THE FREDHOLM ALTERNATIVE FOR PARABOLIC EVOLUTION EQUATIONS WITH INHOMOGENEOUS BOUNDARY CONDITIONS

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Abstract. We study the Fredholm properties of parabolic evolution equations on $\mathbb{R}$ with inhomogeneous boundary values. These problems are transformed into evolution equations with inhomogeneities taking values in certain extrapolation spaces. Assuming that the underlying homogeneous problem is asymptotically hyperbolic, we show the Fredholm alternative for these equations.

1. Introduction

In recent years the Fredholm properties of evolution equations

$$u'(t) = A(t)u(t) + f(t), \qquad t \in \mathbb{R}, \tag{1.1}$$

on a Banach space X have attracted considerable interest. In this work we establish a Fredholm alternative for a large class of parabolic inhomogeneous boundary value problems, see (1.4), which can be transformed into a problem similar to (1.1) with inhomogeneities f taking values in spaces $X^t_{\alpha-1}$ larger than X. Before discussing the contents of our paper, we first want to recall related results concerning (1.1) with $f : \mathbb{R} \to X$.

A main line of research concentrates on parabolic problems, where the operators $A(t)$ generate an evolution family $U(t,s)$, $t \geq s$, having regularity properties similar to those of analytic semigroups. Moreover, it is assumed that (1.1) possesses *maximal regularity* on a space F of functions $f : \mathbb{R} \to X$ (cf. [7]). Roughly speaking, this notion means that the operator $G^0u = -u' + A(\cdot)u$ is closed in F on the 'minimal' domain $D(G^0) = D(d/dt) \cap D(A(\cdot)) = \{u \in F : u(t) \in D(A(t)),\ u', A(\cdot)u \in F\}$. This property typically requires function spaces such as $F = L^p(\mathbb{R}, X)$ or $C^\alpha(\mathbb{R}, X)$ with $p \in (1,\infty)$ or $\alpha \in (0,1)$ (the choice $F = L^p$ leads to additional restrictions on X and $A(t)$). Finally, one supposes that the operators $A(t)$ converge to operators $A_{\pm\infty}$ as $t \to \pm\infty$ in a suitable sense and that $i\mathbb{R}$ belongs to the resolvent sets of $A_{\pm\infty}$, i.e., the problem is 'asymptotically hyperbolic'. It is then known that $U(\cdot,\cdot)$ has an exponential dichotomy on intervals $[T, +\infty)$ and $(-\infty, -T]$ for possibly large $T \geq 0$, see [8], [28], [30].

In this setting, the (Semi–)Fredholmity of G^0 was characterized in terms of properties of the stable and unstable subspaces of $U(t,s)$ at $t = T$, see [1], [14], [15], [16], [23], [26], [27], and the references therein (compare also Theorem 3.6 below). This characterization

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implies that G^0 is Fredholm if the unstable subspaces of $A_{\pm\infty}$ have finite dimensions $d_\pm$ (e.g., if $D(A_{\pm\infty})$ is compactly embedded in X), and then G^0 has the index $d_- - d_+$.

The above setting occurs if one linearizes a nonlinear parabolic problem on a bounded domain along a heteroclinic orbit connecting two hyperbolic equilibria. In this case the Fredholm property of G^0 is crucial to study the bifurcation behaviour of the heteroclinic orbit by means of the Lyapunov–Schmidt reduction, see e.g. [14], [26], [27]. We add that the property of maximal regularity makes it posssible to show the persistence of Fredholm properties under large classes of perturbations, see [16].

If one discards the strong assumption of maximal regularity, then it seems to be most appropriate to define G via the 'mild equation'

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\tau)f(\tau)\,d\tau, \qquad t \geq s, \tag{1.2}$$

for a given exponentially bounded 'evolution family' $U(t,s)$, $t \geq s$, with time interval $\mathbb{R}$ (i.e., (2.3) below holds and $(t,s) \mapsto U(t,s)$ is strongly continuous for $t \geq s$). We say that a function $u \in F$ belongs to the domain $D(G)$ and $Gu = f$ if there is a function $f \in F$ such that (1.2) holds for all $t \geq s$ in $\mathbb{R}$. If the Cauchy problem

$$u'(t) = A(t)u(t), \quad t \geq s, \qquad u(s) = x, \tag{1.3}$$

is well–posed, then G is the closure of G^0 as defined above, where $F = C_0(\mathbb{R}, X)$ or $F = L^p(\mathbb{R}, X)$ with $1 \leq p < \infty$, cf. [11], [29]. In the recent paper [20] it is shown that G is Fredholm on F if and only if $U(\cdot,\cdot)$ has exponential dichotomies on intervals $(-\infty, a]$ and $[b, +\infty)$ and a certain 'node operator' connecting the dichotomies is Fredholm in X. We refer to [21] for somewhat stronger results under stronger assumptions and also to [9]. In fact, the 'if' implication of the results from [9], [20], [21] coincides with the corresponding assertions in [14], [15], [16], see [14, §5.3]. We further mention that the invertibility of G on F is equivalent to the exponential dichotomy of $U(\cdot,\cdot)$ on $\mathbb{R}$, see [11].

In the present paper we study the (Semi–)Fredholm properties of the parabolic inhomogeneous boundary value problem

$$\begin{aligned} u'(t) &= A_m(t)u(t) + g(t), \qquad t \in \mathbb{R}, \\ B(t)u(t) &= h(t), \qquad t \in \mathbb{R}. \end{aligned} \tag{1.4}$$

Here the linear operators $A_m(t)$ and $B(t)$ are defined on a subspace Z_t of X (e.g., $Z_t = W_p^2(\Omega)$ if $X = L^p(\Omega)$), $A_m(t)$ maps Z_t into the state space X, and $B(t)$ maps Z_t into a 'boundary space' Y such as $W_p^{1-1/p}(\partial\Omega)$. The inhomogeneities g and h are continuous with values in X and Y, respectively. Typically, $A_m(t)$ is an elliptic differential operator and $B(t)$ is a differential operator of lower order. It is assumed that the restrictions $A(t)$ of $A_m(t)$ to the kernel of $B(t)$ satisfy the Acquistapace–Terreni conditions stated in (2.1) and (2.2). These conditions are quite flexible in so far they only require a Hölder condition in t and they allow for non–dense and time varying domains $D(A(t))$. Under these conditions the family $A(\cdot)$ generates an evolution family $U(\cdot,\cdot)$ on X having parabolic regularity due to [3] and [4], as described in the following section.

For a fixed operator $A(t)$ and $\alpha \in (0,1)$, we further define the real interpolation spaces X^t_α of order (α,∞) between $D(A(t))$ and X. In Section 2 we also introduce the extrapolation spaces $X^t_{\alpha-1}$ which are larger than X. In general, both X^t_α and $X^t_{\alpha-1}$ depend on t. The operator $A(t)$ possesses an extension $A_{\alpha-1}(t) : X^t_\alpha \to X^t_{\alpha-1}$. We further suppose that the abstract boundary value problem

$$(\omega - A_m(t))v = 0, \qquad B(t)v = \varphi,$$

has a unique solution $v = D(t)\varphi$ for $\varphi \in Y$ and that $Z_t \hookrightarrow X^t_\alpha$ for some $\alpha \in (0,1)$. (Here ω is a fixed large real number.) As we see in Section 4, one can rewrite (1.4) as the evolution equation

$$u'(t) = A_{\alpha-1}(t)u(t) + f(t), \qquad t \in \mathbb{R}, \tag{1.5}$$

where $f := g + (\omega - A_{\alpha-1}(\cdot))D(\cdot)h$. This reformulation of a boundary value problem seems to go back to work in boundary control theory, see e.g. [12], [25]. We also refer to [2], [7], [10], [13], [18] and [22, §5.1] for related results and techniques. We then show that f belongs to the space $E_{\alpha-1}$ for some $\alpha \in (0,1)$ which is the extrapolation space for the multiplication operator $A(\cdot)$ defined on $E := C_0(\mathbb{R}, X)$. It is crucial for our approach that the operators $U(t,s)$ have locally uniformly bounded extensions $U_{\alpha-1}(t,s) : X^s_{\alpha-1} \to X^t_{\alpha-1}$ which map $X^s_{\alpha-1}$ into X with norm less than $c(t-s)^{\alpha-1}$ for $0 < t-s \le 1$, see Proposition 2.1 and Lemma 5.1.

Thus we can define an operator $G_{\alpha-1}$ as in (1.2): A function $u \in E$ belongs to $D(G_{\alpha-1})$ and $G_{\alpha-1}u = f$ if there is an $f \in E_{\alpha-1}$ such that

$$u(t) = U(t,s)u(s) + \int_s^t U_{\alpha-1}(t,\tau)f(\tau)\,d\tau \qquad \forall\, t \ge s \text{ in } \mathbb{R}. \tag{1.6}$$

A function $u \in C(\mathbb{R}, X)$ satisfying (1.6) is called a 'mild solution' of (1.5). In Proposition 2.6 we show that a function u satisfying (1.6) indeed solves (1.5) pointwise in the space $X^t_{\beta-1}$ for every $\beta \in (0,\alpha)$. In so far the 'mild definition' of $G_{\alpha-1}$ is justified. However, in this work we will concentrate on the asymptotic behaviour of (1.5), and we will not study the local regularity of the solutions to (1.5) in further details. These matters are treated in depth in [7, §V.2] assuming that for some $\alpha \in (0,1)$ the spaces X^t_α and $X^t_{\alpha-1}$ do not depend on t, see also [2].

We further suppose that $U(\cdot,\cdot)$ has exponential dichotomies on half lines $(-\infty,-T]$ and $[T,+\infty)$ for some $T \ge 0$. (This property holds in the asymptotically hyperbolic case where the resolvents $R(\omega, A(t))$ converge in norm as $t \to \pm\infty$ to the resolvents of operators $A_{\pm\infty}$ with $i\mathbb{R} \subset \rho(A_{\pm\infty})$, see [30] and also [8], [28]). We prove in Proposition 2.2 that $U_{\alpha-1}(\cdot,\cdot)$ inherits the exponential dichotomies of $U(\cdot,\cdot)$.

Our arguments are based on the properties of the extrapolated evolution family $U_{\alpha-1}(\cdot,\cdot)$, and they are inspried by the techniques of [15] and [16]. The main difference arises from the fact that we work with an 'integral' definition of $G_{\alpha-1}$ instead of the more explicit definition $G^0 = -d/dt + A(\cdot)$. The approach via G^0 would run into severe difficulties here. First, even if we consider homogeneous boundary conditions $h = 0$ in (1.4) (i.e., (1.5) on $E = C_0(\mathbb{R}, X)$ with $\alpha = 1$), we cannot expect that (1.5) has maximal regularity since we work with sup norm in time. This means that G^0 is not closed with a rather complicated domain $D(G) = D(\overline{G^0})$. Second and more importantly, we want to allow for f taking values in time depending extrapolation spaces $X_{\alpha-1}^t$ so that a direct treatment of the differential equation (1.5) is quite unconvenient, cf. Section 5. Fortunately, the mild description (1.6) of $G_{\alpha-1}$ suffices for the questions studied in this paper. On the other hand, the results from [20] or [21] do not apply since we work in extrapolation spaces and $(t,s) \mapsto U(t,s)$ need not to be strongly continuous at $t = s$.

We characterize the (Semi–)Fredholm properties of $G_{\alpha-1}$ in terms of the stable and unstable subspaces of $U(t,s)$ at T in Theorem 3.6. In the asymptotically hyperbolic case, $G_{\alpha-1}$ is Fredholm with index $d_- - d_+$ if the unstable subspaces of $A_{\pm\infty}$ have finite dimensions $d_\pm$. We further describe the kernel and range of $G_{\alpha-1}$ in Propositions 3.5 and 3.8. We point out that our conditions do not involve the extrapolated spaces $X_{\alpha-1}^t$. These results lead to a Fredholm alternative for the mild solutions $u \in C_0(\mathbb{R}, X)$ of (1.5) in Theorem 3.10. This theorem in turn implies a Fredholm alternative for the mild solutions of (1.4) stated in Theorem 4.4. In Example 4.5 we study a variant of this result, namely a diffusion equation formulated in the space $X = C(\overline{\Omega})$.

In the next section we collect the background material for our investigations. We further show several auxiliary facts concerning the extrapolated evolution family $U_{\alpha-1}(t,s)$, its exponential dichotomies, and the bounded solvability of Cauchy problems on half lines. The third section contains our main results on the operator $G_{\alpha-1}$ which are based on a careful analysis of its behaviour of its restrictions to the intervals $[T, +\infty)$ and $(-\infty, T]$. Here the main difficulty comes from the fact that in general $U(t,s)$ only has dichotomies on disjoint intervals $(-\infty, -T]$ and $[T, +\infty)$, see [15], [20], and [30, §4.2] for a discussion of this phenomenon. In Section 4 we translate the results of Section 3 to the boundary value problem (1.4). The last section contains a proof of the regularity result Proposition 2.6. In a forthcoming paper we will treat perturbation results for the Fredholm index.

2. Notations, assumptions, and preliminaries

We denote by $D(A)$, $N(A)$, $R(A)$, $\sigma(A)$, $\rho(A)$ the domain, kernel, range, spectrum and resolvent set of a linear operator A. Moreover, $R(\lambda, A) := (\lambda I - A)^{-1} = (\lambda - A)^{-1}$ for $\lambda \in \rho(A)$ and $\mathcal{L}(X)$ is the space of bounded linear operators on a Banach space X. By $c(\alpha, \cdots)$ we designate a generic constant depending on quantities $\alpha, \cdots$.

We investigate linear operators $A(t)$, $t \in \mathbb{R}$, on a Banach space X subject to the following hypotheses introduced by P. Acquistapace and B. Terreni in [3] and [4]. There are constants $\omega \in \mathbb{R}$, $\theta \in (\pi/2, \pi)$, $K > 0$ and $\mu, \nu \in (0,1]$ such that $\mu + \nu > 1$ and

$$\lambda \in \rho(A(t) - \omega), \qquad \|R(\lambda, A(t) - \omega)\| \leq \frac{K}{1 + |\lambda|}\,, \tag{2.1}$$

$$\|(A(t) - \omega)R(\lambda, A(t) - \omega)\,[R(\omega, A(t)) - R(\omega, A(s))]\| \leq K \frac{|t-s|^\mu}{|\lambda - \omega|^\nu} \tag{2.2}$$

for all $t \in \mathbb{R}$ and $\lambda \in \mathbb{C} \setminus \{0\}$ with $|\arg(\lambda)| \leq \theta$. Observe that the domains $D(A(t))$ are not required to be dense. These conditions imply that the operators $A(\cdot)$ generate an evolution family $U(t,s)$, $t \geq s$, $t, s \in \mathbb{R}$. More precisely, for $t > s$ the map $(t,s) \mapsto$

$U(t,s) \in \mathcal{L}(X)$ is continuous and continuously differentiable in t, $U(t,s)X \subseteq D(A(t))$, and $\partial_t U(t,s) = A(t)U(t,s)$. We further have

$$U(t,s)U(s,r) = U(t,r) \quad \text{and} \quad U(t,t) = I \quad \text{for} \;\; t \geq s \geq r; \tag{2.3}$$

Moreover, for $s \in \mathbb{R}$ and $x \in \overline{D(A(s))}$, the function $t \mapsto u(t) = U(t,s)x$ is continuous at $t = s$ and u is the unique solution in $C([s,\infty),X) \cap C^1((s,\infty),X)$ of the Cauchy problem

$$u'(t) = A(t)u(t), \quad t > s, \qquad u(s) = x.$$

These facts have been established in [3] and [4], see also [2], [7], [22], [31], [32].

Before stating additional regularity properties of $U(t,s)$, we have to introduce the inter- and extrapolation spaces for $A(t)$. We refer to [7], [17], and [22] for proofs and further information. Let A be a sectorial operator on X (i.e., (2.1) holds with $A(t)$ replaced by A) and $\alpha \in (0,1)$. We make use of the real interpolation space

$$X_\alpha^A := \{x \in X : \|x\|_\alpha^A := \sup\nolimits_{r>0} \|r^\alpha (A-\omega)R(r, A-\omega)x\| < \infty\},$$

which is a Banach space endowed with the norm $\|\cdot\|_\alpha^A$. For convenience we further write $X_0^A := X$, $\|x\|_0^A := \|x\|$, $X_1^A := D(A)$ and $\|x\|_1^A := \|(\omega - A)x\|$. We also need the closed subspace $\hat{X}^A := \overline{D(A)}$ of X. Moreover, we define the extrapolation space X_{-1}^A as the completion of $\hat{X}^A$ with respect to the norm $\|x\|_{-1}^A := \|R(\omega, A)x\|$. Then A has a unique continuous extension $A_{-1} : \hat{X}^A \to X_{-1}^A$. The operator A_{-1} satisfies (2.1) in X_{-1}^A, it is densely defined, it has the same spectrum as A, and it generates the semigroup $e^{tA_{-1}}$ on X_{-1}^A being the extension of e^{tA}. As above, we can then define the space

$$X_{\alpha-1}^A := (X_{-1})_\alpha^{A_{-1}} \quad \text{with the norm} \;\; \|x\|_{\alpha-1}^A := \|x\|_\alpha^{A_{-1}} = \sup\nolimits_{r>0} \|r^\alpha R(r, A_{-1} - \omega)x\|.$$

The restriction $A_{\alpha-1} : X_\alpha^A \to X_{\alpha-1}^A$ of A_{-1} is sectorial in $X_{\alpha-1}^A$ with the same type as A, it has the same spectrum as A, and the semigroup $e^{tA_{\alpha-1}}$ on $X_{\alpha-1}^A$ is the extension of e^{tA}. Observe that $\omega - A_{\alpha-1} : X_\alpha^A \to X_{\alpha-1}^A$ is an isometric isomorphism. We will frequently use the continuous embeddings

$$\begin{aligned} &D(A) \hookrightarrow X_\beta^A \hookrightarrow D((\omega - A)^\alpha) \hookrightarrow X_\alpha^A \hookrightarrow \hat{X}^A \subset X, \\ &X \hookrightarrow X_{\beta-1}^A \hookrightarrow D((\omega - A_{-1})^\alpha) \hookrightarrow X_{\alpha-1}^A \hookrightarrow X_{-1}^A \end{aligned} \tag{2.4}$$

for all $0 < \alpha < \beta < 1$, where the fractional powers are defined as usually. In general, $D(A)$ is not dense in the spaces X_α^A and X and X is not dense in $X_{\alpha-1}^A$, but we have the inclusions

$$X_\beta^A \hookrightarrow \overline{D(A)}^{\|\cdot\|_\alpha^A} \qquad \text{and} \qquad X_{\beta-1}^A \hookrightarrow \overline{\hat{X}^A}^{\|\cdot\|_{\alpha-1}^A} \tag{2.5}$$

for $0 < \alpha < \beta < 1$. More precisely, one has the following fact: For $x \in X_{\beta-1}^A$, the vectors $x_n = nR(n, A_{-1})x$, $n > \omega$, belong to $\hat{X}^A$, $\|x_n\|_{\beta-1}^A \leq c\,\|x\|_{\beta-1}^A$ and $x_n \to x$ in $X_{\alpha-1}^A$. Moreover, $\hat{X}^A$ is dense in $D((\omega - A_{-1})^\alpha)$ and X_{-1}^A.

Given operators $A(t)$, $t \in \mathbb{R}$, satisfying (2.1), we set

$$X_\alpha^t := X_\alpha^{A(t)}, \quad X_{\alpha-1}^t := X_{\alpha-1}^{A(t)}, \quad \hat{X}^t := \hat{X}^{A(t)}$$

for $0 \leq \alpha \leq 1$ and $t \in \mathbb{R}$, with the corresponding norms. Then the embeddings in (2.4) hold with constants independent of $t \in \mathbb{R}$. Let $J \subset \mathbb{R}$ be a closed interval. We further define on $E = E(J) := C_0(J, X)$ (the space of continuous functions, vanishing at infinity if J is unbounded) the multiplication operator $A(\cdot)$ by

$$(A(\cdot)f)(t) := A(t)f(t) \text{ for } t \in J, \quad D(A(\cdot)) := \{f \in E : f(t) \in D(A(t)),\ A(\cdot)f \in E\}.$$

It is clear that the operator $A(\cdot)$ is also sectorial. We can thus introduce the spaces

$$E_\alpha := E_\alpha^{A(\cdot)}, \quad E_{\alpha-1} := E_{\alpha-1}^{A(\cdot)}, \quad \text{and} \quad \hat{E} := \overline{D(A(\cdot))}$$

for $\alpha \in [0, 1]$, where $E_0 := E$ and $E_1 := D(A(\cdot))$. We observe that $E_{-1} \subseteq \prod_{t \in J} X_{-1}^t$ and that the extrapolated operator $A(\cdot)_{-1} : \hat{E} \longrightarrow E_{-1}$ is given by $(A(\cdot)_{-1}f)(t) := A_{-1}(t)f(t)$ for $t \in J$ and $f \in E$. Further, $E_{\alpha-1}$ has the norm

$$\|f\|_{\alpha-1} := \sup_{r>0} \sup_{s \in J} \|r^\alpha R(r, A_{-1}(s) - \omega)f(s)\|.$$

Let (2.1) and (2.2) hold. Then there exists a constant $C = C(t_0) > 0$ such that

$$\|(\omega - A(t))^\alpha e^{\tau A(t)}\| \leq C\, \tau^{-\alpha}, \tag{2.6}$$

$$\|U(t, s)x\|_\alpha^t \leq C\, (t - s)^{\beta-\alpha} \|x\|_\beta^s, \tag{2.7}$$

$$\|U(t, s)(\omega - A(s))^\theta y\| \leq C\, (\mu - \theta)^{-1}(t - s)^{-\theta} \|y\|, \tag{2.8}$$

$$\|(\omega - A(s))^\gamma (R(\omega, A(s)) - R(\omega, A(t)))\| \leq C\, (t - s)^\mu, \tag{2.9}$$

$$\|(\omega - A(t))^{\gamma-1} - (\omega - A(s))^{\gamma-1}\| \leq C\, (t - s)^\mu \tag{2.10}$$

for all $t, s \in \mathbb{R}$ and $t_0 > 0$ with $0 < t - s \leq t_0$ and all $0 < \tau \leq t_0$, $0 \leq \beta \leq \alpha \leq 1$, $0 \leq \theta < \mu$, $0 \leq \gamma < \nu$, $x \in X_\beta^s$, and $y \in D((\omega - A(s))^\theta)$. Here, (2.6) is well known, (2.7) follows from [4, Thm.2.3] by interpolation, and (2.8) was proved in [32, Thm.2.1] in a slightly different setting, but the proof also works under the present assumptions. Finally, (2.9) and (2.10) are straightforward consequences of (2.1) and (2.2), cf. [30] and [31]. We state an easy consequence of (2.8) which is crucial for our work, see also Lemma 5.1.

Proposition 2.1. *Assume that* (2.1) *and* (2.2) *hold and let* $1-\mu < \alpha < 1$ *and* $0 \leq \beta \leq 1$. *Then the following assertions hold for* $s < t \leq s + t_0$ *and* $t_0 > 0$ *with constants possibly depending on* t_0.
(i) The operators $U(t, s)$ *have continuous extensions* $U_{\alpha-1}(t, s) : X_{\alpha-1}^s \to X$ *satisfying*

$$\|U_{\alpha-1}(t, s)\|_{\mathcal{L}(X_{\alpha-1}^s, X)} \leq c(\alpha)(t - s)^{\alpha-1}, \tag{2.11}$$

and $U_{\alpha-1}(t, s)x = U_{\gamma-1}(t, s)x$ *for* $1 - \mu < \gamma < \alpha < 1$ *and* $x \in X_{\alpha-1}^s$.
(ii) The map $\{(t, s) : t > s\} \ni (t, s) \longmapsto U_{\alpha-1}(t, s)f(s) \in X$ *is continuous for* $f \in E_{\alpha-1}$.
(iii) For $x \in X_{\alpha-1}^s$ *we have*

$$\|U_{\alpha-1}(t, s)x\|_\beta^t \leq c(\alpha)(t - s)^{\alpha-\beta-1} \|x\|_{\alpha-1}^s. \tag{2.12}$$

Proof. Let $s < t \leq s + t_0$. Due to (2.8), we can uniquely extend $U(t, s)$ to operators from $D((\omega - A_{-1})^{\alpha\pm\varepsilon})$ to X, with norms bounded by $c(t - s)^{\alpha-1\pm\varepsilon}$, where $1 - \mu < \alpha \pm \varepsilon < 1$. Assertion (i) now follows by reiteration employing (2.4) and e.g. Theorem 1.2.15 and Proposition 2.2.15 in [22]. The map $\Phi : (t, s) \longmapsto U_{\alpha-1}(t, s)f(s) \in X$ is continuous for

$t > s$ if $f \in E$. For $f \in E_{\alpha-1}$, the continuity of Φ is shown by approximation using (2.11) and (2.5). Finally, (2.7) and (2.11) yield

$$\begin{aligned}\|U_{\alpha-1}(t,s)x\|_\beta^t &= \|U(t,\tfrac{1}{2}(t+s))U_{\alpha-1}(\tfrac{1}{2}(t+s),s)x\|_\beta^t \\ &\le 2^\beta C\,(t-s)^{-\beta}\,\|U_{\alpha-1}(\tfrac{1}{2}(t+s),s)x\| \le c(\alpha)(t-s)^{\alpha-\beta-1}\,\|x\|_{\alpha-1}^s\end{aligned}$$

for $x \in X_{\alpha-1}^s$. □

Exponential dichotomies are another important tool in our study, cf. [11], [22], [29], [30]. We recall that an evolution family $U(\cdot,\cdot)$ is said to have an *exponential dichotomy* in an interval $J \subset \mathbb{R}$ if there exists a family of projections $P(t) \in \mathcal{L}(X)$, $t \in J$, being strongly continuous with respect to t, and numbers $\delta, N > 0$ such that

$$\begin{aligned}&\text{(a)}\quad U(t,s)P(s) = P(t)U(t,s),\\ &\text{(b)}\quad U(t,s): Q(s)(X) \to Q(t)(X)\ \text{ is invertible with inverse }\ \widetilde{U}(s,t),\\ &\text{(c)}\quad \|U(t,s)P(s)\| \le Ne^{-\delta(t-s)},\\ &\text{(d)}\quad \|\widetilde{U}(s,t)Q(t)\| \le Ne^{-\delta(t-s)},\end{aligned} \tag{2.13}$$

for all $s, t \in J$ with $s \le t$, where $Q(t) := I - P(t)$ is the 'unstable projection.' In the parabolic case one easily obtains regularity properties of the exponential dichotomy, see e.g. [30, Proposition 3.18]. For instance, $\|A(t)Q(t)\| \le c(\eta)$ for $t \in J$, $t-\eta > \inf J$ and each $\eta > 0$ since $A(t)Q(t) = A(t)U(t,t-\eta)\tilde{U}(t-\eta,t)Q(t)$. In the next proposition we state some results concerning extrapolation spaces. We use the convention $\pm\infty + r = \pm\infty$ for $r \in \mathbb{R}$, and we set $J' = J \setminus \{\sup J\}$, i.e., $J = J'$ if J is unbounded from above. Moreover, we write $U_0(t,s) := U(t,s)$, $P_0(t) := P(t)$, and $Q_0(t) := Q(t)$, where $X_0^t = X$ by definition.

Proposition 2.2. *Assume that* (2.1) *and* (2.2) *hold and that $U(t,s)$ has an exponential dichotomy on an interval J. Let $\eta > 0$ and $1-\mu < \alpha \le 1$. Then the operators $P(t)$ and $Q(t)$ admit continuous extensions $P_{\alpha-1}(t) : X_{\alpha-1}^t \to X_{\alpha-1}^t$ and $Q_{\alpha-1}(t) : X_{\alpha-1}^t \to X$, respectively, for $t \in J'$; which are uniformly bounded for $t < \sup J - \eta$. Moreover, the following assertions hold for $t, s \in J'$ with $t \ge s$.*

(a) $Q_{\alpha-1}(t)X_{\alpha-1}^t = Q(t)X$;
(b) $U_{\alpha-1}(t,s)P_{\alpha-1}(s) = P_{\alpha-1}(t)U_{\alpha-1}(t,s)$;
(c) $U_{\alpha-1}(t,s) : Q_{\alpha-1}(s)(X_{\alpha-1}^s) \to Q_{\alpha-1}(t)(X_{\alpha-1}^t)$ *is invertible with inverse* $\widetilde{U}_{\alpha-1}(s,t)$;
(d) $\|U_{\alpha-1}(t,s)P_{\alpha-1}(s)x\| \le N(\alpha,\eta)\max\{(t-s)^{\alpha-1},1\}e^{-\delta(t-s)}\|x\|_{\alpha-1}^s$ *for* $x \in X_{\alpha-1}^s$ *and* $s < t < \sup J - \eta$;
(e) $\|\widetilde{U}_{\alpha-1}(s,t)Q_{\alpha-1}(t)x\| \le N(\alpha,\eta)e^{-\delta(t-s)}\|x\|_{\alpha-1}^t$ *for* $x \in X_{\alpha-1}^t$ *and* $s \le t < \sup J - \eta$.
(f) *Let* $J_0 \subset J'$ *be a closed interval and* $f \in E_{\alpha-1}(J_0)$. *Then* $P(\cdot)f \in E_{\alpha-1}(J_0)$ *and* $Q(\cdot)f \in C_0(J_0, X)$.

Proof. Let $t \in J$ such that $t + \eta < \sup J$, $1-\beta < \theta < \mu$, and $x \in D((\omega - A(t))^\theta)$. The estimates (2.8) and (2.13)(d) imply that

$$\|Q(t)(\omega - A(t))^\theta x\| = \|\widetilde{U}(t,t+\eta)Q(t+\eta)U(t+\eta,t)(\omega - A(t))^\theta x\| \le c(\eta)\|x\|.$$

The embeddings (2.4) thus yield

$$\|Q(t)y\| \le c(\eta)\|(\omega - A_{-1}(t))^{-\theta}y\| = c(\eta)\|(\omega - A_{-1}(t))^{1-\theta}y\|_{-1}^t \le c(\eta)\|y\|_{\beta-1}^t \tag{2.14}$$

for all $y \in X$. Observe that (2.14) is true for $\alpha = \beta$, in particular. Taking $\beta < \alpha$ and using the remarks after (2.5) (with reversed roles of α and β), we see that $Q(t)$ has a uniformly bounded extension $Q_{\alpha-1}(t) : X^t_{\alpha-1} \to X$ for $t < \sup J - \eta$. Then the operator $I - Q_{\alpha-1}(t) \in \mathcal{L}(X^t_{\alpha-1})$ is a uniformly bounded extension of $P(t)$.

Assertion (a) is a consequence of the fact that $Q_{\alpha-1}(t)$ has values in X and that it is a projection. Assertion (b) follows from (2.13)(a) by approximation using (2.5) and (2.11). To show (c), let $y \in Q_{\alpha-1}(t)X^t_{\alpha-1} = Q(t)X$. Due to (2.13)(b), there is a unique vector $x \in Q(s)X = Q_{\alpha-1}(s)X^s_{\alpha-1}$ such that $y = U(t,s)x = U_{\alpha-1}(t,s)x$.

Let $t \geq s+1$ and $x \in X^s_{\alpha-1}$. Using the exponential dichotomy of U and the estimate (2.11), we obtain

$$\|U_{\alpha-1}(t,s)P_{\alpha-1}(s)x\| = \|U(t,s+1)P(s+1)U_{\alpha-1}(s+1,s)x\| \leq ce^{-\delta(t-s)}\|x\|^s_{\alpha-1}.$$

If $0 \leq t-s \leq 1$, assertion (d) follows from (b) and (2.11). Assertion (e) is a consequence of (a), (2.13), and (2.14).

Let $f \in E_{\alpha-1}(J_0)$. Then there are $f_n \in C_0(J_0, X)$ converging to f in $E_{\beta-1}(J_0)$ for $\beta \in (1-\mu, \alpha)$. Then $Q(\cdot)f_n$ converges in $C_0(J_0, X)$ to $Q_{\alpha-1}(\cdot)f$, whence (f) follows. □

We further use the operator family

$$\Gamma_{\alpha-1}(t,s) = \begin{cases} U_{\alpha-1}(t,s)P_{\alpha-1}(s), & t \geq s,\ t,s \in J', \\ -\widetilde{U}_{\alpha-1}(t,s)Q_{\alpha-1}(s), & t < s,\ t,s \in J'. \end{cases} \tag{2.15}$$

In some results we shall assume that $A(\cdot)$ is *asymptotically hyperbolic*, i.e., there are two operators $A_{-\infty} : D(A_{-\infty}) \to X$ and $A_{+\infty} : D(A_{+\infty}) \to X$ satisfying (2.1) and

$$\lim_{t\to-\infty} R(\omega, A(t)) = R(\omega, A_{-\infty}), \quad \lim_{t\to+\infty} R(\omega, A(t)) = R(\omega, A_{+\infty}) \qquad (\text{in } \mathcal{L}(X)); \tag{2.16}$$

$$\sigma(A_{+\infty}) \cap i\mathbb{R} = \sigma(A_{-\infty}) \cap i\mathbb{R} = \emptyset. \tag{2.17}$$

Under assumptions (2.1), (2.2), (2.16), (2.17), there exists $T \geq 0$ such that $U(t,s)$ has exponential dichotomies in $(-\infty, -T]$ and in $[T, +\infty)$. For the interval $[T, +\infty)$, this has been shown in Theorem 4.3 of [30]. The proofs given there extend in a straightforward way to the interval $(-\infty, -T]$. The case of dense domains was treated before in [8] and, for a slightly stronger version of (2.16), in [28]. Moreover, we have

$$\dim Q(t)X = \dim\ Q_{+\infty}X, \quad t \geq T, \qquad \dim Q(t)X = \dim Q_{-\infty}X, \quad t \leq -T, \tag{2.18}$$

by [30, Thm.3.2], where $Q_{\pm\infty}$ are the projections for $A_{\pm\infty}$. Due to Proposition 2.2, our extrapolated evolution family $U_{\alpha-1}(t,s)$ has then exponential dichotomies in $(-\infty, -T)$ and in $[T, +\infty)$. From (2.18) and Proposition 2.2 (a), we conclude that

$$\dim Q_{\alpha-1}(t)X^t_{\alpha-1} = \dim Q_{+\infty}X,\ \ t \geq T, \quad \dim Q_{\alpha-1}(t)X^t_{\alpha-1} = \dim Q_{-\infty}X,\ \ t < -T,$$

if (2.1), (2.2), (2.16), and (2.17) hold.

Remark 2.3. In the proof of Theorem 4.3 of [30], the projections $P(t)$ (for $t \geq T$ and $t \leq -T$, respectively) are obtained as the restriction of projections for a parabolic evolution family having an exponential dichotomy on $J = \mathbb{R}$. Hence, assumptions (2.1), (2.2), (2.16), and (2.17) imply that

$$\begin{aligned} &U(\cdot,\cdot) \text{ has exponential dichotomies on } [T,+\infty) \text{ and } (-\infty,-T] \text{ for some } T\ge 0\\ &\text{and the assertions of Proposition 2.2 are true with } \eta=0. \end{aligned} \tag{2.19}$$

Definition 2.4. *We assume that* (2.1) *and* (2.2) *hold, take* $1-\mu<\alpha\le 1$, *and let* $J\subset\mathbb{R}$ *be a closed interval. Let* $f(t)\in X^t_{\alpha-1}, t\in J$, *such that* $f|[a,b]\in E_{\alpha-1}([a,b])$ *for all subintervals* $[a,b]\subseteq J$. *We say that* $u\in C(J,X)$ *is a* mild solution *of*

$$u'(t) = A_{-1}(t)u(t)+f(t), \quad t\in J, \tag{2.20}$$

if the equation

$$u(t) = U(t,s)u(s)+\int_s^t U_{\alpha-1}(t,\sigma)f(\sigma)\,d\sigma \tag{2.21}$$

holds for all $t\ge s$ *in* J. *If in addition* $u\in E(J)$ *and* $f\in E_{\alpha-1}(J)$, *then we write* $u\in D(G_{\alpha-1})$ *and* $G_{\alpha-1}u=f$, *where* $G_0=: G$. *If* u *is a mild solution of* (2.20) *for* $J=[t_0,+\infty)$, *resp.* $J=(-\infty,t_0]$, *with* $u(t_0)=x$, *then we call* u *a* mild solution *of the initial, resp. final, value problems*

$$u'(t) = A_{-1}(t)u(t)+f(t), \quad t\ge t_0, \qquad u(t_0)=x; \qquad \textit{resp.}, \tag{2.22}$$

$$u'(t) = A_{-1}(t)u(t)+f(t), \quad t\le t_0, \qquad u(t_0)=x. \tag{2.23}$$

Remark 2.5. We make the assumptions stated in Definition 2.4. Then there always exists a unique mild solution of (2.22) with $u(t_0)=x\in\hat{X}^{t_0}$. Moreover, a function $u\in C(J,X)$ can be the mild solution of (2.20) for at most one f, so that $G_{\alpha-1}$ is single–valued. Finally, $G_{\alpha-1}: D(G_{\alpha-1})\subset E(J)\longrightarrow E_{\alpha-1}(J)$ is a closed linear operator.

Proof. The first assertion follows easily from Proposition 2.1. For the second assertion, take f and g such that $f(t),g(t)\in X^t_{\alpha-1}$ for $t\in J$, $f|[a,b], g|[a,b]\in E_{\alpha-1}([a,b])$ for all subintervals $[a,b]\subseteq J$, and (2.21) holds for some $u\in C(J,X)$ and both f and g. Setting $h=f-g$, we thus obtain

$$\int_s^t U_{\alpha-1}(t,\sigma)h(\sigma)\,d\sigma = 0 \qquad \forall\, t,s\in J \ \text{ with } \ t\ge s,$$

and hence $U_{\alpha-1}(t,s)h(s)=0$ for all $t>s$ due to Proposition 2.1(ii). Take $\theta\in(1-\nu,\mu)$ such that $\theta>1-\alpha$. Then $(\omega-A_{-1}(\cdot))^{-\theta}h\in\hat{E}([a,b])$ by (2.4), and thus Lemma 5.1 yields $h=0$, i.e., $f=g$. (We can take any $\overline{\alpha}\in(1-\mu,1-\theta)$ when applying Lemma 5.1. We point out that in the proof of this lemma we use no results established after Proposition 2.1.) The last assertion is a straightforward consequence of (2.11). ☐

The next proposition shows that a mild solution of (2.20) is in fact a differentiable solution of (2.20) in a slightly weaker topology, see Section 5 for the proof. However, it is more convenient for us to work with the integral equation (2.21).

Proposition 2.6. *Assume that* (2.1) *and* (2.2) *hold and that* $f\in E_{\alpha-1}(J)$ *for* $1-\mu<\alpha\le 1$ *and some closed interval* $J\subset\mathbb{R}$. *Let* $u\in C(J,X)$ *be a mild solution of* (2.20) *and* *let* $0 \leq \beta < \min\{\alpha, \nu\}$. *Then* $u(t) \in X^t_\beta$, *the map* $s \mapsto u(s)$ *is differentiable at* $s = t$ *in the norm of* $X^t_{\beta-1}$, *and* (2.20) *holds pointwise in* $X^t_{\beta-1}$, *for each* $t \in J \setminus \inf J$.

Employing exponential dichotomies on halflines, we can derive existence results for forward and backward Cauchy problems with inhomogeneities in extrapolation spaces.

Proposition 2.7. *Assume that* (2.1) *and* (2.2) *hold,* $1-\mu < \alpha \leq 1$, *and that* $U(t,s)$ *has an exponential dichotomy on an interval* $[T, +\infty)$. *Let* $t_0 \geq T$, $f \in E_{\alpha-1}([T,+\infty))$, *and* $x \in \overline{D(A(t_0))}$. *Then the mild solution* $u \in C([t_0,\infty), X)$ *of* (2.22) *is bounded on* $[t_0, \infty)$ *if and only if*

$$Q(t_0)x = -\int_{t_0}^{+\infty} \widetilde{U}_{\alpha-1}(t_0,s)Q_{\alpha-1}(s)f(s)ds, \tag{2.24}$$

in which case u *is given by*

$$u(t) = U(t,t_0)P(t_0)x + \int_{t_0}^{t} U_{\alpha-1}(t,s)P_{\alpha-1}(s)f(s)ds - \int_t^\infty \widetilde{U}_{\alpha-1}(t,s)Q_{\alpha-1}(s)f(s)ds. \tag{2.25}$$

Proof. Let $t_0 \geq T$. The mild solution u of (2.22) satisfies

$$u(t) = U_{\alpha-1}(t,t_0)u(t_0) + \int_{t_0}^t U_{\alpha-1}(t,s)f(s)ds, \quad t \geq t_0.$$

Using Proposition 2.2 and (2.15), we can write this equality as

$$\begin{aligned} u(t) &= U(t,t_0)u(t_0) + \int_{t_0}^t U_{\alpha-1}(t,s)P_{\alpha-1}(s)f(s)ds - \int_t^{+\infty} \widetilde{U}_{\alpha-1}(t,s)Q_{\alpha-1}(s)f(s)ds \\ &\quad + \int_{t_0}^{+\infty} \widetilde{U}_{\alpha-1}(t,s)Q_{\alpha-1}(s)f(s)ds \\ &= U(t,t_0)\Big[u(t_0) + \int_{t_0}^{+\infty} \widetilde{U}_{\alpha-1}(t_0,s)Q_{\alpha-1}(s)f(s)ds\Big] + \int_t^{+\infty} \Gamma_{\alpha-1}(t,s)f(s)ds \end{aligned} \tag{2.26}$$

for $t \geq t_0$. Proposition 2.2 and the boundedness of f on $[t_0, +\infty)$ show that u is bounded if and only if the term in brackets $[\cdots]$ belongs to $P(t_0)X$ which is equivalent to (2.24). In this case, (2.25) follows directly from (2.26). ☐

In the next proposition we may also take $t_0 = -T$ in the situation of Remark 2.3.

Proposition 2.8. *Assume that* (2.1) *and* (2.2) *hold,* $1-\mu < \alpha \leq 1$, *and that* $U(t,s)$ *has an exponential dichotomy on an interval* $(-\infty, -T]$. *Let* $t_0 < -T$, $f \in E_{\alpha-1}((-\infty, t_0])$ *and* $x \in X$. *Then there is a bounded mild solution* $u \in C((-\infty,t_0], X)$ *of* (2.23) *on* $(-\infty, t_0]$ *if and only if*

$$P(t_0)x = \int_{-\infty}^{t_0} U_{\alpha-1}(t_0,s)P_{\alpha-1}(s)f(s)ds, \tag{2.27}$$

in which case u *is given by*

$$u(t) = \widetilde{U}(t,t_0)Q(t_0)x - \int_t^{t_0} \widetilde{U}_{\alpha-1}(t,s)Q_{\alpha-1}(s)f(s)ds + \int_{-\infty}^t U_{\alpha-1}(t,s)P_{\alpha-1}(s)f(s)ds. \tag{2.28}$$

Proof. It is straightforward to check that (2.28) gives a bounded mild solution of (2.23) satisfying (2.27). Let $t_0 < -T$, $s \le t \le t_0$, and let u be a bounded mild solution of (2.23). As in Proposition 2.7, we can write

$$u(t) = U(t,s)\Big[P(s)u(s) - \int_{-\infty}^{s} U_{\alpha-1}(s,\tau)P_{\alpha-1}(\tau)f(\tau)d\tau\Big] + \int_{-\infty}^{t} U_{\alpha-1}(t,\tau)P_{\alpha-1}(\tau)f(\tau)d\tau$$
$$+ U(t,s)Q(s)u(s) + \int_{s}^{t} U_{\alpha-1}(t,\tau)Q_{\alpha-1}(\tau)f(\tau)d\tau.$$

Since $U(t,s)Q(s)u(s) + \int_s^t U_{\alpha-1}(t,\tau)Q_{\alpha-1}(\tau)f(\tau)d\tau = Q(t)u(t)$, we have

$$P(t)u(t) = U(t,s)P(s)\Big[P(s)u(s) - \int_{-\infty}^{s} U_{\alpha-1}(s,\tau)P_{\alpha-1}(\tau)f(\tau)d\tau\Big]$$
$$+ \int_{-\infty}^{t} U_{\alpha-1}(t,\tau)P_{\alpha-1}(\tau)f(\tau)d\tau. \qquad (2.29)$$

Due to Proposition 2.2, the boundedness of u and f implies that the term in $[\cdots]$ is bounded for $s \le t_0$. Therefore, letting $s \to -\infty$ in (2.29), we deduce from (2.13) that

$$P(t)u(t) = \int_{-\infty}^{t} U_{\alpha-1}(t,\tau)P_{\alpha-1}(\tau)f(\tau)d\tau, \qquad (2.30)$$

and in particular the condition (2.27) for $t = t_0$. Moreover, it holds

$$Q(t_0)u(t_0) = U(t_0,t)Q(t)u(t) + \int_{t}^{t_0} U_{\alpha-1}(t_0,\tau)Q_{\alpha-1}(\tau)f(\tau)d\tau,$$
$$Q(t)u(t) = \widetilde{U}(t,t_0)Q(t_0)u(t_0) - \int_{t}^{t_0} \widetilde{U}_{\alpha-1}(t,\tau)Q_{\alpha-1}(\tau)f(\tau)d\tau.$$

The last equation together with (2.30) yield the formula (2.28). □

3. Properties of the operator $G_{\alpha-1}$

In this section we assume that the operators $A(t)$, $t \in \mathbb{R}$, on X satisfy the hypotheses (2.1), (2.2), and (2.19) (where the latter condition follows from (2.16) and (2.17)). Again, $U(t,s)$ is the evolution family on X generated by $A(\cdot)$ and $U_{\alpha-1}(t,s)$ is its extrapolated evolution family on $X^s_{\alpha-1}$. Both families have exponential dichotomies on $(-\infty,-T]$ and $[T,+\infty)$ for some $T \ge 0$ with projections $P(\cdot)$ and $P_{\alpha-1}(\cdot)$, respectively. To study the operator $G_{\alpha-1}$ on $J = \mathbb{R}$, we introduce the stable and unstable subspaces of $U_{\alpha-1}(\cdot,\cdot)$.

Definition 3.1. *Let $t_0 \in \mathbb{R}$. We define the* stable space at t_0 *by*

$$X_s(t_0) := \{x \in X^{t_0}_{\alpha-1} : \lim_{t\to+\infty} \|U_{\alpha-1}(t,t_0)x\| = 0\},$$

and the unstable space at t_0 *by*

$$X_u(t_0) := \{x \in X : \exists \text{ a mild solution } u \in C_0((-\infty,t_0],X) \text{ of (2.23) with } f = 0\}.$$

Observe that the function u in the definition of $X_u(t_0)$ satisfies $u(t) = U(t,s)u(s)$ for $s \le t \le t_0$ and $u(t_0) = x$.

Lemma 3.2. *Assume that the assumptions* (2.1), (2.2), *and* (2.19) *are satisfied and that* $1-\mu<\alpha\le 1$. *Then the following assertions hold.*

(a) $X_s(t_0)=P_{\alpha-1}(t_0)X_{\alpha-1}^{t_0}$ *for* $t_0\ge T$;
(b) $X_u(t_0)=Q(t_0)X$ *for* $t_0\le -T$;
(c) $U_{\alpha-1}(t,t_0)X_s(t_0)\subseteq X_s(t)$ *for* $t\ge t_0$ *in* $\mathbb{R}$;
(d) $U(t,t_0)X_u(t_0)=X_u(t)$ *for* $t\ge t_0$ *in* $\mathbb{R}$;
(e) $X_s(t_0)$ *is closed in* $X_{\alpha-1}^{t_0}$ *for* $t_0\in\mathbb{R}$.

Proof. The inclusions '$\supseteq$' in (a) and (b) are clear. Let $t\ge t_0+1>t_0\ge T$ and $x\in X_s(t_0)$. Due to Proposition 2.2, we obtain

$$\begin{aligned} c &\ge \|U_{\alpha-1}(t,t_0)x\| \ge \|U_{\alpha-1}(t,t_0)Q_{\alpha-1}(t_0)x\| - \|U_{\alpha-1}(t,t_0)P_{\alpha-1}(t_0)x\| \\ &\ge N^{-1}e^{\delta(t-t_0)}\|Q_{\alpha-1}(t_0)x\| - Ne^{-\delta(t-t_0)}\|P_{\alpha-1}(t_0)x\|_{\alpha-1}^{t_0}. \end{aligned}$$

Letting $t\to\infty$, this estimate implies that $Q_{\alpha-1}(t_0)x=0$; i.e., (a) is verified. Let $t\le t_0-1<t_0\le -T$ and $x\in X_u(t_0)$. Let u be as in Definition 3.1. We then have $P(t_0)u(t_0)=U(t_0,t)P(t)u(t)$, and thus

$$\|P(t_0)x\|\le Ne^{-\delta(t_0-t)}\|u(t)\|\le ce^{\delta t}.$$

Letting $t\to-\infty$, we deduce $P(t_0)x=0$ so that (b) holds. The assertions (c) and (d) are easy consequences of Definition 3.1. To show (e), let $t_0\in\mathbb{R}$. If $t_0\ge T$, the closedness of $X_s(t_0)$ in $X_{\alpha-1}^{t_0}$ follows from (a). If $t_0<T$, take $x_n\in X_s(t_0)$ such that $x_n\to x$ in $X_{\alpha-1}^{t_0}$. Then assertions (a) and (c) and estimate (2.11) imply that

$$\|U_{\alpha-1}(t,t_0)x\| = \lim_{n\to\infty}\|U(t,T)P(T)U_{\alpha-1}(T,t_0)x_n\|\le cNe^{-\delta(t-T)}.$$

for $t\ge T$. Thus $x\in X_s(t_0)$. ☐

Let $1-\mu<\alpha\le 1$. The restrictions $G_{\alpha-1}^+$ and $G_{\alpha-1}^-$ of $G_{\alpha-1}$ to the halflines $[T,+\infty)$ and $(-\infty,T]$ are defined as in Definition 2.4: A function $u\in C_0([T,+\infty),X)$ (respectively $u\in C_0((-\infty,T],X)$) belongs to $D(G_{\alpha-1}^+)$ (respectively $D(G_{\alpha-1}^-)$) if there is a function $f\in E_{\alpha-1}([T,+\infty))$ (respectively $f\in E_{\alpha-1}((-\infty,T])$) such that

$$u(t)=U(t,s)u(s)+\int_s^t U_{\alpha-1}(t,\sigma)f(\sigma)d\sigma$$

holds for all $t\ge s\ge T$ (respectively, for all $s\le t\le T$).

As in [15] and [16], we introduce on $E_{\alpha-1}([T,+\infty))$ and on $E_{\alpha-1}((-\infty,T])$ the right inverses $R_{\alpha-1}^+$ and $R_{\alpha-1}^-$ for $G_{\alpha-1}^+$ and $G_{\alpha-1}^-$, respectively, by setting

$$(R_{\alpha-1}^+h)(t) = -\int_t^\infty \widetilde{U}_{\alpha-1}(t,s)Q_{\alpha-1}(s)h(s)ds + \int_T^t U_{\alpha-1}(t,s)P_{\alpha-1}(s)h(s)ds,\quad t\ge T,$$

$$(R_{\alpha-1}^-h)(t) = \begin{cases} \displaystyle\int_{-\infty}^t U_{\alpha-1}(t,s)P_{\alpha-1}(s)h(s)ds - \int_t^{-T}\widetilde{U}_{\alpha-1}(t,s)Q_{\alpha-1}(s)h(s)ds, & t\le -T, \\ \displaystyle\int_{-\infty}^{-T} U_{\alpha-1}(t,s)P_{\alpha-1}(s)h(s)ds + \int_{-T}^t U_{\alpha-1}(t,s)h(s)ds, & -T\le t\le T. \end{cases}$$

Proposition 3.3. *Assume that the assumptions* (2.1), (2.2), *and* (2.19) *are satisfied and that* $1-\mu<\alpha\le 1$. *Then the following assertions hold.*

(a) $R^+_{\alpha-1}: E_{\alpha-1}([T,+\infty)) \to C_0([T,+\infty),X)$ *is bounded and* $G^+_{\alpha-1}R^+_{\alpha-1}h = h$ *for each* $h\in E_{\alpha-1}([T,+\infty))$.

(b) $R^-_{\alpha-1}: E_{\alpha-1}((-\infty,T]) \to C_0((-\infty,T],X)$ *is bounded and* $G^-_{\alpha-1}R^-_{\alpha-1}h = h$ *for each* $h\in E_{\alpha-1}((-\infty,T])$.

Proof. (a) Proposition 2.2 shows that $\sup_{t\ge T}\|R^+_{\alpha-1}h(t)\|_\infty \le c\|h\|_{\alpha-1}$ for a constant $c>0$ and $h\in E_{\alpha-1}$. Moreover, $R^+_{\alpha-1}h\in C_0([T,+\infty),X)$ if $h\in C_0([T,+\infty),X)$. For $h\in E_{\alpha-1}([T,+\infty))$ and $1-\mu<\beta<\alpha$, there are $h_n\in C_0([T,+\infty),X)$ converging to h in $E_{\beta-1}([T,+\infty))$ due to (2.5). Therefore $R^+_{\beta-1}h_n \longrightarrow R^+_{\beta-1}h = R^+_{\alpha-1}h$ in $C_b([T,+\infty),X)$ (the space of bounded continuous functions), and the first part of (a) is shown. For $t\ge s\ge T$, we further compute

$$\begin{aligned}
&U(t,s)R^+_{\alpha-1}h(s) + \int_s^t U_{\alpha-1}(t,\tau)h(\tau)d\tau\\
&= \int_s^t U_{\alpha-1}(t,\tau)h(\tau)ds - \int_s^{+\infty}\widetilde{U}_{\alpha-1}(t,\tau)Q_{\alpha-1}(\tau)h(\tau)d\tau + \int_T^s U_{\alpha-1}(t,\tau)P_{\alpha-1}(\tau)h(\tau)d\tau\\
&= \int_T^t U_{\alpha-1}(t,\tau)P_{\alpha-1}(\tau)h(\tau)d\tau - \int_t^{+\infty}\widetilde{U}_{\alpha-1}(t,\tau)Q_{\alpha-1}(\tau)h(\tau)d\tau = R^+_{\alpha-1}h(t).
\end{aligned}$$

Hence, $R^+_{\alpha-1}h\in D(G^+_{\alpha-1})$ and $G^+_{\alpha-1}R^+_{\alpha-1}h=h$.

(b) The first part of (b) follows similarly as in (a). For $h\in E_{\alpha-1}((-\infty,T])$ and $s\le t\le -T$, we calculate

$$\begin{aligned}
&\int_s^t U_{\alpha-1}(t,\tau)h(\tau)d\tau + U(t,s)R^-_{\alpha-1}h(s)\\
&= \int_s^t U_{\alpha-1}(t,\tau)h(\tau)ds + \int_{-\infty}^s U_{\alpha-1}(t,\tau)P_{\alpha-1}(\tau)h(\tau)d\tau - \int_s^{-T}\widetilde{U}_{\alpha-1}(t,\tau)Q_{\alpha-1}(\tau)h(\tau)d\tau\\
&= \int_{-\infty}^t U_{\alpha-1}(t,\tau)P_{\alpha-1}(\tau)h(\tau)d\tau - \int_t^{-T}\widetilde{U}_{\alpha-1}(t,\tau)Q_{\alpha-1}(\tau)h(\tau)d\tau = R^-_{\alpha-1}h(t).
\end{aligned}$$

For $s\le -T\le t$, it holds

$$\begin{aligned}
&\int_s^t U_{\alpha-1}(t,\tau)h(\tau)d\tau + U(t,s)R^-_{\alpha-1}h(s)\\
&= \int_s^t U_{\alpha-1}(t,\tau)h(\tau)ds + \int_{-\infty}^s U_{\alpha-1}(t,\tau)P_{\alpha-1}(\tau)h(\tau)d\tau - \int_s^{-T}\widetilde{U}_{\alpha-1}(t,\tau)Q_{\alpha-1}(\tau)h(\tau)d\tau\\
&= \int_{-\infty}^{-T} U_{\alpha-1}(t,\tau)P_{\alpha-1}(\tau)h(\tau)d\tau + \int_{-T}^t U_{\alpha-1}(t,\tau)h(\tau)d\tau = R^-_{\alpha-1}h(t).
\end{aligned}$$

Finally, for $-T\le s\le t\le T$, we compute

$$\begin{aligned}
&\int_s^t U_{\alpha-1}(t,\tau)h(\tau)d\tau + U(t,s)R^-_{\alpha-1}h(s)\\
&\qquad= \int_s^t U_{\alpha-1}(t,\tau)h(\tau)ds + \int_{-\infty}^{-T} U_{\alpha-1}(t,\tau)P_{\alpha-1}(\tau)h(\tau)d\tau + \int_{-T}^s U_{\alpha-1}(t,\tau)h(\tau)d\tau
\end{aligned}$$

$$= \int_{-\infty}^{-T} U_{\alpha-1}(t,\tau)P_{\alpha-1}(\tau)h(\tau)d\tau + \int_{-T}^{t} U_{\alpha-1}(t,\tau)h(\tau)d\tau = R^-_{\alpha-1}h(t).$$

As a result, $R^-_{\alpha-1}h \in D(G^-_{\alpha-1})$ and $G^-_{\alpha-1}R^-_{\alpha-1}h = h$. ☐

Lemma 3.4. *Assume that* (2.1), (2.2), *and* (2.19) *hold and that* $1-\mu < \alpha \le 1$. *Let* $x \in Q(T)(X)$. *Then there exists* $u \in D(G_{\alpha-1})$ *such that* $R^+_{\alpha-1}u(T) = x$, $R^-_{\alpha-1}u(T) = 0$, *and* $\|u\|_E + \|G_{\alpha-1}u\|_E \le K\|x\|$, *where* $K \ge 0$ *is a constant independent of* x.

Proof. We fix a test function φ with $\varphi(t) = 0$ for $t \le T$ and $\int_T^\infty \varphi(s)ds = -1$, and define the functions

$$u(t) := \varphi(t)U(t,T)x, \quad t \ge T, \qquad u(t) := 0, \quad t \le T,$$
$$f(t) := \varphi'(t)U(t,T)x, \quad t \ge T, \qquad f(t) := 0, \quad t < T.$$

It is easy to check that $R^+_{\alpha-1}u(T) = x$ and $R^-_{\alpha-1}u(T) = 0$. We further obtain

$$\begin{aligned} u(t) - U(t,s)u(s) &= (\varphi(t) - \varphi(s))\, U(t,T)x \\ &= \int_s^t U_{\alpha-1}(t,\tau)\varphi'(\tau)U(\tau,T)x \, d\tau = \int_s^t U_{\alpha-1}(t,\tau)f(\tau)\, d\tau. \end{aligned}$$

for $t \ge s \ge T$. The case $s < T$ is treated similarly. As a result, $u \in D(G_{\alpha-1})$ and $G_{\alpha-1}u = f$, so that the asserted estimate follows. ☐

We can now describe the range and the kernel of $G_{\alpha-1}$.

Proposition 3.5. *Assume that* (2.1), (2.2), *and* (2.19) *are satisfied and that* $1-\mu < \alpha \le 1$. *Then the following assertions hold for* $G_{\alpha-1}$ *defined on* $E_{\alpha-1} = E_{\alpha-1}(\mathbb{R})$.

(a) $N(G^+_{\alpha-1}) = \{u \in C_0([T,+\infty), X)) : u(t) = U(t,T)x \ \ \forall\, t \ge T, \ x \in P(T)\hat{X}^T\}$;

(b) $N(G^-_{\alpha-1}) = \{u \in C_0((-\infty,T]) : u(t) = U(t,s)u(s) \ \ \forall\, s \le t \le T, \, u(T) \in X_u(T)\}$;

(c) $N(G_{\alpha-1}) = \{u \in C_0(\mathbb{R}, X) : u(t) = U(t,s)u(s) \ \ \forall\, t \ge s, \, u(T) \in P(T)X \cap X_u(T)\}$;

(d) $R(G_{\alpha-1}) = \{f \in E_{\alpha-1} : R^+_{\alpha-1}f(T) - R^-_{\alpha-1}f(T) \in P(T)X + X_u(T)\}$, *where for* $f \in R(G_{\alpha-1})$ *a function* $u \in D(G_{\alpha-1})$ *with* $G_{\alpha-1}u = f$ *is given by* (3.1) *below;*

(e) $\overline{R(G_{\alpha-1})} = \{f \in E_{\alpha-1} : R^+_{\alpha-1}f(T) - R^-_{\alpha-1}f(T) \in \overline{P(T)X + X_u(T)}\}$, *where the closure on the left (right) hand side is taken in* $E_{\alpha-1}$ *(in* X*).*

Proof. Assertions (a), (b) and (c) follow from Lemma 3.2 and $P(T)X \cap X_u(T)P(T)\hat{X}^T \cap X_u(T)$. To show (d), let $G_{\alpha-1}u = f \in E_{\alpha-1}$ for some $u \in D(G_{\alpha-1})$. Then the restrictions of f to $[T,+\infty)$ and $(-\infty,T]$ belong to $R(G^+_{\alpha-1})$ and to $R(G^-_{\alpha-1})$, respectively. Proposition 3.3 shows that the functions

$$v_+ = (u|[T,+\infty)) - R^+_{\alpha-1}f \qquad \text{and} \qquad v_- = (u|(-\infty,T]) - R^-_{\alpha-1}f$$

belong to the kernel of $G^+_{\alpha-1}$ and $G^-_{\alpha-1}$, respectively . Thus

$$(R^+_{\alpha-1}f)(T) - (R^-_{\alpha-1}f)(T) = v_-(T) - v_+(T) \in X_u(T) + P(T)\hat{X}^T$$

by (a) and (b). Conversely, let $f \in E_{\alpha-1}$ with $(R^+_{\alpha-1}f)(T) - (R^-_{\alpha-1}f)(T) = y_s + y_u \in P(T)X + X_u(T)$. Set $x_0 := (R^+_{\alpha-1}f)(T) - y_s = y_u + (R^-_{\alpha-1}f)(T)$ and

$$u(t) := \begin{cases} u_+(t) := -U(t,T)y_s + (R^+_{\alpha-1}f)(t), & t \geq T, \\ u_-(t) := \tilde{v}(t) + (R^-_{\alpha-1}f)(t), & t \leq T, \end{cases} \tag{3.1}$$

where $\tilde{v} \in N(G^-_{\alpha-1})$ such that $\tilde{v}(T) = y_u$. Using Propositions 2.1 and 2.2, one checks that $R^\pm_{\alpha-1}f(T) \in X^T_\varepsilon$ for $0 < \epsilon < \alpha$, so that $y_s \in X^t_\varepsilon \subseteq \hat{X}^T$. Hence, $u \in C_0(\mathbb{R}, X)$. Proposition 3.3 further yields

$$u_\pm(t) = U(t,s)u_\pm(s) + \int_s^t U_{\alpha-1}(t,\tau)f(\tau)d\tau$$

for all $t \geq s \geq T$ and $s \leq t \leq T$, respectively. Let now $s \leq T \leq t$. Since $u_+(T) = u_-(T) = x_0$, we have

$$\begin{aligned} u(t) = u_+(t) &= U(t,T)u_-(T) + \int_T^t U_{\alpha-1}(t,\tau)f(\tau)d\tau \\ &= U(t,T)\Big[U(T,s)u_-(s) + \int_s^T U_{\alpha-1}(T,\tau)f(\tau)d\tau\Big] + \int_T^t U_{\alpha-1}(t,\tau)f(\tau)d\tau \\ &= U(t,s)u(s) + \int_s^t U_{\alpha-1}(t,\tau)f(\tau)d\tau. \end{aligned}$$

Therefore $G_{\alpha-1}u = f$, and (d) is established.

The inclusion '$\subseteq$' in assertion (e) follows from (d) and Proposition 3.3. Take $f \in E_{\alpha-1}$ and $z := (R^+_{\alpha-1}f)(T) - (R^-_{\alpha-1}f)(T)$ such that there is a sequence $z_n \in P(T)X + X_u(T)$ converging to z in X as $n \to \infty$. Set $y_n := z - z_n$ and $x_n := Q(T)y_n$. Lemma 3.4 yields a function $f_n \in D(G_{\alpha-1})$ such that $(R^+_{\alpha-1}f_n)(T) = x_n, (R^-_{\alpha-1}f_n)(T) = 0$, and $\|f_n\|_E \leq K\|x_n\|$ for a constant K independent of n. Then the vector

$$(R^+_{\alpha-1}(f - f_n))(T) - (R^-_{\alpha-1}(f - f_n))(T) = z - x_n = z_n + P(T)y_n$$

belongs to $P(T)X + X_u(T)$, so that $f - f_n \in R(G_{\alpha-1})$ by (d). Since $E \hookrightarrow E_{\alpha-1}$, we can estimate

$$\|f - (f - f_n)\|_{E_{\alpha-1}} \leq c\,\|f_n\|_E \leq cK\,\|x_n\| \leq c\,\|z - z_n\|,$$

and thus assertion (e) is shown. □

Using the above results, we are able to describe other properties of the operator $G_{\alpha-1}$, in particular its Fredholmity, in terms of properties of the subspaces $X_s(T)$ and $X_u(T)$, using similar arguments as in [15], see also [16] for L^p spaces. For the convenience of the readers, we give the complete proof. Recall that subspaces V and W of a Banach space E are called a *semi-Fredholm couple* if $V + W$ is closed and if at least one of the dimensions $\dim(V \cap W)$ and $\operatorname{codim}(V + W)$ is finite. The *index* of (V,W) is defined by $\operatorname{ind}(V,W) := \dim(V \cap W) - \operatorname{codim}(V + W)$. If the index is finite, then (V,W) is a *Fredholm couple*. Observe that in the next theorem the operator $U(T,-T)_{|Q(-T)(X)}$ is trivially injective if $T = 0$.

Theorem 3.6. *Assume that* (2.1)*,* (2.2)*, and* (2.19) *are satisfied and that* $1-\mu < \alpha \le 1$*. Then the following assertions hold for* $G_{\alpha-1}$ *defined on* $E_{\alpha-1} = E_{\alpha-1}(\mathbb{R})$*.*

(a) $R(G_{\alpha-1})$ *is closed in* $E_{\alpha-1}$ *if and only if* $P(T)X + X_u(T)$ *is closed in* X*.*
(b) $G_{\alpha-1}$ *is surjective if and only if* $P(T)X + X_u(T) = X$*.*
(c) *If* $G_{\alpha-1}$ *is injective, then* $P(T)X \cap X_u(T) = \{0\}$*. The converse is true if* $U(T,-T)_{|Q(-T)(X)}$ *is injective, in addition.*
(d) *If* $G_{\alpha-1}$ *is invertible, then* $P(T)X \oplus X_u(T) = X$*. The converse is true if* $U(T,-T)_{|Q(-T)(X)}$ *is injective, in addition.*
(e) $\dim N(G_{\alpha-1}) = \dim(P(T)X \cap X_u(T)) + \dim N(U(T,-T)_{|Q(-T)(X)})$.
If $R(G_{\alpha-1})$ *is closed in* $E_{\alpha-1}$*, then* $\operatorname{codim}(P(T)X + X_u(T)) = \operatorname{codim} R(G_{\alpha-1})$.
(f) *If* $G_{\alpha-1}$ *is a semi-Fredholm operator, then* $(P(T)X, X_u(T))$ *is a semi-Fredholm couple, and* $\operatorname{ind}(P(T)X, X_u(T)) \le \operatorname{ind} G_{\alpha-1}$*. If in addition the kernel of* $U(T,-T)_{|Q(-T)(X)}$ *is finite dimensional, then*

$$\operatorname{ind}(P(T)X, X_u(T)) = \operatorname{ind} G_{\alpha-1} - \dim N(U(T,-T)_{|Q(-T)(X)}). \tag{3.2}$$

Conversely, if $(P(T)X, X_u(T))$ *is a semi-Fredholm couple and the kernel of* $U(T,-T)_{|Q(-T)(X)}$ *is finite dimensional, then* $G_{\alpha-1}$ *is a semi-Fredholm operator and* (3.2) *holds.*

Proof. The 'if' part of assertion (a) is a direct consequence of Proposition 3.5(d) and (e). Assume that $R(G_{\alpha-1})$ is closed in $E_{\alpha-1}$. Take $y_n^s \in P(T)X$ and $y_n^u \in X_u(T)$ with $y_n^s + y_n^u \to y$ in X as $n \to \infty$. Set $z = Q(T)y$. By Lemma 3.4, there is a function $h \in D(G_{\alpha-1})$ such that $R_{\alpha-1}^+ h(T) = z$ and $R_{\alpha-1}^- h(T) = 0$. Since

$$z = \lim_{n\to\infty} Q(T)(y_n^s + y_n^u) = \lim_{n\to\infty} Q(T)y_n^u = \lim_{n\to\infty} (y_n^u - P(T)y_n^u),$$

we obtain $R_{\alpha-1}^+ h(T) - R_{\alpha-1}^- h(T) = z \in \overline{P(T)X + X_u(T)}$. Proposition 3.5 implies that $h \in \overline{R(G_{\alpha-1})} = R(G_{\alpha-1})$, and thus $z \in P(T)X + X_u(T)$ by Proposition 3.5(d). As a result, $y = P(T)y + z \in P(T)X + X_u(T)$, and so (a) holds. The 'if' part of assertion (b) follows from Proposition 3.5(d), and the converse can be shown as in statement (a).

Proposition 3.5(c) yields the first part of (c). For the converse, assume that $U(T,-T)_{|Q(-T)(X)}$ is injective and $P(T)X \cap X_u(T) = \{0\}$, and let $u \in N(G_{\alpha-1})$. Then $u(t) = U(t,s)u(s)$ for all $t \ge s$, and so $u(T) = 0$ by Proposition 3.5(c). From Lemma 3.2(b) we further deduce $u(-T) \in Q(-T)(X)$. Since $0 = u(T) = U(T,-T)_{|Q(-T)(X)} u(-T)$, our assumption yields $u(-T) = 0$ and thus $u(t) = 0$ for $t \ge -T$. Finally, $u(t) = \widetilde{U}(t,-T)u(-T)$ for all $t \le -T$ by Proposition 2.8, so that $u = 0$. We have thus shown (c). Assertion (d) is an easy consequence of (b) and (c).

To show the first equality in (e), we define $\Gamma := \{u \in N(G_{\alpha-1}) : u(t) = 0,\ t \ge T\}$ and the linear mapping

$$K : N(G_{\alpha-1})/\Gamma \longrightarrow P(T)X \cap X_u(T), \quad [u] \longmapsto u(T).$$

Proposition 3.5(c) implies that K is well defined and bijective. Since also $\dim \Gamma = \dim N(U(T,-T)_{|Q(-T)(X)})$, the first identity holds. We next assume that $R(G_{\alpha-1})$ is closed in $E_{\alpha-1}$. Hence, $P(T)X + X_u(T)$ is closed X by (a). Define the linear map

$$J : E_{\alpha-1}/R(G_{\alpha-1}) \longrightarrow X/(P(T)X + X_u(T)), \quad [f] \longmapsto [(R^+_{\alpha-1}f)(T) - (R^-_{\alpha-1}f)(T)].$$

Due to Proposition 3.5(d), J is well defined and injective. Take $x \in X$. By Lemma 3.4 there is a function $f \in E_{\alpha-1}$ such that $(R^+_{\alpha-1}f)(T) - (R^-_{\alpha-1}f)(T) = Q(T)x = x - P(T)x$. Hence, $J[f] = [(R^+_{\alpha-1}f)(T) - (R^-_{\alpha-1}f)(T)] = [x]$. Consequently, J is also surjective and thus the second equality in (e) follows. Assertion (f) is a consequence of (a) and (e). $\square$

Using (2.18) and the same arguments as in [15], we obtain the following sufficient condition for the Fredholmity of $G_{\alpha-1}$.

Corollary 3.7. *Assume that* (2.1), (2.2), (2.16), *and* (2.17) *are satisfied and that* $1-\mu < \alpha \le 1$. *Further suppose that* $\dim Q_{\pm\infty}X < \infty$ *(which holds if* $D(A_{\pm}\infty)$ *are compactly embedded in* X*). Then* $G_{\alpha-1}$ *is Fredholm and* $\operatorname{ind} G_{\alpha-1} = \dim Q_{-\infty}X - \dim Q_{+\infty}X$.

We next characterize the range of $G_{\alpha-1}$ in terms of the dual problem, see Remark 3.9 below. Related results have been shown in [16] and [20] for other settings by different methods. We start with a simple observation. Let $0 \le \theta < \alpha \le 1$. Then $X^t_{\alpha-1}$ is densely embedded in $D((\omega - A_{-1}(t))^\theta)$ by (2.4). Since $D((\omega - A(t)^*)^{1-\theta}) \hookrightarrow [D((\omega - A_{-1}(t))^\theta)]^*$, we deduce that $D((\omega - A(t)^*)^{1-\theta} \hookrightarrow (X^t_{\alpha-1})^*$ for $t \in \mathbb{R}$ with a uniform embedding constant. We denote by $\mathcal{V}$ the space of $v \in C(\mathbb{R}, X^*)$ such that $v(s) = U_{\alpha-1}(t,s)^*v(t)$, $v(t) \in D((\omega - A(t)^*)^{1-\theta})$, and $(\omega - A(\cdot)^*)^{1-\theta}v \in L^1(\mathbb{R}, X^*)$ for all $t \ge s$ in $\mathbb{R}$.

Proposition 3.8. *Assume that* (2.1), (2.2), *and* (2.19) *hold and that* $1 - \mu < \theta < \alpha \le 1$. *Then the closure of* $R(G_{\alpha-1})$ *is equal to the space*

$$\mathcal{E} := \{f \in E_{\alpha-1} : \int_{\mathbb{R}} \langle f(s), v(s)\rangle_{X^s_{\alpha-1}}\, ds = 0 \ \ \textit{for all } v \in \mathcal{V}\}.$$

Proof. We first show that under our assumptions it holds

$$\mathcal{V} = \{v \in L^1(\mathbb{R}, X^*) : v(s) = U_{\alpha-1}(t,s)^*v(t) \ \ \forall\, t \ge s\} =: \mathcal{V}'. \tag{3.3}$$

Clearly, $\mathcal{V} \subset \mathcal{V}'$. Take $v \in \mathcal{V}'$. Then $v \in C(\mathbb{R}, X^*)$ since $U(t,s)$ is norm continuous for $t > s$. We denote by $V(t,s)$ the extension of $U(t,s)(\omega - A(s))^{1-\theta}$ to $\mathcal{L}(X)$. For $x \in D((\omega - A(s))^{1-\theta})$, we then obtain

$$\begin{aligned} \langle(\omega - A(s))^{1-\theta}x, v(s)\rangle &= \langle(\omega - A(s))^{1-\theta}x, U_{\alpha-1}(s+1,s)^*v(s+1)\rangle \\ &= \langle V(s+1,s)x, v(s+1)\rangle, \\ |\langle(\omega - A(s))^{1-\theta}x, v(s)\rangle| &\le c\,\|v(s+1)\|_{X^*}\|x\| \end{aligned} \tag{3.4}$$

due to (2.8). The estimate (3.4) yields

$$v(s) \in D((\omega - A(s)^*)^{1-\theta}) \quad \text{ and } \quad \|(\omega - A(s)^*)^{1-\theta}v(s)\|_{X^*} \le c\,\|v(s+1)\|_{X^*}.$$

Thus $v \in \mathcal{V}$ and (3.3) is true. We now come to the main part of the proof. Proposition 3.5 shows that

$$f \in \overline{R(G_{\alpha-1})} \iff z := R^+_{\alpha-1}f(T) - R^-_{\alpha-1}f(T) \in \overline{P(T)X + X_u(T)}.$$

Employing also [24, Theorem 4.7] and [19, (IV.4.11)], we deduce

$$f \in \overline{R(G_{\alpha-1})} \iff z \in {}^{\perp}((P(T)X + X_u(T))^{\perp}) \iff z \in {}^{\perp}((P(T)X)^{\perp} \cap (X_u(T))^{\perp})$$

where $M^{\perp} := \{x^* \in X^* : \langle x, x^* \rangle = 0 \ \forall\, x \in M\}$ for $M \subseteq X$ and ${}^{\perp}N := \{x \in X : \langle x, x^* \rangle = 0 \ \forall\, x^* \in N\}$ for $N \subseteq X^*$. Straightforward duality arguments imply that $U(t,s)^*$ has an exponential dichotomy on $[T, +\infty)$ and $(-\infty, -T]$ with projections $P(t)^*$ and that

$$(P(T)X)^{\perp} = Q(T)^* X^*, \quad (X_u(T))^{\perp} = N(Q(-T)^* U(T,-T)^*), \tag{3.5}$$

using also $X_u(T) = U(T,-T)Q(-T)X$, see Lemma 3.2. We further compute

$$\begin{aligned}
\langle z,\, y^* \rangle &= -\int_T^{+\infty} \langle \widetilde{U}_{\alpha-1}(T,s) Q_{\alpha-1}(s) f(s),\, y^* \rangle_X \, ds - \int_{-T}^{T} \langle U_{\alpha-1}(T,s) f(s),\, y^* \rangle_X \, ds \\
&\quad - \int_{-\infty}^{-T} \langle U_{\alpha-1}(T,s) P_{\alpha-1}(s) f(s),\, y^* \rangle_X \, ds \\
&= - \int_{\mathbb{R}} \langle f(s),\, v(s) \rangle_{X^s_{\alpha-1}} \, ds
\end{aligned}$$

for all $y^* \in (P(T)X)^{\perp} \cap (X_u(T))^{\perp}$, where v is given by

$$v(s) := \begin{cases} \widetilde{U}(T,s)^* Q(T)^* y^* = \widetilde{U}(T,s)^* y^*, & s \ge T, \\ U(T,s)^* y^*, & -T \le s \le T, \\ U(-T,s)^* P(-T)^* U(T,-T)^* y^* = U(T,s)^* y^*, & s \le -T. \end{cases} \tag{3.6}$$

(Here we have used (3.5).) Summing up, we have shown that $f \in \overline{R(G_{\alpha-1})}$ if and only if

$$\int_{\mathbb{R}} \langle f(s),\, v(s) \rangle_{X^s_{\alpha-1}} \, ds = 0$$

for all v as in (3.6) with $y^* \in Q(T)^* X^* \cap N(Q(-T)^* U(T,-T)^*)$. It remains to show that $\mathcal{V}$ consists precisely of the functions defined in (3.6).

First, one verifies by a duality argument that each function v in (3.6) belongs to $\mathcal{V}' = \mathcal{V}$, recall (3.3). Conversely, let $v \in \mathcal{V}$. Then we have

$$P(T)^* v(T) = U(t,T)^* P(t)^* v(t), \quad \|P(T)^* v(T)\| \le N e^{-\delta(t-T)} \|v(t)\|$$

for $t \ge T$. There is a sequence $t_n \to \infty$ such that $\|v(t_n)\|$ is bounded since $v \in L^1(\mathbb{R}, X^*)$. Therefore, $P(T)^* v(T) = 0$. For $s \le -T$, one obtains

$$Q(s)^* v(s) = U(-T,s)^* Q(-T)^* v(-T) = U(-T,s)^* Q(-T)^* U(T,-T)^* v(T), \tag{3.7}$$

$$\|Q(-T)^* U(T,-T)^* v(T)\| = \|\widetilde{U}(s,-T)^* Q(s)^* v(s)\| \le N e^{-\delta(-T-s)} \|v(s)\|. \tag{3.8}$$

As above, it follows that $Q(-T)^* U(T,-T)^* v(T) = 0$. Consequently, v is of the form (3.6) with $y^* = v(T) \in Q(T)^* X^* \cap N(Q(-T)^* U(T,-T)^*)$. ☐

Remark 3.9. One can see that the functions $v \in \mathcal{V}$, see (3.3), solve the dual evolution equation

$$-v'(s) = A(s)^* v(s), \qquad s \in \mathbb{R}, \tag{3.9}$$

in a weak sense. The function v is a classical solution of (3.9) if also the adjoint operators $A(t)^*$ satisfy the Acquistapace–Terreni conditions (2.1) and (2.2), see [2, Prop.2.9].

Theorem 3.6, Corollary 3.7, and Propositions 3.5 and 3.8 now yield the following Fredholm alternative, where we focus on a simplified setting.

Theorem 3.10. *Assume that* (2.1), (2.2), (2.16) *and* (2.17) *are true, that* $\dim Q_{\pm\infty}X < \infty$, *and that* $1 - \mu < \alpha \le 1$. *Let* $f \in E_{\alpha-1} = E_{\alpha-1}(\mathbb{R})$. *Then there is a mild solution* $u \in C_0(\mathbb{R}, X)$ *of* (2.20) *if and only if*

$$\int_{\mathbb{R}} \langle f(s), w(s) \rangle_{X^s_{\alpha-1}} \, ds = 0$$

for each $w \in L^1(\mathbb{R}, X^*)$ *with* $w(s) = U_{\alpha-1}(t,s)^* w(t)$ *for all* $t \ge s$. *The mild solutions* u *are given by*

$$u(t) = v(t) - U(t,T)y_s + (R^+_{\alpha-1}f)(t), \qquad t \ge T,$$
$$u(t) = v(t) + \tilde{v}(t) + (R^-_{\alpha-1}f)(t), \qquad t \le T,$$

where $R^{\pm}_{\alpha-1}$ *were defined before Proposition 3.3,* $(R^+_{\alpha-1}f)(T) - (R^-_{\alpha-1}f)(T) = y_s + y_u \in P(T)X + X_u(T)$, $\tilde{v} \in C_0((-\infty, T], X)$ *with* $\tilde{v}(T) = y_u$ *and* $\tilde{v}(t) = U(t,s)\tilde{v}(s)$ *for all* $T \ge t \ge s$, *and* $v \in C_0(\mathbb{R}, X)$ *with* $v(t) = U(t,s)v(s)$ *for all* $t \ge s$.

Note that in the above result we obtain mild solutions which are unique modulo the finite dimensional subspace $N(G_{\alpha-1})$. We further remark that if $U(\cdot,\cdot)$ has an exponential dichotomy on $\mathbb{R}$ with projections $P(t)$, $t \in \mathbb{R}$, then we can take $T = 0$ and we have $X_u(0) = (I - P(0))X$. Hence, $G_{\alpha-1}$ is invertible by Theorem 3.6(d). As a result, for each $f \in E_{\alpha-1}$ we obtain a unique mild solution of $u \in C_0(\mathbb{R}, X)$ of (2.20) which is given by

$$u(t) = \int_{\mathbb{R}} \Gamma_{\alpha-1}(t,\tau) f(\tau) \, d\tau, \qquad t \in \mathbb{R},$$

(3.1), cf. [11] for this formula in the case $\alpha = 1$. We conclude this section with two remarks indicating straightforward variants of the results established so far. The details are left to the reader.

Remark 3.11. Note that we allow for the case $\alpha = 1$, i.e., $G_0 = G$ on $E_0 = E = C_0(\mathbb{R}, X)$, in this section. In fact, in this case the results shown in this section remain valid for each exponentially bounded evolution family $U(t,s)$, $t \ge s$, (i.e., (2.3) holds) such that $(t,s) \mapsto U(t,s)$ is strongly continuous for $t \ge s$ and $U(\cdot,\cdot)$ has exponential dichotomies on halflines $(-\infty, -T]$ and $[T, +\infty)$. (Here one sets $\hat{X}^t = X$.)

Remark 3.12. All results established in this and the previous section remain valid with the slightly simplified proofs if we replace the function spaces $C_0(J, X)$ by $C_b(J, X)$ in the assertions and in Definitions 2.4 and 3.1 and set $X_s(t_0) = \{x \in X^{t_0}_{\alpha-1} : \sup_{t \ge t_0+1} \|U_{\alpha-1}(t,t_0)x\| < \infty\}$. Moreover, one can replace throughout the space $X^t_{\alpha-1}$ by the closure of X in $X^t_{\alpha-1}$.

4. Non–autonomous parabolic boundary evolution equations

In this section we study the non–autonomous parabolic boundary evolution equation

$$u'(t) = A_m(t)u(t) + g(t), \qquad t \ge t_0,$$

$$B(t)u(t) = h(t), \qquad t \geq t_0, \tag{4.1}$$
$$u(t_0) = u_0,$$

and its variant on the line

$$\begin{aligned} u'(t) &= A_m(t)u(t) + g(t), \qquad t \in \mathbb{R}, \\ B(t)u(t) &= h(t), \qquad t \in \mathbb{R}. \end{aligned} \tag{4.2}$$

Here $t_0 \in \mathbb{R}$, $u_0 \in X$, and the operators $A_m(t)$ and $B(t)$, $t \in \mathbb{R}$, are defined on a Banach space $Z_t \hookrightarrow X$ and map into the state space X and the 'boundary space' Y, respectively. The inhomogeneities g and h take values in X and Y, respectively. In the typical applications $A_m(t)$ is a differential operator with 'maximal' domain not containing boundary conditions and $B(t)$ are boundary operators. We further introduce the operators

$$A(t)u := A_m(t)u, \quad u \in D(A(t)) := \{u \in Z_t : B(t)u = 0\}.$$

More precisely, we make the following assumptions.

(A1) For every $t \in \mathbb{R}$ there is a Banach space $Z_t \hookrightarrow X$ such that $A_m(t) \in \mathcal{L}(Z_t, X)$. The opererators $B(t) \in \mathcal{L}(Z_t, Y)$ are surjective for $t \in \mathbb{R}$.

(A2) The operators $A(t) = A_m(t)|N(B(t))$, $t \in \mathbb{R}$, satisfy (2.1) and (2.2).

Under these hypotheses, there is an evolution family $(U(t,s))_{t \geq s}$ solving the problem with homogeneous conditions $g = h = 0$. Moreover, by [18, Lemma 1.2] there exists the *Dirichlet map* $D(t)$ for $\omega - A_m(t)$, i.e., $v = D(t)y$ is the unique solution of the abstract boundary value problem

$$(\omega - A_m(t))v = 0, \qquad B(t)v = y,$$

for each $y \in Y$. Fixing $\alpha \in (1-\mu, 1]$ (where μ is given by (2.2)), we further assume that

(A3) $\sup_{t \in \mathbb{R}} \|D(t)\|_{\mathcal{L}(Y, X_\alpha^t)} < \infty$ and $\mathbb{R} \ni t \longmapsto D(t)y$ is continuous in X for each $y \in Y$.

If (A1)–(A3) hold with $\mathbb{R}$ replaced by a closed interval J, we may extend $A_m(t)$, $B(t)$, and Z_t constantly to $t \in \mathbb{R}$, and then (A1)–(A3) hold on $\mathbb{R}$ for this extension. Hypotheses (A1)–(A3) describe one convenient general setting for the application of our results, in particular suited for parabolic problems formulated on L^p or C^β spaces. But our approach is more flexible. So we treat in Example 4.5 an initial boundary value problem on the state space $X = C(\overline{\Omega})$ which does not fit in the above setting. We add a simple observation.

Lemma 4.1. *Assume that assumptions (A1)–(A3) hold and that $h \in C_0(J, Y)$ for a closed interval J. Then $(\omega - A_{-1}(\cdot))D(\cdot)h \in E_{\alpha-1}(J)$.*

Proof. Assumption (A3) yields $D(\cdot)h \in E_\alpha(J)$ which implies the assertion. □

In order to apply the results from the previous sections to the boundary evolution equation (4.1), we write it as the inhomogeneous Cauchy problem

$$\begin{aligned} u'(t) &= A_{-1}(t)u(t) + f(t), \qquad t \geq t_0, \\ u(t_0) &= u_0, \end{aligned} \tag{4.3}$$

setting $f := g + (\omega - A_{-1}(\cdot))D(\cdot)h$. We also consider the evolution equation

$$u'(t) = A_{-1}(t)u(t) + f(t), \qquad t \in \mathbb{R}. \tag{4.4}$$

If $g \in C_0(J,X)$ and $h \in C_0(J,Y)$, then $f \in E_{\alpha-1}(J)$ by Lemma 4.1. As in Definition 2.4, we call a function $u \in C(J,X)$ a *mild solution* of (4.2) and (4.4) on J if the equation

$$u(t) = U(t,s)u(s) + \int_s^t U_{\alpha-1}(t,\sigma)[g(\sigma) + (\omega - A_{-1}(\sigma))D(\sigma)h(\sigma)]\,d\sigma \tag{4.5}$$

holds for all $t \geq s$ in J. The function u is called a *mild solution* of (4.1) and (4.3) if in addition $u(t_0) = u_0$ and $J = [t_0, \infty)$. Mild solutions for the corresponding final value problems are defined in the same way. We note that a function $u \in C^1(J,X)$ with $u(t) \in Z_t$ satisfies (4.1), resp. (4.2), if and only if it satisfies (4.3), resp. (4.4), and then it is given by (4.5). These facts can be shown as in Proposition 4.2 of [13].

Propositions 2.7 and 2.8 immediately imply two results on the existence of bounded mild solutions for forward and backward boundary evolution equations.

Proposition 4.2. *Assume that assumptions (A1)–(A3) hold with* $1-\mu < \alpha \leq 1$ *and that* $U(t,s)$ *has an exponential dichotomy on an interval* $[T,\infty)$. *Let* $t_0 \geq T$, $g \in C_0([T,\infty),X)$, $h \in C_0([T,+\infty),Y)$, *and* $u_0 \in \overline{D(A(t_0))}$. *Then the mild solution* $u \in C([t_0,+\infty),X)$ *of the boundary evolution equation* (4.1) *is bounded on* $[t_0,\infty)$ *if and only if*

$$Q(t_0)u_0 = -\int_{t_0}^{+\infty} \widetilde{U}_{\alpha-1}(t_0,s)Q_{\alpha-1}(s)[g(s) + (\omega - A_{-1}(s))D(s)h(s)]\,ds.$$

In this case u *is given by*

$$\begin{aligned} u(t) = U(t,t_0)P(t_0)u_0 &+ \int_{t_0}^t U_{\alpha-1}(t,s)P_{\alpha-1}(s)[g(s) + (\omega - A_{-1}(s))D(s)h(s)]\,ds \\ &- \int_t^\infty \widetilde{U}_{\alpha-1}(t,s)Q_{\alpha-1}(s)[g(s) + (\omega - A_{-1}(s))D(s)h(s)]\,ds, \quad t \geq t_0. \end{aligned}$$

Proposition 4.3. *Assume that assumptions (A1)–(A3) hold with* $1-\mu < \alpha \leq 1$ *and that* $U(t,s)$ *has an exponential dichotomy on an interval* $(-\infty,-T]$. *Let* $t_0 < -T$, $g \in C_0((-\infty,-T],X)$, $h \in C_0((-\infty,-T],Y)$, *and* $u_0 \in X$. *Then there is a bounded mild solution* $u \in C((-\infty,t_0],X)$ *of the backward boundary evolution equation*

$$\begin{aligned} u'(t) &= A_m(t)u(t) + g(t), \qquad t \leq t_0, \\ B(t)u(t) &= h(t), \qquad t \leq t_0, \\ u(t_0) &= u_0, \end{aligned}$$

if and only if

$$P(t_0)u_0 = \int_{-\infty}^{t_0} U_{\alpha-1}(t_0,s)P_{\alpha-1}(s)[g(s) + (\omega - A_{-1}(s))D(s)h(s)]\,ds.$$

In this case u *is given by*

$$\begin{aligned} u(t) = \widetilde{U}(t,t_0)Q(t_0)u_0 &- \int_t^{t_0} \widetilde{U}_{\alpha-1}(t,s)Q_{\alpha-1}(s)[g(s) + (\omega - A_{-1}(s))D(s)h(s)]\,ds \\ &+ \int_{-\infty}^t U_{\alpha-1}(t,s)P_{\alpha-1}(s)[g(s) + (\omega - A_{-1}(s))D(s)h(s)]\,ds, \quad t \leq t_0. \end{aligned}$$

Moreover, Theorem 3.10 implies the following Fredholm alternative for the mild solutions of (4.2).

Theorem 4.4. *Assume that assumptions (A1)–(A3) hold with* $1-\mu < \alpha \le 1$*, that* (2.16) *and* (2.17) *are true, and that* $\dim Q_{\pm\infty} X < \infty$*. Let* $g \in C_0(\mathbb{R}, X)$ *and* $h \in C_0(\mathbb{R}, Y)$*. Then there is a mild solution* $u \in C_0(\mathbb{R}, X)$ *of* (4.2) *if and only if*

$$\int_{\mathbb{R}} \langle f(s), w(s) \rangle_{X^s_{\alpha-1}} \, ds = 0$$

for $f = g + (\omega - A_{-1}(\cdot))D(\cdot)h$ *and all* $w \in L^1(\mathbb{R}, X^*)$ *with* $w(s) = U_{\alpha-1}(t,s)^* w(t)$ *for all* $t \ge s$*. The mild solutions* u *are given by*

$$\begin{aligned} u(t) &= v(t) - U(t,T)y_s + (R^+_{\alpha-1} f)(t), \qquad t \ge T, \\ u(t) &= v(t) + \tilde{v}(t) + (R^-_{\alpha-1} f)(t), \qquad t \le T, \end{aligned}$$

where $R^{\pm}_{\alpha-1}$ *were defined before Proposition 3.3,* $(R^+_{\alpha-1} f)(T) - (R^-_{\alpha-1} f)(T) = y_s + y_u \in P(T)X + X_u(T)$*,* $\tilde{v} \in C_0((-\infty, T], X)$ *with* $\tilde{v}(T) = y_u$ *and* $\tilde{v}(t) = U(t,s)\tilde{v}(s)$ *for all* $T \ge t \ge s$*, and* $v \in C_0(\mathbb{R}, X)$ *with* $v(t) = U(t,s)v(s)$ *for all* $t \ge s$*.*

We add an example dealing with a parabolic pde in a sup norm context. One could treat more general problems, in particular systems, cf. [16], and one could weaken the regularity assumptions.

Example 4.5. We study the boundary value problem

$$\begin{aligned} \partial_t u(t,x) &= A(t,x,D)u(t,x) + g(t,x), \quad t \in \mathbb{R},\ x \in \Omega, \\ B(t,x,D)u(t,x) &= h(t,x), \quad t \in \mathbb{R},\ x \in \partial\Omega, \end{aligned} \tag{4.6}$$

on a bounded domain $\Omega \subseteq \mathbb{R}^n$ with boundary $\partial\Omega$ of class C^2 and outer unit normal vector $\nu(x)$, employing the differential expressions

$$\begin{aligned} A(t,x,D) &= \sum\nolimits_{k,l} a_{kl}(t,x)\partial_k\partial_l + \sum\nolimits_k a_k(t,x)\,\partial_k + a_0(t,x), \\ B(t,x,D) &= \sum\nolimits_k b_k(t,x)\,\partial_k + b_0(t,x). \end{aligned}$$

We require that $a_{kl} = a_{lk}$ and b_k are real–valued, $a_{kl}, a_k, a_0 \in C^\mu_b(\mathbb{R}, C(\overline{\Omega}))$, $b_k, b_0 \in C^\mu_b(\mathbb{R}, C^1(\partial\Omega))$,

$$\sum_{k,l=1}^n a_{kl}(t,x)\,\xi_k\,\xi_l \ge \eta\,|\xi|^2, \qquad \text{and} \qquad \sum_{k=1}^n b_k(t,x)\nu_k(x) \ge \beta$$

for constants $\mu \in (1/2, 1)$, $\beta, \eta > 0$ and all $\xi \in \mathbb{R}^n$, $k,l = 1, \cdots, n$, $t \in \mathbb{R}$, $x \in \overline{\Omega}$ resp. $x \in \partial\Omega$. (C^μ_b is the space of bounded, globally Hölder continuous functions.) We set $X = C(\overline{\Omega})$,

$$Z_t = \{u \in \bigcap\nolimits_{p>1} W^2_p(\Omega) : A(t,\cdot,D)u \in C(\overline{\Omega})\},$$

$A_m(t)u = A(t,\cdot,D)u$ and $B(t)u = B(t,\cdot,D)u$ for $u \in Z_t$, and $A(t) = A_m(t)|N(B(t))$, i.e.,

$$D(A(t)) = \{u \in \bigcap\nolimits_{p>1} W^2_p(\Omega) : A(t,\cdot,D)u \in C(\overline{\Omega}),\ B(t,\cdot,D)u = 0 \text{ on } \partial\Omega\},$$

for $t \in \mathbb{R}$. It is known that the operators $A(t)$, $t \in \mathbb{R}$, satisfy (2.1) and (2.2), see [4], [22], or [30, Exa.2.9]. Thus $A(\cdot)$ generates an evolution family $U(\cdot,\cdot)$ on X. Let us fix numbers $\alpha \in (1-\mu, 1/2)$ and $p > \frac{n}{2(1-\alpha)}$. Then $X_\alpha^t = C^{2\alpha}(\overline{\Omega})$ with uniformly equivalent constants due to Theorem 3.1.30 in [22]. So Sobolev's embedding theorem yields $W_p^2(\Omega) \hookrightarrow X_\alpha^t$ with a uniform constant. Standard elliptic theory tells us that for each $\varphi \in W_p^{1-1/p}(\partial\Omega)$ there is a unique $D(t)\varphi := u \in W_p^2(\Omega)$ such that

$$(\omega - A(t,\cdot,D))u = 0 \quad \text{on } \Omega, \qquad B(t,\cdot,D)u = \varphi \quad \text{on } \partial\Omega,$$

where $D(t) : W_p^{1-1/p}(\partial\Omega) \to W_p^2(\Omega)$ is bounded uniformly in $t \in \mathbb{R}$, see [6, Thm.15.2]. (The Slobodetskij spaces $W_p^s(\partial\Omega)$ are defined in e.g. [5, §7.51].) For $\varphi \in W_p^{1-1/p}(\partial\Omega)$, the properties of $D(t)$ yield

$$(\omega - A(t,\cdot,D))(D(t)\varphi - D(s)\varphi) = (A(t,\cdot,D) - A(s,\cdot,D))D(s)\varphi,$$
$$B(t,\cdot,D)(D(t)\varphi - D(s)\varphi) = (B(s,\cdot,D) - B(t,\cdot,D))D(s)\varphi,$$

so that [6, Thm.15.2]) implies that

$$\begin{aligned}
&\|D(t)\varphi - D(s)\varphi\|_{W_p^2(\Omega)} \\
&\quad \le c\big(\|(A(t,\cdot,D) - A(s,\cdot,D))D(s)\varphi\|_{L^p(\Omega)} + \|(B(s,\cdot,D) - B(t,\cdot,D))D(s)\varphi\|_{W_p^{1-1/p}(\partial\Omega)}\big) \\
&\quad \le c\,|t-s|^\mu\,\|D(s)\varphi\|_{W_p^2(\Omega)} \le c\,|t-s|^\mu\,\|\varphi\|_{W_p^{1-1/p}(\partial\Omega)}
\end{aligned}$$

for constants independent of $t, s \in \mathbb{R}$ (using [5, §7.51]). So we see that $D(\cdot)h \in E_\alpha$ if $h \in C_0(\mathbb{R}, W_p^{1-1/p}(\partial\Omega))$. Further let $g \in C_0(\mathbb{R}, X)$. We define mild solutions of (4.6) again by (4.5). (Observe that a solution $u \in C^1(\mathbb{R}, C(\overline{\Omega}))$ of (4.6) with $u(t) \in Z_t$ for $t \in \mathbb{R}$ solves (4.6) formulated on $X = L^p(\Omega)$. On this state space, (A1)–(A3) hold with $Z_t = W_p^2(\Omega)$ and $Y = W_p^{1-1/p}(\partial\Omega)$ by the above mentioned results. In this setting we have already justified the concept of mild solutions given (4.5).) We further assume that

$$a_\alpha(t,\cdot) \to a_\alpha(\pm\infty,\cdot) \quad \text{in } C(\overline{\Omega}) \qquad \text{and} \qquad b_j(t,\cdot) \to b_j(\pm\infty,\cdot) \quad \text{in } C^1(\partial\Omega)$$

as $t \to \pm\infty$, where $\alpha = (k,l)$ or $\alpha = j$ for $k, l = 1, \cdots, n$ and $j = 0, \cdots, n$. We define the sectorial operators $A_{\pm\infty}$ in the same way as $A(t)$. As in [16, Exa.5.1] one can check that (2.16) holds. Finally we assume that $i\mathbb{R} \subset \rho(A_{\pm\infty})$. (Observe that the operators $A_{\pm\infty}$ have compact resolvent so that the spectrum consists only of eigenvalues. The spectrum of $A_{\pm\infty}$ was studied in [16, Exa.5.1].) Then the Fredholm alternative Theorem 4.4 holds for mild solutions of (4.6) on $X = C(\overline{\Omega})$ for $g \in C_0(J, X)$ and $h \in C_0(\mathbb{R}, W_p^{1-1/p}(\partial\Omega))$ due to the results from Section 3.

5. Appendix: Proof of Proposition 2.6

We start with a lemma giving an additional estimate on $U_{\alpha-1}(t,s)$.

Lemma 5.1. *Assume that* (2.1) *and* (2.2) *hold. Let* $s < t \le s + t_0$, $t_0 > 0$, $1 - \nu < \theta < \mu$, *and* $1 - \mu < \overline{\alpha} < 1 - \theta$. *Then the operators* $V(t,s) := (\omega - A(t))^{-\theta} U_{\overline{\alpha}-1}(t,s)(\omega - A_{-1}(s))^\theta$ *defined on* X *belong to* $\mathcal{L}(X)$ *with norms bounded by a constant* $c(t_0,\theta)$. *We further set* $V(s,s) := I$. *Then the map* $(t,s) \mapsto V(t,s)f(s)$ *is continuous for* $t \ge s$ *and every*

$f \in \hat{E}(J)$, where $J \subset \mathbb{R}$ is a closed interval. For $1-\mu < \alpha \le 1$ the operators $U_{\alpha-1}(t,s) : X^s_{\alpha-1} \to X^t_{\alpha-1}$ are locally uniformly bounded for $s \le t \le s+t_0$,

Proof. Let $s < t \le s+t_0$, $t_0 > 0$, and $1-\nu < \theta < \mu$. By rescaling, we may assume that (2.1) and (2.2) hold for some $\omega < 0$. Then the Yosida approximations $A_n(t) = nA(t)R(n, A(t))$, $t \in \mathbb{R}$, fulfill (2.1) and (2.2) with $\omega = 0$ and possibly different, but n–independent constants, for sufficiently large $n \in \mathbb{N}$. Thus $A_n(\cdot)$ generates an evolution family $U_n(\cdot,\cdot)$ with estimates independent of n. These evolution families satisfy

$$\begin{aligned} V_n(t,s) &:= (-A_n(t))^{-\theta} U_n(t,s)(-A_n(s))^{\theta} \qquad (5.1)\\ &= e^{(t-s)A_n(s)} + [(-A_n(t))^{-\theta} - (-A_n(s))^{-\theta}](-A_n(s))^{\theta} e^{(t-s)A_n(s)} \\ &\quad + \int_s^t V_n(t,\sigma)(-A_n(\sigma))^{1-\theta} \left[(-A_n(\sigma))^{-1} - (-A_n(s))^{-1}\right] (-A_n(s))^{1+\theta} e^{(\sigma-s)A_n(s)}\, d\sigma. \end{aligned}$$

In view of the above integral equation for $V_n(t,s)$, we introduce the operators

$$\begin{aligned} a_n(t,s) &:= \left[(-A_n(t))^{-\theta} - (-A_n(s))^{-\theta}\right] (-A_n(s))^{\theta} e^{(t-s)A_n(s)} \\ k_n(t,s) &:= (-A_n(t))^{1-\theta} \left[(-A_n(t))^{-1} - (-A_n(s))^{-1}\right] (-A_n(s))^{1+\theta} e^{(t-s)A_n(s)} \end{aligned}$$

The estimates (2.6), (2.9) and (2.10) yield

$$\|a_n(t,s)\| \le c\,(t-s)^{\mu-\theta} \quad \text{and} \quad \|k_n(t,s)\| \le c\,(t-s)^{\mu-\theta-1} \qquad (5.2)$$

with constants $c = c(t_0)$ independent of n. Setting $b_n(t,s) := e^{(t-s)A_n(s)} + a_n(t,s)$, we can rewrite (5.1) as

$$V_n(t,s) = b_n(t,s) + \int_s^t V_n(t,\tau)k_n(\tau,s)d\tau =: b_n(t,s) + (V_n * k_n)(t,s).$$

Theorem II.3.2.2 and Lemma II.3.2.1 of [7] now show that

$$V_n(t,s) = b_n(t,s) + \sum_{j=1}^{\infty} (b_n * [k_n *]^j)(t,s) \quad \text{and} \quad \|V_n(t,s)\| \le c \qquad (5.3)$$

for $s \le t \le s+t_0$ and the j–times ‘convolution’ $[k_n*]^j = k_n * \cdots * k_n$, where

$$\|[k_n*]^j(t,s)\| \le c_j (t-s)^{-\alpha} \quad \text{with} \quad \sum_{j=1}^{\infty} c_j < \infty, \qquad (5.4)$$

and the constants $c = c(t_0)$ and $c_j = c_j(t_0)$ do not depend on n. It is straightforward to show that

$$\begin{aligned} \lim_{n\to\infty} a_n(t,s) &= a(t,s) := \left[(-A(t))^{-\theta} - (-A(s))^{-\theta}\right] (-A(s))^{\theta} e^{(t-s)A(s)}, \\ \lim_{n\to\infty} b_n(t,s) &= b(t,s) := e^{(t-s)A(s)} + \left[(-A(t))^{-\theta} - (-A(s))^{-\theta}\right] (-A(s))^{\theta} e^{(t-s)A(s)}, \\ \lim_{n\to\infty} k_n(t,s) &= k(t,s) := (-A(t))^{1-\theta} \left[(-A(t))^{-1} - (-A(s))^{-1}\right] (-A(s))^{1+\theta} e^{(t-s)A(s)} \end{aligned}$$

in $\mathcal{L}(X)$ locally uniformly for $t > s$, cf. [31, Prop.2.1] and use Lemmas 4.1 and 4.2 of [3]. Moreover, the limit operators satisfy estimates analogous to (5.2) and (5.4). Therefore

(5.3) implies that $V_n(t,s)$ converges in $\mathcal{L}(X)$ and locally uniformly for $t>s$ to an operator $V(t,s)$ satisfying $\|V(t,s)\| \le c(t_0)$ and

$$V(t,s) = e^{(t-s)A(s)} + a(t,s) + \int_s^t V(t,\tau)k(\tau,s)\,d\tau \tag{5.5}$$

for $s<t\le s+t_0$ and $t_0>0$. Since $U_n(t,s) \to U(t,s)$ in $\mathcal{L}(X)$ by e.g. Proposition 2.5 of [30], $V_n(t,s)x$ converges to $(\omega - A(t))^{-\theta}U(t,s)(\omega - A(s))^{\theta}x$ for $x \in D((\omega - A(s))^{\theta})$. We then deduce the first assertion from Proposition 2.1 and embedding (2.4) by approximation. Further, the third assertion follows by the reiteration (see e.g. Theorem 1.2.15 and Proposition 2.2.15 in [22]). The second assertion was shown in Proposition 2.1 for $t>s$. Let $f \in \hat{E}(J)$ and $\varepsilon > 0$. Take $g \in D(A(\cdot))$ with $\|f-g\|_\infty \le \varepsilon$. Using (5.5), we estimate

$$\begin{aligned}\|V(t,s)f(s) - f(r)\| &\le \|(e^{(t-s)A(s)} - I)f(s)\| + \|f(s)-f(r)\| + c\,(t-s)^{\mu-\theta}\\ &\le \|(e^{(t-s)A(s)} - I)g(s)\| + c\varepsilon + \|f(s)-f(r)\| + c\,(t-s)^{\mu-\theta}\,.\end{aligned}$$

This inequality shows that

$$\limsup_{(t,s)\to(r,r)} \|V(t,s)f(s) - f(r)\| \le c\varepsilon,$$

and so the last assertion is established, too. $\square$

Proof of Proposition 2.6. By rescaling, we can assume that (2.1) and (2.2) hold with $\omega = 0$. Let $1-\mu < \alpha \le 1$, $f \in E_{\alpha-1}(J)$, $0 \le \beta < \min\{\alpha,\nu\}$, $s<t$ in J, and let $u \in C(J,X)$ be a mild solution of (2.20). Formulas (2.21) and (2.12) yield

$$\begin{aligned}\|u(t)\|_\beta^t &\le \|U(t,s)u(s)\|_\beta^t + \int_s^t \|U_{\alpha-1}(t,\sigma)f(\sigma)\|_\beta^t d\sigma\\ &\le c(t-s)^{-\beta}\|u(s)\| + c\int_s^t (t-\sigma)^{\alpha-\beta-1}\|f(\sigma)\|_{\alpha-1}^s d\sigma\\ &\le c(t-s)^{-\beta}\|u(s)\| + c(t-s)^{\alpha-\beta}\|f\|_{\alpha-1},\end{aligned}$$

so that $u(t) \in X_\beta^t$. Moreover, we have

$$\frac{1}{h}\,(U(t+h,s) - U(t,s))u(s) \longrightarrow A(t)U(t,s)u(s)$$

in X as $h \to 0$. So it remains to differentiate the term

$$v(t) := \int_s^t U_{\alpha-1}(t,\sigma)f(\sigma)\,d\sigma$$

for $t \in J \setminus \inf J$. Fix θ with $\max\{1-\nu, 1-\alpha\} < \theta < \mu$ and let $h>0$. Then we can write

$$\begin{aligned}&\frac{1}{h}\,(-A(t))^{-\theta}(v(t+h) - v(t))\\ &\quad= (-A(t))^{-\theta}\frac{1}{h}\,(U(t+h,t) - I)v(t) + \frac{1}{h}\int_t^{t+h} (-A(t+h))^{-\theta}U_{\overline{\alpha}-1}(t+h,\sigma)f(\sigma)\,d\sigma\\ &\qquad+ \big((-A(t))^{-\theta} - (-A(t+h))^{-\theta}\big)\,\frac{1}{h}\int_t^{t+h} U_{\alpha-1}(t+h,\sigma)f(\sigma)\,d\sigma\\ &\quad=: S_1 + S_2 + S_3\,,\end{aligned}$$

where we take a number $\overline{\alpha} \in (1-\mu, 1-\theta)$ (thus $\overline{\alpha} < \alpha$). Since $(-A_{-1}(\cdot))^{-\theta} f(\cdot) \in \hat{E}$ by $1-\theta < \alpha$ and (2.4), Lemma 5.1 shows that

$$S_2 = \frac{1}{h} \int_t^{t+h} V(t+h,\sigma)(-A_{-1}(\sigma))^{-\theta} f(\sigma)\, d\sigma \longrightarrow (-A_{-1}(t))^{-\theta} f(t)$$

in X as $h \to 0$. Using (2.10) and (2.11), we estimate

$$\|S_3\| \le ch^{\mu-1} \int_t^{t+h} (t+h-\sigma)^{\alpha-1}\, d\sigma\, \|f\|_{\alpha-1} \le ch^{\alpha+\mu-1} \longrightarrow 0, \qquad h \to 0.$$

We note that (5.5) applied to $x \in X_t^\beta$ can be shown also for $\theta = 0$ (where $V(t,s) = U(t,s)$ and $a(t,s) = 0$) using similar methods, cf. [32, p.347]. The term S_1 can thus be transformed into

$$\begin{aligned} S_1 = {} & \frac{1}{h}\left(e^{hA(t)} - I\right)(-A(t))^{-\theta} v(t) + \frac{1}{h}\int_t^{t+h} V(t+h,\sigma)(-A(\sigma))^{1-\theta} \\ & \qquad \cdot \left[(-A(\sigma))^{-1} - (-A(t))^{-1}\right](-A(t))^{1-\gamma} e^{(\sigma-t)A(t)} (-A(t))^{\gamma} v(t)\, d\sigma \\ & + \left[(-A(t))^{-\theta} - (-A(t+h))^{-\theta}\right]\frac{1}{h}\int_t^{t+h} U_{\alpha-1}(t+h,\sigma)(-A(\sigma))^{\theta} \\ & \quad \cdot (-A(\sigma))^{1-\theta}\left[(-A(\sigma))^{-1} - (-A(t))^{-1}\right](-A(t))^{1-\gamma} e^{(\sigma-t)A(t)} (-A(t))^{\gamma} v(t)\, d\sigma \\ =: {} & S_{11} + S_{12} + S_{13}\,. \end{aligned}$$

Here we take γ with $1-\mu < 1-\theta < \gamma < \min\{\alpha,\nu\}$. Since $v(t) \in X_\gamma^t$, the embedding (2.4) yields that $(-A(t))^{-\theta} v(t) \in D(A(t))$, and hence

$$\lim_{h\to 0} \frac{1}{h}\left(e^{hA(t)} - I\right)(-A(t))^{-\theta} v(t) = A(t)(-A(t))^{-\theta} v(t)$$

in X. Lemma 5.1 and the inequalities (2.6) and (2.9) allow to estimate

$$\|S_{12}\| \le \frac{c}{h}\int_t^{t+h} (\sigma-t)^{\mu}(\sigma-t)^{\gamma-1}\, d\sigma\, \|(-A(t))^{\gamma} v(t)\| = ch^{\gamma+\mu-1}\|(-A(t))^{\gamma} v(t)\| \longrightarrow 0.$$

Finally, we deduce from (2.6), (2.8), (2.9) and (2.10) that

$$\begin{aligned} \|S_{13}\| & \le ch^{\mu-1}\int_t^{t+h} (t+h-\sigma)^{-\theta}(\sigma-t)^{\mu}(\sigma-t)^{\gamma-1}\, d\sigma\, \|(-A(t))^{\gamma} v(t)\| \\ & \le ch^{2\mu+\gamma-\theta-1} \longrightarrow 0, \qquad h \to 0. \end{aligned}$$

Therefore S_1 converges to $A(t)(-A(t))^{-\theta} v(t)$ in X. Summarizing, we have established

$$\lim_{h\to 0}(-A(t))^{-\theta}\frac{1}{h}\big(u(t+h) - u(t)\big) = (-A_{-1}(t))^{-\theta}\big(A_{-1}(t)u(t) + f(t)\big)$$

in X. By (2.4), this limit exists in $X_{-\theta}^t$, and so in $X_{\beta-1}^t$ for $0 \le \beta \le 1-\theta < \min\{\alpha,\nu\}$. $\square$

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