Fractional Excitations in low–dimensional spin systems

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Qui dedit beneficium, taceat. Narret, qui accepit.

– Lucius Annaeus Seneca

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List of publications

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4. R. Thomale and M. Greiter, *Numerical analysis of three-band models for CuO planes as candidates for a spontaneous T violating orbital current phase*, Phys. Rev. B **77**, 094511 (2008).

5. D. F. Schroeter, E. Kapit, R. Thomale, and M. Greiter, *Spin Hamiltonian for which the Chiral Spin Liquid is the Exact Ground State*, Phys. Rev. Lett. **99**, 097202 (2007).

6. M. Greiter and R. Thomale, *No evidence for spontaneous currents in finite size studies of three-band models for CuO planes*, Phys. Rev. Lett. **99**, 027005 (2007).

7. R. Thomale, D. Schuricht, and M. Greiter, *Charge excitations in SU(n) spin chains: exact results for the* $1/r^2$ *model*, Phys. Rev. B **75**, 02445 (2007).

8. R. Thomale, D. Schuricht, and M. Greiter, *Exact two-holon wave functions in the Kuramoto–Yokoyama model*, Phys. Rev. B **74**, 024423 (2006).

Contents

Chapter 1

Introduction

The fractionalization of quantum numbers, where the excitations of a strongly-correlated system carry only a fraction of the quantum numbers of the constituents, is currently of great interest in condensed matter physics. In particular, as it is hard to address this physics by mean field theories, a significant body of recent work has focused on finding solvable theoretical models in which this phenomenon occurs [10, 93, 94, 96, 97]. In addition to its intrinsic interest, the phenomenon of fractionalization may well have a bearing on one of the most vexing problems in condensed matter theory, should the long-standing suggestion of a link between fractionalization and high-T_C superconductivity [4, 83] be established. In particular, it has been shown [65] that the topological degeneracy in these systems might be used to protect quantum bits and be applicable to the emerging field of quantum computing.

Fractional statistics, as a generalization of the idea of quantum statistics based on Berry's phase [130], is a sensible idea only in one or two dimensions, where one can define a winding number. In one dimension, the behavior is known to occur in spin-1/2 antiferromagnets [58], where exactly solvable models exhibiting this behavior exist [57, 59, 60, 114]. One milestone towards the understanding of fractional quantization in one dimension is the $1/r^2$ model independently introduced by Haldane [57] and Shastry [114] in 1988. The model describes a spin 1/2 chain with a Heisenberg interaction which falls off as one over the square of the distance between the sites. The exact ground state is provided by a trial wave function proposed by Gutzwiller [51] as early as in 1963. The Haldane–Shastry Model (HSM) offers the opportunity of studying spinons, *i.e.,* the elementary excitations of one-dimensional spin chains, on the level of explicit and analytical expressions for one and two-spinon wave functions [14], which are at least at present not available for any other model. Kuramoto and Yokoyama [87] generalized the model to allow for mobile holes (*i.e.,* empty lattice sites) with a hopping parameter that also falls off with $1/r^2$ as a function of the distance. The Kuramoto–Yokoyama Model (KYM) hence contains spin and charge degrees of freedom, and accordingly supports spinon and holon excitations, which carry spin $\frac{1}{2}$ but no charge and charge $+e$ but no spin, respectively. It allows for the exact study of the decoupling of a fermionic (hole) degree of freedom into a spinon and a holon, *i.e.,* spin charge separation. This is also a key feature of the Luttinger liquid, which is the effective low energy model of one-dimensional spin chains [55].

Fractionalization of statistics is also known to occur in two dimensions, in the presence of a magnetic field that violates the discrete symmetries of parity (P) and time-reversal (T); this situation is realized in the fractional quantum Hall effect [7, 22, 23, 63, 88, 117] (FQHE). The field started about a quarter of a century ago with the discovery of the fractional quantum Hall effect, which was explained by Laughlin [88] in terms of an incompressible quantum liquid supporting fractionally charged (vortex or) quasiparticle excitations. When formulating a hierarchy of quantized Hall states [44, 56, 63] to explain the observation of quantized Hall states at other filling fractions, Halperin [63] noted that these excitations obey fractional statistics [130], and are hence conceptually similar to the charge-flux tube composites introduced by Wilczek two years earlier [129].

The interest was renewed a few years later, when Anderson [4] proposed that hole-doped Mott insulators, and in particular those described by the t–J model universally believed to describe the CuO planes in high T_c superconductors, can be described in terms of a spin liquid (*i.e.,* a state with strong, local antiferromagnetic correlations but without long range order), which would likewise support fractionally quantized excitations. In this proposal, the excitations are spinons and holons, which carry spin 1/2 and no charge or no spin and charge $+e$, respectively. The fractional quantum number of the spinon is the spin, which is half integer while the Hilbert space (for the undoped system) is built up of spin flips, which carry spin one.

One of the earliest proposals for a spin liquid supporting deconfined spinon and holon excitations is the (abelian) chiral spin liquid (CSL). Following up on an idea by D.H. Lee, Kalmeyer and Laughlin [70] proposed in 1987 that a quantized Hall wave function for bosons could be used to describe the amplitudes for spin-flips on a lattice. The excitations of the liquid—spinons, which carry spin 1/2 but no charge, and holons, which carry charge but no spin—obey fractional statistics. Again, the spinons exhibit quantum-number fractionalization and carry only half the spin of the excitations in conventional magnetically-ordered systems, which carry spin 1. Whereas the spinon is supposed to be the fundamental field describing excitations in two-dimensional antiferromagnetic $S = 1/2$ systems in general, the spinon substructure of the magnons carrying spin 1 is generally suppressed, which makes it a less established concept of description for two dimensions than for one dimension. This can be seen in analogy to quantum chromodynamics, where the quarks constitute the substructure of the nucleons, but cannot be observed as free particles.

In the chiral spin liquid, however, the spinons are deconfined, which thus represents a suitable trial state to study fractional quantization of spinons in two dimensions. Unfortunately, for nearly two decades since its emergence, the chiral spin liquid lacked a microscopic model where it is realized. From today's point of view, the CSL state did not turn out to be relevant to CuO superconductivity, but remains one of very few examples of two-dimensional spin liquids with fractional quantization. The other established examples are the resonating valance bond (RVB) phases of the Rokhsar-Kivelson model [83] on the triangular lattice identified by Moessner and Sondhi [94] and of the Kitaev model [81]. While the explicit detection of spinons in two dimensions still remains an unsolved experimental task, the fractional statistics of the quasiparticle excitations in the FQHE has been observed experimentally very recently [22, 23]. In contrast to the one-dimensional case, however, there has been no definite evidence as to whether fractional statistics occurs in the absence of an external field breaking these symmetries explicitly, which would dictate a path to endeavor fractional excitations in generic spin models where the symmetry is generally preserved.

The present renaissance of interest in fractional quantization is also due to possible applications of states supporting excitations with non-abelian statistics to the rapidly evolving field of quantum computation and cryptography [82]. The paradigm for this universality class is the Pfaffian state introduced by Moore and Read [95] in 1991. The state was proposed to be realized at the experimentally observed fraction $\nu = 5/2$ [131] (*i.e.,* at $\nu = 1/2$ in the second Landau level), a proposal which recently received strong experimental support through the direct measurement of the quasiparticle charge [29]. Remarkable subsequent works further explored the underlying nature of the Pfaffian state and non-Abelian statistics by using exact eigenstates of model Hamiltonians [49] as well as by the study of effective Chern-Simons field theories [35] and the deduction of the concise braiding properties of the Pfaffian state [98]. It turned out that (effective) three-body interaction terms appear to be essential to stabilize the Pfaffian state. A further step in understanding the Pfaffian state was done by Read and Green observing that a p-wave BCS superconductor can be described by it where the excitations are the half quantum vortices and their non-Abelian statistics results from a zero-energy mode in the vortex core [103]. Therein, the non-abelian statistics manifests itself in Majorana modes of zero energy in the vortex core.

Effectively, two Majorana modes form one fermionic degree of freedom. The internal Hilbert space whose configuration is changed upon braiding of the half quantum vortices is given by the occupation numbers of these zero mode fermions, which gives rise to a non-commutativity of the braiding and hence to non-abelian statistics. This was made precise by Ivanov [67] in terms of concrete unitary non-Abelian transformations governing the braiding of the half quantum vortices and was further refined later by Stern, von Oppen, and Mariani [116].

Quantum states with non-Abelian excitations (nonabelions) are of great fundamental and potential practical interest. Non-Abelian statistics can be viewed as a generalization of abelian anyonic statistics. For the latter, the many-particle state acquires a non-trivial fractional phase as two anyonic excitations (anyons) wind around each other. This realization of abelian anyonic statistics can be interpreted as a one-dimensional representation of the braid group [77, 130], which we also encounter for the spinons in the CSL, thus being abelian anyonic excitations. Opposed to this, nonabelions possess internal state degrees of freedom whose configuration is changed as the nonabelions wind around each other, thus being a higher dimensional representation of the braid group [33]. Concerning its potential application to topological quantum computing [26, 82], a non-Abelian phase is highly advocated compared to an anyonic phase. This is due to the fact that whereas in the abelian anyonic phase one can only alter the phase of the state by braiding quasiparticles, exchanging particles can also change the internal space configurations of the particles for the non-Abelian case. This gives such a huge variety of protected topological manipulations of the system that any desired unitary transformation can be realized with arbitrary accuracy [36]. Most importantly, however, the internal state vector is insensitive to local perturbations—it can *only* be manipulated through braiding of the non-abelian excitations. These properties together render non-abelions preeminently suited for applications as protected qubits in quantum computation [26].

In this thesis, we start to approach the field of fractional excitations by discussing the spinon and holon excitations of the Kuramoto–Yokoyama model in Chapter 2. The Haldane–Shastry Model, *i.e.,* the KYM at half filling, offers the opportunity of studying spinons, *i.e.,* the elementary excitations of one-dimensional spin chains, on the level of explicit and analytical expressions for one and two-spinon wave functions [14]. In similar ways, we set up the technical apparatus for the discussion of holon excitations at the same level of explicit wave functions. As the first main result of this thesis, considering the two-holon eigenstates, we find that the holons obey half-Fermi statistics, which manifests itself in fractional momentum shifts of the single holon momenta. Half-Fermi statistics in one dimenssion corresponds to a statistical parameter of 1/2 in terms of the Haldane state counting principle [58]. In general, the statistical parameter for a certain type of excitations is defined as $g = p/q$, where q denotes the number of excitations added to the system and p the number of orbitals fully occupied due to this addition. Accordingly, bosons have a statistical parameter $g_{\mathrm{B}} = 0$, since any orbital can contain arbitrarily many bosons, whereas fermions have $g_{\mathrm{F}} = 1$, since one orbital can only be occupied by one fermion due to Pauli principle. While the half-Fermi statistics of spinons can be easily seen from the above reasoning [58], the situation is more subtle for the holons, which thus renders our finding important to establish the view of holons being half-fermions.

This discussion is generalized to general SU(n) spin symmetry in Chapter 3. We use a similar analytical approach which was set up in the previous Chapter 2. Again, the technical problems associated with the treatment of the holon states are the operators by which the Hilbert space of the fractional excitations is constructed – whereas one makes use of bosonic spin flip operators for the spinons, one needs fermionic creation and annihilation operators for the holons. The question how the statement on the fractional statistics of the holons now generalizes for the KYM with enlarged spin symmetry group to SU(n) [76] is answered in this chapter. We start from the SU(3) KYM chain and develop the exact one-holon and two-holon wave functions. We again find a manifestation of fractional statistics in the fractional shifts of the single holon momenta, and observe that the SU(3) holons obey third Fermi statistics. In particular, we explicitly show that the general SU(n) holon excitations obey a statistical parameter of $g_{\mathrm{ho}} = 1/n$, as it was shown in previous works that the SU(n) spinons obey a statistical parameter of $g_{\mathrm{sp}} = n - 1/n$ in terms of the Haldane state counting principle [58]. This provides an interesting view on spin charge separation in SU(n) spin chains, which is the next major result of the thesis. We see that the decay of a fermionic excitation into a holon and spinon emerges as a conservation of statistical parameters: The sum of the spinon and holon parameter yields 1, *i.e.,* the fermionic parameter, for general SU(n) spin symmetry.

In Chapter 4, we extend our discussion to spinon excitations in two dimensions, and start by considering the Chiral Spin Liquid (CSL) state. We first discuss the general properties of this paradigmatic state for two-dimensional spinons and, as a main result of the thesis, develop a microscopic model whose unique and exact ground state is the CSL. This demands a rather technical analytical approach involving tensor decompositions of operators and a development of a numerical method to treat many-body spin Hamiltonians efficiently with exact diagonalization methods, which we call the Kernel sweeping method. We finally discuss the spectrum of the model and find that this model is highly suited for the first rigorous study of two-dimensional spinons in order to achieve a similar level of understanding for spinons in two dimensions as it is the case in one dimension for the spinons in the Haldane–Shastry model.

For Chapter 5, the preceeding chapter can be viewed as a prelude for the results described therein. As probably the major result of our work on low-dimensional spin liquids, we propose a novel chiral spin liquid state for an $S = 1$ antiferromagnet. Most importantly, the spinon and holon excitations of this state are deconfined, possess fractionalized quantum numbers (*i.e.,* spin 1/2 for the spinons), and obey non-abelian statistics, with the braiding governed by Majorana fermion states. This new kind of excitations, non-Abelian spinons, will be discussed in further detail. The non-Abelian chiral spin liquid state violates time reversal (T) and parity (P), is a spin singlet, can be formulated on any lattice type, and fully respects all the lattice symmetries. The state possesses a 3-fold topological degeneracy on the torus geometry. We provide numerical evidence that the state can be stabilized on the triangular lattice by a local Hamiltonian involving three-spin interactions. This demands a development of a new numerical method to find an optimized Hamiltonian which stabilizes a given wave function as its ground state, which we call Hamiltonian Finder. Finally, we hypothesize that *spinons in spin liquids with spin larger than 1/2 generically obey non-abelian statistics*, but are only deconfined in the chiral spin liquids we introduce and study here.

In Chapter 6, we provide a short survey of developments in experimental realizations of fractional excitations in general, commenting on spinons and holons in one-dimensional spin chains, abelian and non-abelian excitations in the Quantum Hall effect and the Kitaev model, and recent work on two-dimensional spinons in $S = 1/2$ lattice materials. In particular, we try to point out directions of experiments which may promise observation of the analytical findings we present in this thesis.

Finally, we finish the thesis with a conclusion. The main theoretical results are, firstly, a general understanding of fractional statistics of holon excitations in spin chains, secondly, the development of an exact model for the two-dimensional chiral spin liquid, and thirdly, the introduction of a spin liquid with non-Abelian spinon excitations. The main technical accomplishments are a Tensor decomposition scheme to construct exact Hamiltonians, the development of the Kernel sweeping method, and the Hamiltonian Finder method. Beyond ongoing projects, we have also omitted the presentation of our work on orbital currents in cuprate superconductors [48, 122] as well as entanglement entropy of critical SU(n) spin chains [38], in order to focus the thesis on our results for low-dimensional spin liquids.

Chapter 2

Holon excitations in antiferromagnetic spin chains

2.1 Introduction

In this chapter, we study the fractional character of holon excitations in antiferromagnetic spin chains. To address this issue at the level of explicit wave functions, we constrain our attention to the Kuramoto–Yokoyama model [87], which is a generalization of the Haldane–Shastry model [57, 114] to allow for mobile holes, *i.e.,* empty lattice sites, with a hopping parameter that also falls off with $1/r^2$ as a function of the distance. The Kuramoto–Yokoyama Model (KYM) hence contains spin and charge degrees of freedom, and accordingly supports spinon and holon excitations, which carry spin 1/2 but no charge and charge +1 but no spin, respectively. In principle, the KYM allows for a similarly explicit construction of holon wave functions, which so far have only been obtained for states involving a single holon. The reason for this deficit has been of technical nature, related to the commutation relations of the operators used to build the Hilbert space of these fractionally quantized excitations. Whereas for the spinons one can use bosonic spin-flip operators, one needs fermionic creation and annihilation operators for the holons. We address and overcome this technical problem as we construct the explicit wave functions for two-holon excitations of the KYM. In Section 2.2 we review the KYM and its properties. In Section 2.3 and 2.4, we briefly discuss the ground state at half filling and the spinon excitations. We further review the analytic results so far known for the one-holon excitations in Section 2.5 as a preliminary for the construction of the explicit two-holon wave functions to be done in Section 2.6. Therein we derive the exact energies and individual holon momenta, which turn out to be quantized according to

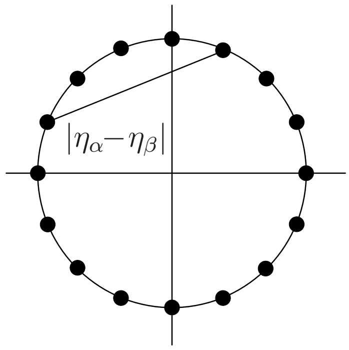

Figure 2.1: Lattice of the Kuramoto–Yokoyama model.

half-Fermi statistics of the holons. The main body of work presented in this chapter is published in

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2.2 Kuramoto–Yokoyama model

The Kuramoto–Yokoyama model [87] is most conveniently formulated by embedding the one-dimensional chain with periodic boundary conditions into the complex plane by mapping it onto the unit circle with the sites located at complex positions $\eta_\alpha = \exp\left(i\frac{2\pi}{N}\alpha\right)$, where N denotes the number of sites and $\alpha = 1, \ldots, N$ (see Fig. 2.1). The sites can be either singly occupied by an up or down-spin electron or empty. The Hamiltonian is given by

$$H_{\mathrm{KY}} = -\frac{2\pi^2}{N^2} \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{N} \frac{P_{\alpha\beta}}{|\eta_\alpha - \eta_\beta|^2}, \tag{2.1}$$

where $P_{\alpha\beta}$ exchanges the configurations on the sites η_α and η_β including a minus sign if both are fermionic. Rewriting (2.1) in terms of spin and electron creation and annihilation operators yields

$$\begin{aligned} H_{\mathrm{KY}} &= \frac{2\pi^2}{N^2} \sum_{\alpha\neq\beta}^{N} \frac{1}{|\eta_\alpha - \eta_\beta|^2} \\ &\quad P_{\mathrm{G}} \left[-\frac{1}{2} \sum_{\sigma=\uparrow\downarrow} \left(c_{\alpha\sigma}^\dagger c_{\beta\sigma} + c_{\beta\sigma}^\dagger c_{\alpha\sigma} \right) + \vec{S}_\alpha \cdot \vec{S}_\beta - \frac{n_\alpha n_\beta}{4} + n_\alpha - \frac{1}{2} \right] P_{\mathrm{G}}, \end{aligned}$$

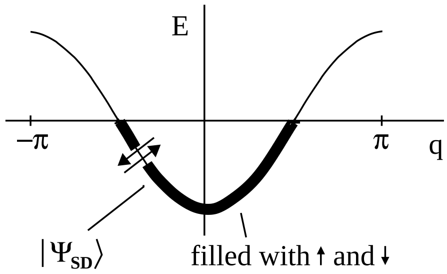


Figure 2.2: N-electron Slater determinant state.

(2.2)

where the Gutzwiller projector $P_{\mathrm{G}} = \prod_\alpha (1 - c^\dagger_{\alpha\uparrow} c^\dagger_{\alpha\downarrow} c_{\alpha\downarrow} c_{\alpha\uparrow})$ enforces at most single occupancy on all sites. The charge occupation and spin operators are given by $n_\alpha = c^\dagger_{\alpha\uparrow} c_{\alpha\uparrow} + c^\dagger_{\alpha\downarrow} c_{\alpha\downarrow}$ and $S^a_\alpha = \frac{1}{2} \sum_{\sigma,\sigma'} c^\dagger_{\alpha\sigma} \tau^a_{\sigma\sigma'} c_{\alpha\sigma'}$, where τ^a, $a = x, y, z$, denote the Pauli matrices.

The interaction strength in (2.1) is an analytic function of the lattice sites by use of

$$\frac{1}{|\eta_\alpha - \eta_\beta|^2} = -\frac{\eta_\alpha \eta_\beta}{(\eta_\alpha - \eta_\beta)^2}. \tag{2.3}$$

The KYM is supersymmetric, *i.e.,* the Hamiltonian (2.1) commutes with the operators $J^{ab} = \sum_\alpha a^\dagger_{\alpha a} a_{\alpha b}$, where $a_{\alpha a}$ denotes the annihilation operator of a particle of species a (a runs over up- and down-spin as well as empty site) at site η_α. The traceless parts of the operators J^{ab} generate the Lie superalgebra su(1|2), which includes in particular the total spin $\vec{S} = \sum_{\alpha=1}^N \vec{S}_\alpha$. In addition, the KYM possesses a super-Yangian symmetry [54], which causes its amenability to rather explicit solution.

2.3 Vacuum state

We first review the ground state at half filling, which is the state containing no excitations (neither spinons nor holons). For N even, this vacuum state is constructed by the Gutzwiller projection of a filled band (or Slater determinant (SD) state) containing a total of N electrons (see Fig. 2.2):

$$|\Psi_0\rangle = P_{\mathrm{G}} \prod_{|q|<q_{\mathrm{F}}} c^\dagger_{q\uparrow} c^\dagger_{q\downarrow} |0\rangle \equiv P_{\mathrm{G}} \left|\Psi^N_{\mathrm{SD}}\right\rangle. \tag{2.4}$$

Taking the fully polarized state $|0_\downarrow\rangle = \prod_\alpha c^\dagger_{\alpha\downarrow}|0\rangle$ as reference state, we can rewrite the vacuum state as

$$|\Psi_0\rangle = \sum_{\{z_i\}} \Psi_0(z_1, \ldots, z_M)\, S^+_{z_1} \ldots S^+_{z_M} |0_\downarrow\rangle\,, \tag{2.5}$$

where $M = N/2$ and the z_i's denote the up-spin coordinates. The sum in (2.5) extends over all possible ways to distribute the coordinates z_i's over the lattice sites η_α. The wave function is given by [57, 114]

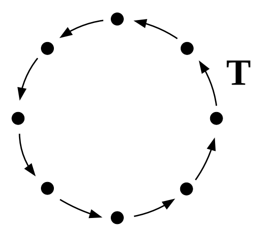
$$\Psi_0(z_1, \ldots, z_M) = \prod_{i<j}^{M} (z_i - z_j)^2 \prod_{i=1}^{M} z_i, \tag{2.6}$$

its energy is

$$E_0 = -\frac{\pi^2}{4N}. \tag{2.7}$$

The total momentum of a state is evaluated by considering the operator $\mathbf{T}$, which translates all arguments of the wave function counterclockwise by one site. $\mathbf{T}$ is related to the momentum operator $\mathbf{P}$ via

$$\mathbf{T} = \exp(\mathrm{i}\mathbf{P}). \tag{2.8}$$



This yields the momentum of $|\Psi_0\rangle$ to equal zero if N is divisible by four and π otherwise. Note that (2.6) represents the ground state of (2.1) only at half filling, *i.e.,* when all sites are occupied. As was shown by Kuramoto and Yokoyama [87], the ground state away from half-filling can be constructed by Gutzwiller projection similar to (2.4).

2.4 Spinon excitations

Let N be odd and $M = (N-1)/2$. A localized spinon at site "η_γ" is constructed by the Gutzwiller projection of an electron inserted in a Slater determinant state of $N+1$ electrons:

$$\left|\Psi^{\mathrm{sp}}_\gamma\right\rangle = P_{\mathrm{G}}\, c_{\gamma\downarrow} \left|\Psi^{N+1}_{\mathrm{SD}}\right\rangle. \tag{2.9}$$

The annihilation of the electron causes an inhomogeneity in the spin and charge degree of freedom. After the projection, however, only the inhomogeneity in the spin survives. The spinon hence possesses spin one-half but

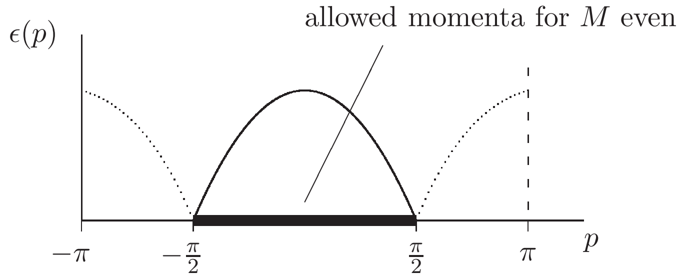


Figure 2.3: One-spinon dispersion relation. The allowed momenta fill the interval $[-\pi/2, \pi/2]$ for $M = (N-1)/2$ even and $[\pi/2, 3\pi/2]$ for M odd.

no charge. The wave function of a localized spinon is given by [59]

$$\Psi_\gamma^{\rm sp}(z_1, \ldots, z_M) = \prod_{i=1}^{M} (\eta_\gamma - z_i)\, \Psi_0(z_1, \ldots, z_M), \tag{2.10}$$

where Ψ_0 is defined in (2.5). Fourier transformation yields the momentum eigenstates

$$|\Psi_m^{\rm sp}\rangle = \sum_{\alpha=1}^{N} (\bar{\eta}_\gamma)^m \left|\Psi_\gamma^{\rm sp}\right\rangle, \tag{2.11}$$

which vanish identically unless $0 \leq m \leq M$. In particular, this implies that the localized one-spinon states (2.9) form an overcomplete set. It is hence not possible to interpret the "coordinate" η_γ literally as the position of the spinon. The momentum eigenstates (2.11) are found to be exact energy eigenstates of the KYM, with its energies given by [59]

$$E_m^{\rm sp} = \frac{2\pi^2}{N^2}\left(\frac{N-1}{2} - m\right) m. \tag{2.12}$$

From there, we can define the one-spinon dispersion relation (see Fig. 2.3)

$$\epsilon(p) = \begin{cases} \dfrac{1}{2}\left(\dfrac{\pi^2}{4} - p^2\right) + \dfrac{\pi^2}{8N^2}, & \text{for } M \text{ even}, \\ \dfrac{1}{2}\left(\dfrac{\pi^2}{4} - (p-\pi)^2\right) + \dfrac{\pi^2}{8N^2}, & \text{for } M \text{ odd}, \end{cases} \tag{2.13}$$

which is linear in p at low energies with velocity $v = \pi/2$. The allowed momenta fill only the inner or outer half of the Brillouin zone, depending on whether $N-1$ is divisible by four or not. This loss of states is reflected in the dynamical spin correlations obtained exactly by Haldane and Zirnbauer [62] and measured in inelastic neutron scattering experiments [27, 119, 120]. The spinons obey half-Fermi statistics, which was first found by the investigation of their state counting rules [58]. The exclusion statistics of spin-polarized spinons can be read off as follows. Consider a chain with $N = 2M + 1$ sites and a single down-spin spinon. If we create the spinon by annihilation of an up-spin electron from a Slater determinant state containing $N + 1$ electrons (2.9), there are according to (2.12) as many single-particle orbitals available to the spinon as there are up-spin electrons in the Slater determinant state, that is, $M + 1$. If we now were to create two additional down-spin spinons, the Slater determinant state would have to contain two more electrons, one of each spin. This implies that there would be one additional orbital, while the two additional spinons would occupy two orbitals, meaning that the number of orbitals available for our original spinon would be reduced by one. The statistical parameter is hence given by $g = 1/2$ [58]. The fractional statistics manifests itself also in the spinon-spinon scattering matrix calculated with the asymptotic Bethe Ansatz. Later it became apparent that the fractional statistics of the spinons manifests itself in the quantization rules for the individual spinon momenta as well [46].

2.5 One-Holon excitations

The charged elementary excitations of the model are holons, the concept of which must be invoked whenever holes and thereby charge carries are doped into the chain. A localized holon at lattice site η_ξ is constructed as

$$\left|\Psi_\xi^{\mathrm{ho}}\right\rangle = c_{\xi\downarrow} P_{\mathrm{G}}\, c_{\xi\downarrow}^\dagger \left|\Psi_{\mathrm{SD}}^{N-1}\right\rangle . \tag{2.14}$$

(Alternatively we could use the operators $c_{\xi\uparrow}$ and $c_{\xi\uparrow}^\dagger$.) Compared to the spinon we eliminate the inhomogeneity in spin while creating an inhomogeneity in the charge distribution after Gutzwiller projection. Thus the holon has no spin but charge $e > 0$ (as the electron charge at site η_ξ is removed). Note that the holon is strictly localized at the holon coordinate ξ, as holon states on neighboring coordinates are orthogonal. In total, there are N independent one-holon states (2.14).

Momentum eigenstates are constructed from (2.14) by Fourier transformation. It turns out that only $(N+3)/2$ of them are energy eigenstates [15]. We will restrict ourselves to this subset in the following. These states are readily described in terms of their wave functions. We take $|0_\downarrow\rangle$ as reference state, and write the one-holon energy eigenstates as [15]

$$\left|\Psi_m^{\text{ho}}\right\rangle = \sum_{\{z_i;h\}} \Psi_m^{\text{ho}}(z_1,\ldots,z_M;h)\, c_{h\downarrow}\, S_{z_1}^+ \ldots S_{z_M}^+ \left|0_\downarrow\right\rangle, \tag{2.15}$$

where the sum extends over all possible ways to distribute the up-spin coordinates z_i and the holon coordinate h over the lattice sites η_α subject to the restriction $z_i \neq h$. The integer m is restricted to $0 \leq m \leq M+1$, where $M = (N-1)/2$ is the number of up-spin coordinates. The one-holon wave function is given by

$$\Psi_m^{\text{ho}}(z_1,\ldots,z_M;h) = h^m \prod_{i=1}^{M}(h - z_i)\, \Psi_0(z_1,\ldots,z_M), \tag{2.16}$$

where Ψ_0 is given by (2.6). Note that as a sum over the coordinates h is included in (2.15), no such sum is required in (2.16). It can be shown that the wave function (2.16) represents an exact energy eigenstate with energy [15]

$$E_m = \frac{2\pi^2}{N^2}\left(m - \frac{N+1}{2}\right) m. \tag{2.17}$$

The one-holon momentum is derived in analogy to the vacuum state to be

$$p_m^{\text{ho}} = \frac{\pi}{2}N + \frac{2\pi}{N}\left(m - \frac{1}{4}\right) \mod 2\pi. \tag{2.18}$$

If we introduce the one-holon dispersion

$$\epsilon^{\text{ho}}(p) = -\frac{1}{2}\left(\frac{\pi^2}{4} - p^2\right) - \frac{\pi^2}{8N^2}, \quad -\frac{\pi}{2} \leq p \leq \frac{\pi}{2}, \tag{2.19}$$

we can rewrite (2.17) with the vacuum energy (2.7) as

$$E_m = E_0 + \epsilon^{\text{ho}}(p_m^{\text{ho}}). \tag{2.20}$$

Opposed to the spinon case, it is less for obvious that the holons obey fractional statistics. While the state counting argument still applies since the Brillouin zone of allowed holon momenta is halved similar to the spinon case, there is no intuitive plausibilization as for the spinon, which was given previously. It is thus even more important to study the fractional character directly by consideration of many-holon states, which we do in the following.

2.6 Two-Holon excitations

2.6.1 Momentum eigenstates

Let N be even and $M = (N-2)/2$. The two-holon state with holons localized at η_{ξ_1} and η_{ξ_2} is constructed in analogy to (2.14) as

$$\left|\Psi^{\mathrm{ho}}_{\xi_1\xi_2}\right\rangle = c_{\xi_1\downarrow}c_{\xi_2\downarrow}P_{\mathrm{G}}\, c^{\dagger}_{\xi_1\downarrow}c^{\dagger}_{\xi_2\downarrow}\left|\Psi^{N-2}_{\mathrm{SD}}\right\rangle . \tag{2.21}$$

In analogy to (2.16), a momentum basis for the two-holon eigenstates is provided by the wave functions

$$\begin{aligned}\Psi^{\mathrm{ho}}_{mn}(z_1,\ldots,z_M;h_1,h_2) \;=\;& (h_1-h_2)(h_1^m h_2^n + h_1^n h_2^m)\\ & \prod_{i=1}^{M}(h_1-z_i)(h_2-z_i)\Psi_0(z_1,\ldots,z_M),\end{aligned} \tag{2.22}$$

where Ψ_0 is again given by (2.6), $h_{1,2}$ denote the holon coordinates, and the integers m and n satisfy

$$0 \le n \le m \le M+1. \tag{2.23}$$

The corresponding state is then given by

$$\left|\Psi^{\mathrm{ho}}_{mn}\right\rangle = \sum_{\{z_i;h_1,h_2\}} \Psi^{\mathrm{ho}}_{mn}(z_1,\ldots,z_M;h_1,h_2)\, c_{h_1\downarrow}\, c_{h_2\downarrow}\, S^+_{z_1}\ldots S^+_{z_M}\left|0_{\downarrow}\right\rangle , \tag{2.24}$$

where the sum extends over all possible ways to distribute the up-spin coordinates z_i and the holon coordinates $h_{1,2}$ over the lattice sites η_α subject to the restriction $z_i \ne h_1 \ne h_2$. The momentum of the states (2.24) is easily found to be

$$p^{\mathrm{ho}}_{mn} = \frac{\pi}{2}N + \frac{2\pi}{N}\big(m+n\big) \mod 2\pi. \tag{2.25}$$

It can further be shown that the states (2.24) are spin singlets, *i.e.*, they are annihilated by $S^{\pm}$ as well as S^z.

2.6.2 Energy eigenstates

Due to the trigonal structure of the Hamiltonian when acting on the Ψ^{ho}_{mn}'s we can derive the energy eigenstates using the Ansatz

$$\left|\Phi^{\mathrm{ho}}_{mn}\right\rangle = \sum_{l=0}^{[\frac{m-n}{2}]} a^{mn}_l \left|\Psi^{\mathrm{ho}}_{m-l\,n+l}\right\rangle , \tag{2.26}$$

which yields the recursion relation

$$a_l^{mn} = -\frac{1}{2l(l-\frac{1}{2}+n-m)} \sum_{k=0}^{l-1} a_k^{mn}(m-n-2k), \quad a_0^{mn} = 1. \tag{2.27}$$

This defines the two-holon energy eigenstates (2.26). The energies are given by

$$E_{mn}^{\text{ho}} = E_0 + \frac{2\pi^2}{N^2}\left[\left(m-\frac{N}{2}\right)m + \left(n-\frac{N}{2}\right)n + \frac{m-n}{2}\right], \tag{2.28}$$

where the momentum quantum numbers satisfy

$$0 \leq n \leq m \leq \frac{N}{2}, \tag{2.29}$$

and the total momentum is given by (2.25).

For the lowest energy state, (2.28) simplifies (up to an additive constant $\pi^2/12N$) to the ground-state energy of the chain doped with two holes, which is a special case of the result by Kuramoto and Yokoyama [87] for the ground state at general filling fraction.

2.7 Statistical parameter

Fractional statistics in one-dimensional systems was originally introduced by Haldane [58] in the context of non-trivial state counting rules. Recently, it was realized by Greiter and Schuricht [46] that the fractional statistics of spinons in the HSM manifests itself also in specific quantization rules for the individual spinon momenta. We will now apply this line of argument to the holon excitations in the KYM.

In this context, we rewrite the two-holon energy (2.28) as

$$E_{mn}^{\text{ho}} = E_0 + \epsilon^{\text{ho}}(p_m) + \epsilon^{\text{ho}}(p_n), \tag{2.30}$$

where we have used the one-holon dispersion (2.19) and introduced the single-holon momenta in (2.30) according to

$$p_m = -\frac{\pi}{2} + \frac{2\pi}{N}\left(m+\frac{1}{4}\right), \qquad p_n = -\frac{\pi}{2} + \frac{2\pi}{N}\left(n-\frac{1}{4}\right). \tag{2.31}$$

The difference in the individual holon momenta is hence given by

$$p_m - p_n = \frac{2\pi}{N}\left(\frac{1}{2} + \text{integer}\right). \tag{2.32}$$

This result is a direct manifestation of the half-Fermi statistics of the holons, as (2.32) is the obvious generalization of the familiar cases of bosons and fermions. Indications of the half-Fermi statistics of the holons have previously been observed in thermodynamic quantities [73, 86] of the KYM as well as the electron addition spectrum [5, 6].

Let us now elaborate on the general implications of this result for holons excitations in antiferromagnetic spin chains. The wave functions we have obtained above are of course eigenstates of the KYM only, which is as idealized as integrable and exactly soluble models tend to be. The quantization rules for the single particle momenta we have obtained for this model, however, have a much broader validity. As mentioned above, the unique feature of the KYM is that the holons are free in the sense that they only interact through their fractional statistics. The single particle momenta of the holons are hence good quantum numbers, which assume fractionally spaced values. For two holons, these are given by (2.29). The crucial observation in this context is that the statistics of the holons is a quantum invariant and as such independent of the details of the model. This implies directly that the fractional spacings are of universal validity as well. If we were to supplement the model we have studied by a potential interaction between the holons, say a Coulomb potential, this interaction would introduce scattering matrix elements between the exact eigenstates we obtained and labeled according to their fractionally spaced single particle momenta. These momenta would hence no longer constitute good quantum numbers. The new eigenstates would be superpositions of states with different single particle momenta, which individually, however, would still possess the fractionally shifted values. In other words, looking at the quantization condition (2.32), the "1/2" on the left of the equation will still be a good quantum number, while the "integer" will turn into a "superposition of integers" in the presence of an interaction between the holons. This line of argument only applies if the quantum numbers describing the holon excitations remain the same, which is why we constrain this argument to antiferromagnetic spin chains [43]. For the XX chain, for example, which can be directly mapped to a free fermion model, no fractionalization occurs and the elementary excitation quantum numbers are fermionic.

2.8 Summary and Outlook

In this chapter, we have studied the two-holon states of the Kuramoto–Yokoyama model. We constructed the explicit two-holon wave functions and derived their momenta and energies. The results display the half-Fermi statistics of the holons, which manifests itself in a shift of $\frac{1}{2}\frac{2\pi}{N}$ in the difference of the individual holon momenta, which appears to be valid beyond the KYM point, given that the quantum numbers of the excitations remain the same in the sense of spin charge separation into spinon and holon degrees of freedom. The crucial task is to provide a measure to observe these statements in experiments, which will be further elaborated on in the following chapters. Additionally, two directions appear promising to us. The first is to use the Kuramoto–Yokoyama model to study the real time evolution of spin charge separation on an analytical footing. In principle, one can start off with the matrix elements of a fermionic annihilation operator acting on the ground state to the one-spinon one-holon eigenstates, and compute local time–dependent quantities like S_i^z or $1 - n_i$, which would relate to a recent approach recently used for describing a sudden change of interaction parameters for the Falicov-Kimball model [32]. A second direction of consideration of us is the explicit computation of a momentum distribution function for fractional excitations. From this perspective, the Haldane–Shastry model, *i.e.,* the KYM at half filling, can be viewed as the free spinon gas, where the spinon only interact through their statistics. By computing this distribution function, it may be possible to constitute a path for identifying experimentally accessible evidence of non-fermionic deviations that can be traced back to the fractional character of the excitations. There, we can make use of a previous work by Greiter and Schuricht [47], where they introduced a Young Tableaux basis of the HSM in terms of many spinon states.

Chapter 3

Holon excitations in SU(n) spin chains

3.1 Introduction

In the previous chapter, we set up an analytical approach for the Haldane–Shastry chain doped with holes, *i.e.,* the Kuramoto–Yokoyama model [87], which also allowed to treat the two-holon excitations, *i.e.,* the elementary charge excitations of a one-dimensional spin chain, at the level of explicit wave functions, which leads to a new understanding of fractional statistics in one dimension in terms of the momentum difference of the fractional excitations extending beyond the exact model where it is solved. The technical problems associated with the treatment of the two-holon states are the operators by which the Hilbert space of the fractional excitations is constructed – whereas one makes use of bosonic spin flip operators for the spinons, one needs fermionic creation and annihilation operators for the holons [123]. Subsequently after the discovery of the HSM in 1988, Kawakami [74] generalized it from SU(2) spins to SU(n), a model in which the spinon excitations obey fractional statistics with statistical parameter $(1 - 1/n)$ $[19, 73, 86, 108, 112, 132, 133]$. The question how this statement for the holons now generalizes for the KYM with enlarged spin symmetry group SU(n) [76] is answered in this chapter. For this, however, no previous work exist on any treatment of holon excitations on an analytical footing.

We analyze the one-holon and two-holon excitations of the SU(n) KYM at the level of explicit wave functions. In Section 3.2 we discuss the SU(3) KYM. We first present the basic properties of the model and review its ground state at half filling where it maps on to the SU(3) HSM, as well as the spinon excitations (colorons). Then we derive the one-holon and two-holon wave functions as well as its exact energies and individual holon momenta. In Section 3.3 we generalize the results to the case of SU(n) and, after reviewing basic properties of the ground state and the SU(n) spinon excitations, derive the one-holon and two-holon wave functions including its energies and holon momenta. This analytical work all condenses into the statements in Section 3.4 where we prove that the holons in the SU(n) KYM obey fractional statistics with statistical parameter $g = 1/n$, which is associated with the non-fermionic spacing difference of the individual holon momenta for the two-holon state. In Section 3.5 we finally discuss the holon excitations and its statistics in the framework of spin-charge separation. The main body of work presented in this chapter is published in

R. Thomale, D. Schuricht, and M. Greiter, *Charge excitations in SU(n) spin chains: Exact results for the* $1/r^2$ *model.* Phys. Rev. B **75**, 024405 (2007).

3.2 SU(3) Kuramoto–Yokoyama model

As before, the Kuramoto–Yokoyama model (KYM) [87] is most conveniently formulated by embedding the one-dimensional chain with periodic boundary conditions into the complex plane by mapping it onto the unit circle with the sites located at the complex positions $\eta_\alpha = \exp\left(i\frac{2\pi}{N}\alpha\right)$, where N denotes the number of sites and $\alpha = 1, \ldots, N$. For the SU(3) case, the sites can be either singly occupied by an SU(3) fermion or empty.

The SU(3) particles transform according to the representation **3** (see Fig. 3.1), in analogy to the familiar color SU(3) from quantum chromodynamics we label the particle types by the colors blue (b), red (r), and green (g). In Sec. 3.2.2 we will see that the elementary spin excitations of the model called colorons transform according to the non-equivalent representation $\bar{\mathbf{3}}$, *i.e.,* they possess the complementary colors yellow (y), cyan (c), and magenta (m).

The SU(3) generators at each lattice site are

$$J_\alpha^a = \frac{1}{2}\sum_{\sigma,\tau=1}^{3} c_{\alpha\sigma}^\dagger \lambda_{\sigma\tau}^a c_{\alpha\tau}, \quad a = 1, \ldots, 8, \tag{3.1}$$

where the λ^a are the Gell-Mann matrices. The operator $c_{\alpha\sigma}^\dagger$ ($c_{\alpha\sigma}$) creates (annihilates) a fermion with color σ at lattice site η_α. The operators (3.1) satisfy the commutation relations

$$\left[J_\alpha^a, J_\beta^b\right] = \delta_{\alpha\beta}\, f^{abc} J_\alpha^c, \quad a, b, c = 1, \ldots, 8, \tag{3.2}$$

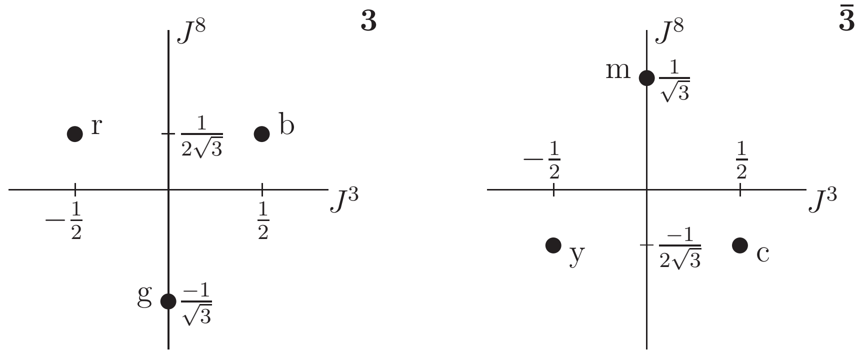


Figure 3.1: Weight diagrams of the three-dimensional representations of SU(3). J^3 and J^8 are the diagonal generators.

(we use the Einstein summation convention) with f^{abc} the structure constants of SU(3), *i.e.,* spin operators on different sites commute.

The Hamiltonian is given by

$$H_{\mathrm{KY}} = -\frac{2\pi^2}{N^2} \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{N} \frac{P_{\alpha\beta}}{|\eta_\alpha - \eta_\beta|^2}, \tag{3.3}$$

where $P_{\alpha\beta}$ exchanges the configurations on the sites η_α and η_β including a minus sign if both are fermionic. Rewriting (3.3) in terms of spin and fermion creation and annihilation operators yields

$$\begin{aligned} H_{\mathrm{KY}} \;&=\; \frac{2\pi^2}{N^2} \sum_{\alpha\neq\beta}^{N} \frac{1}{|\eta_\alpha - \eta_\beta|^2} \\ &\quad P_{\mathrm{G}} \left[-\frac{1}{2} \sum_{\sigma=b,r,g} \left(c^\dagger_{\alpha\sigma} c_{\beta\sigma} + c^\dagger_{\beta\sigma} c_{\alpha\sigma} \right) + \boldsymbol{J}_\alpha \cdot \boldsymbol{J}_\beta - \frac{n_\alpha n_\beta}{3} + n_\alpha - \frac{1}{2} \right] P_{\mathrm{G}}, \end{aligned} \tag{3.4}$$

where we for convenience label the SU(3) spin or color index σ to be either blue (b), red (r), or green (g). The Gutzwiller projector which enforces at most single occupancy on all sites is given by

$$P_{\mathrm{G}} = \prod_{\alpha=1}^{N} \big(n_\alpha - 2\big)\big(n_\alpha - 3\big), \quad n_\alpha \equiv c^\dagger_{\alpha\mathrm{b}} c_{\alpha\mathrm{b}} + c^\dagger_{\alpha\mathrm{r}} c_{\alpha\mathrm{r}} + c^\dagger_{\alpha\mathrm{g}} c_{\alpha\mathrm{g}}, \tag{3.5}$$

where n_α denotes the charge occupation operator. $\boldsymbol{J}_\alpha = \frac{1}{2}\sum_{\sigma\tau} c^\dagger_{\alpha\sigma} \boldsymbol{\lambda}_{\sigma\tau} c_{\alpha\tau}$ is the eight-dimensional SU(3) spin vector, $\boldsymbol{\lambda}$ a vector consisting of the eight

Gell-Mann matrices, and σ and τ are again SU(3) color indices. For all practical purposes, it is convenient to express the spin operator part of (3.4) in terms of color flip operators $e_\alpha^{\sigma\tau} \equiv c_{\alpha\sigma}^\dagger c_{\alpha\tau}$:

$$\begin{aligned} H_{\mathrm{KY}} &= \frac{2\pi^2}{N^2} \sum_{\alpha\neq\beta}^{N} \frac{1}{|\eta_\alpha - \eta_\beta|^2} \\ & P_{\mathrm{G}} \left[-\frac{1}{2} \sum_{\sigma}^{3} \left(c_{\alpha\sigma}^\dagger c_{\beta\sigma} + c_{\beta\sigma}^\dagger c_{\alpha\sigma} \right) + \frac{1}{2} \sum_{\sigma\tau}^{3} e_\alpha^{\sigma\tau} e_\beta^{\tau\sigma} - \frac{n_\alpha n_\beta}{2} + n_\alpha - \frac{1}{2} \right] P_{\mathrm{G}}, \end{aligned} \tag{3.6}$$

where the color double sum includes terms with $\sigma = \tau$.

The KYM is supersymmetric, *i.e.,* the Hamiltonian (3.3) commutes with the operators $J^{ab} = \sum_\alpha a_{\alpha a}^\dagger a_{\alpha b}$, where $a_{\alpha a}$ denotes the annihilation operator of a particle of species a (a runs over color indices as well as empty site) at site η_α. The traceless parts of the operators J^{ab} generate the Lie super-algebra su(1|3), which includes in particular the total spin operators $\boldsymbol{J} = \sum_{\alpha=1}^N \boldsymbol{J}_\alpha$. In addition, the KYM possesses a super-Yangian symmetry [53, 54], which causes its amenability to rather explicit solution.

3.2.1 Vacuum state

We first review the state containing no excitations, *i.e.,* neither spinons nor holons, which is the ground state at half filling where the SU(3) Kuramoto–Yokoyama model reduces to the SU(3) Haldane–Shastry model [111]. The ground state of H_{KY} for $N = 3M$ (M integer) is most easily formulated by Gutzwiller projection of a filled band (or Slater determinant (SD) state) containing a total of N SU(3) particles obeying Fermi statistics

$$|\Psi_0\rangle = P_{\mathrm{G}} \prod_{|q|\leq q_{\mathrm{F}}} c_{q\mathrm{b}}^\dagger c_{q\mathrm{r}}^\dagger c_{q\mathrm{g}}^\dagger |0\rangle \equiv P_{\mathrm{G}} \left|\Psi_{\mathrm{SD}}^N\right\rangle. \tag{3.7}$$

As $\left|\Psi_{\mathrm{SD}}^N\right\rangle$ is an SU(3) singlet by construction and P_{G} commutes with SU(3) rotations, $|\Psi_0\rangle$ is an SU(3) singlet as well.

If one interprets the state $|0_{\mathrm{g}}\rangle \equiv \prod_{\alpha=1}^N c_{\alpha\mathrm{g}}^\dagger \,|\, 0 \,\rangle$ as a reference state and the color flip operators e^{bg} and e^{rg} as "particle creation operators", the ground state (3.7) can be rewritten as [52, 75]

$$|\Psi_0\rangle = \sum_{\{z_i, w_k\}} \Psi_0[z_i; w_k]\, e_{z_1}^{\mathrm{bg}} \dots e_{z_{M_1}}^{\mathrm{bg}} e_{w_1}^{\mathrm{rg}} \dots e_{w_{M_2}}^{\mathrm{rg}} |0_{\mathrm{g}}\rangle, \tag{3.8}$$

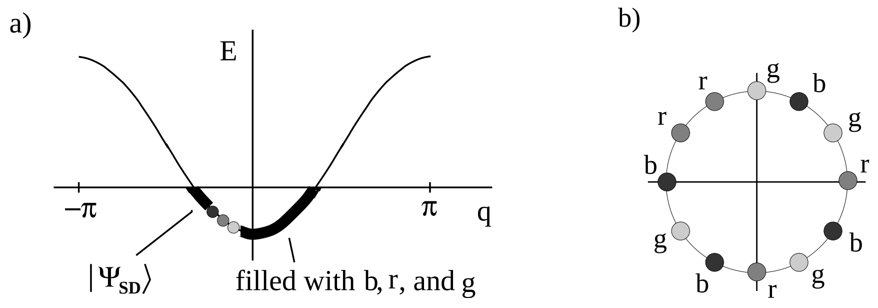


Figure 3.2: a) Total antisymmetric N-particle state. b) Typical configuration in $|\Psi_0\rangle$.

where the sum extends over all possible ways to distribute the positions of the blue particles $z_1, \ldots, z_{M_1}$ and red particles $w_1, \ldots, w_{M_2}$ over the N sites. The ground state wave function is given by

$$\Psi_0[z_i; w_k] \equiv \prod_{i<j}^{M_1}(z_i - z_j)^2 \prod_{k<l}^{M_2}(w_k - w_l)^2 \prod_{i=1}^{M_1}\prod_{k=1}^{M_2}(z_i - w_k) \prod_{i=1}^{M_1} z_i \prod_{k=1}^{M_2} w_k \quad (3.9)$$

with $M_1 = M_2 = M$. The ground state energy is

$$E_0 = -\frac{\pi^2}{36}\left(N + \frac{15}{N}\right). \quad (3.10)$$

The total momentum, as defined through $e^{ip} = \Psi_0[\eta_1 z_i, \eta_1 w_k]/\Psi_0[z_i, w_k]$ with $\eta_1 = \exp(i\frac{2\pi}{N})$, is $p = 0$ regardless of M. For further purposes, it is important to note that the ground state wave function can be equally expressed by any two sets of color variables, as shown in Appendix A.5.

3.2.2 Coloron excitations

Assume $N = 3M+1$. It is shown in [111] that the elementary spin excitations called colorons obey the adjoint representation of the SU(3) lattice spins. Accordingly, a localized coloron at site "η_γ" is constructed by Gutzwiller projection of a fermion annihilation operator of color σ acting on a Slater determinant state of $N+1$ fermions:

$$\left|\Psi^{\mathrm{c}}_{\gamma\bar{\sigma}}\right\rangle = P_{\mathrm{G}} c_{\gamma\sigma} \left|\Psi^{N+1}_{\mathrm{SD}}\right\rangle, \quad (3.11)$$

where $\bar{\sigma}$ shall denote the adjoint or complementary color of the excitation. The annihilation of the electron causes an inhomogeneity in the spin and

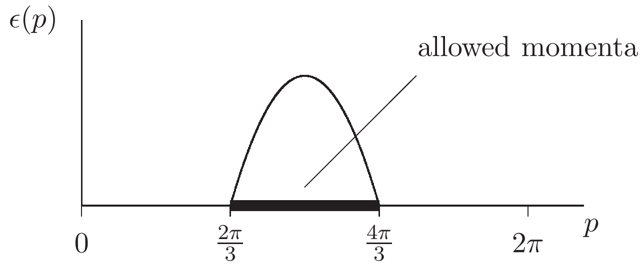


Figure 3.3: One-coloron dispersion relation. The colorons are constrained to one third of the Brillouin zone.

charge degree of freedom. After the projection, however, only the inhomogeneity in the spin survives, the coloron thus possesses a (complementary) color but no charge. The wave function of a localized, e.g. anti-blue or yellow, coloron is given by [111]

$$\Psi_\gamma^{\mathrm{c}}[z_i; w_k] = \prod_{i=1}^{M_1} (\eta_\gamma - z_i)\Psi_0[z_i; w_k], \tag{3.12}$$

with Ψ_0 given by (3.9). Fourier transformation yields the momentum eigenstates

$$|\Psi_n^{\mathrm{c}}\rangle = \frac{1}{N}\sum_{\gamma=1}^{N} (\bar{\eta}_\gamma)^n \left|\Psi_\gamma^{\mathrm{c}}\right\rangle, \tag{3.13}$$

which identically vanish unless $0 \leq n \leq M_1$. The momentum of (3.13) is

$$p_n = \frac{4\pi}{3} - \frac{2\pi}{N}\left(n + \frac{1}{3}\right), \; 0 \leq n \leq M_1. \tag{3.14}$$

It thus yields that the localized one-coloron states (3.11) form an overcomplete set, by which it follows that the interpretation of "η_γ" as the position of the coloron does not exactly fit. The momentum eigenstates (3.13) are found to be exact energy eigenstates of the KYM, its energies are given by [111]

$$E^{\mathrm{c}} = E_0 + \frac{2}{9}\frac{\pi^2}{N^2} + \epsilon^{\mathrm{c}}(p_n), \tag{3.15}$$

where the coloron dispersion is given by

$$\epsilon^{\mathrm{c}}(p_n) = \frac{3}{4}\left(\frac{\pi^2}{9} - (p_n - \pi)^2\right). \tag{3.16}$$

The colorons obey fractional statistics with the statistical parameter 2/3, which from the first view is obvious from state counting rules originally formulated for the SU(2) case [58]. Furthermore, the fractional statistics of the colorons show up in the quantization rules for the individual coloron momenta [46].

As stated before, the elementary coloron excitations are constructed by annihilation of a particle of color σ from an overall color singlet $\left|\Psi_{\mathrm{SD}}^{N+1}\right\rangle$ before Gutzwiller projection. Hence (as hole-like excitations) they transform according to the representation $\bar{\mathbf{3}}$ conjugate to the fundamental representation $\mathbf{3}$ of the original particles on the sites of the chain. This can also be seen by acting with the total SU(3) spin generators on the wave function (3.12). This is consistent with results on the spectrum of the $\mathrm{SU}(3)_1$ Wess–Zumino–Witten model obtained by Bouwknegt and Schoutens [18, 108].

It is straightforward to read off the exclusion statistics [58] of color-polarized colorons. Consider a chain with $N = 3M - 1$ sites and a single yellow coloron. According to (3.11) and (3.13) there are as many single-particle orbitals available to the coloron as there are blue particles in the Slater determinant state, that is, M. If we now were to create three additional yellow colorons, the Slater determinant state would have to contain three more particles, one of each color. This implies that there would be one additional orbital, while the three additional colorons would occupy three orbitals, meaning that the number of orbitals available for our original coloron would be reduced by two. The statistical parameter is hence given by $g = 2/3$. The fractional statistics manifests itself further in the exponents of the algebraic decay of the dynamical structure factor [132, 133], in the thermodynamics of the model [73, 86], as well as the coloron-coloron scattering matrix calculated with the asymptotic Bethe Ansatz [118]. A similar exclusion statistics exists in the conformal field theory spectrum of Wess–Zumino–Witten models [19, 108].

3.2.3 One-Holon excitations

Let N be $N = 3M+1$. Doping the SU(3) fermion chain with holes causes the appearance of holons, *i.e.,* the elementary charge excitations of the system. Opposed to the case of the spinons, setting up the construction scheme for the SU(3) chain is unambiguous and built up in a straight forward way starting from the case of SU(2) [123]. A localized holon state at lattice site "η_ξ" is constructed as

$$\left|\Psi_\xi^{\mathrm{ho}}\right\rangle = c_{\xi\sigma} P_{\mathrm{G}} c_{\xi\sigma}^\dagger \left|\Psi_{\mathrm{SD}}^{N-1}\right\rangle, \tag{3.17}$$

where the color index σ can be chosen arbitrarily. Opposed to the coloron we eliminate the inhomogeneity in color, which exists after Gutzwiller projection, but instead create an inhomogeneity in the charge distribution. Therefore, the state has no color but charge $e > 0$, as the electron charge at site "η_ξ" is annihilated. It should be stressed that there is a difference between the quantum-like "quasi"-coordinate γ of the localized one-coloron wave functions (3.12) and the holon coordinate ξ being a real coordinate of the system. In particular, there exist N independent states (3.17) in total.

Momentum eigenstates are constructed from (3.17) by Fourier transformation. It will be shown in the following that only a subset of them are energy eigenstates, thus being holon excitations, which are most easily described in terms of their wave functions. As for the ground state, we again take $|0_\mathrm{g}\rangle \equiv \prod_{\alpha=1}^{N} c_{\alpha\mathrm{g}}\,|0\rangle$ as a reference state and the color flip operators e^{bg} and e^{rg} as "particle creation operators"

$$\left|\Psi_m^{\mathrm{ho}}\right\rangle = \sum_{z_i,w_k;h} \Psi_m^{\mathrm{ho}}[z_i;w_k;h] c_{h\mathrm{g}} e_{z_1}^{\mathrm{bg}} \dots e_{z_{M_1}}^{\mathrm{bg}} e_{w_1}^{\mathrm{rg}} \dots e_{w_{M_2}}^{\mathrm{rg}} \left|0_\mathrm{g}\right\rangle, \tag{3.18}$$

where the sum extends over all possible ways to distribute the positions of the blue particles $z_1, \dots, z_{M_1}$, the red particles $w_1, \dots, w_{M_2}$, and the holon coordinate h over the N sites under the restriction $h \neq z_i, w_k$. The one-holon wave function is given by

$$\Psi_m^{\mathrm{ho}}[z_i;w_k;h] = h^m \prod_{i=1}^{M_1}(h - z_i) \prod_{k=1}^{M_2}(h - \omega_k)\Psi_0[z_i;w_k], \tag{3.19}$$

where M_1 denotes the number of blue and M_2 the number of red fermions as above (to increase clarity of the terms appearing the upcoming calculations, we at most stages of the paper keep this difference, though we always set M_1, M_2 and M_3, *i.e.,* the number of green fermions to be equal to M at the end since any excitation which we discuss contains equally many blue, red, and green particles). The integer m is restricted to

$$0 \leq m \leq M + 1, \tag{3.20}$$

as will be verified. Note that the localized states (3.17) taken as a basis of (3.18) yield holon excitations only when energy eigenstates are constructed and cannot strictly be interpreted as space-localized holon states. When we take a Fourier transform of the energy eigenstates to localize the holons, the fact there are only M+2 allowed momentum values while there are N sites implies that we will not be able to localize the holons such that they reside exactly on lattice sites, and hence that the true holon states are non-orthogonal in position space.

The momentum of (3.18) is

$$p_m^{\text{ho}} = \frac{2\pi}{3} + \frac{2\pi}{N}\left(m - \frac{1}{3}\right), \tag{3.21}$$

with m restricted to (3.20). The energy yields

$$E_m^{\text{ho}} = E_0 + \epsilon^{\text{ho}}(p_m^{\text{ho}}), \tag{3.22}$$

with the SU(3) chain holon dispersion given by

$$\epsilon^{\text{ho}}(p_m^{\text{ho}}) = -\frac{3}{4}\left(\frac{\pi^2}{9} - (p_m^{\text{ho}} - \pi)^2\right) - \frac{2}{9}\frac{\pi^2}{N^2}, \quad \frac{2\pi}{3} \leq p_m^{\text{ho}} \leq \frac{4\pi}{3}. \tag{3.23}$$

The SU(3) holons thus occupy one third of the Brioullin zone. We now prove that the states (3.18) restricted to (3.20) are the energy eigenstates of the SU(3) Kuramoto–Yokoyama Hamiltonian (3.4) with the energy (3.22).

3.2.4 Two-Holon excitations

Momentum eigenstates

Let N be $N = 3M + 2$ and M again denote the number of variables of one spin color. In analogy to (3.17), the two-holon state localized at η_{ξ_1} and η_{ξ_2} is constructed as

$$\left|\Psi_{\xi_1\xi_2}^{\text{ho}}\right\rangle = c_{\xi_1\downarrow}c_{\xi_2\downarrow}P_{\text{G}}c_{\xi_1\downarrow}^{\dagger}c_{\xi_2\downarrow}^{\dagger}\left|\Psi_{\text{SD}}^{N-2}\right\rangle. \tag{3.24}$$

The momentum eigenstates which are subject to our further consideration are most easily described by their wave functions. Related to (3.19), the two-holon momentum eigenstates are represented by

$$\begin{aligned}\Psi_{mn}^{\text{ho}}[z_i; w_k; h_1, h_2] \;=\; & (h_1 - h_2)(h_1^m h_2^n + h_1^n h_2^m) \\ & \prod_{i=1}^{M_1}(h_1 - z_i)(h_2 - z_i)\prod_{k=1}^{M_2}(h_1 - w_k)(h_2 - w_k)\Psi_0[z_i; w_k],\end{aligned} \tag{3.25}$$

where Ψ_0 is given by (3.9), $h_{1,2}$ denote the holon coordinates, and the integers m and n satisfy

$$0 \leq n \leq m \leq M + 1, \tag{3.26}$$

as will be shown in the following. The state which is represented by (3.25) is given by

$$\left|\Psi_{mn}^{\text{ho}}\right\rangle = \sum_{z_i, w_k; h_1, h_2} \Psi_{mn}^{\text{ho}}[z_i; w_k; h_1, h_2] c_{h_1\text{g}} c_{h_2\text{g}} e_{z_1}^{\text{bg}} \ldots e_{z_{M_1}}^{\text{bg}} e_{w_1}^{\text{rg}} \ldots e_{w_{M_2}}^{\text{rg}} \left|0_{\text{g}}\right\rangle, \tag{3.27}$$

where the sum extends over all possible ways to distribute the positions of the blue particles $z_1, \ldots, z_{M_1}$, the red particles $w_1, \ldots, w_{M_2}$, and the holon coordinates $h_{1,2}$ over the N sites under the restriction $h_{1,2} \neq z_i, w_k$. Again note that the states (3.24) taken as a basis of (3.27) cannot be interpret as true localized two-holon excitations. As we only get $(M+1)^2$ distinct momentum eigenstates which are associated with energy eigenstates thus being interpreted as true holon states, we cannot define true two-holon states which are strictly localized in position, as already explained for the one-holon case.

The total momentum of the states (3.27) is found to be

$$p_{mn}^{\mathrm{ho}} = \frac{4\pi}{3} + \frac{2\pi}{N}\left(m + n - \frac{1}{3}\right) \bmod 2\pi. \tag{3.28}$$

Furthermore, as the Gutzwiller projector commutes with all generators of SU(3), it can be shown that the states (3.27) are spin singlets, *i.e.*, they are annihilated by all components of the SU(3) vector operator $\boldsymbol{J} \equiv \sum_\alpha \boldsymbol{J}_\alpha$.

We thus obtain

$$\begin{aligned} &H_{\mathrm{KY}} \Psi_{mn}^{\mathrm{ho}}[z_i; w_k; h_1, h_2] \\ &= \frac{3\pi^2}{N^2}\left[\left(m - \frac{N+1}{3}\right) m \left(n - \frac{N+1}{3}\right) n + \frac{m-n}{3}\right] \\ &\quad + \frac{2\pi^2}{N^2}(m-n) \sum_{l=1}^{\lfloor \frac{m-n}{2} \rfloor} \Psi_{m-l,n+l}^{\mathrm{ho}} - \frac{\pi^2}{36}\left(N + \frac{3}{N} + \frac{4}{N^2}\right), \end{aligned} \tag{3.29}$$

where in (3.29) we used $\frac{x+y}{x-y}(x^m y^n - x^n y^m) = 2\sum_{l=0}^{m-n} x^{m-l} y^{n+l} - (x^m y^n + x^n y^m)$ and $\lfloor\ \rfloor$ denotes the floor function, *i.e.*, $\lfloor x \rfloor$ is the largest integer $l \leq x$. First, note that the action of the Hamiltonian on Ψ_{mn}^{ho} is trigonal, *i.e.*, the "scattering" in the last line is only to lower values of $m - n$. Second, (3.29) shows that the states Ψ_{mn}^{ho} form a non-orthogonal set out of which we construct an orthogonal basis of eigenfunctions as it is shown in the following.

Energy eigenstates

As the Hamiltonian matrix is trigonal, it is easy to diagonalize. Using the Ansatz

$$\left|\Phi_{mn}^{\mathrm{ho}}\right\rangle = \sum_{l=0}^{\lfloor \frac{m-n}{2} \rfloor} a_l^{mn} \left|\Psi_{m-l\; n+l}^{\mathrm{ho}}\right\rangle, \tag{3.30}$$

we obtain the recursion relation

$$a_p^{mn} = -\frac{1}{3p(p+m-n-\frac{1}{3})} \sum_{k=0}^{p-1} (n - m - 2k) a_k^{mn}, \quad a_0^{mn} = 1, \tag{3.31}$$

which defines the two-holon energy eigenstates (3.30). The corresponding energies are given by

$$\begin{aligned} E^{\mathrm{ho}}_{mn} &= -\frac{\pi^2}{36}\left(N+\frac{3}{N}+\frac{4}{N^2}\right) \\ &\quad +\frac{3\pi^2}{N^2}\left[\left(m-\frac{N+1}{3}\right)m\left(n-\frac{N+1}{3}\right)n+\frac{m-n}{3}\right], \end{aligned} \tag{3.32}$$

where the momentum quantum numbers satisfy

$$0 \leq n \leq m \leq \frac{N+1}{3}, \tag{3.33}$$

in correspondence to (3.20), and the total momentum is given by (3.28).

3.3 Generalization to SU(n)

The expressions derived for the SU(3) model directly generalize to the SU(n) case. Beyond the bare results, only some short remarks about the calculation are made since the decisive methods were already discussed for the SU(3) case.

3.3.1 Hamiltonian

Consider an under-doped chain with at most one particle per lattice site carrying an internal SU(n) quantum number which transforms according to the fundamental representation $\boldsymbol{n}$ of SU(n). Starting from (3.3), the Hamiltonian can be rewritten by

$$\begin{aligned} H^{\mathrm{SU}(n)}_{\mathrm{KY}} &= \left(\frac{2\pi}{N}\right)^2 \sum_{\alpha<\beta}^{N} \frac{1}{|\eta_\alpha-\eta_\beta|^2} \\ &\quad P_{\mathrm{G}}\left\{\boldsymbol{J}^n_\alpha\cdot\boldsymbol{J}^n_\beta-\frac{1}{2}\sum_\sigma\left(c^\dagger_{\alpha\sigma}c_{\beta\sigma}+c^\dagger_{\beta\sigma}c_{\alpha\sigma}\right)-\frac{n-1}{2n}n_\alpha n_\beta+n_\alpha-\frac{1}{2}\right\}P_{\mathrm{G}}, \end{aligned} \tag{3.34}$$

where the summation index σ runs over all spin indices and $\boldsymbol{J}^n_\alpha$ denotes the (n^2-1)-dimensional SU(n) spin vector. (3.34) commutes with the operators $J^{ab}=\sum_\alpha a^\dagger_{\alpha a}a_{\alpha b}$, where $a_{\alpha a}$ denotes the annihilation operator of a particle of species a (a takes values $0,1,\ldots,n$, *i.e.,* it runs over the hole and all n spin indices) at site η_α, the traceless parts of the operators J^{ab} generate the Lie super-algebra su($1|n$), and still possesses a super-Yangian symmetry [54].

3.3.2 Vacuum state

Consider the state containing no excitations. If we use a polarized state of particles of flavor n as reference state and label the coordinates of the particles of flavor $\sigma, 1 \leq \sigma \leq n-1$, by $z_i^\sigma, 1 \leq i \leq M_\sigma$, the wave functions [75]

$$\Psi_0[z_i^\sigma] = \prod_{\sigma=1}^{n-1} \prod_{i<j}^{M_\sigma} (z_i^\sigma - z_j^\sigma)^2 \prod_{\sigma<\tau}^{n-1} \prod_{i=1}^{M_\sigma} \prod_{j=1}^{M_\tau} (z_i^\sigma - z_j^\tau) \prod_{\sigma=1}^{n-1} \prod_{i=1}^{M_\sigma} z_i^\sigma \tag{3.35}$$

constitute exact eigenstates [52] of the Hamiltonian (3.34). For $N = nM$, $M_\sigma = M$, (3.35) is the ground state of (3.34) with energy

$$E_0^n = -\frac{\pi^2}{12} \left(\frac{n-2}{n} N + \frac{2n-1}{N} \right). \tag{3.36}$$

The total momentum of this ground state is $p = (n-1)\pi M \bmod 2\pi$. Note that $p = 0$ only for n odd, otherwise $p = 0$ or π.

3.3.3 Spinon excitations

Localized SU(n) spinons are given by [111]

$$\Psi_\gamma[z_i^\sigma] = \prod_{i=1}^{M_1} (\eta_\gamma - z_i^1)\ \Psi_0[z_i^\sigma] \tag{3.37}$$

for $N = nM - 1$, $M_1 = M - 1$, and $M_2 = \ldots = M_{n-1} = M$, out of which the momentum eigenstates are constructed via Fourier transformation. They transform according to the representation $\bar{\boldsymbol{n}}$. The spinon momenta are given by

$$p = \frac{n-1}{n} \pi N - \frac{2\pi}{N} \left(m + \frac{n-1}{2n} \right) \mod 2\pi, \tag{3.38}$$

which fill the interval $[-\frac{\pi}{n}, \frac{\pi}{n}]$ for n even and M odd, or the interval $[\pi - \frac{\pi}{n}, \pi + \frac{\pi}{n}]$ otherwise (either n odd or M even or both). The energy spectrum is given by

$$E_n^{\text{sp}}(p) = E_0^n + \frac{n^2-1}{12n} \frac{\pi^2}{N^2} + \epsilon_n^{\text{sp}}(p), \tag{3.39}$$

with

$$\epsilon_n^{\text{sp}}(p) = \begin{cases} \dfrac{n}{4} \left(\dfrac{\pi^2}{n^2} - p^2 \right), & \text{if } n \text{ even and } M \text{ odd,} \\[2ex] \dfrac{n}{4} \left(\dfrac{\pi^2}{n^2} - (p-\pi)^2 \right), & \text{otherwise.} \end{cases} \tag{3.40}$$

3.3.4 One-Holon excitations

For all practical purposes in order to generalize the approach for the SU(3) presented previously, we again rewrite (3.34) in terms of spin flip operators, eliminate the spin diagonal contribution of one spin index and thus get

$$\begin{aligned} H_{\mathrm{KY}}^{\mathrm{SU}(n)} &= \left(\frac{2\pi}{N}\right)^2 \sum_{\alpha<\beta}^{N} \frac{1}{|\eta_\alpha - \eta_\beta|^2} P_{\mathrm{G}} \Bigg\{ \sum_{\sigma=1}^{n} c_{\alpha\sigma} c_{\beta\sigma}^\dagger + h_\alpha \sum_{\sigma=1}^{n-1} e_\beta^{\sigma\sigma} - \sum_{\sigma=1}^{n-1} e_\alpha^{\sigma\sigma} \\ &\quad + \sum_{\sigma=1}^{n-1} e_\alpha^{\sigma\sigma} e_\beta^{\sigma\sigma} + \sum_{\sigma<\tau}^{n-1} e_\alpha^{\sigma\sigma} e_\beta^{\tau\tau} + \sum_{\sigma<\tau}^{n} e_\alpha^{\sigma\tau} e_\beta^{\tau\sigma} + \frac{1}{2} - h_\alpha \Bigg\} P_{\mathrm{G}} \,, \end{aligned} \tag{3.41}$$

where the h's again denote hole operators. It yields that the one-holon momentum eigenstates are represented by the wave function

$$\Psi_l^{\mathrm{ho}}[z^\sigma; h] = h^l \prod_{\sigma=1}^{n-1} \prod_{k=1}^{M_\sigma} (h - z_k^\sigma) \Psi_0[z^\sigma], \tag{3.42}$$

with Ψ_0 given by (3.35) and the restriction of the momentum integer quantum number l to be

$$0 \leq l \leq \frac{N+n-1}{n}. \tag{3.43}$$

For momentum eigenvalues p_l^{ho} of (3.42), it yields

$$p_l^{\mathrm{ho}} = \left\{ \frac{n-1}{n} \pi N + \frac{2\pi}{N} \left(l - \frac{n-1}{2n} \right) \right\} \bmod 2\pi, \tag{3.44}$$

which fill the interval $[-\frac{\pi}{n}, \frac{\pi}{n}]$ for n even and M odd, or the interval $[\pi - \frac{\pi}{n}, \pi + \frac{\pi}{n}]$ otherwise (either n odd or M even or both), see Fig. 3.4. The energy is

$$E_n^{\mathrm{ho}}(p_l^{\mathrm{ho}}) = E_0^n - \frac{n^2-1}{12n} \frac{\pi^2}{N^2} + \epsilon_n^{\mathrm{ho}}(p_l^{\mathrm{ho}}), \tag{3.45}$$

with

$$\epsilon_n^{\mathrm{ho}}(p_l^{\mathrm{ho}}) = \begin{cases} -\dfrac{n}{4} \left(\dfrac{\pi^2}{n^2} - (p_l^{\mathrm{ho}})^2 \right), & \text{if } n \text{ even and } M \text{ odd}, \\[2ex] -\dfrac{n}{4} \left(\dfrac{\pi^2}{n^2} - (p_l^{\mathrm{ho}} - \pi)^2 \right), & \text{otherwise.} \end{cases} \tag{3.46}$$

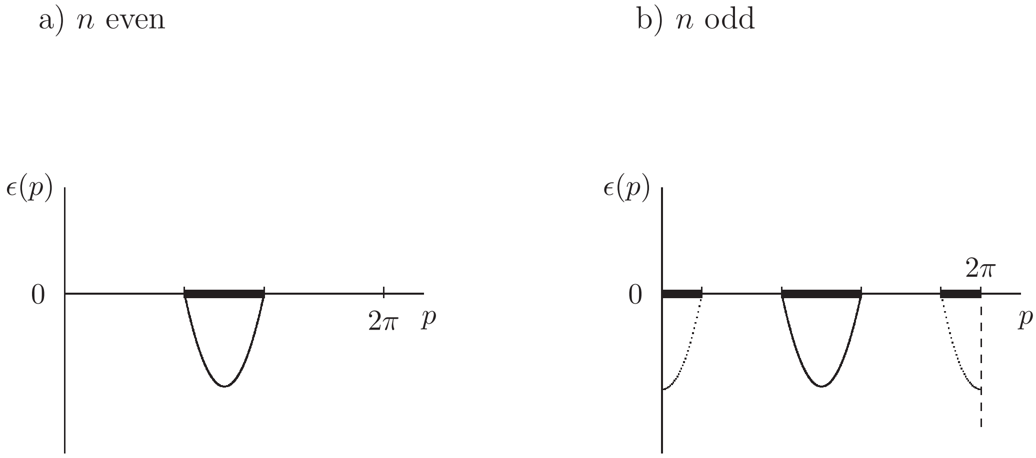


Figure 3.4: SU(n) holon dispersion relations. a) n even. The allowed momenta fill the interval $[\pi - \frac{\pi}{n}, \pi + \frac{\pi}{n}]$ for M even and $[-\frac{\pi}{n}, \frac{\pi}{n}]$ for M odd. b) n odd. The allowed momenta fill the interval $[\pi - \frac{\pi}{n}, \pi + \frac{\pi}{n}]$.

3.3.5 Two-Holon excitations

The two-holon momentum eigenstates are represented by the wave function

$$\Psi^{\mathrm{ho}}_{lm}[z_\sigma; h_1, h_2] = (h_1 - h_2)(h_1^l h_2^m + h_1^m h_2^l) \prod_{\sigma=1}^{n-1} \prod_{k=1}^{M_\sigma} (h_1 - z_k^\sigma)(h_2 - z_k^\sigma) \Psi_0[z_i; w_k], \tag{3.47}$$

where the momentum quantum numbers l and m are restricted to

$$0 \le m \le l \le \frac{N + n - 2}{n}. \tag{3.48}$$

The momentum is given by

$$q^{\mathrm{SU}(n)}_{l,m} = \left\{ \frac{n-1}{n} \pi N + \frac{2\pi}{N} \left(l + m + \frac{2-n}{n} \right) \right\} \bmod 2\pi. \tag{3.49}$$

The momentum eigenstates build an over-complete set out which one constructs the energy eigenstates by the Ansatz

$$\Psi_{lm} = \sum_{p=1}^{\lfloor \frac{l-m}{2} \rfloor} a_p^{lm} \Psi^{\mathrm{ho}}_{l-p,m+p} \,, \tag{3.50}$$

where the recursion relation for the a_p^{lm}'s is found to be

$$a_p^{lm} = \frac{1}{np(m-l+p-\frac{1}{n})} \sum_{k=0}^{p-1} a_k^{lm}(l-m-2k), \quad a_0^{lm} = 1. \tag{3.51}$$

The energy spectrum in terms of the momentum quantum numbers l and m yields

$$\begin{aligned} E_{lm}^{\text{ho}} &= -\frac{\pi^2}{12}\frac{n-2}{n}N - \frac{\pi^2}{12}\frac{2n^2-13n+24}{Nn} + \frac{\pi^2}{3nN^2}(n^2-6n+8) \\ &\quad + n\frac{\pi^2}{N^2}\left[l\left(l - \frac{N+n-2}{n}\right) + m\left(m - \frac{N+n-2}{n}\right) + \frac{l-m}{n}\right]. \end{aligned} \tag{3.52}$$

3.4 Fractional Statistics of holons in SU(n) spin chains

We have seen previously that the fractional statistics of spinons in the Haldane Shastry model manifests itself in specific quantization rules for the individual spinon momenta and was generalized to the statement that SU(n) spinons have a statistical parameter $g = (n-1)/n$, *i.e.,* obey $(n-1)/n$-fermi statistics [112]. Furthermore, we used this line of argument to show the half-fermi statistics of the holon excitations in the SU(2) Kuramoto–Yokoyama model [123], which motivates a general formulation of fractional statistics in one dimension in terms of the differences of the individual particle momenta. In the following we apply this interpretation to the holon excitations of the SU(n) KYM [124].

First, consider the case of SU(3). For that we rewrite the energy (3.32) in the form

$$E_{mn}^{\text{ho}} = E_0 + \epsilon^{\text{ho}}(p_m) + \epsilon^{\text{ho}}(p_n), \tag{3.53}$$

where we have used the one-holon dispersion (3.22), and we have introduced the single-holon momenta in (3.53) according to

$$p_m = \frac{2\pi}{3} + \frac{2\pi}{N}m, \qquad p_n = \frac{2\pi}{3} + \frac{2\pi}{N}\left(n - \frac{1}{3}\right). \tag{3.54}$$

Hence, the difference in the individual holon momenta reads

$$p_m - p_n = \frac{2\pi}{N}\left(\frac{1}{3} + \text{integer}\right). \tag{3.55}$$

Accordingly, we interpret this result as a manifestation of the 1/3-Fermi statistics of the holons, as seen in (3.55). This result is similar to the momentum shifts for the spinon excitations in the HSM [112] as well as the holon excitations in the SU(2) KYM [123], as shown in Chapter 2. The fractional statistics of the holons have also been observed in thermodynamic quantities [73, 86] of the KYM as well as exact results for the electron addition spectrum [5, 6].

Finally, consider the case of SU(n). All statements identically generalize: Using the one-holon dispersion (3.46), we can rewrite the SU(n) two-holon energy (3.52) as

$$E^{\mathrm{ho}}_{lm} = E_0 + \epsilon^{\mathrm{ho}}_n(p_l) + \epsilon^{\mathrm{ho}}_n(p_m), \tag{3.56}$$

where the relative momentum yields

$$p_l - p_m = \frac{2\pi}{N}\left(\frac{1}{n} + \text{integer}\right). \tag{3.57}$$

It thus follows that holons in the SU(n) KYM obey $1/n$-fermi statistics, *i.e.*, have a statistical parameter of $g = 1/n$.

Let us sum up the properties for the KYM approaching the limit $n \to \infty$. As the model is exactly solvable, no aspect of calculation simplifies as we approach the Large-n limit. Second, according to the spinon case, the interval of allowed holon momenta shrinks to zero. Still, this does not mean that the number of holon orbitals vanishes, as in the Large-n limit the number of lattice sites has to grow and hence the momentum spacing tends to zero as well. Third, we find $g \to 0$ as $n \to \infty$, meaning that the holon statistics becomes bosonic in this limit. However, as the holon momenta fill only a small interval and retain the fermionic on-site character, holons still cannot be interpreted as bosons in this limit.

3.5 Spin-charge Separation of SU(n) Fermions

From the considerations of the statistical parameters of spinons and holons for an SU(n) spin symmetry, it emerges the important observation that the sum of both yields the fermionic parameter one

$$g_{\mathrm{sp}} + g_{\mathrm{ho}} = (n-1)/n + 1/n = 1. \tag{3.58}$$

This result is very intuitive from the viewpoint of spin-charge separation which is responsible for the emergence of spinons and holons. Since we create a one-spinon one-holon state by taking out one fermion of parameter $g = 1$, this just means that within the process of spin-charge separation, the statistical parameter of the fermion excitation is conserved and just distributed over the spinon and the holon. Compared to general creation and annihilation processes or decays of excitations, this is a peculiar conservation law. It stresses the difference between a decay of particles and a decoupling process as observed for spin-charge separation in one-dimensional fermion chains. Imagine we take out one fermion at site a and one at site b. Interchanging a and b should thus yield a phase of π. As the fermion excitation decouples to a spinon and a holon, this relation should still hold once one interchanges the spinon-holon pairs which originated from the respective fermions, which thus yields that the statistics of spinons and holon should in total be fermionic, *i.e.,* $g_{\text{sp}} + g_{\text{ho}} = 1$.

A further interesting perspective is the phase space of the fermion chain. Let us compare the hypothetical phase space of a fermion excitation opposed to holon excitations. For the first, the allowed fermion momenta would cover the complete Brillouin zone and the momentum spacings are at least $2\pi/L$. For the SU(n) chain, the spinons and holons fill only one nth of the Brillouin, *i.e.,* in this analogy, the quasi-fermi momentum $p_{\text{max}}^{\text{ho}}$ also shrinks to one nth of the fermi momentum p_{F} for the fermion excitation. However, as the process from the fermion to the holon and spinon is just a separation of different degrees of freedom, the number of states realized in the phase space should be conserved. It is thus clear that interpreting spin charge separation as a decoupling of degrees of freedom, it is plausible that the number of spin and charge associated orbitals remains the same. From this perspective, this is captured by the reduction of the momentum spacing between holons to be at least $2\pi/nL$, *i.e.,* as the momentum space volume is decreased by $1/n$, the momentum density is increased by n. This once more stresses the fundamental characteristic of the spacing shifts of the fractional excitations alluded to above.

3.6 Summary and Outlook

In conclusion, we have derived the wave functions of the one-holon and two-holon excitations of the SU(n) KYM and derived its energies and momenta. We have shown by explicit calculation that the holon excitations of the SU(n) KYM obey $1/n$-fermi statistics, which manifests itself in the difference of the individual holon momenta as $p_l - p_m = \frac{2\pi}{N}\left(\frac{1}{n} + \text{integer}\right)$. We further set it into context of spin-charge separation. As it was the case for SU(2) spin chains, one crucial line of investigation is to realize SU(3) systems experimentally. There are already promising attempts for the realization of SU(3) spin chains in systems of ultra cold atoms [1, 105], and hence a possibility of observing SU(3) holon excitations. Additionally, it appears worthwhile to take the SU(3) KYM as a starting point to study spin-charge separation in SU(3) spin chains in more detail, for which the exact findings in the SU(3) KYM serve as a complementary result for numerical approaches, such as density matrix renormalization group (DMRG) studies. For example, one can discuss the notion of SU(3) color charge separation numerically for the generic SU(3) Hubbard model (repulsive U) and unify it with the above analytical description for the SU(3) KYM being a special t-J model descendant of it.

Chapter 4

Microscopic model for the Chiral Spin Liquid

4.1 Introduction

Fractionalization of statistics is known to occur in two dimensions, in general in the presence of a magnetic field that violates the discrete symmetries of parity (P) and time-reversal (T); this situation is realized in the fractional quantum Hall effect [7, 22, 23, 63, 88, 117] (FQHE). In contrast to the one-dimensional case, however, there has been no definite evidence as to whether fractional statistics occurs in the absence of an external field breaking these symmetries. The paradigmatic state for a spin liquid is the chiral spin liquid introduced by Kalmeyer and Laughlin [70,71], which is constructed to spontaneously violate the symmetries P and T; this violation is generally associated with fractional statistics. The excitations of the liquid—spinons, which carry spin 1/2 but no charge, and holons, which carry charge but no spin—obey fractional statistics. In addition, the spinons exhibit quantum-number fractionalization and carry only half the spin of the excitations in conventional magnetically-ordered systems, which are spin-1. Whereas the spinon may be the fundamental field describing excitations in two-dimensional $S = 1/2$ systems in general, the confinement force is generically so strong that the spinon substructure is often suppressed, in some analogy to quantum chromodynamics where the quark substructure of the nucleons is suppressed at low energy scales. In the chiral spin liquid, however, the spinons are deconfined, which thus represents a suitable spin model to study fractional quantization of spinons in two dimensions. For nearly two decades, however, the chiral spin liquid lacked a microscopic model where it is realized.

In this chapter, we present a spin Hamiltonian for which the chiral spin liquid (CSL) is the exact ground state [109]. In many respects, the Hamiltonian is a generalization of the Haldane-Shastry model [57, 114] (HSM) to two dimensions, and provides an exact spin model in which fractional quantization can be studied. We show that the model has an exact two-fold topologically degenerate CSL ground state for *any* number of lattice sites N, which is an important difference to systems like the Rokhsar-Kivelson model [83], where the degeneracy is only obtained in the thermodynamic limit [66]. We develop an analytical method to construct a parent Hamiltonian for the CSL in explicit detail. Therein, the singlet property of the CSL is used as a key feature, which allows for a spherical Tensor decomposition of the destruction operator we introduce. From there, we construct two different parent Hamiltonians. Both contain 6-body spin operator terms. The next key issue is whether the CSL is the *only* ground state of the models. To address this question we perform exact diagonalization studies of both models for a 16 site square lattice. We introduce an adapted Kernel sweeping method to implement these many body Hamiltonians efficiently. We find that one model has the CSL as its unique ground state, whereas finite size studies indicate that the CSL is not the unique ground state for the other. After providing a construction scheme starting from the one-dimensional Haldane–Shastry model in Section 4.2.1, we continue in Section 4.2 by introducing the chiral spin liquid ground state and the basic properties of spinon and holon excitations. After outlining the general construction of CSL Hamiltonians in Section 4.3, we compose a destruction operator for the CSL state in Section 4.4, where we exploit the spin rotational invariance of the CSL state to decompose the destruction operator into different spherical tensor components which annihilate the CSL state individually. While we previously started with an asserted property of the destruction operator, this assertion is proven in Section 4.5. In Section 4.6, we introduce a Kernel sweeping method to compute the CSL Hamiltonians, where the data is shown and discussed in Section 4.7. The main body of this work is presented in

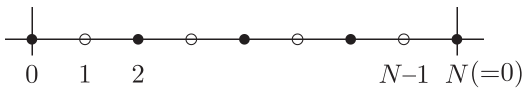
D. F. Schroeter, E. Kapit, R. Thomale, and M. Greiter, *Spin Hamiltonian for which the Chiral Spin Liquid is the Exact Ground state*. Phys. Rev. Lett. **99**, 097202 (2007).

E. Kapit, R. Thomale, D. F. Schroeter, and M. Greiter, *Parent Hamiltonians for the Chiral Spin Liquid*. Submitted to Phys. Rev. B.

4.2 Chiral Spin Liquid state

4.2.1 From the Haldane–Shastry chain to the Chiral Spin Liquid

The chiral spin liquid [70, 71, 90, 126, 137] may be viewed as a brute-force generalization of the Haldane-Shastry wave function to two space dimensions, though the idea for the Chiral Spin Liquid was first composed in 1986 by Dung-Hai Lee, *i.e.,* before the discovery of the Haldane Shastry model in 1988. Consider a periodic one-dimensional lattice on the real axis of a complex plane, with lattice points at integer values:

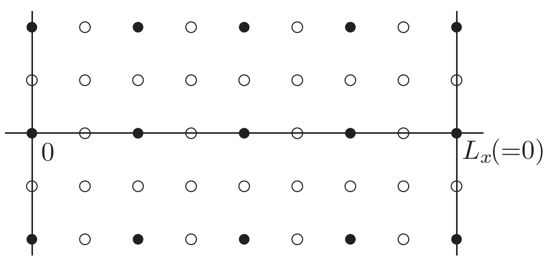


Filled and empty circles represent even and odd integers, with gauge factors $G(z) = -1$ and $G(z) = +1$, respectively. The Haldane-Shastry wave functions is already known from previous Chapter 2 as the Kuramoto–Yokoyama vacuum state at half filling in (2.6), and takes the form

$$\psi_0^{\mathrm{HS}}(z_1, \ldots, z_M) = \prod_{j=1}^{M} G(z_j) \prod_{j<k}^{M} \sin\Big(\frac{\pi}{N}(z_j - z_k)\Big)^2, \tag{4.1}$$

where we took advantage of the fact that now the coordinates z_j are real opposed to the previous implementation.

The chiral spin liquid is obtained by extending the lattice from a circle to a cylinder, or from a segment of the real axis to a strip in the two-dimensional complex plane [45]:



where $G(z) = (-1)^{(x+1)(y+1)}$ for lattice site $z = x + iy$. The wave function for a chiral spin liquid on this cylinder is given by (4.1) multiplied by an exponential factor, which effects that the density of spin flips is $\frac{1}{2}$ for $-L_y/2 < y < L_y/2$, *i.e.*, up to the boundaries of the liquid:

$$\psi^{\mathrm{CSL}}(z_1, \dots, z_M) = \prod_{j=1}^{M} G(z_j) \prod_{j<k}^{M} \sin\left(\frac{\pi}{L_x}(z_j - z_k)\right)^2 \prod_{j=1}^{M} e^{-\pi |y_j|^2}. \tag{4.2}$$

4.2.2 Chiral spin liquid wave function on the torus

The chiral spin liquid takes a more familiar form if we consider open boundary conditions. The wave function for a circular droplet of fluid is simply given by [70, 71]

$$\langle z_1 \cdots z_M | \psi \rangle = \prod_{j<k}^{M} (z_j - z_k)^2 \prod_{j=1}^{M} G(z_j)\, e^{-\frac{\pi}{2}|z_j|^2}. \tag{4.3}$$

The z's in the above expression are the complex positions of the up-spins on the lattice: $z = x + iy$, with x and y integer. $G(z) = (-1)^{(x+1)(y+1)}$ is a gauge factor, which ensures that (4.3) is a spin singlet. Lattice sites not occupied by z's correspond to down-spins. Note that the exponential in (4.2) or (4.3) corresponds to a (fictitious) magnetic field of strength 2π/plaquet.

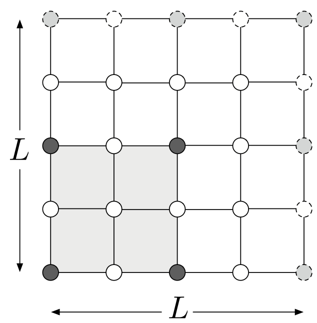


Figure 4.1: The model is defined on a square lattice, length L on a side, with a number of sites $N = L^2$. The image shows the lattice for $\mathcal{N} = 16$. The shaded circles (including the origin) indicate those lattice sites for which $G(z) = -1$ and the open circles those sites for which $G(z) = +1$. The sites on which $G(z) = -1$ define a sub-lattice with twice the original lattice spacing; the doubled unit-cell is shown as the shaded region in the figure surrounding the origin.

For our purposes, it is propitious to choose periodic boundary conditions (PBCs) with equal periods $L_1 = L_2 = L$, L even, and with $N = L^2$ sites.

Following Haldane and Rezayi [61], the wave function for the CSL then takes the form

$$\begin{aligned} \langle z_1 \cdots z_M | \psi \rangle \; = \; & \prod_{\nu=1}^{2} \vartheta_1\Big(\frac{\pi}{L}\,[\mathcal{Z} - Z_\nu]\Big) \prod_{j<k}^{M} \vartheta_1\Big(\frac{\pi}{L}\,[z_j - z_k]\Big)^2 \\ \cdot \; & \prod_{j=1}^{M} G(z_j)\, e^{\frac{\pi}{2}(z_j^2 - |z_j|^2)}, \end{aligned} \tag{4.4}$$

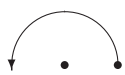
where $M = N/2$ and $\vartheta_1(w) = -\vartheta_1(-w) \equiv \vartheta_1(w|e^{-\pi})$ is the odd Jacobi theta function [2], which is also defined in Appendix B.6. The zeros for the center-of-mass coordinate $\mathcal{Z} = \sum_j z_j$ must lie in the principal region $0 \le \mathrm{Re}(Z_1) < L$, $0 \le \mathrm{Im}(Z_1) < L$ and satisfy $Z_1 + Z_2 = L + iL$; the freedom to choose Z_1 reflects the topological degeneracy and yields two linearly independent ground states for the CSL. These states are spin singlets, are invariant under lattice translations, and are strictly periodic with regard to the PBCs.

4.2.3 Spinon wave functions

It is rather easy to formulate spinon excitations for the chiral spin liquid. In analogy to both the quasiholes in fractionally quantized Hall liquids and the previously discussed spinons (2.10) for the Haldane-Shastry model, we write the wave function for a spinon localized at η is given by

$$\psi^{\mathrm{CSL}}_{\eta\downarrow}(z_1, \ldots, z_M) = \prod_{j=1}^{M} (\eta_\alpha - z_j) \prod_{j=1}^{M} G(z_j) \prod_{j<k}^{M} (z_j - z_k)^2 \prod_{j=1}^{M} e^{-\frac{\pi}{2}|z_j|^2}. \tag{4.5}$$

The spinons obey half-fermi statistics, both in the sense of Haldane's exclusion principle as well as in the sense of the Berry's phases where we adiabatically exchange particles by moving them counterclockwise around each other [130]. This phase is given by $e^{i\theta}$, where θ is the statistical parameter.



$$|\psi\rangle \;\rightarrow\; e^{i\theta}\;|\psi\rangle$$

For $\theta = \pi$ we have fermions, for $\theta = 0$ bosons, and for both $\theta = \frac{\pi}{2}$ and $\theta = -\frac{\pi}{2}$ half-fermions. The choice of sign for θ is physically meaningful, as the allowed values for the relative angular momentum l depend on it:

$$l = \frac{\theta}{\pi} + 2n \quad \text{where } n \text{ integer.} \tag{4.6}$$

Clearly either choice of sign for half-fermions violates parity (P) and time reversal (T). For the spinons in the chiral spin liquid, this result can be found using the adiabatic transport argument of Arovas, Schrieffer, and Wilczek [7]. We obtain $\pi/2$ for (4.5) and $-\pi/2$ for its complex conjugate.

4.2.4 Generation from filled Landau levels

The chiral spin liquid can be generated through projection from the wave functions of a filled lowest Landau level (LLL). If we choose an auxiliary magnetic field with a strength of one half of a Dirac flux quanta per lattice site, the wave function for a circular droplet of $M = \frac{N}{2}$ fermions filling the LLL is given by

$$\phi[z_i] = \prod_{i<j}^{M} (z_i - z_j) \prod_{i=1}^{M} e^{-\frac{\pi}{4}|z_i|^2}. \tag{4.7}$$

The chiral spin liquid state corresponding to (4.3), where the "particles" z_i describe spin flips S_α^+ acting on a "vacuum" state with all the spins $\downarrow$, and $G(\eta_\alpha)$ is as above, can be generated by Gutzwiller projection of the LLL (4.7) filled once with $\uparrow$ and once with $\downarrow$ spin fermions [45, 90]:

$$|\psi_0^{\text{CSL}}\rangle = \sum_{\{z,w\}} \phi[z_i]\, \phi[w_j]\, c_{z_1\uparrow}^\dagger \dots c_{z_M\uparrow}^\dagger\, c_{w_1\downarrow}^\dagger \dots c_{w_M\downarrow}^\dagger\, |\, 0\, \rangle\,, \tag{4.8}$$

where the sum extends over all partitions of the lattice sites into z's and w's and the $c^\dagger$'s are fermion creation operators. This functional equivalence is shown in Appendix B.9. It is often convenient to write (4.8) as

$$|\psi^{\text{CSL}}\rangle \;=\; \text{P}_{\text{GW}} \left|\psi_{\text{SD}}^{N}\right\rangle \tag{4.9}$$

where the Gutzwiller projector

$$\text{P}_{\text{GW}} \equiv \prod_{i=1}^{N} \left(1 - c_{i\uparrow}^\dagger c_{i\uparrow}\, c_{i\downarrow}^\dagger c_{i\downarrow}\right) \tag{4.10}$$

eliminates doubly occupied sites and $\left|\psi_{\text{SD}}^{N}\right\rangle$ is the Slater determinant wave function for the lowest Landau level filled once with $M = \frac{N}{2}$ $\uparrow$-spin and once with M $\downarrow$-spin electrons. The construction is now evidently analogous to the Haldane-Shastry chain in one dimension, which again stresses the strong relation between these states in different dimensions.

The spin singlet property of the chiral spin liquid mentioned above is now evident. Clearly the Slater determinant $\left|\psi_{\text{SD}}^{N}\right\rangle$ is a singlet. Since the Gutzwiller projector commutes with the spin operator on each site,

$$[\text{P}_{\text{GW}}, \boldsymbol{S}_i] = 0, \tag{4.11}$$

$|\psi^{\text{CSL}}\rangle$ must be a singlet as well. The chiral spin liquid, when formulated as a Gutzwiller projection of filled Landau levels, is directly seen to be invariant under lattice transformations. This allows us to define the chiral spin liquid for any lattice of interest like square, but also triangular lattices.

4.2.5 Holon excitations

The Gutzwiller form (4.9) also allows a rather elegant formulation of the spinon excitations. For example, a state with L $\downarrow$-spin spinons at sites $\eta_1, \ldots, \eta_L$ is given by

$$\left|\psi^{\text{CSL}}_{\eta_1\downarrow\ldots\eta_L\downarrow}\right\rangle = \mathrm{P}_{\text{GW}}\, c_{\eta_1\uparrow}\ldots c_{\eta_L\uparrow}\left|\psi^{N+L}_{\text{SD}}\right\rangle, \tag{4.12}$$

where $N+L=2M$ must be an even integer. This form nicely illustrates the (fractional) spin $\frac{1}{2}$ of the spinon. The electron annihilation operators create inhomogeneities in spin and charge before projection. The projector enforces one particle per site and hence restores the homogeneity in the charge distribution, but commutes with spin. We are left with a neutral object of spin $\frac{1}{2}$.

Holon excitations, which carry a positive unit charge and no spin, are constructed from spinon excitations by annihilation of an electron at the spinon site, which must now coincide with a lattice point. For example, a chiral spin liquid with L holons at sites $\eta_1, \ldots, \eta_L$ is given by

$$\left|\psi^{\text{CSL}}_{\eta_1\circ\ldots\eta_L\circ}\right\rangle = c_{\eta_1\downarrow}\ldots c_{\eta_L\downarrow}\,\mathrm{P}_{\text{GW}}\, c_{\eta_1\uparrow}\ldots c_{\eta_L\uparrow}\left|\psi^{N+L}_{\text{SD}}\right\rangle, \tag{4.13}$$

where $N+L=2M$ must again be an even integer. In the following sections when we develop a microscopic model for the chiral spin liquid, we constrain our attention to half filling and do not consider the holon excitations any further. Still, however, there is much reason to suppose that the functional structure will probably resemble the excitations of the Kuramoto–Yokoyama model which we encountered for the one-dimensional case.

4.3 Construction of a Parent Hamiltonian

In order to construct a parent Hamiltonian for the chiral spin liquid, one first derives the destruction operators for the ground state. In our formulation, the destruction operators are constructed from a set of operators ω_j where $j=1,\ldots,\mathcal{N}$ indexes the lattice sites. The operators ω_j, to be introduced in Section 4.4 below, are not themselves destruction operators but have the property that, acting on the ground state, they produce a result independent of the site index j: $\omega_i \left|\psi\right\rangle = \omega_j \left|\psi\right\rangle$. Therefore, once the above result is established in Section 4.5, it follows that the difference of any two of the operators is a destruction operator for the ground state: $d_{ij} = \omega_i - \omega_j$.

In order to construct a sensible parent Hamiltonian, one must minimally demand that it be a translationally-invariant scalar operator. In order to put the Hamiltonian in this form, it is shown in Appendix B.2 that the operators may be written as $\omega_j = \Omega_j^0 + \mho_j^0$ where $\boldsymbol{\Omega}_j$ and $\boldsymbol{\mho}_j$ are vector and third-rank spherical tensor operators respectively and where the 0 superscript indicates the component in spherical notation. The operators $\boldsymbol{\Omega}_j$ and $\boldsymbol{\mho}_j$ are given explicitly in terms of spin operators in Sections 4.4.1 and 4.4.2.

As is discussed in detail in Section 4.4, the Wigner-Eckhart theorem guarantees that all components of the operators $\mathbf{D}_{ij} = \boldsymbol{\Omega}_i - \boldsymbol{\Omega}_j$ as well as $\boldsymbol{\mathcal{D}}_{ij} = \boldsymbol{\mho}_i - \boldsymbol{\mho}_j$ are destruction operators for the chiral spin liquid ground state so long as the reducible tensor operator d_{ij} is. One can then construct Hamiltonians based on either set of operators:

$$H = \sum_{\langle i\, j \rangle} \mathbf{D}_{ij}^{\dagger} \cdot \mathbf{D}_{ij} \tag{4.14}$$

for the vector destruction operators or

$$H = \sum_{\langle i\, j \rangle} \sum_{\nu=-3}^{3} \left(\mathcal{D}_{ij}^{\nu}\right)^{\dagger} \mathcal{D}_{ij}^{\nu} \tag{4.15}$$

for the rank-3 spherical tensor operators. Either Hamiltonian is a scalar and is translationally invariant, both of these properties guaranteed by the construction. Additionally, since the Hamiltonians are positive semi-definite, the chiral spin liquid is a ground state of the model. It should be noted that these models are not themselves unique as one could include any coefficients J_{ij} into the sums of Eqs. 4.14 and 4.15 and remove the restriction that only nearest-neighbor sites are summed over. These two models do, however, represent the simplest models from each class.

In Section 4.6, a numerical method is developed for performing the exact diagonalization of these Hamiltonians that can handle the large number of interactions efficiently. This method is used in Section 4.7 to show that the model given by (4.14) has exactly two ground states, as expected due to the topological degeneracy of the chiral spin liquid on a torus, and that these states are precisely the chiral spin liquid ground states given in Section 4.2 above. Adopting the same procedure, the Hamiltonian given in (4.15) is shown to have a larger ground-state manifold which is not exhausted by the chiral spin liquid ground states.

4.4 Destruction operators

The Hamiltonian which stabilizes the chiral spin liquid is generated by first finding a set of operators ω_i, where i is a site index. These operators are not themselves destruction operators, but the bond operators $\omega_i - \omega_j$, where i and j are any two distinct sites, will be shown to destroy the CSL ground state. The operators may be written as $\omega_j = \omega_j^+ - \omega_j^-$ where $\omega_j^+ = T_j + V_j$ and

$$T_j = \frac{1}{2} \sum_{i\,k \neq j}{}^{\prime} K_{ijk}\, S_j^+\, S_k^- \left(\frac{1}{2} + S_i^z\right) \tag{4.16}$$

$$V_j = \sum_{i \neq j} U_{ij} \left(\frac{1}{2} + S_i^z\right)\left(\frac{1}{2} + S_j^z\right) . \tag{4.17}$$

The two sets of coefficients U_{ij} and K_{ijk} are defined in Section 4.4.3 below and the prime on the sum indicates that one must exclude the coincidences of i and k.

The operator ω_j^- is related to ω_j^+ by a $\pi/2$ rotation about the x-axis that maps S_z and S_y into $-S_z$ and $-S_y$. This means that the entire operator ω_j is given by

$$\omega_j = \sum_{i\,k \neq j}{}^{\prime} K_{ijk} \left[\frac{1}{2\,i} \left(\boldsymbol{S}_j \times \boldsymbol{S}_k\right)^z + \left(\boldsymbol{S}_j \cdot \boldsymbol{S}_k\right) S_i^z - S_i\, S_j\, S_k^z\right] + \sum_{i \neq j} U_{ij}\, S_i^z\, . \tag{4.18}$$

In writing down (4.18), the fact that $\sum_{i \neq j} U_{ij} = 0$, has been employed. This will be demonstrated in Section 4.4.3 below. While the operators ω_i are not themselves destruction operators for the CSL ground state, it will be shown in Section 4.5 that $d_{ij} = \omega_i - \omega_j$ is a destruction operator for the ground state for any choice of i and j.

The operators ω_j are reducible and can be decomposed into irreducible tensor operators, in this case of ranks 1 and 3. From (4.18) it is clear that every term except for the $S_i^z\, S_j^z\, S_k^z$ term is the 0 (or z) component of a rank-1 (vector) operator. This final term can be decomposed into rank-3 and vector components.

It is straightforward to show that if an operator d is a destruction operator for the CSL ground state, then each of its irreducible components are as well. This is because the Wigner-Eckhart theorem tells us that acting with an operator T_m^j on a state $|n\, q\, m_q\rangle$ with angular momentum q and z-component

m_q gives

$$T^j_m\ |n\, q\, m_q\rangle = \sum_{j'\, m'} C^{m\, m_q\, m'}_{j\ q\ j'}\ |n'\, j'\, m'\rangle\ , \tag{4.19}$$

where n and n' are any quantum numbers other than angular momentum. Since the CSL is a spin singlet: $q = m_q = 0$, it follows that there is only a single non-zero term in the above sum corresponding to $j' = j$ and $m' = m$. This means that by decomposing the destruction operator for the ground state d into its tensor components, which may be written $d = \sum_j a_j\, T^j_0$, acting on the ground state to obtain

$$0 = d\ |\psi\rangle = \sum_j a_j\ |n'\, j\, 0\rangle\ , \tag{4.20}$$

and noting that states with different values of j are necessarily orthogonal, it immediately follows that each of the states in the sum are themselves zero and hence the operators T^j are destruction operators for the ground state. In Sections 4.4.1 and 4.4.2 we give two classes of operators that are obtained from the reducible tensor operator ω_j in (4.18).

4.4.1 Vector destruction operator

As shown in Appendix B.2, the operator $S^z_i\, S^z_j\, S^z_k$ may be written as the sum of the 0-components of a vector and a third-rank tensor. The vector component is given by

$$\frac{1}{5}\ \left[(\boldsymbol{S}_i \cdot \boldsymbol{S}_j)\ \boldsymbol{S}^z_k + (\boldsymbol{S}_j \cdot \boldsymbol{S}_k)\ \boldsymbol{S}^z_i + (\boldsymbol{S}_k \cdot \boldsymbol{S}_i)\ \boldsymbol{S}^z_j\right] \tag{4.21}$$

and, working from (4.18), the vector operator $\boldsymbol{\Omega}_j$ is given by

$$\begin{aligned}\boldsymbol{\Omega}_j \;\; = \;\; & \sum_{i,k\neq j}{}' K_{ijk} \left[\frac{1}{2\,i}\,(\boldsymbol{S}_j \times \boldsymbol{S}_k) + \frac{4}{5}\,(\boldsymbol{S}_j \cdot \boldsymbol{S}_k)\ \boldsymbol{S}_i - \frac{1}{5}\,(\boldsymbol{S}_k \cdot \boldsymbol{S}_i)\ \boldsymbol{S}_j \right. \\ & \left. -\frac{1}{5}\,(\boldsymbol{S}_i \cdot \boldsymbol{S}_j)\ \boldsymbol{S}_k\right] + \sum_{i\neq j} U_{ij}\, \boldsymbol{S}_i\,. \end{aligned} \tag{4.22}$$

Since $\boldsymbol{\Omega}_i - \boldsymbol{\Omega}_j$ is a destruction operator for the ground state, it immediately follows that one may construct a Hamiltonian for which the chiral spin liquid is the exact ground state as

$$H = \sum_{\langle i\, j\rangle} (\boldsymbol{\Omega}_i - \boldsymbol{\Omega}_j)^\dagger \cdot (\boldsymbol{\Omega}_i - \boldsymbol{\Omega}_j)\ , \tag{4.23}$$

where the sum runs over all nearest-neighbors on the lattice. By construction, the Hamiltonian is a scalar operator and translationally invariant.

However, note that there is nothing restricting possible models to run only over next-nearest neighbors. Rather one can consider any combination of bond-operators (including arbitrary coefficients so long as one maintains positive semi-definiteness in H) in constructing a parent Hamiltonian for the CSL.

4.4.2 Tensor destruction operator

It is also possible to create a set of third-rank tensor destruction operators. As shown in Appendix B.2, the operator $S_i^z\, S_j^z\, S_k^z$ may be fully decomposed into the 0-components of a vector operator (given in (4.21)) and a third-rank tensor operator, which is necessarily just the difference between $S_i^z\, S_j^z\, S_k^z$ and the operator in (4.21). This gives a destruction operator whose 0-component is

$$\begin{aligned} \mho_j^0 \quad = \quad & -\frac{1}{\sqrt{10}} \sum_{i,k\neq j}' K_{ijk} \Big[(\boldsymbol{S}_i \cdot \boldsymbol{S}_j)\; S_k^z + (\boldsymbol{S}_j \cdot \boldsymbol{S}_k)\; S_i^z + (\boldsymbol{S}_k \cdot \boldsymbol{S}_i)\; S_j^z \\ & -5\, S_i^z\, S_j^z\, S_k^z \Big] . \end{aligned} \tag{4.24}$$

The other components are straightforward to obtain (see Appendix B.2) and one may again use these operators to form a Hamiltonian for the chiral spin liquid according to

$$H = \sum_{\langle i\, j\rangle} \sum_{\nu=-3}^{3} \left(\mho_i^\nu - \mho_j^\nu\right)^\dagger \left(\mho_i^\nu - \mho_j^\nu\right) . \tag{4.25}$$

The Hamiltonian in (4.25) has two significant advantages over the model in (4.23): it depends only on one set of coefficients (K_{ijk} but not U_{ij}) and, because the operator in the sum in (4.24) is symmetric under interchange of i and k, one may replace K_{ijk} by $A_{ijk} = (K_{ijk} + K_{kji})\,/2$ where the new coefficients are manifestly symmetric in interchange of the first and third indices. From a general perspective, this model has additional features which make it both more and less attractive for further study, as will be discussed in detail in Section 4.7.

4.4.3 Coefficients

The coefficients appearing in (4.18) are functions of the distance between the sites of the form $K_{ijk} = K(z_k - z_j, z_i - z_j)$ where

$$K(x, y) = \frac{1}{N/2 - 1} \lim_{R\to\infty} \sum_{0 \leq z_0 \leq R} \frac{P(x - z_0, y)}{x - z_0}, \tag{4.26}$$

and the sum over z_0 is a sum over all lattice translations: $z_0 = (m + i\, n)\, L$ for m and n integer. This sum guarantees that the function $K(x, y)$ is periodic in its first argument.

The coefficients $U_{ij} = \pi\, U(\pi\, [z_j - z_i]\,/L)\, L$ are given by

$$\begin{aligned} \frac{\pi}{L} U\Big(\frac{\pi}{L}\, z\Big) = \frac{\pi}{L} W\Big(\frac{\pi}{L}\, z\Big) + \frac{1}{N-2}\Big[\, \frac{d}{dx} P(x, -z)\Big|_0 \\ + \lim_{R\to 0} \sum_{0 < |z_0| \leq R} \frac{P(z_0, -z)}{z_0}\Big], \end{aligned} \tag{4.27}$$

where $W(z)$ is the periodic extension of $1/z$ to the torus [89] and also related to the logarithmic derivatives of the theta functions:

$$\frac{\pi}{L} W\Big(\frac{\pi}{L}\, z\Big) = \frac{d}{dz} \ln \vartheta\Big(\frac{\pi}{L}\, z\Big) + \frac{\pi}{L}\, \frac{z - z^*}{2\, L}\,. \tag{4.28}$$

The function $P(x, y)$ is given by

$$P(x, y) = \lim_{R\to\infty} \sum_{0 \leq |z_0 - y| \leq R} \frac{\mathrm{Co}\big(\frac{\pi}{2\,L}\, [z_0 - y]\big)}{\mathrm{Co}\big(\frac{\pi}{2\,L}\, [z - (y - z_0)]\big)}\, \frac{e^{-\frac{\pi}{L^2}\, |z_0 - y|^2}}{n(y)}, \tag{4.29}$$

where $\mathrm{Co}(x) = \cos x + \cosh x$ and where $n(y)$ is a normalization factor chosen such that $P(0, y) = 1$ which entails the choice

$$n(z) = \vartheta_3\Big(\frac{\pi}{L}\,\mathrm{Re}\,[z]\Big|\, i\Big)\ \vartheta_3\Big(\frac{\pi}{L}\,\mathrm{Im}[z]\Big|\, i\Big)\,. \tag{4.30}$$

While the form of the coefficients as given by Eqs. 4.26–4.28 are essential for forming a Hamiltonian that stabilizes the CSL, there is significant freedom in how one chooses the function $P(x, y)$. The only requirements are that it be a periodic function of y, fall off faster than $1/x$ with increasing x, and be analytic apart from 1^{st} order poles that occur at the coincidence of the two arguments: $x = y$. It is straightforward to show that $U(z)$ is an odd function; this in turn guarantees that $\sum_i U_{ij} = 0$ and lets this sum be dropped, as was done in writing down (4.18).

4.5 Proof

In order to prove that either of the Hamiltonians given in Eqs. 4.14 and 4.15 are true parent Hamiltonians for the chiral spin liquid, we must demonstrate that $\omega_j\,|\psi\rangle = \omega_i\,|\psi\rangle$ which we will demonstrate by first showing that

$$\langle z_1 \cdots z_M | \,\omega_j\, |\psi\rangle = f(\mathcal{Z})\, \langle z_1 \cdots z_M \rangle\, \psi\,, \tag{4.31}$$

where $f(\mathcal{Z})$ is a function only of the center of mass: $\mathcal{Z} = \sum_{i=1}^{M} z_i$. This identity in turn follows from the fact that

$$\frac{\langle z_1 \cdots z_M | \,\omega_j^+\, |\psi\rangle}{\langle z_1 \cdots z_M \rangle\, \psi} = \begin{cases} f(\mathcal{Z}) & z_j \in \{z_1 \cdots z_M\} \\ 0 & \text{otherwise}\,, \end{cases} \tag{4.32}$$

and the result that the function $f(\mathcal{Z})$ is both odd and periodic. To see this, recall that one can write $\omega_j^- = U\,\omega_j^+\,U^\dagger$ where U performs the $\pi/2$ rotation about the x-axis as discussed in Section 4.4 above. The CSL ground state is invariant under such a rotation so that

$$\begin{aligned} \langle z_1 \cdots z_M | \,\omega_j^+\, |\psi\rangle &= \langle z_1 \cdots z_M | \,U^\dagger \omega_j^- U\, |\psi\rangle \\ &= \langle w_1 \cdots w_M | \,\omega_j^-\, |\psi\rangle\,, \end{aligned} \tag{4.33}$$

where $\{w_i\}$, the locations of the down spins on the lattice, is the complement of $\{z_i\}$. It then follows from (4.32) that

$$\frac{\langle z_1 \cdots z_M | \,\omega_j^-\, |\psi\rangle}{\langle z_1 \cdots z_M \rangle\, \psi} = \begin{cases} 0 & z_j \in \{z_1 \cdots z_M\} \\ f(\mathcal{W}) & \text{otherwise}\,. \end{cases} \tag{4.34}$$

Assuming that the origin of the lattice is chosen such that the sites occupy positions $z_i = (\ell + i\,m)$ for ℓ and m integer, it is straightforward to show that

$$\mathcal{Z} + \mathcal{W} = \frac{L\,(L-1)}{2}\,(1+i)\;L, \tag{4.35}$$

and since L is even it follows that the sum of $\mathcal{Z}$ and $\mathcal{W}$ is equivalent to a translation of the lattice z_0. Because the function $f(\mathcal{Z})$ is periodic and odd, both properties will be shown below, it immediately follows that $f(\mathcal{W}) = f(z_0 - \mathcal{Z}) = -f(\mathcal{Z})$. Combining this fact with (4.34) completes the proof that (4.32) entails (4.31).

4.6 Kernel Sweeping Method

To implement the Hamiltonians given in (4.23) and (4.25), one has to take into account that 6-body terms appear in the Hamiltonians. If a Hamiltonian of that kind is taken as a whole to act on a basis for a given spin lattice, this turn out to be technically impracticable. Instead, we apply an adapted form of a Kernel sweeping method. Consider for simplicity translationally invariant Hamiltonians of the form

$$\mathcal{H} = \sum_{i=1}^{N} \mathcal{H}_i, \tag{4.36}$$

with periodic boundary conditions (PBC's) and $\mathcal{H}_i$ being the contribution acting on lattice site i. As an example, we consider some two-body interaction

$$\mathcal{H}_i = \sum_{\substack{j=1 \\ j\neq i}}^{N} a_{ij}\, \mathcal{J}^{ij}, \tag{4.37}$$

where $\mathcal{J}^{ij}$ be a quadratic operator term and a_{ij} is its amplitude (or weight). For the translational invariant system we may rewrite a_{ij} as a_{i-j}.

The main idea of the kernel sweeping method is that the Hamiltonian kernel (*e.g.* $\mathcal{J}^{ij}$ in (4.37)) is the same for all possible configurations $\{i, j\}$ and therefore should only be stored once and be applied to all lattice configurations by *sweeping* with the kernel over the whole lattice basis. The kernel is a matrix of dimension μ^n for a system with a μ-dimensional local basis and n–body interaction. In general, we thus consider an n-site Hamiltonian version of (4.37), where as before $\mathcal{J}^{j_1 j_2 j_3 \cdots j_n}$ denotes the n-site operator acting on the sites $j_1 j_2 j_3 \cdots j_n$ and $a_{j_1 j_2 j_3 \cdots j_n}$ is the associated weight.

For demonstration, we constrain our attention to the vector Hamilton operator (4.23). Setting $\boldsymbol{\Omega}_{ij} = \boldsymbol{\Omega}_i - \boldsymbol{\Omega}_j$, we split up the Hamiltonian into

$$H = \sum_{\langle ij\rangle} \boldsymbol{\Omega}_{ij}^{z,\dagger} \boldsymbol{\Omega}_{ij}^{z} + \frac{1}{2}\left(\boldsymbol{\Omega}_{ij}^{+,\dagger} \boldsymbol{\Omega}_{ij}^{+} + \boldsymbol{\Omega}_{ij}^{-,\dagger} \boldsymbol{\Omega}_{ij}^{-}\right), \tag{4.38}$$

where the z-component as well as the ladder components of the vector operators can be written out in terms of spin operators S^z, $S^+ = S^x + iS^y$, and $S^- = S^x - iS^y$. As the treatment is very similar, we constrain our attention to the contribution $\sum_{\langle ij\rangle} \boldsymbol{\Omega}_{ij}^{+,\dagger} \boldsymbol{\Omega}_{ij}^{+}$, where for clarity we again write out the + ladder operator explicitly:

$$\begin{aligned} \boldsymbol{\Omega}_j^+ \;&=\; \sum_{i,k\neq j}{}' K_{ijk} \left[\frac{1}{4\,i}\left(S_j^z S_k^+ - S_j^+ S_k^z\right) + \frac{4}{5}\left(\boldsymbol{S}_j \cdot \boldsymbol{S}_k\right)\, S_i^+ - \frac{1}{5}\left(\boldsymbol{S}_k \cdot \boldsymbol{S}_i\right)\, S_j^+ \right. \\ &\quad \left. -\frac{1}{5}\left(\boldsymbol{S}_i \cdot \boldsymbol{S}_j\right)\, S_k^+ \right] + \sum_{i\neq j} U_{ij}\, S_i^+ \,. \end{aligned} \tag{4.39}$$

We apply the previously defined notation

$$\mathbf{\Omega}_j^+ = \sum_{\substack{i \\ i\neq j}} \left[\sum_{\substack{k \\ k\neq i,j}} K_{ijk}\, \mathcal{J}_{\mathrm{k}}^{ijk,+} + U_{ij} \mathcal{J}_{\mathrm{u}}^{i,+} \right], \tag{4.40}$$

where $\mathcal{J}_{\mathrm{k}}^{ijk,+}$ and $\mathcal{J}_{\mathrm{u}}^{i,+}$ are the first and second line operators of (4.39), respectively. We create one action list for each kernel, where the weights K_{ijk} and U_{ij} are computed according to the formulae (4.26) and (4.27). Then for each lattice basis state, the three-site and the one-site kernel sweep over the lattice and generate the matrix element contributions. The implementation of the Tensor Hamilton operator (4.25) is completely analogous.

4.7 Numerics

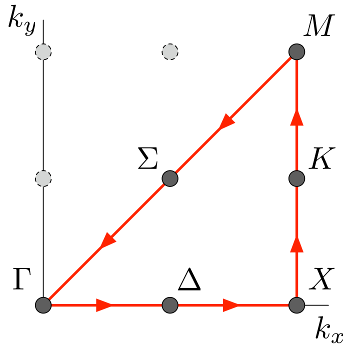


Figure 4.2: A plot of the symmetry points in the first Brillouin zone. The arrows show the path taken in plotting the energy spectra in Figure 4.3, starting from the origin at $\Gamma = (0,0)$.

We discuss a 4 by 4 square lattice and PBCs. As the models (4.23) and (4.25) conserve S^z_{tot}, the Hamiltonian matrix splits up into subspaces for $S^z_{\mathrm{tot}} = 0, \pm 1, \pm 2, \ldots$, where for ground state considerations the $S^z_{\mathrm{tot}} = 0$ subspace is our predominant interest. The lattice basis thus corresponds to $S^z_{\mathrm{tot}} = 0$ at half filling, *i.e.*, 8 up and 8 down spins, which can be further reduced by a 4-fold translational symmetry in x and y direction. Accordingly we can address a path of non-equivalent points in the Brillouin zone shown in Fig. 4.2.

We start with the consideration of the Vector Hamiltonian (4.23). The spectrum is depicted in Fig 4.3.

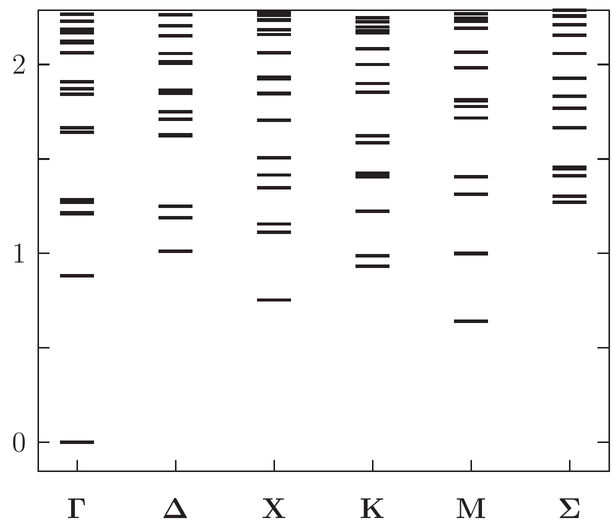


Figure 4.3: Low energy spectrum of the Hamiltonian (4.23), scaled down to order of unity. There are two $E = 0$ eigenvalues at the Γ point.

We find the spectrum to be positive semi-definite, with a doubly degenerate 0 eigenvalue at the Γ point. The rest of the spectrum is well separated by a gap, which is substantial and not due to finite size effects as it exceeds the finite size level splitting of the spectrum by a factor of ~ 15. This fits well in the interpretation of a Hamiltonian having the CSL as its ground state: The elementary excitations should be spinons and as such should have an excitation gap, in the case of the $S^z = 0$ sub-block a gap between the ground state and a two spinon excitation state. It is a promising task to explicitly discuss the spinon excitations of this model, which is an important question that will be addressed in future work. We now discuss the two orthogonal 0 eigenvalue eigenstates which are supposed to be the CSL state of two-fold topological degeneracy. For comparison, we construct the CSL state (4.4) explicitly. For the torus geometry of PBCs, we indeed find a two-dimensional subspace of functions with the center of mass variable being treated as an external parameter. We compute the overlap of the Hamiltonian ground state subspace and the CSL subspace. We find that they match perfectly. This thus shows that the doubly degenerate ground state of the Hamiltonian is indeed the topologically two-fold degenerate CSL state. Additionally, we have *only two* 0 eigenvalues, by which follows that the CSL state is the *only* ground state of the model, a statement which previously could not be achieved analytically.

For the Tensor Hamiltonian, however, we find that the 0 energy subspace is largely degenerate, which of course contains the CSL, but also many additional states. While the restriction to small system sizes prevents us from studying the thermodynamic limit precisely, our numerical findings indicate that the Hamiltonian (4.25) does not stabilize the CSL state as the unique ground state, which thus singles out the model (4.23) to be subject of further study.

4.8 Summary and Outlook

In this work we have shown a method for constructing parent Hamiltonians for the chiral spin liquid. We have computed the spectra of the Hamiltonians by use of a Kernel sweeping method in exact diagonalization. There, for the Hamiltonian operator composed of the spherical vector component of the CSL destruction operator, we observe that the CSL states are the only ground states of the model. We suggest that this model is a promising candidate to also study the elementary excitations of the model, *i.e.,* spinons, and many other question in the field of two-dimensional fractionalization of quantum numbers in spin systems. As a concrete perspective, it is important to discuss the spinon excitations numerically, where there is some indication that the elementary one-spinon or even two-spinon wave function correspond to exact eigenstates of the model, which can be addressed by exact diagonalization. As a further step, it is worthwhile studying systems away from half filling where the holon excitations appear, for which the model described above gives a promising starting point. Additionally, one may be interested in making the relation to the Haldane–Shastry model more explicit. This could be accomplished by a thin torus expansion of the model described above, which should yield the Haldane–Shastry Hamiltonian to leading order. This approach has already been successfully applied to address various questions in the Quantum Hall effect [13]. As in the one-dimensional case, it would be desirable to use this potentially analytically accessible model to derive statements on the spinon momentum distribution function, which may be linked to experiments.

Chapter 5

Non-Abelian Chiral Spin Liquid

5.1 Introduction

Starting from the Chiral Spin Liquid discussed in the previous chapter, we now introduce a new spin liquid state for a spin 1 and higher spin lattices. So far, we have only dealt with spinons possessing *abelian* fractional statistics in the sense that the wave function acquires a non-trivial phase when spinons are braided around each other. The present renaissance of interest in fractional quantization, however, is due to possible applications of states supporting excitations with non-abelian statistics to the rapidly evolving field of quantum computation and cryptography [82]. The paradigm for this universality class is the Pfaffian state introduced by Moore and Read [95] in 1991. The state was proposed to be realized at the experimentally observed fraction $\nu = 5/2$ [131] (*i.e.,* at $\nu = 1/2$ in the second Landau level) by Greiter, Wen, and Wilczek [49], a proposal which recently received strong experimental support through the direct measurement of the quasiparticle charge [29]. Pfaffian-type states are further conjectured to be realized for one-dimensional bosons with three-body hard core interactions in general [100]. The Moore–Read state possesses $p+ip$ wave pairing correlations. The flux quantum of the vortices is one half of the Dirac quantum, which implies a quasiparticle charge of $e/4$. Like the vortices in a p-wave superfluid, these quasiparticles possess Majorana-fermion states [103] at zero energy (*i.e.,* one fermion state per pair of vortices, which can be occupied or unoccupied). A Pfaffian state with $2L$ spatially separated quasiparticle excitations is hence 2^L fold degenerate, in accordance with the dimension of the internal space spanned by the zero energy states. While adiabatic interchanges of quasiparticles yield only overall phases in abelian quantized Hall states, braiding of half vortices of the Pfaffian state will in general yield non-trivial changes in the occupations of the zero energy states [67, 116], which render the interchanges non-commutative or non-abelian. Most importantly, however, the internal state vector is insensitive to local perturbations—it can *only* be manipulated through braiding of the vortices. These properties together render non-abelions preeminently suited for applications as protected qubits in quantum computation [26]. Non-abelian anyons further appear in certain other quantum Hall states described by Jack polynomials [16, 115] including the Read-Rezayi states [104], in the non-abelian phase of the Kitaev model [81], and in the Yao-Kivelson model [30, 107, 134].

In this chapter, we propose a novel chiral spin liquid state for an $S = 1$ antiferromagnet. The spinon and holon excitations of this state are deconfined and obey non-abelian statistics, with the braiding governed by Majorana fermion states. The state violates time reversal (T) and parity (P), is a spin singlet, can be formulated on any lattice type, and fully respects all the lattice symmetries. The state possesses a 3-fold topological degeneracy on the torus geometry. We provide numerical evidence that the state can be stabilized on the triangular lattice by a local Hamiltonian involving three-spin interactions. Finally, we hypothesize that *spinons in spin liquids with spin larger than 1/2 generically obey non-abelian statistics*, but are only deconfined in the chiral spin liquids we introduce and study here. The main body of this work is presented in a further upcoming publication and

M. Greiter and R. Thomale, *Non-Abelian Statistics in a Quantum Antiferromagnet*, submitted to Phys. Rev. Lett.

5.2 Non-abelian chiral spin liquid state

The state we propose is most easily written down for a circular droplet with open boundary conditions occupying N sites of a triangular or square lattice $S = 1$ antiferromagnet. The wave function for re-normalized spin flips,

$$\psi_0[z_i] = \mathrm{Pf}\left(\frac{1}{z_j - z_k}\right) \prod_{i<j}^{N} (z_i - z_j) \prod_{i=1}^{N} G(z_i)\, e^{-\frac{\pi}{2}|z_i|^2} \tag{5.1}$$

is simply given by a bosonic Pfaffian state in the complex coordinates $z \equiv x + iy$ supplemented by a gauge phase $G(\eta_\alpha)$. The Pfaffian is given by the fully antisymmetrized sum over all possible pairings of the N coordinates,

$$\mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \equiv \mathcal{A}\left\{\frac{1}{z_1 - z_2} \cdot \ldots \cdot \frac{1}{z_{N-1} - z_N}\right\}. \tag{5.2}$$

The "particles" z_i represent re-normalized spin flips acting on a vacuum with all spins in the $S^{\mathrm{z}} = -1$ state,

$$|\psi_0\rangle = \sum_{\{z_1,\ldots,z_N\}} \psi_0(z_1,\ldots,z_N)\ \tilde{S}^+_{z_1}\ldots\tilde{S}^+_{z_N}\ |-1\rangle_N\,, \tag{5.3}$$

where the sum extends over all possibilities of distributing the N "particles" over the N lattice sites allowing for double occupation, and

$$\tilde{S}^+_\alpha \equiv \frac{S^{\mathrm{z}}_\alpha + 1}{2} S^+_\alpha, \quad |-1\rangle_N \equiv \otimes_{\alpha=1}^N |1,-1\rangle_\alpha\,. \tag{5.4}$$

The lattice may be anisotropic; we have chosen the lattice constants such that the area of the unit cell spanned by the primitive lattice vectors is set to unity. For a triangular or square lattice with lattice positions given by $\eta_{n,m} = na+mb$, where a and b are the primitive lattice vectors in the complex plane, $G(\eta_{n,m}) = (-1)^{(n+1)(m+1)}$ [70, 137]. In comparison to the chiral spin liquid state discussed in Chapter 4, some similarities are observed. The first is the general functional form of the wave function, being a Jastrow factor to second power for the CSL and a single power Pfaffian and Jastrow form for the non-abelian CSL. However, as seen from (5.4), the construction of the Hilbert space already contains subtleties not present for the spin 1/2 CSL. We will encounter further similarities, but also strong differences in the following.

5.2.1 Singlet property

While the topological properties, and in particular the non-abelian statistics of the fractionalized excitations of (5.2), are highly suggestive to those familiar with Pfaffian states, the invariance under spin rotation and lattice symmetries is less so. We content ourselves here with a direct proof of the singlet property, which at the same time serves to motivate the necessity for the re-normalization of the spin-flip operators (5.4).

Since $S^z_{\mathrm{tot}}\,|\psi_0\rangle = 0$ by construction, it is sufficient to show $S^-_{\mathrm{tot}}\,|\psi_0\rangle = 0$. Note first that when we substitute (5.1) with (5.2) into (5.3), we may omit the antisymmetrization $\mathcal{A}$ in (5.2), as it is taken care automatically by the commutativity of the bosonic operators $\tilde{S}_\alpha$. (Throughout this chapter, we do not keep track of overall normalization factors.) Let $\tilde{\psi}_0$ be ψ_0 without the operator $\mathcal{A}$ in (5.2). Since $\tilde{\psi}_0[z_i]$ is still symmetric under interchange of pairs, we may assume that a spin flip operator S^-_α acting on $|\tilde{\psi}_0\rangle$ will act on the pair (z_1, z_2):

$$\begin{aligned}
S_\alpha^- |\tilde{\psi}_0\rangle =& \sum_{\{z_3,\dots,z_N\}} \Bigg\{ \sum_{z_2(\neq\eta_\alpha)} \tilde{\psi}_0(\eta_\alpha, z_2, z_3, \dots)\, S_\alpha^-\, \tilde{S}_\alpha^+ \tilde{S}_{z_2}^+ \\
&\qquad + \sum_{z_1(\neq\eta_\alpha)} \tilde{\psi}_0(z_1, \eta_\alpha, z_3, \dots)\, S_\alpha^-\, \tilde{S}_{z_1}^+ \tilde{S}_\alpha^+ \\
&\qquad + \tilde{\psi}_0(\eta_\alpha, \eta_\alpha, z_3, \dots)\, S_\alpha^-\, (\tilde{S}_\alpha^+)^2 \Bigg\} \tilde{S}_{z_3}^+ \dots |{-1}\rangle_N \\
=& \sum_{\{z_3,\dots,z_N\}} \Bigg\{ \sum_{z_2} 2\tilde{\psi}_0(\eta_\alpha, z_2, z_3, \dots)\, \tilde{S}_{z_2}^+ \Bigg\} \tilde{S}_{z_3}^+ \dots |{-1}\rangle_N
\end{aligned}$$

where we have used

$$S_\alpha^-\, (\tilde{S}_\alpha^+)^n\, |1,-1\rangle_\alpha = n\, (\tilde{S}_\alpha^+)^{n-1}\, |1,-1\rangle_\alpha\,.$$

This implies $S_{\mathrm{tot}}^- \,|\psi_0\rangle = \sum_{\alpha=1}^N S_\alpha^-\, |\psi_0\rangle = 0$ if and only if

$$\sum_{\alpha=1}^N \tilde{\psi}(\eta_\alpha, z_2, z_3, \dots) = 0 \qquad \forall\; z_2, z_3, \dots z_N.$$

This, however, is just a special case of the Perelomov identity [101], which holds for lattice sums of $e^{-\frac{\pi}{2}|\eta_\alpha|^2} G(\eta_\alpha)$ times any analytic function of η_α. An insightful proof of this sum rule is given in terms of Jacobi theta functions in Appendix B.3.

5.2.2 Generation from filled Landau levels

We saw in the previous chapter that the spin 1/2 chiral spin liquid can be written as

$$|\psi_0^{\mathrm{CSL}}\rangle = \sum_{\{z,w\}} \phi[z_i]\, \phi[w_j]\, c_{z_1\uparrow}^\dagger \dots c_{z_M\uparrow}^\dagger\, c_{w_1\downarrow}^\dagger \dots c_{w_M\downarrow}^\dagger\, |\,0\,\rangle\,, \tag{5.5}$$

where the sum extends over all partitions of the lattice sites into z's and w's, the $c^\dagger$'s are fermion creation operators, and ϕ denote a Landau level filled with one certain species of spins. Trivially, we can rewrite the CSL state vector in terms of Schwinger bosons $a^\dagger$ and $b^\dagger$ (see Appendix B.8),

$$|\psi_0^{\mathrm{CSL}}\rangle = \Psi^{\mathrm{CSL}}\big[c_\uparrow^\dagger, c_\downarrow^\dagger\big]\, |\,0\,\rangle = \Psi^{\mathrm{CSL}}\big[a^\dagger, b^\dagger\big]\, |\,0\,\rangle\,, \tag{5.6}$$

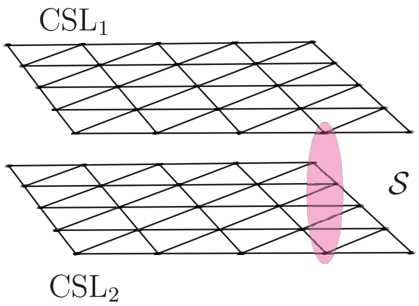


Figure 5.1: Schematic construction of the Pfaffian Liquid. Starting with two CSL lattices (depicted CSL$_1$ and CSL$_2$ above), a spin 1 wave function is constructed by symmetrization over the sites.

provided we define $\Psi_0^{\text{CSL}}\left[c_\uparrow^\dagger, c_\downarrow^\dagger\right]$ such that the operators are ordered according to a fixed labeling of the lattice sites. The non-abelian CSL state (5.1) is then given by

$$|\psi_0\rangle = \left(\Psi^{\text{CSL}}\left[a^\dagger, b^\dagger\right]\right)^2 |\, 0\, \rangle \,. \tag{5.7}$$

To verify (5.7), use $\frac{1}{\sqrt{2}}(a^\dagger)^n(b^\dagger)^{(2-n)} \,|\, 0\, \rangle = (\tilde{S}^+)^n \,|\, 0\, \rangle$ and

$$\mathcal{S}\prod_{i<j,1}^{M}(z_i - z_j)^2 \prod_{i<j,M+1}^{2M}(z_i - z_j)^2 = \text{Pf}\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}^{2M}(z_i - z_j), \tag{5.8}$$

where $\mathcal{S}$ indicates symmetrization and (5.8) holds due to the Frobenius identity going back to 1882 [37]. Since the LLL states (4.7) are (on compact surfaces) translationally and rotationally invariant modulo gauge transformations in the auxiliary magnetic field, and (4.8) is manifestly gauge covariant, both the CSL states (4.3) from the previous chapter and (5.1) are invariant under lattice transformations. Note that this projective construction also implies the singlet property of the CSL states. It can be used to formulate the CSL states on any lattice, and to generalize them to arbitrary spin:

$$\left|\psi_0^{\text{Spin } S}\right\rangle = \left(\Psi^{\text{CSL}}\left[a^\dagger, b^\dagger\right]\right)^{2S} |\, 0\, \rangle \,. \tag{5.9}$$

Written in terms of (then differently) re-normalized spin flip "particles", the wave function generalizes from a bosonic Pfaffian state for $S = 1$ to bosonic Read-Rezayi states [104] for $S > 1$. This is a very important result, which, however, we do not elaborate on here in detail. It means that we can generate not only Pfaffian-like non-abelian spinons, but spinons of any non-abelian statistics type so far known in the Quantum Hall effect. Rephrased in other words, we hence provide a representation of the complete Jack polynomial series on spin lattices recently developed by Bernevig and Haldane [16]. It is important to observe that the single site dimensionality on the spin lattice has to increase with higher Jack series terms, which supports the view that the internal Hilbert space structure gets more complicated.

5.3 Non-abelian spinon and holon excitations

The spinon excitations of (5.1) are completely analogous to the half vortex quasiparticles of the Moore-Read quantum Hall state [95]. For example, to create 4 $\downarrow$ spin spinons at locations η_1, η_2, η_3, and η_4, we simply insert half quantum vortices inside the Pfaffian (5.2), which then becomes

$$\mathrm{Pf}\left(\frac{(z_i-\eta_1)(z_j-\eta_2)(z_i-\eta_3)(z_j-\eta_4)+(i\leftrightarrow j)}{z_i-z_j}\right). \tag{5.10}$$

The braiding properties of the spinons are insensitive to the spinon spin, and exactly those of the Moore-Read quasiparticles, which has been discussed in explicit detail in the literature [67,103,116]. The proof of the singlet property given above can be extended to show that a pair of $\downarrow$ spin spinons transforms as an $S=1$ triplet excitation, which implies that each spinon carries spin $S=\frac{1}{2}$. With the implicit assumption that the $S=1$ spins on each lattice site consist of two electrons in triplet configurations, we can create holon excitations by annihilating $\downarrow$ spin electrons on sites with $\downarrow$ spin spinons. The braiding properties of the holons are equivalent to those of the spinons. The braiding analogy to half quantum vortices, however, only accounts for the subspace of spin polarized spinons. For the general case, however, we find no analogous state in the FQHE, since the chirality of the vortices is fixed therein due to the external magnetic field, which is not necessarily the case for the spinons of the non abelian CSL state.

5.4 Microscopic model - Hamiltonian Finder method

The first question with regard to possible applications of our state to quantum computation is whether a state belonging to the universality class described by (5.1) can be stabilized through a local Hamiltonian. While we are short of a definite answer, we have done our best to address the question numerically. To begin with, we have written out the state (5.1) for an isotropic, triangular lattice with 16 sites and periodic boundary conditions, which imply a three-fold topological degeneracy [50]. For given translational symmetries, the CSL

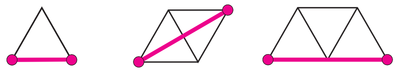
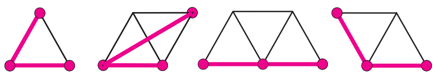
operator	configurations	coefficients
S_iS_j		1,931 0,079
$S_i(S_j \times S_k)$		0,970 0,344
$(S_iS_j)^2$		-0,513 -0,241 -0,086
$(S_iS_j)(S_iS_k)$		-0,137 -0,023 -0,089 -0,017

Figure 5.2: The eleven interaction terms included in our trial Hamiltonian with the numerically optimized coefficients (see text). Note that the coefficients fall off rapidly with the distance.

states are situated at the Γ point. Additionally, we have a two-fold spin reflection symmetry for the system, where the CSL states acquire no phase under the symmetry operation. We then numerically optimized the coefficients of a set of local spin interaction terms (see Fig. 5.2) such that the ground state of our trial Hamiltonian is energetically closest to a suitable linear combination of the three (in the thermodynamic limit degenerate) Pfaffian states, which we then compare to the exact eigenstates. This is the key step performed by the Hamiltonian Finder method. To outline the principal structure, it starts with an overlap optimization of a collection of Hamiltonian terms without any further constraint with respect to a given input state, which in our case is the non-abelian CSL state. This problem can be solved by computation of the scattering overlap of the states resulting from different Hamiltonian terms acting on the input state. However, this just optimizes the overlap and there is no control which spectral state is optimized a priori. Instead, what we want to have in our problem is no adjustment to maximize some excited state overlap with the input state, but exclusively the ground state, *i.e.,* we want the lowest energy state of the resulting spectrum to have the maximum overlap with the input state. What we do numerically is that for each Hamiltonian term entering the optimization, we first identify where the input state is situated in the spectrum of solely this term and how strongly it is

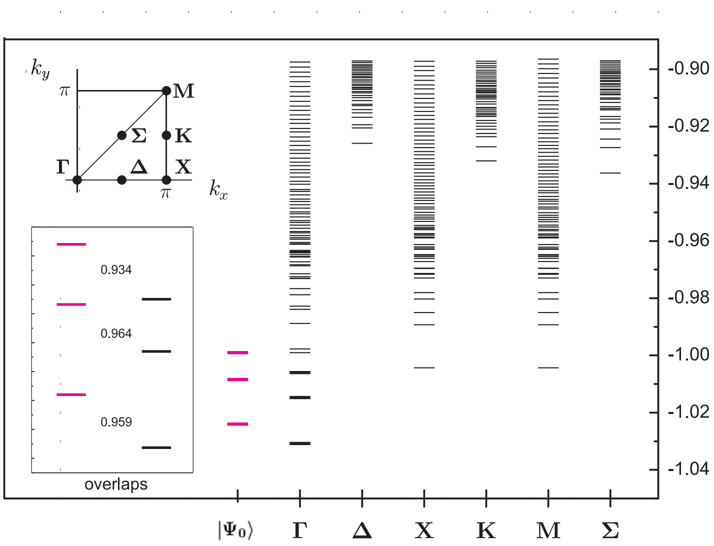


Figure 5.3: (Color online) Spectral plot of our trial Hamiltonian in comparison with the energy expectations values for the three (in the infinite system topologically degenerate) Pfaffian ground states states at the Γ point. The inset shows the overlap of the Pfaffian states with the three lowest states of our Hamiltonian.

scattered upon action of term, *i.e.,* what is the self overlap of the input state with respect to this Hamiltonian term. From there, we compute optimization precondition weights to enter the overlap optimization commented on previously. In doing so, we can incorporate the information into the optimization that is needed to push the input state down in the resulting spectrum, and, upon iteration, finally to become ground state. Of course, one must keep in mind that the Hamiltonian terms entering the optimization are educates guesses from the structure of the input state, and the input for Hamiltonian Finder method has to be chosen carefully for different input states. As shown in Fig. 5.3, the three lowest energy eigenstates of our trial Hamiltonian have a significant overlap with the Pfaffian states (*i.e.,* 0.959, 0.964, and 0.934 in a Hilbert space which contains 227,475 singlets), which suggests that the exact states belong to the same universality class. (Our evidence is unfortunately not conclusive as the three states are not separated by a large gap from the remainder of the spectrum, which indicates that the system we can access numerically is too small to settle the question unambiguously.)

5.5 General scope of non-Abelian spinons

Efforts to understand high T_c superconductivity in terms of an RVB spin liquid have revealed a general connection between d-wave superconductors and $S = \frac{1}{2}$ spin liquids on the square lattice [3, 68]. In particular, a wide class of (undoped) $S = \frac{1}{2}$ spin liquids can be obtained by Gutzwiller projection from the wave function of a d-wave superconductor with suitably chosen parameters. This suggests a general connection between the (abelian) vortices of the superconductor and the (abelian) spinons in the spin liquid. If one Gutzwiller projects a $d + id$ wave superconductor with suitably chosen parameter on a square lattice, one obtains exactly the CSL state (4.3).

The $p + ip$ pairing correlations in the non-abelian CSL state (5.1) introduced above suggest a similar correspondence between the non-abelian vortices of the superconductor and the non-abelian spinon excitations (5.10). As in the abelian case $S = \frac{1}{2}$, the P and T violation of the state appears to be necessary for the spinon to be deconfined, but does not seem essential to the topological properties. We are hence led to conjecture that there is a general connection between p-wave superfluids and $S = 1$ spin liquids, in that the non-abelian braiding properties of the vortices of the superfluid are also general properties of the spinons in $S = 1$ (and higher spin) antiferromagnets. True, the spinons will only be free under very special circumstances, and the propensity to be confined will only increase with the spin S. Even in an ordered antiferromagnet, however, spinons (and holons) are the fields appropriate for describing the physics at sufficiently high energy scales, *i.e.*, energies above the ordering temperature.

5.6 Summary and Outlook

In this chapter, we have constructed an $S = 1$ CSL and demonstrated that its spinon and holon excitations obey non-abelian statistics. Our analysis suggests that spinons in spin liquids with Spin $S > 1$ obey non-abelian statistics in general. Of course, we have to multiply validate this statement in the future. As an additional direction of research, it is obvious that we have to analyze this new kind of excitations in more detail, which could first be addressed again by exact diagonalization methods. Additionally, it may be possible to consider further types of spin liquids that can be constructed by the symmetrization procedure outline above. Already at this stage, we can apply the Hamiltonian Finder method to deduce a microscopic model for the Spin 1 chirality liquid introduced by Greiter in 2002 [45]. Finally, one can go one step back and retell the whole construction scheme for one dimension, yielding a wave function in one dimension that has a Pfaffian structure. Introducing already many directions of further research by itself, it has been additionally recently accomplished by Greiter in a current ongoing work to establish a class of Hamiltonians on Spin S chains for which the Spin S chain states are the *exact* ground states [42], which we have already analyzed numerically and will further investigate in the future.

Chapter 6

Experimental observation of fractional excitations

6.1 Introduction

Experimental observation and analysis of fractional excitations has been the main predominantly unresolved part of the field since its discovery in the early 80's [88, 128, 129]. Apparently many systems in which fractional excitations could be studied analytically and in explicit detail were impossible to realize experimentally, and for a long time it remained elusive to close this gap. In the 90's, significant progress was made in both the regime of the quantum Hall effect [41] and of one-dimensional spin chains [79], which will be further commented on below. In contrast, two-dimensional fractional excitations in spin models could not be studied yet, whereas the theoretical picture of anyons in two dimensions is generally much clearer than for one-dimension [77, 130]. Only very recently, since the rise of ultra-cold atoms in optical lattices [34] and new materials [20], it has become possible to study anyonic spin excitations in more detail, and the field is evolving rapidly. From this perspective, this chapter only intends to shortly comment on recent developments in the field. Rather, the main intention is to support the view that many questions associated with fractional excitations in one and two dimensions may be possible to address experimentally in the very near future.

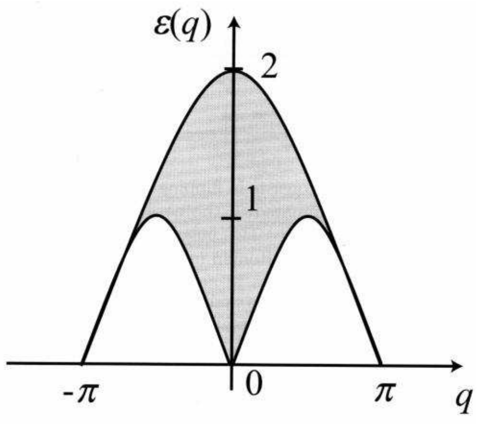


Figure 6.1: Two-spinon continuum, schematic plot taken from [40]. As before, the individual spinons only occupy one half of the Brillouin zone.

6.2 Spinons and Holons in one dimension

6.2.1 State of the field

Very important measurements of the spinon-excitation continuum were first performed using neutron scattering in $KCuF_3$ [119]. $KCuF_3$ provides a good example of a quasi one-dimensional antiferromagnet. It is nearly tetragonal and due to Jahn-Teller distortion, the fluorines are displaced such that it effectively yields quasi-1D strongly coupled chains. The observations can be explained up to a high precision by the exactly known spin structure factor in the Haldane Shastry model [62], and can thus be explained as an excitation continuum composed of two-spinon excitations (see Fig. 6.1). Beyond observing the spinon substructure yielding an excitation continuum, the results also confirm that the spinon dispersion only occupies half of the Brillouin zone. Concerning the exactly solvable model of spinons discussed previously, this leads to the important observation that the elementary properties of spinons found therein directly relate to generic and experimentally realized $S = 1/2$ Heisenberg chains. Further detailed neutron scattering experiments have been performed on quasi-1D Cs_2CuCl_4 [25]. In particular, the square root singularity of the spin structure factor could be observed, which can be attributed to the half-Fermi statistics of the spinons. A further seminal step was taken by performing angel-resolved photoemission spectroscopy (ARPES) on $SrCuO_2$ [79]. In principle, $SrCuO_2$ is a 1D antiferromagnetic (Mott) insulator, being modelled by chains of atoms with one electron per site, and doubly occupied sites are suppressed due to strong on-site Coulomb

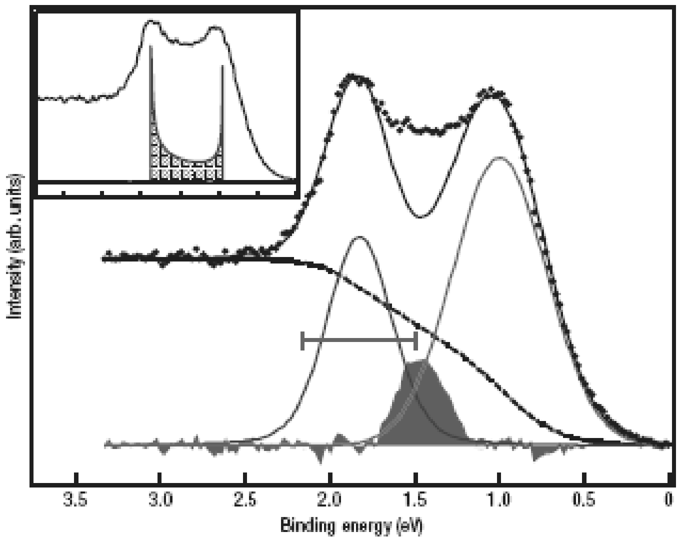


Figure 6.2: Analysis of the spectral function, plot taken from [78]. The raw data (big dots) are fitted with two Gaussian peaks relating to spinon and holon, respectively. The upper left inset shows the expected Bethe-Ansatz result for the Hubbard model with infinite Coulomb interaction. The peaks for the two separate holon and spinon excitations are thus observed explicitly.

repulsion. In ARPES, the electrons are excited above the vacuum level by incident (monochromatic) photons, followed by the measurement of the energy and the emission angle of the emitted electrons by an analyzer. For the one-dimensional structures given here, it is especially easy to deduce the electron momentum inside the material from the emission angle, which allows a detailed resolution of energy E versus quasi-momentum k along the chain. By ARPES, it is thus explicitly studied how spin-charge separation emerges: The photon takes out one fermionic degree of freedom, thus creating a spinon holon pair. The spectral results directly relate to this finding. It is observed that the spectrum contains a region where a continuum is found, which can only be explained in terms of a spinon-holon continuum. Subsequent works also studied the temperature dependence of the spinon and holon excitations [91]. The explicit two-peak structure, one corresponding to the holon and one corresponding to the spinon peak, has been observed recently by more refined ARPES measurements [78, 84] (see Fig. 6.2). A third range of experiments concentrates on so-called "particle-hole" probes, such as optical spectroscopy, resonant inelastic x-ray scattering (RIXS), and electron energy-loss spectroscopy (EELS). In these experiments, no charge is removed from the system. Instead, the scattering yields a move of an electron to another site, inducing a doubly occupied site and an empty site. This, by contrast to ARPES, decays into a spinon and an antiholon, which has been studied in detail and set into theoretical context by analytical and numerical analysis [12, 64, 80]. Recently, spin-charge separation has also been studied by tunneling experiments between two wires in the ballistic regime [9].

6.2.2 Influence of fractional statistics

From the viewpoint of the theoretical results pointed out in Chapter 2, the comments above show that certain properties of fractional spinon and holon excitations have been measured experimentally, such as the spinon-spinon and spinon-holon continuum, as well as the halfed Brillouin zone for the spinons and holons and the square root divergency in the spin structure factor. However, the *direct* testing of individual fractional excitations of spin chains remains an open problem. In particular, it is unclear whether the fractional half-fermionic spacing which we find for spinons and holons in the Haldane-Shastry and Kuramoto–Yokoyama model leads to an effect which can be observed experimentally. The first step to address this problem may be a thermodynamic description of the Haldane–Shastry model in terms of multiple spinon states, which is one of our ongoing considerations.

6.3 Fractional excitations in the Quantum Hall effect

6.3.1 State of the field

Abelian excitations

The fractional quantum Hall effect is the first system where fractional excitations emerged. Initiated by the fundamental work of Laughlin in 1983 [88], it was soon realized that the phenomenology of certain Quantum Hall plateaus at fractional filling can be suitably captured in the language of fractional excitations, the Laughlin quasiholes and quasielectrons. Again, however, the concise experimental measurement of the fractional quantum numbers and fractional statistics remained an unsolved problem for more than a decade. In 1995, Goldman and Su measured the fractional charge $e/3$ of the $m = 3$ Lauglin state at $\nu = 1/3$ filling in resonant tunneling experiments in the Quantum Hall regime [41]. However, the charges still could not be addressed individually, which was finally accomplished by shot noise experiments performed by De-Picciotto *et al.* [102]. Still, there was a remaining ambiguity whether the shot noise experiments at $\nu = 1/3$ filling really measured the charges themselves relating to $e/3$ or just the conductance relating to the filling factor. This could be checked by performing the same shot noise measurement for the Quantum Hall plateau at filling $\nu = 2/5$, where the fractional charge is supposed to be $e/5$. Reznikov *et al.* [106] showed that it is indeed the charge which is measured. The subsequent question to be addressed is how the fractional statistics of the excitations can be measured explicitly. Opposed to one-dimension elaborated on previously, the winding operation yielding non-fermionic phases upon interchange of abelian fractional excitations can be unambiguously defined, suggesting quantum inteference measurements to dectect the effect of fractional statistics. Beyond other proposals like combining the Aharonov Bohm effect with noise measurments raised by Kane [72], Camino, Zhou, and Goldman recently suggested a laughlin quasiparticle interferometer, which should enable the acquired winding phase to be studied explicitly [22, 24] (see Fig. 6.3). They realized an interferometer where an $e/3$ Laughlin quasiparticle executes a closed path around an island of the $2/5$ fractional quantum Hall fluid, from which they can deduce the acquired phases.

Non-Abelian excitations: Pfaffian state

While many other proposals for non-Abelian fractional quantum Hall states exist in theory, experimental efforts have nearly exclusively concentrated on the Pfaffian state at $\nu = 5/2$ filling, *i.e.,* with the lowest Landau level filled and the next level half filled. This is mainly due to the experimental accessibility, as it is by far the most clearly detectable and accessible state. For this state, it is predicted by theory that the elementary excitations are half quantum vortices of charge $e/4$, which obey non-trivial winding rules. In particular, the winding of the vortices does not only induce a non-trivial winding phase as is the case for abelian excitations, but changes the topology of state, which render the winding non-commutative or non-abelian. The interest in this state even increased further when Das Sarma, Freedman, and Nayak supposed that a topologically protected qubit can be formed from it [26]. One milestone has been achieved recently by Dolev *et al.* [29], by considering statistical current fluctuations (shot noise) in the $\nu = 5/2$ state, as similar theoretical proposals existed before [11]. This gives a very strong indication of non-Abelian statistics being realized in this state, since this perspective correctly predicts the $e/4$ quasiparticle charge of the half quantum vortices. In analogy to the evolution of experiments for the Abelian

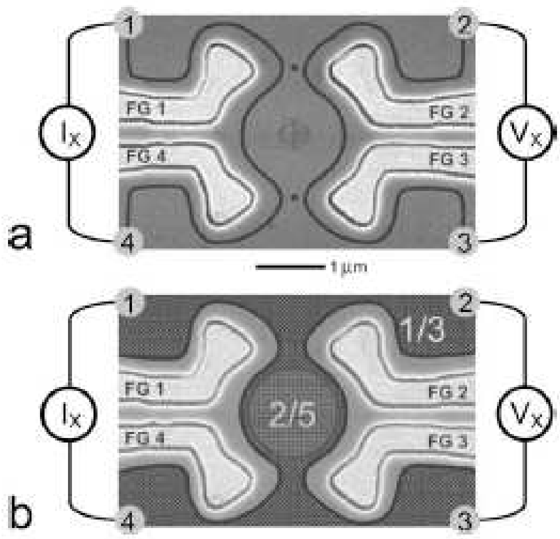


Figure 6.3: Laughlin quasiparticle interferometer, plot taken from [136]. Four Au/Ti front gates define the central island in a) and b) separated from the 2D bulk by two wide constrictions. The electrons constrained to 2D are depleted near the gates. a) shows the scenario for quantum Hall filling $f = 1$, where the chiral edge channels follow equipotential lines. b) is the important setting, which illustrates the $f = 2/5$ island surrounded by an $f = 1/3$ fractional quantum Hall fluid.

states, however, this is only a first step. In particular, explicitly measuring the non-Abelian statistics of the quasiparticles will be a complicated task, since a simple measurement of a winding phase will not be sufficient. This is indicated in a recent theoretical proposal using interferometry [17].

6.4 Spinons in two dimensions

6.4.1 State of the field

Whereas we have seen above that the experimental progress in detecting fractional excitations already started in the 90's in the fractional quantum hall effect, it took more time for the field to detect fractional spin excitations in two dimensions by similar means. Several conceptual problems arise. Firstly, until today, the basic notion of spinons in two-dimensional antiferromagnets is a very contentious issue, and even for the Chiral Spin Liquid, the paradigmatic state of spinons in two dimensions, a microscopic model was not found until recently by us, which would have allowed to endeavor the physical properties of spinons. Secondly, even for models like the Rokhsar-Kivelson dimer model where the excitations can be interpreted as being half-fermionic spinon excitations as well, the Hamiltonians did not allow for experimental realization. In contrast, generic antiferromagnetic models like the two-dimensional Heisenberg model, appear to be efficiently described by magnon excitations carrying spin 1, meaning that opposed to the one-dimensional case, the confinement of the spinons too strong for the substructure to be resolved. In analogy, for scattering energies lower than the quark confinement, it is impossible to resolve the quark substructure of the nucleons in high energy physics. Thirdly, no spin model in which non-Abelian excitations are supposed to appear was known, not to speak of the non-Abelian spinons which are introduced by us in the preceding Chapter 5. The cure for the first and partly the third point was given by the seminal work of Kitaev in 2006 [81]. Therein, he introduced a model which allows anyonic excitations to be established in a very accessible manner. Experimentally, the important progress made by Micheli *et al.* [92] establishes a way to engineer arbitary Hamiltonians by use of polar molecules in optical lattices. The combination of microwave excitations with dipole-dipole interactions and spin-rotation coupling enables the construction of a complete toolbox for effective two-spin interactions, in particular with designable range, spatial anisotropy, and coupling strengths significantly larger than relevant decoherence rates. Additionally, Büchler *et al.* have shown that polar molecules in optical lattices driven by microwave fields naturally give rise to models with nearest-neighbor three body interactions [21], which appears to be generally essential for non-Abelian statistics to emerge, as cofirmed by our microscopic model for non-Abelian chiral spin liquid involving three-spin interaction terms, as well as for the $\nu = 5/2$ fractional quantum hall state. It is important to note that in the systems considered by Büchler *et al.*, the two-body interactions can still be tuned independently of the three-body interactions by use of external fields, which in total makes it a formidable experimental ground to endeavor anyonic phases.

One recent development related to point two, which can be interpreted in close analogy to the one-dimensional spinon case we have considered previously, is given by high-precision neutron scattering experiments on $CsCoBr_3$ by Braun *et al.* [20]. The material $CsCoBr_3$ can be interpreted as an antiferromagnetic anisotropic spin 1/2 chain, being close to the Ising limit. Along the dominant spin direction, the lowest excitations can be interpreted as domain walls separating between regions of the two (different) degenerate ground states. Quantum fluctuations due to the transverse coupling destroys the high dengeneracy of the solitons which form a band, with two degenerate soliton minima carrying different chirality quantum numbers, as left-right

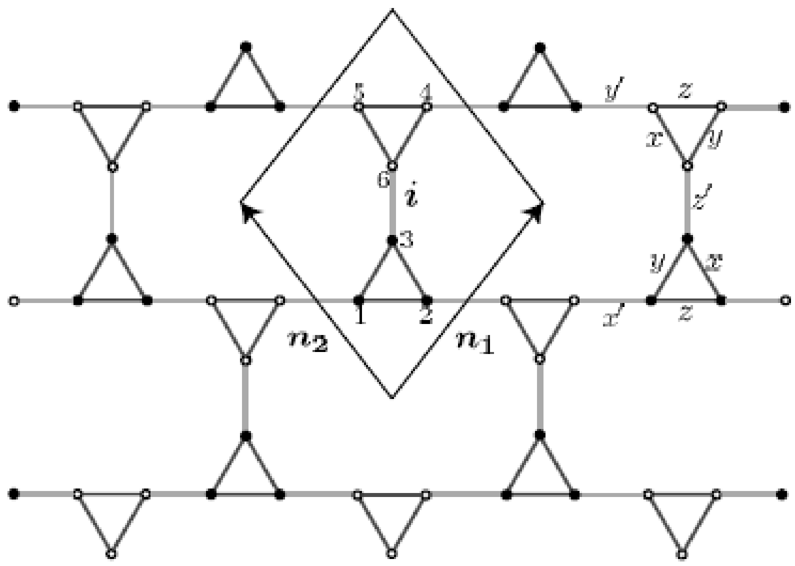


Figure 6.4: A piece of the triangle-honeycomb lattice taken from [31]. A unit cell with lattice vector denoted by n_1 and n_2 contains 6 sites. The different types of links are distinguished by light and dark shading.

symmetry (P) is still preserved by the system. Although the phenomenology is slightly different to the spinons we have encountered in the Chiral spin liquid, one may equally call these soliton excitations spinons in the sense that they both carry fractional quantum numbers. Neutron scattering can only induce the creation of pairs of solitons, which leads to a soliton excitation continuum. Remarkably, Braun *et al.* managed to lift the P symmetry of the model and detect the chirality quantum number of the solitons explicitly. Analogously, the spinons in the Chiral Spin Liquid carry a chirality quantum number. This work dictates the path to discuss spinons in two-dimensional antiferromagnets in more detail.

6.4.2 Half-Fermi Excitations in the Kitaev model

Beyond other versions of topological models in [81], Kitaev proposed a two-dimensional spin 1/2 system defined on a triangle-Honeycomb lattice, described by

$$H = -\sum_{\alpha}\Big[\sum_{\alpha-\text{links}} J_\alpha \sigma_i^\alpha \sigma_j^\alpha + \sum_{\alpha'-\text{links}} J'_\alpha \sigma_i^\alpha \sigma_j^\alpha\Big], \tag{6.1}$$

where the σ_i^α's are the usual Pauli matrices at site i and α take on values x, y, or z. Links of type x, y, z or x', y', z' are illustrated in Fig. 6.4. The model contains anyonic excitations with both abelian and non-abelian excitations, depending on the parameter regime of the J_α's. While the lattice is hard to realize in natural magnetic materials, it is techniqually easily achievable in optical lattices. Most importantly, the Hamiltonian terms themselves are only of nearest neighbor two-body form, and thus again easy to engineer experimentally in optical lattices. There are two experimental groups associated with Zoller and Das Sarma who have recently reported on fundamental progress in establishing the Kitaev model in optical lattices [69, 135]. The possible detection of anyons in the Kitaev model has been recently proposed by Dusuel *et al.* [30] by explicit braiding of the vortices. The crucial bottleneck for this operation is the potential creation of further uncontrolled vortices while performing the braiding operation. Considering state of the art technique in optical lattices [34, 125], a sufficient experimental scenario may be created in the near future.

6.5 Summary and Outlook

We could only shortly discuss a small selection of recent developments in the field of experimental observation of fractional excitations. To mention some very recent proposals, there may also be a way to engineer quantum entangled states out of equilibrium, which is proposed in the yet unpublished works by Diehl *et al.* [28] and Kraus *et al.* [85] . In addition there is even a recent proposal by Norman and Micklitz to detect spinon Fermi surfaces in spin liquids by looking for oscillatory couplings between two ferromagnets via a spin liquid spacer [99]. In the very near future, we expect that the strongest progress in the field of experimental measurement of anyonic excitations in spin models will be done for the Kitaev model, where many aspects have already been accomplished. However, as fruitful as the discussion of anyons in the Kitaev model may be, it does not bring us further to discuss abelian and non-abelian spinons in spin 1/2 and spin 1 lattices experimentally. It has to be investigated whether the microscopic models for these states shown in the preceding chapters can be engineered in optical lattices, for example. Additionally, it also of interest to think about quantities measurable by scattering experiments that may enable us to detect the spinon substructure in generic spin 1/2. Most importantly, the potentially present substructure of spin 1 materials, as indicated by our results presented in Chapter 5, must be addressed experimentally by deep inelastic neutron scattering, for example.

Chapter 7

Conclusion

In this thesis, we have discussed various new aspects of fractional quantization and fractional statistics in one and two-dimensional spin systems.

In Chapter 2 we have studied the holon excitations of the Kuramoto–Yokoyama model. For the standard case of SU(2) spin symmetry, we constructed the explicit two-holon wave functions not known previously and derived their momenta and energies. The results display the half-Fermi statistics of the holons, which manifests itself in a shift of $\frac{1}{2}\frac{2\pi}{N}$ in the difference of the individual holon momenta. It is a main task to (if possible) discuss the implications for experimental observations.

In Chapter 3, we generalize this view to spin chains of general SU(n) symmetry and have derived the wave functions of the one-holon and two-holon excitations of the SU(n) KYM as well as its energies and momenta. We have shown by explicit calculation that the holon excitations of the SU(n) KYM obey $1/n$-fermi statistics, which manifests itself in the difference of the individual holon momenta as $p_l - p_m = \frac{2\pi}{N}\left(\frac{1}{n} + \text{integer}\right)$. We further set it into the context of spin-charge separation. This result supports the view that fractional statistics in one dimension manifests itself not only through a fractional exclusion principle [58], but decisively in a fractional shift of the individual holon momenta. This can be set into the general context of color charge separation for SU(3) spin chains, which promises significance in future experiments of ultra cold atoms in chains of optical lattices.

In Chapter 4, we turned to fractional spin excitations in two dimensions and have shown a method for constructing parent Hamiltonians for the chiral spin liquid. We have computed the spectra of the Hamiltonians by use of a novel Kernel method in exact diagonalization, and applied a Tensor decomposition technique to find the Hamiltonian. There, for the Hamiltonian operator composed of the spherical vector component of the CSL destruction operator, we observe that the CSL states are the only ground states of the model. We conclude that this model is a promising candidate to also study the elementary excitations of the model, *i.e.,* spinons, and many other question in the field of two-dimensional fractionalization of quantum numbers in spin systems.

In Chapter 5, we have constructed an $S = 1$ non-abelian chiral spin liquid and demonstrated that its spinon and holon excitations obey non-abelian statistics. Our analysis suggests that spinons in spin liquids with Spin $S > 1$ obey non-abelian statistics in general, which is a fundamental statement that possibly can and must be considered experimentally and by further theoretical study.

Finally, in Chapter 6, we review recent progress to detect aspects of fractional quantization in various experimental systems, and outline certain directions that may lead to a detection of theoretical findings presented in this thesis.

In conclusion, we believe that the field of fractional excitations is on the rise. Recent developments like quantum cryptography, quantum computation, topological phases and many other aspects from the theoretical side as well as tremendous progress on the experimental side will render fractional quantization as one of the most important fields in condensed matter physics of strongly correlated systems in the very soon future.

Appendix A

Holons in the Kuramoto–Yokoyama model

A.1 Two-Holon excitations in the SU(2) KYM

In the following we will construct the two-holon energy eigenstates starting from (2.22). First, we introduce the auxiliary wave functions

$$\varphi_{mn}(z_1,\ldots,z_M;h_1,h_2)=h_1^m h_2^n\prod_{i=1}^{M}(h_1-z_i)(h_2-z_i)\,\Psi_0(z_1,\ldots,z_M). \quad \text{(A.1)}$$

The action of the Hamiltonian on the states (2.22) will be obtained later via

$$\Psi^{\text{ho}}_{mn}=\varphi_{m+1,n}+\varphi_{n+1,m}-\varphi_{m,n+1}-\varphi_{n,m+1}. \quad \text{(A.2)}$$

Second, we rewrite the Hamiltonian (2.2) in analogy to [15] as

$$H_{\text{KY}}=\frac{2\pi^2}{N^2}\left(H_{\text{S}}^{\text{ex}}+H_{\text{S}}^{\text{Is}}+H_{\text{V}}+H_{\text{C}}^{\uparrow}+H_{\text{C}}^{\downarrow}\right), \quad \text{(A.3)}$$

where we separate the spin-exchange, spin-Ising, potential, $\uparrow$-charge kinetic term, and $\downarrow$-charge kinetic terms. In the following we treat each term separately. For the spin-exchange term we begin by observing that $[S_\alpha^+S_\beta^-\varphi_{nm}](z_1,\ldots,z_M;h_1,h_2)$ is identically zero unless one of the arguments $z_1,...,z_M$ equals η_α. We have

$$\left[H_{\text{S}}^{\text{ex}}\varphi_{mn}\right](z_1,\ldots,z_M;h_1,h_2)\equiv\left[\sum_{\alpha\neq\beta}^{N}\frac{P_{\text{G}}S_\alpha^+S_\beta^-P_{\text{G}}}{|\eta_\alpha-\eta_\beta|^2}\varphi_{mn}\right](z_1,\ldots,z_M;h_1,h_2)$$

$$=\sum_{j=1}^{M}\sum_{\beta\neq j}^{N}\frac{\eta_\beta}{|z_j-\eta_\beta|^2}\frac{\varphi_{nm}(z_1,\ldots,z_{j-1},\eta_\beta,z_{j+1},\ldots,z_M;h_1,h_2)}{\eta_\beta}$$

$$
\begin{aligned}
&= \sum_{j=1}^{M}\sum_{l=0}^{N-1}\sum_{\beta\neq j}^{N} \frac{\eta_\beta(\eta_\beta - z_j)^l}{l!|z_j-\eta_\beta|^2}\frac{\partial^l}{\partial z_j^l}\left(\frac{\varphi_{nm}(z_1,\ldots,z_M;h_1,h_2)}{z_j}\right)\\
&= \sum_{j=1}^{M}\sum_{l=0}^{N-1} A_l \frac{z_j^{l+1}}{l!}\frac{\partial^l}{\partial z_j^l}\frac{\varphi_{mn}}{z_j}\\
&= \sum_{j=1}^{M}\left(\frac{(N-1)(N-5)}{12}z_j - \frac{N-3}{2}z_j^2\frac{\partial}{\partial z_j} + \frac{1}{2}z_j^3\frac{\partial^2}{\partial z_j^2}\right)\frac{\varphi_{mn}}{z_j}\\
&= \left\{\frac{M}{12}(5-2N)h_1^m h_2^n - h_1^m h_2^n \sum_{i\neq j}^{M}\frac{1}{|z_i-z_j|^2} - h_1^m h_2^{n+2}\frac{\partial^2}{\partial h_2^2} - h_1^{m+2}h_2^n\frac{\partial^2}{\partial h_1^2}\right.\\
&\qquad + \frac{N-3}{2}\left(h_1^m h_2^{n+1}\frac{\partial}{\partial h_2} + h_1^{m+1}h_2^n\frac{\partial}{\partial h_1}\right) + \frac{h_1^m h_2^{n+2}}{h_1-h_2}\frac{\partial}{\partial h_2}\\
&\qquad \left. -\frac{h_1^{m+2}h_2^n}{h_1-h_2}\frac{\partial}{\partial h_1}\right\}\frac{\varphi_{mn}}{h_1^m h_2^n},
\end{aligned}
\tag{A.4}
$$

where we have introduced the coefficients $A_l = -\sum_{\alpha=1}^{N-1}\eta_\alpha^2(\eta_\alpha - 1)^{l-2}$. Evaluation of the latter yields $A_0 = (N-1)(N-5)/12$, $A_1 = -(N-3)/2$, $A_2 = 1$, and $A_l = 0$ for $2 < l \leq N-1$ [14].

For the spin-Ising term we obtain

$$
\begin{aligned}
&\left[H_{\mathrm{S}}^{\mathrm{Is}}\varphi_{mn}\right](z_1,\ldots,z_M;h_1,h_2) \equiv \left[\sum_{\alpha\neq\beta}^{N}\frac{P_{\mathrm{G}}S_\alpha^z S_\beta^z P_{\mathrm{G}}}{|\eta_\alpha-\eta_\beta|^2}\varphi_{mn}\right](z_1,\ldots,z_M;h_1,h_2)\\
&= \left\{\sum_{i\neq j}^{M}\frac{1}{|z_i-z_j|^2} + \sum_{i=1}^{M}\frac{1}{|z_i-h_1|^2} + \sum_{i=1}^{M}\frac{1}{|z_i-h_2|^2} + \frac{1/2}{|h_1-h_2|^2}\right.\\
&\qquad \left. - N\frac{N^2-1}{48}\right\}\varphi_{mn}.
\end{aligned}
\tag{A.5}
$$

The potential term yields

$$
\begin{aligned}
&\left[H_{\mathrm{V}}\varphi_{mn}\right](z_1,\ldots,z_M;h_1,h_2)\\
&\qquad \equiv \left[\sum_{\alpha\neq\beta}^{N}\frac{P_{\mathrm{G}}\left(-\frac{1}{4}n_\alpha n_\beta + n_\alpha - \frac{1}{2}\right)P_{\mathrm{G}}}{|\eta_\alpha-\eta_\beta|^2}\varphi_{mn}\right](z_1,\ldots,z_M;h_1,h_2)\\
&= \left\{-\frac{1}{2}\frac{1}{|h_1-h_2|^2} - \frac{N^2-1}{12} + \frac{N}{4}\frac{N^2-1}{12}\right\}\varphi_{mn}.
\end{aligned}
\tag{A.6}
$$

The charge kinetic terms deserve particular care as new techniques are required. For the $\uparrow$-charge kinetic term, we first observe that

$[c_{\beta\uparrow}c^{\dagger}_{\alpha\uparrow}\varphi_{mn}](z_1,...,z_M;h_1,h_2)$ is identically zero unless one of the arguments $z_1,...,z_M$ equals η_α. We thus find

$$\begin{aligned}
\Big[H^{\uparrow}_{\mathrm{C}}\varphi_{mn}\Big](z_1,\dots,z_M;h_1,h_2) &\equiv \left[\sum_{\alpha\neq\beta}^{N}\frac{P_{\mathrm{G}}c_{\beta\uparrow}c^{\dagger}_{\alpha\uparrow}P_{\mathrm{G}}}{|\eta_\alpha-\eta_\beta|^2}\varphi_{mn}\right](z_1,\dots,z_M;h_1,h_2)\\
= &\sum_{\alpha=h_1,h_2}\sum_{\beta\neq\alpha}^{N}\frac{1}{|\eta_\alpha-\eta_\beta|^2}\varphi_{mn}\\
= &\sum_{\beta\neq h_2}^{N}\frac{\varphi_{mn}(z_1,\dots,z_M;h_1,\eta_\beta)}{|h_2-\eta_\beta|^2}+\sum_{\beta\neq h_1}^{N}\frac{\varphi_{mn}(z_1,\dots,z_M;\eta_\beta,h_2)}{|h_1-\eta_\beta|^2}\\
= &\sum_{l=0}^{M}\sum_{\beta\neq h_2}^{N}\frac{\eta_\beta^n(\eta_\beta-h_2)^l}{l!|h_2-\eta_\beta|^2}\frac{\partial^l}{\partial\eta_\beta^l}\left(\frac{\varphi_{mn}(z_1,\dots,z_M;h_1,\eta_\beta)}{\eta_\beta^n}\right)\Bigg|_{\eta_\beta=h_2}\\
&+\sum_{l=0}^{M}\sum_{\beta\neq h_1}^{N}\frac{\eta_\beta^m(\eta_\beta-h_1)^l}{l!|h_1-\eta_\beta|^2}\frac{\partial^l}{\partial\eta_\beta^l}\left(\frac{\varphi_{mn}(z_1,\dots,z_M;\eta_\beta,h_2)}{\eta_\beta^m}\right)\Bigg|_{\eta_\beta=h_1}\\
= &\sum_{l=0}^{M}B_l^m\frac{\partial^l}{\partial h_1}\left(\frac{\varphi_{mn}}{h_1^m}\right)+\sum_{l=0}^{M}B_l^n\frac{\partial^l}{\partial h_2}\left(\frac{\varphi_{mn}}{h_2^n}\right)\\
= &\left\{\left(\frac{N^2-1}{6}+\frac{m(m-N)}{2}+\frac{n(n-N)}{2}\right)h_1^m h_2^n\right.\\
&-\left(\frac{N-1}{2}-m\right)h_1^{m+1}h_2^n\frac{\partial}{\partial h_1}-\left(\frac{N-1}{2}-n\right)h_1^m h_2^{n+1}\frac{\partial}{\partial h_2}\\
&\left.+\frac{1}{2}h_1^{m+2}h_2^n\frac{\partial^2}{\partial h_1^2}+\frac{1}{2}h_1^m h_2^{n+2}\frac{\partial^2}{\partial h_2^2}\right\}\frac{\varphi_{mn}(z_1,\dots,z_M;h_1,h_2)}{h_1^m h_2^n},
\end{aligned}\tag{A.7}$$

where we have introduced the coefficients $B_l^n=\sum_{\beta\neq h}^{N}\eta_\beta^n(\eta_\beta-h)^l/l!|h-\eta_\beta|^2$, which are evaluated in Appendix A.7. (A.7) is valid if and only if $0\leq n,m\leq(N+2)/2$, which finally leads to the restriction (2.23) for the actual Ψ^{ho}_{mn}'s.

For the treatment of the $\downarrow$-charge kinetic term we avail ourselves of the fact that φ_{mn} can be equally expressed by the up-spin or down-spin variables, as we show in Appendix A.2. If we denote the down-spin coordinates by w_i, we obtain

$$\begin{aligned}
\Big[H^{\downarrow}_{\mathrm{C}}\varphi_{mn}\Big](z_1,\dots,z_M;h_1,h_2) &\equiv \left[\sum_{\alpha\neq\beta}^{N}\frac{P_{\mathrm{G}}c_{\beta\downarrow}c^{\dagger}_{\alpha\downarrow}P_{\mathrm{G}}}{|\eta_\alpha-\eta_\beta|^2}\varphi_{mn}\right](z_1,\dots,z_M;h_1,h_2)\\
= &\left\{\left(\frac{N^2-1}{6}+\frac{m(m-N)}{2}+\frac{n(n-N)}{2}\right)h_1^m h_2^n\right.
\end{aligned}$$

$$
- \left(\frac{N-1}{2} - m \right) h_1^{m+1} h_2^n \frac{\partial}{\partial h_1} - \left(\frac{N-1}{2} - n \right) h_1^m h_2^{n+1} \frac{\partial}{\partial h_2}
+ \frac{1}{2} h_1^{m+2} h_2^n \frac{\partial^2}{\partial h_1^2} + \frac{1}{2} h_1^m h_2^{n+2} \frac{\partial^2}{\partial h_2^2} \Bigg\} \frac{\varphi_{mn}(w_1, \ldots, w_M; h_1, h_2)}{h_1^m h_2^n}. \qquad \text{(A.8)}
$$

Using identities verified in Appendix A.2.1 for the derivatives with respect to the z_i's and $h_{1,2}$'s, the total charge-kinetic term becomes

$$
\begin{aligned}
&\left[\sum_{\alpha \neq \beta}^{N} \frac{H_{\mathrm{C}}^{\downarrow} + H_{\mathrm{C}}^{\uparrow}}{|\eta_\alpha - \eta_\beta|^2} \varphi_{mn} \right] (z_1, \ldots, z_M; h_1, h_2) \\
&= \Bigg\{ \Bigg[\frac{N^2-1}{3} + m(m-N) + n(n-N) - C_2 - C_1^2 + m \left(C_1 - \frac{1}{2} \right) \\
&\quad + n \left(C_1 - \frac{1}{2} \right) - \frac{h_1 + h_2}{h_1 - h_2} \left(\frac{m-n}{2} \right) \Bigg] h_1^m h_2^n + h_1^{m+2} h_2^n \frac{\partial^2}{\partial h_1^2} \\
&\quad + h_1^m h_2^{n+2} \frac{\partial^2}{\partial h_2^2} + h_1^m h_2^{n+1} \frac{h_2}{(h_1 - h_2)} \frac{\partial}{\partial h_2} + h_1^{m+1} h_1^n \frac{h_1}{h_2 - h_1} \frac{\partial}{\partial h_1} \\
&\quad + C_1 h_1^{m+1} h_2^n \frac{\partial}{\partial h_1} + C_1 h_1^m h_2^{n+1} \frac{\partial}{\partial h_2} + h_1^m h_2^n \sum_i^M \frac{h_2^2}{(z_i - h_2)^2} \\
&\quad + h_1^m h_2^n \sum_i^M \frac{h_1^2}{(z_i - h_1)^2} + \frac{h_1^2 + h_2^2}{(h_1 - h_2)^2} \Bigg\} \frac{\varphi_{mn}(z_1, \ldots, z_M; h_1, h_2)}{h_1^m h_2^n},
\end{aligned} \qquad \text{(A.9)}
$$

with the constants $C_1 = \sum_{\alpha=1}^{N-1} 1/(1-\eta_\alpha) = (N-1)/2$ and $C_2 = \sum_{\alpha=1}^{N-1} 1/(1-\eta_\alpha)^2 = (6N - 5 - N^2)/12$ introduced and evaluated in [14]. Summing up all terms, we finally obtain the action of the Hamiltonian (2.2) on the auxiliary wave functions φ_{mn}:

$$
\begin{aligned}
H_{\mathrm{KY}} \varphi_{mn} &= \frac{2\pi^2}{N^2} \Bigg\{ \frac{8-9N}{8} + m(m-N) + n(n-N) + m\frac{N-2}{2} + n\frac{N-2}{2} \\
&\quad - \frac{1}{2} \frac{h_1 + h_2}{h_1 - h_2} (m-n) + \frac{h_1^2 + h_2^2}{(h_1 - h_2)^2} \Bigg\} \varphi_{mn}.
\end{aligned} \qquad \text{(A.10)}
$$

With (A.2) this implies

$$
\begin{aligned}
H_{\mathrm{KY}} \Psi_{mn}^{\mathrm{ho}} &= \frac{2\pi^2}{N^2} \Bigg[\left(-\frac{8+N}{8} + 1 + \left(m - \frac{N}{2} \right) m + \left(n - \frac{N}{2} \right) n \right) \Psi_{mn}^{\mathrm{ho}} \\
&\quad + \frac{m-n}{2} (h_1 - h_2) \frac{h_1 + h_2}{h_1 - h_2} (h_1^m h_2^n - h_1^n h_2^m) \Psi_0 \Bigg]
\end{aligned}
$$

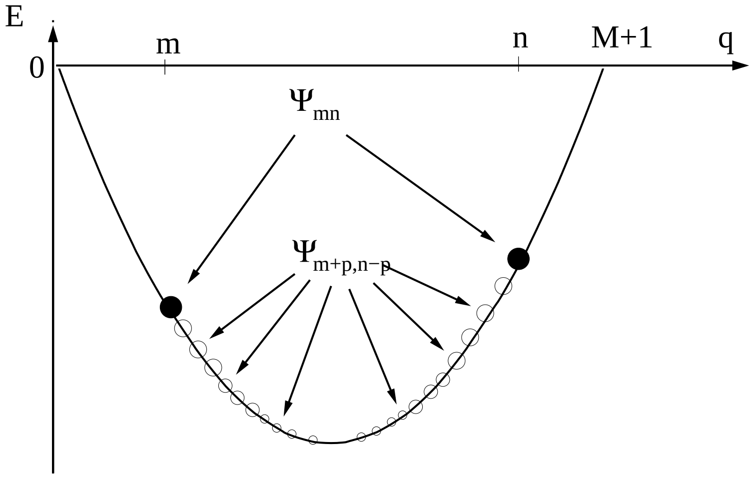


Figure A.1: The scattering of Ψ_{nm} to lower energy states

$$= \frac{2\pi^2}{N^2}\left[-\frac{N}{8} + \left(m - \frac{N}{2}\right)m + \left(n - \frac{N}{2}\right)n + \frac{m-n}{2}\right]\Psi^{\text{ho}}_{mn}$$
$$+\frac{2\pi^2}{N^2}(m-n)\sum_{l=1}^{\lfloor\frac{m-n}{2}\rfloor}\Psi^{\text{ho}}_{m-l,n+l}, \tag{A.11}$$

where we have used $\frac{x+y}{x-y}(x^m y^n - x^n y^m) = 2\sum_{l=0}^{m-n} x^{m-l}y^{n+l} - (x^m y^n + x^n y^m)$. The symbol $\lfloor\ \rfloor$ denotes the floor function, *i.e.*, $\lfloor x \rfloor$ is the largest integer $l \leq x$. First, note that the action of the Hamiltonian on the Ψ^{ho}_{mn}'s is trigonal, *i.e.*, the "scattering" in the last line is only to lower values of $m-n$ (see Fig. A.1). Second, (A.11) shows that the states Ψ^{ho}_{mn} form a non-orthogonal set.

A.2 SU(2) Wave function in ↑- and ↓-spin coordinates

The wave functions Ψ^{ho}_{mn} can be equally expressed either in up-spin (z) or down-spin coordinates (w):

$$\frac{\Psi^{\text{ho}}_{mn}(z_1,...,z_M;h_1,h_2)}{h_1^m h_2^n(h_1-h_2)}$$
$$= (-1)^{\frac{1}{2}M(M+1)}\frac{\prod_j^M(h_1-z_j)(h_2-z_j)z_j\prod_{i\neq j}^M(z_i-z_j)\prod_{l,j}^M(w_l-z_j)}{\prod_{l,j}^M(w_l-z_j)}$$
$$= \frac{\Psi^{\text{ho}}_{mn}(w_1,...,w_M;h_1,h_2)}{h_1^m h_2^n(h_1-h_2)}. \tag{A.12}$$

This identity applies to the auxiliary wave functions φ_{mn}, as the prepolynomial contains only the coordinates $h_{1,2}$.

A.2.1 SU(2) derivative identity

The necessary relation for the $\downarrow$-charge kinetic term is

$$
\begin{aligned}
&\sum_{i\neq j} \frac{h_2^2}{(w_i-h_2)(w_j-h_2)} \\
= \;& -C_2 + C_1^2 + 2\sum_i \frac{h_2^2}{(z_i-h_2)^2} + \sum_{i\neq j}\frac{h_2}{z_i-h_2}\frac{h_2}{z_j-h_2} + 2\frac{h_2^2}{(h_1-h_2)^2} \\
& +2C_1\sum_i \frac{h_2}{z_i-h_2} + 2C_1\frac{h_2}{h_1-h_2} + 2\frac{h_2}{h_1-h_2}\sum_i\frac{h_2}{z_i-h_2}, \qquad \text{(A.13)}
\end{aligned}
$$

with C_1 and C_2 defined as above. (A.13) is also valid for $h_1 \leftrightarrow h_2$.

A.3 One-holon excitations in the SU(3) KYM

To evaluate the action of H_{KY} on $\left|\Psi_m^{\mathrm{ho}}\right\rangle$, we first replace $e_\alpha^{\mathrm{gg}}e_\beta^{\mathrm{gg}}$ by $(1-h_\alpha - e_\alpha^{\mathrm{rr}} - e_\alpha^{\mathrm{bb}})(1-h_\beta - e_\beta^{\mathrm{rr}} - e_\beta^{\mathrm{bb}})$, where h_α denotes the hole occupation operator $h_\alpha = 1-n_\alpha$, and rewrite the Hamiltonian in a way to easily treat terms separately in the following:

$$
\begin{aligned}
H_{\mathrm{KY}} \;=\; & \frac{2\pi^2}{N^2}\sum_{\alpha\neq\beta}^N \frac{1}{|\eta_\alpha-\eta_\beta|^2}\left(e_\alpha^{\mathrm{bg}}e_\beta^{\mathrm{gb}} + e_\alpha^{\mathrm{rg}}e_\beta^{\mathrm{gr}} + e_\alpha^{\mathrm{br}}e_\beta^{\mathrm{rb}}\right) \\
& + \frac{2\pi^2}{N^2}\sum_{\alpha\neq\beta}^N \frac{1}{|\eta_\alpha-\eta_\beta|^2}\left(e_\alpha^{\mathrm{bb}}e_\beta^{\mathrm{bb}} + e_\alpha^{\mathrm{rr}}e_\beta^{\mathrm{rr}} + e_\alpha^{\mathrm{bb}}e_\beta^{\mathrm{rr}}\right) \\
& - \frac{2\pi^2}{N^2}\sum_{\alpha\neq\beta}^N \frac{1}{|\eta_\alpha-\eta_\beta|^2}\left(e_\alpha^{\mathrm{bb}} + e_\alpha^{\mathrm{rr}}\right) + \frac{2\pi^2}{N^2}\sum_{\alpha\neq\beta}^N \frac{1}{|\eta_\alpha-\eta_\beta|^2}\left(n_\alpha - \frac{1}{2}\right) \\
& + \frac{2\pi^2}{N^2}\sum_{\alpha\neq\beta}^N \frac{1}{|\eta_\alpha-\eta_\beta|^2}\left(e_\alpha^{\mathrm{rr}} + e_\alpha^{\mathrm{bb}}\right)(1-n_\beta) \\
& + \frac{2\pi^2}{N^2}\sum_{\alpha\neq\beta}^N \frac{1}{|\eta_\alpha-\eta_\beta|^2}\left[\frac{1}{2}(c_{\alpha\mathrm{r}}c_{\beta\mathrm{r}}^\dagger + c_{\alpha\mathrm{g}}c_{\beta\mathrm{g}}^\dagger) + \frac{1}{2}(c_{\alpha\mathrm{b}}c_{\beta\mathrm{b}}^\dagger + c_{\alpha\mathrm{r}}c_{\beta\mathrm{r}}^\dagger)\right. \\
& \left. + \frac{1}{2}(c_{\alpha\mathrm{b}}c_{\beta\mathrm{b}}^\dagger + c_{\alpha\mathrm{g}}c_{\beta\mathrm{g}}^\dagger)\right]. \qquad \text{(A.14)}
\end{aligned}
$$

The first term $[e_\alpha^{\rm bg} e_\beta^{\rm gb} \Psi_m^{\rm ho}][z_i; w_k; h]$, which vanishes unless one of the z_i's is equal to η_α, yields through Taylor expansion

$$\begin{aligned}
&\left[\sum_{\alpha\neq\beta}^{N} \frac{e_\alpha^{\rm bg} e_\beta^{\rm gb}}{|\eta_\alpha-\eta_\beta|^2}\Psi_m^{\rm ho}\right][z_i; w_k; h] \\
=& \sum_{i=1}^{M_1}\sum_{\beta\neq i}^{N} \frac{\eta_\beta}{|z_i-\eta_\beta|^2}\frac{\Psi_m^{\rm ho}[\ldots, z_{i-1}, \eta_\beta, z_{i+1}, \ldots; w_k; h]}{\eta_\beta} \\
=& \sum_{i=1}^{M_1}\sum_{m=0}^{N-1} \frac{A_m z_i^{m+1}}{m!}\frac{\partial^m}{\partial z_i^m}\frac{\Psi_m^{\rm ho}}{z_i} \\
=& \frac{M_1}{12}(N^2+8M_1^2-6M_1(N+1)+3)\,\Psi_m^{\rm ho} \qquad \text{(A.15)}\\
&-\frac{N-3}{2}\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\frac{z_i}{z_i-w_k}\Psi_m^{\rm ho} + \sum_{i\neq j}^{M_1}\frac{z_i^2}{(z_i-z_j)^2}\Psi_m^{\rm ho} \qquad \text{(A.16)}\\
&+2\sum_{i\neq j}^{M_1}\sum_{k=1}^{M_2}\frac{z_i^2}{(z_i-z_j)(z_i-w_k)}\Psi_m^{\rm ho} \qquad \text{(A.17)}\\
&+\frac{1}{2}\sum_{i=1}^{M_1}\sum_{k\neq l}^{M_2}\frac{z_i^2}{(z_i-w_k)(z_i-w_l)}\Psi_m^{\rm ho} \qquad \text{(A.18)}\\
&+\sum_{\substack{i,j=1\\ i\neq j}}^{M_1}\frac{2z_i^2}{(z_i-z_j)(z_i-h)}\Psi_m^{\rm ho} - \frac{N-3}{2}\sum_{i=1}^{M_1}\frac{z_i}{z_i-h}\Psi_m^{\rm ho} \qquad \text{(A.19)}\\
&+\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\frac{z_i^2}{(z_i-\omega_k)(z_i-h)}\Psi_m^{\rm ho}\,, \qquad \text{(A.20)}
\end{aligned}$$

where we have used $\deg_{z_i}\Psi_m^{\rm ho}[z_i; w_k; h] = N-1$ and $A_m \equiv -\sum_{\alpha=1}^{N-1}\eta_\alpha^2(\eta_\alpha - 1)^{m-2}$. The evaluation of the latter yields $A_0 = (N-1)(N-5)/12$, $A_1 = -(N-3)/2$, $A_2 = 1$, and $A_m = 0$ for $2 < m \leq N-1$, as previously used in Sec. A.1. Furthermore, we have used

$$\frac{x^2}{(x-y)(x-z)} + \frac{y^2}{(y-x)(y-z)} + \frac{z^2}{(z-x)(z-y)} = 1, \quad x, y, z \in \mathbb{C}. \qquad \text{(A.21)}$$

The second term $[e_\alpha^{\rm rg} e_\beta^{\rm gr}\Psi_m^{\rm ho}][z_i; w_k]$ can be treated in the same way. It generates terms which together with the first term in (A.16) yield

$$-\frac{N-3}{2}\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\frac{z_i}{z_i-w_k} + \frac{N-3}{2}\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\frac{w_k}{z_i-w_k} = -\frac{N-3}{2}M_1 M_2,$$

and with one part of (A.17) and (A.18)

$$\sum_{\substack{i,j=1\\i\neq j}}^{M_1}\sum_{k=1}^{M_2}\left(\frac{z_i^2}{(z_i-z_j)(z_i-w_k)}+\frac{1}{2}\frac{w_k^2}{(z_i-w_k)(z_j-w_k)}\right)=\frac{1}{2}M_1(M_1-1)M_2,$$

as well as similar expressions for $z_i \leftrightarrow w_k$.

For the third term of (A.14) we obtain

$$\begin{aligned}
&\left[\sum_{\alpha\neq\beta}^{N}\frac{e_\alpha^{\rm br}e_\beta^{\rm rb}}{|\eta_\alpha-\eta_\beta|^2}\Psi_m^{\rm ho}\right][z_i;w_k]\\
=\;&\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\frac{z_iw_k}{(z_i-w_k)^2}\prod_{j\neq i}^{M_1}\left(1+\frac{z_i-w_k}{z_j-z_i}\right)\prod_{l\neq k}^{M_2}\left(1-\frac{z_i-w_k}{w_l-w_k}\right)\Psi_m^{\rm ho}\\
=\;&\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\frac{z_iw_k}{(z_i-w_k)^2}\Psi_m^{\rm ho}-\sum_{i\neq j}^{M_1}\sum_{k=1}^{M_2}\frac{z_iw_k}{(z_i-z_j)(z_i-w_k)}\Psi_m^{\rm ho} &\text{(A.22)}\\
&-\sum_{i=1}^{M_1}\sum_{k\neq l}^{M_2}\frac{z_iw_k}{(w_k-z_i)(w_k-w_l)}\Psi_m^{\rm ho} &\text{(A.23)}\\
&+\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\sum_{m=2}^{M_1-1}\frac{1}{m!}\sum_{\{a_j\}}\frac{z_iw_k(z_i-w_k)^{m-2}}{(z_{a_1}-z_i)\cdots(z_{a_m}-z_i)}\Psi_m^{\rm ho} &\text{(A.24)}\\
&+\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\sum_{n=2}^{M_2-1}\frac{(-1)^n}{n!}\sum_{\{b_l\}}\frac{z_iw_k(z_i-w_k)^{n-2}}{(w_{b_1}-w_k)\cdots(w_{b_n}-w_k)}\Psi_m^{\rm ho} &\text{(A.25)}\\
&+\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\sum_{m=1}^{M_1-1}\sum_{n=1}^{M_2-1}\frac{(-1)^n}{m!n!}\\
&\times\sum_{\{a_j\}\{b_l\}}\frac{z_iw_k(z_i-w_k)^{m+n-2}}{(z_{a_1}-z_i)\cdots(z_{a_m}-z_i)(w_{b_1}-w_k)\cdots(w_{b_n}-w_k)}\Psi_m^{\rm ho}, &\text{(A.26)}
\end{aligned}$$

where $\{a_j\}$ ($\{b_l\}$) is a set of integers between 1 and M_1 (M_2). The summations run over all possible ways to distribute the z_{a_j} (w_{b_l}) over the blue (red) coordinates, where z_i (w_k) is excluded. The two terms (A.24) and (A.25) vanish due to

Theorem A.1 *Let $M\geq 3$, $z\in\mathbb{C}$, and $z_1,\ldots,z_M\in\mathbb{C}$ distinct. Then,*

$$\sum_{i=1}^{M}\frac{z_i(z_i-z)^{M-3}}{\prod_{j\neq i}^{M}(z_j-z_i)}=0. \tag{A.27}$$

A proof of Theorem A.1 is given in [111]. The last term (A.26) can be simplified using a theorem due to Ha and Haldane [52]:

Theorem A.2 *Let $\{a_j\}$ be a set of distinct integers between* 1 *and M_1, and $\{b_l\}$ a set of distinct integers between* 1 *and M_2. Then,*

$$\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\sum_{m=1}^{M_1-1}\sum_{n=1}^{M_2-1}\sum_{\{a_j\}\{b_l\}}\frac{(-1)^n}{m!n!}\frac{z_i w_k(z_i-w_k)^{m+n-2}}{(z_{a_1}-z_i)\cdots(z_{a_n}-z_i)(w_{b_1}-w_k)\cdots(w_{b_m}-w_k)}$$
$$=-\sum_{m=1}^{\min(M_1,M_2)}(M_1-m)(M_2-m).$$

Furthermore, the two terms in line (A.23), together with the remainder of (A.17) and the corresponding expression from the second term of the Hamiltonian, can be simplified to yield $M_1M_2(M_1+M_2-2)\Psi_m^{\mathrm{ho}}/2$. The diagonal contributions, *i.e.,* the 2nd and 3rd line of (A.14), yield

$$\begin{aligned}
&\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{N}\frac{e_\alpha^{\mathrm{bb}}e_\beta^{\mathrm{bb}}+e_\alpha^{\mathrm{rr}}e_\beta^{\mathrm{rr}}+e_\alpha^{\mathrm{bb}}e_\beta^{\mathrm{rr}}-e_\alpha^{\mathrm{rr}}-e_\beta^{\mathrm{bb}}+n_\alpha-\frac{1}{2}}{|\eta_\alpha-\eta_\beta|^2}\Psi_m^{\mathrm{ho}}[z_i;w_k;h]\\
&=\quad\left(\frac{1}{2}M_1(M_1-1)+\frac{1}{2}M_2(M_2-1)\right)\Psi_m^{\mathrm{ho}}-\sum_{\substack{i,j=1\\ i\neq j}}^{M_1}\frac{z_i^2}{(z_i-z_j)^2}\Psi_m^{\mathrm{ho}}\\
&\quad-\sum_{\substack{k,l=1\\ k\neq l}}^{M_2}\frac{\omega_k^2}{(\omega_k-\omega_l)^2}\Psi_m^{\mathrm{ho}}-\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\frac{\omega_k z_i}{(z_i-\omega_k)^2}\Psi_m^{\mathrm{ho}}\\
&\quad-\frac{N^2-1}{12}\left(\frac{N}{2}+M_1+M_2-1\right)\Psi_m^{\mathrm{ho}},
\end{aligned}\tag{A.28}$$

by which the remainder of (A.16) as well as (A.22) are cancelled. The remaining spin terms yield

$$\begin{aligned}
&\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{N}\frac{\left(e_\alpha^{\mathrm{rr}}+e_\alpha^{\mathrm{bb}}\right)(1-n_\beta)}{|\eta_\alpha-\eta_\beta|^2}\Psi_m^{\mathrm{ho}}[z_i;w_k;h]\\
&=\left(\sum_{i=1}^{M_1}\frac{1}{|z_i-h|^2}+\sum_{k=1}^{M_2}\frac{1}{|w_k-h|^2}\right)\Psi_m^{\mathrm{ho}}.
\end{aligned}\tag{A.29}$$

We now turn to the treatment of the remaining charge kinetic term contributions which are technically new compared to the calculation methods introduced in previous works for the colorons in the SU(3) HSM [111] and holons in the SU(2) KYM [123].

Again we use a Taylor expansion technique in the way that derivative operators act on the analytic extension of the wave function. In this case, it is crucial that the fermion creation and annihilation operators appearing in the expansion match with the variables of the holon wave function which is analytically extended. As it is the case for the ground state, the one-holon wave function (3.19) can be equally expressed by an arbitrary pair of sets of color variables. In (A.14) we have thus written the charge terms in a symmetric way that respectively puts two pairs of fermion creation and annihilation operators together. For the first term we get

$$
\begin{aligned}
&\left[\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{N} \frac{1}{2}\frac{c_{\alpha\mathrm{b}}c^{\dagger}_{\beta\mathrm{b}}+c_{\alpha\mathrm{r}}c^{\dagger}_{\beta\mathrm{r}}}{|\eta_\alpha-\eta_\beta|^2}\right]\Psi^{\mathrm{ho}}_{m}[z_i;w_k;h]\\
=\;&\left[\sum_{\alpha\neq\beta}^{N}\frac{c_{\alpha v}c^{\dagger}_{\beta v}}{|\eta_\alpha-\eta_\beta|^2}\right]\Psi^{\mathrm{ho}}_{m}[v_1,\ldots,v_{M_1+M_2};h]\\
=\;&\sum_{\substack{\beta=1\\ \beta\neq h}}^{N}\frac{\eta_\beta^m}{|h-\eta_\beta|^2}\frac{\Psi^{\mathrm{ho}}_{m}[v_1,\ldots,v_{M_1+M_2};\eta_\beta]}{\eta_\beta^n}\\
=\;&\sum_{l=0}^{M_1+M_2}\sum_{\substack{\beta=1\\ \beta\neq h}}^{N}\frac{\eta_\beta^n(\eta_\beta-h)^l}{l!|h-\eta_\beta|^2}\frac{\partial^l}{\partial\eta_\beta^l}\left(\frac{\Psi^{\mathrm{ho}}_{m}[v_1,\ldots,v_{M_1+M_2};\eta_\beta]}{\eta_\beta^m}\right)\Big|_{\eta_\beta=h}\\
=\;&\sum_{l=0}^{M_1+M_2}\frac{B_l^m}{l!}\frac{\partial^l}{\partial h^l}\left(\frac{\Psi^{\mathrm{ho}}_{m}[z_1,\ldots,z_{M_1};\omega_1,\ldots,\omega_{M_2};h]}{h^n}\right)\\
=\;&\Bigg\{\Big(\frac{N^2-1}{12}+\frac{n(n-N)}{2}\Big)h^n-\Big(\frac{N-1}{2}-n\Big)h^{n+1}\frac{\partial}{\partial h}\\
&+\frac{1}{2}h^{n+2}\frac{\partial^2}{\partial h^2}\Bigg\}\frac{\Psi^{\mathrm{ho}}_{n}[z_1,\ldots,z_{M_1};\omega_1,\ldots,\omega_{M_2};h]}{h^n}\\
=\;&-\Big(\frac{N-1}{2}-n\Big)\Big[\sum_{i=1}^{M_1}\frac{h}{h-z_i}+\sum_{k=1}^{M_2}\frac{h}{h-\omega_k}\Big]\Psi^{\mathrm{ho}}_{m}\\
&+\frac{1}{2}\Bigg[\sum_{\substack{i,j\\ i\neq j}}^{M_1}\frac{h^2}{(h-z_i)(h-z_j)}+2\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\frac{h}{h-z_i}\frac{h}{h-\omega_k}
\end{aligned}
$$

$$+\sum_{\substack{k,l\\k\neq l}}^{M_2}\frac{h^2}{(h-\omega_k)(h-\omega_l)}\Bigg]\Psi_m^{\mathrm{ho}}+\Big(\frac{N^2-1}{12}+\frac{n(n-N)}{2}\Big)\Psi_m^{\mathrm{ho}},\tag{A.30}$$

where we introduced $B_l^m=h^{n+l}\sum_\beta^{N-1}\eta_\beta^{m+1}(\eta_\beta-1)^{l-2}$. The evaluation of the latter is shown in Appendix A.7 and yields $B_0^m=(N^2-1)/12+m(m-N)/2$, $B_1^m=m-(N-1)/2$, $B_2^m=1$, and $B_l=0$ for $3\leq l$ and $0\leq m\leq(N+2)/3$, by which (3.20) follows.

For the other charge term contributions we expound the fact that the holon wave function can be expressed by either pairs of sets of color variables as shown in Appendix A.5 and the same formalism is used as before. The appearing terms involving the green color variables are rewritten in terms of the blue and red variables, *i.e.*, z's and w's, by identities given in Appendix A.5.1. It thus yields

$$\begin{aligned}&\left[\sum_{\substack{\alpha,\beta=1\\\alpha\neq\beta}}^{N}\frac{1}{2}\frac{c_{\alpha\mathrm{b}}c^\dagger_{\beta\mathrm{b}}+c_{\alpha\mathrm{g}}c^\dagger_{\beta\mathrm{g}}}{|\eta_\alpha-\eta_\beta|^2}\right]\Psi_m^{\mathrm{ho}}[z_i;w_k;h]\\
=\;&\Big(\frac{N^2-1}{12}+\frac{n(n-N)}{2}\Big)\Psi_m^{\mathrm{ho}}-\Big(\frac{N-1}{2}-n\Big)\Bigg[C_1-\sum_{k=1}^{M_2}\frac{h}{h-\omega_k}\Bigg]\Psi_m^{\mathrm{ho}}\\
&+\frac{1}{2}\Bigg[C_1^2-C_2-2C_1\sum_{k=1}^{M_2}\frac{h}{h-\omega_k}+2\sum_{k=1}^{M_2}\frac{h^2}{(h-\omega_k)^2}\\
&+\sum_{\substack{k,l\\k\neq l}}^{M_2}\frac{h^2}{(h-\omega_k)(h-\omega_l)}\Bigg]\Psi_m^{\mathrm{ho}},\end{aligned}\tag{A.31}$$

as well as

$$\begin{aligned}&\left[\sum_{\substack{\alpha,\beta=1\\\alpha\neq\beta}}^{N}\frac{1}{2}\frac{c_{\alpha\mathrm{r}}c^\dagger_{\beta\mathrm{r}}+c_{\alpha\mathrm{g}}c^\dagger_{\beta\mathrm{g}}}{|\eta_\alpha-\eta_\beta|^2}\right]\Psi_m^{\mathrm{ho}}[z_i;w_k;h]\\
=\;&\Big(\frac{N^2-1}{12}+\frac{n(n-N)}{2}\Big)\Psi_m^{\mathrm{ho}}-\Big(\frac{N-1}{2}-n\Big)\Big[C_1-\sum_{i=1}^{M_1}\frac{h}{h-z_i}\Big]\Psi_m^{\mathrm{ho}}\\
&+\frac{1}{2}\Big[C_1^2-C_2-2C_1\sum_{i=1}^{M_1}\frac{h}{h-z_i}+2\sum_{i=1}^{M_1}\frac{h^2}{(h-z_i)^2}\end{aligned}$$

$$+\sum_{\substack{i,j\\i\neq j}}^{M_1}\frac{h^2}{(h-z_i)(h-z_j)}\Big]\Psi_m^{\mathrm{ho}}, \tag{A.32}$$

where we have defined the constants $C_1=\sum_\alpha^{N-1}1/(1-\eta_\alpha)=(N-1)/2$ and $C_2=\sum_\alpha^{N-1}1/(1-\eta_\alpha)^2=(6N-5-N^2)/12$ as was derived in [14]. Due to

$$\sum_{\substack{i,j=1\\i\neq j}}^{M_1}\frac{2z_i^2}{(z_i-z_j)(z_i-h)}+\sum_{\substack{i,j=1\\i\neq j}}^{M_1}\frac{h^2}{(h-z_i)(h-z_j)}=M_1(M_1-1), \tag{A.33}$$

which gives $M_2(M_2-1)$ for the analogous contributions $z_i\leftrightarrow w_k$, and

$$\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\frac{z_i^2}{z_i-\omega_k}\frac{1}{z_i-h}+\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\frac{\omega_k^2}{\omega_k-z_i}\frac{1}{\omega_k-h}$$
$$+\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}\frac{h}{h-z_i}\frac{h}{h-\omega_k}=M_1M_2, \tag{A.34}$$

the remainders (A.19) and (A.20) are cancelled. Finally, (A.29) cancels the last non-diagonal terms of (A.31) and (A.32), and summing up all terms, we finally get

$$H_{\mathrm{KY}}\left|\Psi_m^{\mathrm{ho}}\right\rangle=E_m\left|\Psi_m^{\mathrm{ho}}\right\rangle, \tag{A.35}$$

$$E_m=\frac{2\pi^2}{N^2}\left[-\frac{1}{72}N^3-\frac{3}{72}N+\frac{4}{72}+\frac{3}{2}m(m-M-1)\right], \tag{A.36}$$

by which the one-holon energy (3.22) follows.

A.4 Two Holon excitations in the SU(3) KYM

Now we will construct the two-holon energy eigenstates starting from (3.24). The strategy is similar to the construction of the two-holon states in the SU(2) model as presented in [123]. We define the auxiliary wave functions

$$\begin{aligned}\varphi_{mn}[z_i;w_k;h_1,h_2] &= h_1^m h_2^n\prod_{i=1}^{M_1}(h_1-z_i)(h_2-z_i)\\ &\quad \prod_{k=1}^{M_2}(h_1-w_k)(h_2-w_k)\Psi_0[z_i;w_k]\\ &= \psi_{h_1h_2}\Psi_0[z_i;w_k].\end{aligned} \tag{A.37}$$

The action of the Hamiltonian (3.4) on the states (3.27) will be later obtained by

$$\Psi_{mn}^{\text{ho}} = \varphi_{m+1,n} + \varphi_{n+1,m} - \varphi_{m,n+1} - \varphi_{n,m+1}. \tag{A.38}$$

We rewrite the Hamiltonian as for the one-holon case (A.14) and treat each term separately. As some terms of (A.14) only generate contributions familiar from the one-holon calculation, we concentrate on the others. In all cases, the basic Taylor expansion technique is applied identically.

The contributions of $[e_\alpha^{\text{bg}} e_\beta^{\text{gb}} \varphi_{mn}^{\text{ho}}][z_i; w_k; h_1, h_2]$ split up into contributions of the ground state polynomial which we are already familiar with from the one-holon calculation and terms generated by the action on the prepolynomial ψ_{h_1,h_2}:

$$\begin{aligned}
\left[\sum_{\alpha\neq\beta}^{N} \frac{e_\alpha^{\text{bg}} e_\beta^{\text{gb}}}{|\eta_\alpha - \eta_\beta|^2} \varphi_{mn}^{\text{ho}}\right][z_i; w_k; h_1, h_2] &= \sum_{i=1}^{M_1} \sum_{m=0}^{N-1} \frac{A_m z_i^{m+1}}{m!} \frac{\partial^m}{\partial z_i^m} \frac{\varphi_{mn}^{\text{ho}}}{z_i} \\
= \quad & \frac{M_1}{12}(N^2 + 8M_1^2 - 6M_1(N+1) + 3)\, \varphi_{mn}^{\text{ho}} \\
& - \frac{N-3}{2} \sum_{i=1}^{M_1} \sum_{k=1}^{M_2} \frac{z_i}{z_i - w_k} \varphi_{mn}^{\text{ho}} + \sum_{i\neq j}^{M_1} \frac{z_i^2}{(z_i - z_j)^2} \varphi_{mn}^{\text{ho}} \\
& + 2 \sum_{i\neq j}^{M_1} \sum_{k=1}^{M_2} \frac{z_i^2}{(z_i - z_j)(z_i - w_k)} \Psi_m^{\text{ho}} + \frac{1}{2} \sum_{i=1}^{M_1} \sum_{k\neq l}^{M_2} \frac{z_i^2}{(z_i - w_k)(z_i - w_l)} \varphi_{mn}^{\text{ho}} \\
& + \Psi_0 \sum_{i=1}^{M_1} \left(\frac{1}{2} z_i^2 \frac{\partial^2}{\partial z_i^2} + \sum_{\substack{i,j=1 \\ i\neq j}}^{M_1} \frac{2 z_i^2}{z_i - z_j} \frac{\partial}{\partial z_i} - \frac{N-3}{2} z_i \frac{\partial}{\partial z_i} \right) \psi_{h_1,h_2} \qquad \text{(A.39)} \\
& + \Psi_0 \sum_{i=1}^{M_1} \sum_{k=1}^{M_2} \frac{z_i^2}{z_i - w_k} \frac{\partial}{\partial z_i} \psi_{h_1,h_2}, \qquad \text{(A.40)}
\end{aligned}$$

with notations as before. We explicitly write out (A.39) and (A.40) and thus get

$$\begin{aligned}
& \sum_{i=1}^{M_1} (1 - \frac{h_1^2}{(h_1 - h_2)(h_1 - z_i)} - \frac{h_2^2}{(h_2 - h_1)(h_2 - z_i)}) \varphi_{mn}^{\text{ho}} \\
& + \sum_{\substack{i,j=1 \\ i\neq j}}^{M_1} (1 - \frac{h_1^2}{(h_1 - z_i)(h_1 - z_j)}) \varphi_{mn}^{\text{ho}} + \sum_{\substack{i,j=1 \\ i\neq j}}^{M_1} (1 - \frac{h_2^2}{(h_2 - z_i)(h_2 - z_j)}) \varphi_{mn}^{\text{ho}} \\
& - \frac{N-3}{2} \sum_{i}^{M_1} (1 - \frac{h_1}{h_1 - z_i}) \varphi_{mn}^{\text{ho}} - \frac{N-3}{2} \sum_{i}^{M_1} (1 - \frac{h_2}{h_2 - z_i}) \varphi_{mn}^{\text{ho}}
\end{aligned}$$

$$+\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}(1-\frac{w_k^2}{(w_k-z_i)(w_k-h_1)}-\frac{h_1^2}{(h_1-z_i)(h_1-w_k)})\varphi_{mn}^{\text{ho}}$$
$$+\sum_{i=1}^{M_1}\sum_{k=1}^{M_2}(1-\frac{w_k^2}{(w_k-z_i)(w_k-h_2)}-\frac{h_2^2}{(h_2-z_i)(h_2-w_k)})\varphi_{mn}^{\text{ho}}, \qquad \text{(A.41)}$$

which identically holds for the term $[e_\alpha^{\text{rg}}e_\beta^{\text{gr}}\varphi_{mn}^{\text{ho}}][z_i;w_k;h_1,h_2]$ by interchanging w_k and z_i.

The spin term $[e_\alpha^{\text{br}}e_\beta^{\text{rb}}\varphi_{mn}^{\text{ho}}][z_i;w_k;h_1,h_2]$ yields the same contributions as for the one-holon case and thus cancels the same non-diagonal terms as previously mentioned, so do the trivial spin-diagonal terms. Furthermore, we have

$$\begin{aligned}&\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{N}\frac{\left(e_\alpha^{\text{rr}}+e_\alpha^{\text{bb}}\right)(1-n_\beta)}{|\eta_\alpha-\eta_\beta|^2}\varphi_{mn}^{\text{ho}}\\ =&\left(\sum_{i=1}^{M_1}\frac{1}{|z_i-h_1|^2}+\sum_{k=1}^{M_2}\frac{1}{|w_k-h_2|^2}+[h_1\leftrightarrow h_2]\right)\varphi_{mn}^{\text{ho}}.\end{aligned} \qquad \text{(A.42)}$$

For the charge kinetic terms of the two-holon states, we combine the technique previously explained for the SU(3) one-holon with the calculation for the SU(2) two-holon states:

$$\begin{aligned}&\left[\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{N}\frac{1}{2}\frac{c_{\alpha\text{b}}c_{\beta\text{b}}^\dagger+c_{\alpha\text{r}}c_{\beta\text{r}}^\dagger}{|\eta_\alpha-\eta_\beta|^2}\varphi_{mn}^{\text{ho}}\right][z_i;w_k;h_1,h_2]\\
=&\left[\sum_{\alpha=h_1,h_2}\sum_{\substack{\beta=1\\ \alpha\neq\beta}}^{N}\frac{c_{\alpha v}c_{\beta v}^\dagger}{|\eta_\alpha-\eta_\beta|^2}\varphi_{mn}^{\text{ho}}\right][v_1,\ldots,v_{M_1+M_2};h_1,h_2]\\
=&\sum_{l=0}^{M_1+M_2}\frac{B_l^m}{l!}\frac{\partial^l}{\partial h_1^l}\left(\frac{\varphi_{mn}^{\text{ho}}}{h_1^m}\right)+\sum_{l=0}^{M_1+M_2}\frac{B_l^n}{l!}\frac{\partial^l}{\partial h_2^l}\left(\frac{\varphi_{mn}^{\text{ho}}}{h_2^n}\right)\\
=&\left\{\Big(\frac{N^2-1}{6}+\frac{n(n-N)}{2}+\frac{m(m-N)}{2}\Big)h_1^nh_2^m\right.\\
&-\Big(\frac{N-1}{2}-m\Big)h_1^{m+1}h_2^n\frac{\partial}{\partial h_1}-\Big(\frac{N-1}{2}-n\Big)h_1^mh_2^{n+1}\frac{\partial}{\partial h_2}\\
&\left.+\frac{1}{2}h_1^mh_2^{n+2}\frac{\partial^2}{\partial h_2^2}+\frac{1}{2}h_1^{m+2}h_2^n\frac{\partial^2}{\partial h_1^2}\right\}\frac{\varphi_{mn}^{\text{ho}}}{h_1^mh_2^n}\end{aligned}$$

$$
\begin{aligned}
= \; & \Bigg\{ \Big(\frac{N^2-1}{6} + \frac{n(n-N)}{2} + \frac{m(m-N)}{2}\Big) - \\
& + \Big(\frac{N-1}{2} - n\Big) \left(\sum_{i=1}^{M_1} \frac{h_1}{h_1 - z_i} + \sum_{k=1}^{M_2} \frac{h}{h - w_k} \right) \\
& - \Big(\frac{N-1}{2} - m\Big) \left(\sum_{i}^{M_1} \frac{h_2}{h_2 - z_i} + \sum_{k}^{M_2} \frac{h_2}{h_2 - w_k} \right) \\
& + \frac{1}{2} \Big[\sum_{\substack{i,j \\ i \neq j}}^{M_1} \frac{h_1^2}{(h_1 - z_i)(h_1 - z_j)} + 2 \sum_{i=1}^{M_1} \sum_{k=1}^{M_2} \frac{h_1}{h_1 - z_i} \frac{h_1}{h_1 - w_k} \\
& + \sum_{\substack{k,l \\ k \neq l}}^{M_2} \frac{h_1^2}{(h_1 - w_k)(h_1 - w_l)} \Big] + [h_1 \leftrightarrow h_2] \Bigg\} \varphi_{mn}^{\mathrm{ho}}, \qquad \text{(A.43)}
\end{aligned}
$$

where we again introduced $B_l^m = h^{m+l} \sum_\beta^{N-1} \eta_\beta^{m+1} (\eta_\beta - 1)^{l-2}$, which we already used for the one-holon case. The evaluation of the latter again is discussed in Appendix A.7 and, for the case of the two-holon states, yields $B_0^m = (N^2-1)/12 + m(m-N)/2$, $B_1^m = m - (N-1)/2$, $B_2^m = 1$, and $B_l = 0$ for $3 \leq l$ and $0 \leq m \leq (N+4)/3$, by which (3.26) follows.

For the other charge term contributions, we expound the fact that the two-holon wave function can be expressed by either pairs of sets of color variables as shown in Appendix A.5. The appearing terms which involve the green color variables are rewritten in terms of the blue and red variables, *i.e.*, z's and w's, by identities given in Appendix A.5.1. Adding all three charge kinetic terms, it finally yields

$$
\begin{aligned}
& \left[\sum_{\substack{\alpha,\beta=1 \\ \alpha \neq \beta}}^{N} \frac{\sum_\sigma c_{\sigma\alpha} c_{\sigma\beta}^\dagger}{|\eta_\alpha - \eta_\beta|^2} \varphi_{mn}^{\mathrm{ho}} \right] [z_i; w_k; h_1, h_2] \\
= \; & \Bigg\{ \frac{N^2-1}{2} + \frac{3}{2} n(n-N) + \frac{3}{2} m(m-N) + (n+m)(N-2) - 2C_1^2 - 2C_2 \\
& - (m-n) \frac{h_1 + h_2}{h_1 - h_2} + \Big[- \frac{N-1}{2} \left(\sum_i \frac{h_1}{h_1 - z_i} + \sum_k \frac{h_1}{h_1 - w_k} \right) \\
& + \sum_k \frac{h_1^2}{(h_1 - w_k)^2} + \frac{h_1}{h_1 - h_2} \left(\sum_i \frac{h_1}{h_1 - z_i} + \sum_k \frac{h_1}{h_1 - w_k} \right)
\end{aligned}
$$

$$+\sum_i \frac{h_1^2}{(h_1-z_i)^2} + \sum_{i\neq j} \frac{h_1^2}{(h_1-z_i)(h_1-z_j)} + \sum_{i,k} \frac{h_1^2}{(h_1-z_i)(h_1-w_k)}$$
$$\left. + \sum_{k\neq l} \frac{h_1^2}{(h_1-w_k)(h_1-w_l)} + 2\frac{h_1^2}{(h_1-h_2)^2} + [h_1 \leftrightarrow h_2] \right\} \varphi^{\mathrm{ho}}_{nm}, \tag{A.44}$$

with definitions as introduced before.

Most non-diagonal terms cancel by direct observation. Summing up all contributions, we finally derive the action of H_{KY} on the auxiliary wave functions $\varphi^{\mathrm{ho}}_{mn}$ to be

$$H^{\mathrm{SU(3)}}_{\mathrm{KY}} \varphi^{\mathrm{ho}}_{mn} = \frac{2\pi^2}{N^2} \left\{ \frac{1}{72}(-40+33N-N^3) + \frac{3}{2}n(n-N) + \frac{3}{2}m(m-N) \right.$$
$$\left. +(n+m)(N-2) + 2\frac{h_1^2+h_2^2}{(h_1-h_2)^2} - (m-n)\frac{h_1+h_2}{h_1-h_2} \right\} \varphi^{\mathrm{ho}}_{mn}. \tag{A.45}$$

Using (A.38) we thus obtain

$$\begin{aligned} & H_{\mathrm{KY}} \Psi^{\mathrm{ho}}_{mn}[z_i; w_k; h_1, h_2] \\ = \;& +\frac{3\pi^2}{N^2}\left[\left(m-\frac{N+1}{3}\right) m \left(n - \frac{N+1}{3}\right) n + \frac{m-n}{3}\right] \\ & + \frac{2\pi^2}{N^2}(m-n) \sum_{l=1}^{\lfloor \frac{m-n}{2} \rfloor} \Psi^{\mathrm{ho}}_{m-l,n+l} - \frac{\pi^2}{36}\left(N + \frac{3}{N} + \frac{4}{N^2}\right), \end{aligned} \tag{A.46}$$

where in (A.46) we again used $\frac{x+y}{x-y}(x^m y^n - x^n y^m) = 2\sum_{l=0}^{m-n} x^{m-l} y^{n+l} - (x^m y^n + x^n y^m)$ and $\lfloor \; \rfloor$ denotes the floor function, *i.e.*, $\lfloor x \rfloor$ is the largest integer $l \leq x$. First, note that the action of the Hamiltonian on Ψ^{ho}_{mn} is trigonal, *i.e.*, the "scattering" in the last line is only to lower values of $m-n$. Second, (A.46) shows that the states Ψ^{ho}_{mn} form a non-orthogonal set out of which we construct an orthogonal basis of eigenfunctions as it is shown in Chapter 3.

A.5 SU(3) wave function transformations

It is used that the wave functions appearing in the calculation can, beyond an irrelevant minus sign, be equally expressed by any two sets of color variables. For the ground state wave function, this is proved by

$$\Psi_0[z_1, \ldots, z_{M_1}; \omega_1, \ldots, \omega_{M_2}]$$

$$
\begin{aligned}
&= (-1)^{M_1\frac{M_1-1}{2}} \prod_{\substack{i,j\\ i\neq j}}^{M_1}(z_i-z_j)\prod_{\substack{k,l\\ k<l}}^{M_2}(\omega_k-\omega_l)^2\prod_{i=1}^{M_1}\prod_{k=1}^{M_2}(z_i-\omega_k)\prod_{i=1}^{M_1}z_i\prod_{k=1}^{M_2}\omega_k \\
&\overset{\text{(i)}}{=} (-1)^{M_1\frac{M_1-1}{2}}(-1)^{M_1M_2}\frac{\prod_{i=1}^{M_1}z_i\prod_{k=1}^{M_2}\omega_k\prod_{\substack{k,l\\ k<l}}^{M_2}(\omega_k-\omega_l)^2\prod_{i=1}^{M_1}\frac{N}{z_i}}{\prod_{i=1}^{M_1}\prod_{m=1}^{M_3}(u_m-z_i)} \\
&= (-1)^{M_1\frac{M_1-1}{2}}(-1)^{M_1M_2}\frac{\prod_{k=1}^{M_2}\omega_k\prod_{\substack{k,l\\ k<l}}^{M_2}(\omega_k-\omega_l)^2N^{M_1}}{\prod_{i=1}^{M_1}\prod_{m=1}^{M_3}(u_m-z_i)}\ ,
\end{aligned}
\tag{A.47}
$$

where (i) follows by (A.62). Accordingly, we find for the ground state wave function, being expressed by the green (u) and red (w) variables:

$$
\begin{aligned}
&\Psi_0[u_1,\ldots,u_{M_3};\omega_1,\ldots,\omega_{M_2}] \\
&= (-1)^{M_3\frac{M_3-1}{2}}\prod_{\substack{n,m\\ n\neq m}}^{M_3}(u_n-u_m)\prod_{\substack{k,l\\ k<l}}^{M_2}(\omega_k-\omega_l)^2\prod_{n=1}^{M_3}\prod_{k=1}^{M_2}(u_n-\omega_k)\prod_{n=1}^{M_3}u_n\prod_{k=1}^{M_2}\omega_k \\
&\overset{\text{(i)}}{=} (-1)^{M_3\frac{M_3-1}{2}}(-1)^{M_3M_2}\frac{\prod_{n=1}^{M_3}u_n\prod_{k=1}^{M_2}\omega_k\prod_{\substack{k,l\\ k<l}}^{M_2}(\omega_k-\omega_l)^2\prod_{n=1}^{M_3}\frac{N}{u_n}}{\prod_{i=1}^{M_1}\prod_{m=1}^{M_3}(u_m-z_i)} \\
&= (-1)^{M_3\frac{M_3-1}{2}}(-1)^{M_3M_2}(-1)^{M_1M_3}\frac{\prod_{k=1}^{M_2}\omega_k\prod_{\substack{k,l\\ k<l}}^{M_2}(\omega_k-\omega_l)^2N^{M_3}}{\prod_{i=1}^{M_1}\prod_{m=1}^{M_3}(u_m-z_i)} \\
&\overset{\text{(ii)}}{=} (-1)^{M^2}\Psi_0^{\mathrm{SU(3)}}[z_1,\ldots,z_{M_1};\omega_1,\ldots,\omega_{M_2}],
\end{aligned}
\tag{A.48}
$$

where (i) again follows by (A.62) and (ii) sets $M_1 = M_2 = M_3 = M$, which is the case for the ground state.

Using (A.62) yields the same line of argument for the one-holon and two-holon wave functions. For the first, we have

$$
\begin{aligned}
&\Psi_m^{\mathrm{ho}}[z_i;w_k;h] = h^n\prod_{i=1}^{M_1}(h-z_i)\prod_{k=1}^{M_2}(h-\omega_k)\Psi_0[z_i;w_k] \\
&= (-1)^{M_1\frac{M_1-1}{2}}(-1)^{M_1M_2}h^n\frac{\prod_{k=1}^{M_2}(h-\omega_k)\prod_{k=1}^{M_2}\omega_k\prod_{\substack{k,l\\ k<l}}^{M_2}(\omega_k-\omega_l)^2N^{M_1}}{\prod_{i=1}^{M_1}\prod_{m=1}^{M_3}(u_m-z_i)},
\end{aligned}
$$

$$(A.49)$$

whereas starting in the $[u;w]$ representation, *i.e.,* the green and red variables, leads to

$$\begin{aligned}
\Psi_m^{\mathrm{ho}}[u_n; w_k] = h^n \prod_{n=1}^{M_3}(h-u_n)\prod_{k=1}^{M_2}(h-\omega_k)\prod_{\substack{n,m\\n<m}}^{M_3}(u_n-u_m)^2\prod_{\substack{k,l\\k<l}}^{M_2}(\omega_k-\omega_l)^2 \\
\prod_{n=1}^{M_3}\prod_{k=1}^{M_2}(u_n-\omega_k)\prod_{n=1}^{M_3}u_n\prod_{k=1}^{M_2}\omega_k \\
= (-1)^{M_3\frac{M_3-1}{2}+M_3M_2+M_1M_3}h^n\frac{\prod_{k=1}^{M_2}(h-\omega_k)\prod_{k=1}^{M_2}\omega_k\prod_{\substack{k,l\\k<l}}^{M_2}(\omega_k-\omega_l)^2N^{M_3}}{\prod_{i=1}^{M_1}\prod_{m=1}^{M_3}(u_m-z_i)} \\
= (-1)^{M^2}\Psi_n^{\mathrm{ho}}[z_1,\ldots,z_{M_1};\omega_1,\ldots,\omega_{M_2};h], \qquad (A.50)
\end{aligned}$$

by comparing the result with (A.49). The same argument applies for the two-holon case yielding

$$\Psi_{mn}^{\mathrm{ho}}[z_i; w_k; h_1, h_2] = (-1)^{M^2}\Psi_{mn}^{\mathrm{ho}}[u_l; w_k; h_1, h_2]. \qquad (A.51)$$

Thus the appearing holon wave functions can be equally re-expressed by two sets of arbitrary color indices up to an irrelevant global minus sign if M is odd. All statements trivially generalize to SU(n).

A.5.1 SU(3) derivative identities

We mention some identities to rewrite expressions in terms of the blue and red spin variables used for the evaluation of the charge kinetic terms. Using

$$\frac{N-1}{2} = \sum_{\alpha=1}^{N}\frac{1}{1-\eta_\alpha} = \sum_{i=1}^{M_1}\frac{h}{h-z_i} + \sum_{k=1}^{M_2}\frac{h}{h-\omega_k} + \sum_{n=1}^{M_3}\frac{h}{h-u_n}, \qquad (A.52)$$

for the one-holon case, one gets

$$\sum_{\substack{n,m\\n\neq m}}^{M_3}\frac{h^2}{(h-u_m)(h-u_n)} = \sum_{\substack{n,m\\n\neq m}}^{M_3}\frac{h^2}{(h-u_m)(h-u_n)} - \sum_{m}^{M_3}\frac{h^2}{(h-u_m)^2}$$

$$
\begin{aligned}
= \quad & C_1^2 - C_2 + 2\sum_{i=1}^{M_1}(\frac{h}{h-z_i})^2 + 2\sum_{k=1}^{M_2}(\frac{h}{h-\omega_k})^2 \\
& -2C_1\sum_{i=1}^{M_1}\frac{h}{h-z_i} - 2C_1\sum_{k=1}^{M_2}\frac{h}{h-\omega_k} + 2\sum_{i=1}^{M_1}\frac{h}{h-z_i}\sum_{k=1}^{M_2}\frac{h}{h-\omega_k} \\
& +\sum_{\substack{k,l\\k\neq l}}^{M_2}\frac{h^2}{(h-\omega_k)(h-\omega_l)} + \sum_{\substack{i,j\\i\neq j}}^{M_1}\frac{h^2}{(h-z_i)(h-z_j)},
\end{aligned}
\tag{A.53}
$$

with definitions of $C_{1,2}$ as mentioned before. For the two-holon case, we apply the identity

$$
\begin{aligned}
& \sum_{l\neq m}\frac{h_1^2}{(h_1-u_l)(h_1-u_m)} = \sum_{l,m}\frac{h_1^2}{(h_1-u_l)(h_1-u_m)} - \sum_m\frac{h_1^2}{(h_1-u_m)^2} \\
= \quad & -C_2 + \sum_i\frac{h_1^2}{(h_1-z_i)^2} + \sum_k\frac{h_1^2}{(h_1-w_k)^2} + \frac{h_1^2}{(h_1-h_2)^2} \\
& + \left(C_1 - \sum_i\frac{h_1}{h_1-z_i} - \sum_k\frac{h_1}{h_1-w_k} - \frac{h_1}{h_1-h_2}\right) \\
& \cdot\left(C_1 - \sum_j\frac{h_1}{h_1-z_j} - \sum_l\frac{h_1}{h_1-w_l} - \frac{h_1}{h_1-h_2}\right) \\
= \quad & -C_2 + C_1^2 + \sum_{i\ neqj}\frac{h_1^2}{(h_1-z_i)(h_1-z_j)} + \sum_{k\neq l}\frac{h_1^2}{(h_1-w_k)(h_1-w_l)} \\
& +2\sum_i\frac{h_1^2}{(h_1-z_i)^2} + 2\sum_k\frac{h_1^2}{(h_1-w_k)^2} + 2\frac{h_1^2}{(h_1-h_2)^2} - 2C_1\frac{h_1}{h_1-h_2} \\
& +2\sum_{i,k}\frac{h_1^2}{(h_1-w_k)(h_1-z_i)} - 2C_1\sum_k\frac{h_1}{h_1-w_k} - 2C_1\sum_i\frac{h_1}{h_1-z_i} \\
& +2\frac{h_1}{h_1-h_2}\left(\sum_j\frac{h_1}{h_1-z_j} + \sum_k\frac{h_1}{h_1-w_k}\right).
\end{aligned}
\tag{A.54}
$$

For the case of SU(n), the above derivation easily generalizes. Starting with the generalized formula of (A.52) for n spin indices, the identities follow by straight forward calculation.

A.6 Gell-Mann matrices

The Gell-Mann matrices are given by [39]

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

They are normalized as $\mathrm{tr}\left(\lambda^a \lambda^b\right) = 2\delta_{ab}$ and satisfy the commutation relations $\left[\lambda^a, \lambda^b\right] = 2f^{abc}\lambda^c$. The structure constants f^{abc} are totally antisymmetric and obey Jacobi's identity

$$f^{abc}f^{cde} + f^{bdc}f^{cae} + f^{dac}f^{cbe} = 0.$$

Explicitly, the non-vanishing structure constants are given by $f^{123} = i$, $f^{147} = f^{246} = f^{257} = f^{345} = -f^{156} = -f^{367} = i/2$, $f^{458} = f^{678} = i\sqrt{3}/2$, and 45 others obtained by permutations of the indices. The SU(3) spin operators are expressed in terms of the color-flip operators by

$$\boldsymbol{J}_\alpha \cdot \boldsymbol{J}_\beta \equiv \sum_{a=1}^{8} J_\alpha^a J_\beta^a = \frac{1}{2}\sum_{\sigma\tau}^{3} e_\alpha^{\sigma\tau}\, e_\beta^{\tau\sigma} - \frac{1}{6} + \frac{1}{6}(h_\alpha + h_\beta - h_\alpha h_\beta),$$

where h_α denotes the hole occupation operator $h_\alpha = 1 - n_\alpha$.

A.7 B-series

Evaluation of the series

$$\begin{aligned} B_l^n &= \sum_{\beta \neq h}^{N} \frac{\eta_\beta^n (\eta_\beta - h)^l}{l! |h - \eta_\beta|^2} \\ &= -\frac{1}{l!} h^{n+l} \sum_{\beta=1}^{N-1} \eta_\beta^{n+1} (\eta_\beta - 1)^{l-2}, \end{aligned} \tag{A.55}$$

with l restricted to $0 \le l \le M = (N-2)/2$ yields

$$B_0^n = \frac{N^2-1}{12} + \binom{n(n-N)}{2} \quad \text{for} \quad 0 \le n \le N, \tag{A.56}$$

$$B_1^n = \begin{cases} n - \dfrac{N-1}{2} & \text{for} \quad 0 \le n < N, \\ -\dfrac{N-1}{2} & \text{for} \quad n = N, \end{cases} \tag{A.57}$$

$$B_2^n = \begin{cases} 1 & \text{for} \quad 0 \le n \le N-2, \; n = N, \\ 1-N & \text{for} \quad n = N-1, \end{cases} \tag{A.58}$$

$$B_l^n = \begin{cases} 0 & \text{for} \quad l \ge 3, \; 0 \le n \le \dfrac{N+2}{2}, \\ N\dbinom{l-2}{N-n-1} & \text{for} \quad l \ge 3, \; \dfrac{N+2}{2} < n \le N. \end{cases} \tag{A.59}$$

Proof: B_0^n, B_1^n, and B_2^n are found by straight forward evaluation of the respective sums. For (A.59) consider

$$\begin{aligned} B_l^n &= -\sum_{\alpha=1}^{N-1} \eta_\alpha^{n+1} \sum_{k=0}^{l-2} \binom{l-2}{k} (-1)^{l-k-2} \eta_\alpha^k \\ &= \sum_{k=0}^{l-2} \binom{l-2}{k} (-1)^{l-k-1} \Big(1 - \sum_{\alpha=1}^{N} \eta_\alpha^{k+l+1}\Big) \\ &= \begin{cases} -\displaystyle\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} = 0 & \text{for} \quad 3 \le l, 0 \le n \le (N+2)/2, \\ \displaystyle\sum_{k=0}^{l-2} \binom{l-2}{k} N\delta_{k,N-1-n} = N\binom{l-2}{N-n-1} & \text{otherwise.} \end{cases} \end{aligned}$$

For the last steps note that the the binomial coefficients of even and odd sites equal each other. The B series appearing in the charge kinetic terms is thus the point in the calculation which yields the restriction of the holon momentum quantum number to one half (third) of the Brillouin zone for the case of SU(2) (SU(3)). The above presentation can be easily generalized to SU(n) yielding that the holon momenta occupy only one nth of the Brillouin zone.

A.8 Useful formulas

Some of the results presented in this appendix can be found in [14] as well as proven in [110, 112, 121].

1.
$$\eta_\alpha^N = 1, \quad \sum_{\alpha=1}^{N} \eta_\alpha^m = N\,\delta_{0m}, \quad \prod_{\alpha=1}^{N} \eta_\alpha = (-1)^{N-1}. \tag{A.60}$$

2.
$$\prod_{\alpha=1}^{N} (z - \eta_\alpha) = z^N - 1. \tag{A.61}$$

3. Accordingly, it holds
$$\prod_{\substack{\alpha \\ \alpha\neq\beta}} (\eta_\alpha - \eta_\beta) = \lim_{\eta\to\eta_\beta} \frac{\eta^N - 1}{\eta - \eta_\beta} = \frac{N}{\eta_\beta}\,. \tag{A.62}$$

4.
$$\sum_{\alpha=1}^{N} \frac{\eta_\alpha}{z - \eta_\alpha} = \frac{N}{z^N - 1}. \tag{A.63}$$

5. The previous statement implies
$$\sum_{\alpha=1}^{N} \frac{1}{\eta - \eta_\alpha} = \frac{N\eta^{N-1}}{\eta^N - 1}. \tag{A.64}$$

6.
$$\sum_{\alpha=1}^{N-1} \frac{\eta_\alpha^m}{\eta_\alpha - 1} = \frac{N+1}{2} - m, \quad 1 \le m \le N. \tag{A.65}$$

7.
$$\sum_{\alpha=1}^{N-1} \frac{\eta_\alpha^m}{|\eta_\alpha - 1|^2} = \frac{N^2-1}{12} - \frac{m(N-1)}{2} + \frac{m(m-1)}{2}, \quad 0 \le m \le N. \tag{A.66}$$

8. For
$$A_m = -\sum_{\alpha=1}^{N-1} \eta_\alpha^2 (\eta_\alpha - 1)^{m-2} \tag{A.67}$$
we have: $A_0 = (N-1)(N-5)/12$ by (A.66), $A_1 = -(N-3)/2$ by (A.65), and $A_2 = 1$ by $\sum_\alpha^N \eta_\alpha^m = N\delta_{m0}$. Furthermore,
$$A_m = -\sum_{\alpha=1}^{N-1} \eta_\alpha^2 \sum_{k=0}^{m-2} \binom{m-2}{k} (-1)^{m-k-2} \eta_\alpha^k$$

$$
\begin{aligned}
&= \sum_{k=0}^{m-2} \binom{m-2}{k} (-1)^{m-k} \Big(1 - \sum_{\alpha=1}^{N} \eta_\alpha^{k+2}\Big) \\
&= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} = 0, \quad 2 < m \leq N-1.
\end{aligned}
$$

as the sums of the binomial coefficients of even sites and odd sites equal each other.

Appendix B

Chiral Spin Liquid

B.1 Operator action the Spin $1/2$ CSL state

B.1.1 Action of T_j

In order to prove (4.32), we first consider the off-diagonal terms in the operator ω_j^+ which come from T_j defined in (4.16). We consider a general element of the vector $T_j\,|\psi\rangle$:

$$\langle z_1 \cdots z_M |\, T_j \,|\psi\rangle = \frac{1}{2} \sum_{i,k\neq j}' K_{ijk} \left\langle z_1 \cdots z_M \left| S_j^+ \, S_k^- \left(\frac{1}{2} + S_i^z \right) \right| \psi \right\rangle . \tag{B.1}$$

The element is clearly zero unless $z_j \in \{z_1 \cdots z_M\}$. When this is satisfied, acting onto the bra on the right-hand side of the equation with the spin operators wipes out the matrix element unless $z_i \in \{z_1 \cdots z_M\}$ and replaces z_j with z_k:

$$\langle z_1 \cdots z_M |\, T_j \,|\psi\rangle = \frac{1}{2} \sum_{i\neq j}^{M} \sum_{k\neq j}^{\mathcal{N}} K_{ijk} \, \langle z_1 \cdots z_{j-1} \, z_k \, z_{j+1} \cdots z_M \rangle \, \psi \, . \tag{B.2}$$

The upper limit of M (rather than $\mathcal{N}$) on the sum on i indicates that z_i must be a member of the up-spins. Rewriting $K_{ijk} = K(z_k - z_j, z_i - z_j)$ and defining $z = z_k - z_j$, this may be rewritten as

$$\langle z_1 \cdots z_M |\, T_j \,|\psi\rangle = \frac{1}{2} \sum_{i\neq j}^{M} \sum_{z\neq 0} K(z, z_i - z_j) \, \langle z_1 \cdots z_j + z \cdots z_M \rangle \, \psi \, . \tag{B.3}$$

Using the definition of the coefficient K from (4.26), this can be rewritten as

$$
\begin{aligned}
\langle z_1 \cdots z_M | \, T_j \, | \psi \rangle \;\; &= \;\; \frac{1}{N-2} \sum_{i \neq j}^{M} \sum_{z \neq 0} \left(\lim_{R \to \infty} \sum_{0 \leq z_0 < R} \frac{P(z - z_0, z_i - z_j)}{z - z_0} \right) \\
& \quad \langle z_1 \cdots z_j + z \cdots z_M \rangle \, \psi \, . \qquad \text{(B.4)}
\end{aligned}
$$

Since the wave function itself is periodic, the two sums over z and over z_0 may be combined into a single sum that runs over the entire infinite lattice for which we use the variable $x = z - z_0$. However, since the point $z = 0$ is missing from the original sum, all of its images in the infinite lattice will be missing from the second sum and this must be subtracted off, giving

$$
\begin{aligned}
& \langle z_1 \cdots z_M | \, T_j \, | \psi \rangle \\
= \;\; & \frac{1}{N-2} \sum_{i \neq j}^{M} \left(\lim_{R \to \infty} \sum_{0 < |x| < R} \frac{P(x, z_i - z_j)}{x} \, \langle z_1 \cdots z_j + x \cdots z_M \rangle \, \psi \right) \\
& - \frac{1}{N-2} \sum_{i \neq j}^{M} \sum_{z_0} \frac{P(-z_0, z_i - z_j)}{-z_0} \, \langle z_1 \cdots z_M \rangle \, \psi \, . \qquad \text{(B.5)}
\end{aligned}
$$

Dividing both sides of the equation by $\langle z_1 \cdots z_M \rangle \, \psi$ and rewriting the ratio of elements in terms of the analytic function of x, $A(x)$ given in Appendix B.4 gives

$$
\begin{aligned}
\frac{\langle z_1 \cdots z_M | \, T_j \, | \psi \rangle}{\langle z_1 \cdots z_M \rangle \, \psi} \;\; &= \;\; -\frac{1}{N-2} \sum_{i \neq j}^{M} \lim_{R \to \infty} \sum_{0 < |x| < R} \frac{P(x, z_i - z_j)}{x} \, A(x) \; G(x) \; e^{-\frac{\pi}{2} |x|^2} \\
& \quad - \frac{1}{N-2} \sum_{i \neq j}^{M} \sum_{z_0} \frac{P(z_0, z_i - z_j)}{z_0} \, . \qquad \text{(B.6)}
\end{aligned}
$$

Note that $A(x)$ is an analytic function only of x, and not of the remaining $\{z_i\}$ on which it also depends.

The first sum in (B.6) may be evaluated with the corollary to the Singlet Sum Rule, Eq. B.36. A derivation of the sum rule and the necessary corollary is given in Appendix B.3. The function $P(x, y)$ falls off exponentially with increasing x while the quantity $A(x) \; G(x) \; e^{-\frac{\pi}{2} |x|^2}$ is essentially constant due to the periodicity of the wave function. This guarantees that the sum is absolutely convergent and the sum rule may be applied. Additionally, the product $P(x, z_i - z_j) \; A(x)$ is itself an analytic function of x. As a function of x, the function $P(x, z_i - z_j)$ necessarily has poles. However, these occur when when $x = z_i - z_j$ and the function $A(x)$ has second-order zeroes at these locations since this corresponds to a coincidence of up-spins. Since the product is analytic and the sum is absolutely convergent, the Singlet Sum Rule may be applied to give

$$\begin{aligned}\frac{\langle z_1 \cdots z_M | \, T_j \, |\psi\rangle}{\langle z_1 \cdots z_M \rangle \, \psi} &= -\frac{1}{N-2} \sum_{i \neq j}^{M} \frac{d}{dx} \left[P(x, z_i - z_j) \; A(x) \right] \bigg|_0 \\ &\quad -\frac{1}{N-2} \sum_{i \neq j}^{M} \sum_{z_0} \frac{P(z_0, z_i - z_j)}{z_0} . \end{aligned} \tag{B.7}$$

Using the fact that $A(0) = P(0, z_i - z_j) = 1$ and the relation for dA/dx given in (B.45), this becomes

$$\begin{aligned} &\frac{\langle z_1 \cdots z_M | \, T_j \, |\psi\rangle}{\langle z_1 \cdots z_M \rangle \, \psi} \\ &= -\frac{1}{N-2} \sum_{i \neq k, j}^{M} \left\{ \sum_{\nu=1}^{2} \frac{\pi}{L} W\!\left(\frac{\pi}{L} \, [\mathcal{Z} - Z_\nu] \right) + 2 \sum_{\ell \neq j}^{M} \frac{\pi}{L} W\!\left(\frac{\pi}{L} \, [z_j - z_\ell] \right) \right. \\ &\quad \left. + \frac{d}{dx} P(x, z_i - z_j) \bigg|_0 \right\} - \frac{1}{N-2} \sum_{i \neq j, k}^{M} \sum_{z_0} \frac{P(z_0, z_i - z_j)}{z_0} . \end{aligned} \tag{B.8}$$

The sum on i may be completed for the terms containing the W functions (picking up a factor of $M = N/2 - 1$) and this gives, renaming ℓ as i,

$$\begin{aligned} \frac{\langle z_1 \cdots z_M | \, T_j \, |\psi\rangle}{\langle z_1 \cdots z_M \rangle \, \psi} &= f(\mathcal{Z}) - \sum_{i \neq j}^{M} \frac{\pi}{L} W\!\left(\frac{\pi}{L} \, [z_j - z_i] \right) \\ &- \frac{1}{N-2} \sum_{i \neq j}^{M} \left[\sum_{z_0} \frac{P(z_0, z_i - z_j)}{z_0} + \frac{d}{dx} P(x, z_i - z_j) \bigg|_0 \right] , \end{aligned} \tag{B.9}$$

where

$$f(\mathcal{Z}) = -\frac{1}{2} \sum_{\nu=1}^{2} \frac{\pi}{L} W\!\left(\frac{\pi}{L} \, [\mathcal{Z} - Z_\nu] \right) . \tag{B.10}$$

The fact that $f(\mathcal{Z})$ is both odd and periodic, required for the proof of (4.31) above, follows from these same properties of the W function. Comparison with (4.27) shows that

$$\frac{\langle z_1 \cdots z_M | \, T_j \, |\psi\rangle}{\langle z_1 \cdots z_M \rangle \, \psi} = f(\mathcal{Z}) - \sum_{i \neq j}^{M} U_{ij} \tag{B.11}$$

if z_j is an element of the up-spins and zero otherwise.

B.1.2 Action of V_j

The action of the operator V_j on the CSL ground state is straightforward to compute. Proceeding in an analogous manner we have

$$\langle z_1 \cdots z_M | \, V_j \, |\psi\rangle = \sum_{i \neq j}^{\mathcal{N}} U_{ij} \, \langle z_1 \cdots z_M | \left(\frac{1}{2} + S_i^z \right) \left(\frac{1}{2} + S_j^z \right) |\psi\rangle \ . \tag{B.12}$$

The matrix element vanishes unless both z_i and z_j are elements of $\{z_1 \cdots z_M\}$. Therefore, the diagonal contribution to the operator ω_j gives

$$\frac{\langle z_1 \cdots z_M | \, V_j \, |\psi\rangle}{\langle z_1 \cdots z_M \rangle \, \psi} = \sum_{i \neq j}^{M} U_{ij} \tag{B.13}$$

if $z_j \in \{z_1 \cdots z_M\}$ and 0 otherwise. Combining Eqs. B.13 and B.11 proves (4.32) and therefore proves that the chiral spin liquid is an exact ground state of either of the Hamiltonians in Eqs. 4.23 or 4.25.

B.2 Tensor decomposition

The operators ω_j introduced in Section 4.4 may be decomposed into irreducible spherical tensors of ranks 1 and 3. We write these irreducible operators as T^q_m; q and m correspond to angular momentum and its z component respectively. We wish to write $\omega = \sum c_q T^q$, where T^q is the collection of all operators which transform as a spherical tensor of rank q. Here we have suppressed the site index on the operator ω.

The operator in (4.18) that is not manifestly the component of a vector is $S_i^z \, S_j^z \, S_k^z$, which is a component of a third-rank Cartesian tensor. In order to keep the notation manageable, we start by considering the direct product of two operators U and V with angular momentum j_1 and j_2 respectively. An element in the direct product space of these operators may be written as

$$U^{j_1}_{m_1} V^{j_2}_{m_2} = \sum_{j_{12}=|j_2-j_1|}^{j_1+j_2} \sum_{m_{12}=-j_{12}}^{j_{12}} C^{m_1\, m_2\, m_{12}}_{j_1\; j_2\; j_{12}} T^{j_{12}}_{m_{12}} \tag{B.14}$$

in terms of irreducible spherical tensors $T^{j_{12}}_{m_{12}}$ carrying angular momentum j_{12} with z-component $m_{12} = m_1 + m_2$. Eq. B.14 may be inverted to give

$$T^{j_{12}}_{m_{12}} = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} C^{m_1\, m_2\, m_{12}}_{j_1\; j_2\; j_{12}} U^{j_1}_{m_1} V^{j_2}_{m_2} \,. \tag{B.15}$$

Using these equations, one may construct corresponding expressions for the product of three vector operators by applying (B.14) twice:

$$\begin{aligned} U^{j_1}_{m_1} V^{j_2}_{m_2} W^{j_3}_{m_3} &= \sum_{j_{12}=-|j_1-j_2|}^{j_1+j_2} \sum_{m_{12}=-j_{12}}^{j_{12}} C^{m_1\, m_2\, m_{12}}_{j_1\; j_2\; j_{12}} T^{j_{12}}_{m_{12}} W^{j_3}_{m_3} \\ &= \sum_{j_{12}=-|j_1-j_2|}^{j_1+j_2} \sum_{m_{12}=-j_{12}}^{j_{12}} C^{m_1\, m_2\, m_{12}}_{j_1\; j_2\; j_{12}} \sum_{j=|j_{12}-j_3|}^{j_{12}+j_3} \sum_{m=-j}^{j} C^{m_{12}\, m_3\, m}_{j_{12}\; j_3\; j} T^{j(j_{12})}_{m} \,. \end{aligned} \tag{B.16}$$

The second superscript on the tensor T in the last line distinguishes between the different tensors of the same rank that appear when combining three vector operators; since $1 \otimes 1 \otimes 1 = 3 \oplus 2 \oplus 2 \oplus 1 \oplus 1 \oplus 1 \oplus 0$, there are two rank-2 spherical tensors and three vector operators that can be formed. For the case of interest $m_1 = m_2 = m_3 = 0$ and $j_1 = j_2 = j_3 = 1$, the expression reduces to

$$\begin{aligned} U^z V^z W^z &= \sum_{j_{12}=0}^{2} \sum_{j=|j_{12}-1|}^{j_{12}+1} C^{000}_{11j_{12}} C^{0\;\;00}_{j_{12}1j} T^{j(j_{12})}_0 \\ &= -\frac{1}{\sqrt{3}} T^{1(0)}_0 - \frac{2}{\sqrt{15}} T^{1(2)}_0 + \sqrt{\frac{2}{5}} T^3_0 \,, \end{aligned} \tag{B.17}$$

which shows that the operator contains only vector and rank-3 tensor components, but no scalar or rank-2 tensor components. Note that the second index on the rank-3 tensor has been suppressed since the construction of this object is unambiguous.

Applying (B.15) twice, the rank-3 tensor component is

$$\begin{aligned} T_0^3 &= \sum_{m_{12}=-2}^{2} \sum_{m_3=-1}^{1} C_{2\ \ 1\ \ 3}^{m_{12} m_3 0} T_{m_{12}}^2 W_{m_3}^1 \\ &= \sum_{m_1,m_2,m_3=-1}^{1} C_{2\ \ 1\ \ 3}^{-m_3 m_3 0} C_{1\ \ 1\ \ 2}^{m_1 m_2 -m_3} U_{m_1}^1 V_{m_2}^1 W_{m_3}^1 \\ &= \frac{5\, U^z V^z W^z - (\boldsymbol{U}\cdot\boldsymbol{V})\, W^z - (\boldsymbol{V}\cdot\boldsymbol{W})\, U^z - (\boldsymbol{W}\cdot\boldsymbol{U})\, V^z}{\sqrt{10}}, \end{aligned} \tag{B.18}$$

where we have used the fact that the dot product is $\mathbf{U}\cdot\mathbf{V} = \sum_m (-1)^m U_m^1 V_{-m}^1$ in the spherical representation. A similar construction can be used to find the vector operator or, one may note from Eqs. B.17 and B.18 that the vector component is equivalent to

$$U^z V^z W^z - \sqrt{\frac{2}{5}}\, T_0^3 = \frac{(\boldsymbol{U}\cdot\boldsymbol{V})\, W^z + (\boldsymbol{V}\cdot\boldsymbol{W})\, U^z + (\boldsymbol{W}\cdot\boldsymbol{U})\, V^z}{5} \tag{B.19}$$

as used in writing down (4.22).

Construction of the remaining (x and y) components of the vector operator in (B.19) is straightforward since one merely replaces z with either x or y. In order to construct the remaining six components of the rank-3 tensor operator one simply applies (B.15) twice without specifying $m = 0$:

$$T_m^3 = \sum_{m_1,m_2,m_3=-1}^{1} C_{2\ \ \ \ \ 1\ \ 3}^{m-m_3\, m_3\, m} C_{1\ \ 1\ \ 2}^{m_1 m_2\, m-m_3} U_{m_1}^1 V_{m_2}^1 W_{m_3}^1 . \tag{B.20}$$

The explicit form of these components are

$$\begin{aligned} T_1^3 &= -\frac{1}{2\sqrt{30}} \Big[(5\, V^z W^z - \boldsymbol{V}\cdot\boldsymbol{W})\, U^+ + (5\, U^z W^z - \boldsymbol{U}\cdot\boldsymbol{W})\, V^+ \\ &\quad + (5\, U^z V^z - \boldsymbol{U}\cdot\boldsymbol{V})\, W^+ \Big] \end{aligned} \tag{B.21}$$

$$T_2^3 = \frac{1}{2\sqrt{3}} \left[U^+ V^+ W^z + U^+ V^z W^+ + U^z V^+ W^+ \right] \tag{B.22}$$

$$T_3^3 = -\frac{1}{2\sqrt{2}} U^+ V^+ W^+ , \tag{B.23}$$

with the remaining three components obtained from $T_{-m}^q = (-1)^m (T_m^q)^\dagger$.

B.3 Sum Rule

The sum rule used in Section 4.5, on which the proof that ω destroys the ground state hinges, is given by

$$\lim_{R\to\infty} \sum_{0\le|z|<R} G(z)\; z^n\, e^{-\frac{\pi}{2}|z|^2} = 0\,. \tag{B.24}$$

The sum rule has been previously obtained by Laughlin [89]; in this appendix we show how to obtain the sum rule by application of Liouville's theorem. We first consider the related sum

$$F(c) = \lim_{R\to 0} \sum_{0\le|z|<R} G(z)\, \exp\left[\frac{1}{2}\,c\,z - \frac{\pi}{2}\,|z|^2\right]. \tag{B.25}$$

In order to prove the sum rule in (B.24), we will first show that $F(c) = 0$ for any value of the parameter c and then use this to prove (B.24) by taking derivatives of the function $F(c)$.

In order to show that $F(c) = 0$, we use the gauge function $G(z)$ to write (B.25) as two sums, one over the entire lattice and one over the points z' on the lattice for which $G(z') = -1$. As shown in Figure 4.1, these sites defines a sublattice with twice the original lattice spacing.

$$F(c) = \sum_z e^{\frac{1}{2}\,c\,z - \frac{\pi}{2}\,|z|^2} - 2\sum_{z'} e^{\frac{1}{2}\,c\,z' - \frac{\pi}{2}\,|z'|^2}\,. \tag{B.26}$$

Setting $z' = 2\,z$ we can write this as

$$F(c) = \sum_z e^{\frac{1}{2}\,c\,z - \frac{\pi}{2}\,|z|^2} - 2\sum_z e^{c\,z - 2\,\pi\,|z|^2}\,, \tag{B.27}$$

where both sums now run over the entire lattice. Writing $z = x + i\,y$ this function can be factored into four sums over the integers x and y:

$$\begin{aligned} F(c) \;\; &= \;\; \left(\sum_x e^{\frac{1}{2}\left(c\,x - \pi\,x^2\right)}\right)\left(\sum_y e^{\frac{1}{2}\left(i\,c\,y - \pi\,y^2\right)}\right) \\ &\quad -2\left(\sum_x e^{c\,x - 2\,\pi\,x^2}\right)\left(\sum_y e^{i\,c\,y - 2\,\pi\,y^2}\right). \end{aligned} \tag{B.28}$$

In terms of the third Jacobi theta function [2]

$$\theta_3(z|\tau) = \sum_{n=-\infty}^{\infty} e^{i\,\pi\,n^2\,\tau}\, e^{2\,i\,n\,z}\,, \tag{B.29}$$

this function may be recast as

$$\begin{aligned} F(c) &= \vartheta_3\left(-i\,\frac{c}{4}\middle|\,\frac{i}{2}\right)\,\vartheta_3\left(\frac{c}{4}\middle|\,\frac{i}{2}\right) \\ &\quad -2\,\vartheta_3\left(-i\,\frac{c}{2}\middle|\,2\,i\right)\,\vartheta_3\left(\frac{c}{2}\middle|\,2\,i\right)\,. \end{aligned} \tag{B.30}$$

The fact that the two terms in this expression precisely cancel is a result of Liouville's theorem [2, 127],

$$\theta_3(z|\tau) = \frac{1}{\sqrt{-i\,\tau}}\,e^{z^2/i\,\pi\,\tau}\,\vartheta_3\left(\pm\frac{z}{\tau}\middle|-\frac{1}{\tau}\right)\,, \tag{B.31}$$

and the fact that the third Jacobi theta function is even. Application of this identity to either product of theta functions in (B.30) shows that the two terms precisely cancel, proving that $F(c) = 0$. This in turn proves the $n = 0$ case of (B.24) by simply setting $c = 0$. The other instances of the sum rule are obtained by noting that

$$\frac{1}{m!}\frac{d^m}{dc^m}F(c) = \lim_{R\to\infty}\sum_{0\le|z|<R} G(z)\;z^m\,e^{c\,z}\,e^{-\frac{\pi}{2}\,|z|^2}\,. \tag{B.32}$$

Since $F(c)$ is 0 for all values of c (within the radius of convergence), setting $c = 0$ in the above expression gives the desired result in (B.24).

B.3.1 Corollary to Sum Rule

We now consider the case where we wish to evaluate a sum of the form

$$\lim_{R\to\infty}\sum_{0<|z|<R}\frac{1}{z}\,A(z)\;G(z)\;e^{-\frac{\pi}{2}\,|z|^2} \tag{B.33}$$

where $A(z)$ is an analytic function of z. Since it is analytic, we can expand the function $A(z)$ in a Taylor series:

$$A(z) = \sum_{\ell=0}^{\infty}\frac{1}{\ell!}\;\frac{d^\ell A}{dz^\ell}\bigg|_0\;z^\ell \tag{B.34}$$

and, so long as the sum in (B.33) is absolutely convergent, we can interchange the order of the two infinite sums to obtain

$$\sum_\ell\frac{1}{\ell!}\;\frac{d^\ell A}{dz^\ell}\bigg|_0\left(\lim_{R\to\infty}\sum_{0<|z|<R} z^{\ell-1}\,G(z)\;e^{-\frac{\pi}{2}\,|z|^2}\right)\,. \tag{B.35}$$

All terms for which $\ell > 2$ immediately vanish from the interior sum due to the sum rule in (B.24). The term with $\ell = 0$ also vanishes because in that case the summand is an odd function summed over the entire lattice. Finally the term with $\ell = 1$ can be evaluated using the sum rule and is simply the negative of the value of the summand at $z = 0$ (which is not included in this sum but is included in (B.24)). Therefore, so long as $A(z)$ is chosen so that the sum itself is absolutely convergent,

$$\lim_{R\to\infty} \sum_{0<|z|<R} \frac{1}{z}\, A(z)\; G(z)\; e^{-\frac{\pi}{2}\,|z|^2} = \left.\frac{dA}{dz}\right|_0 . \tag{B.36}$$

B.4 The function $A(z)$

The ratio of wave function coefficients appearing in (B.5),

$$\frac{\langle z_1 \cdots z_j + x \cdots z_M \rangle\, \psi}{\langle z_1 \cdots z_j \cdots z_M \rangle\, \psi}\,, \tag{B.37}$$

can be written in terms of the Gauge function $G(x)$, the Gaussian $e^{-\frac{\pi}{2}\,|x|^2}$, and an analytic function of x, $A(x)$. To see this we note that this ratio may be written explicitly as

$$\prod_{\nu=1}^{2} \frac{\vartheta\big(\frac{\pi}{L}\;[\mathcal{Z} + x - Z_\nu]\big)}{\vartheta\big(\frac{\pi}{L}\;[\mathcal{Z} - Z_\nu]\big)} \prod_{i\neq j}^{M} \frac{\vartheta^2\big(\frac{\pi}{L}\;[z_j + x - z_i]\big)}{\vartheta^2\big(\frac{\pi}{L}\;[z_j - z_i]\big)}\, \frac{G(z_j + x)}{G(z_j)\; G(x)}$$
$$\frac{e^{\frac{\pi}{2}\left[(z_j+x)^2 - |z_j+x|^2\right]}}{e^{\frac{\pi}{2}\left[z_j^2 - |z_j|^2\right]}\, e^{\frac{\pi}{2}\left[x^2 - |x|^2\right]}}\, G(x)\; e^{\frac{\pi}{2}\left(x^2 - |x|^2\right)} . \tag{B.38}$$

This simplifies by noting that the exponential terms obey an addition formula

$$\frac{e^{\frac{\pi}{2}\left[(z_j+x)^2 - |z_j+x|^2\right]}}{e^{\frac{\pi}{2}\left[z_j^2 - |z_j|^2\right]}\, e^{\frac{\pi}{2}\left[x^2 - |x|^2\right]}} = e^{\frac{\pi}{2}\left[(x-x^*)\, z_j + x\left(z_j - z_j^*\right)\right]}, \tag{B.39}$$

and the Gauge function obeys an addition formula given by

$$\frac{G(z_j + x)}{G(z_j)\; G(x)} = -e^{\frac{\pi}{2}\left(z_j^*\, x^* - z_j\, x\right)} . \tag{B.40}$$

Since the terms involving x^* cancel on multiplying the two expressions in Eqs. B.39 and B.40, the ratio of coefficients is

$$\frac{\langle z_1 \cdots z_j + x \cdots z_M \rangle\, \psi}{\langle z_1 \cdots z_j \cdots z_M \rangle\, \psi} = -A(x)\; G(x)\; e^{-\frac{\pi}{2}\,|x|^2} \tag{B.41}$$

where $A(x)$ is an analytic function of x:

$$A(x) = \prod_{\nu=1}^{2} \frac{\vartheta\left(\frac{\pi}{L}\,[\mathcal{Z}+x-Z_\nu]\right)}{\vartheta\left(\frac{\pi}{L}\,[\mathcal{Z}-Z_\nu]\right)} \prod_{i\neq j}^{M} \frac{\vartheta^2\left(\frac{\pi}{L}\,[z_j+x-z_i]\right)}{\vartheta^2\left(\frac{\pi}{L}\,[z_j-z_i]\right)}\, e^{\frac{\pi}{2}\left[x^2+x\left(z_j-z_j^*\right)\right]}\,. \tag{B.42}$$

The derivative of this function is given by

$$\begin{aligned} \frac{dA}{dx} \;&=\; \left\{\sum_{\nu=1}^{2} \frac{d}{dx} \ln\vartheta\left(\frac{\pi}{L}\,[\mathcal{Z}+x-Z_\nu]\right) + 2\sum_{i\neq j}^{M} \frac{d}{dx} \ln\vartheta\left(\frac{\pi}{L}\,[z_j-z_i+x]\right)\right. \\ &\left. +\frac{\pi}{2}\left(2\,x+z_j-z_j^*\right)\right\} A(x)\;. \end{aligned} \tag{B.43}$$

Evaluating this at $x=0$ and noting that $A(0)=1$ from (B.42) gives

$$\begin{aligned} \left.\frac{dA}{dx}\right|_0 \;&=\; \sum_{\nu=1}^{2} \frac{d}{d\mathcal{Z}} \ln\vartheta\left(\frac{\pi}{L}\,[\mathcal{Z}-Z_\nu]\right) + 2\sum_{i\neq j}^{M} \frac{d}{dz_j} \ln\vartheta\left(\frac{\pi}{L}\,[z_j-z_i]\right) \\ &+N\,\frac{\pi}{L}\,\frac{z_j-z_j^*}{2\,L}\,. \end{aligned} \tag{B.44}$$

In terms of the function $W(z)$ introduced in (4.28) this may be written as

$$\left.\frac{dA}{dx}\right|_0 = \sum_{\nu=1}^{2} \frac{\pi}{L}\, W\left(\frac{\pi}{L}\,[\mathcal{Z}-Z_1]\right) + 2\sum_{i\neq j}^{M} \frac{\pi}{L} W\left(\frac{\pi}{L}\,[z_j-z_i]\right)\;. \tag{B.45}$$

The final expression follows from the fact that the center of mass zeroes are constrained to satisfy $\sum_\nu Z_\nu = 0$ as pointed out in Section 4.2.

B.5 Pfaffian function

Given an antisymmetric matrix M_{ij} the *Pfaffian* of M is defined to be

$$\mathrm{Pf}(M) \equiv \sum_{pairings} \pm \prod_{\text{pairs (ab)}} M_{\text{ab}} \tag{B.46}$$

where the sign associated with each term is positive if the permutation needed to bring the indices back to their original order is even, and negative if the required permutation is odd. Thus for example the Pfaffian of a 4×4 matrix is

$$M_{12}M_{34} - M_{13}M_{24} + M_{14}M_{23}.$$

The most important fact about the Pfaffian used in our calculations is that its square is equal to the determinant,

$$\mathrm{Pf}(M)^2 = \det(M). \tag{B.47}$$

B.6 Theta functions

The ϑ functions are used to rephrase the plane boundary wave functions to the torus geometry of the PBCs. They are defined by

$$\vartheta_1(z,\tau) = \sum_{n=-\infty}^{+\infty} e^{\pi i(n+\frac{1}{2})^2\tau}\, e^{2\pi i(n+\frac{1}{2})(z+\frac{1}{2})} \tag{B.48}$$

where the other theta function follow out of (B.48) by

$$\begin{aligned} \vartheta_2(z,\tau) &= \vartheta_1(z+\frac{1}{2},\tau) \\ \vartheta_3(z,\tau) &= M\vartheta_1(z+\frac{(1+\tau)}{2},\tau) \\ \vartheta_4(z,\tau) &= M\vartheta_1(z+\frac{\tau}{2},\tau), \end{aligned} \tag{B.49}$$

where $M \equiv e^{i\pi\tau/4}e^{i\pi z}$. For our purposes the most important properties of the ϑ functions are the periodicities

$$\begin{aligned} \vartheta_1(z+1) &= -\vartheta_1(z) \\ \vartheta_2(z+1) &= -\vartheta_2(z) \\ \vartheta_3(z+1) &= +\vartheta_3(z) \\ \vartheta_4(z+1) &= +\vartheta_4(z), \end{aligned} \tag{B.50}$$

$$\begin{aligned} \vartheta_1(z+\tau) &= -e^{-i\pi\tau}e^{-i2\pi z}\vartheta_1(z) \\ \vartheta_2(z+\tau) &= +e^{-i\pi\tau}e^{-i2\pi z}\vartheta_2(z) \\ \vartheta_3(z+\tau) &= +e^{-i\pi\tau}e^{-i2\pi z}\vartheta_3(z) \\ \vartheta_4(z+\tau) &= -e^{-i\pi\tau}e^{-i2\pi z}\vartheta_4(z)\ , \end{aligned} \tag{B.51}$$

the reflection relations

$$\vartheta_1(-z) = -\vartheta_1(z)$$

$$\begin{aligned}
\vartheta_2(-z) &= \vartheta_2(z) \\
\vartheta_3(-z) &= \vartheta_3(z) \\
\vartheta_4(-z) &= \vartheta_4(z)\ ,
\end{aligned} \tag{B.52}$$

and the fact that ϑ_1 is a holomorphic function whose only zeroes are simple ones occurring at the lattice points $m + n\tau$, where m, n are integers.

B.7 Octopuzz theorem

B.7.1 Plane geometry

We start with the case of open boundaries. For the $S = 1/2$ lattice setup for the CSL with $z_1, \dots z_M$ denoting the positions of the up spins and $M = N/2$ with N being the number of sites, the following theorem holds asymptotically for large N [71]:

$$\prod_{\alpha,\eta_\alpha \neq z_j}^{N} (z_j - \eta_\alpha) = G(z_j) e^{\frac{\pi}{2}|z_j|^2} \prod_{\alpha=1}^{N} e^{\frac{\pi}{2N}|\eta_\alpha|^2} \cdot \text{const.}, \tag{B.53}$$

where the first sum runs over all lattice sites except z_j. This can be shown by defining

$$h_j(\zeta) = \prod_{\alpha,\eta_\alpha \neq z_j} (\zeta - \eta_\alpha), \tag{B.54}$$

with ζ being a continuous variable. Now, one considers the limit $\zeta \to z_j$ for

$$\ln h_j(\zeta) = \sum_{\alpha=1}^{N} \ln|\zeta - \eta_\alpha| - \ln|\zeta - z_j| + i \arg[h_j(\zeta)], \tag{B.55}$$

which can be interpreted as a two-dimensional Coulomb gas energy problem with a lattice of unit charges and a test charge approaching the lattice point z_j. The singularities of the first and second term as $\zeta \to z_j$ cancel each other, where to leading order the energy contribution from the test charge with the average charge density background as well as sub-leading self energy corrections of the lattice charges remain:

$$\sum_{\alpha=1}^{N} \ln|\zeta - \eta_\alpha| - \ln|\zeta - z_j| \sim \frac{\pi}{2}|\zeta|^2 + \sum_{\alpha=1}^{N} \frac{\pi}{2N}|\eta_\alpha|^2 + \text{const.} \tag{B.56}$$

As generating an additional phase π by propagating from one site to a nearest neighbor position in the average charge density background [71], the third term of (B.55) can be shown to essentially give an alternating sign by exponentiation, which is just the gauge factor $G(z_j)$. In total, it thus yields (B.53).

B.7.2 Torus geometry

For PBCs, one can find a theorem similar to (B.53). For simplicity, consider a standard torus surface supercell with principal region given by unit length 1 and τ, where N denotes the number of sites and the site coordinates η_α are parameterized by

$$\eta_\alpha = l_\alpha a + m_\alpha b, \tag{B.57}$$

with $aL_1 = 1$, $bL_2 = \tau$, and $L_1 L_2 = N$. In analogy to the Jastrow factors in (B.53), we consider the function

$$P[z_j] = \prod_{\alpha=1,\eta_\alpha \neq z_j}^{N} \vartheta_1(\eta_\alpha - z_j, \tau), \tag{B.58}$$

where we use the first theta function from (B.48) and from now on omit the index one in this paragraph. We will now show that the following relation holds:

$$\frac{P(l_j a + m_j b)}{P(0)} = (-1)^{L_2 l_j + m_j} e^{\pi i \tau L_1 m_j (1 - \frac{m_j+1}{L_2})} \tag{B.59}$$

For this, we first consider

$$\frac{P(l_j a + m_j b)}{P(m_j b)} = \frac{\mathcal{P}_2(0) \ldots \mathcal{P}_2(l_j - 1)}{\mathcal{P}_2(L_1) \ldots \mathcal{P}_2(L_1 + l_j - 1)}, \tag{B.60}$$

where $\mathcal{P}_2(l)$ is defined as the lth line contribution in τ direction to $P(l_j a + m_j b)$:

$$\mathcal{P}_2(l) = \prod_{m=0}^{L_2-1} \vartheta((l - l_j)a + (m - m_j)b, \tau). \tag{B.61}$$

Due to the periodicity of the first theta function (B.48), we obtain

$$\mathcal{P}_2(l)/\mathcal{P}_2(L_1 + l) = (-1)^{L_1}, \tag{B.62}$$

and thus find

$$\frac{P(l_j a + m_j b)}{P(m_j b)} = (-1)^{L_2 l_j}. \tag{B.63}$$

Similarly, we consider a shift in τ direction

$$\frac{P(m_j b)}{P(0)} = \frac{\mathcal{P}_1(0) \ldots \mathcal{P}_1(m_j - 1)}{\mathcal{P}_1(L_2) \ldots \mathcal{P}_2(L_2 + m_j - 1)}, \tag{B.64}$$

where $\mathcal{P}_1(m)$ is defined as the mth line contribution in 1 direction to $P(m_j b)$, *i.e.*, $\mathcal{P}_1(m) = \prod_{l=0}^{L_1-1} \vartheta((la + (m - m_j)b, \tau)$. Recalling the quasi-periodicity relation (B.48) of $\vartheta(\eta, \tau)$ in τ direction yields

$$\frac{\mathcal{P}_1(m)}{\mathcal{P}_1(L_2 + m)} = (-1)e^{\pi i \tau L_1} e^{2\pi i \tau \frac{L_1}{L_2}(m - m_l)}, \tag{B.65}$$

so that for (B.64) it holds

$$\begin{aligned} \frac{P(m_j b)}{P(0)} &= \prod_{m=0}^{m_j - 1} \frac{\mathcal{P}_1(m)}{\mathcal{P}_1(L_2 + m)} \\ &= (-1)^{m_j} e^{\pi i \tau L_1 m_j} e^{-\pi i \tau \frac{L_1}{L_2}(m_j + 1) m_j}. \end{aligned} \tag{B.66}$$

Combining (B.63) and (B.66) thus yields (B.59).

B.8 Schwinger Bosons

Schwinger bosons [8,113] constitute a way to formulate spin-S representations of an SU(2) algebra. The spin operators

$$\begin{aligned} S^x + iS^y &= S^+ = a^\dagger b \\ S^x - iS^y &= S^- = b^\dagger a \\ S^z &= \tfrac{1}{2}(a^\dagger a - b^\dagger b) \end{aligned} \tag{B.67}$$

are given in terms of boson creation and annihilation operators which obey the usual commutation relations

$$\begin{gathered} \left[a, a^\dagger\right] = \left[b, b^\dagger\right] = 1 \\ [a, b] = \left[a, b^\dagger\right] = \left[a^\dagger, b\right] = \left[a^\dagger, b^\dagger\right] = 0. \end{gathered} \tag{B.68}$$

It is readily verified with (B.68) that

$$\left[S^i, S^j\right] = i\epsilon^{ijk} S^k \quad \text{where} \quad i, j, k = x, y, \text{or } z. \tag{B.69}$$

The spin quantum number S is given by half the number of bosons,

$$2S = a^\dagger a + b^\dagger b, \tag{B.70}$$

and the usual spin states (simultaneous eigenstates of $\boldsymbol{S}^2$ and S^z) are given by

$$|s, m\rangle = \frac{(a^\dagger)^{s+m}}{\sqrt{(s+m)!}} \frac{(b^\dagger)^{s-m}}{\sqrt{(s-m)!}} \,|\, 0 \,\rangle\,. \tag{B.71}$$

In particular, the spin-$\frac{1}{2}$ states are given by

$$|\uparrow\rangle = a^\dagger\,|\,0\,\rangle = c^\dagger_\uparrow\,|\,0\,\rangle \qquad |\downarrow\rangle = b^\dagger\,|\,0\,\rangle = c^\dagger_\downarrow\,|\,0\,\rangle\,, \tag{B.72}$$

i.e., $a^\dagger$ and $b^\dagger$ act just like the fermion creation operators $c^\dagger_\uparrow$ and $c^\dagger_\downarrow$ in this case. The difference shows up only when two (or more) creation operators act on the same site or orbital. The fermion operators create an antisymmetric or singlet configuration (in accordance with the Pauli principle),

$$|0,0\rangle = c^\dagger_\uparrow c^\dagger_\downarrow\,|\,0\,\rangle\,, \tag{B.73}$$

while the Schwinger bosons create a totally symmetric or triplet (or higher spin if we create more than two bosons) configuration,

$$\begin{aligned} |1,1\rangle &= \tfrac{1}{\sqrt{2}}(a^\dagger)^2\,|\,0\,\rangle \\ |1,0\rangle &= a^\dagger b^\dagger\,|\,0\,\rangle \\ |1,-1\rangle &= \tfrac{1}{\sqrt{2}}(b^\dagger)^2\,|\,0\,\rangle\,. \end{aligned} \tag{B.74}$$

B.9 Z-W symmetry for Chiral Spin Liquid

B.9.1 Plane geometry

To begin with, we use $M = N/2$ to rewrite the CSL wave function as

$$\psi^{\mathrm{CSL}}(z_1,\ldots,z_M) = \prod_{j=1}^{M}\left(G(z_j)\,e^{-\frac{M\pi}{N}|z_j|^2}\prod_{\substack{k=1\\(k\neq j)}}^{M}(z_j - z_k)\right). \tag{B.75}$$

Let w_l with $l = 1, 2, \ldots, N-M$ be the lattice sites not occupied by the z_j's. The arguments $(z_1,\ldots,z_M)$ and $(w_1,\ldots,w_M)$ are from now on abbreviated by $\{z\}$ and $\{w\}$ in this paragraph. The previously derived octopuzz theorem (B.53) implies

$$\begin{aligned} \psi^{\mathrm{CSL}}(\{z\}) &= \prod_{j=1}^{M}\left(\prod_{l=1}^{M}\frac{1}{z_j - w_l}\right)\prod_{j=1}^{M} e^{+\frac{(N-M)\pi}{2N}|z_j|^2} \\ &\quad\times\prod_{l=1}^{M} e^{+\frac{M\pi}{2N}|w_l|^2}. \end{aligned} \tag{B.76}$$

Let

$$\mathcal{S}[z,w] \equiv \langle\,0\,|\,c_{z_1\downarrow}\ldots c_{z_M\downarrow}\,c_{w_1\downarrow}\ldots c_{w_{N-M}\downarrow}\big|\underbrace{\downarrow\downarrow\ldots\ldots\downarrow}_{\text{all } N \text{ spins } \downarrow}\big\rangle, \tag{B.77}$$

be the sign associated with ordering the z's and w's according to their lattice positions. Then we can use

$$\prod_{j<k}^{M}(z_j - z_k)\prod_{j=1}^{M}\prod_{l=1}^{N-M}(z_j - w_l)\prod_{l<m}^{N-M}(w_l - w_m)$$
$$\times\prod_{j=1}^{M}e^{-\frac{\pi}{2}|z_j|^2}\prod_{l=1}^{N-M}e^{-\frac{\pi}{2}|w_l|^2} = \mathcal{S}[z,w]\cdot\text{const.} \tag{B.78}$$

to rewrite (B.76) as

$$\psi^{\text{CSL}}(\{z\}) = \mathcal{S}[z,w]\,\phi(\{z\})\,\phi(\{w\}) \tag{B.79}$$

where

$$\phi(\{z\}) = \prod_{j<k}^{M}(z_j - z_k)\prod_{j=1}^{M}e^{-\frac{M\pi}{2N}|z_j|^2} \tag{B.80}$$

is simply the wave function for a filled Landau level in a (fictitious) magnetic field with flux $\frac{2\pi M}{N}$/plaquet. If we rewrite (B.75) in terms of (B.80) and compare it to (B.79), we obtain the lattice particle-hole symmetry

$$\prod_{j=1}^{M}G(z_j)\,\phi(\{z\}) = \mathcal{S}[z,w]\,\phi(\{w\})\cdot\text{const.}, \tag{B.81}$$

which holds for any M. The chiral spin liquid ground state, where $M = N/2$, is according to (B.79) simply given by

$$|\psi^{\text{CSL}}\rangle = \sum_{\{z_1,\ldots,z_M;w_1,\ldots,w_M\}}\phi(z_1,\ldots,z_M)\,\phi(w_1,\ldots,w_M)$$
$$\times\, c^\dagger_{z_1\uparrow}\ldots c^\dagger_{z_M\uparrow}c^\dagger_{w_1\downarrow}\ldots c^\dagger_{w_M\downarrow}\,|\,0\,\rangle\,, \tag{B.82}$$

where the sum extends over all possible ways to distribute the coordinates z_j and w_l on mutually distinct lattice sites.

B.9.2 Torus geometry

A similar particle-hole separation can be equivalently shown for the CSL wave function with PBCs on the torus geometry. For simplicity consider again the standard supercell primary region on the torus as in (B.57) with one lattice point in the center. We reparametrize the lattice point in a symmetric form as

$$\eta_\alpha = (-\frac{L_1 - 1}{2} + l_\alpha)a + (-\frac{L_2 - 1}{2} + m_\alpha)b, \tag{B.83}$$

with $aL_1 = 1$, $bL_2 = \tau$, and $(l/m)_\alpha = 0, \ldots, L_{1/2} - 1$. In its most convenient form for our purposes, the CSL wave function on the torus is given by

$$\begin{aligned}\Psi^{\mathrm{CSL}}(\{z\}) &= \vartheta_1(Z - Z_1)\vartheta_1(Z + Z_1)\prod_{i,j}^{M}\vartheta_1(z_i - z_j)^2 \\ &\quad\times\prod_{i}^{M}G(z_i)\exp(-\frac{\pi}{2}|z_i|^2 + \frac{\pi}{2}z_i^2),\end{aligned} \tag{B.84}$$

where G is the gauge factor, $M = N/2$, Z is the center of mass $Z = \sum_{i=1}^{M} z_i$, and z denotes the position of the up spins as before. Again the index one at the theta functions will be omitted in the following. There are two choices of the parameter Z_1 that give orthogonal wave functions so that the ground state is doubly degenerate, *i.e.,* the topological degeneracy of the CSL on the torus. We define

$$\phi(\{z\}) = \vartheta(Z - Z_1)\prod_{i<j}^{M}\vartheta(z_i - z_j)\prod_{i}^{M}e^{-\frac{M\pi}{2N}(|z_i|^2 - z_i^2)} \tag{B.85}$$

with its complementary counterpart

$$\phi(\{w\}) = \vartheta(W - Z_1)\prod_{k<l}^{M}\vartheta(w_k - w_l)\prod_{k}^{M}e^{-\frac{(N-M)\pi}{2N}(|w_k|^2 - w_k^2)}, \tag{B.86}$$

which is the wave function for a Landau band in symmetric gauge and a magnetic field of 2π/plaquet. We now rewrite (B.86) in terms of up spin variables z. First note that as one lattice site is a the origin, $Z{=}{-}W$ and thus

$$\vartheta(W - Z_1) = -\vartheta(Z + Z_1). \tag{B.87}$$

Furthermore we rewrite the normalization by $\prod_{\alpha=1}^{N}\exp(\frac{\pi(N-M)}{2N}(|\eta_\alpha|^2 - \eta_\alpha^2)) =$ const. to get

$$\begin{aligned}\prod_{k}e^{-\frac{\pi(N-M)}{2}|w_k|^2} &= \text{const.}\prod_{i}^{M}e^{-\frac{M\pi}{2N}(|z_i|^2 - z_i^2)}\prod_{i}^{M}e^{\frac{\pi}{2}|z_i|^2 - z_i^2} \\ &= \text{const.}\prod_{i=1}^{M}(-1)^{l_i m_i}\prod_{i=1}^{M}e^{\pi\frac{b}{a}(-m_i(L_2-1)+m_i^2)} \\ &\quad\times\prod_{i}^{M}e^{-\frac{M\pi}{2N}(|z_i|^2 - z_i^2)}.\end{aligned} \tag{B.88}$$

As the next step, we use the previously defined Octopuzz theorem to rewrite the theta functions. Using $\prod_{\alpha<\beta}^{N} \vartheta(\eta_\alpha - \eta_\beta) = \text{const.}$ we get (where c denotes a constant)

$$\begin{aligned}
\prod_{k<l}^{M} \vartheta(w_k - w_l) &= \frac{\mathcal{S}[z,w]c}{\prod_{i<j}^{M} \vartheta(z_i - z_j)} \prod_{i=1}^{M} \prod_{k=1}^{N-M} \frac{1}{\vartheta(z_i - w_k)} \\
&= \mathcal{S}[z,w]c \prod_{i<j}^{M} \vartheta(z_i - z_j) \prod_{i=1}^{M} \prod_{\alpha=1}^{N} \frac{1}{\vartheta(z_i - \eta_\alpha)} \\
&= \mathcal{S}[z,w]c \prod_{i<j}^{M} \vartheta(z_i - z_j) \\
&\quad \prod_{i=1}^{M} e^{-\pi \frac{b}{a}(-m_i(L_2-1)+m_i^2)} (-1)^{l_i+m_i+1},
\end{aligned} \tag{B.89}$$

where the second line above cancels the last term in (B.88). Putting all together, the particle hole symmetry for the CSL on the torus becomes manifest:

$$\Psi^{\mathrm{CSL}}(\{z\}) = \mathcal{S}[z,w]\phi(\{z\})\phi(\{w\}). \tag{B.90}$$

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