1) be true. We consider $x_0, y_0 \in I$ and we like to show

$$(2.2) \qquad (\varphi+\psi)\left(\frac{x_0+y_0}{2}\right) = \frac{(\varphi+\psi)(x_0)+(\varphi+\psi)(y_0)}{2}.$$

We assume $x_0 \le y_0$ and define recursively

$$(2.3) \qquad x_n = \min\{(\varphi+\psi)^{-1}(\varphi(x_{n-1})+\psi(y_{n-1})), (\varphi+\psi)^{-1}(\varphi(y_{n-1})+\psi(x_{n-1}))\},$$

$$(2.4) \qquad y_n = \max\{(\varphi+\psi)^{-1}(\varphi(x_{n-1})+\psi(y_{n-1})), (\varphi+\psi)^{-1}(\varphi(y_{n-1})+\psi(x_{n-1}))\}$$

$(n = 1, 2, 3, \dots)$. We get

$$[x_0, y_0] \supseteq [x_1, y_1] \supseteq [x_2, y_2] \supseteq \dots,$$

hence

$$x_n \uparrow \bar{x},\ y_n \downarrow \bar{y},\ \bar{x} \le \bar{y}.$$

When adding (2.3), (2.4), then (1.1) implies $x_n + y_n = x_{n-1} + y_{n-1}$. Therefore we have $x_n + y_n = x_0 + y_0 \quad (n = 1, 2, 3, \dots)$, in the limit

$$\bar{x} + \bar{y} = x_0 + y_0. \tag{2.5}$$

(2.3), (2.4) also can be written as

$$\varphi(x_n) + \psi(x_n) = \min\{\varphi(x_{n-1}) + \psi(y_{n-1}), \varphi(y_{n-1}) + \psi(x_{n-1})\}, \tag{2.6}$$

$$\varphi(y_n) + \psi(y_n) = \max\{\varphi(x_{n-1}) + \psi(y_{n-1}), \varphi(y_{n-1}) + \psi(x_{n-1})\}. \tag{2.7}$$

Adding them we get

$$(\varphi + \psi)(x_n) + (\varphi + \psi)(y_n) = (\varphi + \psi)(x_{n-1}) + (\varphi + \psi)(y_{n-1}),$$

hence

$$(\varphi + \psi)(x_n) + (\varphi + \psi)(y_n) = (\varphi + \psi)(x_0) + (\varphi + \psi)(y_0) \tag{2.8}$$

$(n = 1, 2, 3, \dots)$. Now $n \to \infty$ in (2.6), (2.7) gives

$$\varphi(\bar{x}) + \psi(\bar{y}) = \varphi(\bar{x}) + \psi(\bar{x}) \quad \text{or} \quad \varphi(\bar{x}) + \psi(\bar{y}) = \varphi(\bar{y}) + \psi(\bar{y}).$$

In both cases we get $\bar{x} = \bar{y}$, and because of (2.5) we have

$$\bar{x} = \bar{y} = \frac{x_0 + y_0}{2}.$$

Then $n \to \infty$ in (2.8) leads to (2.2).

Equation (2.2) is true for arbitrary $x_0, y_0 \in I$, and this means that $\varphi + \psi : I \to I\!R$ is a solution of the Jensen functional equation. Furthermore $\varphi + \psi$ is continuous and strictly increasing, therefore we get (2.1) with some $a > 0,\ b \in I\!R$; cf., e.g., Aczél [1] or Kuczma [6].

On the other hand, if continuous, strictly increasing functions $\varphi, \psi : I \to I\!R$ satisfy (2.1), then (1.1) can easily be verified.

Remark. *Consider $a_0, b_0 \in I$, and replace* (2.3), (2.4) *by the formulas*

$$a_n = (\varphi + \psi)^{-1}(\varphi(a_{n-1}) + \psi(b_{n-1})), \tag{2.9}$$

$$b_n = (\varphi + \psi)^{-1}(\varphi(b_{n-1}) + \psi(a_{n-1})) \tag{2.10}$$

$(n = 1, 2, 3, \dots)$. *Then*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \frac{a_0 + b_0}{2}. \tag{2.11}$$

Proof. For $x_0 = \min\{a_0, b_0\},\ y_0 = \max\{a_0, b_0\}$ the x_n, y_n from (2.3), (2.4) are just

$$x_n = \min\{a_n, b_n\}, \quad y_n = \max\{a_n, b_n\}.$$

This, together with $x_n \uparrow \frac{1}{2}(x_0 + y_0),\ y_n \downarrow \frac{1}{2}(x_0 + y_0)$, and $x_0 + y_0 = a_0 + b_0$ leads to (2.11).

Using Theorem 1 we are able to describe our solutions of equation (1.1) more precisely.

Theorem 2. *The continuous, strictly increasing* $\varphi, \psi : I \to \mathbb{R}$ *solving* (1.1) *can be obtained in the following way:*

I) *We start with an arbitrary Lipschitz-continuous, strictly increasing function* $\varphi : I \to \mathbb{R}$*; let* λ_φ *denote its smallest Lipschitz-constant.*

II) *We determine* $\psi : I \to \mathbb{R}$ *by means of* (2.1), *where* $b \in \mathbb{R},\ a \geq \lambda_\varphi$ *are arbitrary, but where the last inequality has to be replaced by* $a > \lambda_\varphi$*, if on a non-degenerate sub-interval of* I *the function* φ *is linear with slope* λ_φ.

Proof. Let us begin with continuous, strictly increasing $\varphi, \psi : I \to \mathbb{R}$ solving (1.1). By Theorem 1 we have (2.1), and for $x, y \in I, x \leq y$ we get

$$\varphi(y) - \varphi(x) = a(y - x) - \psi(y) + \psi(x) \leq a(y - x).$$

So a is a Lipschitz-constant for φ, hence $\lambda_\varphi \leq a$.

Let us suppose φ to be linear with slope λ_φ on some interval $[p, q] \subseteq I$ (where $p < q$). Then we have $\varphi(q) - \varphi(p) = \lambda_\varphi(q - p)$ and taking (2.1) for $x = q,\ x = p$ and subtracting we get

$$\lambda_\varphi(q - p) + \psi(q) - \psi(p) = a(q - p).$$

Because of $\psi(p) < \psi(q)$, this implies $\lambda_\varphi < a$.

These considerations show that φ, ψ are included in the procedure given by I), II). Conversely it is easy to see that all functions φ, ψ obtained by I), II) are continuous and strictly increasing on I (in fact, it only remains to show that $\psi : I \to \mathbb{R}$ is strictly increasing). According to the construction they fulfil (2.1) with some $a, b \in \mathbb{R}$, hence they solve (1.1).

3. Background. The functional equation (1.1) is related to the question of invariance of a quasi-arithmetic mean with respect to a mean-type mapping defined by two quasi-arithmetic means. This question leads to

$$\varphi^{-1}\left(\frac{\varphi(x) + \varphi(y)}{2}\right) + \psi^{-1}\left(\frac{\psi(x) + \psi(y)}{2}\right) = x + y \qquad (x, y \in I), \tag{3.1}$$

where the unknown functions $\varphi, \psi : I \to I\!R$ are continuous and strictly increasing. Sutô [9] determined the analytic solutions of (3.1). The same solutions then had been found in [7] under the assumption of twice continuous differentiability and after this by Daróczy and Páles [3] in the general case.

Jarczyk and Matkowski [5], motivated by a more general invariance problem for weighted quasi-arithmetic means, considered the functional equation

$$(3.2)\qquad \begin{aligned} p\varphi^{-1}(q\varphi(x) + (1-q)\varphi(y)) + (1-p)\psi^{-1}(r\psi(x) + (1-r)\psi(y)) \\ = px + (1-p)y \quad (x, y \in I), \end{aligned}$$

where $p, q, r \in]0, 1[$ are arbitrarily given and the unknown $\varphi, \psi : I \to I\!R$ again are continuous and strictly increasing. They determined the twice continuously differentiable solutions of (3.2). Then Jarczyk [4] got the same solutions in the general case.

The means

$$(3.3)\qquad A_{\varphi,\psi}(x, y) = (\varphi + \psi)^{-1}(\varphi(x) + \psi(y)) \qquad (x, y \in I)$$

had been introduced in [8]; we also can write them as

$$A_{\varphi,\psi}(x, y) = \left(\frac{\varphi + \psi}{2}\right)^{-1} \left(\frac{\varphi(x) + \psi(y)}{2}\right) \qquad (x, y \in I).$$

For $\varphi = \psi$ we get the quasi-arithmetic mean generated by φ, and because of this we call $A_{\varphi,\psi}$ a *quasi-arithmetic mean with two generators* (φ and ψ). Let us observe that weighted quasi-arithmetic means are special cases of the means (3.3).

Using (3.3) we can write a functional equation considered by Baják and Páles [2] as

$$(3.4)\qquad A_{\varphi_1,\psi_1}(x, y) + A_{\varphi_2,\psi_2}(x, y) = x + y \qquad (x, y \in I).$$

They determine all four times continuously differentiable solutions $\varphi_1, \psi_1, \varphi_2, \psi_2 : I \to I\!R$ such that $\varphi_1'(x), \psi_1'(x), \varphi_2'(x), \psi_2'(x) > 0$ $(x \in I)$. Our functional equation (1.1) is a special case of (3.4), namely it can be written as

$$(3.5)\qquad A_{\varphi,\psi}(x, y) + A_{\psi,\varphi}(x, y) = x + y \qquad (x, y \in I).$$

It should be mentioned that all the solutions of (1.1) which are given by our Theorem 1 already were known to Baják and Páles [2].

Let us finally observe that, when dividing (3.5) by two, this equation can be interpreted as invariance of the arithmetic mean with respect to the mean-type mapping $(A_{\varphi,\psi}, A_{\psi,\varphi}) : I^2 \to I^2$.

4. An application. Suppose the continuous, strictly increasing functions $\varphi, \psi : I \to \mathbb{R}$ solve (1.1), and let $f : I \to \mathbb{R}$ be arbitrary. Writing (3.5) instead of (1.1), we then get

$$f(\tfrac{1}{2}(A_{\varphi,\psi}(x,y) + A_{\psi,\varphi}(x,y))) = f\left(\frac{x+y}{2}\right) \qquad (x, y \in I).$$

This means that $F : I^2 \to \mathbb{R}$ defined by

$$F(x,y) = f\left(\frac{x+y}{2}\right) \qquad (x, y \in I) \tag{4.1}$$

fulfils the functional equation

$$F(A_{\varphi,\psi}(x,y), A_{\psi,\varphi}(x,y)) = F(x,y) \qquad (x, y \in I). \tag{1.2}$$

Now we shall see that under some continuity assumptions the solutions $F : I^2 \to \mathbb{R}$ of (1.2) have the form (4.1).

Theorem 3. *Let $\varphi, \psi : I \to \mathbb{R}$ be continuous, strictly increasing functions solving* (1.1). *Suppose $F : I^2 \to \mathbb{R}$ to be continuous in the points of the diagonal $\{(x,x) \mid x \in I\}$. Then F solves the functional equation* (1.2) *if and only if there is a continuous $f : I \to \mathbb{R}$ such that* (4.1) *holds.*

Proof. So let $F : I^2 \to \mathbb{R}$ be a solution of (1.2) which is continuous in the points (x,x) from I^2. We consider $(x,y) \in I^2$, and for $a_0 = x,\ b_0 = y$ we define a_n, b_n $(n = 1, 2, 3, \dots)$ as in the Remark after Theorem 1. Observe that (2.9), (2.10) can be written as

$$a_n = A_{\varphi,\psi}(a_{n-1}, b_{n-1}),\ b_n = A_{\psi,\varphi}(a_{n-1},\ b_{n-1}) \quad (n = 1, 2, 3, \dots).$$

Therefore we get from (1.2)

$$F(a_n, b_n) = F(a_{n-1},\ b_{n-1}) \quad (n = 1, 2, 3, \dots),$$

and this implies

$$F(a_n, b_n) = F(a_0, b_0) = F(x,y) \qquad (n = 1, 2, 3 \dots). \tag{4.2}$$

Because of (2.11) we have $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = \frac{1}{2}(a_0 + b_0) = \frac{1}{2}(x+y)$, and $n \to \infty$ in (4.2) leads to

$$F\left(\frac{x+y}{2}, \frac{x+y}{2}\right) = F(x,y). \tag{4.3}$$

Let us define $f : I \to \mathbb{R}$ by $f(x) = F(x,x)$ $(x \in I)$, then f is continuous, and (4.1) follows from (4.3).

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References

[1] J. Aczél, *Lectures on functional equations and their applications.* Academic Press New York 1966.

[2] Szabolcs Baják and Zsolt Páles, *Invariance equation for generalized quasi-arithmetic means.* Aequationes Math., to appear.

[3] Zoltán Daróczy and Zsolt Páles, *Gauss-composition of means and the solution of the Matkowski-Sutô problem.* Publ. Math. Debrecen **61**, 157-218 (2002).

[4] Justyna Jarczyk, *Invariance of weighted quasi-arithmetic means with continuous generators.* Ibid. **71**, 279-294 (2007).

[5] – and Janusz Matkowski, *Invariance in the class of weighted quasi-arithmetic means.* Ann. Polon. Math. **88**, 39-51 (2006).

[6] Marek Kuczma, *An introduction to the theory of functional equations and inequalities, Cauchy's equation and Jensen's inequality.* Państwowe Wydawnictwo Naukowe Warszawa 1985.

[7] Janusz Matkowski, *Invariant and complementary quasi-arithmetic means.* Aequationes Math. **57**, 87-107 (1999).

[8] –, *Remark 1* (at the Second Debrecen-Katowice Winter Seminar on Functional Equations and Inequalities Hajdúszoboszló 2002). Ann. Math. Silesianae **16**, 93 (2003).

[9] Onosaburô Sutô, *Studies on some functional equations, II.* Tôhoku Math. J. **6**, 82-101 (1914).

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