Gamma distributions for stationary Poisson flat processes

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Abstract

We consider a stationary Poisson process Φ of k-flats in $\mathbb{R}^d$ with intensity measure Λ and a random closed set S of k-flats depending on $F_1, \dots, F_n \in \Phi$, $x \in \mathbb{R}^d$, and Φ in a specific equivariant way. If $(F_1, \dots, F_n, x)$ is properly sampled, then $\Lambda(S)$ has a gamma distribution. This result is generalizing and unifying earlier work in [5], [9], and [13]. As a new example we will show that the volume of the fundamental region of a typical j-face of a stationary Poisson Voronoi-tessellation is conditionally gamma-distributed.

Keywords: stochastic geometry, gamma distribution, k-flat, Poisson process, Palm distribution, stopping set, Voronoi-tessellation, typical face

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1 Introduction

Since the seminal work in [5] and [9] it is known, that the (generalized) integral-geometric contents of several closed sets constructed on stationary Poisson processes of flats are (conditionally) gamma-distributed. It is also known (see [13]) that the (intensity) measure of certain stoppings sets defined on Poisson processes is conditionally gamma-distributed. These are some of the rare cases in stochastic geometry where the distribution of non-trivial geometric functionals of Poisson processes is explicitly known. The aim of this paper is to generalize these results using a unified framework combining stopping sets with Palm distributions.

We consider here a *stationary Poisson process* Φ *of* k*-flats* in $\mathbb{R}^d$, where $d \geq 1$ and $k \in \{0, \dots, d-1\}$. This is a Poisson process on the space of all k-flats, whose distribution is invariant under translation of the flats. Its distribution is determined by the intensity γ (see (2.1)) and a probability measure $\mathbb{Q}$ on the space of all k-dimensional linear subspaces of $\mathbb{R}^d$, see (2.2). The intensity measure of Φ equals $\gamma\Lambda$, where Λ is the intensity measure of a stationary Poisson process of k-flats with intensity 1. Note that Φ is *isotropic*, if and

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only if $\mathbb{Q}$ is the uniform distribution. If $k = 0$ then Φ is just a stationary Poisson process on $\mathbb{R}^d$. We refer to [11] and [10] for these and other fundamental facts from stochastic geometry.

We study a (random) set

$$S \equiv S(F_1, \dots, F_n, x, \Phi^!_{F_1,\dots,F_n}),$$

of k-flats depending on pairwise different $F_1, \dots, F_n \in \Phi$, $x \in \mathbb{R}^d$, and $\Phi^!_{F_1,\dots,F_n} := \Phi \setminus \{F_1, \dots, F_n\}$, where $n \geq 1$ is a fixed integer. We assume that $S(F_1, \dots, F_n, x, \cdot)$ is a *stopping set* for all k-flats $F_1, \dots, F_n$ and $x \in \mathbb{R}^d$, see [13] and Subsection 2.3 for the concept of a stopping set. We are also assuming that S is *equivariant* in the sense that a joint scaling or translation of all arguments leads to the same scaling or translation of the elements of S. For a momentarily fixed realization of Φ we are interested in the distribution of the intensity measure $\Lambda(S)$ of S under the following sampling scheme for the tuple $(F_1, \dots, F_n, x)$. First we select only those tuples satisfying a given condition. This condition has to be invariant with respect to scaling and translation (including Φ) and to satisfy some measurabililty condition with respect to $S(F_1, \dots, F_n, x, \cdot)$ for all $(F_1, \dots, F_n, x)$. Any n-tuple $(F_1, \dots, F_n)$ for which there is a $x \in B$ satisfying the condition is taken with the same deterministic weight. Here $B \subset \mathbb{R}^d$ is a “large” compact set. Once $F_1, \dots, F_n$ are selected, x is distributed in

$$Z(F_1, \dots, F_n) \cap \{y \in B : \mathrm{card}(\Phi^!_{F_1,\dots,F_n} \cap S(F_1, \dots, F_n, y, \Phi^!_{F_1,\dots,F_n})) = m\}$$

according to j-dimensional Hausdorff measure, where $Z(F_1, \dots, F_n)$ is a j-dimensional equivariant closed subset of $\mathbb{R}^d$ and $j \geq 0$ and $m \geq 0$ are given integers. More generally we can allow here for weights that depend on $(F_1, \dots, F_n, x, \Phi^!_{F_1,\dots,F_n})$ in an invariant way. We assume that the total measure of all samples $(F_1, \dots, F_n, x)$ is finite. For any $t \in [0, \infty]$ we can then study the measure $M(t)$ of all samples such that $\Lambda(S(F_1, \dots, F_n, x, \Phi^!_{F_1,\dots,F_n}))$ is smaller than t. If $M(\infty)$ has even a finite mean, then the (random!) distribution function $M(t)/M(\infty)$ converges almost surely, as B expands to $\mathbb{R}^d$, to $\Gamma(m+n-(d-j)/(d-k), \gamma)$, i.e. a gamma distribution with shape parameter $m+n-(d-j)/(d-k)$ and scale parameter γ. The result can also be formulated in case $n = 0$. Then $Z = \mathbb{R}^d$ and $j = d$. We will also show that quantities depending on $(F_1, \dots, F_n, x, \Phi^!_{F_1,\dots,F_n})$ in a translation- and scale-invariant way are (asymptotically) independent of $\Lambda(S)$.

If S does not depend on the last argument, the above results provide slight generalizations of Miles' complementary theorem in [5] (see also [9]) in case $j = 0$ and of the results on subprocesses in [9] in case $j = d$. The case where S does not depend on the first n arguments was essentially treated in [13]. In all other cases the result seems to be new. Moreover, we are not aware of gamma distributions occuring in stationary Poisson flat processes that are not explained by our theorem.

The structure of this paper is as follows. Section 2 contains some technical prerequisites that are required for a sound mathematical treatment of our topic. Section 3 provides the exact formulation of our results (Theorem 3.1 and Theorem 3.3) and their proofs. The assumptions will be more general than described above. Instead of relying on ergodicity arguments indicated above, we will use the more elegant (standard) approach via Palm probabilities. In Section 4 we will discuss (and generalize) the classical special cases mentioned above. In Section 5 we will present new specific examples in case $k = 0$, i.e.

in case Φ is a stationary Poisson process. We will consider area-biased and area-unbiased versions of the *typical j-face* of the *Voronoi-tessellation* based on Φ. The volume of the *fundamental region* of this face has a conditional gamma distribution given the number of Φ-points in this region. The area-unbiased version for $j = d$ can be found in [7] and [9], while the area-unbiased version for $j = 1$ can be found in [1]. The classical case $j = 0$ has been treated in [6]. All other cases are new. The appendix contains some material on stopping sets that cannot be found elsewhere.

2 Preliminaries

2.1 Geometrical preliminaries

We work in Euclidean d-space $\mathbb{R}^d, d \geq 1$, equipped with the Euclidean norm $|\cdot|$ and the Borel σ-field $\mathcal{B}^d$. The closed ball with radius $r \geq 0$ centred at $x \in \mathbb{R}^d$ is denoted by $B(x, r)$ while $B^0(x, r)$ denotes the corresponding open ball. The unit ball $B(0, 1)$ centred at the origin $0 \in \mathbb{R}^d$ is denoted by B^d. We write $\mathcal{H}^k$ for the k-dimensional Hausdorff measure in $\mathbb{R}^d$ and let $\kappa_d := \mathcal{H}^d(B^d)$.

The system of all closed subsets of $\mathbb{R}^d$ is denoted by $\mathcal{F}$. For any $K \subset \mathbb{R}^d$ we write

$$\mathcal{F}_K := \{F \in \mathcal{F} : F \cap K \neq \emptyset\}.$$

We make $\mathcal{F}$ a measurable space by introducing the smallest σ-field containing $\mathcal{F}_K$ for all compact K. This is actually the Borel σ-field associated with the topology of closed convergence on $\mathcal{F}$, see [10] for more details. If $K \subset \mathbb{R}^d$ is compact, then $\mathcal{F}_K$ is compact with respect to this topology. Conversely, any compact subset of $\mathcal{F} \setminus \{\emptyset\}$ is contained in $\mathcal{F}_K$ for some compact K. Let $k \in \{0, \dots, d-1\}$. A *k-flat* is a k-dimensional affine subspace F of $\mathbb{R}^d$. The space of all such flats is denoted by $\mathcal{E}^k$. Any $F \in \mathcal{E}^k$ can be uniquely written as $F = L + x$, where L is an element of the space $\mathcal{L}^k$ of all k-dimensional linear subspaces of $\mathbb{R}^d$ and x is in the orthogonal complement $L^\perp$ of L. Therefore $\mathcal{E}^k$ can also be considered as a subset of $\mathbb{R}^{d+d^2}$. The sets $\mathcal{E}^k \cup \{\emptyset\}$ and $\mathcal{L}^k$ are compact subsets of $\mathcal{F}$. Therefore $\mathcal{E}^k$ is a locally compact, second countable Hausdorff space. The space of all closed subsets of $\mathcal{E}^k$ is denoted by $\mathcal{F}(\mathcal{E}^k)$. Again this space can be equipped with the topology of closed convergence. In particular, $\mathcal{F}(\mathcal{E}^k)$ becomes a measurable space in its own right.

2.2 Stationary Poisson processes of flats

We let $\mathbf{N}^k$ denote the space of all locally finite subsets of $\mathcal{E}^k$. Any $\varphi \in \mathbf{N}^k$ is identified with the counting measure $A \mapsto \varphi(A) := \operatorname{card} \varphi \cap A$ on $\mathcal{E}^k$. The σ-field $\mathcal{N}^k$ is the smallest σ-field on $\mathbf{N}^k$ making the mappings $\varphi \mapsto \varphi(A)$ measurable for all Borel sets $A \subset \mathcal{E}^k$. A *point process* of k-flats is a point process on $\mathcal{E}^k$, i.e. a measurable mapping Φ from some abstract probability space $(\Omega, \mathcal{A}, \mathbb{P})$ into $\mathbf{N}^k$. It is called *stationary* if $\Phi + x$ has the same distribution as Φ for all $x \in \mathbb{R}^d$. Here $\varphi + x := \{F + x : F \in \varphi\}$ for $\varphi \in \mathbf{N}^k$. Consider a stationary point process of k-flats and assume that its *intensity*

$$\gamma := \frac{1}{\kappa_{d-k}} \mathbb{E} \operatorname{card}\{F \in \Phi : F \cap B^d \neq \emptyset\} \tag{2.1}$$

is finite and positive. Then the *intensity measure* of Φ can be written as

$$\mathbb{E}\Phi(A) = \gamma\Lambda(A) \tag{2.2}$$

where

$$\Lambda(\cdot) := \int_{\mathcal{L}^k} \int_{F^\perp} \mathbf{1}\{F + x \in \cdot\} \mathcal{H}^{d-k}(dx) \mathbb{Q}(dF) \tag{2.3}$$

for some uniquely determined probability measure $\mathbb{Q}$ on $\mathcal{L}^k$, see again [10]. In case $k = 0$ we may identify $\mathcal{E}^k$ with $\mathbb{R}^d$, and Λ becomes Lebesgue measure. A Poisson process Φ on $\mathcal{E}^k$ (see e.g. [3] for a definition of general Poisson processes) is stationary if and only if its intensity measure is a multiple of (2.3) for some $\mathbb{Q}$. Such stationary Poisson processes of k-flats are the subject of this paper. If $k = 0$ then Φ is just a stationary Poisson process on $\mathbb{R}^d$.

For any closed set $K \subset \mathbb{R}^d$ we define

$$\mu_\Lambda(K) := \Lambda(\{F \in \mathcal{E}^k : F \cap K \neq \emptyset\}). \tag{2.4}$$

In case $k = 0$ this is just the volume of K. If $\mathbb{Q}$ is the uniform distribution and K has some further properties, then this is an *integral-geometric* contents of K. If, for instance, K is a compact and convex set, then $\mu_\Lambda(K)$ is proportional to the surface area of K in case $k = 1$ and proportional to the *mean breadth* in case $k = d - 1$, see [11] and [10] for more details. For general $\mathbb{Q}$ we may interpret (2.4) as a generalized integral-geometric contents of K.

2.3 Stopping sets

Let $A \subset \mathcal{E}^k$ be measurable and let $\mathcal{N}^k_A$ be the σ-field generated by the mapping $\varphi \mapsto \varphi \cap A$. A *stopping set* (defined on $\mathbf{N}^k$) is a mapping $T : \mathbf{N}^k \to \mathcal{F}(\mathcal{E}^k)$ such that

$$\{\varphi \in \mathbf{N}^k : T(\varphi) \subset K\} \in \mathcal{N}^k_K, \quad K \in \mathcal{F}(\mathcal{E}^k). \tag{2.5}$$

This is essentially the definition from [13]. Note however, that we are not restricting T to be compact. Moreover, due to the special properties of the domain $\mathbf{N}^k$ we can derive (in the appendix) some more specific properties of a stopping set. The *stopping σ-field* associated with a stopping set T is defined by

$$\mathcal{N}^k_T := \{A \in \mathcal{N}^k : A \cap \{T \subset K\} \in \mathcal{N}^k_K \text{ for all } K \in \mathcal{F}(\mathcal{E}^k)\}. \tag{2.6}$$

For the main purpose of this paper it would be sufficient to concentrate on a generic type of a stopping set constructed as follows. For any closed set $K \subset \mathbb{R}^d$ we define a measurable mapping $\pi'_K : \mathbf{N}^k \to \mathbf{N}^k$ by

$$\pi'_K(\varphi) := \{F \in \varphi : F \cap K \neq \emptyset\}. \tag{2.7}$$

If $T' : \mathbf{N}^k \to \mathcal{F}$ is measurable, then $\mathcal{E}_{T'} : \mathbf{N}^k \to \mathcal{F}(\mathcal{E}^k)$ defined by

$$\mathcal{E}_{T'}(\varphi) := \{F \in \varphi : F \cap T'(\varphi) \neq \emptyset\} \tag{2.8}$$

is also measurable. It turns out that $\mathcal{E}_{T'}$ is a stopping set if and only if T' has the following natural stopping set property.

Proposition 2.1. *Let $T' : \mathbf{N}^k \to \mathcal{F}$ be measurable. Then $\mathcal{E}_{T'}$ is a stopping set iff*

$$\{T' \subset K\} \in {\pi'_K}^{-1}(\mathcal{N}^k), \quad K \in \mathcal{F}. \tag{2.9}$$

In this case

$$\mathcal{N}^k_{\mathcal{E}_{T'}} = \{A \in \mathcal{N}^k : A \cap \{T' \subset K\} \in {\pi'_K}^{-1}(\mathcal{N}^k) \textit{ for all } K \in \mathcal{F}\}. \tag{2.10}$$

The proof of this result and further details on a more general concept of a stopping set are given in the appendix. If $k = 0$ then $\mathcal{E}_{T'} = T'$. In this case the literature has numerous examples of stopping sets (see e.g. [13]). Here is a simple example that applies for any k.

Example 2.2. Let $i \in \mathbb{N}$ and define a measurable mapping $\tau_i : \mathbf{N}^k \to [0, \infty)$ by

$$\tau_i(\varphi) := \inf\{r \geq 0 : \text{card}\{F \in \varphi : F \cap rB^d \neq \emptyset\} \geq i\}.$$

It is easy to show that $T'(\varphi) := \tau_i(\varphi)B^d$ (i.e. $T'(\varphi) := \mathbb{R}^d$ in case $\tau_i(\varphi) = \infty$) gives a measurable mapping saisfying (2.9).

2.4 Stationary random measures and mark distributions

We work on the probability space $(\mathbf{N}^k, \mathcal{N}^k, \mathbb{P})$, where $\mathbb{P}$ is the distribution of a stationary Poisson process of k-flats with intensity $\gamma > 0$. The identity on $\mathbf{N}^k$ is denoted by Φ. A *random measure* M on $\mathbb{R}^d$ (see e.g. [3]) is a random variable taking its values in the space $\mathbf{M}$ of all locally finite measures α on $\mathbb{R}^d$ equipped with the σ-field $\mathcal{M}$ generated by the mappings $\alpha \mapsto \alpha(B)$, $B \in \mathcal{B}^d$. We write $M(\varphi, B) := M(\varphi)(B)$. Note that $\mathbf{N}^0$ is a measurable subset of $\mathbf{M}$. An element of $\mathbf{N}^0$ is also called *simple* counting measure on $\mathbb{R}^d$. A (simple) *point process* on $\mathbb{R}^d$ is a random measure M satisfying $\mathbb{P}(M(\Phi) \in \mathbf{N}^0) = 1$. A random measure M on $\mathbb{R}^d$ is called *stationary* if

$$M(\varphi, B + x) = M(\varphi - x, B), \quad \varphi \in \mathbf{N}^k, x \in \mathbb{R}^d, B \in \mathcal{B}^d. \tag{2.11}$$

If M is a stationary random measure then the distribution of $M(\cdot + x)$ is the same for any $x \in \mathbb{R}^d$.

Let $(\mathbf{X}, \mathcal{X})$ be a measurable space and let $M_{\mathbf{X}}$ denote the space of all measures α on $\mathbb{R}^d \times \mathbf{X}$ such that $\alpha(\cdot \times \mathbf{X})$ is locally finite. The σ-field $\mathcal{M}_{\mathbf{X}}$ on $M_{\mathbf{X}}$ is defined exactly as above. A *marked random measure* M on $\mathbb{R}^d$ with mark space $\mathbf{X}$ is a $M_{\mathbf{X}}$-valued random variable. It is called *stationary* if $M(\cdot \times C)$ is stationary for all $C \in \mathcal{X}$. If M is such a stationary random measure, then $\mathbb{E}M(B \times C) = \mathcal{H}^d(B)\mathbb{V}'(C)$ for some measure $\mathbb{V}'$ on $\mathbf{X}$. In case the *intensity* $\lambda_M := \mathbb{V}'(\mathbf{X}) = \mathbb{E}M([0,1]^d \times \mathbf{X})$ of M is positive and finite we may normalize $\mathbb{V}'$ to obtain the (Palm) *mark distribution* $\mathbb{V}$ of M. We then have the *refined Campbell theorem*

$$\mathbb{E}\int f(x,y)M(\Phi, d(x,y)) = \lambda_M \iint f(x,y)dx\mathbb{V}(dy) \tag{2.12}$$

for all measurable $f : \mathbb{R}^d \times \mathbf{X} \to [0, \infty)$, where dx means integration with respect to Lebesgue measure $\mathcal{H}^d$. It is common to call $\mathbb{V}$ the distribution of the *typical mark* of M.

Let N be stationary random measure on $\mathbb{R}^d$. Then

$$M(\varphi, \cdot) := \int \mathbf{1}\{(x, \varphi - x) \in \cdot\} N(\varphi, dx)$$

is a stationary marked random measure with mark space $\mathbf{N}^k$. If the intensity $\lambda_N := \mathbb{E}N([0,1]^d)$ of N is positive and finite, then the mark distribution of this M is the *Palm probability measure* $\mathbb{P}_N^0$ of N. The refined Campbell theorem takes the form

$$\mathbb{E} \int f(\Phi - x, x) N(\Phi, dx) = \lambda_N \iint f(\varphi, x) dx \mathbb{P}_N^0(d\varphi) \tag{2.13}$$

for all measurable $f : \mathbf{N}^k \times \mathbb{R}^d \to [0, \infty)$. The measure $\mathbb{P}_N^0$ is describing the statistical behaviour of Φ as seen from a *typical point* of N.

3 Formulation and proof of the main result

For the remainder of the paper we let $(\mathbf{N}^k, \mathcal{N}^k, \mathbb{P})$ be the underlying probability space, see Subsection 2.4.

We consider an integer $n \geq 0$, and measurable mappings $S : (\mathcal{E}^k)^n \times \mathbb{R}^d \times \mathbf{N}^k \to \mathcal{F}(\mathcal{E}^k)$, $Z : (\mathcal{E}^k)^n \to \mathcal{F}$, and $R : (\mathcal{E}^k)^n \times \mathbb{R}^d \times \mathbf{N}^k \to [0, \infty)$ with the following properties. The mapping $S(F_1, \dots, F_n, x, \cdot)$ is assumed to be a stopping set for all $F_1, \dots, F_n \in \mathbf{N}^k$ and $x \in \mathbb{R}^d$. Accordingly we define $\mathcal{N}^k_{F_1,\dots,F_n,x}$ as the associated stopping σ-field. Furthermore we assume for all $F_1, \dots, F_n \in \mathbf{N}^k$, $x, y \in \mathbb{R}^d$, and $\varphi \in \mathbf{N}^k$ the *equivariance* property

$$S(F_1 + y, \dots, F_n + y, x + y, \varphi + y) = S(F_1, \dots, F_n, x, \varphi) + y. \tag{3.1}$$

Here and later we use for $A \subset \mathcal{E}^k$, $y \in \mathbb{R}^d$ and $c > 0$ the notation $A + y := \{F + y : F \in A\}$ and $cA := \{cF : F \in A\}$. If $n = 0$, Z is assumed to equal $\mathbb{R}^d$. Otherwise Z is assumed to satisfy the equivariance property

$$Z(F_1 + y, \dots, F_n + y) = Z(F_1, \dots, F_n) + y. \tag{3.2}$$

Furthermore

$$B \mapsto \mathcal{H}^j(Z(F_1, \dots, F_n) \cap B), \quad B \in \mathcal{B}^d, \tag{3.3}$$

is assumed to be a locally finite measure. This means that $j = d$ in case $n = 0$. Note that the assumption on (3.3) is no further restriction in case $j = d$. The mapping R has to satisfy the invariance property

$$R(F_1 + y, \dots, F_n + y, x + y, \varphi + y) = R(F_1, \dots, F_n, x, \varphi), \tag{3.4}$$

and $R(F_1, \dots, F_n, x, \cdot)$ is assumed to be $\mathcal{N}^k_{F_1,\dots,F_n,x}$-measurable for all $F_1, \dots, F_n \in \mathcal{E}^k$ and $x \in \mathbb{R}^d$. Let $m \geq 0$ and define

$$R_m(F_1, \dots, F_n, x, \varphi) := \mathbf{1}\{\varphi(S(F_1, \dots, F_n, x, \varphi)) = m\} R(F_1, \dots, F_n, x, \varphi) \tag{3.5}$$

for $F_1,\dots,F_n\in\mathcal{E}^k$, $x\in\mathbb{R}^d$, and $\varphi\in\mathbf{N}^k$. We assume the scale-invariance

$$\begin{aligned}&\mathbb{E}R_m(cF_1,\dots,cF_n,cx,c\Phi)\mathbf{1}\{cx\in Z(cF_1,\dots,cF_n)\} \qquad (3.6)\\ &\quad=\mathbb{E}R_m(F_1,\dots,F_n,x,\Phi)\mathbf{1}\{x\in Z(F_1,\dots,F_n)\},\quad F_1,\dots,F_n\in\mathcal{E}^k,x\in\mathbb{R}^d,c>0.\end{aligned}$$

This holds if S and Z are equivariant under scaling and R is invariant under scaling. Recalling (2.3), we now define for any $\varphi\in\mathbf{N}^k$ a measure $M(\varphi,\cdot)$ on $\mathbb{R}^d\times[0,\infty]$ by

$$\begin{aligned}M(\varphi,B\times C):=\iint&\mathbf{1}\{x\in Z(F_1,\dots,F_n)\cap B\}\mathbf{1}\{\Lambda(S(F_1,\dots,F_n,x,\varphi^!_{F_1,\dots,F_n}))\in C\}\\ &R_m(F_1,\dots,F_n,x,\varphi^!_{F_1,\dots,F_n})\mathcal{H}^j(dx)\varphi^{(n)}(d(F_1,\dots,F_n)),\qquad (3.7)\end{aligned}$$

where $\varphi^!_{F_1,\dots,F_n}:=\varphi\setminus\{F_1,\dots,F_n\}$ and $\varphi^{(n)}$ is the measure on $(\mathcal{E}^k)^n$ such that integration with respect to this measure is just summation over all tuples $(F_1,\dots,F_n)\in\varphi^n$ with pairwise different entries. In case $n=0$ this has to be interpreted as

$$M(\varphi,B\times C):=\int_B\mathbf{1}\{\Lambda(S(x,\varphi))\in C\}R_m(x,\varphi)dx. \qquad (3.8)$$

We asume that $M(\varphi,\cdot\times[0,\infty])$ is locally finite and prove below that $M\equiv M(\cdot,\cdot)$ is then actually a marked random measure. The equivariance and invariance assumptions (3.1), (3.2), (3.4) easily imply that M is stationary in the sense of Subsection 2.4.

Theorem 3.1. *Assume that M defined by* (3.7) *has a positive and finite intensity. Then the mark distribution of M is $\Gamma(m+n-(d-j)/(d-k),\gamma)$, where $j=d$ in case $n=0$.*

Remark 3.2. Let $\tilde S:(\mathcal{E}^k)^n\times\mathbb{R}^d\times\mathbf{N}^k\to\mathcal{F}$ be measurable and define $S:=\mathcal{E}_{\tilde S}$ as in (2.8). Equivariance of S under translation is then equivalent to the corresponding equivariance of $\tilde S$. A similar statement applies to equivariance (resp. invariance) of S under scaling. The random measure M takes the form

$$\begin{aligned}M(\varphi,B\times C):=\iint&\mathbf{1}\{x\in Z(F_1,\dots,F_n)\cap B\}\mathbf{1}\{\mu_\Lambda(\tilde S(F_1,\dots,F_n,x,\varphi^!_{F_1,\dots,F_n}))\in C\}\\ &R_m(F_1,\dots,F_n,x,\varphi^!_{F_1,\dots,F_n})\mathcal{H}^j(dx)\varphi^{(n)}(d(F_1,\dots,F_n)),\end{aligned}$$

where the generalized integral-geometric contents μ_Λ is defined by (2.4). By Proposition 2.1 the stopping set property of S amounts to assuming that $\tilde S(F_1,\dots,F_n,x,\cdot)$ has the property (2.9) for all $F_1,\dots,F_n\in\mathbf{N}^k$ and $x\in\mathbb{R}^d$.

To state a second result we consider a measurable mapping G on $(\mathcal{E}^k)^n\times\mathbf{N}^k$ taking values in some measurable space $(\mathbf{X},\mathcal{X})$. We assume that G is invariant under scaling. For any $\varphi\in\mathbf{N}^k$, $B\in\mathcal{B}^d$ and measurable $C\subset[0,\infty]\times\mathbf{X}$ we define

$$\begin{aligned}&M_G(\varphi,B\times C) \qquad (3.9)\\ &:=\iint\mathbf{1}\{(\Lambda(S(F_1,\dots,F_n,x,\varphi^!_{F_1,\dots,F_n})),G(F_1-x,\dots,F_n-x,\varphi^!_{F_1,\dots,F_n}-x))\in C\}\\ &\qquad\mathbf{1}\{x\in Z(F_1,\dots,F_n)\cap B\}R_m(F_1,\dots,F_n,x,\varphi^!_{F_1,\dots,F_n})\mathcal{H}^j(dx)\varphi^{(n)}(d(F_1,\dots,F_n)),\end{aligned}$$

with the obvious interpretation in case $n = 0$, cf. (3.8). Again we obtain a stationary marked random measure M_G, this time with mark space $[0, \infty] \times \mathbf{X}$. Note the equality $M(B \times C) = M_G(B \times (C \times \mathbf{X}))$. The following result generalizes the independence assertions of Theorem 3 and 4 in [9].

Theorem 3.3. *Under the hypothesis of Theorem 3.1 the mark distribution of M_G is a product of a gamma distribution and a distribution on $\mathbf{X}$.*

In proving our results we extend and unify ideas from [9] and [13]. Moreover, we will use some specific properties of stopping sets defined on $\mathbf{N}^k$ which are listed in the appendix.

First we recall that the identity Φ on $\mathbf{N}^k$ is a Poisson process of k-flats with intensity measure $\gamma\Lambda$. For any $\rho > 0$ we let $\mathbb{P}_\rho$ denote the distribution of a Poisson process of k-flats with intensity measure $\rho\Lambda$. The expectation w.r.t. $\mathbb{P}_\rho$ is denoted by $\mathbb{E}_\rho$. We have the useful scaling property

$$\mathbb{P}_\rho = \mathbb{P}_1(\rho^{-1/(d-k)}\Phi \in \cdot). \tag{3.10}$$

Indeed, by the well-known mapping theorem $\rho^{-1/(d-k)}\Phi$ is a Poisson process under $\mathbb{P}_1$ and an easy computation shows that its intensity measure is given by $\rho\Lambda$. Note that $\mathbb{P} = \mathbb{P}_\gamma$. The following result is an analogon of Theorem 1 in [9] and equation (9) in [13].

Proposition 3.4. *Let $T : \mathbf{N}^k \to \mathcal{F}(\mathcal{E}^k)$ be a stopping set and $\rho > 0$. Then we have for all measurable $g : \mathbf{N}^k \to [0, \infty)$ that*

$$\mathbb{E}_\rho \mathbf{1}\{\Phi(T) = m\} g(\Phi \cap T) = \frac{\rho^m}{\gamma^m} \mathbb{E}\Big[\exp((\gamma - \rho)\Lambda(T))\mathbf{1}\{\Phi(T) = m\} g(\Phi \cap T)\Big].$$

Proof. Assume $m \geq 1$ and let $f : \mathbf{N}^k \times \mathcal{F}(\mathcal{E}^k) \to [0, \infty)$ be measurable. Using Lemma 6.4 along with an iterated version of Mecke's Satz 3.1 in [4] we get

$$\mathbb{E} f(\Phi \cap T, T)\mathbf{1}\{\Phi(T) = m\} = \frac{\gamma^m}{m!} \int f(\{F_1, \dots, F_m\}, T(\{F_1, \dots, F_m\})) \tag{3.11}$$

$$\exp(-\gamma\Lambda(T(\{F_1, \dots, F_m\})))\mathbf{1}\{F_1, \dots, F_m \in T(\{F_1, \dots, F_m\})\}\Lambda^m(d(F_1, \dots, F_m)).$$

In case $m = 0$ we have instead

$$\mathbb{E} f(\Phi \cap T, T)\mathbf{1}\{\Phi(T) = 0\} = f(\emptyset, T(\emptyset)) \exp(-\gamma\Lambda(T(\emptyset))). \tag{3.12}$$

Applying (3.11) with ρ instead of γ yields

$$\mathbb{E}_\rho \mathbf{1}\{\Phi(T) = m\} g(\Phi \cap T) = \frac{\rho^m}{\gamma^m}\frac{\gamma^m}{m!} \int g(\{F_1, \dots, F_m\}) \exp(-\gamma\Lambda(T(\{F_1, \dots, F_m\})))$$

$$\mathbf{1}\{\Lambda(T(\{F_1, \dots, F_m\})) < \infty\} \exp(-(\rho - \gamma)\Lambda(T(\{F_1, \dots, F_m\})))$$

$$\mathbf{1}\{F_1, \dots, F_m \in T(\{F_1, \dots, F_m\})\}\Lambda^m(d(F_1, \dots, F_m)).$$

Another application of (3.11) gives

$$\mathbb{E}_\rho \mathbf{1}\{\Phi(T) = m\} g(\Phi \cap T)$$

$$= \frac{\rho^m}{\gamma^m} \mathbb{E}\big[\mathbf{1}\{\Phi(T) = m\} g(\Phi \cap T)\mathbf{1}\{\Lambda(T) < \infty\} \exp(-(\rho - \gamma)\Lambda(T))\big].$$

As (3.11) does also apply that

$$\mathbb{P}(\Lambda(T) = \infty, \Phi(T) = m) = 0,$$

we obtain the assertion for $m \geq 1$. In case $m = 0$ the assertion follows in the same way from (3.12). □

Proof of Theorem 3.1. We prove the theorem in case $n \geq 1$. The proof in case $n = 0$ is similar but simpler. First we have to show that $M(\cdot, B \times C)$ is measurable for all Borel sets $B \subset \mathbb{R}^d$ and $C \subset [0, \infty]$. As the measurability of $(F_1, \ldots, F_n, \varphi) \mapsto \varphi^!_{F_1,\ldots,F_n}$ can easily be established, it follows from Fubini's theorem that $\Lambda(S(F_1, \ldots, F_n, x, \varphi^!_{F_1,\ldots,F_n}))$ is a measurable function of $(F_1, \ldots, F_n, x, \varphi)$. Consider the measurable function

$$\tilde{R}(F_1, \ldots, F_n, x, \varphi) := R_m(F_1, \ldots, F_n, x, \varphi^!_{F_1,\ldots,F_n})\mathbf{1}\{\Lambda(S(F_1, \ldots, F_n, x, \varphi^!_{F_1,\ldots,F_n})) \in C\}.$$

Since the measure (3.3) is assumed to be locally finite, we can apply Corollary 2.1.4 in [12] to obtain that

$$h(F_1, \ldots, F_n, \varphi) := \int \mathbf{1}\{x \in Z(F_1, \ldots, F_n) \cap B\}\tilde{R}(F_1, \ldots, F_n, x, \varphi)\mathcal{H}^j(dx)$$

is measurable. Therefore

$$M(\varphi, B \times C) = \int h(F_1, \ldots, F_n, \varphi)\varphi^{(n)}(d(F_1, \ldots, F_n))$$

is a measurable function of φ.

Let $\mathbb{V}$ be the mark distribution of M. Take $s < \gamma$ and define $\rho := (\gamma - s)/\gamma$ and $C_\rho := \rho^{-1/(d-k)}[0, 1]^d$. Let

$$L(s) := \int \exp[st]\mathbb{V}(dt)$$

be the Laplace-transform of $\mathbb{V}$ evaluated at s. From the definition (3.7) of M and the refined Campbell theorem (2.12) we have that

$$\lambda_M L(s) = \rho^{d/(d-k)}\mathbb{E}\iint \exp[s\Lambda(S(F_1, \ldots, F_n, x, \Phi^!_{F_1,\ldots,F_n}))]R_m(F_1, \ldots, F_n, x, \Phi^!_{F_1,\ldots,F_n})$$
$$\mathbf{1}\{x \in Z(F_1, \ldots, F_n) \cap C_\rho\}\mathcal{H}^j(dx)\Phi^{(n)}(d(F_1, \ldots, F_n)).$$

From the multivariate version of Mecke's formula we obtain that

$$\lambda_M L(s) = \rho^{d/(d-k)}\gamma^n\mathbb{E}\iint \exp[s\Lambda(S(F_1, \ldots, F_n, x, \Phi))]R_m(F_1, \ldots, F_n, x, \Phi)$$
$$\mathbf{1}\{x \in Z(F_1, \ldots, F_n) \cap C_\rho\}\mathcal{H}^j(dx)\Lambda^n(d(F_1, \ldots, F_n)).$$

Since the measure (3.3) is σ-finite, we can apply Fubini's theorem to obtain that

$$\lambda_M L(s) = \rho^{d/(d-k)}\gamma^n \iint \mathbb{E}\Big[\exp[s\Lambda(S(F_1, \ldots, F_n, x, \Phi))]R_m(F_1, \ldots, F_n, x, \Phi)\Big]$$
$$\mathbf{1}\{x \in Z(F_1, \ldots, F_n) \cap C_\rho\}\mathcal{H}^j(dx)\Lambda^n(d(F_1, \ldots, F_n)). \quad (3.13)$$

We have assumed that $S(F_1,\dots,F_n,x,\cdot)$ is a stopping set for all $F_1,\dots,F_n\in\mathcal{E}^k$ and $x\in\mathbb{R}^d$. Moreover, we have from Lemma 6.2 (i) that

$$R(F_1,\dots,F_n,x,\Phi)=R(F_1,\dots,F_n,x,\Phi\cap S(F_1,\dots,F_n,x,\Phi))$$

since $R(F_1,\dots,F_n,x,\cdot)$ is assumed to be $\mathcal{N}^k_{F_1,\dots,F_n,x}$-measurable. Taking into account the definition (3.5) of R_m, we obtain from Proposition 3.4 and (3.13) that

$$\begin{aligned}\lambda_M L(s)=&\rho^{d/(d-k)}\gamma^n\frac{\gamma^m}{\rho^m\gamma^m}\iint\mathbb{E}_{\rho\gamma}\Big[\exp[(\rho\gamma-\gamma+s)\Lambda(S(F_1,\dots,F_n,x,\Phi))]\\&R_m(F_1,\dots,F_n,x,\Phi)\Big]\mathbf{1}\{x\in Z(F_1,\dots,F_n)\cap C_\rho\}\mathcal{H}^j(dx)\Lambda^n(d(F_1,\dots,F_n)).\end{aligned}$$

Since $\rho\gamma=\gamma-s$ this simplifies to

$$\begin{aligned}\lambda_M L(s)=\rho^{d/(d-k)}\frac{\gamma^n}{\rho^m}\iint&\mathbb{E}_{\rho\gamma}[R_m(F_1,\dots,F_n,x,\Phi)]\mathbf{1}\{x\in Z(F_1,\dots,F_n)\cap C_\rho\}\\&\mathcal{H}^j(dx)\Lambda^n(d(F_1,\dots,F_n)).\end{aligned}$$

The scaling property (3.10) implies that $\mathbb{P}_{\rho\gamma}=\mathbb{P}(\rho^{-1/(d-k)}\Phi\in\cdot)$. Using the scale invariance (3.6), we arrive at

$$\begin{aligned}\lambda_M L(s)=&\rho^{d/(d-k)}\frac{\gamma^n}{\rho^m}\iint\mathbb{E}[R_m(\rho^{1/(d-k)}F_1,\dots,\rho^{1/(d-k)}F_n,\rho^{1/(d-k)}x,\Phi)]\\&\mathbf{1}\{\rho^{1/(d-k)}x\in Z(\rho^{1/(d-k)}F_1,\dots,\rho^{1/(d-k)}F_n)\cap[0,1]^d\}\mathcal{H}^j(dx)\Lambda^n(d(F_1,\dots,F_n)).\end{aligned}$$

Recalling the definition (2.3) of Λ, this means that

$$\begin{aligned}\lambda_M L(s)=&\rho^{d/(d-k)}\frac{\gamma^n}{\rho^m}\iiint\mathbb{E}[R_m(F_1+\rho^{1/(d-k)}x_1,\dots,F_n+\rho^{1/(d-k)}x_n,\rho^{1/(d-k)}x,\Phi)]\\&\mathbf{1}\{\rho^{1/(d-k)}x\in Z(F_1+\rho^{1/(d-k)}x_1,\dots,F_n+\rho^{1/(d-k)}x_n)\cap[0,1]^d\}\\&\mathbf{1}\{x_1\in F_1^\perp,\dots,x_n\in F_n^\perp\}\mathcal{H}^j(dx)(\mathcal{H}^{d-k})^n(d(x_1,\dots,x_n))\mathbb{Q}^n(d(F_1,\dots,F_n)).\end{aligned}$$

For any fixed linear subspaces $F_1,\dots,F_n$ the transformation

$$(y_1,\dots,y_n,y):=\rho^{1/(d-k)}(x_1,\dots,x_n,x)$$

has the Jacobian $\rho^{-(n(d-k)+j)/(d-k)}$. Therefore

$$\lambda_M L(s)=\rho^{d/(d-k)}\gamma^n\rho^{-(m+n+j/(d-k))}a,$$

where

$$a:=\iint\mathbb{E}[R_m(F_1,\dots,F_n,y,\Phi)]\mathbf{1}\{y\in Z(F_1,\dots,F_n)\cap[0,1]^d\}\mathcal{H}^j(dy)\Lambda^n(d(F_1,\dots,F_n)).$$

Putting $s=0$ yields $\rho=1$ and therefore $\lambda_M=\gamma^n a$. It follows that

$$L(s)=\rho^{-(m+n+j/(d-k))+d/(d-k)}=\left(\frac{\gamma}{\gamma-s}\right)^{m+n-(d-j)/(d-k)},\tag{3.14}$$

which is the Laplace-transform of $\Gamma(m+n-(d-j)/(d-k), \gamma)$. $\square$

Proof of Theorem 3.3. Let $\mathbb{W}$ be the mark distribution of M_G and let $f : \mathbf{X} \to [0, \infty)$ be measurable. Replacing in the above proof R_m by the translation- and scale invariant function $R_m(F_1, \dots, F_n, x, \varphi) f(G(F_1 - x, \dots, F_n - x, \varphi - x))$ we obtain exactly as before that

$$\lambda_M \int \exp[st] f(z) \mathbb{W}(d(t,z)) = \gamma^n \rho^{-(m+n-(d-j)/(d-k))} a_f, \tag{3.15}$$

where

$$a_f := \iint \mathbb{E}[R_m(F_1, \dots, F_n, y, \Phi) f(G(F_1 - y, \dots, F_n - y, \Phi - y))]$$
$$\mathbf{1}\{y \in Z(F_1, \dots, F_n) \cap [0,1]^d\} \mathcal{H}^j(dy) \Lambda^n(d(F_1, \dots, F_n)).$$

The choice $s = 0$ yields

$$\lambda_M \int f(z) \mathbb{W}(d(t,z)) = \gamma^n a_f.$$

Combining this with (3.15) and (3.14) gives

$$\int \exp[st] f(z) \mathbb{W}(d(t,z)) = \int \exp[st] \mathbb{W}(d(t,z)) \int f(z) \mathbb{W}(d(t,z)).$$

This is sufficient for concluding the assertion. $\square$

Remark 3.5. Assume that S does not depend on the last argument and that $k = 0$. Then the measurability assumptions on S can be weakened. Indeed, the above proof shows that it is sufficient to assume that $(x_1, \dots, x_n, x, y) \mapsto \mathbf{1}\{y \in S(x_1, \dots, x_n, x)\}$ is measurable. In particular, $S(x_1, \dots, x_n, x)$ need not be closed.

In the remainder of this section we give an alternative, slightly more succinct, formulation of the above theorems. To do so, we need to assume the existence of measurable mappings $g_1, \dots, g_n : \mathbf{N}^k \to \mathbb{R}^d$ such for all $\varphi \in \mathbf{N}^k$ and all pairwise different $F_1, \dots, F_n \in \varphi$ we have

$$\{g_1(\varphi - x) + x, \dots, g_n(\varphi - x) + x\} = \{F_1, \dots, F_n\} \tag{3.16}$$

for $\mathcal{H}^j$-a.e. $x \in Z(F_1, \dots, F_n)$ with $R_m(F_1, \dots, F_n, x, \varphi^!_{F_1,\dots,F_n}) > 0$. This says that for all "interesting" tuples $(F_1, \dots, F_n, x, \varphi)$ the flats $F_1, \dots, F_n$ are determined by φ and x in a measurable and translation-equivariant way. Using these functions we define

$$S^*(\varphi) := S(g_1(\varphi), \dots, g_n(\varphi), 0, \varphi \setminus \{g_1(\varphi), \dots, g_n(\varphi)\}), \quad \varphi \in \mathbf{N}^k. \tag{3.17}$$

We also consider a measurable scale-invariant mapping G on $(\mathcal{E}^k)^n \times \mathbf{N}^k$ taking values in some measurable space $(\mathbf{X}, \mathcal{X})$. Let

$$G^*(\varphi) := G(g_1(\varphi), \dots, g_n(\varphi), \varphi \setminus \{g_1(\varphi), \dots, g_n(\varphi)\}), \quad \varphi \in \mathbf{N}^k.$$

Finally we define a stationary random measure M' by $M'(\varphi, B) := M(\varphi, B \times [0,\infty])$, i.e.

$$M'(\varphi, B) := \iint \mathbf{1}\{x \in Z(F_1,\dots,F_n) \cap B\}$$
$$R_m(F_1,\dots,F_n,x,\varphi^!_{F_1,\dots,F_n})\mathcal{H}^j(dx)\varphi^{(n)}(d(F_1,\dots,F_n)). \qquad (3.18)$$

Theorem 3.6. *Let the above assumptions and the assumptions of Theorem 3.1 be satisfied. Then $\Lambda(S^*)$ and G^* are independent under the Palm probability measure $\mathbb{P}^0_{M'}$ and $\Lambda(S^*)$ has a $\Gamma(m+n-(d-j)/(d-k),\gamma)$-distribution, where $j = d$ in case $n = 0$.*

Proof. From (3.16) we easily obtain that the random measure M_G defined by (3.9) can be written as

$$M_G(\varphi, B \times C) = \int \mathbf{1}\{x \in B\}\mathbf{1}\{(\Lambda(S^*(\varphi - x)), G^*(\varphi - x)) \in C\}M'(\varphi, dx).$$

Hence the refined Campbell theorem (2.13) shows that $\mathbb{P}^0_{M'}((\Lambda(S^*(\Phi)), G^*(\Phi)) \in \cdot)$ is the mark distribution of M_G, and the assertions follow from Theorems 3.1 and 3.3. ☐

Remark 3.7. The results of this paper can be generalized to *independently marked* Poisson processes of flats as follows. Let $(\mathbf{X},\mathcal{X})$ be a measurable space and let $\mathbf{N}^k_{\mathbf{X}}$ be the space of all counting measures ψ on $\mathcal{E}^k \times \mathbf{X}$ such that $\psi(\cdot \times \mathbf{X})$ is a locally finite simple counting measure. The space $\mathbf{N}^k_{\mathbf{X}}$ can be equipped with a σ-field $\mathcal{N}^k_{\mathbf{X}}$ as before. On $\mathbf{N}^k_{\mathbf{X}}$ we may then consider the distribution of a Poisson process on $\mathcal{E}^k \times \mathbf{X}$ with intensitiy measure $\gamma\Lambda \otimes \mathbb{L}$, where $\mathbb{L}$ is a distribution on $\mathbf{X}$. A stopping set is now a measurable mapping $T : \mathbf{N}^k_{\mathbf{X}} \to \mathcal{F}(\mathcal{E}^k)$ such that

$$\{\psi \in \mathbf{N}^k_{\mathbf{X}} : T(\psi) \subset K\} \in \mathcal{N}^k_K, \quad K \in \mathcal{F}(\mathcal{E}^k),$$

where $\mathcal{N}^k_K := \sigma(\pi_K)$ and $\pi_K(\psi) := \{(F,z) \in \psi : F \in K\}$. The results in the appendix apply in this setting, and the above theorems can be extended in the obvious way.

4 Some special cases

4.1 Miles' complementary theorem

In this subsection we assume that S and R do not depend on the last argument and that $j = 0$ and $n \geq 1$. Under the assumptions of Theorem 3.1 (in particular (3.6)) the typical mark of the stationary marked random measure M defined by (3.7) has a $\Gamma(m+n-d/(d-k),\gamma)$-distribution. This is a slight generalization of the so-called complementary theorem derived in [5], see also [9]. In its original formulation S is given as in Remark 3.2, $R \equiv 1$ and $Z(F_1,\dots,F_n) = \{z(F_1,\dots,F_n)\}$ is a singleton. The *centroid* $z : (\mathcal{E}^k)^n \to \mathbb{R}^d$ is assumed measurable and equivariant with respect to translation and scaling. Taking a measurable and scale-invariant mapping G on $(\mathcal{E}^k)^n \times \mathbf{N}^k$, the random marked measure (3.9) takes the form

$$M_G(\varphi, B \times \cdot) := \int \mathbf{1}\{z(F_1,\dots,F_n) \in B\}\mathbf{1}\{(\Lambda(S'(F_1,\dots,F_n)), G'(F_1,\dots,F_n)) \in \cdot\}$$
$$\mathbf{1}\{\varphi(S'(F_1,\dots,F_n)) = m\}\varphi^{(n)}(d(F_1,\dots,F_n)),$$

where $S'(F_1, \dots, F_n) := S(F_1, \dots, F_n, z(F_1, \dots, F_n))$ and

$$G'(F_1, \dots, F_n) := G(F_1 - z(F_1, \dots, F_n), \dots, F_n - z(F_1, \dots, F_n)).$$

The scale-invariance (3.6) holds if S (and hence also S') is equivariant under scaling. By Theorem 3.3 the mark distribution of M_G is a product measure.

The papers [5] and [9] present many examples where the complementary theorem applies. Problems related to the integrability assumption $\lambda_M < \infty$ are reported in [2].

4.2 Subprocesses

Again we will consider a case, where S and R do not depend on the last argument. Let $H \in \mathcal{N}^k$ and $S' : \mathbf{N}^k \to \mathcal{F}(\mathcal{E}^k)$ be measurable. Then

$$\begin{aligned} M(\varphi, B \times C) := \iint & \mathbf{1}\{x \in B\}\mathbf{1}\{\Lambda(S'(\{F_1 - x, \dots, F_n - x\})) \in C\} \\ & \mathbf{1}\{\{F_1 - x, \dots, F_n - x\} \in H\} \\ & \mathbf{1}\{\varphi^!_{F_1,\dots,F_n}(S'(\{F_1 - x, \dots, F_n - x\}) + x) = m\} dx \varphi^{(n)}(d(F_1, \dots, F_n)), \end{aligned} \tag{4.1}$$

is of the form (3.7). Indeed we can take $S(F_1, \dots, F_n, x) := S'(\{F_1 - x, \dots, F_n - x\}) + x$, $R(F_1, \dots, F_n, x) := \mathbf{1}\{\{F_1 - x, \dots, F_n - x\} \in H\}$, and $j = d$. Using stationarity and Fubini's theorem we get

$$\mathbb{E}M(\Phi, B \times C) = \lambda_{m,n} \mathcal{H}^d(B) \mathbb{V}_{m,n}(C)$$

where

$$\lambda_{m,n} := \mathbb{E} \int \mathbf{1}\{\{F_1, \dots, F_n\} \in H\}\mathbf{1}\{\Phi^!_{F_1,\dots,F_n}(S'(\{F_1, \dots, F_n\})) = m\}\Phi^{(n)}(d(F_1, \dots, F_n)),$$

and

$$\begin{aligned} \mathbb{V}_{m,n} := \lambda^{-1}_{m,n} \mathbb{E} \int & \mathbf{1}\{\Lambda(S'(\{F_1, \dots, F_n\})) \in \cdot\}\mathbf{1}\{\{F_1, \dots, F_n\} \in H\} \\ & \mathbf{1}\{\Phi^!_{F_1,\dots,F_n}(S'(\{F_1, \dots, F_n\})) = m\}\Phi^{(n)}(d(F_1, \dots, F_n)), \end{aligned} \tag{4.2}$$

where we assume that $\lambda_{m,n} > 0$. We are also assuming that S' and H are equivariant resp. invariant under scaling. Then (3.6) holds and Theorem 3.1 implies $\mathbb{V}_{m,n} = \Gamma(m + n, \gamma)$.

Under additional assumptions the previous result can be expressed in a different way. So we assume that there is a point process $\Psi : \mathbf{N}^k \to \mathbf{N}^k$ which is a.s. uniquely determined by the conditions $\Psi \subset \Phi$, $\Psi \in H$, and $(\Phi \setminus \Psi)(S'(\Psi)) = m$. We may call Ψ a *subprocess* of Φ. From (4.2) it is obvious that

$$\mathbb{V}_{m,n} = \mathbb{P}(\Lambda(S'(\Psi)) \in \cdot | \Psi(\mathcal{E}^k) = n), \tag{4.3}$$

so that the conditional distribution of $\Lambda(S'(\Psi))$ given $\Psi(\mathcal{E}^k) = n$ is $\Gamma(m + n, \gamma)$. We may also consider the scaled point process

$$\Psi' := (\Lambda(S'(\Psi)))^{-1/(d-k)}\Psi$$

where $a^{-1} := 0$ if $a = 0$. Using the scaling properties of S', H, and Λ it is easy to see that a.s. $\Psi' = G(\Phi)$ for a scale-invariant measurable function $G : \mathbf{N}^k \to \mathbf{N}^k$. Using this G, we can define a stationary marked random measure M_G by (3.9). The associated mark distribution equals $\mathbb{P}((\Lambda(S'(\Psi)), \Psi') \in \cdot | \Psi(\mathcal{E}^k) = n)$. Therefore we obtain from Theorem 3.3 that $\Lambda(S'(\Psi))$ and Ψ' are conditionally independent given $\Psi(\mathcal{E}^k) = n$. In case $m = 0$ and the setting of Remark 3.2 these results are the content of Theorem 3 in [9].

4.3 The measure of stopping sets

In this subsection we consider the case $n = 0$, so that we can then work within the setting of Theorem 3.6. The stationary random measure M' defined by (3.18) is then given by

$$M'(\varphi, B) := \int \mathbf{1}\{x \in B\}\mathbf{1}\{\varphi(S'(\varphi - x) + x) = m\}R'(\varphi - x)dx,$$

where $S'(\varphi) := S(0, \varphi)$ and $R'(\varphi) := R(0, \varphi)$. From stationarity we get for any $A \in \mathcal{N}^k$ that

$$\mathbb{E}\int_B \mathbf{1}\{\Phi - x \in A\}M'(\Phi, dx) = \mathcal{H}^d(B)\mathbb{E}\mathbf{1}\{\Phi \in A\}\mathbf{1}\{\Phi(S') = m\}R'.$$

Assuming that

$$0 < \lambda_{M'} = \mathbb{E}\mathbf{1}\{\Phi(S') = m\}R' < \infty$$

we derive that

$$\mathbb{P}_m := (\mathbb{E}\mathbf{1}\{\Phi(S') = m\}R')^{-1}\mathbb{E}\mathbf{1}\{\Phi \in A\}\mathbf{1}\{\Phi(S') = m\}R'$$

is the Palm probability measure of M'. The same arguments show that assumption (3.6) takes the form

$$\mathbb{E}\mathbf{1}\{c\Phi(S'(c\Phi)) = m\}R'(c\Phi) = \mathbb{E}\mathbf{1}\{\Phi(S'(\Phi)) = m\}R'(\Phi), \quad c > 0.$$

Under this assumption Theorem 3.3 implies that $\Lambda(S')$ is $\Gamma(m, \gamma)$-distributed under $\mathbb{P}_m$. Moreover, any scale-invariant random variable G is $\mathbb{P}_m$-independent of $\Lambda(S')$. In case $R' \equiv 1$ this is a special case of Theorem 2 in [13]. However, Remark 3.2 sheds new light on this result.

5 Fundamental regions in Voronoi-tessellation

5.1 Voronoi-tessellations

In this section we apply Theorem 3.6 in case $k = 0$ to the Voronoi-tessellation based on the Poison process Φ. We start with a few basic definitions and refer the reader to [8] and Section 6.2 in [10] for more details. Let $\varphi \in \mathbf{N}^0$. The *Voronoi cell* $C(\varphi, x)$ of $x \in \varphi$ consists of all points $y \in \mathbb{R}^d$ satisfying $|y - x| \le \min\{|y - z| : z \in \varphi\}$. Then $\mathcal{S}_d(\varphi) := \{C(\varphi, x) : x \in \varphi\}$ is the Voronoi-tessellation based on φ. A cell is refered to as a d-face. Let $j \in \{0, \dots, d - 1\}$. A *$j$-face* (of $\mathcal{S}_d(\varphi)$) is a j-dimensional convex set which is the intersection of (at least) $d - j + 1$ Voronoi cells $C(\varphi, x_1), \dots, C(\varphi, x_{d-j+1})$, where $x_1, \dots, x_{d-j+1}$ are points of φ.

Let $j \in \{0, \dots, d\}$, $x_1, \dots, x_{d-j+1} \in \mathbb{R}^d$ and assume in case $j < d$ that these points are in *general position*, i.e. not contained in some affine space of dimension $d - j - 1$. For such points we define

$$Z_j(x_1, \dots, x_{d-j+1}) := \{y \in \mathbb{R}^d : |y - x_j| = |y - x_1|, j = 2, \dots, d - j + 1\}.$$

If the closed set

$$L_j(x_1, \dots, x_{d-j+1}, \varphi) := \{x \in Z_j(x_1, \dots, x_{d-j+1}) : \varphi \cap B^0(x, |x - x_1|) = \emptyset\}$$

has non-empty relative interior, then $L_j(x_1, \dots, x_{d-j+1}, \varphi)$ is a j-face of the Voronoi-tessellation based on $\varphi \cup \{x_1, \dots, x_{d-j+1}\}$. If $L_j(x_1, \dots, x_{d-j+1}, \varphi)$ is compact and non-empty, we define

$$S_j(x_1, \dots, x_{d-j+1}, \varphi) := \bigcup B(y, |y - x_1|),$$

where the union is over all vertices y of $L_j(x_1, \dots, x_{d-j+1}, \varphi)$, see Figure 5.1. Otherwise we let $S_j(x_1, \dots, x_{d-j+1}, \varphi) := \mathbb{R}^d$. If $x_1, \dots, x_{d-j+1} \in \mathbb{R}^d$ are not in general position, we set

$$L_j(x_1, \dots, x_{d-j+1}, \varphi) := Z_j(x_1, \dots, x_{d-j+1}) := \emptyset, \qquad S_j(x_1, \dots, x_{d-j+1}, \varphi) := \mathbb{R}^d.$$

It is quite standard to show that Z_j, L_j, and S_j are measurable mappings. The proof of the following geometrically obvious properties is also left to the reader. For any $x_1, \dots, x_{d-j+1} \in \mathbb{R}^d$ and $\varphi, \psi \in \mathbf{N}^0$ with $\psi \subset S_j(x_1, \dots, x_{d-j+1}, \varphi)^c$ we have

$$L_j(x_1, \dots, x_{d-j+1}, \varphi) = L_j(x_1, \dots, x_{d-j+1}, \varphi \cap S_j(x_1, \dots, x_{d-j+1}, \varphi) \cup \psi), \tag{5.1}$$

$$S_j(x_1, \dots, x_{d-j+1}, \varphi) = S_j(x_1, \dots, x_{d-j+1}, \varphi \cap S_j(x_1, \dots, x_{d-j+1}, \varphi) \cup \psi), \tag{5.2}$$

$$x \in L_j(x_1, \dots, x_{d-j+1}, \varphi) \Longrightarrow B(x, |x - x_1|) \subset S_j(x_1, \dots, x_{d-j+1}, \varphi). \tag{5.3}$$

Because of (5.1) we may call $S_j \equiv S_j(x_1, \dots, x_{d-j+1}, \varphi)$ the *fundamental region* of the face L_j: changing the underlying configuration outside S_j has no influence on the shape of L_j.

Lemma 5.1. *For any $x_1, \dots, x_{d-j+1} \in \mathbb{R}^d$ the mapping $S_j(x_1, \dots, x_{d-j+1}, \cdot) : \mathbf{N}^0 \to \mathcal{F}$ is a stopping set and $L_j(x_1, \dots, x_{d-j+1}, \cdot) : \mathbf{N}^0 \to \mathcal{F}$ is measurable w.r.t $\mathcal{N}_{S_j(x_1, \dots, x_{d-j+1}, \cdot)}$.*

Proof. The second assertion follows by (5.1) and Lemma 6.2 (i). To prove the first one, we omit $x_1, \dots, x_{d-j+1}$ in the argument of S_j. Let $\varphi, \psi \in \mathbf{N}^0$ with $\psi = \varphi \cap S_j(\psi)$. Using $\psi \subset S_j(\psi)$ and $\varphi = \psi \cup (\varphi \cap S_j(\psi)^c)$ and (5.2), we obtain $S_j(\psi) = S_j(\varphi)$. Equation (5.2) implies $S_j(\varphi) = S_j(\varphi \cap S_j(\varphi))$. By Proposition 6.3, S_j is a stopping set. □

Lemma 5.2. *For any $x_1, \dots, x_{d-j+1}, x \in \mathbb{R}^d$ the mapping*

$$R(x_1, \dots, x_{d-j+1}, x, \cdot) : \varphi \mapsto \mathbf{1}\{\varphi(B^0(x, |x - x_1|)) = 0,\ x \in Z_j(x_1, \dots, x_{d-j+1})\}$$

is measurable w.r.t. $\mathcal{N}_{S_j(x_1, \dots, x_{d-j+1}, \cdot)}$.

Proof. We fix $x, x_1, \dots, x_{d-j+1} \in \mathbb{R}^d$ and omit them in the argument of R, S_j and L_j. Obviously $R(\varphi) \le R(\varphi \cap S_j(\varphi))$. Assume $R(\varphi \cap S_j(\varphi)) = 1$. Then we have by (5.1) $x \in L_j(\varphi \cap S_j(\varphi)) = L_j(\varphi)$, and by (5.3)

$$\emptyset = \varphi \cap S_j(\varphi) \cap B^0(x, |x - x_1|) = \varphi \cap B^0(x, |x - x_1|).$$

Hence $R(\varphi) = 1$. This shows $R(\varphi) = R(\varphi \cap S_j(\varphi))$ so that the assertion follows from Lemma 6.2 (i). □

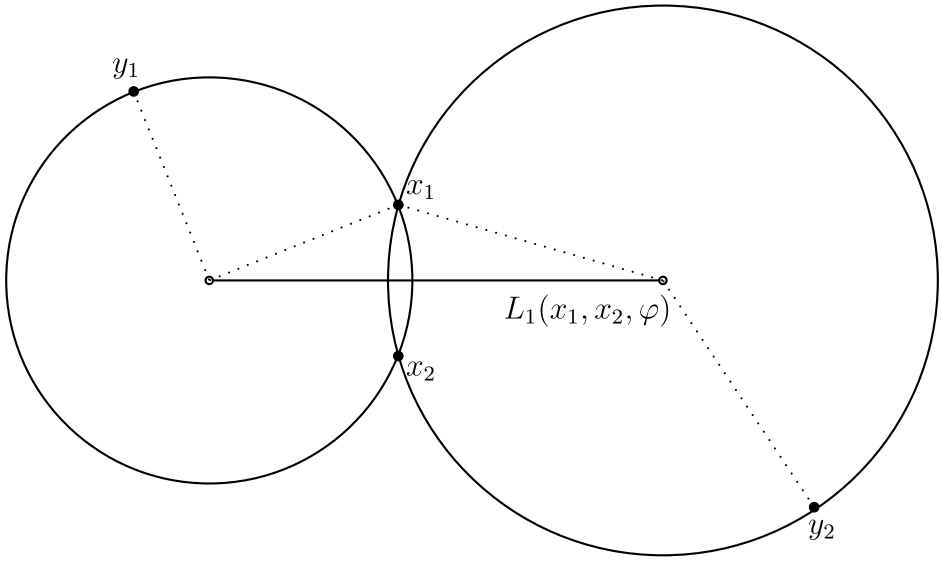

Figure 5.1: *In this picture we consider $\varphi \in \mathbf{N}^0$ with $y_1, y_2 \in \varphi$ and possibly other points outside the union of the two balls. The edge $L_1(x_1, x_2, \varphi)$ is the intersection of Voronoi cells $C(\varphi', x_1)$ and $C(\varphi', x_2)$, where $\varphi' := \varphi \cup \{x_1, x_2\}$. The fundamental region of the edge $L_1(x_1, x_2, \varphi)$ is given by the union of the two balls. The vertices of $L_1(x_1, x_2, \varphi)$ are the intersection of $C(\varphi, x_1)$, $C(\varphi, x_2)$ and $C(\varphi, y_1)$ resp. $C(\varphi, x_1)$, $C(\varphi, x_2)$ and $C(\varphi, y_2)$.*

5.2 The fundamental region of the area-biased typical j-face

With $m \geq 0$ and $j \geq 0$ fixed as in the previous sections we define stationary measures $M_{j,m}$ and M_j by

$$M_{j,m}(\varphi, \cdot) := \iint \mathbf{1}\{x \in Z_j(x_1, \ldots, x_{d-j+1}) \cap \cdot\} \mathbf{1}\{\varphi^!_{x_1,\ldots,x_{d-j+1}}(B^0(x, |x - x_1|)) = 0\}$$
$$\mathbf{1}\{\varphi^!_{x_1,\ldots,x_{d-j+1}}(S_j(x_1, \ldots, x_{d-j+1}, \varphi^!_{x_1,\ldots,x_{d-j+1}})) = m\}$$
$$\mathcal{H}^j(dx)\varphi^{(d-j+1)}(d(x_1, \ldots, x_{d-j+1}))$$

and

$$M_j(\varphi, \cdot) := \iint \mathbf{1}\{x \in Z_j(x_1, \ldots, x_{d-j+1}) \cap \cdot\} \mathbf{1}\{\varphi^!_{x_1,\ldots,x_{d-j+1}}(B^0(x, |x - x_1|)) = 0\}$$
$$\mathcal{H}^j(dx)\varphi^{(d-j+1)}(d(x_1, \ldots, x_{d-j+1})).$$

For sufficiently irregular $\varphi \in \mathbf{N}^0$ the measure $M_{j,m}(\varphi, \cdot)$ is the restriction of $\mathcal{H}^j$ onto the union of all j-faces of $\mathcal{S}_d(\varphi)$ having exactly m $(j-1)$-faces in their boundary.

Lemma 5.3. *For any $\varphi \in \mathbf{N}^0$, the measure $M_j(\varphi, \cdot)$ is locally finite.*

Proof. Let $\varphi \in \mathbf{N}^0$. Note that for all $x_1, \ldots, x_{d-j+1} \in \mathbb{R}^d$ we have

$$\mathcal{H}^j(Z_j(x_1, \ldots, x_{d-j+1}) \cap B^d) \leq \kappa_j,$$

where κ_j denotes the volume of the unit ball in $\mathbb{R}^j$. Fix $y \in \varphi$ and let $x_1, \dots, x_{d-j+1} \in \varphi$ be in general position. If $B^0(x, |x - x_1|) \cap \varphi = \emptyset$ for some $x \in B^d \cap Z_j(x_1, \dots, x_{d-j+1})$, then

$$B^0(x, |x - x_i|) \cap \varphi = \emptyset, \quad i = 1, \dots, d - j + 1.$$

In particular we have $y \notin B^0(x, |x - x_i|)$ for $i = 1, \dots, d - j + 1$. Hence

$$|y| + 1 \geq |y - x| \geq |x_i - x| \geq |x_i| - 1, \quad i = 1, \dots, d - j + 1,$$

and we conclude $|x_i| \leq |y| + 2$ for $i = 1, \dots, d - j + 1$. We obtain

$$M_j(\varphi, B^d) \leq \kappa_j \varphi(B(0, |y| + 2))^{d-j+1} < \infty.$$

The assertion follows by $M_j(\varphi, B + x) = M_j(\varphi - x, B)$ for $x \in \mathbb{R}^d$ and $B \in \mathcal{B}^d$. □

The stationary random measure M_j has positive and finite intensity λ_{M_j}, see e.g. formula (1.2) in [1]. The intensity of $M_{j,m}$ is positive iff one of the following cases hold: $j = 0$ and $m = 0$, $j = 1$ and $m = 2$ or $j \in \{2, \dots, d\}$ and $m \geq j + 1$. In the following we will only consider these cases.

Let $\varphi \in \mathbf{N}^0$ and $x \in \mathbb{R}^d$. If 0 is in the relative interior of some j-face L of $\mathcal{S}_d(\varphi)$ and if L is the intersection of exactly $d - j + 1$ Voronoi cells $C(\varphi, x_1), \dots, C(\varphi, x_{d-j+1})$, where $x_1, \dots, x_{d-j+1}$ are lexicographically ordered points of φ, then we define $L_j(\varphi) := L$ and

$$g_1(\varphi) := x_1, \dots, g_{d-j+1}(\varphi) := x_{d-j+1}.$$

Otherwise we set $L_j(\varphi) := \{0\}$ and $g_i(\varphi) := 0$, $i = 1, \dots, d - j + 1$. We define

$$S_j^*(\varphi) := S_j(g_1(\varphi), \dots, g_{d-j+1}(\varphi), \varphi \setminus \{g_1(\varphi), \dots, g_{d-j+1}(\varphi)\}).$$

Under $\mathbb{P}^0_{M_j}$ we can interpret L_j as the *(area-biased) typical j-face* of the Poisson Voronoi-tessellation $\mathcal{S}_d(\Phi)$. The refined Campbell theorem (2.13) easily implies that $\lambda_{M_{j,m}} = \lambda_{M_j} \mathbb{P}^0_{M_j}(\Phi(S_j^*) = m + d - j + 1)$ and

$$\mathbb{P}^0_{M_{j,m}} = \mathbb{P}^0_{M_j}(\cdot \mid \Phi(S_j^*) = m + d - j + 1). \tag{5.4}$$

Using Lemmata 5.1 and 5.2, we may now apply Theorem 3.6 with $n = d - j + 1$ to obtain the following result.

Theorem 5.4. *The distribution of the volume of S_j^* under $\mathbb{P}^0_{M_{j,m}}$ is $\Gamma(d + m - j + j/d, \gamma)$.*

Remark 5.5. The probability measure $\mathbb{P}^0_{M_0}$ describes Φ as seen from the *typical vertex* $L_0 = \{0\}$. Its fundamental region is the ball centred in 0 and having the $d + 1$ nearest Φ-neighbours of 0 in its boundary. It is a classical fact (see [6]) that the volume of this ball has a $G(d, \gamma)$-distribution (see also [1]).

Remark 5.6. In case $j = d$ we have that M_d is proportional to Lebesgue measure and $\mathbb{P}^0_{M_d} = \mathbb{P}$. Under $\mathbb{P}$ the face L_d is the (stationary) 0-*cell*, i.e. the cell containing the origin. The volume of its fundamental region S_d^* is by Theorem 5.4 conditionally $G(m + 1, \gamma)$-distributed, given that L_d has $m \geq d + 1$ faces of dimension $(d - 1)$. To our surprise we were not able to find this result in the literature. (The case $d = 1$ is clearly well-known.) The case $1 \leq j \leq d - 1$ does also seem to be new.

5.3 The fundamental region of the area-debiased typical j-face

For $x_1,\dots,x_{d-j+1}\in\mathbb{R}^d$ in general position we denote by $z(x_1,\dots,x_{d-j+1})$ the center of the uniquely determined $(d-j)$-dimensional ball having $x_1,\dots,x_{d-j+1}$ in its boundary. If $x_1,\dots,x_{d-j+1}\in\mathbb{R}^d$ are not in general position we set $z(x_1,\dots,x_{d-j+1}):=\infty$, where ∞ is some point outside $\mathbb{R}^d$. We define a stationary measure $N_{j,m}$ by

$$N_{j,m}(\varphi,\cdot):=\int \mathbf{1}\{L_j(x_1,\dots,x_{d-j+1},\varphi^!_{x_1,\dots,x_{d-j+1}})\neq\emptyset\}\mathbf{1}\{z(x_1,\dots,x_{d-j+1})\in\cdot\} \qquad (5.5)$$

$$\mathbf{1}\{\varphi^!_{x_1,\dots,x_{d-j+1}}(S_j(x_1,\dots,x_{d-j+1},\varphi^!_{x_1,\dots,x_{d-j+1}}))=m\}\varphi^{(d-j+1)}(d(x_1,\dots,x_{d-j+1})),$$

if

$$N_j(\varphi,\cdot):=\int \mathbf{1}\{L_j(x_1,\dots,x_{d-j+1},\varphi^!_{x_1,\dots,x_{d-j+1}})\neq\emptyset\}\mathbf{1}\{z(x_1,\dots,x_{d-j+1})\in\cdot\} \qquad (5.6)$$

$$\varphi^{(d-j+1)}(d(x_1,\dots,x_{d-j+1}))$$

is a locally finite measure. Otherwise we let $N_{j,m}(\varphi,\cdot)$ and $N_j(\varphi,\cdot)$ be the zero-measure. The right-hand side of (5.5) is in general neither simple (see Subsection 2.4) nor locally finite. However, the right-hand side of (5.6) is for $\mathbb{P}$-a.e. φ simple as well as locally finite, and N_j has a positive and finite intensity, see e.g. Subsection 2.3 in [1]. The intensity of $N_{j,m}$ is positive iff that of $M_{j,m}$ is positive, see the previous subsection.

If $N_j(\varphi,\cdot)$ is simple and has the origin in its support, then there exist a uniquely determined j-face $C_j(\varphi)$ of $\mathcal{S}_d(\varphi)$ and lexicographically ordered points $x_1,\dots,x_{d-j+1}\in\varphi$ with $C_j(\varphi)=L_j(x_1,\dots,x_{d-j+1},\varphi)$ and $z(x_1,\dots,x_{d-j+1})=0$. If, in addition, the points $x_1,\dots,x_{d-j+1}\in\varphi$ are uniquely determined by $C_j(\varphi)$, then we define

$$h_1(\varphi):=x_1,\dots,h_{d-j+1}(\varphi):=x_{d-j+1}.$$

Otherwise we set $h_i(\varphi):=0$, $i=1,\dots,d-j+1$. We define

$$T^*_j(\varphi):=S_j(h_1(\varphi),\dots,h_{d-j+1}(\varphi),\varphi\setminus\{h_1(\varphi),\dots,h_{d-j+1}(\varphi)\}).$$

Under $\mathbb{P}^0_{N_j}$ we can call C_j the *(area-debiased) typical j-face* of the Poisson Voronoi-tessellation $\mathcal{S}_d(\Phi)$. As at (5.4) we have $\lambda_{N_{j,m}}=\lambda_{N_j}\mathbb{P}^0_{N_j}(\Phi(T^*_j)=m+d-j+1)$ and

$$\mathbb{P}^0_{N_{j,m}}=\mathbb{P}^0_{N_j}(\cdot\mid\Phi(T^*_j)=m+d-j+1). \qquad (5.7)$$

Using Lemma 5.1, we may now apply Theorem 3.6 with $n=d-j+1$ and j replaced with 0 to obtain the following result.

Theorem 5.7. *The distribution of the volume of T^*_j under $\mathbb{P}^0_{N_{j,m}}$ is $\Gamma(d+m-j,\gamma)$.*

Remark 5.8. Under $\mathbb{P}_d$ the face L_d is the *typical cell* of $\mathcal{S}_d(\Phi)$. The volume of its fundamental region T^*_d is conditionally $G(m,\gamma)$-distributed, given that L_d has $m\geq d+1$ faces of dimension $(d-1)$. This special case of Theorem 5.7 is well-known, see [7] and [9]. Note the difference in the shape parameter when compared with the stationary 0-cell alluded to in Remark 5.6. This phenomenon can be best explained in case $d=1$. The classical special case $j=0$ has already been discussed in Remark 5.5. The probability measure $\mathbb{P}^0_{N_1}$ describes Φ as seen from the *typical edge* L_1 of $\mathcal{S}_d(\Phi)$. Using different methods, it has been shown in [1] that the volume of the fundamental region of L_1 (see Figure 5.1) has a $G(d+1,\gamma)$-distribution. This is in accordance with Theorem 5.7. The remaining special cases of Theorem 5.7 are all new.

6 Appendix: Stopping sets

In this section we present some basic results on stopping sets in a general setting. Let $\mathbf{Y}$ be a locally compact second countable Hausdorff space. The system of all closed subsets of $\mathbf{Y}$ is denoted by $\mathcal{F}(\mathbf{Y})$. On $\mathcal{F}(\mathbf{Y})$ we consider the smallest σ-field containing $\mathcal{F}_K := \{F \in \mathcal{F}(\mathbf{Y}) : F \cap K \neq \emptyset\}$ for all compact $K \subset \mathbf{Y}$.

Let $(\mathbf{W}, \mathcal{W})$ be a measurable space and π_K, $K \in \mathcal{F}(\mathbf{Y})$, a family of measurable mappings $\pi_K : \mathbf{W} \to \mathbf{W}$ with

$$\pi_{K_1} \circ \pi_{K_2} = \pi_{K_1}, \quad K_1 \subset K_2,\ K_1, K_2 \in \mathcal{F}(\mathbf{Y}). \tag{6.1}$$

Two examples of such mappings π_K are given in (6.5) and (6.7). The σ-field generated by π_K is $\mathcal{W}_K := \pi_K^{-1}(\mathcal{W})$. By (6.1) we have $\mathcal{W}_{K_1} \subset \mathcal{W}_{K_2}$ for $K_1, K_2 \in \mathcal{F}(\mathbf{Y})$ with $K_1 \subset K_2$. A *stopping set* (defined on $\mathbf{W}$, w.r.t. to $(\mathcal{W}_K)_{K \in \mathcal{F}(\mathbf{Y})}$) is a mapping $T : \mathbf{W} \to \mathcal{F}(\mathbf{Y})$ such that

$$\{\varphi \in \mathbf{W} : T(\varphi) \subset K\} \in \mathcal{W}_K, \quad K \in \mathcal{F}(\mathbf{Y}).$$

The *stopping σ-field* associated with a stopping set T is defined by

$$\mathcal{W}_T := \{A \in \mathcal{W} : A \cap \{T \subset K\} \in \mathcal{W}_K \text{ for all } K \in \mathcal{F}(\mathbf{Y})\}.$$

It is easy to check that a stopping set T is $\mathcal{W}_T$-measurable. Indeed, if $U \subset \mathbf{Y}$ is open then

$$\{T \cap U = \emptyset\} \cap \{T \subset K\} = \{T \subset K \setminus U\} \in \mathcal{W}_{K \setminus U} \subset \mathcal{W}_K.$$

Lemma 6.1. *Let $(\mathbf{X}, \mathcal{X})$ be a Borel space and $f : \mathbf{W} \to \mathbf{X}$ a measurable function. Then f is $\mathcal{W}_K$-measurable iff $f = f \circ \pi_K$.*

Proof. Let f be measurable w.r.t. $\mathcal{W}_K$. By Lemma 1.13 in [3] there exists some measurable mapping $h : \mathbf{W} \to \mathbf{X}$ with $f = h \circ \pi_K$. Hence $f \circ \pi_K = f$ by (6.1). □

For a function $T : \mathbf{W} \to \mathcal{F}(\mathbf{Y})$ we define $\pi_T : \mathbf{W} \to \mathbf{W}$ by

$$\pi_T(\varphi) := \pi_{T(\varphi)}(\varphi), \quad \varphi \in \mathbf{W}.$$

Lemma 6.2. *Let T be a stopping set. Then the following assertions hold:*

(i) *$\mathcal{W}_T = \sigma(\pi_T)$ if π_T is measurable.*

(ii) *$T(\varphi) = T(\pi_T(\varphi))$ for all $\varphi \in \mathbf{W}$.*

(iii) *Let $\varphi \in \mathbf{W}$ and $K \in \mathcal{F}(\mathbf{Y})$. Then $T(\varphi) \subset K$ iff $T(\pi_K(\varphi)) \subset K$.*

(iv) *Let $\varphi \in \mathbf{W}$, $K \in \mathcal{F}(\mathbf{Y})$ and assume $T(\varphi) \subset K$. Then $T(\varphi) = T(\pi_K(\varphi))$.*

Proof. To prove $\mathcal{W}_T \subset \sigma(\pi_T)$, let $A \in \mathcal{W}_T$, $K \in \mathcal{F}(\mathbf{Y})$, $\varphi \in \mathbf{W}$. By definition of $\mathcal{W}_T$ the function $\mathbf{1}_A \mathbf{1}\{T \subset K\}$ is measurable w.r.t $\mathcal{W}_K$. Using Lemma 6.1 we get

$$\mathbf{1}_A(\varphi)\mathbf{1}\{T(\varphi) \subset K\} = \mathbf{1}_A(\pi_K(\varphi))\mathbf{1}\{T(\pi_K(\varphi)) \subset K\}. \tag{6.2}$$

Setting $A = \mathbf{W}$ yields the third assertion, i.e.

$$\mathbf{1}\{T(\varphi) \subset K\} = \mathbf{1}\{T(\pi_K(\varphi)) \subset K\}. \tag{6.3}$$

Set $K := T(\varphi)$ and use (6.2), (6.3) to obtain $\mathbf{1}_A(\varphi) = \mathbf{1}_A(\pi_T(\varphi))$, which in turn implies $A = \pi_T^{-1}(A) \in \pi_T^{-1}(\mathcal{W}) = \sigma(\pi_T)$.

Recall that T is $\mathcal{W}_T$-measurable. Using (6.2) with $A = T^{-1}(\{T(\varphi)\}) \in \mathcal{W}_T$ and (6.3), we have in case of $T(\varphi) \subset K$

$$1 = \mathbf{1}\{T(\varphi) = T(\varphi)\} = \mathbf{1}\{T(\pi_K(\varphi)) = T(\varphi)\}.$$

Hence the second and fourth assertion are also true.

Now we prove $\sigma(\pi_T) \subset \mathcal{W}_T$. Let $A \in \mathcal{W}$, $K \in \mathcal{F}(\mathbf{Y})$, $\varphi \in \mathbf{W}$ and π_T be measurable. By (6.1), (6.3) and assertion (iv) we get

$$\mathbf{1}\{\pi_T(\varphi) \in A\}\mathbf{1}\{T(\varphi) \subset K\} = \mathbf{1}\{\pi_T(\pi_K(\varphi)) \in A\}\mathbf{1}\{T(\pi_K(\varphi)) \subset K\},$$

which implies $\mathcal{W}_K$-measurability of $\{\pi_T \in A\} \cap \{T \subset K\}$. Hence $\{\pi_T \in A\} \in \mathcal{W}_T$, finishing the proof of the first assertion. □

Proposition 6.3. *A measurable function $T : \mathbf{W} \to \mathcal{F}(\mathbf{Y})$ is a stopping set iff $T(\varphi) = T(\pi_T(\varphi))$ for all $\varphi \in \mathbf{W}$ and the following implication holds for all for all $\varphi, \psi \in \mathbf{W}$:*

$$\psi = \pi_{T(\psi)}(\varphi) \quad \Longrightarrow \quad T(\psi) = T(\varphi). \tag{6.4}$$

Proof. Let T be a stopping set, $\psi, \varphi \in \mathbf{W}$ with $\psi = \pi_{T(\psi)}(\varphi)$ and set $K := T(\psi)$. Using Lemma 6.2 (iii), $\psi = \pi_K(\varphi)$ and $T(\pi_K(\varphi)) = T(\psi) \subset K$, we obtain $T(\varphi) \subset K$. Invoking Lemma 6.2 (iv), we see

$$T(\varphi) = T(\pi_K(\varphi)) = T(\psi).$$

Note that $T(\varphi) = T(\pi_T(\varphi))$ holds by Lemma 6.2 (ii).

Now assume $T(\varphi) = T(\pi_T(\varphi))$ and (6.4) for all $\varphi, \psi \in \mathbf{W}$. Let $\varphi \in \mathbf{W}$, $K \in \mathcal{F}(\mathbf{Y})$ with $T(\varphi) \subset K$ and define $\psi := \pi_T(\varphi)$. We have $T(\psi) = T(\varphi) \subset K$, so that by (6.1)

$$\psi = \pi_{T(\psi)}(\varphi) = \pi_{T(\psi)}(\pi_K(\varphi)).$$

If vice versa there exists $\psi \in \mathbf{W}$ with $T(\psi) \subset K$ and $\psi = \pi_{T(\psi)}(\pi_K(\varphi))$, then $\psi = \pi_{T(\psi)}(\varphi)$ by (6.1), and (6.4) yields $T(\varphi) = T(\psi) \subset K$. Hence

$$T(\varphi) \subset K \quad \Longleftrightarrow \quad \text{there exists } \psi \in \mathbf{W} \text{ with } T(\psi) \subset K \text{ and } \psi = \pi_{T(\psi)}(\pi_K(\varphi)).$$

Therefore $\mathbf{1}\{T(\varphi) \subset K\} = \mathbf{1}\{T(\pi_K(\varphi)) \subset K\}$. As $\{T \subset K\} \in \mathcal{W}$ we get in fact $\{T \subset K\} \in \mathcal{W}_K$, so that T is a stopping set. □

Proposition 6.3 is the right tool for proving Proposition 2.1.

Proof of Proposition 2.1. Let $T' : \mathbf{N}^k \to \mathcal{F}$ be measurable and assume that (2.9) holds. Then T' is a stopping set w.r.t. $({\pi'_K}^{-1}(\mathcal{N}^k))_{K \in \mathcal{F}}$, where π'_K is defined in (2.7), i.e.

$$\pi'_K(\varphi) = \{F \in \varphi : F \cap K \neq \emptyset\}, \quad \varphi \in \mathbf{N}^k, K \in \mathcal{F}. \tag{6.5}$$

Defining $T := \mathcal{E}_{T'}$ we obtain from Proposition 6.3

$$T'(\varphi) = T'(\pi'_{T'}(\varphi)) = T'(\varphi \cap T(\varphi)) \tag{6.6}$$

and the implication

$$\psi = \varphi \cap T(\psi) \quad \Longrightarrow \quad T'(\varphi) = T'(\psi)$$

for all $\varphi, \psi \in \mathbf{N}^k$. Note that $T'(\varphi) = T'(\psi)$ implies $T(\varphi) = T(\psi)$ and (6.6) implies $T(\varphi) = T(\varphi \cap T(\varphi))$. By Proposition 6.3 T is a stopping set w.r.t. $(\pi_K^{-1}(\mathcal{N}^k))_{K\in\mathcal{F}(\mathcal{E}^k)}$, where π_K is defined by

$$\pi_K(\varphi) := \varphi \cap K, \quad \varphi \in \mathbf{N}^k, K \in \mathcal{F}(\mathcal{E}^k). \tag{6.7}$$

Conversely, assume that $T' : \mathbf{N}^k \to \mathcal{F}$ is measurable and $T := \mathcal{E}_{T'}$ is a stopping set w.r.t. $(\pi_K^{-1}(\mathcal{N}^k))_{K\in\mathcal{F}(\mathcal{E}^k)}$. If $K' \in \mathcal{F}$ then $\mathcal{E}_{K'} := \{F \in \mathcal{E}^k : F \cap K' \neq \emptyset\}$ is closed and we have

$$\{T' \subset K'\} = \{\mathcal{E}_{T'} \subset \mathcal{E}_{K'}\} \in \pi_{\mathcal{E}_{K'}}^{-1}(\mathcal{N}^k) = (\pi'_{K'})^{-1}(\mathcal{N}^k).$$

Therefore T' is a stopping set w.r.t. to $({\pi'_{K'}}^{-1}(\mathcal{N}^k))_{K'\in\mathcal{F}}$.

From $\pi'_{T'} = \pi_T$ and Lemma 6.2 (i) we obtain

$$\mathcal{N}^k_{T'} = \sigma(\pi'_{T'}) = \sigma(\pi_T) = \mathcal{N}^k_T,$$

which proves (2.10). □

The next result leads to an easy and transparent proof of Proposition 3.4.

Lemma 6.4. *Set $\pi_K(\varphi) := \varphi \cap K$, $\varphi \in \mathbf{N}^k$, $K \in \mathcal{F}(\mathcal{E}^k)$. Let $\varphi \in \mathbf{N}^k$, $T : \mathbf{N}^k \to \mathcal{F}(\mathcal{E}^k)$ be a stopping set and $f : \mathbf{N}^k \times \mathcal{F}(\mathcal{E}^k) \to [0, \infty)$ be a measurable function. Then*

$$\begin{aligned} f(\pi_T(\varphi), T(\varphi))\mathbf{1}\{\varphi(T(\varphi)) = m\} = \frac{1}{m!} \int & f(\{F_1, \dots, F_m\}, T(\{F_1, \dots, F_m\})) \\ & \mathbf{1}\{F_1, \dots, F_m \in T(\{F_1, \dots, F_m\})\} \\ & \mathbf{1}\{\varphi^!_{F_1,\dots,F_m}(T(\{F_1, \dots, F_m\})) = 0\}\varphi^{(m)}(d(F_1, \dots, F_m)) \end{aligned}$$

for $m \geq 1$ and

$$f(\pi_T(\varphi), T(\varphi))\mathbf{1}\{\varphi(T(\varphi)) = 0\} = f(\emptyset, T(\emptyset))\mathbf{1}\{\varphi(T(\emptyset)) = 0\}.$$

Proof. We assume that $m \geq 1$. In case $m = 0$ the proof is similar but simpler. For any $\varphi \in \mathbf{N}^k$ we have

$$\begin{aligned} & f(\pi_T(\varphi), T(\varphi))\mathbf{1}\{\varphi(T(\varphi)) = m\} \\ & = \frac{1}{m!} \int f(\{F_1, \dots, F_m\}, T(\varphi))\mathbf{1}\{\{F_1, \dots, F_m\} = \pi_T(\varphi)\}\varphi^{(m)}(d(F_1, \dots, F_m)). \end{aligned}$$

From Proposition 6.3 we obtain

$$\{F_1, \dots, F_m\} = \pi_T(\varphi) \quad \Longleftrightarrow \quad \{F_1, \dots, F_m\} = \pi_{T(\{F_1,\dots,F_m\})}(\varphi),$$

as well as $T(\varphi) = T(\{F_1, \dots, F_m\})$, if the equations above are true. We conclude

$$
\begin{aligned}
&f(\pi_T(\varphi), T(\varphi))\mathbf{1}\{\varphi(T(\varphi)) = m\} \\
&\quad = \frac{1}{m!}\int f(\{F_1, \dots, F_m\}, T(\{F_1, \dots, F_m\})) \\
&\qquad\qquad \mathbf{1}\{\{F_1, \dots, F_m\} = \pi_{T(\{F_1,\dots,F_m\})}(\varphi)\}\varphi^{(m)}(d(F_1, \dots, F_m)) \\
&\quad = \frac{1}{m!}\int f(\{F_1, \dots, F_m\}, T(\{F_1, \dots, F_m\}))\mathbf{1}\{F_1, \dots, F_m \in T(\{F_1, \dots, F_m\})\} \\
&\qquad\qquad \mathbf{1}\{\varphi^!_{F_1,\dots,F_m}(T(\{F_1, \dots, F_m\})) = 0\}\varphi^{(m)}(d(F_1, \dots, F_m)). \qquad \Box
\end{aligned}
$$

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