“Ghost” ILDM-Manifolds and their Identification

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Summary. One of the popular methods (Intrinsic Low-Dimensional Manifolds – ILDM)) of decomposition of multiscale systems into fast and slow sub-systems for reduction of their complexity is considered in the present paper. The method successfully locates a position of slow manifolds of considered system and as any other numerical approach has its own disadvantages. In particular, an application of the ILDM-method produces so-called “ghost”-manifolds that do not have any connection to the true dynamics of the system. It is shown analytically that for two-dimensional singularly perturbed system (for which the fast-slow decomposition has been already done in analytical way) the “ghost”-manifolds appear. The problem of discrimination/identification of the “ghost”-manifolds is under consideration and two numerical criteria for their identification are proposed. A number of analyzed examples demonstrate efficiency of the suggested approach.

1 Introduction

In this paper, following Maas and Pope [16] we consider Intrinsic Low- Dimensional Manifolds Method (ILDM) for systems of ordinary differential equations. The main aim of this paper is to demonstrate that application of the conventional ILDM machinery can produce additional artificial objects (“ghost” manifolds) that do not have any connection to the true slow invariant manifold of considered system. Two various approaches for the “ghost” manifolds identification/discrimination are suggested and their application is demonstrated.

The paper is organized as follows. In Sect. 2, we give a review of several reduction methods, which are used in combustion and chemical kinetics problems. In Sect. 3, we give the examples of the “ghost” manifolds phenomenon. In Sect. 4, we suggest two criteria for identification/discrimination of the “ghost” objects. In Sect. 5, we conclude the results.

2 Theoretical Background

In this section the method of invariant manifolds, iterative method of Fraser, Inflector-method, Intrinsic Low-Dimensional Manifolds (ILDM) method and its modification (TILDM) will be briefly described.

2.1 Method of Invariant Manifolds (MIM)

Consider a singularly perturbed system of ordinary differential equations

$$\epsilon \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}, \mathbf{y}, \epsilon) \tag{1}$$

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \mathbf{g}(\mathbf{x}, \mathbf{y}, \epsilon) \tag{2}$$

Here $\mathbf{x} \in \Re^m$, $\mathbf{y} \in \Re^n$ are vectors in Euclidean space, $t \in (t_0, +\infty)$ is a time-like variable, $0 < \epsilon < \epsilon_0 << 1$, functions $\mathbf{f}: \Re^m \times \Re^n \to \Re^m$, $\mathbf{g}: \Re^m \times \Re^n \to \Re^n$ are supposed to be sufficiently smooth for all $\mathbf{x} \in \Re^m$, $\mathbf{y} \in \Re^n$, $0 < \epsilon < \epsilon_0$. The values $|\mathbf{f_i}(\mathbf{x}, \mathbf{y}, \epsilon)|$, $|\mathbf{g_i}(\mathbf{x}, \mathbf{y}, \epsilon)|$, $(i = 1, ...m;\ j = 1, ...n)$ are assumed to be comparable with the unity as $\epsilon \to 0$.

Definition 1. *A smooth surface in the phase space $M \in \Re^m \times \Re^n \times \Re$ is called an invariant manifold of the system (1)-(2), if any phase trajectory $(\mathbf{x}(t, \epsilon), \mathbf{y}(t, \epsilon))$ such that $(\mathbf{x}(t_1, \epsilon), \mathbf{y}(t_1, \epsilon)) \in M$ belongs to M for any $t > t_1$. If the last condition holds only for $t \in [t_1, T]$, then M is called a local invariant manifold.*

The simplest examples of invariant manifolds are phase trajectory and phase space.

The manifold's existence leads to the fact that the analysis of the system's behaviour can be considerably simplified by reducing a dimension of the system. We are interested in the invariant manifolds of dimension m (the dimension of the slow variable) that can be represented as a graph of the vector-valued function:

$$\mathbf{x} = \mathbf{h}(\mathbf{y}, \epsilon). \tag{3}$$

The invariant manifolds mentioned above are called manifolds of slow motions (this term was adopted from the nonlinear mechanics). The system's dynamics on this manifold is described by the equation

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \mathbf{g}(\mathbf{h}(\mathbf{y}, \epsilon), \mathbf{y}, \epsilon). \tag{4}$$

If $\mathbf{y}(t, \epsilon)$ is a solution of the Eq.(4), then the pair $\mathbf{x}(t, \epsilon), \mathbf{y}(t, \epsilon)$ where $\mathbf{x}(t, \epsilon) = \mathbf{h}(\mathbf{y}(t, \epsilon), \epsilon)$ is a solution of the original system (1)-(2), since it determines a trajectory on the invariant manifold.

A usual approach in the qualitative study of (1)-(2) is to consider first the degenerate system, which is obtained by substituting $\epsilon = 0$ into the system

$$0 = \mathbf{f}(\mathbf{x}, \mathbf{y}, 0) \tag{5}$$

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \mathbf{g}(\mathbf{x}, \mathbf{y}, 0), \tag{6}$$

and then to draw conclusions for the qualitative behaviour of the full system for sufficiently small ϵ. The Eq.(5) determines the slow surface. The slow surface is the zeroth approximation of the slow invariant manifold. It is assumed that the Eq.(5) has an isolated smooth solution $\mathbf{x} = \mathbf{h_0}(\mathbf{y})$. Moreover, the next relation should take place

$$lim_{\epsilon \to 0} \mathbf{h}(\mathbf{y}, \epsilon) = \mathbf{h_0}(\mathbf{y})$$

In addition, only these manifolds are important here that are stable (attractive). By the famous Tikhonov's theorem, the question of stability of an invariant manifold can be reduced to study of its zeroth approximation stability.

Invariant manifold $\mathbf{x} = \mathbf{h}(\mathbf{y}, \epsilon)$ of the system (1)-(2) is stable, if the real parts of all eigenvalues of the matrix $D_{\mathbf{x}}\mathbf{f}(\mathbf{h_0(y)}, \mathbf{y}, 0)$ are negative.

Points of the slow surface determined by (5) are sub-divided into two types: *standard* points and *turning* points. A point $(\mathbf{x}, \mathbf{y})$ is a standard point of the slow surface if in some neighborhood of this point the surface can be represented as a graph of a function $\mathbf{x} = \mathbf{h_0}(\mathbf{y})$ such that $\mathbf{f}(\mathbf{h_0}(\mathbf{y}), \mathbf{y}, 0) = 0$. It means that the condition of the Implicit Function Theorem $D_{\mathbf{x}}\mathbf{f}(\mathbf{h_0}(\mathbf{y}), \mathbf{y}, 0) \neq 0$ holds and the slow surface has the dimension of slow variable. Points where this condition does not hold are turning points of the slow surface. In other words, turning points are defined as solutions of the following system

$$\mathbf{f}(\mathbf{x}, \mathbf{y}, 0) = 0$$

$$\mathbf{f}_x(\mathbf{x}, \mathbf{y}, 0) = 0$$

The asymptotic method described below can not be applied there.

Problems of existence, uniqueness and stability of invariant manifolds have been studied by many authors. The main results of these studies can be summarized in the following theorems.

Theorem 1. *(Mitropolsky and Lykova, 1973) Let the system (1)-(2) satisfies the following conditions:*

(i) The equation $\mathbf{f}(\mathbf{x}, \mathbf{y}, 0) = 0$ *has an isolate solution* $\mathbf{x} = \mathbf{h_0}(\mathbf{y})$ *in some domain* $G = \{(\mathbf{x}, \mathbf{y}, \epsilon) : \ \mathbf{y} \in \Re^n, 0 < \epsilon < \epsilon_0, ||\mathbf{x} - \mathbf{h_0}(\mathbf{y})|| \leq \rho\}$.

(ii) The functions $\mathbf{f}, \mathbf{g}, \mathbf{h_0}$ *and their first and second partial derivatives are uniformly continues and bounded in* G.

(iii) The eigenvalues $\lambda_i(\mathbf{y}),\ i = 1, 2, ..., n$ *of the matrix* $D_{\mathbf{x}}\mathbf{f}(\mathbf{h_0(y)}, \mathbf{y}, 0)$ *satisfy the condition* $Re[\lambda_i(\mathbf{y})] \leq -\beta,\ i = 1, 2, ..., n,\ \mathbf{y} \in \Re^n$ *for some* $\beta > 0$.

Then there exists an $\epsilon_1 : \ 0 < \epsilon_1 < \epsilon_0$*, such that for every* $\epsilon : \ 0 < \epsilon < \epsilon_1$ *the system (1)-(2) has a unique invariant manifold* $\mathbf{x} = \mathbf{h}(\mathbf{y}, \epsilon)$*, where the function* $\mathbf{h}$ *satisfies the equality* $\mathbf{h}(\mathbf{y}, 0) = \mathbf{h_0}(\mathbf{y})$.

Theorem 2. *(Strygin and Sobolev, 1988) Let the assumptions (i)-(iii) of the previous theorem hold. Then there exists an* $\epsilon_1 : \ 0 < \epsilon_1 < \epsilon_0$*, such that for every* $\epsilon : \ 0 < \epsilon < \epsilon_1$ *the invariant manifold* $\mathbf{x} = \mathbf{h}(\mathbf{y}, \epsilon)$ *is stable.*

In general situations the determination of the exact form and location of the slow invariant manifold is impossible. Therefore, methods of approximation are necessary. One of them finds the slow invariant manifold as a power series with respect to the small parameter ϵ:

$$\mathbf{h}(\mathbf{y}, \epsilon) = \mathbf{h_0}(\mathbf{y}) + \Sigma \epsilon^i \mathbf{h_i}(\mathbf{y})$$

Theorem 3. *(Strygin and Sobolev, 1988) Let the assumptions of the previous theorem hold. Then the invariant manifold* $\mathbf{x} = \mathbf{h}(\mathbf{y}, \epsilon)$ *can be represented as*

$$\mathbf{h}(\mathbf{y}, \epsilon) = \mathbf{h_0}(\mathbf{y}) + \Sigma_{i=1}^{k} \epsilon^i \mathbf{h_i}(\mathbf{y}) + \mathbf{h}^*(\mathbf{y}, \epsilon) \tag{7}$$

for some k*, where* $\mathbf{h}^*(\mathbf{y}, \epsilon)$ *is a smooth function with a bounded norm, such that* $|\mathbf{h}^*(\mathbf{y}, \epsilon)| = O(\epsilon^{k+1})$ *for all* $\mathbf{y} \in \Re^n$*.*

It is not hard to see from the Eq. 7 that the slow surface $\mathbf{x} = \mathbf{h_0}(\mathbf{y})$ is $O(\epsilon)$ approximation of the slow invariant manifold, except the turning points. Thus, the general scheme of application of this technique for singularly perturbed system can be subdivided to analysis of the fast and slow motions. The analysis can be considerably simplified by this decomposition and reducing the dimension of the system to the dimension of the slow variable $\mathbf{y}$ and to the dimension of the fast variable $\mathbf{x}$. It means that in $O(\epsilon)$ approximation of the slow invariant manifold, the analysis of the original system can be reduced to the analysis of system's dynamics on the slow surface. On the slow surface the changes of the slow and fast variables are comparable (i.e. the fast and the slow processes are balanced). Beyond the slow surface the slow variables are fixed (quasi-stationary). Hence, each system's trajectory can be approximated by fast motions (which are beyond the slow manifold) that are described by the fast sub-system

$$\epsilon \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}, \mathbf{y^0}, \epsilon);\ \mathbf{y} = \mathbf{y^0} = const,$$

and slow motions (which are on the slow manifold) that are given by the slow subsystem (4) with $\mathbf{h}(\mathbf{y}, \epsilon) = \mathbf{h_0}(\mathbf{y})$.

The method of invariant manifolds has been used for study of singularly perturbed systems of ordinary differential equations by many authors (see, for example [4], [12], [32]). The asymptotics of the slow invariant manifold are given explicitly, for example, in [17], [20].

2.2 Iterative Method of Fraser

The method of functional iteration for finding slow manifold was supposed in [5], further developed and applied to enzyme kinetics in [3], [18], [26], [27], [25]. In [13] there was done the asymptotic analysis of the method and comparison with the ILDM-method.

The method was inspired by the phase space geometry of an enzyme kinetics model involving a fast and a slow species, where the slow manifold is a curve in the phase plane, and extended naturally to multidimensional systems with higher-dimensional slow manifolds.

The idea of the method is as follows. Consider the planar dynamical system

$$\dot{x} = f(x, y)$$

$$\dot{y} = g(x, y),$$

where x can be considered as a slow variable and y as a fast one. Taking $g = 0$ as a zeroth iteration the procedure matches the slope of the slow manifold. From the trajectory equation

$$y'(x)f(x,y) = g(x,y)$$

there is obtained functional equation

$$y = \varphi(x, y')$$

and from here iterative scheme

$$y_{n+1} = \varphi(x, y'_n).$$

The procedure is explicit if the vector field is linear in the fast variable and implicit otherwise. In [13] there was considered general singularly perturbed system of ordinary differential equations, which is linear for the fast variable

$$\dot{\mathbf{y}} = \mathbf{f_1}(\mathbf{y}, \epsilon)\mathbf{z} + \mathbf{f_2}(\mathbf{y}, \epsilon),$$

$$\epsilon\dot{\mathbf{z}} = \mathbf{g_1}(\mathbf{y}, \epsilon)\mathbf{z} + \mathbf{g_2}(\mathbf{y}, \epsilon)$$

It was shown that for such system the iterative method generates, term by term, the asymptotic expansion of the slow invariant manifold. Starting from the slow surface, the i-th iteration of the algorithm yields the correct expansion coefficient at $O(\epsilon^i)$. Thus, after l applications, the expansion is accurate up to and including the terms of $O(\epsilon^l)$.

2.3 Inflector method

In this sub-section we describe very briefly the definition of inflector and some its properties. This object was introduced by Japanese mathematician Masami Okuda in the early eighties [21], [22], [23]. This investigation is interest for us because the Inflector can be considered as some prediction of Intrinsic Low-Dimensional Manifolds. The study deals with two-dimensional dynamical systems, but can be naturally generalized for higher dimensional problems.

Definitions of Inflector, A-inflector and R-inflector

Here we remind the definitions of inflector, A-inflector and R-inflector. Consider two-dimensional dynamical system of the type

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \tag{8}$$

where

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \ \mathbf{F}(\mathbf{x}) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}.$$

Let $A = A(\mathbf{x})$ be a Jacobian matrix of $\mathbf{F} = \mathbf{F}(\mathbf{x})$:

$$A = \frac{\partial \mathbf{F}}{\partial \mathbf{x}} = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}.$$

Let $\lambda_i = \lambda_i(x)$ $(i = 1, 2)$ be the eigenvalues of A, and assume $|\lambda_1| \leq |\lambda_2|$. For the dynamical system (8) the author defined three sets: C (inflector), C_a (A-inflector), C_r (R-inflector) by

$$C = \{\mathbf{x} \mid (A - \lambda_i \mathbf{I})\mathbf{F} = 0,\ i = 1\ or\ 2\}, \tag{9}$$

$$C_a = \{\mathbf{x} \mid \lambda_2 < 0,\ \mid \lambda_1/\lambda_2 \mid < 1,\ (A - \lambda_1 \mathbf{I})\mathbf{F} = 0\}, \tag{10}$$

$$C_r = \{\mathbf{x} \mid \lambda_2 > 0,\ \mid \lambda_1/\lambda_2 \mid < 1,\ (A - \lambda_1 I)\mathbf{F} = 0\}, \tag{11}$$

where I is the unit matrix. The definition (10) means that the A-inflector is found as all the points in the phase plane where the vector field is parallel to the slow eigenvector. Notice here that A-inflector (R-inflector) was called attractor (repellor) in the previous author's study (1976).

Eliminating λ_i from Eq. (9), one can obtain another expression for C

$$C = \{\mathbf{x} \mid f(g_x f + g_y g) - g(f_x f + f_y g) = 0\} \tag{12}$$

It is obvious that

$$C_a \subset C, \quad C_r \subset C,$$

but $C_a \cup C_r$ is not always C.

The relation between the A-inflector (R-inflector) and the attracting (repelling) naive trajectory

In [21], [23] and [22] the author investigated properties of the inflector. Let us remind here very briefly one of them concerning asymptotics of the inflector.

Consider according [23] the system

$$\dot{x} = u(x, y, \epsilon) \tag{13}$$

$$\epsilon \dot{y} = v(x, y, \epsilon), \tag{14}$$

where $\epsilon > 0$ and the functions u and v have the power-series expansions in powers of ϵ. It is assumed that the trajectory equation

$$\epsilon u(x, y, \epsilon)\mathrm{d}y = v(x, y, \epsilon)\mathrm{d}x \tag{15}$$

has solution $y = Y^0(x, \epsilon)$ in the neighborhood of $\epsilon = 0$ in some region Ω in the phase plane. The author denoted this trajectory as

$$T^0(\epsilon) = \{\mathbf{x} \mid y = Y^0(x, \epsilon)\} \tag{16}$$

and called $T^0(\epsilon)$ *a naive trajectory* (NT) in the neighborhood of $\epsilon = 0$.

¿From the definition of the function $Y^0(x, \epsilon)$ follows that it has the power-series expansion

$$Y^0(x, \epsilon) = \Sigma_{i=0}^{\infty} \psi_j(x) \epsilon^j \tag{17}$$

which converges in the neighborhood of $\epsilon = 0$ uniformly in x in $\cap_{|\epsilon| < \epsilon_0} \{x \mid \mathbf{x} \in T^0(\epsilon)\}$ with some $\epsilon_0 > 0$. For calculation of the functions $\psi_j(x)$ in [23] the standard procedure was used:substitution of Eq. (17) into Eq. (15) and equating coefficients of like powers of ϵ. Then we have

$$v_0(x, \psi_0(x)) = 0, \tag{18}$$

$$\psi_1(x) = -(\bar{u}_0 \bar{v}_{0x} + \bar{v}_1 \bar{v_{0y}})/\bar{v}_{0y}^2, \tag{19}$$

where $\bar{u}_0 = u_0(x, \psi_0(x))$, $\bar{v}_{0y} = v_{0y}(x, \psi_0(x))$, etc.

An attracting part and a repelling part of the naive trajectory was defined as follows:

$$T_a^0(\epsilon) = \{\mathbf{x} \mid y = Y^0(x, \epsilon),\ D(\mathbf{x}) < 0\} \tag{20}$$

$$T_r^0(\epsilon) = \{\mathbf{x} \mid y = Y^0(x, \epsilon),\ D(\mathbf{x}) > 0\}, \tag{21}$$

where $D(\mathbf{x})$ is so-called *a repulsion rate*. This object was introduced by the author in [22] for stability analysis in transient states. In that article he gave the mathematical expression for the repulsion rate and found two properties of the inflector with relation to it. The repulsion rate $D(\mathbf{x})$ has the following properties for the dynamical system (8) [22]: (i) Let $T(\mathbf{x})$ be a section of the trajectory passing through a point $\mathbf{x}$. If $D(\mathbf{x}) < 0$, then any state point in the neighborhood of $\mathbf{x}$ will approach $T(\mathbf{x})$ at that point of time, and if $D(\mathbf{x}) > 0$ it will go away from $T(\mathbf{x})$. (ii) If $\mathbf{x}$ is a regular point belonging to the A-inflector C_a (R-inflector C_r), then $D(\mathbf{x}) < 0$ ($D(\mathbf{x}) > 0$).

Let $\Omega^0(\chi)$ be the region $\{\mathbf{x} \mid \ \mid v_y^0 \mid \geq \chi\}$, where χ is an arbitrary positive constant independent of ϵ and $v^0 \equiv v(x, y, 0) = v_0(x, y)$. Then the repulsive rate can be written as

$$D(\mathbf{x}) = \bar{v}_{0y}\epsilon^{-1} + O(1) \tag{22}$$

for a regular point $\mathbf{x} \in T^0(\epsilon)$ as $\epsilon \to 0$ in the region $\Omega^0(\chi)$.

The following important property was proved in [23] (Property 1.1): The A-inflector (R-inflector) is a first order approximation to $T_a^0(\epsilon)$ ($T_r^0(\epsilon)$) for sufficiently small ϵ in $\Omega^0(\chi)$ except singular points.

2.4 Intrinsic Low-Dimensional Manifold Method (ILDM)

Let us describe here very briefly the essential steps of the ILDM method. Consider differential system

$$\frac{\mathrm{d}\mathbf{Z}}{\mathrm{d}t} = \mathbf{F}(\mathbf{Z}) \tag{23}$$

Assume that this system can be represented locally as a multi-scale system for a corresponding choice of a local basis. The last depends on the choice of an arbitrary point $\mathbf{Z}$ in the n-dimensional Euclidean space $\mathbf{\Re^n}$. It means that in this local basis a separation of variables in accordance with their rates of changes is possible (i.e. the considered system can be rewritten in this local basis for some neighborhood of the point $\mathbf{Z}$ as singularly perturbed system). According to the assumption, the system can be subdivided locally into fast relaxing and slow or non-relaxing subsystems. Suppose that the fast sub-system has the same dimension n_f ($n_f < n$) at any point $\mathbf{Z} \subseteq \mathbf{\Re^n}$.

For typical situations a set of all steady states of the fast subsystem represents an n_s-dimensional slow manifold ($n_s = n - n_f$) and our aim is to determine its location. The authors of ILDM suggested that the dynamics of the overall system from arbitrary initial condition should decay very quickly onto this n_s-dimensional manifold. The ILDM allows to identify approximately (as a set of separate points) the slow invariant manifolds (so-called intrinsic low-dimensional manifolds – ILDM-manifolds). These manifolds can be found in the following manner [16]. Suppose a local basis of the original phase space is formed by the invariant subspaces of the Jacobi matrix M_j of the vector field $\mathbf{F}$ at an arbitrary point $\mathbf{Z_0}$. If the set of eigenvalues λ_i can be sub-divided into two groups

$$max\{Re[\lambda_i], i = 1, ..., n_f\} << \tau < min\{Re[\lambda_i], i = n_f + 1, ..., n\} \tag{24}$$

(where $\tau < 0$) one can introduce invariant sub-spaces T_f and T_s. The sub-space T_f is spanned by the eigenvectors corresponding to eigenvalues with large negative (fast) real parts. In turn, the sub-space T_s is spanned by the eigenvectors corresponding to eigenvalues with small negative or positive (slow) real parts. Therefore, the new basis $Q(\mathbf{Z})$, which is constructed from the eigenvectors of the Jacobi matrix and transition matrix from the standard basis to this local basis $Q^{-1}(\mathbf{Z})$ can be written like two block matrices

$$Q = \begin{pmatrix} Q_f & Q_s \end{pmatrix}; \; Q^{-1} = \begin{pmatrix} \tilde{Q}_f \\ \tilde{Q}_s \end{pmatrix} \tag{25}$$

where matrices Q_f and Q_s correspond to the fast and slow subspaces (Q_f is $n \times n_f$ matrix of the fast eigenvectors, Q_s is $n \times n_s$ matrix of the slow eigenvectors, $\tilde{Q}_f$ is $n_f \times n$ matrix and $\tilde{Q}_s$ is $n_s \times n$ matrix). The parameter τ is a time scale splitting parameter. This splitting parameter determines the dimensions of the slow (n_s) and fast (n_f) sub-spaces.

Using a standard lineaization of the RHS of (23) at the point $\mathbf{Z_0}$ we get

$$\frac{\mathrm{d}\mathbf{Z}}{\mathrm{d}t} = \mathbf{F}(\mathbf{Z}) \approx \mathbf{F}(\mathbf{Z_0}) + \frac{\partial \mathbf{F}}{\partial \mathbf{Z}} \big|_{\mathbf{Z}=\mathbf{Z_0}} (\mathbf{Z} - \mathbf{Z_0}) \tag{26}$$

The Jacobian at the point $\mathbf{Z_0}$ can be represented as a product of three matrices: the transition matrix Q, a two-blocks representation J_{M_J} of the Jacobian in the eigenvectors basis and inverse of the transition matrix Q^{-1}

$$\frac{\partial \mathbf{F}}{\partial \mathbf{Z}} \big|_{\mathbf{Z}=\mathbf{Z_0}} = M_J(\mathbf{Z_0}) = Q J_{M_J} Q^{-1} = M_J = \begin{pmatrix} Q_f & Q_s \end{pmatrix} \begin{pmatrix} J_{M_f} & 0 \\ 0 & J_{M_s} \end{pmatrix} \begin{pmatrix} \tilde{Q}_f \\ \tilde{Q}_s \end{pmatrix} \tag{27}$$

The square $(n \times n)$ matrix J_{M_J} is decomposed into a two-block matrix. The blocks J_{M_f}, J_{M_s} correspond to fast and slow invariant sub-spaces. The matrix J_{M_f} is $n_f \times n_f$ and the matrix J_{M_s} is $n_s \times n_s$.

Introduce the intermediate variable $\boldsymbol{\phi} = \mathbf{Z} - \mathbf{Z_0}$ and rewrite the expression (26) in the form

$$\frac{\mathrm{d}\boldsymbol{\phi}}{\mathrm{d}t} = \mathbf{F}(\mathbf{Z_0}) + M_J(\mathbf{Z_0})\boldsymbol{\phi} = \mathbf{F}(\mathbf{Z_0}) + Q(\mathbf{Z_0}) J_{M_J}(\mathbf{Z_0}) Q^{-1}(\mathbf{Z_0})\boldsymbol{\phi} \tag{28}$$

Multiply both sides of (28) by the inverse matrix $Q^{-1}(\mathbf{Z_0})$

$$Q^{-1}(\mathbf{Z_0}) \frac{\mathrm{d}\boldsymbol{\phi}}{\mathrm{d}t} = Q^{-1}(\mathbf{Z_0})\mathbf{F}(\mathbf{Z_0}) + J_{M_J}(\mathbf{Z_0}) Q^{-1}(\mathbf{Z_0})\boldsymbol{\phi}$$

and introduce new variable (this is a point of the transition from the original basis to the new one, which allows the decomposition into fast and slow motions)

$$\boldsymbol{\Psi} = Q^{-1}(\mathbf{Z_0})\boldsymbol{\phi}$$

With respect to the new variable the equation can be written in the form

$$\frac{\mathrm{d}\boldsymbol{\Psi}}{\mathrm{d}t} = \boldsymbol{\phi} \frac{\mathrm{d}Q^{-1}}{\mathrm{d}t}(\mathbf{Z_0}) + Q^{-1}(\mathbf{Z_0})\mathbf{F}(\mathbf{Z_0}) + J_{M_J}(\mathbf{Z_0})\boldsymbol{\Psi}$$

One can show that the first term in the RHS of the last equation is negligible under certain special conditions [16]. The equation is reduced to the simple equation in the form

$$\frac{d\mathbf{\Psi}}{dt} \approx Q^{-1}(\mathbf{Z_0})\mathbf{F}(\mathbf{Z_0}) + J_{M_J}(\mathbf{Z_0})\mathbf{\Psi}$$

According to the original algorithm of Maas and Pope (1992) the Intrinsic Low-Dimensional Manifold (ILDM) is determined by the following system of equations

$$\tilde{Q}_f(\mathbf{Z})\mathbf{F}(\mathbf{Z}) = \mathbf{0} \tag{29}$$

This definition means that the fast component of the original vector field $\mathbf{F}(\mathbf{Z})$, that corresponds to the ("big") fast block $\mathbf{J_{M_f}}$ of the Jacoby matrix representation, is vanished.

ILDM-algorithm for singularly perturbed system

Suppose that we initially have differential system in singularly perturbed form and we are interested in asymptotic expansion of ILDM equation in order to compare it with the invariant manifold. In this case the transition matrix Q, its inverse Q^{-1} and the vector field have the following representation

$$Q = \begin{pmatrix} Q_{ff} & Q_{sf} \\ Q_{fs} & Q_{ss} \end{pmatrix},\ \tilde{Q} = \begin{pmatrix} \tilde{Q}_{ff} & \tilde{Q}_{fs} \\ \tilde{Q}_{sf} & \tilde{Q}_{ss} \end{pmatrix},\ \mathbf{F} = \begin{pmatrix} \epsilon^{-1}\mathbf{f} \\ \mathbf{g} \end{pmatrix}, \tag{30}$$

where $\tilde{Q}_{ff}$ is $n_f \times n_f$ matrix, $\tilde{Q}_{fs}$ is $n_f \times n_s$ matrix, $\tilde{Q}_{sf}$ is $n_s \times n_f$ matrix and $\tilde{Q}_{ss}$ is $n_s \times n_s$ matrix.

The ILDM-equation gets the form

$$\tilde{Q}_{ff}\mathbf{f} + \epsilon\tilde{Q}_{fs}\mathbf{g} = \mathbf{0}$$

In the zero approximation $\epsilon \to 0$ the equation is

$$\tilde{Q}_{ff}\mathbf{f} = \mathbf{0}.$$

If $\det \tilde{Q}_{ff} = 0$, then the last equation gets additional solutions ("ghost" manifolds) except the slow manifold $\mathbf{f} = \mathbf{0}$. This is one of reasons for "ghost" manifolds appearance. The others will be considered in the future works of the authors.

Connection of the ILDM and the C_a-inflector

Consider a two-dimensional system (8):

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}),\ \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix},\ \mathbf{F}(\mathbf{x}) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}.$$

Assume that $|\lambda_1| < |\lambda_2|$ and $\lambda_2 < 0$ hold in some domain D of the phase plane. According to the definition of the C_a-inflector (10) its equation can be written as

$$fg_x + g(g_y - \lambda_1) = 0 \tag{31}$$

According to the ILDM-method, λ_1 is a slow eigenvalue and λ_2 is a fast eigenvalue in D. The equation for the ILDM in this case looks as

$$\frac{1}{\det(Q)}(fg_x + g(g_y - \lambda_1)) = 0, \tag{32}$$

where Q is the new basis matrix, which is built from the eigenvectors of the Jacobi matrix of the system.

¿From the direct calculation we get

$$\det(Q) = g_x(\lambda_2 - \lambda_1) \tag{33}$$

Equations (31) and (32) show that in D ILDM coincides with C_a-inflector up to the expression g_x.

Remarks about possible non-coincidence of ILDM and slow invariant manifold

The interesting fact is that the concept of the Intrinsic Low-Dimensional Manifolds is well known and widely used in the reduction methods [21]-[23], [25]. The following hints of "ghost"-manifolds existence are given in these studies:

(i) Consider the definition of the A-inflector (Eq.10) Sect. 2.3. We see that the points where the eigenvalues of the Jacobian are equal are out of that definition. In our study (see, for example, [6]) we show that the original algorithm cannot treat these points (curves, surfaces). It can be easily shown by formula (33). Namely, the expression $1/\det(Q)$ is always involved into an ILDM-equation and $\lambda_1 - \lambda_2 = 0$ is one of possibilities to the determinant to vanish. Therefore numerical application of the ILDM-algorithm yields "ghost"-objects in the points $\lambda_1 - \lambda_2 = 0$.

(ii) Note, that the asymptotic comparison between the A-inflector (R-inflector) and $T_a^0(\epsilon)$ ($T_r^0(\epsilon)$) in [23] was performed in the region $\Omega^0(\chi) = \{\mathbf{x} \mid \mid v_y^0 \mid \geq \chi\}$. One can show that the original algorithm does not work in the zones where $v_y^0 = 0$ and their neighborhoods.

(iii) In some cases it can be shown for a two-dimensional singularly perturbed system that in turning zones eigenvalues of Jacobi matrix are complex. It means that their real parts are identical. By the definition, $v_y^0 = 0$ in turning points. ¿From the above we can conclude that turning zones are problematic for ILDM-method. The analysis of the algorithm shows that existence of complex eigenvalues is one of the main problems of the method.

($iiii$) It should be noticed that the ILDM-method was used in [25] for analysis of fast-slow planar dynamical systems. In this study the Intrinsic Low-Dimensional Manifold was called a *slow tangent manifold*. It was defined as the curve on which the slow eigenvector is parallel to the velocity field (this definition coincides with the ILDM definition, see [16], [13]). It was shown that the slow tangent manifold lies close to the slow invariant manifold. In our study we demonstrate that in some situations the ILDM does not coincide with the slow invariant manifolds, and different disruptions of the original algorithm are reasoned by different types of non-linearity of a vector field of the considered ODE system.

2.5 TILDM

The remarks (i)-($iiii$) show that the ILDM-algorithm has several disadvantages and some improved version is needed.

TILDM-method [2] is a modified version of the original ILDM approach of Maas and Pope. The additional letter "T" comes from the word "Transpose". The basic difference between the algorithms is that the TILDM uses the symmetric matrix $T = J \cdot J^t$ instead of the Jacobi matrix J. It is known that any symmetric matrix has real eigenvalues and orthogonal eigenvectors. This solves one of the main problems of the ILDM-approach (complex eigenvalues with a large negative real part of the Jacobi matrix) and also problems connected to non-orthogonality of the eigenvectors. Note, that idea to exploit properties of a symmetrized (in some special sense) matrix was suggested in [9, 10].

Consider the differential system

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, \mathbf{z}, \epsilon)$$

$$\epsilon\dot{\mathbf{z}} = \mathbf{g}(\mathbf{y}, \mathbf{z}, \epsilon),$$

where $\mathbf{y} \in K_1 \subset \Re^m$, $\mathbf{z} \in K_2 \subset \Re^n$, $0 < \epsilon \leq \epsilon_0$, functions $\mathbf{f}$, $\mathbf{g}$ derivatives are proportional to the unity when $\epsilon \to 0$.

Fix an arbitrary point $(\mathbf{y}, \mathbf{z})$. The Jacoby matrix is

$$J = \begin{pmatrix} D_y\mathbf{f} & D_y\mathbf{g} \\ \epsilon^{-1}D_z\mathbf{f} & \epsilon^{-1}D_z\mathbf{g} \end{pmatrix}$$

The corresponding symmetric matrix T is

$$T = J \cdot J^t = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

T_{11} is $m \times m$ matrix with the elements proportional to $O(\epsilon^0)$, T_{12} is $m \times n$ matrix with the elements proportional to $O(\epsilon^{-1})$, T_{21} is $n \times m$ matrix with the elements proportional to $O(\epsilon^{-1})$, T_{22} is $n \times n$ matrix with the elements proportional to $O(\epsilon^{-2})$. For arbitrary point $(\mathbf{y}, \mathbf{z})$ the matrix T has positive eigenvalues and orthogonal eigenvectors. The eigenvalues of T fall into two distinct groups: n fast eigenvalues (proportional to $O(\epsilon^{-2})$ and m slow ones (proportional to $O(\epsilon^0)$). ¿From linear algebra we know that in some orthonormal basis Q the matrix T has a diagonal form with its eigenvalues in the diagonal. The eigenvalues can appear along the diagonal in any desirable order.

$$T = QT_dQ^t,$$

where

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}, \quad Q^t = \begin{pmatrix} Q_{11}^t & Q_{21}^t \\ Q_{12}^t & Q_{22}^t \end{pmatrix}$$

Here $\begin{pmatrix} Q_{11} \\ Q_{21} \end{pmatrix}$ is the orthonormal basis of the fast sub-space, $\begin{pmatrix} Q_{12} \\ Q_{22} \end{pmatrix}$ is the orthonormal basis of the slow sub-space. T_d is a following diagonal matrix

$$T_d = \begin{pmatrix} \Lambda_f & 0 \\ 0 & \Lambda_s \end{pmatrix}$$

Here Λ_f is a fast block ($n \times n$ block of the fast eigenvalues), Here Λ_s is a slow block ($m \times m$ block of the slow eigenvalues). By the definition, the equation for TILDM manifold is multiplication of the fast part of the matrix Q^t by the vector field $\mathbf{F} = (\mathbf{f},\ \epsilon^{-1}\mathbf{g})^t$:

$$Q_{11}^t\mathbf{f} + \epsilon^{-1}Q_{21}^t\mathbf{g} = \mathbf{0}. \tag{34}$$

Asymptotic analysis with respect to the small parameter ϵ shows that zeroth approximation of the TILDM coincides with the zeroth approximation of the slow invariant manifold (slow surface $\mathbf{g} = \mathbf{0}$). It must be noticed that the turning points are not problematic for the TILDM-algorithm and this fact is one of the most important advantages of the method.

3 "Ghost" ILDM-Manifolds Examples

In this section the examples of "ghost"-manifolds appearance will be demonstrated. The examples 1-3 are theoretical ones; the example 4 is practical one. All presented systems are written in the singularly perturbed form. Nevertheless the "ghost" objects appear when we apply the ILDM method.

Example 1. This example will demonstrate appearance of a large number of "ghost" manifolds because of non-correct fast direction defined by the ILDM method. It should be noticed that the slow manifold of this system does not have turning points and also it is stable. Consequently according to the conjecture [24] the ILDM manifold should coincide with the invariant manifold, but this statement is not true for this example. In other words, the present example can be considered as a counterexample for the conjecture suggested in [24].

Consider the following system of differential equations with small parameter ϵ :

$$\epsilon \dot{x} = -x - \sin(x) - \sin(y)$$

$$\dot{y} = -y$$

The slow manifold (the manifold of critical points) is given by the equation

$$-x - \sin(x) - \sin(y) = 0 \tag{35}$$

The slow manifold is shown as the central object on Fig.1(below). Application of the ILDM method for this example provides us with two equations for domains with different hierarchy of the eigenvalues $\lambda_{1,2}$:

$$-x - \sin(x) - \sin(y) + \frac{\epsilon y \cos(y)}{-1 + \epsilon - \cos(x)} = 0, \ |\lambda_1| > |\lambda_2|$$

$$y = 0, \ |\lambda_2| > |\lambda_1|$$

Fig.1(upper) demonstrates two ILDM manifolds (solid lines)and a system's trajectory (thick dashed line). Fig.1(below) demonstrates the slow curve (solid line)and a system's trajectory (thick dashed line). On the figure we can see that the trajectory with arbitrary initial conditions approaches the ILDM curve passing through "ghost" manifolds (fast motion, almost parallel to the x axe). it must be noticed that one of the ILDM-manifolds (the central part of Fig.1(upper)) is very close to the slow manifold.

Example 2. This example will demonstrate the essential perturbations produced by the ILDM algorithm on unique slow manifold. Consider the following system of differential equations with small parameter ϵ

$$\epsilon \dot{x} = -x - \sin(x) - \sin(y) + 10$$

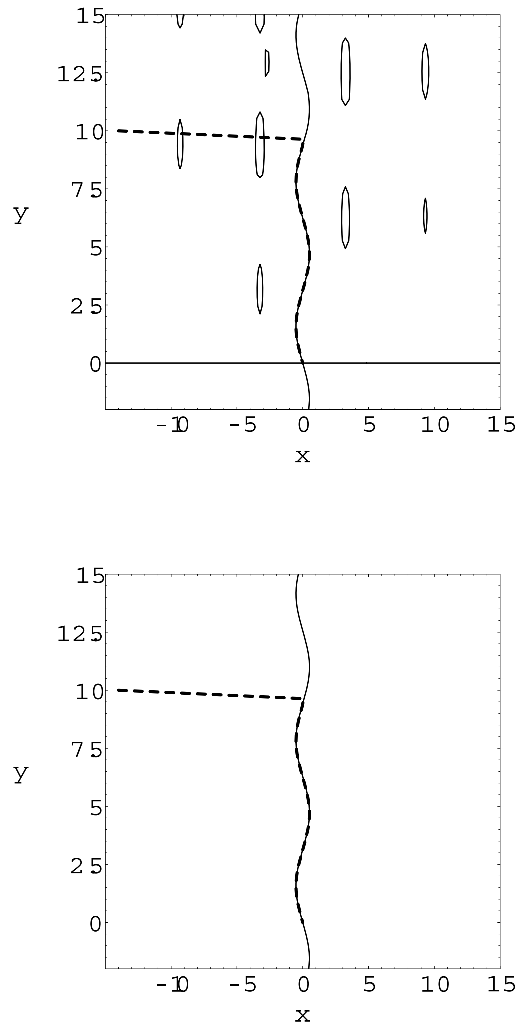

Fig. 1: Example 1. Upper graph – ILDM and trajectory, lower graph – slow curve and trajectory

$$\dot{y} = -2y - \sin(y)$$

The method of invariant manifolds provides us with the slow manifold as follows (dashed line on Fig.2).

$$-x - \sin(x) - \sin(y) + 10 = 0 \tag{36}$$

As in the previous example we get two ILDM-equations (it depends on which of the eigenvalues is "fast" in the considered domain) applying the algorithm. Fig.2

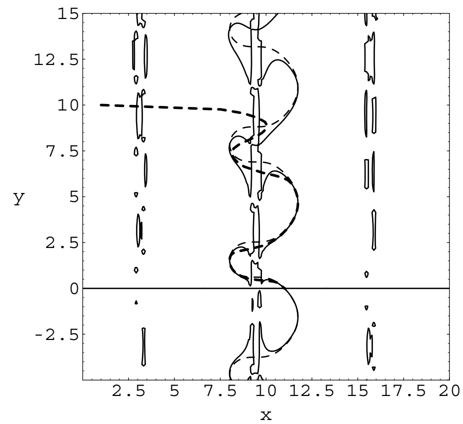

Fig. 2: Application of ILDM algorithm for Theoretical Example 2

demonstrates two ILDM manifolds (solid lines), the slow curve (central dashed line) and a system's trajectory (thick dashed line).

Example 3. Consider the following system of differential equations with small parameter ϵ

$$\epsilon\dot{x} = -x/2 - \sin(x) - \sin(xy)$$
$$\dot{y} = -y$$

We will show that this example is pathological for the ILDM algorithm in some sense. The method of invariant manifolds provides us with the slow manifold

$$-x/2 - \sin(x) - \sin(xy) = 0$$

The analysis of the eigenvalues shows that either $|\lambda_1| >> |\lambda_2|$ or $|\lambda_1| = O(|\lambda_2|)$ and the last relation holds in almost all points of the phase plane. On Fig.3 the curve is depicted, on which $\lambda_1 = \lambda_2$. Then, in some small vicinity of this curve the eigenvalues are comparable. We see that the curve fills up the whole plane and has a very interesting form. Let us remark that the system is written in the singularly perturbed form with explicit small parameter.

Fig. 4 shows that the ILDM method approximates the slow manifold very well.

Example 4. Consider classical model of thermal explosion in a gas. The dimensionless model reads as

$$\epsilon\frac{\mathrm{d}\theta}{\mathrm{d}t} = \eta\exp\left(\frac{\theta}{1+\beta\theta}\right) - \alpha\theta = f(\theta,\eta) \tag{37}$$

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = -\eta\exp\left(\frac{\theta}{1+\beta\theta}\right) = g(\theta,\eta) \tag{38}$$

$$\theta(0) = 0,\ \eta(0) = 1 \tag{39}$$

Here θ is a dimensionless temperature, η is a dimensionless concentration, α is a dimensionless heat loss parameter, ϵ is a reciprocal of the dimensionless adiabatic

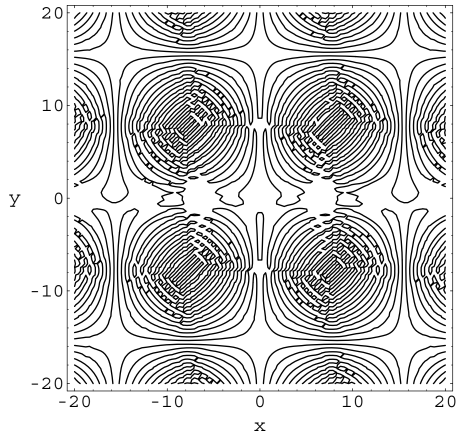

Fig. 3: The curve on which the eigenvalues of the Jacobian in Example 3 are equal one to another

temperature rise, β is a dimensionless ambient temperature. For realistic combustible gas mixtures typical values of ϵ lie in the interval $(0.01,\ 0.1)$ and the following relation is satisfied: $\beta^2 < \epsilon < \beta$. Therefore this system can be considered as a singularly perturbed system with small parameter ϵ, where θ is a fast variable, η is a slow variable.

Note that the dynamics of the system is known very well, see, for example, [2], [6], [7], [8]. In particular, in [7], [8] the dynamics of the system was analyzed in framework of the method of invariant manifolds (MIM); in [6] the detailed analysis of the ILDM algorithm application to the system (37)-(39) was performed; in [2] the modification of ILDM (TILDM) was applied. Here we remind only basic results of the ILDM-method application.

According to the ILDM-method, the Jacobian of the system is

$$J = \begin{pmatrix} \epsilon^{-1} f_\theta & \epsilon^{-1} f_\eta \\ g_\theta & g_\eta \end{pmatrix}$$

The eigenvalues are

$$\lambda_{1,2} = 1/2(\epsilon^{-1} f_\theta + g_\eta \pm \sqrt{D(\theta, \eta)}),$$

where

$$D(\theta, \eta) = (\epsilon^{-1} f_\theta + g_\eta)^2 - 4\epsilon^{-1}(f_\theta g_\eta - f_\eta g_\theta)$$

There are three possibilities depending on the sign of the discriminant $D(\theta, \eta)$:
a) $D(\theta, \eta) > 0$. The Jacobi matrix provides us with two real different eigenvalues. Depending on order of magnitude of the eigenvalues two ILDM equations are obtained for different domains of the phase space.
b) $D(\theta, \eta) = 0$. The Jacobian provides us with two identical eigenvalues. In this case one of the main assumptions of the ILDM approach does not hold, namely, the eigenvalues can not be sub-divided into two different groups (24). It means that there is no splitting on fast and slow eigenvalues and the ILDM-method can not be

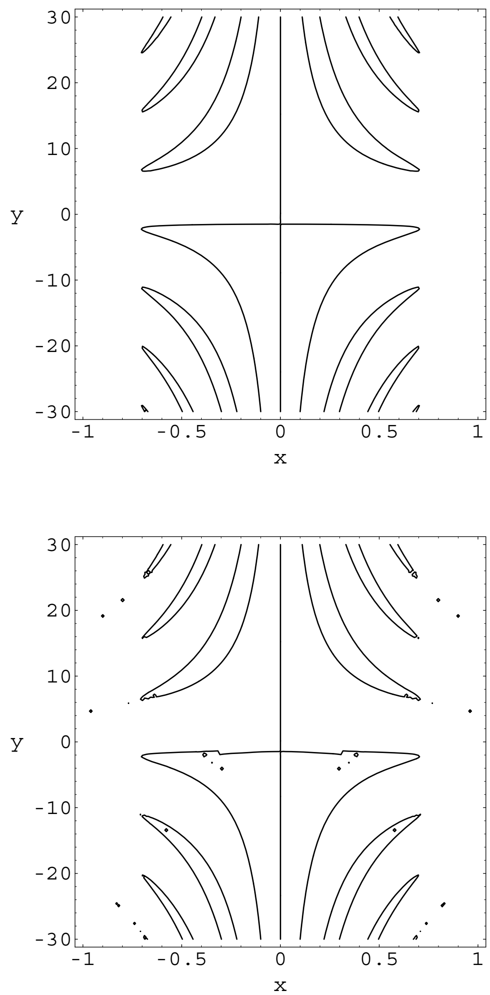

Fig. 4: Example 3. Upper graph – the slow manifold, lower graph - ILDM

applied.
c) $D(\theta, \eta) < 0$. The Jacobian provides us with two complex eigenvalues. It means that their real parts are identical. We can repeat the previous argument to conclude that the original technique does not work in this case. The region in the phase plane corresponding to this case is the domain between the curves Y_+ and Y_- (see Fig. 5).

Fig. 5 shows all the curves $(M1, M2, Y_\pm)$ obtained by the ILDM-algorithm (thick solid lines) and the slow manifold (dashed line).

The functions $Y_{\pm}(\theta)$ are the solutions of the equation $D(\theta, Y_{\pm}(\theta)) = 0$ and they have their own sense. These functions serve as the separating lines on the phase plane between domains of real and complex eigenvalues.

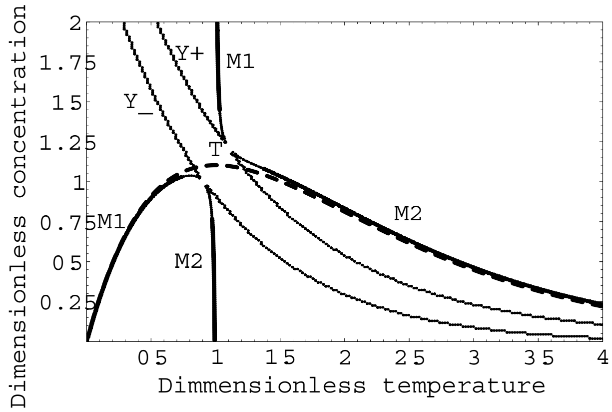

Fig. 5: ILDM and the slow curve for the Semenov's model

Let us now illustrate briefly the basic steps of the system's analysis by the method of invariant manifolds.

In accordance with Sect. 2.1, the slow curve of the system (37)-(39) is given by

$$f(\theta, \eta) \equiv \eta \exp\left(\frac{\theta}{1+\beta\theta}\right) - \alpha\theta = 0 \tag{40}$$

Eq.(40) has a unique isolated solution $\theta(\eta)$ for all η, except at the turning points, at which $f = 0,\ f_\theta = 0$. The slow curve has two turning points. On Fig. 5 we can see one of them T. The second point has a very big θ-coordinate for reasonable values of the system's parameters. On the slow curve the relative rates of the processes are comparable, and the system's dynamics is governed by the reduced system on the slow curve:

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = -\eta \exp\left(\frac{\theta(\eta)}{1+\beta\theta(\eta)}\right)$$

where $\theta(\eta)$ is given by (40).

The first approximation of the slow invariant manifold reads

$$\eta = \alpha\theta \exp\left(-\frac{\theta}{1+\beta\theta}\right) + \epsilon \frac{\theta(1+\beta\theta)}{\theta - (1+\beta\theta)^2}$$

Fig. 6 represents both the slow curve (dashed line) and the first approximation (solid line) of the exact manifold.

If we compare Fig. 5 and Fig. 6 we see that the first approximation of the invariant manifold and the intrinsic low-dimensional manifolds are identical, except the lines $Y_{\pm}(\theta)$. These lines separate in the phase plane domains of real and complex

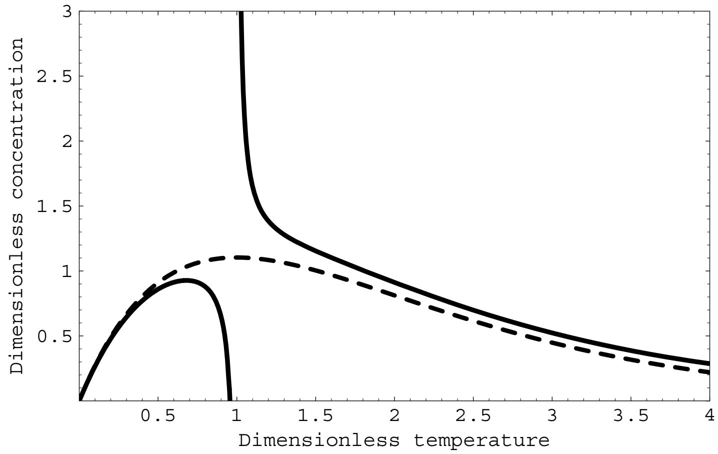

Fig. 6: The zeroth and the first approximations of the slow invariant manifold for the Semenov's model

eigenvalues. Between Y_- and Y_+ the "transition zone" is located (the "gray zone" in [6]), which is large in this example, because of the existence of complex eigenvalues far from the "transition point". In this zone the ILDM method does not work and it confirms [7] that there is no division into fast and slow processes in "transition zone".

3.1 Conclusions

We can see that there exist "ghost" manifolds as a result of ILDM-method application. The first example demonstrated that even for two-dimensional singularly perturbed system the slow manifold of which is stable and does not have turning points the ILDM does not coincide with the invariant manifold. The second example demonstrated appearance of "ghost"- manifolds in neighborhoods of the turning points. It is known (see, for example[1], [2], [6]) that in these zones the original ILDM method doesn't work. Let us remind that the C_a-inflector (see Sect. 2.3) is not defined in zones containing turning points. The third example is pathological for the ILDM-algorithm in some sense. In spite of the original system of equations is done in the standard singularly perturbed form the processes involved are comparable in almost all phase plane. Nevertheless, the algorithm locates the slow manifold well. The application of the algorithm gives "ghost" manifolds. The forth example demonstrated one of the main ILDM-method's problems: existence of complex eigenvalues of the Jacobi matrix.

4 Criteria for "ghost"-manifolds identification

In Sect. 3 we demonstrated that application of the original Maas and Pope algorithm produces so-called "ghost"-manifolds. In this section we suggest two criteria that allow to distinguish the "ghost"-manifolds from the correct ones.

4.1 Criterion 1: "Normal vector"

The idea of the criterion "Normal vector" can be described as follows. Fix an arbitrary point that belongs to the invariant manifold of the system (1)-(2). In this point the vector field $\mathbf{F} = (\epsilon^{-1} f,\ g)^T$ and vector normal to the invariant manifold $\mathbf{n}$ are ϵ-close to orthogonal pair, i.e. the value of $(\mathbf{F},\ \mathbf{n})$ is comparable with ϵ. If a point is far from the invariant manifold then the vector field and the normal have some angle $\mid \alpha - \pi/2 \mid \sim O(1)$ and $(\mathbf{F},\ \mathbf{n})$ cannot be small.

Apply the suggested criterion for discrimination of "ghost"-manifolds in theoretical example 1. The slow manifold for this system is exactly known (Eq. (35), Fig.1). Fig.7 demonstrates result of application of the criterion. The horizontal axe is x-coordinate of the checked point, the vertical axe shows values of $\log(\mathbf{F}, \mathbf{n})$ for different x. For $x \in (-1,\ 1)$ we have $\log(\mathbf{F}, \mathbf{n}) = O(1)$. It means that $(\mathbf{F}, \mathbf{n}) = O(\epsilon)$. According to the suggested criterion the point belongs to the correct ILDM-branch. For x from any other zone we have $(\mathbf{F}, \mathbf{n}) = O(1)$. That is, the point belongs to "ghost"-manifold.

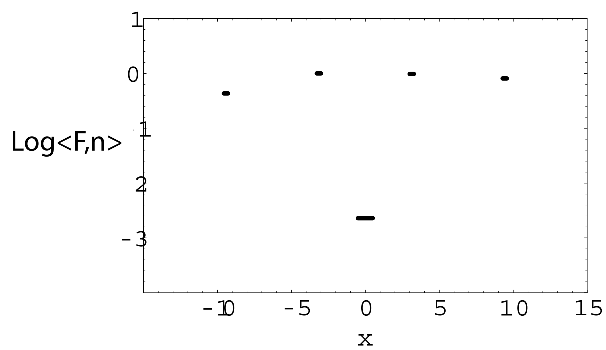

Fig. 7: Application of criterion "Normal vector" for Theoretical Example 1

The obtained results are confirmed by Fig.1. For $x \in (-1,\ 1)$ the ILDM coincides with the slow manifold (we do not see "ghost" manifolds in this zone); for x out of this interval there is only "ghost" ILDM.

Apply the suggested criterion for discrimination of "ghost"-manifolds in theoretical example 2. The slow manifold for this system is exactly known (Eq. (36), Fig.2).

Application of the criterion is shown on Fig.8. Difference of values $\log(\mathbf{F}, \mathbf{n})$ is easily seen. According to Fig.8 points from $x \in (11,\ 12)$ belong to the real ILDM, because $\log(\mathbf{F}, \mathbf{n}) = O(1)$ and so $(\mathbf{F}, \mathbf{n}) = O(\epsilon)$. Points from other intervals belong to "artificial" ILDM-branches. This result is confirmed by Fig.2. We can see that for $x \in (11,\ 12)$ the ILDM coincides with the slow manifold. For x out of this interval there is only "ghost" ILDM.

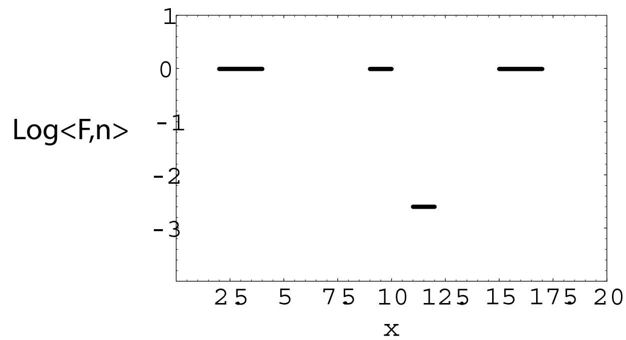

Fig. 8: Application of criterion "Normal vector" for Theoretical Example 2

4.2 Criterion 2: "Slow Matrix"

Consider the system of ordinary differential equations (23). Suppose that this system can be represented as a singularly perturbed system (1)-(2) in some coordinate system. In this sub-section we find an invariant that does not depend on a choice of coordinate system and can distinguish between ILDM-manifolds that correspond to true invariant manifold and ILDM-manifolds that are far from any invariant manifold. Our analysis is asymptotic one.

We know from Sect. 2.3 that that ILDM-manifolds are solutions of Eq. (29). Denote the intrinsic low-dimensional manifold by S. Let us analyze values of $\tilde{Q}_s(\mathbf{Z})\mathbf{F}(\mathbf{Z}) = \mathbf{0}$ for different points $(\mathbf{x},\ \mathbf{y})$. The matrix $\tilde{Q}_s$ can be represented as $\tilde{Q}_s = (\tilde{Q}_{sf}\ \ \tilde{Q}_{ss})$ see Eq. (30). That is, we have

$$\tilde{Q}_s\mathbf{F} = \frac{1}{\epsilon}\tilde{Q}_{sf}\mathbf{f} + \tilde{Q}_{ss}\mathbf{g} \tag{41}$$

Let us remind that zeroth approximation of the slow invariant manifold is defined by $\mathbf{f} = \mathbf{0}$. If the ILDM-manifold S belongs to ϵ-neighborhood of the slow invariant manifold, then the term $\epsilon^{-1}\mathbf{f}$ has the order $O(1)$ on S. If the ILDM-manifold is far from the slow invariant manifold, then the term $\epsilon^{-1}\mathbf{f}$ is comparable with the value $O(\epsilon^{-1})$ on S.

¿From (41) we can conclude that

(i) If ILDM-manifold S belongs to ϵ-neighborhood of slow invariant manifold, then $\tilde{Q}_s(\mathbf{Z})\mathbf{F}(\mathbf{Z})$ has the order $O(|\ \mathbf{g}\ |)$ on S.

(ii) If ILDM-manifold is far from any slow invariant manifold then $\tilde{Q}_s(\mathbf{Z})\mathbf{F}(\mathbf{Z}) >>|\ \mathbf{g}\ |$ on S.

Then, the described criterion suggests to use values of $\tilde{Q}_s(\mathbf{Z})\mathbf{F}(\mathbf{Z})$ for discrimination of "ghost"-manifolds.

Apply the suggested criterion for discrimination of "ghost" manifolds in the theoretical Example 1. The eigenvalues of the Jacobi matrix are $\lambda_1 = (-1 - cos(x))/\epsilon$, $\lambda_2 = -1$. Consider any artificial branch, for example, $x \in (2,\ 4)$. The eigenvalues analysis shows that in this region $|\lambda_2| > |\lambda_1|$. Then, $\tilde{Q}_f\mathbf{F} = -y$ and

Table 1: Application of criterion "Slow matrix" for Theoretical Example 1

x	y	$\tilde{Q}_s\mathbf{F}$
2.588	0	-311.375
3.439	0	-314.597
2.89	0	-313.92

$\tilde{Q}_s\mathbf{F} = -x - \sin(x) - \sin(y) + \frac{\epsilon y \cos(y)}{-1+\epsilon-\cos(x)}$. The result of application of the criterion is given in Table 1.

Table 1 shows that the values of $\tilde{Q}_s\mathbf{F}$ are much bigger than g. Therefore according to the suggested criterion the points from this region belong to "ghost" manifold.

Check the points of the ILDM that belong to ϵ-neighborhood of the slow invariant manifold, $x \in (-1,\ 1)$. Then, $\tilde{Q}_f\mathbf{F} = -x - \sin(x) - \sin(y) + \frac{\epsilon y \cos(y)}{-1+\epsilon-\cos(x)}$ and $\tilde{Q}_s\mathbf{F} = -y \equiv g$. Therefore for these points $|\tilde{Q}_s\mathbf{F}| = O(|g|)$. According to the criterion this means that all the points from the considered interval belong to correct ILDM manifold.

Results of the suggested criterion are conformed by the method of invariant manifolds, criterion 1 and Fig.1.

5 Conclusions

The present paper represents a natural continuation of the authors work on a comparative analysis of the two powerful asymptotic methods ILDM and MIM.

As any other algorithm, ILDM has its own restrictions, which were partly demonstrated in the present paper on a number of examples. It was shown, that ILDM can not treat the regions of the phase space, where the leading eigenvalues of the Jacobi matrix are equal. In particular, it means, that the ILDM approach may face problems in the vicinity of the turning surfaces, where the leading eigenvalues are normally complex (their real values are equal and there is no splitting in rates of change of the processes involved). As a result of the ILDM application in these regions of the phase space, so called ghost manifolds can appear. It is illustrated by a number of examples.

The problem of the determination and elimination of the ghost manifolds is of high importance. A numerical criterion allowing distinguishing the ghost manifolds from the true ones is suggested in the present paper. The criterion is based on the unique properties of the true invariant manifolds. The efficiency of the suggested criterion is demonstrated on the number of the examples introduced earlier.

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