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Efficient strategies for goal-oriented error estimation and mesh adaptation in structural dynamics

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Abstract

This contribution deals with spatially adaptive schemes in finite element computations in structural dynamics based on a semi-discrete approach. Both discretization steps, the spatial finite element discretization and the temporal discretization by use of a suitable time stepping scheme introduce different errors into the numerical solution, which can be controlled by adaptive approaches. Here we restrict ourselves focussing on the spatial discretization error resulting from the finite element discretization of the spatial domain. The spatial discretization is carried out by use of low order 'Solid-Shell'-elements. Since low order element suffer from locking an assumed strain interpolation is used which has to be taken into account in the error estimation procedure.

For the estimation of the spatial discretization error in an arbitrary quantity of interest an appropriate adjoint or dual problem has to be introduced. As the dual problem is a backward problem in time with initial conditions at the current time and characterizes the spatial and temporal transport of the spatial discretization error, the numerical evaluation of this dual problem for the error estimation results in a large numerical effort which might exceed the effort for the computation of the problem on a finer spatial discretization. Thus a particular focus of our contribution is on the reduction of the numerical effort for the goal-oriented error estimation. For this purpose we discuss which parts of the total error can be neglected in the error representation, proposing an error indicator which then serves as the basis for an adaptive mesh refinement scheme. A numerical example shows the efficiency of the proposed error estimator and the mesh adaptation scheme.

1 INTRODUCTION

The standard semi-discrete approach in finite element computations in structural dynamics consists of the spatial discretization with finite elements and the time integration of the resulting system of ordinary differential equations. Both discretization steps contribute to the total discretization error of the numerical solution. Thus, the total discretization error can be split into the spatial discretization error $\boldsymbol{e}_s(\boldsymbol{x},t)$ and the time integration error $\boldsymbol{e}_t(\boldsymbol{x},t)$:

$$\begin{aligned} \boldsymbol{e}(\boldsymbol{x},t) &= \boldsymbol{u}(\boldsymbol{x},t) - \boldsymbol{u}_{h,k}(\boldsymbol{x},t) \\ &= \underbrace{(\boldsymbol{u}(\boldsymbol{x},t) - \boldsymbol{u}_h(\boldsymbol{x},t))}_{\boldsymbol{e}_s(\boldsymbol{x},t)} + \underbrace{(\boldsymbol{u}_h(\boldsymbol{x},t) - \boldsymbol{u}_{h,k}(\boldsymbol{x},t))}_{\boldsymbol{e}_t(\boldsymbol{x},t)} \end{aligned} \tag{1}$$

Herein $\boldsymbol{u}(\boldsymbol{x},t)$ denotes the exact solution of the continuous equation of motion, $\boldsymbol{u}_h(\boldsymbol{x},t)$ is the exact solution of the spatially discretized problem and $\boldsymbol{u}_{h,k}(\boldsymbol{x},t)$ the numerical solution obtained by the applied time integration scheme. As the time integration error is in most cases far smaller than the error due to the spatial discretization our contribution is restricted to the error estimation of the spatial discretization error. Here we focus on the estimation of the spatial discretization error in a particular quantity of interest. For this so-called goal-oriented error estimation an adjoint or dual problem is introduced. The dual problem is a backward problem in time and describes the spatial and temporal transport of the error. For the mathematical foundation of goal-oriented error estimation we refer to Becker and Rannacher [2], Bangerth and Rannacher [1] and Oden and Prudhomme [7].

The spatial discretization is performed by use of 'Solid-Shell'-Elements with bilinear ansatz-functions, see e.g. [6]. Since low order shell elements suffer from transversal shear locking the well-known assumed strain approach (ANS) [4] is used. The ANS-method is a mesh dependent reduction of the compatible strain field:

$$\boldsymbol{\varepsilon}^{ANS} = R_h(\boldsymbol{\varepsilon}(\boldsymbol{u})) \quad \text{ with } \boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(Grad\boldsymbol{u} + Grad^T\boldsymbol{u}) \tag{2}$$

For a proper incorporation of this modificaton into the error estimation procedure we refer to [3, 8].

The paper is organized as follows. First suitable error representation formulas which are the basis of the error estimation are derived. Then the standard approach of goal-oriented error estimation using the full dual backward problem is described and tested with a numerical example. Then a simplified error estimator is derived which is taken as the basis for a mesh adaptation scheme. The simplified error estimator and the mesh adaptation are also tested with the same numerical example as the standard error estimator.

2 ERROR REPRESENTATION

For brevity we will restrict the following derivations to the equation of motion without damping. Damping terms could however easily be introduced into the error equations. The variational form of the exact problem can be stated as:

$$\begin{aligned}
\rho_0(\ddot{\boldsymbol{u}}, \boldsymbol{w}) + a(\boldsymbol{u}, \boldsymbol{w}) &= \mathcal{F}_u(\boldsymbol{w}) \quad \forall \boldsymbol{w} \in W \\
\text{with } a(\boldsymbol{u}, \boldsymbol{w}) &= \int_{\mathcal{B}_0} \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{C} : \boldsymbol{\varepsilon}(\boldsymbol{w}) dV, \\
(\ddot{\boldsymbol{u}}, \boldsymbol{w}) &= \int_{\mathcal{B}_0} \ddot{\boldsymbol{u}} \cdot \boldsymbol{w} dV, \\
\mathcal{F}_u(\boldsymbol{w}) &= \int_{\mathcal{B}_0} \rho_0 \boldsymbol{b} \cdot \boldsymbol{w} dV + \int_{\partial\mathcal{B}_N} \boldsymbol{t} \cdot \boldsymbol{w} dA
\end{aligned} \tag{3}$$

and initial conditions at time $t = 0$. The semi-discrete equation applying low order element with assumed strains reads:

$$\begin{aligned}
\rho_0(\ddot{\boldsymbol{u}}_h, \boldsymbol{w}_h) \quad + \quad & a_h(\boldsymbol{u}_h, \boldsymbol{w}_h) = \mathcal{F}_u(\boldsymbol{w}_h) \quad \forall \boldsymbol{w}_h \in W^h \\
\text{with} \qquad & a_h(\boldsymbol{u}_h, \boldsymbol{w}_h) = \int_{\mathcal{B}_0} \boldsymbol{\varepsilon}^{ANS}(\boldsymbol{u}_h) : \boldsymbol{C} : \boldsymbol{\varepsilon}^{ANS}(\boldsymbol{w}_h) dV,
\end{aligned} \tag{4}$$

i.e. for the stiffness term the mesh dependent modification 2 has to be applied. Since in the semi-discrete solution a mesh dependent bilinear form $a_h(\cdot, \cdot)$ is used the exact solution is in general not a solution of the discrete problem, i.e.:

$$\rho_0(\ddot{\boldsymbol{u}}, \boldsymbol{w}_h) + a_h(\boldsymbol{u}, \boldsymbol{w}_h) \neq \mathcal{F}_u(\boldsymbol{w}_h) \quad \forall \boldsymbol{w}_h \in W^h, \tag{5}$$

Therefore for finite elements with assumed strains the well-known Galerkin-orthogonality condition does not hold [3]. In order to construct suitable error representations the weak form of the differential equation of the spatial discretization error is needed. There are now two ways to construct the weak form. One is to define the weak form of the differential equation by use of the unmodified bilinear form $a(\cdot,\cdot)$ the second option is to apply the mesh dependent bilinear form $a_h(\cdot,\cdot)$ for the error equation. Since both alternatives are used for the following error estimators both strategies shall be briefly presented. Both error representations are based on the weak form of the residual:

$$\mathcal{R}_u(\boldsymbol{w}) = \mathcal{F}_u(\boldsymbol{w}) - \rho_0(\ddot{\boldsymbol{u}}_h, \boldsymbol{w}) - a_h(\boldsymbol{u}_h, \boldsymbol{w}) = \rho_0(\ddot{\boldsymbol{u}}, \boldsymbol{w}) + a(\boldsymbol{u}, \boldsymbol{w}) - \rho_0(\ddot{\boldsymbol{u}}_h, \boldsymbol{w}) - a_h(\boldsymbol{u}_h, \boldsymbol{w}) \quad (6)$$

2.1 Error representation based on the unmodified problem

In order to formulate the weak form of the error equation by means of the unmodified bilinear form $a(\cdot,\cdot)$ equation (6) is reformulated as follows:

$$\begin{aligned} \mathcal{R}_u(\boldsymbol{w}) \quad = \quad & \rho_0(\ddot{\boldsymbol{u}}, \boldsymbol{w}) - \rho_0(\ddot{\boldsymbol{u}}_h, \boldsymbol{w}) + a(\boldsymbol{u}, \boldsymbol{w}) - a(\boldsymbol{u}_h, \boldsymbol{w}) \\ & + \underbrace{a(\boldsymbol{u}_h, \boldsymbol{w}) - a_h(\boldsymbol{u}_h, \boldsymbol{w})}_{-\mathcal{R}_c(\boldsymbol{w})} \end{aligned} \quad (7)$$

Herein the consistency term $\mathcal{R}_c(\boldsymbol{w}) = a_h(\boldsymbol{u}_h, \boldsymbol{w}) - a(\boldsymbol{u}_h, \boldsymbol{w})$ denotes the part of the residual arising from the modification of the bilinear form in the discrete variational problem (4). Equation (7) can now be transformed into the weak form of the spatial discretization error:

$$\rho_0(\ddot{\boldsymbol{e}}_S, \boldsymbol{w}) + a(\boldsymbol{e}_S, \boldsymbol{w}) = \mathcal{R}_u(\boldsymbol{w}) + \mathcal{R}_c(\boldsymbol{w}) \qquad \forall w \in W \quad (8)$$

We now introduce the exact solution $\boldsymbol{z}$ of the so-called dual or adjoint problem as test function for the error equation (8) while the dual problem itself is generally defined as:

$$\rho_0(\ddot{\boldsymbol{z}}, \boldsymbol{w}) + a^*(\boldsymbol{z}, \boldsymbol{w}) = \mathcal{F}_z(\boldsymbol{w}) \quad \forall \boldsymbol{w} \in W \quad (9)$$

The dual problem runs backward in time and the initial conditions are therefore defined at $t = t_n$. In our case $a^*(\boldsymbol{z}, \boldsymbol{w}) = a(\boldsymbol{z}, \boldsymbol{w})$ holds. The partial integration of (8) over the time domain yields the following identity:

$$\begin{aligned} \left[\rho_0(\dot{\boldsymbol{e}}_S, \boldsymbol{z}) - \rho_0(\boldsymbol{e}_S, \dot{\boldsymbol{z}})\right]_0 \quad + \quad & \int_0^{t_n} \mathcal{R}_u(\boldsymbol{z}) + \mathcal{R}_c(\boldsymbol{z}) dt \\ = \left[\rho_0(\boldsymbol{z}, \dot{\boldsymbol{e}}_S) - \rho_0(\dot{\boldsymbol{z}}, \boldsymbol{e}_S)\right]^{t_n} \quad + \quad & \int_0^{t_n} \mathcal{F}_z(\boldsymbol{e}_S) dt \end{aligned} \quad (10)$$

Equation (10) is arranged such that the dual solution $\boldsymbol{z}$ serves as weighting function of the discretization error on the left hand side of the equation and occurs as trial function with the weighting function $\boldsymbol{e}_S$ on the right hand side. Therefore the right hand side of equation (10) is used for the appropiate definition of the dual problem while the left hand side is employed for the error estimation.

We now choose a homogeneous dual problem , i.e. $\mathcal{F}_z(\boldsymbol{w}) = \mathcal{F}_z(\boldsymbol{e}_S) = 0$ and assume homogeneous initial conditions for the primal problem, i.e. $\boldsymbol{e}_S(\boldsymbol{x}, 0) = \dot{\boldsymbol{e}}_S(\boldsymbol{x}, 0) = 0$. Equation (10) then reduces to:

$$\int_0^{t_n} \mathcal{R}_u(\boldsymbol{z}) + \mathcal{R}_c(\boldsymbol{z}) dt = \left[\rho_0(\boldsymbol{z}, \dot{\boldsymbol{e}}_S) - \rho_0(\dot{\boldsymbol{z}}, \boldsymbol{e}_S)\right]^{t_n} \quad (11)$$

The right hand side of equation (11) is now used to define the initial conditions of the dual problem with regard to the quantity of interest. We restrict ourselves to the estimation of single point displacements. If we want to control the error in the displacements at point $\boldsymbol{x}_i$ in direction j we can define the quantity of interest

$$Q(\boldsymbol{u}) = (\boldsymbol{u}(\boldsymbol{x}, t_n), \boldsymbol{\delta}_j(\boldsymbol{x}_i)) \tag{12}$$

by use of the Dirac delta function $\boldsymbol{\delta}_j(\boldsymbol{x}_i)$. The error in the displacement component of interest is then defined as:

$$E(\boldsymbol{u}, \boldsymbol{u}_h) = Q(\boldsymbol{u}) - Q(\boldsymbol{u}_h) = (\underbrace{\boldsymbol{u}(\boldsymbol{x}, t_n) - \boldsymbol{u}_h(\boldsymbol{x}, t_n)}_{\boldsymbol{e}_S(\boldsymbol{x}, t_n)}, \boldsymbol{\delta}_j(\boldsymbol{x}_i)) = (\boldsymbol{e}_S, \boldsymbol{\delta}_j(\boldsymbol{x}_i)) \tag{13}$$

Consequently, setting $\boldsymbol{z}(t_n) = 0$ equation (13) together with the right hand side of equation (11) yields the definition of the initial velocities of the dual problem.

$$-\rho_0(\dot{\boldsymbol{z}}(t_n), \boldsymbol{w}) = (\boldsymbol{\delta}_j(\boldsymbol{x}_i, t_n), \boldsymbol{w}) \quad \forall \boldsymbol{w} \in W \tag{14}$$

With this specification of the dual problem the error representation applying the left hand side of equation (11) reads:

$$E(\boldsymbol{u}, \boldsymbol{u}_h) = \int_0^{t_n} \mathcal{R}_u(\boldsymbol{z}) + \mathcal{R}_c(\boldsymbol{z}) dt = \int_0^{t_n} \mathcal{F}_u(\boldsymbol{z}) - \rho_0(\ddot{\boldsymbol{u}}_h, \boldsymbol{z}) - a(\boldsymbol{u}_h, \boldsymbol{z}) dt \tag{15}$$

An equivalent error representation can also be derived by using the velocities of the dual problem (9) as testfunction. This yields different initial conditions of the dual problem which might be easier to evaluate for a different quantity of interest, such as local stresses.

Equation (15) will serve as basis for the error estimation in section 3.1.

2.2 Error representation based on the modified problem

As an alternative to equation (8) the weak form of the differential equation of the spatial discretization error can be formulated applying the mesh dependent modification. This yields a mesh dependent dual problem which also applies the assumed strain modification. The corresponding error representation can now be derived in the same fashion as in the preceding section.

For our purpose it is more suitable to use the velocities $\dot{\boldsymbol{z}}$ of the homogeneous dual problem as weighting function. Then the partial integration of the modified error equation yields the second identity:

$$\begin{aligned} E(\boldsymbol{u}, \boldsymbol{u}_h) &= [\rho_0(\dot{\boldsymbol{e}}_S, \dot{\boldsymbol{z}}) + a_h(\boldsymbol{e}_S, \boldsymbol{z})]_0 + \int_0^{t_n} \mathcal{R}_u(\dot{\boldsymbol{z}}) + \mathcal{R}_c(\dot{\boldsymbol{z}}) dt \\ &= [\rho_0(\dot{\boldsymbol{z}}, \dot{\boldsymbol{e}}_S) + a_h(\boldsymbol{z}, \boldsymbol{e}_S)]^{t_n}, \end{aligned} \tag{16}$$

which will serve as the basis for the error estimation in section 3.2. Setting $\dot{\boldsymbol{z}}(t_n) = 0$ the initial displacements of the dual problem at t_n for the estimation of a single point displacement read:

$$a_h(\boldsymbol{z}(t_n), \boldsymbol{w}) = (\boldsymbol{\delta}_j(\boldsymbol{x}_i), \boldsymbol{w}) \quad \forall \mathbf{w} \in W \tag{17}$$

That means the dual problem is reduced to a static problem at the time t_n. Since for the derivation of the initial conditions of the dual problem the mesh dependent modifications (2) have to be taken into account the present error representation seems to be suitable if the dual problem shall be computed on the same mesh as the primal problem.

3 ERROR ESTIMATION

3.1 Error estimation with full backward integration

The usual approach in goal oriented error estimation is the numerical evaluation of an error representation formula. Here we take equation (15). Since the residual $\mathcal{R}_u(\boldsymbol{z}_h) = 0 \ \forall \boldsymbol{z}_h \in W^h$ the dual problem cannot be computed on the same spatial discretization as the primal problem. Furthermore in order to capture the consistency part of the error a suitable approximation of $\mathcal{R}_c(\boldsymbol{z})$ is needed. One option is to compute the dual problem on the same mesh with higher order interpolation, see e.g. [1, 5]. In our case we introduce a reference mesh with mesh size $H = \frac{h}{2}$ and introduce the dual problem:

$$\rho_0(\ddot{\boldsymbol{z}}_H, \boldsymbol{w}_H) + a_H(\boldsymbol{z}_H, \boldsymbol{w}_H) = \mathcal{F}_z(\boldsymbol{w}_H) \ \ \forall \boldsymbol{w}_H \in W^H \ 0 \leq t \leq t_n. \tag{18}$$

The error representation is then replaced by the approximation:

$$E(\boldsymbol{u}, \boldsymbol{u}_h) \approx \int_0^{t_n} \mathcal{F}_u(\boldsymbol{z}_H) - \rho_0(\ddot{\boldsymbol{u}}_h, \boldsymbol{z}_H) - a_H(\boldsymbol{u}_h, \boldsymbol{z}_H) dt \tag{19}$$

which can be written in the discrete formulation as the sum over all time steps. Here, as a suitable choice the time integral is replaced by a one-point quadrature rule, i.e. the dual-weighted residual has to be computed in the middle of each time step Δt. The error estimator then reads:

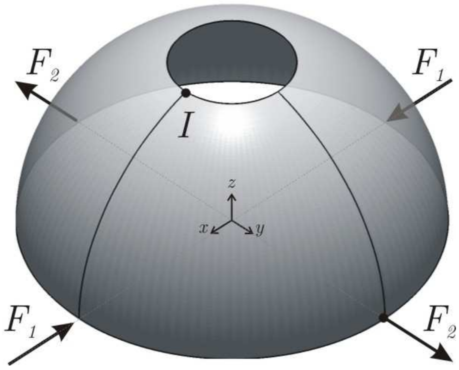
$$E(\boldsymbol{u}, \boldsymbol{u}_h) \approx \sum_{j=1}^{n} \Delta t_j \cdot \left(\sum_{i=1}^{n_{el}} (\boldsymbol{p}_H, \boldsymbol{z}_H)_{\mathcal{B}_i} - (\ddot{\boldsymbol{u}}_h, \boldsymbol{z}_H)_{\mathcal{B}_i} - a_H(\boldsymbol{u}_h, \boldsymbol{z}_H)_{\mathcal{B}_i} \right)_{j-\frac{1}{2}} \tag{20}$$

Herein $\boldsymbol{p}_H$ denotes the interpolation of the external forces in the reference mesh.

3.1.1 Numerical example

The error estimator (20) shall now be tested with a numerical example, now including damping. We consider the hemisphere with a hole depicted in figure 1. The hemisphere is subjected to two pairs of single forces, the temporal evolution of the forces is depicted in figure 2.

Time integration is performed with the standard Newmark algorithm with a constant time step size $\Delta t = 0.005$. Due to symmetry only a quarter of the hemisphere is discretized with bilinear Solid-Shell elements. For the error estimation two uniform meshes are considered. The first mesh consists of $n_{el,1} = 256$ elements with $n_{dof,1} = 1632$ degrees of freedom, mesh 2 consist of $n_{el,2} = 1024$ elements with $n_{dof,2} = 6336$ degrees of freedom. The reference solution is computed with a uniform mesh with $n_{el,ref} = 16348$ elements and $n_{dof,ref} = 99072$.



radius of hemisphere:	R	$=$	10
thickness:	t	$=$	0.04
radius of the hole:	r	$=$	3
modulus of elasticity:	E	$=$	$6,8 \cdot 10^7$
Poisson ratio:	ν	$=$	$0,3$
density:	ρ	$=$	5
damping parameters:	c_m	$=$	$0,0003$
	c_k	$=$	$0,0001$

Fig. 1 Example: Hemisphere with hole

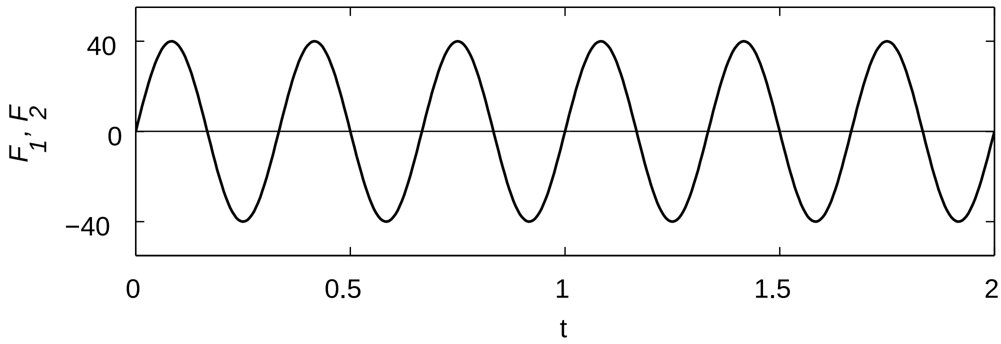


Fig. 2 Hemisphere with hole: Loading function

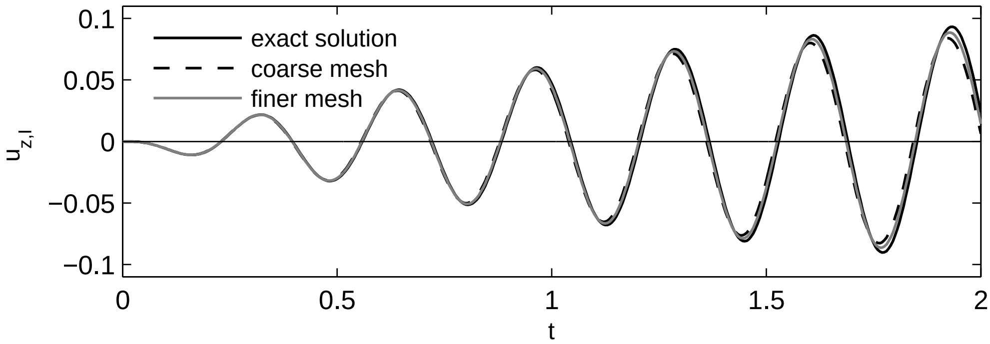


Fig. 3 Hemisphere with hole: vertical displacement of point $\boldsymbol{I}$ for the different spatial discretizations

Figure 3 shows the vertical displacement of point $\boldsymbol{I}$ for the two meshes and the reference solution.

Figures 4 and 5 show the estimated vs. the exact error in the quantity of interest. In both cases the error estimation works rather well. Especially the temporal evolution of the error can be well captured. Nevertheless there is a remarkable phase shift between the estimated and the exact error, which mainly results from the phase error of the dual solution due to its numerical approximation. As a consequence the error estimation is better for the finer mesh 2. The main drawback of using the error estimator (20) is the nearly unsurmountable numerical effort which arises from the error estimation procedure. For each time t_j for which the error shall be

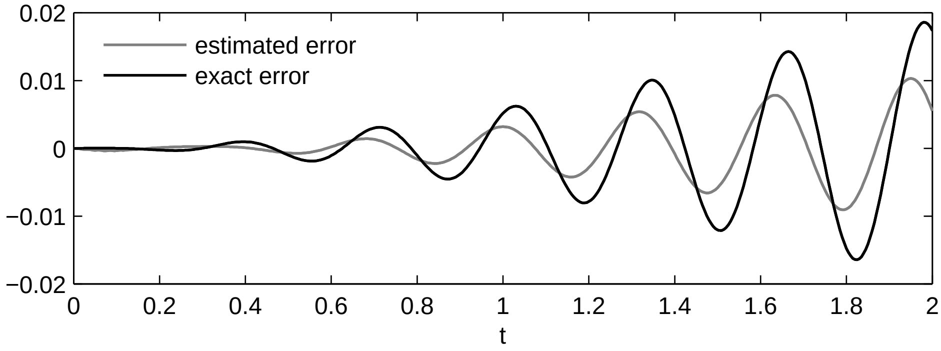


Fig. 4 Estimated vs. exact error in the displacement of point $\boldsymbol{I}$ – coarse mesh ($n_{dof,1} = 1632$)

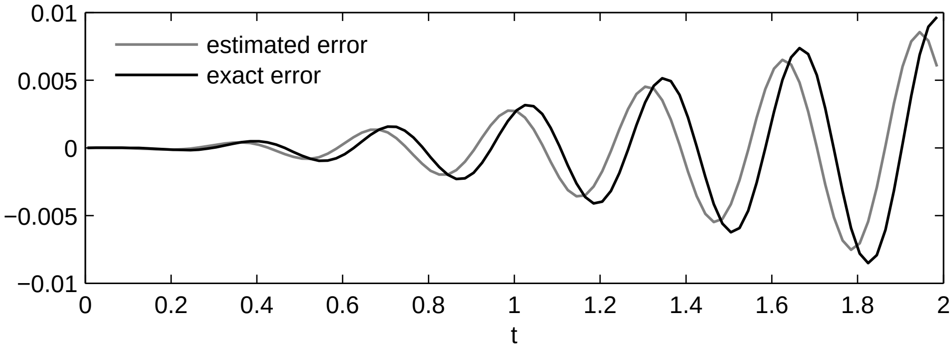


Fig. 5 Estimated vs. exact error in the displacement of point $\boldsymbol{I}$ – finer mesh ($n_{dof,2} = 6336$)

estimated the whole temporal coupling of the primal and the dual problem has to be carried out. Furthermore the whole primal and dual problem has to be stored over the whole time domain which results in huge memory requirements even for rather small numerical models.

The main issue of the following section is therefore the reduction of the numerical effort of the error estimation procedure.

3.2 Error estimation without backward integration

In order to reduce the numerical effort of the error estimation procedure we first perform a modal decomposition of the spatial discretization error. The exact solution can be stated in modal form:

$$\boldsymbol{u}(\boldsymbol{x}, t) = \sum_{i=1}^{\infty} \boldsymbol{U}_i(\boldsymbol{x}) \cdot f_i(t) \tag{21}$$

with $\boldsymbol{U}_i(\boldsymbol{x})$ being the natural modes of the exact eigenvalue problem:

$$div\boldsymbol{\sigma}(\boldsymbol{U}) + \rho\omega^2\boldsymbol{U} = \boldsymbol{0}. \tag{22}$$

The corresponding representation of the discrete solution reads:

$$\boldsymbol{u}_h(\boldsymbol{x}, t) = \sum_{i=1}^{n_{dof}} \boldsymbol{U}_i^h(\boldsymbol{x}) \cdot f_i^h(t), \tag{23}$$

The total spatial discretization error can now be split in the following form:

$$\begin{aligned} \boldsymbol{e}_S(\boldsymbol{x},t) &= \sum_{i=1}^{n_{dof}} \underbrace{\left(\boldsymbol{U}_i(\boldsymbol{x}) - \boldsymbol{U}_i^h(\boldsymbol{x})\right)}_{\boldsymbol{E}_i(\boldsymbol{x})} \cdot f_i^h(t) + \boldsymbol{U}_i(\boldsymbol{x}) \cdot \underbrace{\left(f_i(t) - f_i^h(t)\right)}_{e_{\varphi,i}} \\ &+ \sum_{i=n_{dof}+1}^{\infty} \boldsymbol{U}_i(\boldsymbol{x}) \cdot f_i(t), \end{aligned} \tag{24}$$

which means that the total spatial discretization error consists of the errors $\boldsymbol{E}_i$ of the spatial approximation of the natural modes, the phase errors $\boldsymbol{U}_i \cdot e_{\varphi,i}$ due to the approximation of the natural frequencies and the cut-off error of higher modes which are not included in the numerical model. For suitably chosen meshes the cut-off error can usually be neglected.

A closer look at equation (24) shows that the phase error mainly depends on the time and that the phase error might be dominant even if the natural frequencies are captured sufficiently. Therefore, controlling the phase error usually yields overly fine spatial discretizations. The numerical example in section 3.1 also exhibits that due to the phase error the maximum values of the errors usually occur at points in time where the quantity of interest is nearly zero. For practical applications it might be rather interesting to judge whether the spatial discretization is suitable to represent the extreme values of the amplitudes of the quantity of interest.

So our approach is now to simply neglect the phase error in the error representation. That means we assume that the spatial discretization error can be restated as:

$$\boldsymbol{e}_S(\boldsymbol{x},t) = \sum_{i=1}^{n_{dof}} \boldsymbol{E}_i(\boldsymbol{x}) \cdot f_i^h(t) \tag{25}$$

In other words, we assume that the exact solution can be reached by a pure spatial enhancement of the discrete solution $\boldsymbol{u}_h$. Furthermore with the restriction in equation (25) we implicitly assume that the temporal evolution of the spatial discretization error is known.

In the error estimation procedure in section 3.1 the backward integration and the numerical evaluation of the dual weighted residual was necessary since the temporal distribution of the spatial discretization error was unknown. Therefore the computable right hand side of the differential equation of the error had to be used for the error estimation. The boundary conditions at time t_n have only been employed for the proper definition of the dual problem. Now in our case the temporal distribution of the spatial discretization error is assumed to be known which renders the backward integration in time unnecessary. The error estimation can now be performed simply on the basis of the boundary terms at time t_n, i.e. we now employ the right hand side of equation (16) not only for the proper definition of the dual problem but also for the error estimation. The error representation now reads:

$$E(\boldsymbol{u},\boldsymbol{u}_h) = \left[\rho_0(\dot{\boldsymbol{e}}_S,\boldsymbol{z}) + a_h(\boldsymbol{e}_S,\boldsymbol{z})\right]^{t_n} \tag{26}$$

Since we have to apply the same strain interpolation scheme for the dual as for the primal problem we first compute the numerical solution $\boldsymbol{z}_h(t_n)$ on the mesh of the primal problem. Then we split the dual solution as $\boldsymbol{z}(t_n) = \boldsymbol{z}_h(t_n) + \boldsymbol{e}_Z(t_n)$. The error representation for the absolute value can then be formulated as:

$$|E(\boldsymbol{u},\boldsymbol{u}_h)| = \left|\rho_0(\dot{\boldsymbol{e}}_Z,\dot{\boldsymbol{e}}_S) + a_h(\boldsymbol{e}_Z,\boldsymbol{e}_S) + \rho_0(\dot{\boldsymbol{z}}_h,\dot{\boldsymbol{e}}_S) + a_h(\boldsymbol{z}_h,\boldsymbol{e}_S)\right|^{t_n} \tag{27}$$

Since $\mathcal{R}_c(\boldsymbol{z}_h) \neq 0$ the last two terms in equation (27) do not vanish as in the case of a standard Galerkin-scheme. Neglecting the last two terms means that the consistency error will not be captured with the error estimator. Nevertheless the application of Cauchy-Schwarz inequality to equation (27) would yield a huge overestimation of the consistency parts of the error, see e.g. Diez, Morana and Huerta [3]. Fortunately, the numerical examples in [3] show that an adaptive scheme which is based solely on the error estimation without the consistency error is suitable to reduce the total discretization error. Therefore we neglect the consistency part of the error in the following and obtain the simplified error indicator:

$$|E(\boldsymbol{u}, \boldsymbol{u}_h)| \approx \rho_0 ||\dot{\boldsymbol{e}}_S||_{L_2} \cdot ||\dot{\boldsymbol{e}}_Z||_{L_2} + ||\boldsymbol{e}_S||_{a,h} \cdot ||\boldsymbol{e}_Z||_{a,h} \tag{28}$$

with the L_2-norm of the errors in the velocities and the mesh dependent energy norms of the errors in the displacements of the dual and primal problem. If we now restrict the error estimation to displacements of single points the velocity parts vanish and only the strain parts remain in the error estimation procedure. The mesh dependent energy norms of the errors are now estimated by use of the well-known error estimator by Zienkiewicz and Zhu [11] which is based on recovered stresses. For the evaluation of the Zienkiewicz-Zhu error estimator the mesh dependent modifications have to be considered:

$$||\boldsymbol{e}_S||_{a,h} = \left(\int\limits_{\mathcal{B}_0} (\boldsymbol{\sigma}^* - \boldsymbol{\sigma}(\boldsymbol{\varepsilon}^{ANS})) : \boldsymbol{C}^{-1} : (\boldsymbol{\sigma}^* - \boldsymbol{\sigma}(\boldsymbol{\varepsilon}^{ANS})) dV \right)^{1/2} \tag{29}$$

Herein $\boldsymbol{\sigma}^*$ denotes a smoothed stress field which is obtained by employing the so-called superconvergent patch recovery concept. In case of wave propagation problems the main characteristics of the underlying physical problem lies in the temporal and spatial transport of the wave. So neglecting the temporal transport of the error in wave propagation problems means that a main part of important information is neglected. So the suitable application of the error estimator is restricted to predominantly vibration type problems.

4 MESH ADAPTATION SCHEME

The simplified error estimator (28) shall now serve as basis of a mesh refinement scheme. We use a hierarchical mesh refinement scheme, i.e. all meshes throughout the computation contain the first spatial mesh. To ensure compatible meshes transition elements are introduced at the transition from coarser to finer discretized domains. The adaptive scheme is now as follows. At each point in time t_j at which a prescribed error tolerance is exceeded a mesh refinement is performed. Then the current data is transfered onto the new mesh and the computation continues. The transfer of the data is done by use of the scheme proposed by Radovitzky and Ortiz in [9]. This procedure consists of two steps:

- geometric interpolation of the data at the beginning of the current time step at time t_{j-1}
- computation of the state variables on the new mesh at the end of the time step (t_j) by use of the time integration scheme. This yields an admissible state at the end of the time step.

4.1 Numerical example

The mesh adaptation scheme is now applied to the hemisphere with hole which has already been mentioned in section 3.1. The quantity of interest is once again the vertical displacement of the point $\boldsymbol{I}$, see figure 1. The tolerance in our adaptive scheme for the quantity of interest is $e_{tol} = 4 \cdot 10^{-4}$. Since the error estimator neglects parts of the error representation we can expect an underestimation of the true error, i.e. the true error resulting from the mesh adaptation scheme might be larger than e_{tol}.

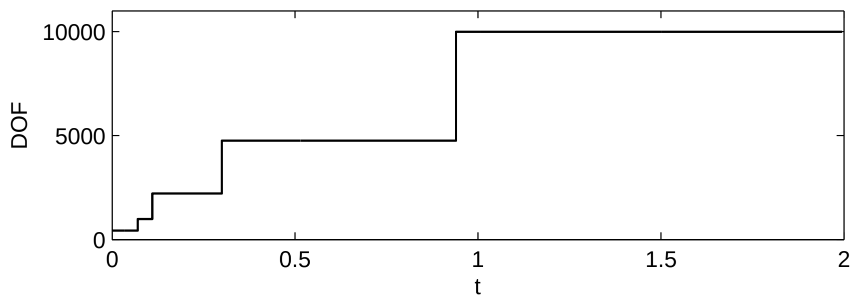


Fig. 6 Hemisphere with hole: Evolution of the number of equations throughout the computation

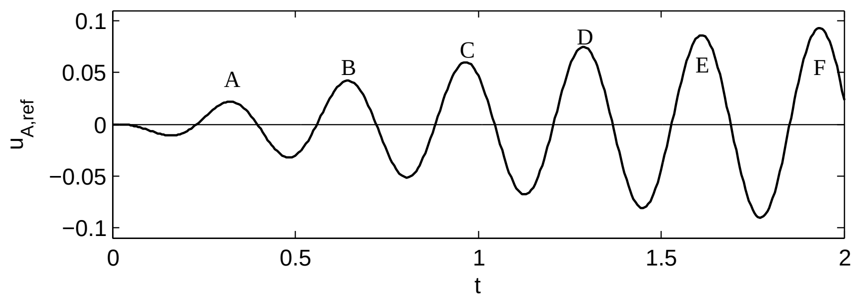
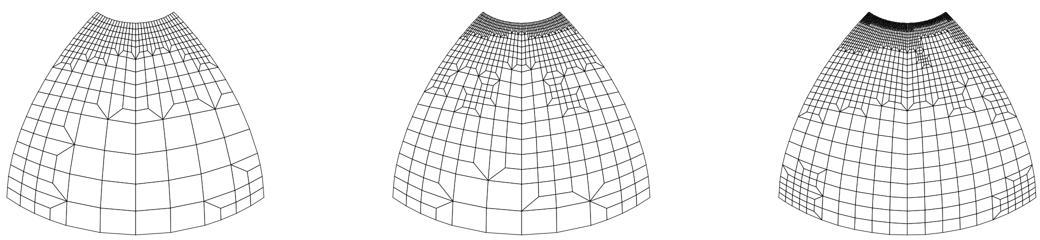


Fig. 7 Hemisphere with hole: reference solution $u_{A,ref}$ of vertical displacement of point $\boldsymbol{I}$



mesh at A– $n_{dof} = 2216$ mesh at B – $n_{dof} = 4754$ mesh at C–F – $n_{dof} = 9996$

Fig. 8 Hemisphere with hole: Sequence of adaptively generated meshes

Figure (6) shows the evolution of the numbers of equations throughout the computation; in figure 8 the corresponding adaptive meshes are depicted. One can see clearly the strong refinement of the mesh in the region of the quantity of interest.

We are now interested if the simplified error estimator and the adaptive mesh refinement scheme are suitable to improve the solution in the quantity of interest. Therefore the maximum values of the displacements of point $\boldsymbol{I}$ at the times A – F depicted in figure 7 are compared.

time		uniform mesh $n_{dof} = 6336$		uniform mesh $n_{dof} = 24960$		adaptive meshes		
	$u_{A,ref}$	e_A	%	e_A	%	e_A	%	FHG
A	$0,02174$	$1,31 \cdot 10^{-4}$	$0,6\%$	$5,4 \cdot 10^{-5}$	$0,24\%$	$4,5 \cdot 10^{-5}$	$0,20\%$	2216
B	$0,04209$	$4,52 \cdot 10^{-4}$	$1,1\%$	$1,75 \cdot 10^{-4}$	$0,4\%$	$2,9 \cdot 10^{-4}$	$0,7\%$	4754
C	$0,0601$	$9,6 \cdot 10^{-4}$	$1,6\%$	$3,5 \cdot 10^{-4}$	$0,6\%$	$5,6 \cdot 10^{-4}$	$0,9\%$	9996
D	$0,075$	$1,78 \cdot 10^{-3}$	$2,4\%$	$6,7 \cdot 10^{-4}$	$0,9\%$	$9,6 \cdot 10^{-4}$	$1,3\%$	9996
E	$0,086$	$3,11 \cdot 10^{-3}$	$3,6\%$	$1,13 \cdot 10^{-3}$	$1,3\%$	$1,3 \cdot 10^{-3}$	$1,6\%$	9996
F	$0,093$	$4,7 \cdot 10^{-3}$	$5,0\%$	$1,63 \cdot 10^{-3}$	$1,7\%$	$1,7 \cdot 10^{-3}$	$1,8\%$	9996

Tab. 1 Hemisphere with hole: Comparison of the error in the amplitudes of the vertical displacement of point $\boldsymbol{I}$ for two uniform meshes and for the adaptive scheme

Table 1 shows the comparison of the errors for two uniform meshes with 6336 and 24960 degrees of freedom and for the adaptive scheme. The maximum number of degrees of freedom of the adapted meshes is 9996 and the error in the quantity of interest is comparable to the error for the finer uniform mesh.

time		uniform mesh – $n_{dof} = 24960$		adaptive mesh – $n_{dof} = 9996$	
	$u_{A,ref}$	e_A	%	e_A	%
A	$0,02174$	$5,4 \cdot 10^{-5}$	$0,24\%$	$-6,3 \cdot 10^{-5}$	$0,3\%$
B	$0,04209$	$1,75 \cdot 10^{-4}$	$0,4\%$	$-1,0 \cdot 10^{-4}$	$0,3\%$
C	$0,0601$	$3,5 \cdot 10^{-4}$	$0,6\%$	$-1,2 \cdot 10^{-4}$	$0,2\%$
D	$0,075$	$6,7 \cdot 10^{-4}$	$0,9\%$	$-7,8 \cdot 10^{-5}$	$0,1\%$
E	$0,086$	$1,13 \cdot 10^{-3}$	$1,3\%$	$-2,1 \cdot 10^{-5}$	$0,02\%$
F	$0,093$	$1,63 \cdot 10^{-3}$	$1,7\%$	$1,1 \cdot 10^{-4}$	$0,1\%$

Tab. 2 Hemisphere with hole: Comparison of the error in the amplitudes of the vertical displacement of point $\boldsymbol{I}$ for a uniform mesh with $n_{dof} = 24960$ and for the computation of the whole problem with the last adaptive mesh with $n_{dof} = 9996$

If we now take the last mesh of the adaptive procedure and restart the computation with this mesh we obtain a solution which is rather close to the reference solution which has been computed on a mesh with 99072 degrees of freedom.

5 CONCLUSIONS

This present contribution deals with goal-oriented error estimation and mesh-adaptivity in structural dynamics. Two error estimation techniques have been presented. Besides the standard goal oriented error estimation a strongly simplified error estimator which neglects the phase error due to the spatial discretization has been considered. Instead of solving the complete dual problem in time only a static problem has to be solved which results in a very efficient error estimation procedure. This simplified error estimator was used as the basis of an adaptive mesh refinement scheme. The numerical example shows that the simplified error estimator – within its limitation to vibration type problems– seems to be a suitable tool for the generation of efficient spatial discretizations.

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