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Closest point projection in contact mechanics: existence and uniqueness for different types of surfaces

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In most contact algorithms especially in case of large deformations the closest point projection (CPP) procedure is necessarily involved in order to check the contact conditions. Despite the number of publications on numerical contact algorithms there are no complete results about existence and uniqueness of the CPP procedure for arbitrary surfaces. The current contribution is aimed to fill this void.

1 Introduction

The CPP procedure is introduced as a numerical scheme to compute coordinates of a point projected onto a surface. In variational formulations for contact problems it appears due to the split of contact displacements into a normal and a tangential part. Hallquist et.al. [1] considered the split into normal and tangential direction via the projection operation as a so-called "master-slave" approach for a dynamic contact analysis. Nowadays, the CPP procedure is an essential part for almost all contact enforcement methods: penalty method, Lagrange multiplier method, augmented Lagrangian method. Despite the enormous number of publications on contact mechanics, there are only a few publications covering to some completeness the problem of uniqueness and existence of the CPP procedure for the surfaces of arbitrary geometry as well as describing effective numerical algorithms to overcome problems. The problem of non-uniqueness and non-existence of the projection for e.g. bi-linear approximations of a surface by finite elements is known since the first publications, see Hallquist et.al. [1], and mostly reported in theoretical manuals of popular commercial codes. Heegaard and Curnier in [2] mentioned that geometrical parameters of a surface can be used to determine existence and uniqueness of the projection for smooth surfaces. Some techniques dealing with the non-existence of the projection in certain cases are well known in contact mechanics: descriptions of various rather heuristic approaches can be found in the books of Wriggers [3] and Laursen [4].

The current contribution deals with analytical tools allowing to create, a-priori, *proximity domains of contact surfaces* from which a given contact point is always uniquely projected. This approach is based on the geometrical properties of contact surfaces exploiting the covariant description for contact problems as developed in Konyukhov and Schweizerhof [5], [6]. First, all C^2-continuous surfaces are classified according to their differential properties allowing a unique projection. Then, *proximity domains* are created for C^1-continuous surfaces. Finally, *proximity domains* are proposed for globally C^0-continuous surfaces covering all practical approximations. The projection scheme in the latter case is further generalized including the projection onto geometrical objects of lower order (curved edges, corner points). In such cases the corresponding proximity domains are created by a geometrical analysis of those objects.

2 Analysis of CPP and projection domains for surfaces

Let us assume that the general parameterization (e.g. by a finite element approximation, or a spline approximation or by a NURBS approximation) is given for the "master" surface via Gaussian coordinates as $\boldsymbol{\rho}(\xi^1, \xi^2)$ and $\mathbf{r}_S$ is a "slave" point. The CPP procedure is then formulated as an extremal problem

$$\mathbf{F}(\xi^1, \xi^2) = \frac{1}{2}||\mathbf{r} - \boldsymbol{\rho}(\xi^1, \xi^2)|| \to \min, \quad \longrightarrow (\mathbf{r} - \boldsymbol{\rho}) \cdot (\mathbf{r} - \boldsymbol{\rho}) \to \min, \tag{1}$$

which is solved then mostly numerically. As is known, the convexity of the function $\mathbf{F}$ leads to existence and uniqueness of the extremal problem (1). This analysis is performed in a 3D spatial coordinate system related to the surface coordinate system, which is introduced as follows:

$$\mathbf{r}(\xi^1, \xi^2, \xi^3) = \boldsymbol{\rho} + \mathbf{n}\xi^3. \tag{2}$$

The second derivative of the function $\mathbf{F}$ is computed and transformed in this local coordinate system as

$$\mathbf{F}'' = \begin{bmatrix} a_{11} - \xi^3 h_{11} & a_{12} - \xi^3 h_{12} \\ a_{21} - \xi^3 h_{21} & a_{22} - \xi^3 h_{22} \end{bmatrix}, \tag{3}$$

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where a_{ij} are components of the metric tensor, h_{ij} are components of the curvature tensor for the master surface. The convexity requirement for the function $\mathbf{F}$ in eqn. (1) is becoming then the positivity requirement for the matrix $\mathbf{F}''$ in eqn. (3). The latter is analyzed with the Sylvester criterion, namely

$$\begin{array}{c}(a_{11} - \xi^3 h_{11}) > 0 \\ \det[(a_{ij} - \xi^3 h_{ij})] > 0.\end{array} \tag{4}$$

Remarkably, the second equation in (4) is similar to that which is used in differential geometry for the analysis of the surface structure.

Projection domains surrounding a given surface, from which a point can be a-priori uniquely projected onto the surface can be constructed for C^2-continuous surfaces in the local coordinate system in eqn. (2). The structure is different for an elliptic, a hyperbolic and a parabolic point and is defined as follows:

$$\Omega(\xi^1, \xi^2, \xi^3) := \{\mathbf{r} = \boldsymbol{\rho} + \xi^3 \mathbf{n}, \;\; where \;\; \xi^3 \in Q\}. \tag{5}$$

Thus e.g. for an elliptic point $Q \equiv (0, \min(\frac{1}{R_1}, \frac{1}{R_2}))$ (with R_i as radii of curvature) for the locally convex part with $\xi^3 > 0$ and $Q \equiv [-\infty, 0)$ for the locally concave part with $\xi^3 < 0$. Other cases are constructed in a similar fashion, see Fig. 1.

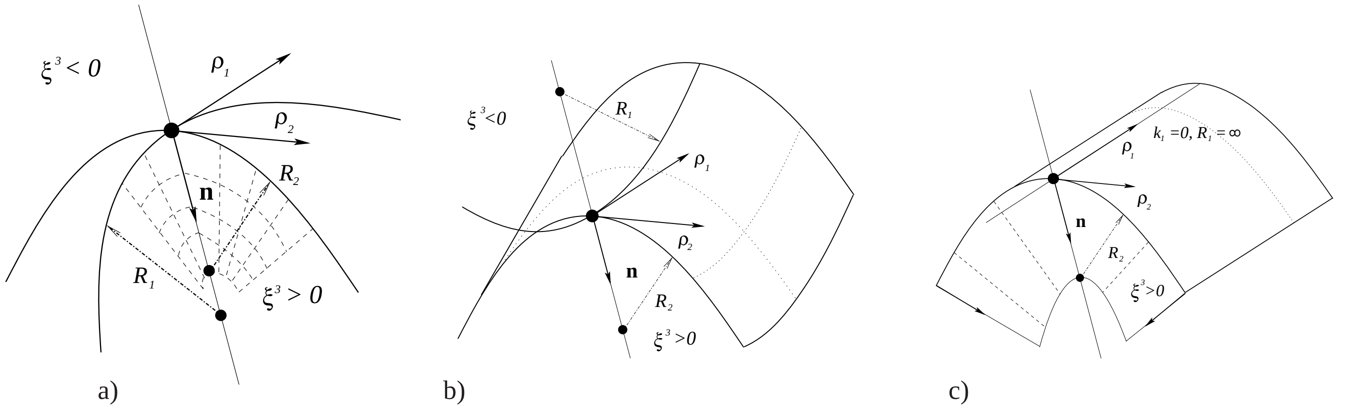

Fig. 1 Structure of projection domains for various cases: a) elliptic point. b) hyperbolic point. c) parabolic point.

In order to describe the points which can not be described in the local surface coordinate system (2) the surface CPP should be generalized and include then a point-to-edge CPP and a angular point CPP. The solvability of the point-to-edge CPP is analyzed then in the Serret-Frenet frame exploiting the differential properties of the corresponding curve.

3 Conclusion

In this contribution fundamental problems of existence and uniqueness of the closest point projection procedure are investigated. The analysis is given in a surface coordinate system, which has also been a basis for a covariant description of the contact formulation. The consideration of the differential properties of smooth surfaces allows to create ”projection domains” from which a projection of e.g. a slave node is uniquely defined. For arbitrary C^0 continuous surfaces, however, the projection procedure should be generalized to include projections not only onto surfaces, but also onto objects of lower geometrical dimension, such as curved lines and points. The corresponding criteria of existence and uniqueness and, therefore, projection domains are then constructed in the Serret-Frenet frame and in the local frame connected to the angular point.

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