UNIQUENESS FOR ELLIPTIC OPERATORS ON $L^p(\mathbb{R}^N)$ WITH UNBOUNDED COEFFICIENTS

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Abstract. Let A denote a general second order differential operator in divergence form, with real coefficients satisfying only local boundedness conditions. Then, A can be viewed as an unbounded operator A_p on $L^p(\mathbb{R}^N)$ with maximal domain $D(A_p)$. The main result of this paper is a criterion for the injectivity of A_p when $p \in (1, \infty)$. This criterion is next used to establish the denseness of $C_0^\infty(\mathbb{R}^N)$ in $D(A_p)$ or, under more general conditions, in a smaller natural domain. When $p = 2$, this also yields a positive answer to a selfadjointness question raised by Kato in 1981.

1. Introduction

Second order elliptic operators on $\mathbb{R}^N$ with unbounded coefficients arise in the study of Kolmogoroff and Schrödinger equations. In several respects, their properties differ markedly from those known in the case of bounded coefficients. For example, test functions need not form a core of the maximal domain $D(A_p)$ of A in $L^p(\mathbb{R}^N)$ and it may happen that $A + \lambda$ is not injective on $D(A_p)$ for any $\lambda > 0$. In this paper, we provide conditions about the coefficients of A ensuring that A is injective and that test functions are a core of $D(A_p)$. In particular, by extending techniques from [12], we are able to treat the limiting cases of some problems that were left open in the works [2] and [9].

Throughout this paper, A denotes the second order linear differential operator on $\mathbb{R}^N$

$$A := -\sum_{i,j=1}^{N} \partial_i(a_{ij}(x)\partial_j) + \sum_{i=1}^{N} b_i(x)\partial_i + c(x), \tag{1.1}$$

with *real* coefficients $a_{ij} = a_{ji}, b_i \in W^{1,\infty}_{loc} := W^{1,\infty}_{loc}(\mathbb{R}^N)$ and $c \in L^\infty_{loc} := L^\infty_{loc}(\mathbb{R}^N)$. More generally, for every $m \in \mathbb{N} \cup \{0\}$ and $q \in [1, \infty]$, we shall always use the convenient shortcut $W^{m,q}_{(loc)}$ to denote the Sobolev space $W^{m,q}_{(loc)}(\mathbb{R}^N)$ ($L^q_{(loc)}$ when $m = 0$). Consistent with this notation, $|| \cdot ||_{m,q}$ is the norm of $W^{m,q}$. If $\Omega \subsetneq \mathbb{R}^N$ is an open subset, $|| \cdot ||_{m,q,\Omega}$ refers to the norm of the space $W^{m,q}(\Omega)$.

Given $p \in (1, \infty)$ and $u \in L^p$, the assumptions about the coefficients imply that Au is well defined as a distribution on $\mathbb{R}^N$. Accordingly, we may and shall denote by A_p the unbounded operator with domain

$$D(A_p) := \{u \in L^p : Au \in L^p\}, \tag{1.2}$$

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given by

$$A_p u := Au, \qquad \forall u \in D(A_p). \tag{1.3}$$

It is our goal to establish various properties, starting with the injectivity, of the operators A_p under additional assumptions about the coefficients. Uniqueness results for Kolmogorov operators (i.e., when $c = 0$) can also be found in Eberle's monograph [5].

First and foremost, we shall assume ellipticity, i.e., that for every $x \in \mathbb{R}^N$, there is a constant $\alpha(x) > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \alpha(x)|\xi|^2, \qquad \forall \xi := (\xi_1, \cdots, \xi_N) \in \mathbb{R}^N, \tag{1.4}$$

where, as usual, $|\cdot|$ is the euclidian norm on $\mathbb{R}^N$. Since $\alpha(x)$ above can be chosen to coincide with the smallest eigenvalue of the symmetric matrix $(a_{ij}(x))$ and since this smallest eigenvalue depends continuously upon x, (1.4) is equivalent to the local uniform ellipticity condition

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \alpha_K|\xi|^2, \qquad \forall x \in K, \quad \forall \xi := (\xi_1, \cdots, \xi_N) \in \mathbb{R}^N,$$

where K is an arbitrary compact subset of $\mathbb{R}^N$ and $\alpha_K > 0$. It then follows from elliptic regularity that

$$D(A_p) \subset W^{2,p}_{loc}. \tag{1.5}$$

In addition, from the classical a priori estimates (see for instance Gilbarg and Trudinger [8]), for every $R > 0$, there is a constant $C_R > 0$ independent of $u \in W^{2,p}_{loc}$ such that

$$||u||_{2,p,B_R} \leq C_R \left(||u||_{0,p,B_{2R}} + ||Au||_{0,p,B_{2R}} \right), \tag{1.6}$$

where B_R denotes the ball in $\mathbb{R}^N$ with center 0 and radius R. Together with (1.3) and (1.5), this shows at once that the operator A_p above is closed. Equivalently, $D(A_p)$ is a Banach space for the graph norm

$$||u||_{D(A_p)} := ||u||_{0,p} + ||Au||_{0,p}. \tag{1.7}$$

For future use, note that $C_0^\infty (:= C_0^\infty(\mathbb{R}^N)) \subset D(A_p)$, so that $D(A_p)$ is dense in L^p.

The additional hypotheses that we shall make about the coefficients of A in (1.1) are not standard and involve an auxiliary C^N function $\rho \geq 1$ on $\mathbb{R}^N$ and a parameter $s > 0$. Even though auxiliary functions have already been used in related matters (mostly the selfadjointness issue when $p = 2$; see for instance Evans [6], Kato [9]), our approach, based on Federer's coarea formula, follows a much different line. The hypotheses recently introduced in Rabier [12] when $p = 2$ and $a_{ij} \in L^\infty$ also involve a parameter s, but the method fails to work when $p \neq 2$. Thus, while the results of this paper may be viewed as a (partial) generalization of those in [12] and in spite of notable similarities, the arguments are, of necessity, markedly different.

From now on,

$$a(x, \xi, \eta) := \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\eta_j, \qquad \forall \xi, \eta \in \mathbb{R}^N \tag{1.8}$$

and

$$b(x) := (b_1(x), \cdots, b_N(x)) \tag{1.9}$$

is the vector of the first order coefficients of A.

The main result of this paper (sufficient condition for the injectivity of A_p) is given in Theorem 3.2. The denseness of C_0^∞ in $D(A_p)$ equipped with the graph norm is subsequently obtained as a by-product under slightly more general assumptions (Theorem 4.2). A generalization of the denseness property is given in Theorem 4.4, where $D(A_p)$ is replaced by a possibly smaller domain. A special case (Corollary 4.5) allows for an easy comparison with recent results of the same nature in [2].

To complete the paper, a simple "concrete" application of Theorem 4.2 is given in Section 5, where we answer in particular a question left open by Kato in [9] when $p = 2$ and $b = 0$.

2. Preliminary estimates

If $\rho \geq 0$ is a C^1 function on $\mathbb{R}^N$ with $\lim_{|x|\to\infty} \rho(x) = \infty$ and if A is a differential operator (1.1) with $a_{ij} = a_{ji}, b_i \in L^\infty_{loc}$, we set, for every $r \geq 1$ and every $\lambda > 1$,

$$M_\lambda(\rho, r) := \sup_{\lambda^{-1} r \leq \rho(x) \leq \lambda^2 r} \{1 + |b(x) \cdot \nabla\rho(x)| + a(x, \nabla\rho(x), \nabla\rho(x))\}, \tag{2.1}$$

where a and b are given by (1.8) and (1.9), respectively. Note that $M_\lambda(\rho, r) < \infty$ (in particular, observe that the set $\{\lambda^{-1} r \leq \rho \leq \lambda^2 r\}$ is compact since $\lim_{|x|\to\infty} \rho(x) = \infty$).

Lemma 2.1. *Given $p \in (1, \infty)$ and $\lambda > 1$ as above, there is a constant $C(\lambda, p) > 0$ such that, for every C^1 function $\rho \geq 0$ on $\mathbb{R}^N$ with $\lim_{|x|\to\infty} \rho(x) = \infty$ and every elliptic operator A in (1.1) with real coefficients $a_{ij} = a_{ji}, b_i \in W^{1,\infty}_{loc}$ and $c \in L^\infty_{loc}$ such that $c - \frac{\nabla\cdot b}{p} \geq 0$, the following inequality holds for every $u \in W^{2,p}_{loc}$ and every $r \geq 1$:*

$$\int_{\{r\leq\rho\leq\lambda r\}} |u|^{p-2} a(\cdot, \nabla u, \nabla u)\chi_{\{u\neq0\}} + \left(c - \frac{\nabla\cdot b}{p}\right)|u|^p \leq C(\lambda, p)\left(\int_{\{\lambda^{-1}r<\rho<\lambda^2 r\}} |Au|^p + M_\lambda(\rho, r)|u|^p\right), \tag{2.2}$$

where $M_\lambda(\rho, r)$ is given by (2.1) and $\chi_{\{u\neq0\}}$ is the characteristic function of the set $\{u \neq 0\}$.

Proof. Choose a smooth real-valued function ψ_λ on $[0, \infty)$ such that $0 \leq \psi_\lambda \leq 1, \psi_\lambda(t) = 0$ if $0 \leq t \leq \lambda^{-1}$ or $t \geq \lambda^2$ and $\psi_\lambda(t) = 1$ if $1 \leq t \leq \lambda$. For $r > 0$, set $\psi_{\lambda,r}(t) := \psi_\lambda(t/r)$, so that $0 \leq \psi_{\lambda,r} \leq 1, \psi_{\lambda,r}(t) = 0$ if $0 \leq t \leq \lambda^{-1}r$ or $t \geq \lambda^2 r$ and $\psi_{\lambda,r}(t) = 1$ if $r \leq t \leq \lambda r$. Next, let $\eta_{\lambda,r}(x) := \psi_{\lambda,r}(\rho(x))$ for $x \in \mathbb{R}^N$. Observe that $0 \leq \eta_{\lambda,r} \leq 1$ and that $\eta_{\lambda,r} \in C_0^\infty$ since $\lim_{|x|\to\infty} \rho(x) = \infty$. That $\eta_{\lambda,r}$ has compact support is used implicitly in various places to ensure that all the integrals over $\mathbb{R}^N$ below are well defined.

If $u \in W^{2,p}_{loc}$, then $Au \in L^p_{loc}$ and, by multiplying Au by $\eta^2_{\lambda,r}|u|^{p-2}u$ and integrating by parts, we obtain

$$\begin{aligned}(2.3)\quad \int_{\mathbb{R}^N} (Au)\eta^2_{\lambda,r}|u|^{p-2}u &= (p-1)\int_{\mathbb{R}^N} \eta^2_{\lambda,r}|u|^{p-2}a(\cdot,\nabla u,\nabla u)\chi_{\{u\neq 0\}} \\ &\quad + 2\int_{\mathbb{R}^N} \eta_{\lambda,r}|u|^{p-2}ua(\cdot,\nabla u,\nabla \eta_{\lambda,r}) \\ &\quad + \int_{\mathbb{R}^N}\left(c-\frac{\nabla\cdot b}{p}\right)\eta^2_{\lambda,r}|u|^p - \frac{2}{p}\int_{\mathbb{R}^N}\eta_{\lambda,r}|u|^p b\cdot\nabla\eta_{\lambda,r}.\end{aligned}$$

Above, we used the fact that the integrals over $\mathbb{R}^N$ are in fact integrals over a ball with large enough radius. Also, while (2.3) is routine when[1] $p \geq 2$, it is not trivial when $p \in (1,2)$ due to the singularities of $|u|^{p-2}$ at the points where u vanishes. However, the related technical difficulties have been resolved in Metafune and Spina [10].

By the Cauchy-Schwarz inequality for the bilinear form $a(x,\cdot,\cdot)$ (since A is elliptic),

$$\left|\int_{\mathbb{R}^N}\eta_{\lambda,r}|u|^{p-2}ua(\cdot,\nabla u,\nabla\eta_{\lambda,r})\right| \leq \int_{\mathbb{R}^N}\eta_{\lambda,r}|u|^{p-1}a(\cdot,\nabla u,\nabla u)^{1/2}a(\cdot,\nabla\eta_{\lambda,r},\nabla\eta_{\lambda,r})^{\frac{1}{2}}.$$

Thus, by using $|u|^{p-1} = |u|^{\frac{p-2}{2}}|u|^{\frac{p}{2}}$ when $u \neq 0$ and the Cauchy-Schwarz inequality in L^2, it follows that

$$\begin{aligned}&\left|2\int_{\mathbb{R}^N}\eta_{\lambda,r}|u|^{p-2}ua(\cdot,\nabla u,\nabla\eta_{\lambda,r})\right| \\ &\quad\leq 2\left(\int_{\mathbb{R}^N}\eta^2_{\lambda,r}|u|^{p-2}a(\cdot,\nabla u,\nabla u)\chi_{\{u\neq 0\}}\right)^{1/2}\left(\int_{\mathbb{R}^N}|u|^p a(\cdot,\nabla\eta_{\lambda,r},\nabla\eta_{\lambda,r})\right)^{1/2} \\ &\quad\leq \gamma\int_{\mathbb{R}^N}\eta^2_{\lambda,r}|u|^{p-2}a(\cdot,\nabla u,\nabla u)\chi_{\{u\neq 0\}} + \gamma^{-1}\int_{\mathbb{R}^N}|u|^p a(\cdot,\nabla\eta_{\lambda,r},\nabla\eta_{\lambda,r}),\end{aligned}$$

for every $\gamma > 0$. By substitution into (2.3), we obtain

$$\begin{aligned}&(p-1-\gamma)\int_{\mathbb{R}^N}\eta^2_{\lambda,r}|u|^{p-2}a(\cdot,\nabla u,\nabla u)\chi_{\{u\neq 0\}} + \int_{\mathbb{R}^N}\left(c-\frac{\nabla\cdot b}{p}\right)\eta^2_{\lambda,r}|u|^p \\ &\quad\leq \int_{\mathbb{R}^N}(Au)\eta^2_{\lambda,r}|u|^{p-2}u + \int_{\mathbb{R}^N}\left(\frac{2}{p}\eta_{\lambda,r}b\cdot\nabla\eta_{\lambda,r} + \gamma^{-1}a(\cdot,\nabla\eta_{\lambda,r},\nabla\eta_{\lambda,r})\right)|u|^p.\end{aligned}$$

Above, choose $\gamma > 0$ such that $p-1-\gamma > 0$. Since $c - \frac{\nabla\cdot b}{p} \geq 0$ and $\eta_{\lambda,r}(x) = 1$ when $r \leq \rho(x) \leq \lambda r$, this yields

$$\begin{aligned}&\min\{p-1-\gamma,1\}\left(\int_{\{r\leq\rho\leq\lambda r\}}|u|^{p-2}a(\cdot,\nabla u,\nabla u)\chi_{\{u\neq 0\}} + \left(c-\frac{\nabla\cdot b}{p}\right)|u|^p\right) \\ &\quad\leq \int_{\mathbb{R}^N}(Au)\eta^2_{\lambda,r}|u|^{p-1} + \int_{\mathbb{R}^N}\left(\frac{2}{p}\eta_{\lambda,r}b\cdot\nabla\eta_{\lambda,r} + \gamma^{-1}a(\cdot,\nabla\eta_{\lambda,r},\nabla\eta_{\lambda,r})\right)|u|^p.\end{aligned}$$

[1] If so, $\chi_{\{u\neq 0\}}$ is not needed in the formula, which is obvious if $p > 2$. If $p = 2$, this is due to the well known fact (see Gilbarg and Trudinger [8, Lemma 7.7], Stampacchia [13]) that $\nabla u = 0$ a.e. on the set $\{u = 0\}$.

Now, in the right-hand side, use $0 \leq \eta_{\lambda,r} \leq 1, \eta_{\lambda,r}(x) = 0$ if $\rho(x) \leq \lambda^{-1}r$ or $\rho(x) \geq \lambda^2 r$ and $\nabla\eta_{\lambda,r} = \psi'_{\lambda,r}(\rho)\nabla\rho = r^{-1}\psi'_\lambda(\rho/r)\nabla\rho$ to get

$$\begin{aligned}
&\min\{p-1-\gamma,1\}\left(\int_{\{r\leq\rho\leq\lambda r\}}|u|^{p-2}a(\cdot,\nabla u,\nabla u)\chi_{\{u\neq 0\}} + \left(c-\frac{\nabla\cdot b}{p}\right)|u|^p\right)\\
&\quad\leq \int_{\{\lambda^{-1}r<\rho<\lambda^2 r\}}|Au||u|^{p-1}\\
&\qquad+\int_{\{\lambda^{-1}r<\rho<\lambda^2 r\}}\left(\frac{2\|\psi'_\lambda\|_\infty}{p}r^{-1}|b\cdot\nabla\rho| + \gamma^{-1}\|\psi'_\lambda\|_\infty^2 r^{-2}a(\cdot,\nabla\rho,\nabla\rho)\right)|u|^p.
\end{aligned}$$

Lastly, observe that $|Au||u|^{p-1} \leq \frac{1}{p}|Au|^p + \frac{p-1}{p}|u|^p$ (since $st \leq \frac{1}{p}s^p + \frac{p-1}{p}t^{\frac{p}{p-1}}$ for every $s,t \geq 0$), so that if $r \geq 1$, then

$$\begin{aligned}
&\min\{p-1-\gamma,1\}\left(\int_{\{r\leq\rho\leq\lambda r\}}|u|^{p-2}a(\cdot,\nabla u,\nabla u)\chi_{\{u\neq 0\}} + \left(c-\frac{\nabla\cdot b}{p}\right)|u|^p\right)\\
&\quad\leq \frac{1}{p}\int_{\{\lambda^{-1}r<\rho<\lambda^2 r\}}|Au|^p\\
&\qquad+\int_{\{r\leq\rho\leq\lambda r\}}\left(\frac{p-1}{p}+\frac{2\|\psi'_\lambda\|_\infty}{p}r^{-1}|b\cdot\nabla\rho| + \gamma^{-1}\|\psi'_\lambda\|_\infty^2 r^{-2}a(\cdot,\nabla\rho,\nabla\rho)\right)|u|^p\\
&\quad\leq \max\left\{\frac{1}{p},\frac{p-1}{p},\frac{2\|\psi'_\lambda\|_\infty}{p},\gamma^{-1}\|\psi'_\lambda\|_\infty^2\right\}\left(\int_{\{\lambda^{-1}r<\rho<\lambda^2 r\}}|Au|^p + M_\lambda(\rho,r)|u|^p\right).
\end{aligned}$$

Since the above inequality is valid whenever $0<\gamma<p-1$, it follows that (2.2) holds with

$$C(\lambda,p) := \inf_{0<\gamma<p-1}\left(\frac{\max\left\{\frac{1}{p},\frac{p-1}{p},\frac{2\|\psi'_\lambda\|_\infty}{p},\gamma^{-1}\|\psi'_\lambda\|_\infty^2\right\}}{\min\{p-1-\gamma,1\}}\right).$$

□

It is noteworthy that the constant $C(\lambda,p)$ in Lemma 2.1 is "universal", i.e., independent of the function ρ and the operator A satisfying the required conditions.

Lemma 2.2. *Assume that the operator A in (1.1) (with coefficients $a_{ij}=a_{ji}, b_i \in W^{1,\infty}_{loc}$ and $c \in L^\infty_{loc}$) satisfies the ellipticity condition (1.4). Suppose also that for some $p\in(1,\infty)$, $c-\frac{\nabla\cdot b}{p}\geq 0$ and there are $s>0, \lambda>1$ and a C^1 function $\rho\geq 1$ with $\lim_{|x|\to\infty}\rho(x)=\infty$ such that*

$$M_{\lambda,s,p}(\rho) := \sup_{r\geq 1} e^{-psr}M_\lambda(\rho,r) < \infty, \tag{2.4}$$

where $M_\lambda(\rho,r)$ is given by (2.1).
Then, for every $u\in D(A_p)$,

$$\int_{\mathbb{R}^N} e^{-ps\rho}|u|^{p-2}a(\cdot,\nabla u,\nabla u)\chi_{\{u\neq 0\}} \leq 3C(\lambda,p)\int_{\mathbb{R}^N}\left(|Au|^p + M_{\lambda,s,p}(\rho)|u|^p\right), \tag{2.5}$$

with $C(\lambda,p)$ from Lemma 2.1. In particular, $e^{-ps\rho}|u|^{p-2}a(\cdot,\nabla u,\nabla u)\chi_{\{u\neq 0\}}\in L^1$.

Proof. Recall that $D(A_p) \subset W^{2,p}_{loc}$ under the standing assumptions (see (1.5)). Let then $u \in D(A_p)$ be given. Since $\rho \geq 1$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} e^{-ps\rho}|u|^{p-2}a(\cdot,\nabla u,\nabla u)\chi_{\{u\neq 0\}} \\
&\quad = \sum_{n=0}^{\infty}\int_{\{\lambda^n\leq\rho<\lambda^{n+1}\}} e^{-ps\rho}|u|^{p-2}a(\cdot,\nabla u,\nabla u)\chi_{\{u\neq 0\}} \\
&\quad \leq \sum_{n=0}^{\infty} e^{-ps\lambda^n}\int_{\{\lambda^n\leq\rho\leq\lambda^{n+1}\}} |u|^{p-2}a(\cdot,\nabla u,\nabla u)\chi_{\{u\neq 0\}}.
\end{aligned}$$

By Lemma 2.1 and the assumption $c - \frac{\nabla\cdot b}{p} \geq 0$,

$$\begin{aligned}
&\int_{\{\lambda^n\leq\rho\leq\lambda^{n+1}\}} |u|^{p-2}a(\cdot,\nabla u,\nabla u)\chi_{\{u\neq 0\}} \\
&\qquad \leq C(\lambda,p)\int_{\{\lambda^{n-1}<\rho<\lambda^{n+2}\}} \left(|Au|^p + M_\lambda(\rho,\lambda^n)|u|^p\right),
\end{aligned}$$

so that, by (2.4),

$$\begin{aligned}
&\int_{\mathbb{R}^N} e^{-ps\rho}|u|^{p-2}a(\cdot,\nabla u,\nabla u)\chi_{\{u\neq 0\}} \\
&\quad \leq C(\lambda,p)\sum_{n=0}^{\infty}\int_{\{\lambda^{n-1}\leq\rho<\lambda^{n+2}\}} \left(e^{-ps\lambda^n}|Au|^p + e^{-ps\lambda^n}M_\lambda(\rho,\lambda^n)|u|^p\right) \\
&\quad \leq C(\lambda,p)\sum_{n=0}^{\infty}\int_{\{\lambda^{n-1}\leq\rho<\lambda^{n+2}\}} \left(|Au|^p + M_{\lambda,s,p}(\rho)|u|^p\right).
\end{aligned}$$

The desired inequality (2.5) now follows by writing the integral over $\{\lambda^{n-1} \leq \rho < \lambda^{n+2}\}$ as the sum of the integrals over the sets $\{\lambda^{n-1} \leq \rho < \lambda^n\}, \{\lambda^n \leq \rho < \lambda^{n+1}\}$ and $\{\lambda^{n+1} \leq \rho < \lambda^{n+2}\}$. □

More generally, Lemma 2.2 remains valid if ρ is bounded from below by some positive constant. This can be seen by a straightforward modification of the above proof.

3. The injectivity of A_p

Given $p \in (1,\infty)$, the injectivity of the operator A_p is established in Theorem 3.2 below, under suitable assumptions about the coefficients of A. The proof is based on the estimate of Lemma 2.2 and on the following result.

Lemma 3.1. *Let ρ be a C^N function on $\mathbb{R}^N$ such that $\sup\rho = \infty$ and let $g : \mathbb{R}^N \to \mathbb{R}$ be Lebesgue measurable. If $g|\nabla\rho| \in L^1$, there is a sequence (r_n) of regular values of ρ with $\lim r_n = \infty$ such that $\{\rho = r_n\} \neq \emptyset$ and $\lim\int_{\{\rho=r_n\}} g d\mathcal{H}^{N-1} = 0$, where $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure.*

Proof. By Federer's coarea formula, the function $G(r) := \int_{\{\rho=r\}} g d\mathcal{H}^{N-1}$ is in $L^1(\mathbb{R})$ and $\int_{\mathbb{R}^N} g|\nabla\rho| = \int_{\mathbb{R}}\left(\int_{\{\rho=r\}} g d\mathcal{H}^{N-1}\right) dr$. More precisely, since ρ is Lipschitz continuous on the bounded subsets of $\mathbb{R}^N$, then if $E \subset \mathbb{R}^N$ is bounded and Lebesgue measurable, we have

$$\int_E g|\nabla\rho| = \int_{\mathbb{R}} dr \int_{\{\rho=r\}\cap E} g d\mathcal{H}^{N-1}$$

(Federer [7, 3.2.22], Ambrosio, Fusco and Pallara [1, Theorem 2.93 and Remark 2.94]; see also Ohtsuka [11]). The claim follows by using this formula with $E := \{\rho < R\}$ and letting $R \to \infty$ after splitting g into its positive and negative parts.

Since ρ is of class C^N, it follows from Sard's theorem that the set of critical values of ρ has Lebesgue measure 0, so that the function $\int_{\{\rho=r\}} g d\mathcal{H}^{N-1}$ may be replaced by ∞ for every critical value r of ρ without affecting the relation $G \in L^1(\mathbb{R})$. As is well known, the latter implies the existence of a sequence (r_n) with $\lim r_n = \infty$ such that $\lim G(r_n) = 0$, i.e., $\lim \int_{\{\rho=r_n\}} g d\mathcal{H}^{N-1} = 0$. Clearly, r_n must then be a regular value of ρ for n large enough. □

Theorem 3.2. *Assume that the operator A in (1.1) (with coefficients $a_{ij} = a_{ji}, b_i \in W^{1,\infty}_{loc}$ and $c \in L^\infty_{loc}$) satisfies the ellipticity condition (1.4) and that for some $p \in (1,\infty)$, there are $0 < s' < s$ and a C^N function[2] $\rho \geq 1$ with $\lim_{|x|\to\infty} \rho(x) = \infty$ and $\nabla\rho \neq 0$ a.e. on $\mathbb{R}^N$ such that:*
(i) $c - \frac{\nabla\cdot b}{p} \geq 0$.
(ii) $c^{(s)} - \frac{\nabla\cdot b^{(s)}}{p} \geq 0$,
where b is defined in (1.9) and

$$b_i^{(s)} := b_i - 2s\sum_{j=1}^N a_{ji}\partial_j\rho, \tag{3.1}$$

$$c^{(s)} := c + s\left(b\cdot\nabla\rho - \sum_{i,j=1}^N \partial_j a_{ij}\partial_i\rho - \sum_{i,j=1}^N a_{ij}\partial^2_{ij}\rho\right) - s^2 a(\cdot,\nabla\rho,\nabla\rho). \tag{3.2}$$

(iii) $e^{-ps'\rho}a(\cdot,\nabla\rho,\nabla\rho) \in L^\infty$ (see (1.8)).
(iv) $e^{-ps'\rho}b\cdot\nabla\rho \in L^\infty$.
Then, $u \in D(A_p)$ and $Au = 0$ imply $u = 0$.

Proof. Let $u \in D(A_p)$ be such that $Au = 0$. Throughout this proof, we set

$$v := e^{-s\rho}u. \tag{3.3}$$

so that $v \in L^p \cap W^{2,p}_{loc}$ (see (1.5)) and $A^{(s)}v = 0$, where $A^{(s)} := e^{-s\rho}A(e^{s\rho}\cdot)$. Since $u = 0$ if and only if $v = 0$, it suffices to show that $v = 0$ to establish the injectivity of A_p. The first step consists in noticing that $A^{(s)}$ is actually the differential operator

$$A^{(s)} := -\sum_{i,j=1}^N \partial_i(a_{ij}(x)\partial_j) + \sum_{i=1}^N b_i^{(s)}(x)\partial_i + c^{(s)}(x), \tag{3.4}$$

with $b_i^{(s)}$ and $c^{(s)}$ given by (3.1) and (3.2), respectively.

[2]If $N \geq 2$. When $N = 1$, it must also be assumed that ρ' is locally Lipschitz continuous, so that $c^{(s)} \in L^\infty_{loc}$ in (3.2), a property implicitly used in the proof.

Let $r > 0$ be a regular value of ρ. When nonempty, the set $\{\rho = r\}$ is a C^N hypersurface of $\mathbb{R}^N$ and the boundary of the bounded[3] open set $\{\rho < r\}$. Furthermore, the outward unit normal vector $\nu(x)$ along $\{\rho = r\}$ is well defined and given by

$$\nu(x) = \nabla\rho(x)/|\nabla\rho(x)|. \tag{3.5}$$

Suppose first that $p \geq 2$. By multiplying $A^{(s)}v(=0)$ by $|v|^{p-2}v$ and integrating by parts over the set $\{\rho < r\}$ with $r > 0$ as above, it follows from (3.4) that (if $p \in (1,2)$, this procedure must once again be justified by [10])

$$\begin{aligned}
0 &= \int_{\{\rho<r\}} (A^{(s)}v)|v|^{p-2}v \\
&= (p-1)\int_{\{\rho<r\}} |v|^{p-2}a(\cdot,\nabla v,\nabla v)\chi_{\{v\neq 0\}} + \int_{\{\rho<r\}} \left(c^{(s)} - \frac{\nabla\cdot b^{(s)}}{p}\right)|v|^p \\
&\quad + \int_{\{\rho=r\}} \left(\frac{|v|^p b^{(s)}\cdot\nu}{p} - |v|^{p-2}va(\cdot,\nabla v,\nu)\right) d\mathcal{H}^{N-1},
\end{aligned}$$

where we have used the standard fact that $\mathcal{H}^{N-1}$ coincides with the $(N-1)$-dimensional Lebesgue measure on C^1 hypersurfaces of $\mathbb{R}^N$. Using (3.3), $\nabla v = e^{-s\rho}(\nabla u - su\nabla\rho)$, which in turn may be used to rewrite the boundary integral in terms of u in the above formula. Together with (3.5), this yields

$$\begin{aligned}
&(p-1)\int_{\{\rho<r\}} |v|^{p-2}a(\cdot,\nabla v,\nabla v)\chi_{\{v\neq 0\}} + \int_{\{\rho<r\}} \left(c^{(s)} - \frac{\nabla\cdot b^{(s)}}{p}\right)|v|^p \\
&= \int_{\{\rho=r\}} \frac{e^{-ps\rho}}{|\nabla\rho|}\left[|u|^{p-2}ua(\cdot,\nabla u,\nabla\rho) - \left(sa(\cdot,\nabla\rho,\nabla\rho) + \frac{b^{(s)}\cdot\nabla\rho}{p}\right)|u|^p\right] d\mathcal{H}^{N-1} \\
&\leq \int_{\{\rho=r\}} \frac{e^{-ps\rho}}{|\nabla\rho|}\left[|u|^{p-1}|a(\cdot,\nabla u,\nabla\rho)| + \left(sa(\cdot,\nabla\rho,\nabla\rho) + \frac{|b^{(s)}\cdot\nabla\rho|}{p}\right)|u|^p\right] d\mathcal{H}^{N-1}.
\end{aligned}$$

By writing

$$\begin{aligned}
|u|^{p-1}|a(\cdot,\nabla u,\nabla\rho)| &\leq |u|^{p-1}a(\cdot,\nabla u,\nabla u)^{1/2}a(\cdot,\nabla\rho,\nabla\rho)^{1/2} \\
&= |u|^{\frac{p-2}{2}}a(\cdot,\nabla u,\nabla u)^{1/2}|u|^{\frac{p}{2}}a(\cdot,\nabla\rho,\nabla\rho)^{1/2} \\
&\leq \frac{1}{2}|u|^{p-2}a(\cdot,\nabla u,\nabla u) + \frac{1}{2}|u|^p a(\cdot,\nabla\rho,\nabla\rho),
\end{aligned}$$

when $u \neq 0$, the above inequality becomes

$$\begin{aligned}
&(p-1)\int_{\{\rho<r\}} |v|^{p-2}a(\cdot,\nabla v,\nabla v)\chi_{\{v\neq 0\}} + \int_{\{\rho<r\}} \left(c^{(s)} - \frac{\nabla\cdot b^{(s)}}{p}\right)|v|^p \\
&\quad \leq \int_{\{\rho=r\}} g\,d\mathcal{H}^{N-1},
\end{aligned} \tag{3.6}$$

where

$$g := \frac{e^{-ps\rho}}{|\nabla\rho|}\left[\frac{1}{2}|u|^{p-2}a(\cdot,\nabla u,\nabla u)\chi_{\{u\neq 0\}} + \left(\frac{2s+1}{2}a(\cdot,\nabla\rho,\nabla\rho) + \frac{1}{p}|b^{(s)}\cdot\nabla\rho|\right)|u|^p\right]$$

[3]Since $\lim_{|x|\to\infty}\rho(x) = \infty$.

In fact, (3.6) holds when r is any regular value of ρ, i.e., also when $\{\rho = r\} = \emptyset$, for then both sides are 0 (by the intermediate value theorem and the assumption $\lim_{|x|\to\infty} \rho(x) = \infty$, it follows that $\{\rho < r\} = \emptyset$ whenever $\{\rho = r\} = \emptyset$).

The function g is well defined a.e. on $\mathbb{R}^N$ and Lebesgue measurable since $\nabla\rho \neq 0$ a.e. and

$$g|\nabla\rho| = \frac{e^{-ps\rho}}{2}|u|^{p-2}a(\cdot, \nabla u, \nabla u)\chi_{\{u\neq 0\}} + e^{-ps\rho}\left(\frac{2s+1}{2}a(\cdot, \nabla\rho, \nabla\rho) + \frac{|b^{(s)} \cdot \nabla\rho|}{p}\right)|u|^p.$$

We claim that, above, $e^{-ps\rho}|u|^{p-2}a(\cdot, \nabla u, \nabla u)\chi_{\{u\neq 0\}} \in L^1$. To avoid disrupting the proof with technicalities, this is shown in Lemma 3.3 below. On the other hand, $e^{-ps\rho}\left(\frac{2s+1}{2}a(\cdot, \nabla\rho, \nabla\rho) + \frac{|b^{(s)}\cdot\nabla\rho|}{p}\right)|u|^p \in L^1$ from the hypotheses (iii) and (iv) since $u \in L^p$. In that regard, note that (iii) and (iv) imply $e^{-ps'\rho}b^{(s)} \cdot \nabla\rho \in L^\infty$ (see (3.1)), whence $e^{-ps\rho}b^{(s)} \cdot \nabla\rho \in L^\infty$ since $s > s'$. Likewise, $e^{-ps\rho}a(\cdot, \nabla\rho, \nabla\rho) \in L^\infty$ by (iv).

This shows that $g|\nabla\rho| \in L^1$, so that Lemma 3.1 ensures the existence of a sequence (r_n) of regular values of ρ with $\lim r_n = \infty$ such that the right-hand side of (3.6) with $r = r_n$ tends to 0 as $n \to \infty$. Since $c^{(s)} - \frac{\nabla\cdot b^{(s)}}{p} \geq 0$ by the hypothesis (ii), it follows from (3.6) that

$$\lim \int_{\{\rho<r_n\}} |v|^{p-2}a(\cdot, \nabla v, \nabla v)\chi_{\{v\neq 0\}} = 0.$$

Since $r_n \to \infty$, Fatou's lemma shows that $\int_{\mathbb{R}^N} |v|^{p-2}a(\cdot, \nabla v, \nabla v)\chi_{\{v\neq 0\}} = 0$, so that $|v|^{p-2}a(\cdot, \nabla v, \nabla v)\chi_{\{v\neq 0\}} = 0$ a.e.. By the ellipticity of A, this implies that $\nabla v = 0$ a.e. on the set $\{v \neq 0\}$. Since also $\nabla v = 0$ a.e. on the set $\{v = 0\}$, we infer that $\nabla v = 0$ a.e.. Thus, v is constant, so that $v = 0$ since $v \in L^p$. This completes the proof. □

We now justify the claim that $e^{-ps\rho}|u|^{p-2}a(\cdot, \nabla u, \nabla u)\chi_{\{u\neq 0\}} \in L^1$ for every $u \in D(A_p)$, used in a crucial way in the above proof.

Lemma 3.3. *Under the assumptions of Theorem 3.2,*

$$u \in D(A_p) \Rightarrow e^{-ps\rho}|u|^{p-2}a(\cdot, \nabla u, \nabla u)\chi_{\{u\neq 0\}} \in L^1.$$

Proof. The assertion follows from Lemma 2.2 and the remark that, in that lemma, $M_{\lambda,s,p}(\rho) < \infty$ with the choice $\lambda := \sqrt{s/s'}$ (> 1). To see this, let $r \geq 1$ be fixed. By (2.1) and the compactness of the set $\{\lambda^{-1}r \leq \rho(x) \leq \lambda^2 r\}$ (and the continuity of the coefficients a_{ij} and b_i), there is $x_r \in \mathbb{R}^N$ such that

$$\lambda^{-1}r \leq \rho(x_r) \leq \lambda^2 r \tag{3.7}$$

and that

$$M_\lambda(\rho, r) = 1 + |b(x_r) \cdot \nabla\rho(x_r)| + a(x_r, \nabla\rho(x_r), \nabla\rho(x_r)).$$

On the other hand, $r \geq \lambda^{-2}\rho(x_r)$ by (3.7), so that $e^{-psr} \leq e^{-ps\lambda^{-2}\rho(x_r)} = e^{-ps'\rho(x_r)}$ since $\lambda^2 = s/s'$. Therefore,

$$\begin{aligned} e^{-psr}M_\lambda(\rho, r) &\leq e^{-ps'\rho(x_r)}(1 + |b(x_r) \cdot \nabla\rho(x_r)| + a(x_r, \nabla\rho(x_r), \nabla\rho(x_r))) \\ &\leq 1 + ||e^{-ps'\rho}|b \cdot \nabla\rho|||_{0,\infty} + ||e^{-ps'\rho}a(\cdot, \nabla\rho, \nabla\rho)||_{0,\infty} < \infty, \end{aligned}$$

by the hypotheses (iii) and (iv) of Theorem 3.2. Thus, $M_{\lambda,s,p}(\rho) < \infty$ by (2.4). □

The hypothesis (i) of Theorem 3.2 and a somewhat weaker form of (ii) and (iv) are already used in [12, Theorem 3.3] when $p = 2$ and $a_{ij} \in L^\infty$, under the extra assumption that $e^{-t\rho}\nabla\rho \in L^\infty$ for every $t > 0$ (so that (iii) holds). Although the general idea to prove injectivity is the same, the technicalities are very different. In particular, no use of the coarea formula is made in [12] . Also, since the class of admissible functions ρ is broader (except for smoothness), the results of this paper have value even when $p = 2$ and $a_{ij} \in L^\infty$.

Remark 3.1. *It may be useful to point out that Theorem 3.2 remains true when ρ is C^2 irrespective of N and without the requirement that $\nabla\rho \neq 0$ a.e. in (at least) two cases. The first one is when $\nabla\rho$ does not vanish outside a ball, for then Lemma 3.1 remains obviously true without using Sard's theorem. The second case is when the coefficients a_{ij} are C^1, the coefficient c is C^0 and (ii) holds in the slightly stronger form $c^{(s)} - \frac{\nabla\cdot b^{(s)}}{p} > 0$. If so, it follows easily from the classical results on the approximation by real-analytic functions (Whitney [14]) that the hypotheses of Theorem 3.2 continue to hold when ρ is replaced by a close enough real-analytic approximation $\widetilde{\rho}$. In particular, $\nabla\widetilde{\rho} \neq 0$ a.e. since $\widetilde{\rho}$ is not constant.*

Remark 3.2. *To check condition (ii) of Theorem 3.2 in practice, note that, in light of (3.1) and (3.2),*

$$c^{(s)} - \frac{\nabla\cdot b^{(s)}}{p} = c - \frac{\nabla\cdot b}{p} - s^2 a(\cdot,\nabla\rho,\nabla\rho) + s\left[b\cdot\nabla\rho + \left(\frac{2}{p}-1\right)\left(\sum_{i,j=1}^N \partial_j a_{ij}\partial_i\rho + \sum_{i,j=1}^N a_{ij}\partial_{ij}^2\rho\right)\right] \tag{3.8}$$

where $\partial_{ij}^2\rho = \partial_{ji}^2\rho$ was used (since ρ is at least C^2 when $N \geq 2$). If $p = 2$, this reduces to

$$c^{(s)} - \frac{\nabla\cdot b^{(s)}}{2} = c - \frac{\nabla\cdot b}{2} + sb\cdot\nabla\rho - s^2 a(\cdot,\nabla\rho,\nabla\rho). \tag{3.9}$$

From Theorem 3.2, one can deduce concrete conditions on the coefficients by choosing the function ρ. For example, if $\rho(x) = Log(1 + |x|)$ for $|x|$ large enough, then (iii) and (iv) of Theorem 3.2 hold when the coefficients a and b are polynomially bounded in $|x|$ if p is large enough; see Corollary 4.5 and the examples following that corollary.

4. The denseness of C_0^∞ in $D(A_p)$

Given $p \in (1,\infty)$ we shall show that under hypotheses slightly weaker than those of Theorem 3.2, C_0^∞ is a core of A_p, that is, dense in $D(A_p)$ equipped with the graph norm. In what follows, A^* denote the formal adjoint of the differential operator A in (1.1), given by

$$A^* := -\sum_{i,j=1}^N \partial_i(a_{ij}(x)\partial_j) - \sum_{i=1}^N \partial_i(b_i(x)\cdot) + c(x). \tag{4.1}$$

Observe that A^* may be written in the form

$$A^* := -\sum_{i,j=1}^N \partial_j(a_{ij}^*(x)\partial_i) + \sum_{i=1}^N b_i^*(x)\partial_i + c^*(x), \tag{4.2}$$

with

$$a^*_{ij}(x) = a_{ij} \in W^{1,\infty}_{loc}, b^*_i = -b_i \in W^{1,\infty}_{loc} \text{ and } c^* = c - \nabla \cdot b \in L^\infty_{loc}, \tag{4.3}$$

so that there is a well defined operator $(A^*)_{p'}$ obtained by replacing A by A^* and p by p'.

We shall need the following surjectivity result.

Lemma 4.1. *Assume that A satisfies the ellipticity condition (1.4) and that, for some $p \in (1, \infty)$, there is a constant $\beta_p > 0$ such that*

$$c - \frac{\nabla \cdot b}{p} \geq \beta_p. \tag{4.4}$$

Then, $A_p : D(A_p) \to L^p$ is onto.

Proof. For $n \in \mathbb{N}$, let B_n denote the open ball with center 0 and radius n in $\mathbb{R}^N$ and, given $f \in L^p$, let $u_n \in W^{2,p}(B_n) \cap W^{1,p}_0(B_n)$ be the unique solution of

$$Au_n = f \text{ in } B_n. \tag{4.5}$$

Multiply both sides of (4.4) by $|u_n|^{p-2}u_n$ and integrate by parts (with the help of [10] when $p \in (1, 2)$). This yields

$$(p-1)\int_{B_n} |u_n|^{p-2} a(\cdot, \nabla u_n, \nabla u_n)\chi_{\{u_n \neq 0\}} + \int_{B_n} \left(c - \frac{\nabla \cdot b}{p}\right)|u_n|^p = \int_{B_n} f|u_n|^{p-2}u_n.$$

Therefore, by (4.4) and the ellipticity of A,

$$\beta_p ||u_n||^p_{0,p,B_n} \leq \int_{B_n} f|u_n|^{p-2}u_n \leq ||f||_{0,p,B_n}||u_n||^{p-1}_{0,p,B_n} \leq ||f||_{0,p}||u_n||^{p-1}_{0,p,B_n}$$

It follows that $||u_n||_{0,p,B_n} \leq \beta_p^{-1}||f||_{0,p}$, so that the sequence (u_n) is bounded in L^p, where u_n is identified with its extension by 0 outside B_n. Therefore, there are $u \in L^p$ and a subsequence (u_{n_ℓ}) such that $u_{n_\ell} \overset{w}{\rightharpoonup} u$ in L^p. Furthermore, if $\varphi \in C^\infty_0$, then $\operatorname{Supp}\varphi \subset B_{n_\ell}$ for ℓ large enough, so that $\langle Au_{n_\ell}, \varphi\rangle = \int_{\mathbb{R}^N} f\varphi$ for ℓ large enough by (4.5). Since Au_{n_ℓ} tends to Au as a distribution (note that $Au_{n_\ell} \overset{w}{\rightharpoonup} Au$ in $W^{-2,p}$ due to the hypotheses made about the coefficients of A), this shows that $Au = f$. Thus, $u \in D(A_p)$ and the proof is complete. □

Theorem 4.2. *Assume that the operator A in (1.1) (with coefficients $a_{ij} = a_{ji}, b_i \in W^{1,\infty}_{loc}$ and $c \in L^\infty_{loc}$) satisfies the ellipticity condition (1.4) and that for some $p \in (1, \infty)$, there are $0 < s' < s$ and a C^N function[4] $\rho \geq 1$ with $\lim_{|x|\to\infty} \rho(x) = \infty$ and $\nabla\rho \neq 0$ a.e. on $\mathbb{R}^N$ such that:*
(i) $\inf\left(c - \frac{\nabla \cdot b}{p}\right) > -\infty$.
(ii) $\inf\left(c^{(\pm s)} - \frac{\nabla \cdot b^{(\pm s)}}{p}\right) > -\infty$,
where $b^{(s)}$ and $c^{(s)}$ are defined by (3.1) and (3.2), respectively.
(iii) $e^{-ps'\rho} b \cdot \nabla\rho \in L^\infty$ *(see (1.9)).*
(iv) $e^{-ps'\rho} a(\cdot, \nabla\rho, \nabla\rho) \in L^\infty$ *(see (1.8)).*
Then, C^∞_0 is a core of $D(A_p)$.

[4]If $N \geq 2$. When $N = 1$, it must also be assumed that ρ' is locally Lipschitz continuous, so that $c^{(s)} \in L^\infty_{loc}$ in (3.2), a property implicitly used in the proof.

Proof. Evidently, $D(A_p)$ is unchanged if A is changed into $A+\lambda$ and $\lambda \in \mathbb{R}$. By choosing $\lambda > 0$ large enough, it follows from (i) and (ii) that we may assume that

$$c - \frac{\nabla \cdot b}{p} \geq 1 \text{ and } c^{(\pm s)} - \frac{\nabla \cdot b^{(\pm s)}}{p} > 0. \tag{4.6}$$

Indeed, the change of A into $A+\lambda$ does not affect the coefficients a_{ij} and b, hence has no impact on (iii) and (iv) above, while c becomes $c+\lambda$, which increases both $c - \frac{\nabla \cdot b}{p}$ and $c^{(\pm s)} - \frac{\nabla \cdot b^{(\pm s)}}{p}$ by λ.

With the extra assumption (4.6) replacing (i) and (ii), it follows from Theorem 3.2 that A_p is one to one on $D(A_p)$ while, by (4.6) and Lemma 4.1, A_p is onto L^p. Thus, A_p is an isomorphism of $D(A_p)$ onto L^p. Now, a straightforward verification shows that the hypotheses of Theorem 4.2 are unchanged upon replacing A by A^* and p by p'. In addition, such a change does not affect (4.6) either, because

$$c^* - \frac{\nabla \cdot b^*}{p'} = c - \frac{\nabla \cdot b}{p} \text{ and } c^{*(s)} - \frac{\nabla \cdot b^{*(s)}}{p'} = c^{(-s)} - \frac{\nabla \cdot b^{(-s)}}{p}.$$

As a result, Theorem 3.2 for A^* and p' shows that $(A^*)_{p'}$ is one to one on $D((A^*)_{p'})$. In turn, this implies that $A(C_0^\infty)$ is dense in L^p. Otherwise, there is some $h \in L^{p'}$ with $h \neq 0$ and $\int_{\mathbb{R}^N} hA\varphi = 0$ for every $\varphi \in C_0^\infty$, i.e., $\langle h, A\varphi \rangle = 0$ for every $\varphi \in C_0^\infty$. In other words, the (well defined, by (4.2) and (4.3)) distribution A^*h is 0. But $h \in L^{p'}$ and $A^*h = 0$ together mean that $h \in D((A^*)_{p'})$ and $(A^*)_{p'}h = 0$, so that $h = 0$, which is a contradiction. This proves the claim.

In summary, $A_p(C_0^\infty)$ is dense in L^p and A_p is an isomorphism of $D(A_p)$ to L^p. It follows that the inverse isomorphism $(A_p)^{-1}$ maps $A_p(C_0^\infty)$ onto a dense subset of $D(A_p)$. Since $(A_p)^{-1}(A_p(C_0^\infty)) = C_0^\infty$, this amounts to saying that C_0^∞ is dense in $D(A_p)$. ☐

From the proof given above, the hypotheses of Theorem 4.2 also suffice for C_0^∞ to be dense in $D((A^*)_{p'})$.

Remark 4.1. *If the coefficients a_{ij} and b_i are C^1 and c is C^0, Theorem 4.2 remains true when ρ is C^2 irrespective of N and without the requirement that $\nabla\rho \neq 0$ a.e. (use Remark 3.1 in the proof; that b_i is C^1 is needed to ensure that c^* is C^0 and hence that Remark 3.1 can be used with A^*).*

In Theorem 4.2, the hypothesis $\inf\left(c^{(s)} - \frac{\nabla \cdot b^{(s)}}{p}\right) > -\infty$ is used to ensure that A_p is an isomorphism of $D(A_p)$ onto L^p after replacing A by $A+\lambda$ for sufficiently large $\lambda \geq 0$. In fact, if the requirement that the domain coincides with the "maximal" domain $D(A_p)$ is relaxed, the following result by Arendt, Metafune and Pallara [2, Theorem 3.1] shows that this hypothesis is not needed.

Theorem 4.3. *Suppose that $c - \frac{\nabla \cdot b}{p} \geq 0$. Given $p \in (1,\infty)$, there exists a unique resolvent positive operator $A_p' \subset A_p$ which is minimal among the resolvent positive restrictions of A_p, i.e., if $B_p \subset A_p$ is resolvent positive, then $R(\mu, B_p) \geq R(\mu, A_p')$ for every $\mu > 0$.*

Even though extra smoothness and boundedness are assumed of the coefficients of A in [2], the proof of Theorem 4.3 does not require more than the standing assumptions $a_{ij} = a_{ji}, b_i \in W^{1,\infty}_{loc}$ and $c \in L^\infty_{loc}$. We also refer to [2] (where A_p' and A_p are called A_p and $A_{p,\max}$, respectively) for the terminology "resolvent positive". Here, the more important point is that, by Theorem 4.3, the condition $\inf\left(c - \frac{\nabla \cdot b}{p}\right) > -\infty$ alone ensures that $A + \lambda$ is an isomorphism from $D(A'_p) \subset D(A_p)$ onto L^p, if $\lambda \geq 0$ is large enough. Of course, in this statement, $D(A'_p)$ is equipped with the graph norm. By using this in the proof of Theorem 4.2, we obtain the following generalization:

Theorem 4.4. *In Theorem 4.2, replace condition (ii) by the weaker*
(ii') $\inf\left(c^{(-s)} - \frac{\nabla \cdot b^{(-s)}}{p}\right) > -\infty$.
Then, C_0^∞ is a core of $D(A'_p)$.

As a special case of Theorem 4.4, we mention the following corollary.

Corollary 4.5. *Assume that $a_{ij} = a_{ji} \in C_b^1$ (i.e., C^1 bounded with bounded first derivatives), $b_i \in C^1$ and $c \in C^0$. Assume also that there is a nonnegative function $z \in C^2$ such that $\lim_{|x| \to \infty} z(x) = \infty$, that the first and second derivatives of $Log(1+z)$ are bounded on $\mathbb{R}^N$ and that*

$$b \cdot \nabla z \leq M(1+z)\left(1 + c - \frac{\nabla \cdot b}{p}\right) \text{ in } \mathbb{R}^N \backslash B_R, \tag{4.7}$$

$$\sup \frac{|b \cdot \nabla z|}{(1+z)^{q+1}} < \infty, \tag{4.8}$$

for some constants $M, q \geq 0$ and some open ball $B_R \subset \mathbb{R}^N$. Then, C_0^∞ is a core of $D(A'_p)$ if $c - \frac{\nabla \cdot b}{p} \geq 0$ in $\mathbb{R}^N \backslash B_R$ and either (a) $p > 1$ and $\sup\left(c - \frac{\nabla \cdot b}{p}\right) < \infty$ or (b) $p > \max\{Mq, 1\}$ and $\sup\left(c - \frac{\nabla \cdot b}{p}\right) = \infty$.

Proof. Since the coefficients a_{ij} and their first derivatives are bounded and the first and second derivatives of $Log(1+z)$ are also bounded, it follows from (4.7) that the condition (ii') of Theorem 4.4 is satisfied with $\rho := Log(1+z)$ and any $s > 0$ if $\sup\left(c - \frac{\nabla \cdot b}{p}\right) < \infty$ or with $s = M^{-1}$ if $\sup\left(c - \frac{\nabla \cdot b}{p}\right) = \infty$ (use (3.8) with s replaced by $-s$ and Remark 4.1 since ρ is only C^2).

Also, the conditions (i) and (iv) of Theorem 4.4 (i.e., 4.2) hold trivially, the latter with any $0 < s' < s$, while, by (4.8), condition (iii) holds with $s' = q/p \in [0, s)$, provided only that s is chosen large enough in case (a). Thus, Theorem 4.4 yields the claimed core property. □

More generally, Corollary 4.5 remains true if (4.7) is replaced by

$$b \cdot \nabla z \leq M(1+z)\left(K + c - \frac{\nabla \cdot b}{p}\right) \text{ in } \mathbb{R}^N \backslash B_R,$$

where $K \geq 0$ is another constant. Corollary 4.5 should be compared with [2, Theorem 4.1], where the denseness of C_0^∞ in $D(A'_p)$ is obtained under the condition that the first derivatives of $Log(1+z)$ are bounded (the boundedness of the second derivatives is not required), that $c - \frac{\nabla \cdot b}{p} > 0$ and that

$$b \cdot \nabla z \leq M(1+z)\left(1 + c - \frac{\nabla \cdot b}{p}\right)^\alpha, \tag{4.9}$$

for some constants $M \geq 0$ and $\alpha \in [0, 1)$. Since (4.7) corresponds to the case $\alpha = 1$ in (4.9), it is clearly more general than the latter. In fact, if $\sup\left(c - \frac{\nabla \cdot b}{p}\right) = \infty$, then (4.9) implies (4.7) with M replaced by any $\varepsilon > 0$ after increasing R if necessary.

Thus, upon replacing (4.7) by (4.9), it follows from Corollary 4.5 that C_0^∞ is a core of $D(A'_p)$ for every $p > 1$ such that $c - \frac{\nabla \cdot b}{p} > 0$ outside a ball, regardless of the value of q in (4.8). However, since (4.8) is not retained at all in [2, Theorem 4.1], this is not a full generalization.

Roughly speaking, (4.8) limits the growth of b to exponential (since z grows at most exponentially in Corollary 4.5). On the other hand, Theorem 4.4 is much more general than its special case in Corollary 4.5 and, in particular, it is valid without any boundedness condition about the coefficients a_{ij}. This is not the case in [2, Theorem 4.1], where the boundedness of a_{ij} is used in the proof.

Remark 4.2. *If $N \geq 2$ and z is C^N with $\nabla z \neq 0$ a.e., then Corollary 4.5 is valid when $a_{ij} = a_{ji} \in W^{1,\infty}, b_i \in W^{1,\infty}_{loc}$ and $c \in L^\infty_{loc}$. Also, if (4.7) is replaced by the stronger condition $|b \cdot \nabla z| \leq M(1+z)\left(1 + c - \frac{\nabla \cdot b}{p}\right)$, then Theorem 4.2 is applicable, so that A'_p can be replaced by A_p in Corollary 4.5.*

Example 4.1. *Let $A := -\Delta + |x|^{\beta-1}x \cdot \nabla + c_0|x|^\gamma$ where $c_0 > 0, \beta \geq 1$ and $\gamma \geq 0$. Then, C_0^∞ is a core of $D(A_p)$ if $\gamma > \beta - 1$ or if $\gamma = \beta - 1$ and $c_0 > p^{-1}(N + \beta - 1)$. This follows from Corollary 4.5 with $z(x) := |x|^\alpha$ for large $|x|$ and $\alpha > p^{-1}(\beta - 1)$ and from the second part of Remark 4.2. If $N = 1, \beta = 3$ and $\gamma = 2$, this justifies the statement in* [2, Proposition 6.3], *whose argument for this case (and this case only) is incorrect. In addition, that A'_p may be replaced by A_p does not follow from* [2].

Example 4.2. *In Example 4.1, change the drift term into its negative, i.e., let $A := -\Delta - |x|^{\beta-1}x \cdot \nabla + c_0|x|^\gamma$. Then, C_0^∞ is a core of $D(A'_p)$ for* every *$\beta \geq 1$ and every $\gamma \geq 0$. This follows from Corollary 4.5 with the same choice $z(x) := |x|^\alpha, \alpha > p^{-1}(\beta - 1)$ as before. However, the second part of Remark 4.2 to the effect that A'_p may be replaced by A_p in this statement is only valid under the same restriction $\gamma > \beta - 1$ or $\gamma = \beta - 1$ and $c_0 > p^{-1}(N + \beta - 1)$ as in Example 4.1.*

Other natural choices of z in the above examples, such as $z(x) := e^{a|x|}$ $(a > 0)$ lead to weaker results.

5. On a problem of Kato

We shall illustrate Theorem 4.2 with the simple case when $p = 2$ and

$$A := -\sum_{i,j=1}^{N} \partial_i(a_{ij}(x)\partial_i) + c(x), \tag{5.1}$$

(so that $b = 0$) and $a_{ij} = a_{ji} \in W^{1,\infty}_{loc}$ and $c \in L^\infty_{loc}$ satisfy the conditions

$$\sum_{i,j=1}^{N} a_{ij}(x)\frac{x_i x_j}{r^2} \leq k_0(1+r)^\tau \tag{5.2}$$

and

$$c(x) \geq c_0 r^\theta, \tag{5.3}$$

for $r := |x| > 0$ large enough, where $k_0, c_0 > 0$ and $\tau, \theta \geq 0$ are constants. Of course, we also assume that the ellipticity condition (1.4) holds.

Theorem 5.1. *Under the above assumptions, C_0^∞ is a core of $D(A_2)$ if $0 \leq \tau \leq \theta + 2$.*

Proof. The hypotheses (i) to (iv) of Theorem 4.2 depend only on the values of ρ and of the coefficients for large r, so that we may (and shall) choose ρ such that

$$\rho(x) := Log(1+r) \tag{5.4}$$

for $r > 0$ large enough. It is obvious that this can be done in such a way that ρ is C^∞, $\rho \geq 1$ and $\nabla\rho$ vanishes only when $x = 0$.

Since conditions (i) and (iii) of Theorem 4.2 hold irrespective of (p and) s', it suffices to check the validity of (ii) and (iv). For $p = 2$, the latter amounts to

$$k_0(1+r)^{\tau-2-2s'} \in L^\infty,$$

which happens if and only if $s' \geq \frac{\tau}{2} - 1$. In turn, such a choice of s' with $0 < s' < s$ is possible if and only if

$$s > \max\left\{0, \frac{\tau}{2} - 1\right\}. \tag{5.5}$$

It follows from (3.9) with $b = 0$, (5.2) and (5.3) that condition (ii) of Theorem 4.2 holds with $p = 2$ if

$$c_0 r^\theta - k_0 s^2 (1+r)^{\tau-2} \geq 0, \tag{5.6}$$

for $r > 0$ large enough. Since $c_0 > 0$, this inequality is true irrespective of s if $\tau < \theta + 2$. Thus, if $\tau < \theta + 2$, the hypotheses of Theorem 4.2 are satisfied upon choosing any $s > 0$ satisfying (5.5). Accordingly, C_0^∞ is a core of $D(A_2)$ in this case.

To complete the proof, we now consider the case $\tau = \theta + 2$. If so, (5.5) amounts to $s > \frac{\theta}{2}$ while (5.6) reduces to assuming that $k_0^{-1}c_0 > s^2$. Clearly, such an $s > 0$ can be found provided that $\theta^2 < 4k_0^{-1}c_0$ and so this condition alone suffices to ascertain that C_0^∞ is a core of A_2. However, with a little extra work, this restriction can be removed, as we now show.

Upon writing any function u on $\mathbb{R}^N$ in the form $u(x) = v(x/\varepsilon)$ for some $\varepsilon > 0$, it appears that it suffices to prove the denseness of C_0^∞ in $D(\widetilde{A}_2)$ when A is replaced by the operator $\widetilde{A}$ with coefficients

$$\widetilde{a}_{ij}(x) := \varepsilon^{-1} a_{ij}(\varepsilon x) \qquad \text{and} \qquad \widetilde{c}(x) := c(\varepsilon x),$$

respectively. The ellipticity condition (1.4) is not affected by this change, while the hypotheses (5.2) and (5.3) above take the form (using $\tau = \theta + 2$)

$$\sum_{i,j=1}^N \widetilde{a}_{ij}(x) \frac{x_i x_j}{r^2} \leq k_0 \varepsilon^{-1}(1+\varepsilon r)^{\theta+2} \qquad \text{and} \qquad \widetilde{c}(x) \geq c_0 \varepsilon^\theta r^\theta,$$

for $r > 0$ large enough, respectively. As a result, with the same choice (5.4) of ρ, condition (ii) of Theorem 4.2 for $\widetilde{A}$ holds if

$$c_0 \varepsilon^\theta r^\theta - k_0 s^2 \varepsilon^{-1} (1+\varepsilon r)^{\theta+2} (1+r)^{-2} \geq 0$$

for r large enough, which is the case if

$$\varepsilon s^2 < k_0^{-1} c_0.$$

Next, by the arguments used above with the operator A, condition (iv) of Theorem 4.2 for $\widetilde{A}$ is satisfied if and only if $s > \frac{\theta}{2}$. Therefore, Theorem 4.2 yields the denseness of C_0^∞ in $D(\widetilde{A}_2)$ if $\varepsilon\theta^2 < 4k_0^{-1}c_0$. Since $k_0, c_0 > 0$, this inequality always holds if ε is chosen small enough in the first place. This proves that C_0^∞ is a core of A_2 when $\tau = \theta + 2$. □

It is well known that the denseness of C_0^∞ in $D(A_2)$ implies that A_2 is selfadjoint. When $0 \leq \tau < \theta + 2$, the selfadjointness of A_2 was already obtained by Kato [9], where the limiting case $\tau = \theta + 2$ was explicitly left open when $N \geq 2$. Theorem 5.1 resolves the issue in the affirmative (for locally regular coefficients) and the result is sharp since there are counterexamples to essential selfadjointness when $\theta = 0$ and $\tau > 2$ (see Davies [3, Example 3.5] or Chapter 2, §1 of Eberle [5] and the references therein). On the other hand, a special structure of the diffusion matrix (a_{ij}) implies selfadjointness regardless of any growth condition (Devinatz [4]).

It is of course possible to generalize Theorem 5.1 when $p \neq 2$ by using Theorem 4.2, but this requires making more assumptions than just (5.2) and (5.3) since the terms $\sum_{i,j=1}^N \partial_j a_{ij} \partial_i \rho$ and $a_{ij}\partial_{ij}^2 \rho$ are involved (see (3.8)). Thus, the discussion depends in part upon what is assumed about $\partial_j a_{ij}$. We shall not explore the various options.

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