VERY WEAK SOLUTIONS WITH BOUNDARY SINGULARITIES FOR SEMILINEAR ELLIPTIC DIRICHLET PROBLEMS IN DOMAINS WITH CONICAL CORNERS

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Abstract. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with a cone-like corner at $0 \in \partial\Omega$. We prove existence of at least two positive unbounded very weak solutions of the problem $-\Delta u = u^p$ in Ω, $u = 0$ on $\partial\Omega$, which have a singularity at 0, for any p slightly bigger that the generalized Brezis-Turner exponent p^*. On an example of a planar polygonal domain the actual size of the p-interval on which the existence result holds is computed. The solutions are found variationally as perturbations of explicitly constructed singular solutions in cones. This approach also makes it possible to find numerical approximations of the two very weak solutions on Ω following a gradient flow of a suitable functional and using the mountain-pass algorithm. Two-dimensional examples are presented.

1. Introduction and main result

In this paper we prove existence of positive, unbounded very weak solutions of the boundary value problem

$$-\Delta u = u^p \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1}$$

where Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, with a cone-like corner. To describe our main result let us assume for the moment that Ω is smooth except for one corner, where it locally coincides with a cone of cross-section $\omega \subset \mathbb{S}^{n-1}$. Such domains will be in the class of *domains with a conical boundary piece*, cf. Definition 7 below. Let $(\tilde\lambda_1, \tilde\psi_1)$ be the first Dirichlet eigenvalue, Dirichlet eigenfunction of the Laplace-Beltrami operator $-\Delta_B$ on ω and define the exponent

$$p^* = \frac{n+\gamma^*}{n+\gamma^*-2} \text{ where } \gamma^* = \frac{2-n}{2} + \sqrt{\Big(\frac{n-2}{2}\Big)^2 + \tilde\lambda_1}. \tag{2}$$

Note that p^* depends on ω. Now we can state our main result.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with a conical boundary piece of cross section $\omega \subset \mathbb{S}^{n-1}$ at $0 \in \partial\Omega$. Then there is $\epsilon > 0$ such that for $p \in (p^*, p^* + \epsilon)$ there exist at least two positive, unbounded, very weak solutions of* (1) *blowing up at* 0.

The concept of a very weak solution of (1) goes back to Stampacchia [18]. It is a special kind of distributional solution. The precise definition is given below in Definition 3 and Definition 4.

The exponent p^* is called generalized Brezis-Turner exponent, cf. [11]. The original Brezis-Turner exponent $p_{BT} = \frac{n+1}{n-1}$ appeared in the work of Brezis, Turner [4] on uniform a-priori bounds for H^1_0-solutions of boundary value problems similar to (1) on smooth domains. Recently, Quittner and Souplet [16] explained the precise role of p_{BT}. For

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smooth domains they showed that p_{BT} governs uniform a-priori bounds and regularity for problems of the type

$$-\Delta u = f(x,u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{3}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory-function. Let us point out two of their results.

(i) If $|f(x,s)| \leq C(1+|s|^p)$ with a constant $C > 0$ and $1 < p < p^*$ then every very weak solution of (1) is bounded and therefore classical.

(ii) If additionally $f(x,s) \geq -C + \lambda s$ for some $C > 0$ and $\lambda > \lambda_1$ (the first Dirichlet eigenvalue of $-\Delta$ on Ω) then there is a uniform L^∞-bound for every positive solution of (3).

These results were recently generalized to domains with conical boundary pieces by McKenna, Reichel [11]. Instead of the Brezis-Turner exponent $p_{BT} = \frac{n+1}{n-1}$ it was shown that the statements (i), (ii) above are true provided $1 < p < p^*$ with p^* being the generalized Brezis-Turner exponent of (2). Unlike p_{BT} the generalized Brezis-Turner exponent p^* depends on the geometry of $\partial\Omega$ and is determined by the conical corner with smallest cross-section $\omega \subset \mathbb{S}^{n-1}$, where smallness is measured by $\tilde{\lambda}_1$ being large. Note that for locally flat boundary pieces $\gamma^* = 1$ and thus the exponents $p^* = \frac{n+1}{n-1} = p_{BT}$ all coincide.

A second major result in the understanding of very weak solutions was achieved by Souplet [17]. For a given smooth domain and for $p > p_{BT}$ he constructed a nonlinearity $f(x,s) = a(x)s^p$ with $0 \leq a \in L^\infty(\Omega)$ and a corresponding positive, very weak, but unbounded solution of (3). This showed that the Brezis-Turner exponent p_{BT} is truly a critical exponent. In the recent paper of McKenna and Reichel [11] the corresponding result for domains with conical boundary pieces was proved which shows that also the generalized Brezis-Turner exponent p^* is a truly critical exponent.

Until recently, one of the open questions that remained unsolved, was whether positive, unbounded very weak solutions for the constant coefficient problem (1) do exist. A first result in this direction on smooth domains has recently been proved by del Pino, Musso, Pacard [9]. They showed the existence of $\epsilon > 0$ such that for $p \in [p_{BT}, p_{BT} + \epsilon)$ an unbounded, positive, very weak solution of (1) exists which blows up at a prescribed point of $\partial\Omega$. Their method is based on a fixed point argument which also allows the construction of solutions blowing up on k-dimensional subsets of $\partial\Omega$ with $0 \leq k \leq n-2$.

In this paper we follow a different approach, which has e.g. in the two dimensional case the advantage of being constructive in the following sense: for a given planar polygonal domain we can give an actual value $\epsilon > 0$ for which Theorem 1 holds, cf. Table 1 and moreover we can find numerical approximations for the solutions predicted by Theorem 1. The solutions of Theorem 1 are found variationally as perturbations of explicitly constructed singular solutions in cones. They are of the form $u = w + h + z$, where w is the explicitly constructed singular solution in an infinite cone, h a harmonic function with boundary values $-w$ except at the singularity and z is found variationally as a local minimizer and as a mountain pass of a suitable functional on $H_0^1(\Omega)$. This approach is then also used for the second major result of this paper: finding numerical approximations for the solutions of Theorem 1 in the two dimensional case. This is achieved in a finite element setting by following the analytical ansatz and by finding critical points z of a suitable functional via steepest decent method and the mountain pass algorithm.

We note that the idea of perturbing an explicitly known singular solution appears already in Pacard's work [14]; see also Bidaut-Véron, Ponce, Véron [2] for an explicit construction of singular half-space solutions. In a recent paper [15] Quittner and Reichel used a very similar variational approach for the construction of unbounded very weak solutions of a problem with nonlinear Neumann boundary conditions.

Finally let us point out an outstanding open problem: the existence of unbounded very weak solutions of (1) for all exponents above the critical is open both in case of smooth domains and domains with conical corners. Our numerical results in Section 7 indicate for the two dimensional case that in practice the actual value of ϵ is considerably bigger than the value predicted in Table 1. However, as p increases further from p^* substantial numerical difficulties arise. Thus, for large p it remains unsolved both to prove existence of unbounded very weak solutions of (1) and to find their numerical approximations.

The paper is organized as follows. In Section 2 we give the exact definition of very weak solutions and we also provide some background information on the asymptotic behavior of solutions to linear problems near conical corners. In Section 3 the construction of singular solutions to (1) on infinite cones is carried out. A further ingredient, on which our main theorem is based, are Hardy and Hardy-Sobolev inequalities with singularity on the boundary, cf. Lemma 12. They are proved in Section 4. The proof of Theorem 1 is given in detail in Section 5. In Section 6 we consider the case $n = 2$ and we give a lower bound for the actual value ϵ from our main theorem. Finally, in Section 7 the analytical results are accompanied by a numerical method, which is suitable to find numerical approximations for unbounded, very weak solutions of (1).

2. Definitions and background material

Let $\delta(x) := \min\{|x-y|, y \in \partial\Omega\}$ be the distance function to $\partial\Omega$ and set $D = \sup_{x\in\Omega} |x|$. We denote by (λ_1, ϕ_1) the first Dirichlet eigenvalue, Dirichlet eigenfunction of the operator $-\Delta$ on Ω and we assume $\phi_1(x) > 0$ in Ω.

Definition 2. *For a given domain $\Omega \subset \mathbb{R}^n$ let $m : \Omega \to [0, \infty]$ be measurable and $1 \le p < \infty$. Let $L^p_m(\Omega) = \{v : \Omega \to \overline{\mathbb{R}}$ measurable $: \int_\Omega |v|^p m\, dx < \infty\}$ with the norm $\|v\|_{p,m} = \left(\int_\Omega |v|^p m\, dx\right)^{1/p}$.*

Brezis et al. [3] have given the following definition for very weak solutions on smooth domains of the linear problem

$$-\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{4}$$

Definition 3. *Let Ω be a bounded $C^{2,\alpha}$-domain. A function $u : \Omega \to \bar{\mathbb{R}}$ is called a very weak solution of (4) if $u \in L^1(\Omega)$, $f \in L^1_\delta(\Omega)$ and*

$$-\int_\Omega u\Delta\psi\, dx = \int_\Omega f\psi\, dx \quad \forall \psi \in C^2(\overline{\Omega}) \text{ with } \psi|_{\partial\Omega} = 0.$$

For Lipschitz domains there are various reasons why the above definition needs to be modified, cf. [11] Section 6. The following extension of Definition 3, which is in particular suitable for Lipschitz domains, was given in [11]. For $C^{2,\alpha}$-domains both definitions are equivalent.

Definition 4. *Let Ω be a bounded Lipschitz domain. A function $u : \Omega \to \bar{\mathbb{R}}$ is called a very weak solution of (4) if $u, f \in L^1_{\phi_1}(\Omega)$ and*

$$\int_\Omega u\eta\, dx = \int_\Omega f(-\Delta)^{-1}\eta\, dx$$

for all measurable functions $\eta : \Omega \to \mathbb{R}$ with $\|\eta/\phi_1\|_\infty < \infty$. Here $(-\Delta)^{-1} : L^2(\Omega) \to W_0^{1,2}(\Omega)$.

Remark. Note that $|\eta| \le \text{const.}\, \phi_1$ implies that $|(-\Delta)^{-1}\eta| \le \text{const.}\, \phi_1$ by the maximum principle. Hence $\int_\Omega f(-\Delta)^{-1}\eta\, dx$ is well defined because $f \in L^1_{\phi_1}(\Omega)$. Both definitions include what is meant by a very weak solution of (1) on the corresponding smooth and Lipschitz domains. Note also that a weak $H_0^1(\Omega)$ solution of (1) is also automatically a very weak solution provided $1 \le p \le \frac{n+2}{n-2}$.

Remark. Nonnegative very weak solutions of (1) are defined by replacing f in the above definitions by u^p.

Lemma 5 (Maximum principle)**.** *Let Ω be a bounded Lipschitz domain and let $g \in L^1_{\phi_1}(\Omega)$ with $g \ge 0$ a.e. in Ω. Suppose $v \in L^1_{\phi_1}(\Omega)$ is a very weak solution of $-\Delta v = g$ in Ω with $v = 0$ on $\partial\Omega$. Then $v \ge 0$ a.e. in Ω.*

Proof. The conclusion follows from $\int_\Omega v\eta\, dx \ge 0$ for all non-negative η with $\eta/\phi_1 \in L^\infty(\Omega)$. Hence $v \ge 0$ a.e. in Ω. ☐

Definition 6 (Cone, conical piece)**.** *For $x \in \mathbb{R}^n$ let $(r,\theta) \in [0,\infty) \times \mathbb{S}^{n-1}$ be the spherical-coordinates of x abbreviated by $x = (r,\theta)$. Given an open Lipschitz cross section $\omega \subset \mathbb{S}^{n-1}$ let*

$$C_\omega = \bigcup_{r>0} r\omega = \{x = (r,\theta) : r > 0, \theta \in \omega\}$$

be the corresponding infinite cone. The set

$$C_\omega^R = C_\omega \cap B_R(0)$$

is called a conical piece with cross-section ω and radius R.

Definition 7. *A bounded Lipschitz domain $\Omega \subset C_\omega$ is called a* domain with a conical boundary piece *if there exists a conical piece C_ω^R such that $\Omega \cap B_R(0) = C_\omega^R$.*

Lemma 8. *Let C_ω^R be a conical piece and let $(\tilde\psi_i)_{i\in\mathbb{N}}$ be an $L^2(\omega)$-complete orthonormal set of Dirichlet eigenfunctions of $-\Delta_B$ on ω with corresponding eigenvalues $\tilde\lambda_i$. Define $\beta_i = \sqrt{(\frac{n-2}{2})^2 + \tilde\lambda_i}$ and $\gamma_i = \frac{2-n}{2} + \beta_i$. Let $v \in C^2(C_\omega^R) \cap C(\overline{C}_\omega^R)$ solve $-\Delta v = \lambda v$ in C_ω^R with $v = 0$ on $\partial C_\omega^R \cap \partial C_\omega$ and assume that $\lambda \ge 0$. If $g(\theta) := v(R,\theta) \not\equiv 0$ then the series-expansion*

$$v(x) = \begin{cases} \left(\dfrac{|x|}{R}\right)^{\frac{2-n}{2}} \displaystyle\sum_{i=1}^\infty (g,\tilde\psi_i)_{L^2} \frac{J_{\beta_i}(\sqrt{\lambda}|x|)}{J_{\beta_i}(\sqrt{\lambda}R)} \tilde\psi_i(\theta) & \text{if } \lambda > 0, \\ \displaystyle\sum_{i=1}^\infty \left(\dfrac{|x|}{R}\right)^{\gamma_i} (g,\tilde\psi_i)_{L^2} \tilde\psi_i(\theta) & \text{if } \lambda = 0 \end{cases} \tag{5}$$

converges uniformly for $|x| \le R' < R$. Hence $v(x) = (g,\tilde\psi_1)_{L^2} (\frac{|x|}{R})^{\gamma_1} \tilde\psi_1(\theta)(1 + o(1))$ as $x \to 0$.

Proof. We provide the proof for $\lambda > 0$; the case $\lambda = 0$ is a simple adaptation of the following argument. Note first that $r^{\frac{2-n}{2}} J_{\beta_i}(\sqrt{\lambda} r)\tilde\psi_i(\theta)$ with $r = |x|$ solves the equation $-\Delta v = \lambda v$ in C_ω^R with $v = 0$ on $\partial C_\omega^R \cap \partial C_\omega$. Hence (5) is the correct L^2-convergent expansion of v. Recall that

$$J_\nu(y) = \left(\frac{y}{2}\right)^\nu \sum_{k=0}^\infty \frac{(-1)^k}{k!\Gamma(\nu+k+1)} \left(\frac{y}{2}\right)^{2k}.$$

Hence

$$\frac{J_{\beta_i}(\sqrt{\lambda}|x|)}{J_{\beta_i}(\sqrt{\lambda}R)}\left(\frac{R}{|x|}\right)^{\beta_i} \to 1 \text{ as } i\to\infty$$

uniformly with respect to $x \in \mathbb{R}^n$. Furthermore, standard regularity (Moser iteration) implies that $\|\tilde{\psi}_i\|_\infty \le C\tilde{\lambda}_i\|\tilde{\psi}_i\|_{H^1(\omega)} = C\tilde{\lambda}_i^{3/2}\|\tilde{\psi}_i\|_{L^2(\omega)} = C\tilde{\lambda}_i^{3/2}$. Therefore the series in (5) is dominated by $\|g\|_{L^2(\omega)}\sum_{i=1}^\infty(\frac{|x|}{R})^{\sqrt{\tilde{\lambda}_i}}\tilde{\lambda}_i^{3/2}$. Weyl's asymptotic formula, cf. Davies [7] Theorem 6.3.1, states that $C_1 i^{\frac{2}{n-1}} \le \tilde{\lambda}_i \le C_2 i^{\frac{2}{n-1}}$ for some constants $0 < C_1 < C_2$. In particular, the multiplicity of the i-th eigenvalue is at most $C_3 i$, with $C_3 = (\frac{C_2}{C_1})^{\frac{n-1}{2}} - 1$. Hence, the convergence behavior of the series is the same as $\sum_{i=1}^\infty(\frac{|x|}{R})^{i^{\frac{1}{n-1}}} i^{\frac{n+2}{n-1}}$ which converges uniformly for $|x| \le R' < R$. ☐

3. Singular solutions on infinite cones

In this section we shall construct a singular solution to the problem

$$(6) \qquad -\Delta w = w^p \text{ in } C_\omega, \quad w = 0 \text{ on } \partial C_\omega \setminus \{0\}$$

in the infinite cone C_ω with cross-section $\omega \subset \mathbb{S}^{n-1}$ for $p > p^*$ given by (2). The idea for the construction of a singular solution of (6) is to look for a function of the form $w = |x|^\alpha \phi(\theta)$ with $\alpha = \frac{-2}{p-1}$. The above ansatz leads to the following equation for ϕ

$$-\Delta_B\phi - \alpha(\alpha+n-2)\phi = \phi^p \text{ in } \omega, \quad \phi = 0 \text{ on } \partial\omega.$$

Lemma 9. *Let $1 < p < \infty$ if $n = 2,3$, $1 < p < \frac{n+1}{n-3}$ if $n \ge 4$ and $\lambda < \tilde{\lambda}_1$. Then the boundary value problem*

$$-\Delta_B\phi - \lambda\phi = \phi^p \text{ in } \omega, \quad \phi = 0 \text{ on } \partial\omega$$

has a positive solution $\phi \in H_0^1(\omega) \cap L^\infty(\omega)$ with

$$\phi = \left(\frac{\tilde{\lambda}_1 - \lambda}{c_p}\right)^{\frac{1}{p-1}}(\tilde{\psi}_1 + o(1)),$$

as $\lambda \nearrow \tilde{\lambda}_1$. Here $c_p = \int_\omega \tilde{\psi}_1^{p+1}\,d\theta$ and the expansion holds with respect to the $H_0^1(\omega)$-norm.

Proof. Existence for $\lambda < \tilde{\lambda}_1$ may be obtained via the mountain pass theorem. The expansion for λ near $\tilde{\lambda}_1$ follows from the standard theory of bifurcation from a simple eigenvalue. ☐

Theorem 10. *Let $p^* < p < \infty$ if $n = 2,3$, $p^* < p < \frac{n+1}{n-3}$ if $n \ge 4$. Then (6) has a singular positive solution $w(r,\theta) = r^{-\frac{2}{p-1}}\phi(\theta)$ such that $\|\phi\|_\infty \to 0$ as $p \searrow p^*$.*

Proof. The statement is a consequence of Lemma 9 and the fact that for a subcritical problem on ω the $L^\infty(\omega)$-norm of ϕ is controlled by the $H_0^1(\omega)$-norm through the standard bootstrap scheme. ☐

Lemma 11. *Let $p^* < p < \infty$ if $n = 2,3$, $p^* < p < \frac{n+1}{n-3}$ if $n \ge 4$ and let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with a conical boundary piece. If $w(r,\theta)$ is the solution from Theorem 10 then $w^p \in L^1_{\phi_1}(\Omega)$. Moreover, if h is a bounded harmonic function in Ω with $h = 0$ on $\partial\Omega \cap \overline{B}_\rho(0)$ then $w^p h \in L^1(\Omega)$.*

Proof. The statement follows from Lemma 8, the estimate $w(x)^p|h(x)| \le \text{const.}\,|x|^{\frac{-2p}{p-1}+\gamma^*}$ and $\frac{-2p}{p-1} + n - 1 + \gamma^* > -1$. ☐

4. Hardy's inequality

The standard Hardy inequality states that $\int_\Omega \frac{u^2}{|x|^2}\,dx \le \frac{4}{(n-2)^2}\int_\Omega |\nabla u|^2\,dx$ for all $u \in H_0^1(\Omega)$, cf. Opic, Kufner [12]. For space-dimension $n=2$ the inequality is trivial because the right-hand side is infinite. However, if Ω is a domain with a conical corner at $0 \in \partial\Omega$ then the following lemma provides an improvement of the classical Hardy inequality.

Lemma 12. *Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be a bounded Lipschitz domain with a conical boundary piece at $0 \in \partial\Omega$ with cross-section $\omega \subset \mathbb{S}^{n-1}$. Then*

$$\int_\Omega \frac{u^2}{|x|^2}\,dx \le C_H \int_\Omega |\nabla u|^2\,dx \text{ for all } u \in H_0^1(\Omega)$$

with

$$C_H = \left(\frac{(n-2)^2}{4} + \tilde{\lambda}_1\right)^{-1}.$$

Proof. By an approximation argument we may assume $u \in C_0^\infty(\Omega)$. The following identity is the basis of the proof:

$$(7)\quad 0 = \oint_{\partial\Omega} u^2 \xi\cdot\nu\,d\sigma = \int_\Omega (u^2 \operatorname{div}\xi + 2u\nabla u\cdot\xi)\,dx \le \int_\Omega \left(u^2(\operatorname{div}\xi + |\xi|^2) + |\nabla u|^2\right)dx$$

for $u \in C_0^\infty(\Omega)$ and a vector-field ξ such that $\xi \in L^2_{loc}(\Omega) \cap W^{1,1}_{loc}(\Omega)$. Similar identities have been used by Barbatis, Filippas and Tertikas [1] for the proof of various other Hardy inequalities. In our case the choice of the vector-field ξ is done as follows. Recall from Definition 7 that Ω is contained in the cone C_ω. For points $x = (r,\theta)$ in the cone C_ω define $\psi(r,\theta) = r^{\frac{2-n}{2}}\tilde{\psi}_1(\theta)$. Then ψ satisfies

$$-\Delta\psi = \frac{\tilde{\lambda}_1 + \frac{(n-2)^2}{4}}{|x|^2}\psi \text{ in } C_\omega.$$

We set $\xi = \nabla\psi/\psi$ so that

$$\operatorname{div}\xi + |\xi|^2 = \frac{\Delta\psi}{\psi} = -\frac{\tilde{\lambda}_1 + \frac{(n-2)^2}{4}}{|x|^2}.$$

Inserting this into (7) we obtain the result. $\square$

For $s \ge 0$ and $1 \le q < \infty$ let $L^q_{|x|^{-s}}(\Omega)$ be the weighted L^q-space with weight $m = |x|^{-s}$ as introduced in Definition 2.

Corollary 13 (Hardy-Sobolev inequality)**.** *Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be a bounded Lipschitz domain with a conical boundary piece at $0 \in \partial\Omega$. Let $0 \le s < n$ and suppose $0 < q < \frac{2(n-s)}{n-2}$ if $n \ge 3$ and $0 < q < \infty$ if $n = 2$. Then there exists a constant $C > 0$ such that*

$$\left(\int_\Omega \frac{|u|^q}{|x|^s}\,dx\right)^{2/q} \le C\int_\Omega |\nabla u|^2\,dx \text{ for all } u \in H_0^1(\Omega).$$

If $s \le 2$ and $n \ge 3$ the inequality also holds for $q = \frac{2(n-s)}{n-2}$. If additionally $1 \le q < \infty$ if $n = 2$ or $1 \le q < \frac{2(n-s)}{n-2}$ if $n \ge 3$ then the embedding $H_0^1(\Omega) \to L^q_{|x|^{-s}}(\Omega)$ is compact.

Proof. Suppose $n \ge 3$. Let $0 \le s \le q$ and notice that our assumptions on q imply $s \le 2$ in this case. We use the splitting

$$\frac{|u|^q}{|x|^s} = \frac{|u|^s}{|x|^s}\cdot|u|^{q-s}$$

together with a Hölder-inequality and obtain

$$\int_\Omega \frac{|u|^q}{|x|^s}\,dx \le \Big(\int_\Omega \frac{u^2}{|x|^2}\,dx\Big)^{s/2}\Big(\int_\Omega |u|^{\frac{2(q-s)}{2-s}}\,dx\Big)^{(2-s)/2}. \tag{8}$$

By the assumption on q we have $\frac{2(q-s)}{2-s} \le \frac{2n}{n-2}$. Hence the first term is estimated by the Hardy inequality of Lemma 12 and the second term by the Sobolev embedding theorem.

Now assume $q < s < n$. In this case the assumptions on q imply $0 < q < 2$. Therefore we can use the splitting

$$\frac{|u|^q}{|x|^s} = \frac{|u|^q}{|x|^q}\cdot\frac{1}{|x|^{s-q}}$$

and Hölder's inequality to find

$$\int_\Omega \frac{|u|^q}{|x|^s}\,dx \le \Big(\int_\Omega \frac{u^2}{|x|^2}\,dx\Big)^{q/2}\Big(\int_\Omega \frac{1}{|x|^{\frac{2s-2q}{2-q}}}\,dx\Big)^{1-q/2}. \tag{9}$$

The first term in (9) can be estimated by the Hardy inequality of Lemma 12. The assumption on q implies $\frac{2s-2q}{2-q} < n$ and therefore the second integral is convergent. In both cases the dimension $n = 2$ poses no further restriction.

The proof of the compactness of the embedding is standard. It follows the pattern of the above proof and builds on the compactness of Sobolev embedding in the subcritical cases. □

5. Proof of Theorem 1

The goal of this section is to prove Theorem 1, i.e., the existence of unbounded very weak solutions of (1). Note that any very weak solution u of

$$-\Delta u = u_+^p \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{10}$$

satisfies by Lemma 5 automatically $u \ge 0$ and is thus a very weak solution of (1). Here $u_+(x) = \max\{u(x), 0\}$ for $x \in \Omega$.

Since Ω is a domain with a conical boundary piece there exists a cone C_ω and a radius $\rho > 0$ such that $\Omega \cap B_\rho(0) = C_\omega \cap B_\rho(0)$. The proof of Theorem 1 is done variationally by perturbing the explicitly known singular solution in the cone C_ω. A similar idea already appears in Pacard's work [14] and was used in a recent paper of Quittner, Reichel [15]. Let w be the singular solution in the cone C_ω obtained in Theorem 10. Let h be a harmonic function satisfying

$$-\Delta h = 0 \text{ in } \Omega, \quad h = 0 \text{ on } \partial\Omega \cap \overline{B}_\rho(0), \quad h = -w \text{ on } \partial\Omega \setminus \overline{B}_\rho(0). \tag{11}$$

Our solution ansatz for (10) is $u = w + h + z$ with $z \in H_0^1(\Omega)$. For u to be a positive very weak solution of (10) we need to solve

$$-\Delta z = (w + h + z)_+^p - w^p \text{ in } \Omega, \quad z = 0 \text{ on } \partial\Omega. \tag{12}$$

Such solutions can be found as critical points of the functional

$$J[z] = \int_\Omega \frac{1}{2}|\nabla z|^2 - G(x, z)\,dx$$

where

$$G(x, s) = \frac{(w(x) + h(x) + s)_+^{p+1}}{p+1} - w(x)^p s - \frac{(w(x) + h(x))^{p+1}}{p+1} \text{ for } (x, s) \in \Omega \times \mathbb{R}.$$

Clearly, $J[tz] \to -\infty$ if $t \to \infty$ and if $0 < z \in H_0^1(\Omega)$. One of the solutions of Theorem 1 will be a local minimizer and the other will be a mountain pass point of the functional J.

The remaining structural prerequisites needed for existence of these two critical points are given in Lemma 16 (Frechét differentiability, weak sequential lower semi-continuity and Palais-Smale condition) and Lemma 17 (existence of a local minimizer).

We begin by stating some elementary properties of the harmonic function h.

Lemma 14. *The function $w+h$ is a nonnegative very weak solution of $-\Delta(w+h)=w^p$ in Ω, $w+h=0$ on $\partial\Omega$. Moreover we have the estimate*

$$0 \le -h(x) \le \phi(\theta)|x|^\kappa \rho^{-\frac{2}{p-1}-\kappa} \text{ in } \Omega$$

where

$$\kappa = \frac{2-n}{2} + \sqrt{\left(\frac{n-2}{2}\right)^2 - \frac{2}{p-1}\left(n-2-\frac{2}{p-1}\right)}$$

and $\kappa>0$ provided $\frac{n}{n-2}>p>p^*$.

Proof. By Lemma 5 one has $w+h\ge 0$ a.e. in Ω. This implies the estimate $0\le -h(x)\le \phi(\theta)\rho^{\frac{-2}{p-1}}$ on the set $\partial B_\rho(0)\cap\Omega$. Moreover, the function $K(x):=\phi(\theta)|x|^\kappa$ satisfies

$$-\Delta K = \big((-\kappa(\kappa+n-2)+\lambda)\phi+\phi^p\big)|x|^{\kappa-2} \text{ in } C_\omega.$$

By the above choice κ is the larger of the two roots of the equation $\kappa(\kappa+n-2)=\lambda=\frac{-2}{p-1}\big(\frac{-2}{p-1}+n-2\big)$. Hence K is superharmonic and therefore $\phi(\theta)|x|^\kappa\rho^{\frac{-2}{p-1}-\kappa}$ is an upper bound on $-h$ in Ω as claimed. $\square$

The next lemma provides some basic estimates for $G(x,s)$ and the function $H(x,s):=G(x,s)-\frac{p}{2}w(x)^{p-1}s^2$.

Lemma 15. *Let $p>1$ and $2<q<p+1$. There exists a constant $M>0$ such that the following estimates hold for all $(x,s)\in\Omega\times\mathbb{R}$:*

$$-pw(x)^{p-1}|h(x)||s| \le G(x,s) \le 2^{p-1}p\Big(\frac{|s|^{p+1}}{p+1}+\frac{w(x)^{p-1}s^2}{2}\Big)+pw(x)^{p-1}|h(x)||s|, \tag{13}$$

$$qG(x,s)-\frac{\partial G}{\partial s}(x,s)s \le Mw(x)^{p-1}s^2+(q-1)pw(x)^{p-1}|h(x)||s|. \tag{14}$$

The constant M in (14) may be chosen as follows

$$M = \begin{cases} \frac{p}{2}(q-1) & \text{if } 1<p\le 2,\\ \frac{p}{2}(p-1)\left(\frac{p-1}{p+1-q}\right)^{p-2} & \text{if } p\ge 2. \end{cases} \tag{15}$$

Moreover, if $1<p\le 2$ then

$$|H(x,s)| \le \frac{1}{p+1}|s|^{p+1}+2^{p+1}|h(x)|w(x)^p+2^p|h(x)||s|^p \tag{16}$$

and if $p>2$ then

$$|H(x,s)| \le 2^{p-2}(p-1)p\Big(\frac{|s|^{p+1}}{p(p+1)}+\frac{w(x)^{p-2}|s|^3}{6}\Big)+2^{p+1}|h(x)|w(x)^p+2^p|h(x)||s|^p \tag{17}$$

Proof. For the proof of (13) let first

$$g(x,s) = \frac{(w(x)+h(x)+s)_+^{p+1}}{p+1} - (w(x)+h(x))^p s - \frac{(w(x)+h(x))^{p+1}}{p+1} \tag{18}$$
$$= \int_0^s \big((w(x)+h(x)+t)_+^p - (w(x)+h(x))^p\big)\, dt.$$

By convexity $g(x,s) \geq 0$ for all $s \in \mathbb{R}$. And since

$$(w+h+t)_+^p - (w+h)^p \begin{cases} \leq p(w+h+t)^{p-1}t \leq 2^{p-1}p(t^p + (w+h)^{p-1}t) & \text{if } t \geq 0, \\ \geq p(w+h)^{p-1}t & \text{if } t \leq 0 \end{cases} \tag{19}$$

we obtain from (18), (19) that

$$0 \leq g(x,s) \leq 2^{p-1}p\Big(\frac{|s|^{p+1}}{p+1} + \frac{(w(x)+h(x))^{p-1}s^2}{2}\Big). \tag{20}$$

Note that $G(x,s) = g(x,s) + ((w(x)+h(x))^p - w(x)^p)s$ and $0 \geq (w(x)+h(x))^p - w(x)^p \geq pw(x)^{p-1}h(x)$ due to the negativity of h. Hence we obtain from (20) the desired inequality (13).

To prove (14) we proceed by showing the existence of a suitably large constant M. The choice of M given in (15) will be explained in Lemma 20 in the Appendix. First we claim that there exists $M > 0$ such that

$$qg(x,s) - \frac{\partial g}{\partial s}(x,s)s \leq M(w(x)+h(x))^{p-1}s^2 \leq Mw(x)^{p-1}s^2 \tag{21}$$

where g is defined in (18). By homogeneity, the first inequality in (21) amounts to

$$\frac{q}{p+1}\Big(\frac{(1+t)_+^{p+1} - 1 - (p+1)t}{t^2}\Big) \leq \frac{(1+t)_+^p - 1 - pt}{t} + M + p. \qquad \forall t \in \mathbb{R}\setminus\{0\}. \tag{22}$$

The last relation holds as $t \to +\infty$ and, provided $M > 0$, also for $t \to -\infty$. Moreover it holds as $t \to 0$ provided $M > pq/2 - p$. Hence, by choosing M sufficiently large, (22) holds for all $t \in \mathbb{R}$. Recall that $G(x,s) = g(x,s) + ((w(x)+h(x))^p - w(x)^p)s$. Hence

$$\begin{aligned} qG(x,s) - \frac{\partial G}{\partial s}(x,s)s &= qg(x,s) - \frac{\partial g}{\partial s}(x,s)s + (q-1)\Big((w(x)+h(x))^p - w(x)^p\Big)s \\ &\leq qg(x,s) - \frac{\partial g}{\partial s}(x,s)s + (q-1)pw(x)^{p-1}|h(x)||s| \end{aligned}$$

and (14) follows from (21).

The estimate for H is based on the splitting $H(x,s) = k_1(x,s) + k_2(x,s)$ where

$$\begin{aligned} k_1(x,s) &= \frac{(w(x)+s)_+^{p+1} - w(x)^{p+1}}{p+1} - w(x)^p s - \frac{p}{2}w(x)^{p-1}s^2, \\ k_2(x,s) &= \frac{(w(x)+h(x)+s)_+^{p+1} - (w(x)+s)_+^{p+1}}{p+1} + \frac{w(x)^{p+1} - (w(x)+h(x))^{p+1}}{p+1}. \end{aligned}$$

First we observe that

$$k_1(x,s) = p\int_0^{|s|}\int_0^t \big((w(x)+\tau\,\mathrm{sign}\, s)_+^{p-1} - w(x)^{p-1}\big)\, d\tau\, dt.$$

If $1 < p \leq 2$ then clearly $|k_1(x,s)| \leq \int_0^{|s|}\int_0^t p\tau^{p-1}\, d\tau\, dt = \frac{1}{p+1}|s|^{p+1}$. If $p > 2$ then using (19) with p replaced by $p-1$ we get

$$|k_1(x,s)| \leq p(p-1)2^{p-2}\int_0^{|s|}\left(\frac{t^p}{p} + \frac{w(x)^{p-2}t^2}{2}\right) dt.$$

Both estimates for $k_1(x,s)$ lead to the first terms in (16), (17). It remains to estimate $k_2(x,s)$. This is done due to

$$(w(x)+h(x)+s)_+^{p+1}-(w(x)+s)_+^{p+1}\begin{cases}\le 0,\\ \ge (p+1)(w(x)+s)_+^p h(x)\end{cases}$$

for all $(x,s)\in\Omega\times\mathbb{R}$ and hence

$$|(w(x)+h(x)+s)_+^{p+1}-(w(x)+s)_+^{p+1}|\le (p+1)|h(x)|2^p(w(x)^p+|s|^p).$$

Applying this estimate twice we find

$$|k_2(x,s)|\le 2^{p+1}|h(x)|w(x)^p+2^p|h(x)||s|^p$$

which is the remaining term in (16), (17). This finishes the proof of the lemma. $\square$

Lemma 16. *For $n=2$ let $p^*<p<\infty$ and for $n\ge 3$ let $p^*<p<\frac{n+2}{n-2}$. The functional J is well-defined on $H_0^1(\Omega)$ and continuously Frechét-differentiable. If p is sufficiently close to p^* then J is weakly sequentially lower semi-continuous and satisfies the Palais-Smale condition.*

Proof. Well-defined: Recall that $w(x)=\phi(\theta)|x|^{-2/(p-1)}$. Lemma 15 shows that for J to be well-defined one needs to verify that

$$\int_\Omega |z|^{p+1}\,dx<\infty,\qquad \int_\Omega \frac{z^2}{|x|^2}\,dx<\infty,\qquad \int_\Omega \frac{|z||h|}{|x|^2}\,dx<\infty\qquad \text{for } z\in H_0^1(\Omega).$$

Since $p<\frac{n+2}{n-2}$ the number $p+1$ is smaller than the critical Sobolev embedding number $\frac{2n}{n-2}$. Therefore the first integral is finite. The second integral is finite due to Hardy's inequality in Lemma 12 and the third integral is finite (after applying the Cauchy-Schwarz inequality) provided $\int_\Omega h^2/|x|^2\,dx<\infty$. This follows directly from Lemma 8.

Frechét-differentiability: It is sufficient to prove the differentiability of $\int_\Omega G(x,z)\,dx$. The Mean Value Theorem implies

$$G(x,z+v)-G(x,z)-\frac{\partial G}{\partial s}(x,z)v=\int_0^1(1-t)p(w(x)+h(x)+z+tv)_+^{p-1}v^2\,dt.$$

Since $|w(x)+h(x)+z+tv|^{p-1}\le C(w(x)^{p-1}+|h(x)|^{p-1}+|z|^{p-1}+|v|^{p-1})$ it follows from Hölder's and Hardy's inequalities (mentioned above) and the Sobolev embedding that

$$\Big|\int_\Omega (G(x,z+v)-G(x,z))\,dx-A(z)v\Big|\le C\|v\|^2,$$

where $A(z)v:=\int_\Omega \frac{\partial G}{\partial s}(x,z)v\,dx$ and $\|\cdot\|$ is the $H_0^1(\Omega)$-norm. With a similar argument one sees that the mapping $z\mapsto A(z)$ is a continuous map from $H_0^1(\Omega)$ into its dual, hence $\int_\Omega G(x,z)\,dx$ is continuously Fréchet differentiable.

Weak sequential lower semi-continuity: The functional J can be written in the form $J=J_0-J_1$, where $J_0[z]=\frac12\|z\|^2-\frac p2\int_\Omega w^{p-1}z^2\,d\sigma$, and $J_1[z]=\int_\Omega H(x,z)\,dx$. The functional J_0 represents the square of an equivalent norm in $H_0^1(\Omega)$ if p is close to p^*. Hence J_0 is weakly lower semi-continuous. It remains to consider J_1. Note first that the term $|h(x)|w(x)^p\in L^1(\Omega)$ by Lemma 11. If $1\le p\le 2$ then (16) of Lemma 15 shows that J_1 depends continuously on $z\in L^{p+1}(\Omega)$ where $2<p+1<\frac{2n}{n-2}$. Due to the compactness of the Sobolev embedding, J_1 is weakly sequentially continuous in $H_0^1(\Omega)$.

If $p \geq 2$ then (17) of Lemma 15 shows that J_1 depends continuously on $z \in L^{p+1}(\Omega) \cap L^3_{|x|^{-s}}(\Omega)$ with $s = \frac{2(p-2)}{p-1}$. The weak sequential continuity of J_1 follows from the compactness of the embedding $H_0^1(\Omega) \to L^3_{|x|^{-s}}(\Omega)$, cf. Corollary 13, provided

$$3 < \frac{2(n-s)}{n-2}, \quad \text{i.e.,} \quad p < \frac{n+2}{n-2}.$$

which is fulfilled by our assumption.

Palais-Smale condition: Let $(z_n)_{n\in\mathbb{N}}$ be a Palais-Smale sequence, i.e., $J[z_n]$ is bounded and $J'[z_n] \to 0$ as $n \to \infty$. Hence if $2 < q < p+1$ it follows from (14) of Lemma 15 that

$$o(\|z_n\|) = qJ[z_n] - J'[z_n]z_n \geq \Big(\frac{q}{2} - 1\Big) \int_\Omega |\nabla z_n|^2\,dx - \int_\Omega w^{p-1}(Mz_n^2 + (q-1)p|h||z_n|)\,dx.$$

Next we use that $w = |x|^{-2/(p-1)}\phi(\theta)$ and $\|\phi\|_\infty \to 0$ as $p \searrow p^*$. With the help of Lemma 14 we conclude

$$\begin{aligned}
o(\|z_n\|) &\geq \Big(\frac{q}{2} - 1\Big)\|z_n\|^2 - \|\phi\|_\infty^{p-1} \int_\Omega \frac{Mz_n^2 + (q-1)p|h||z_n|}{|x|^2}\,dx \\
&\geq \Big(\frac{q}{2} - 1\Big)\|z_n\|^2 - \|\phi\|_\infty^{p-1} MC_H\|z_n\|^2 \\
&\quad - (q-1)p\|\phi\|_\infty^p \rho^{\frac{-2}{p-1}-\kappa}\left(\int_\Omega |x|^{2\kappa-2}\,dx\right)^{1/2}\sqrt{C_H}\|z_n\| \\
&\geq \Big(\frac{q}{2} - 1\Big)\|z_n\|^2 - \|\phi\|_\infty^{p-1} MC_H\|z_n\|^2 \\
&\quad - (q-1)p\|\phi\|_\infty^p \rho^{\frac{-2}{p-1}-\kappa}\left(\frac{D^{2\kappa+n-2}|\omega|}{2\kappa+n-2}\right)^{1/2}\sqrt{C_H}\|z_n\|,
\end{aligned}$$

where $D = \sup_{x\in\Omega}|x|$. Since $q > 2$ the latter inequality implies that the sequence $(z_n)_{n\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$ provided $\|\phi\|_\infty$ is sufficiently small, i.e., p is sufficiently close to p^*. After passing to a weakly convergent subsequence one can show the strong convergence of this subsequence in a straight-forward manner, cf. Struwe [19], Chapter II, Proposition 2.2. ☐

Lemma 17. *For p larger but sufficiently close to p^* there exists $\alpha, \beta > 0$ such that $J[z] \geq \alpha$ if $\|z\| = \beta$. In particular, for such p the functional J attains a local minimum inside the ball $B_\beta(0) \subset H_0^1(\Omega)$ and the local minimizer is non-trivial.*

Proof. By (13) and Lemma 14

$$\begin{aligned}
J[z] &\geq \int_\Omega \left(\frac{|\nabla z|^2}{2} - 2^{p-1}p\Big(\frac{|z|^{p+1}}{p+1} + w(x)^{p-1}\frac{z^2}{2}\Big) - pw(x)^{p-1}|h(x)||z|\right)dx \\
&\geq \frac{1}{2}\|z\|^2 - 2^{p-1}p\left(\frac{C_p^{p+1}}{p+1}\|z\|^{p+1} + \frac{C_H}{2}\|\phi\|_\infty^{p-1}\|z\|^2\right) \\
&\quad - p\|\phi\|_\infty^p \rho^{\frac{-2}{p-1}-\kappa}\left(\frac{D^{2\kappa+n-2}|\omega|}{2\kappa+n-2}\right)^{1/2}\sqrt{C_H}\|z\|,
\end{aligned}$$

where C_p is a constant appearing in the Sobolev embedding inequality $\|z\|_{L^{p+1}} \leq C_p\|z\|$. Recall that $\|\phi\|_\infty \to 0$ as $p \searrow p^*$. Hence, if p is sufficiently close to p^* we have

$$J[z] \geq \frac{1}{4}\|z\|^2 - d_1\|z\|^{p+1} - d_2\|\phi\|_\infty^p\|z\| = \|z\|^2\left(\frac{1}{4} - d_1\|z\|^{p-1} - \frac{d_2\|\phi\|_\infty^p}{\|z\|}\right)$$

with appropriate constants $d_1, d_2 > 0$. Choosing $\|z\| = \beta := (\frac{1}{8d_1})^{1/(p-1)}$ and assuming $\|\phi\|_\infty$ sufficiently small we obtain $J[z] \geq \alpha > 0$ as claimed. $\square$

6. Results for $n = 2$

Lemma 18. *For $\lambda < \pi^2/\omega^2$ let ϕ be the positive solution of*

$$-\phi'' = \lambda\phi + \phi^p \text{ in } (0,\omega), \qquad \phi(0) = \phi(\omega) = 0 \tag{23}$$

with $\|\phi\|_\infty = \phi(\frac{\omega}{2}) = \alpha$. If $\alpha < 1$ then

$$\alpha \leq \left[\frac{p+1}{2}\left(\frac{\pi^2}{\omega^2} - \lambda\right)\right]^{\frac{1}{p-1}}$$

Proof. Let $G(s) = \frac{\lambda s^2}{2} + \frac{s^{p+1}}{p+1}$. Then

$$\frac{\omega}{2} = \int_0^\alpha \frac{dt}{\sqrt{2G(\alpha) - 2G(t)}} = \int_0^1 \frac{\alpha\, ds}{\sqrt{2G(\alpha)}\sqrt{1 - \frac{G(\alpha s)}{G(\alpha)}}}.$$

Since by assumption $0 \leq s \leq \alpha \leq 1$ it follows that $G(\alpha s) \leq s^2 G(\alpha)$. Hence $\frac{\omega}{2} \leq \frac{\alpha}{\sqrt{2G(\alpha)}} \cdot \frac{\pi}{2}$, which leads to the estimate on α as claimed. $\square$

Theorem 19. *Suppose $\Omega \subset \mathbb{R}^2$ is a domain with a conical boundary piece of cross-section $\omega \subset \mathbb{S}^1 = (0, 2\pi)$ at $0 \in \partial\Omega$ and let $D = \sup_{x\in\Omega}|x|$. Then $p^* = \frac{2\omega+\pi}{\pi} \in (1,5)$. Recall the definition of d_1, d_2 in the proof of Lemma 17. Here, their values are given by*

$$d_1 = 2^{p-1}p\left(\frac{1}{2\sqrt{\pi}}\right)^{p+1}(p+1)^{\frac{p+1}{2}}|\Omega|, \qquad d_2 = \frac{p}{2}\rho^{\frac{-4}{p-1}}D^{\frac{2}{p-1}}\big((p-1)\omega\big)^{1/2}\frac{\omega}{\pi}. \tag{24}$$

The existence of two unbounded very weak solutions of (1) holds provided $p > p^*$ satisfies

$$\frac{p+1}{2}\left(\frac{\pi^2}{\omega^2} - \frac{4}{(p-1)^2}\right) \leq \min\left\{\frac{1}{p}\cdot\frac{\pi^2}{\omega^2}, \frac{q-2}{2M}\cdot\frac{\pi^2}{\omega^2}, \frac{1}{p2^p}\cdot\frac{\pi^2}{\omega^2}, (8d_2)^{\frac{1-p}{p}}(8d_1)^{\frac{-1}{p}}\right\}, \tag{25}$$

where M is defined as in Lemma 15. Due to monotonicity with respect to p the inequality (25) holds for $p \in (p^*, p^* + \epsilon)$ for some $\epsilon > 0$.

Proof. Note that $C_H = 1/\tilde{\lambda}_1 = \omega^2/\pi^2$ and $\kappa = \frac{2}{p-1}$ stems from Lemma 14. This explains the value of d_2 in (24). The constant d_1 is defined as $d_1 = \frac{2^{p-1}p}{p+1}C_p^{p+1}$ and C_p is the Sobolev-embedding constant $\|z\|_{L^{p+1}} \leq C_p\|z\|$. By a classical result (combining Lemma 7.12 and 7.14 in Gilbarg, Trudinger [10]) we have in $n = 2$

$$C_p = \frac{1}{2\sqrt{\pi}}(p+1)^{\frac{1}{2}+\frac{1}{p+1}}|\Omega|^{\frac{1}{p+1}}$$

and hence the value of d_1 in (24).

To show the sufficiency of (25) note first that by Lemma 18 the left-hand side of (25) is just an upper estimate on $\|\phi\|_\infty^{p-1}$ with ϕ as in Lemma 9. Now we follow the steps of Lemma 16 and Lemma 17.

Weak sequential lower semi-continuity: Here we need $pC_H\|\phi\|_\infty^{p-1} < 1$. This amounts to the first part of the inequality (25).

Palais-Smale condition: This requires $MC_H\|\phi\|_\infty^{p-1} < \frac{q}{2} - 1$, which amounts to the second part of the inequality (25).

Boundary point	ω	ρ	D	p^*	$p^* + \epsilon$
P_1	π	$\frac{\sqrt{3}}{2}$	$\sqrt{2}$	3	$3.013242\ldots$
P_2	$\frac{\pi}{2}$	1	2	2	$2.007394\ldots$
P_3	$\frac{\pi}{3}$	$\frac{2}{\sqrt{3}}$	$\sqrt{5}$	$\frac{5}{3} = 1.\bar{6}$	$1.671374\ldots$
P_4	$\frac{2\pi}{3}$	1	$\sqrt{\frac{16}{3} - \frac{4}{\sqrt{3}}}$	$\frac{7}{3} = 2.\bar{3}$	$2.344770\ldots$

TABLE 1. Table with values of p^*, $p^* + \epsilon$ at different corners

Existence of local minimizer: As a first step in Lemma 17 one needs $2^{p-1}pC_H\|\phi\|_\infty^{p-1} < \frac{1}{2}$, which explains the third part of the inequality (25). Finally, with $\|z\| = \beta = (\frac{1}{8d_1})^{1/(p-1)}$ a positive lower bound for J on the sphere of radius β is guaranteed provided $1/8 > d_2\|\phi\|_\infty^p(8d_1)^{\frac{1}{p-1}}$. This explains the fourth and final part of the inequality (25). $\square$

From now on we consider the special polygonal domain $\Omega \subset \mathbb{R}^2$ shown in Figure 1 with interior opening angles $\pi, \frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}$ at the boundary points P_1, P_2, P_3, P_4, respectively.

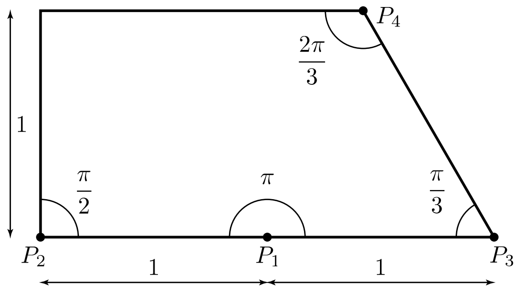

FIGURE 1. Domain for numerical experiments

Table 1 gives information on the values $p^*, p^*+\epsilon$ corresponding to the different boundary points. The table is based on the result of Theorem 19 and was computed with MAPLE. We have chosen q as the arithmetic mean of 2 and $p+1$. The value ρ gives the size of the ball $B_\rho(P_i)$ around the boundary point P_i such that $\Omega \cap B_\rho(P_i)$ equals a conical piece. The value D gives the biggest distance to P_i inside $\overline{\Omega}$. In all cases the size of ϵ is of the order 10^{-2}.

7. NUMERICAL EXAMPLES FOR $n = 2$

In this section we describe how numerical approximations for solutions of (1) can be obtained. We follow the analytical approach and decompose $u = w+h+z$. First we show how one can find numerical approximations for $w = |x|^{\frac{-2}{p-1}}\phi$ and h. Then we explain how one can obtain z once as a local minimizer of the functional J through a steepest decent method and once by the mountain pass algorithm. Finally, both the local minimizer and the mountain pass are improved through an application of Newton's method.

7.1. **The boundary value problem on the cross-section.** The boundary value problem (23) for ϕ is reformulated as a system

$$(26)\qquad \left\{ \begin{array}{rclcrcl} \phi_1' &=& \phi_2, & & \phi_1(0) &=& 0,\\ \phi_2' &=& -\frac{4}{(p-1)^2}\phi_1 - \phi_1^p, & & \phi_2(\frac{\omega}{2}) &=& 0. \end{array} \right.$$

Matlab `bvp4c` function is used to solve (26) on $[0, \frac{\omega}{2}]$ and to compute ϕ on grid-points $\{\frac{i}{k}\frac{\omega}{2}\}_{i=0}^{k}$. Then ϕ is extended symmetrically to grid points of the interval $[\frac{\omega}{2}, \omega]$ and interpolated by cubic splines between the grid points. In order to evaluate $w(x, y)$ at point $(x, y) \in \Omega$ one determines the polar-coordinates (r, θ) and computes $r^{\frac{-2}{p-1}}\phi(\theta)$.

7.2. **The finite element method.** We use a standard approach as described, e.g., in [6]. In order to approximate H^1-functions on Ω we take piecewise linear finite elements on a triangulation $\mathcal{T}^\tau = \{T_i\}_i$ of domain Ω shown in Figure 1 where τ characterizes the size of the triangles. The triangulation of Ω is done by Matlab's PDE Toolbox. The numerical results shown below were computed on a triangulation consisting of 138,240 triangles and 69,585 vertices with an average length of one side of a triangle $\tau = 1/160$.

We use the following standard finite element spaces:

$$V^\tau = \{\varphi \in H^1(\Omega) : \varphi \text{ linear on each triangle } T_i\}, \qquad V_0^\tau = V^\tau \cap H_0^1(\Omega).$$

Next we introduce the notation for the vertices of the triangles. The notation will change depending on which of the four boundary points $P \in \{P_1, \dots, P_4\}$ is chosen. Let $\Gamma_0 = \partial\Omega \cap \overline{B}_\rho(P)$ with ρ given in Table 1 and let $\Gamma_1 = \partial\Omega \setminus \Gamma_0$. Denote:

- I : index set of triangle vertices in $\overline{\Omega}$,
- I_{int} : index set of interior triangle vertices (in Ω),
- I_{Γ_0} : index set of boundary triangle vertices lying on Γ_0,
- I_{Γ_1} : index set of boundary triangle vertices lying on $\partial\Omega$ which are not in I_{Γ_0}.

The vertex set I is the union of three disjoint sets $I = I_{\text{int}} \cup I_{\Gamma_0} \cup I_{\Gamma_1}$. The coordinates of the triangle vertices are denoted by$\{(x_k, y_k)\}_{k\in I}$. Let $\{\varphi_k\}_{k\in I} \subset V^\tau$ be a set of basis elements such that φ_k equals 1 at k-th vertex (x_k, y_k) of the triangulation $\mathcal{T}^\tau$ and 0 at all other vertices. Then $v \in V^\tau$ can be written as follows

$$(27)\quad v = \sum_{k\in I} v_k\varphi_k, \quad \bar{v} = (v_k)_{k\in I}, \quad \bar{v}_{\text{int}} = (v_k)_{k\in I_{\text{int}}}, \quad \bar{v}_{\Gamma_0} = (v_k)_{k\in I_{\Gamma_0}}, \quad \bar{v}_{\Gamma_1} = (v_k)_{k\in I_{\Gamma_1}}.$$

The stiffness matrix, which can be computed explicitly, is given by

$$(28)\qquad K = \left(\int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j \, dx\, dy\right)_{i,j\in I_{\text{int}}}.$$

The finite element version of problem (12) is to find $z \in V_0^\tau$ such that

$$(29)\qquad J'[z]\varphi_k = (K\bar{z})_k - \int_\Omega \big((w+h+z)_+^p - w^p\big)\varphi_k \, dx\, dy = 0 \quad \forall k \in I_{\text{int}},$$

where we write $\bar{z}$ instead of $\bar{z}_{\text{int}}$ for the sake of simplicity, and the functional J will be evaluated as follows

$$(30)\qquad J[z] = \frac{1}{2}\bar{z}^T K\bar{z} - \int_\Omega \left(\frac{1}{p+1}\big((w+h+z)_+^{p+1} - (w+h)^{p+1}\big) - w^p z\right) dx\, dy.$$

The integrands in (29), (30) need to be integrated numerically over each triangle. This is done using a Fortran algorithm described in [8].

7.3. **Representation of the harmonic correction h.** Recall from the definition of h as a solution of (11) that h attains nonzero values on parts of the boundary. We reformulate (11) as follows. Choose $h^{(2)} \in V^\tau$ as an arbitrary function coinciding with w in vertices on Γ_1 and being 0 on Γ_0, i.e.,

$$h_i^{(2)} = w(x_i, y_i) \text{ for } i \in I_{\Gamma_1}, \qquad h_i^{(2)} = 0 \text{ for } i \in I_{\Gamma_0}. \tag{31}$$

To find h in the form $h = h^{(1)} - h^{(2)}$ let $h^{(1)}$ be the FEM solution of

$$-\Delta \tilde{h} = -\Delta h^{(2)} \text{ in } \Omega, \quad \tilde{h} = 0 \text{ on } \partial\Omega \tag{32}$$

which leads to

$$K\bar{h}^{(1)}_{\text{int}} = K\bar{h}^{(2)}_{\text{int}} + \left(\int_\Omega \nabla\varphi_k \cdot \nabla\varphi_i \, dx\, dy\right)_{k\in I_{\text{int}}, i \in I_{\Gamma_1}} \bar{h}^{(2)}_{\Gamma_1}. \tag{33}$$

Therefore the basis coefficients $\bar{h}$ of h are given by

$$\begin{bmatrix} \bar{h}_{\text{int}} \\ \bar{h}_{\Gamma_0} \\ \bar{h}_{\Gamma_1} \end{bmatrix} = \begin{bmatrix} \bar{h}^{(1)}_{\text{int}} - \bar{h}^{(2)}_{\text{int}} \\ 0 \\ -\bar{h}^{(2)}_{\Gamma_1} \end{bmatrix} = \begin{bmatrix} K^{-1}\Big(\sum_{i\in I_{\Gamma_1}} w(x_i, y_i) \int_\Omega \nabla\varphi_k \cdot \nabla\varphi_i \, dx\, dy\Big)_{k\in I_{\text{int}}} \\ 0 \\ -\left(w(x_i, y_i)\right)_{i \in I_{\Gamma_1}} \end{bmatrix}. \tag{34}$$

Note that h depends only on the values of w in the vertices on Γ_1 but does not depend on the choice of $h^{(2)}$. As in the case of the stiffness matrix, the integrals in (34) can be computed explicitly.

7.4. **Steepest decent, mountain pass algorithm and Newton's method.** The steepest descent method (SDM) and the mountain-pass algorithm (MPA) are both based on the flow defined by $g := \nabla J[z] \in H_0^1(\Omega)$, the gradient of J at $z \in H_0^1(\Omega)$ (which is the Riesz representation of the linear functional $J'[z] \in H^{-1}(\Omega)$). From (29) it follows that in the discretized case it is computed as

$$\bar{g} = \bar{z} - K^{-1}\left(\int_\Omega \big((w+h+z)_+^p - w^p\big)\varphi_k \, dx\, dy\right)_{k\in I_{\text{int}}} \qquad \text{for } z \in V_0^\tau. \tag{35}$$

SDM solves numerically (using a forward Euler scheme) the initial value problem

$$\frac{d}{dt}\zeta(t) = -\nabla J[\zeta(t)], \qquad \zeta(0) = \zeta_0 \in V_0^\tau \tag{36}$$

for some initial function ζ_0. Lemma 17 states that J attains a local minimum in a small ball centered at 0. Hence $\zeta_0 = 0$ is a suitable choice. We denote $z_{\min}$ the function SDM converges to.

MPA has first been described in [5]. Here we give a very brief description only. We take a discretized path $\{e_\ell\}_{\ell=0}^L \subset V_0^\tau$ consisting of $L+1$ points which connects $e_0 := z_{\min}$ with e_L such that $J[e_L] < J[e_0]$. The endpoint e_L can be chosen as a large enough positive function since $\lim_{t\to\infty} J[tz] = -\infty$ for $z > 0$ as noted in Section 5. We find the maximum of J along the path. The point e_ℓ where the maximum occurs is moved a small distance in the direction of $-\nabla J[e_\ell]$. This deforms the path and lowers the maximum of J along it. The deforming of the path is repeated until the maximum cannot be lowered any more, i.e., until a critical point is reached. We denote this critical point z_{MP}.

Newton's method is used to improve an initial guess (usually the output of MPA). Its goal is to find a solution of

$$\mathcal{F}(\bar{z}) := K\bar{z} - \left(\int_\Omega \big((w+h+z)_+^p - w^p\big)\varphi_k \, dx\, dy\right)_{k\in I_{\text{int}}} = 0 \tag{37}$$

by computing recursively $\bar{z}_{m+1} = \bar{z}_m - (\frac{d\mathcal{F}}{d\bar{z}}(\bar{z}_m))^{-1}\mathcal{F}(\bar{z}_m)$, where the derivative of $\mathcal{F}$ is

$$\frac{d\mathcal{F}}{d\bar{z}}(\bar{z}) = K - \left(\int_\Omega p(w+h+z)_+^{p-1}\varphi_i\varphi_j\,dx\,dy\right)_{i,j\in I_{\text{int}}}. \tag{38}$$

7.5. **Numerical results.** Figures 2–5 show numerical solutions z of (12) and u of (1) with the singularity of w placed at a boundary point $P \in \{P_1, P_2, P_3, P_4\}$ of the domain Ω shown in Figure 1. For each of the four points P a particular value $p > p^*$ is fixed and a local minimizer $z_{\min}$ and a mountain-pass point z_{MP} of the functional J are computed and improved by Newton's method. Their graphs are in the first row of Figures 2–5. The second row of these figures shows contour lines of $u = w + h + z$ where z is the local minimizer $z_{\min}$ or the mountain pass z_{MP}, respectively. The actual values of p, for which solutions are produced are considerably bigger than the ones predicted in Table 1.

We first observe that the numerical minimizer $z_{\min}$ is a negative function on Ω in all four cases. For p close to p^* its magnitude is rather small, e.g., of order 10^{-3} in Figure 5. With growing p the contour lines become more dense close to the corner P and the magnitude gets larger, e.g., of order 1 in Figure 3.

The mountain pass z_{MP} is positive for p close to p^*, cf. Figure 4 and 5. With growing p the contour lines become more dense close to the corner P and z_{MP} becomes negative on a subset of Ω, cf. nodal lines in Figures 2 and 3.

We observed that the shape of the graph of numerical solutions in Figures 4 and 5 does not change visibly under the refinement of triangulation. In case of solutions in Figures 2 and 3 the contour lines become more dense close to the corner P for finer triangulations. The nodal line of the mountain-pass solution also moves visibly closer to P. Trying to increase p even more we ran into substantial numerical difficulties. Therefore, we cannot conjecture (and it remains completely open) whether (1) possesses unbounded very weak solutions for large values of p.

Figure 6 shows the singular solution w of (6) restricted to the computational domain Ω for all the four choices of the cone vertex P and the corresponding choice of p. From these graphs and those of u it can be seen that u is indeed a perturbation of w.

Appendix

Lemma 20. *Let $2 < q < p + 1$. With the choice of the value M given by*

$$M = \begin{cases} \dfrac{p}{2}(q-1) & \text{if } 1 < p \le 2, \\ \dfrac{p}{2}(p-1)\left(\dfrac{p-1}{p+1-q}\right)^{p-2} & \text{if } p \ge 2. \end{cases}$$

the following inequality holds

$$(1+t)_+^p \le \left(1 - \frac{q}{p+1}\right)(1+t)_+^{p+1} + (q-1)t + Mt^2 + \frac{q}{p+1} \tag{39}$$

which is equivalent to (22) *in the proof of Lemma 15.*

Proof. Let $l(t) = (1+t)_+^p$ and $r(t) = (1 - \frac{q}{p+1})(1+t)_+^{p+1} + (q-1)t + Mt^2 + \frac{q}{p+1}$ denote the left and right-hand sides of (39), respectively. Note that $l(0) = r(0)$ and $l'(0) = r'(0)$. Let us first show (39) in the case $p \ge 2$. This will hold if $l''(t) \le r''(t)$ for all $t \in \mathbb{R}$, i.e.,

$$p(p-1)(1+t)_+^{p-2} \le p(p+1-q)(1+t)_+^{p-1} + 2M. \tag{40}$$

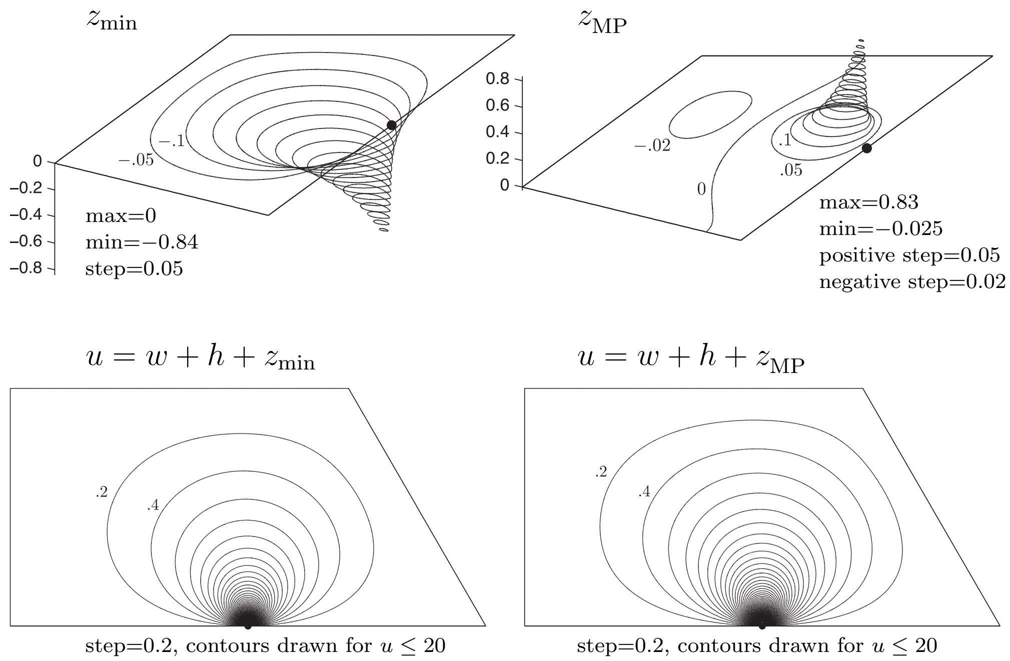

FIGURE 2. Numerical solutions with a singularity placed at P_1 for $p = 3.5$.

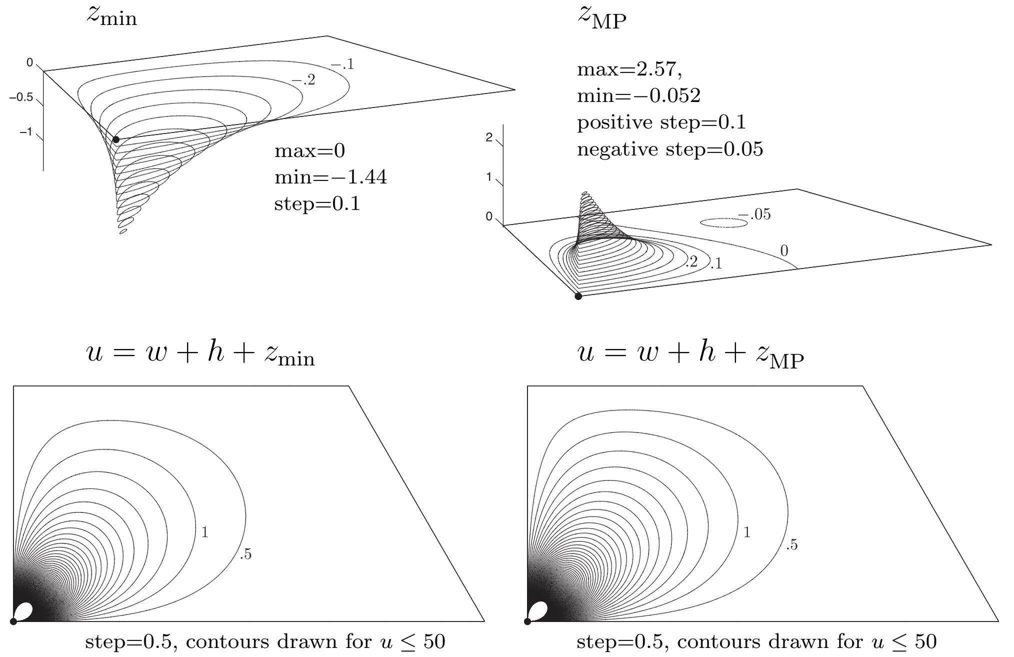

FIGURE 3. Numerical solutions with a singularity placed at P_2 for $p = 2.35$.

Let $t_0 = \frac{q-2}{p+1-q}$ be the value where $p-1 = (p+1-q)(1+t_0)$. For $t \ge t_0$ the inequality (40) holds automatically, while for $t \le t_0$ it holds provided

$$p(p-1)(1+t_0)^{p-2} \le 2M,$$

which is true with equality due the choice of M above. Now consider the case $1 < p \le 2$. For $t \ge 0$ (40) holds provided $p(p-1) \le p(p+1-q) + 2M$, i.e., provided $p(q-2) \le 2M$. For $t \le 0$ we argue with first derivatives instead of second derivatives, i.e, we show

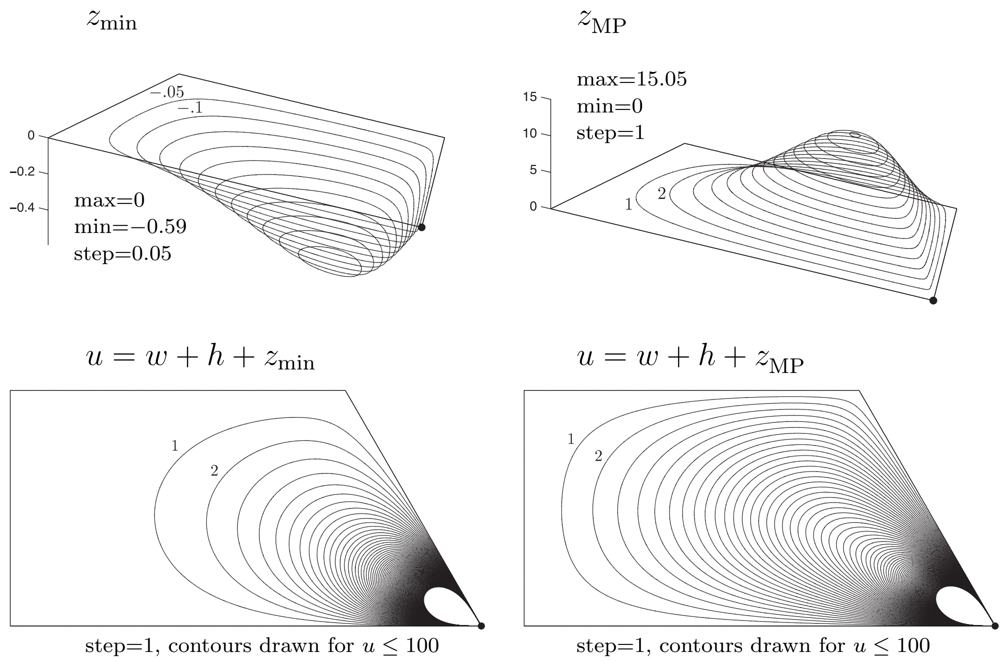

Figure 4. Numerical solutions with a singularity placed at P_3 for $p = 1.9$.

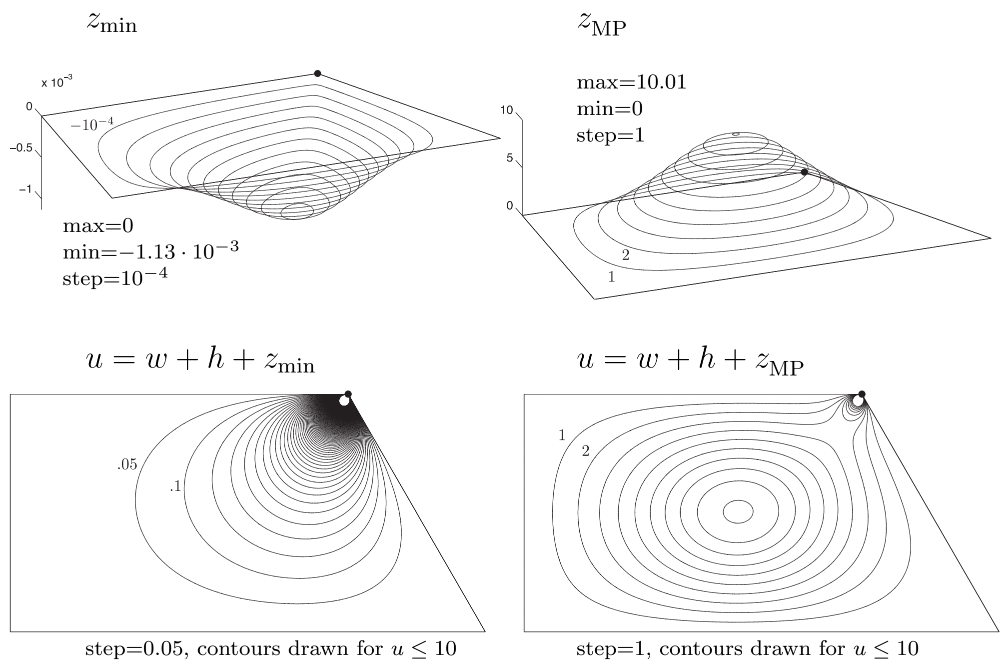

Figure 5. Numerical solutions with a singularity placed at P_4 for $p = 2.35$.

$l'(t) \geq r'(t)$ for $t \leq 0$. This amounts to

$$p(1+t)_+^{p-1} \geq (p+1-q)(1+t)_+^p + q - 1 + 2Mt \text{ for } t \leq 0. \tag{41}$$

Combining the two inequalities

$$\begin{array}{rcll} (1-q)(1+t)_+^p + q - 1 & \leq & (1-q)pt \leq -2Mt & \text{for } t \leq 0 \text{ if } p(q-1) \leq 2M, \\ (1+t)_+^p & \leq & (1+t)_+^{p-1} & \text{for } t \leq 0. \end{array}$$

we obtain (41) provided $p(q-1) \leq 2M$. This is guaranteed by the above choice of M. □

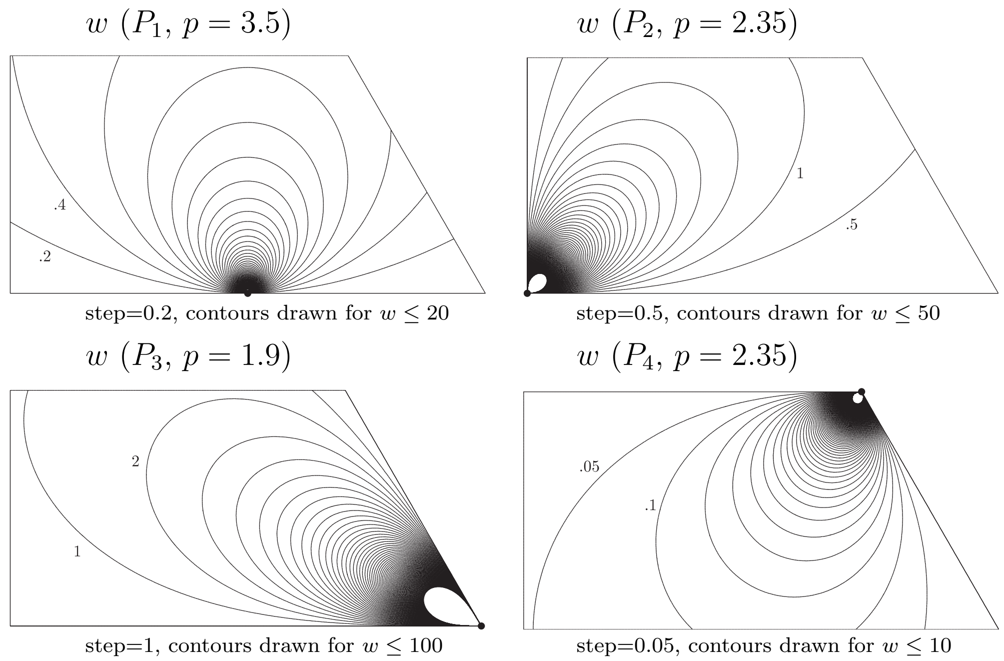

Figure 6. Solution w of (6) for cones with vertices and cross-sections listed in Table 1 restricted to the domain shown in Figure 1 for various values of p.

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