ON CRITICAL VALUES OF RANKIN-SELBERG CONVOLUTIONS

HENDRIK KASTEN AND CLAUS-GÜNTHER SCHMIDT

Abstract. For a pair (π, σ) of cuspidal automorphic representations of GL_n and GL_{n-1}, both of non-vanishing cohomology with possibly non-trivial coefficients, we show algebraicity properties of critical values of the associated Rankin-Selberg L-function twisted by finite order characters. A certain non-vanishing assumption about an associated archimedean Rankin-Selberg pairing on the cohomology is established for $n = 3$.

0 Introduction

In this article we intend to generalise and continue previous investigations of special values of L-functions of Rankin-Selberg type attached to pairs of cuspidal automorphic representations of GL_n and GL_{n-1}. These L-functions, introduced by Jacquet, Piatetski-Shapiro, and Shalika in [JPS4], are entire functions in the complex variable s and satisfy a functional equation, when s goes to $1-s$. Under the assumption, that the representations occur in cohomology, we show algebraicity properties of special values similar to those predicted by Deligne for motivic L-functions (cf. [Del]). For cohomology with constant coefficients this has been done earlier: for $n = 2$ by Mazur and Swinnerton-Dyer in [MS], for $n = 3$ by one of the authors in [Schm1], and for arbitrary n by Kazhdan, Mazur, and one of the authors in [KMS]. In this paper we generalise this approach to cohomology with not necessarily constant coefficients. The case $n = 2$ had already been settled in the context of modular forms by Manin (cf. [Man]).

To be more precise, let (π, σ) be a pair of cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ resp. $\mathrm{GL}_{n-1}(\mathbb{A})$ over the adèle ring $\mathbb{A}$ of the field $\mathbb{Q}$ of rational numbers. We assume π to have non-vanishing cohomology with coefficients in a finite-dimensional, rational representation M_μ of GL_n of highest weight μ, so that for the infinity component π_∞ of π the representation $\pi_\infty \otimes M_\mu$ has non-trivial Lie algebra cohomology. We make the analogous assumption for σ with a suitable representation M_ν of GL_{n-1}. To each weight there is attached an integer $\mathsf{w} = \mathsf{wt}(\mu)$ resp. $\mathsf{w}' = \mathsf{wt}(\nu)$ relating weights with their duals, and it turns out that the half integer $\kappa = \frac{1}{2}(\mathsf{w} + \mathsf{w}' + 1)$ is critical for the Rankin-Selberg L-function $L(\pi, \sigma; s)$ if and only if w and w' have the same parity.

Like in [KMS] for a fixed pair (π, σ) we are interested in the package of critical values $L(\pi \otimes \chi, \sigma; \kappa)$, where *chi* runs through all finite Dirichlet characters. We want to show, that the function $\chi \mapsto L(\pi \otimes \chi, \sigma; \kappa)$, after division by an appropriate period depending only on the sign of χ, takes algebraic numbers as values. Furthermore we want to understand in general certain arithmetic properties of these values, in particular how they behave p-adically, when χ varies over finite Dirichlet characters of p-power conductor for a fixed prime number p. Once the algebraicity of the special values is settled, the techniques developped in [Schm2] immediately supply associated p-adic measures and p-adic L-functions interpolating p-adically those special values. So in this

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article we concentrate on the algebraicity properties for characters of p-power conductor,[1] thus remaining as close as possible to the situation in [KMS].

The proof of algebraicity properties of special values relies on the choice of a linear form λ on the tensor product of the coefficient systems M_μ and M_ν. We have to analyse a certain product

$$\Lambda(s) = P_{\lambda,\infty}(s) \cdot L(\pi \otimes \chi, \sigma; s)$$

with an entire function $P_{\lambda,\infty}(s)$ determined by λ and the zeta integrals on the Whittaker models of π_∞ and σ_∞. Our first main result, Theorem A, expresses the special value $\Lambda(\kappa)$ in terms of algebraic numbers resulting from pairings of cohomology groups via λ. As in [KMS] we are faced with the question, if $P_{\lambda,\infty}(\kappa)$ is zero. This is obviously vital for the desired conclusion to algebraicity statements for special values of the L-function itself.

Our second main result, Theorem B, deals with this problem in the case $n = 3$. We show for arbitrary coefficient systems M_μ and M_ν that there is a linear form λ with appropriate rationality properties, such that $P_{\lambda,\infty}(\kappa)$ does not vanish indeed. The idea of the proof is to carefully keep track of the pairing of the cohomology of $\pi_\infty \otimes M_\mu$ with the cohomology $\sigma_\infty \otimes M_\nu$, whose image consists of $P_{\lambda,\infty}(\kappa) \cdot \mathbb{C}$ and which is induced by the Rankin-Selberg pairing of the respective Whittaker spaces of π_∞ and σ_∞. The important key result here is, that this pairing of the Whittaker spaces remains non-trivial when restricted to minimal K-types as long as these K-types "fit together".

1 Representations with non–vanishing cohomology

1.1. The coefficient systems. Let $n \in \mathbb{N}$ be a natural number. Throughout this paper we write Z_n for the centre of GL_n and set $K_{n,\infty} = \mathrm{SO}_n(\mathbb{R})Z_n^+(\mathbb{R})$, where by $Z_n^+(\mathbb{R})$ we mean the set of elements of $Z_n(\mathbb{R})$ with positive determinant. We will always use small letters to identify the respective Lie algebras $\mathfrak{gl}_n$, $\mathfrak{sl}_n$, $\mathfrak{so}_n$, $\mathfrak{z}_n$, $\mathfrak{k}_{n,\infty}$, etc. of $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{SO}_n(\mathbb{R})$, $Z_n(\mathbb{R})$, $K_{n,\infty}$, etc.

As we will prove in Section 1.2 being cohomological is a local property at infinity:

Lemma 1.1. *An irreducible cuspidal automorphic representation π occurs in cohomology in the sense of* [KMS] *if and only if*

$$H^\bullet(\mathfrak{gl}_n, K_{n,\infty}; \pi_\infty) \neq 0.$$

We will use a more general notion (cf. [Har], p. 60ff). In order to do this we need to introduce some new vocabulary. So let $B_n = T_n U_n$ denote the group of upper triangular matrices in GL_n. Here, T_n resp. U_n is the standard maximal torus in GL_n resp. the unipotent radical of B_n. Let $X(T_n)$ be the set of algebraic characters $\nu : T_n \to \mathbb{G}_m$ of T_n. Then $X^+(T_n)$ denotes the set of dominant weights in $X(T_n)$. We identify $\mathbb{Z}^n$ with $X(T_n)$ by sending $\mu = (\mu_i)$ to $t \mapsto \prod_i t_i^{\mu_i}$. For a weight $\mu \in X^+(T_n)$ we denote by (ϱ_μ, M_μ) the irreducible algebraic representation of $\mathrm{GL}_n(\mathbb{Q})$ of highest weight μ. Note, that such a representation always exists, and is unique up to equivalence. Since by [Clo], p. 122, the representation $\varrho_\mu : \mathrm{GL}_n(\mathbb{Q}) \to \mathrm{GL}(M_\mu)$ is defined over $\mathbb{Q}$, we may assume that M_μ is a $\mathbb{Q}$-vector space. For any extension $E/\mathbb{Q}$ we set $M_{\mu,E} := M_\mu \otimes E$.

Definition 1.2. *Let $\mathrm{Coh}(\mathrm{GL}_n, \mu)$ denote the set of all irreducible cuspidal automorphic representations π of $\mathrm{GL}_n(\mathbb{A})$ satisfying*

$$H^\bullet(\mathfrak{gl}_n, K_{n,\infty}; \pi_\infty \otimes M_{\mu,\mathbb{C}}) \neq 0 \tag{1.1}$$

for the relative Lie algebra cohomology, where π_∞ is the infinity component of π.

[1] not a serious restriction

Lemma 1.1 now says: $\mathrm{Coh}(\mathrm{GL}_n, 0)$ is the set of all representations that occur in cohomology in the sense of [KMS]. In that case, $\mu = 0$ is the dominant weight and ϱ_μ is the trivial representation.

Note, that not every dominant weight occurs as maximal weight in the coefficient system of a cohomological representation. More precisely, by Section 3.1.1 of [Mah] we have $\mu \in X_0^+(T_n)$, the set of all dominant weights in $X(T_n)$ satisfying

$$\mu + w_{\mathrm{GL}_n}\mu = (\mathsf{wt}(\mu), \dots, \mathsf{wt}(\mu)) \tag{1.2}$$

for some $\mathsf{wt}(\mu) \in \mathbb{Z}$, where w_{GL_n} is the longest element of the Weyl group $W_{\mathrm{GL}_n} = W_{\mathrm{GL}_n}(T_n)$ of GL_n. If we denote by $\check{\mu} \in X^+(T_n)$ the dual weight of μ, i. e. the highest weight of the contragredient representation $(\check{\varrho}_\mu, \check{M}_\mu)$, we have $\check{\mu} = -w_{\mathrm{GL}_n}\mu$. Hence (1.2) amounts to saying that μ is self-contragredient up to twist.

1.2. A closer look at relative Lie algebra cohomology. For later use we are interested in a submodule of $H^\bullet(\mathfrak{gl}_n, K_{n,\infty}; \pi_\infty \otimes M_{\mu,\mathbb{C}})$ that is one-dimensional as a $\mathbb{C}$-vector space and easy to describe. We know that for the *lower cohomological bound*

$$b_n = \frac{1}{4}(n^2 - n + 2\lfloor\frac{n}{2}\rfloor)$$

the cohomology space $H^{b_n}(\mathfrak{gl}_n, K_{n,\infty}; \pi_\infty \otimes M_{\mu,\mathbb{C}})$ is one or two dimensional, so that we expect to find a suitable submodule there. We provide a formula for those cohomology modules by the following

Proposition 1.3. *For $\pi \in \mathrm{Coh}(\mathrm{GL}_n, \mu)$ we have*

$$H^\bullet(\mathfrak{gl}_n, K_{n,\infty}; \pi_\infty \otimes M_{\mu,\mathbb{C}}) = \left(\bigwedge^{\bullet}(\mathfrak{sl}_n/\mathfrak{so}_n)^* \otimes \pi_\infty \otimes M_{\mu,\mathbb{C}}\right)^{\mathrm{SO}_n(\mathbb{R})}.$$

Proof. Since $K_{n,\infty}$ is connected, by section I.5 of [BW][2] we may write

$$H^\bullet(\mathfrak{gl}_n, K_{n,\infty}; \pi_\infty \otimes M_{\mu,\mathbb{C}}) = H^\bullet(\mathfrak{gl}_n, \mathfrak{k}_{n,\infty}; \pi_\infty \otimes M_{\mu,\mathbb{C}}).$$

Consider the complex

$$C^\bullet(\mathfrak{gl}_n, \mathfrak{k}_{n,\infty}; \pi_\infty \otimes M_{\mu,\mathbb{C}}) \overset{\text{[BW], I.1.2}}{=} \mathrm{Hom}_{\mathfrak{k}_{n,\infty}}(\bigwedge^{\bullet} \mathfrak{gl}_n/\mathfrak{k}_{n,\infty}, \pi_\infty \otimes M_{\mu,\mathbb{C}}).$$

By Theorem I.5.3 of [loc. cit.], and since $\pi \in \mathrm{Coh}(\mathrm{GL}_n, \mu)$, the central character of π_∞ equals the one of $\check{\varrho}_\mu$, implying that $\pi_\infty \otimes M_{\mu,\mathbb{C}}$ has trivial central character. Recall that π_∞ and M_μ both are irreducible representations of $\mathrm{GL}_n(\mathbb{R})$. By the triviality of the central character of $\pi_\infty \otimes M_{\mu,\mathbb{C}}$ the latter uniquely corresponds to the tensor product of the irreducible representations of $\mathrm{SL}_n^\pm(\mathbb{R})$ given by restriction. We will identify the respective modules and denote them the same.

Because of $\mathfrak{k}_{n,\infty} = \mathfrak{so}_n \oplus \mathfrak{z}_n$ the vector spaces $\mathfrak{gl}_n/\mathfrak{k}_{n,\infty}$ and $\mathfrak{sl}_n/\mathfrak{so}_n$ are the same, so that we have

$$\begin{aligned} C^\bullet(\mathfrak{gl}_n, \mathfrak{k}_{n,\infty}; \pi_\infty \otimes M_{\mu,\mathbb{C}}) &= \mathrm{Hom}_{\mathfrak{so}_n}(\bigwedge^{\bullet} \mathfrak{sl}_n/\mathfrak{so}_n, \pi_\infty \otimes M_{\mu,\mathbb{C}}) \\ &= C^\bullet(\mathfrak{sl}_n, \mathfrak{so}_n; \pi_\infty \otimes M_{\mu,\mathbb{C}}), \end{aligned}$$

whence

$$H^\bullet(\mathfrak{gl}_n, K_{n,\infty}; \pi_\infty \otimes M_{\mu,\mathbb{C}}) = H^\bullet(\mathfrak{sl}_n, \mathfrak{so}_n; \pi_\infty \otimes M_{\mu,\mathbb{C}}).$$

[2]Note, that since the central action is by a scalar, the $K_{n,\infty}$-invariant submodules of $\pi_\infty \otimes M_{\mu,\mathbb{C}}$ are just the same as the SO_n-invariant ones. Therefore, we may apply the results of [BW] on $K_{n,\infty}$, even if the latter is not compact. In the results we cite, the maximality of the compact subgroup is never needed.

Now, since $\mathrm{SO}_n(\mathbb{R})$ is connected, all that is left to show is

$$H^\bullet(\mathfrak{sl}_n, \mathfrak{so}_n; \pi_\infty \otimes M_{\mu,\mathbb{C}}) = \mathrm{Hom}_{\mathfrak{so}_n}(\bigwedge^\bullet \mathfrak{sl}_n/\mathfrak{so}_n, \pi_\infty \otimes M_{\mu,\mathbb{C}}). \tag{1.3}$$

But this follows directly from Proposition II.3.1 of [BW], we just have to verify that we are allowed to use it. In order to do that we choose $\mathrm{SL}_n(\mathbb{R})$ as connected, reductive Lie group and $\mathrm{SO}_n(\mathbb{R})$ as its maximal compact subgroup. We have to guarantee that $d\pi_\infty(C)$ and $d\varrho_{\mu,\mathbb{C}}(C)$ are scalar operators, where C is the Casimir element of the envelopping algebra $\mathfrak{U}(\mathfrak{sl}_n)$, and $d\pi_\infty$ and $d\varrho_{\mu,\mathbb{C}}$ are the respective induced mappings on $\mathfrak{U}(\mathfrak{sl}_n)$. By Schur's Lemma (cf. [Kna2], Proposition 5.1), and since C is in the centre of $\mathfrak{U}(\mathfrak{sl}_n)$, it would suffice to show that π_∞ and $M_{\mu,\mathbb{C}}$ are irreducible $\mathrm{SL}_n(\mathbb{R})$-modules. Obviously, this does not hold in general, but since all representations are irreducible as $\mathrm{SL}_n^\pm(\mathbb{R})$-modules, we may use

Lemma 1.4. *Let $\varrho : \mathrm{SL}_n^\pm(\mathbb{R}) \to \mathrm{GL}(V)$ be an irreducible $\mathrm{SL}_n^\pm(\mathbb{R})$-module and $d\varrho$ the induced mapping on $\mathfrak{U}(\mathfrak{sl}_n)$. Then there is a scalar r such that $d\varrho(C) = r \cdot \mathrm{id}$.*

By applying the lemma on π_∞ and on $M_{\mu,\mathbb{C}}$ we may use Proposition II.3.1 of [BW] now. Since $\pi \in \mathrm{Coh}(\mathrm{GL}_n, \mu)$, we get $d\pi_\infty(C) = d\varrho_{\mu,\mathbb{C}}(C)$, and therefore (1.3). This concludes the proof of Proposition 1.3. □

Proof of Lemma 1.4. If V is still irreducible as $\mathrm{SL}_n(\mathbb{R})$-module, there is nothing to show. So assume that V decomposes into a direct sum of (irreducible) $\mathrm{SL}_n(\mathbb{R})$-modules (ϱ_1, V_1) and (ϱ_2, V_2). Note, that since the index is 2 this is the only other case. If V_1 and V_2 are isomorphic, still there is nothing to show. So assume that V_1 and V_2 are not isomorphic as $\mathrm{SL}_n(\mathbb{R})$-modules. Choose $g \in \mathrm{SL}_n^\pm(\mathbb{R})$ and $v_1 \in V_1$ with $gv_1 \notin V_1$. Then gV_1 is not contained in V_1. Since for $h \in \mathrm{SL}_n(\mathbb{R})$ and $v_1 \in V_1$ we have

$$h(gv_1) = (gg^{-1})h(gv_1) = g(g^{-1}hg)v_1 \in gV_1, \tag{1.4}$$

gV_1 is a $\mathrm{SL}_n(\mathbb{R})$-module. Note, that $\mathrm{SL}_n(\mathbb{R})$ is normal because of its index 2 in $\mathrm{SL}_n^\pm(\mathbb{R})$.

Since $V_1 \not\cong V_2$, the only $\mathrm{SL}_n(\mathbb{R})$-submodules of V are $0, V_1, V_2$, and V, so that gV_1 is isomorphic to V_2. Then (1.4) tells us how the module structures of V_1 and V_2 are related: Clearly it is enough to proof $g^{-1}Cg = C$ to get $d\varrho_1(C) = d\varrho_2(C)$.

We may write $C = \sum_i X_i X_i^*$, where the X_i resp. the X_i^* form a basis of $\mathfrak{sl}_n$, dual to each other via the Killing form κ of $\mathfrak{sl}_n$. It holds

$$\kappa(g^{-1}X_ig, g^{-1}X_j^*g) = \kappa(X_i, X_j^*) \quad \forall g \in \mathrm{SL}_n^\pm(\mathbb{R}),$$

so that the basis formed by the $g^{-1}X_ig$ and the one formed by the $g^{-1}X_i^*g$ are also dual to each other. The lemma follows because of $g^{-1}Cg = \sum_i g^{-1}X_igg^{-1}X_i^*g$ and the independence of the Casimir element of its basis. □

Now let $\pi_\infty^{(\mathrm{O}_n)}$ denote the space of $\mathrm{O}_n(\mathbb{R})$-finite vectors in the representation space of π_∞. Note, that since $M_{\mu,\mathbb{C}}$ is of finite dimension, we have $M_{\mu,\mathbb{C}}^{(\mathrm{O}_n)} = M_{\mu,\mathbb{C}}$. Let further $H^{\mathrm{O}_n(\mathbb{R})} = H_+$ and H_- denote the respective (± 1)-eigenspaces with respect to the $\mathrm{O}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$-action of any $\mathrm{SO}_n(\mathbb{R})$-invariant module H, and write ω_{π_∞} for the central character of π_∞. We get the following corollary, which will be useful in Section 3.6.

Corollary 1.5. (a) *For even n the space $H^{b_n}(\mathfrak{gl}_n, K_{n,\infty}; \pi_\infty \otimes M_{\mu,\mathbb{C}})_\varepsilon$ is one-dimensional for both $\varepsilon \in \{+,-\}$.*

(b) *For odd n the space $H^{b_n}(\mathfrak{gl}_n, K_{n,\infty}; \pi_\infty \otimes M_{\mu,\mathbb{C}})_\varepsilon$ is one-dimensional, if $\varepsilon = \mathrm{sgn}(\omega_\pi(-1)(-1)^{\mathrm{wt}(\mu)/2})$, and zero otherwise.*

(c) $H^{b_n}(\mathfrak{gl}_n, K_{n,\infty}; \pi_\infty \otimes M_{\mu,\mathbb{C}})_\pm = \left(\bigwedge^{b_n}(\mathfrak{sl}_n/\mathfrak{so}_n)^* \otimes \pi_\infty^{(\mathrm{O}_n)} \otimes M_{\mu,\mathbb{C}}\right)_\pm^{\mathrm{SO}_n(\mathbb{R})}$

Proof. By Proposition 1.3 and [BW], I.5 we have

$$H^{b_n}(\mathfrak{gl}_n, K_{n,\infty}; \pi_\infty \otimes M_{\mu,\mathbb{C}})^{\mathrm{O}_n(\mathbb{R})} = \left(\bigwedge^{b_n}(\mathfrak{sl}_n/\mathfrak{so}_n)^* \otimes \pi_\infty^{(\mathrm{O}_n)} \otimes M_{\mu,\mathbb{C}}\right)^{\mathrm{O}_n(\mathbb{R})}.$$

Assertion (c) follows, since $H^{b_n}(\mathfrak{gl}_n, K_{n,\infty}; \pi_\infty \otimes M_{\mu,\mathbb{C}})$ is the direct sum of its (± 1)-eigenspaces. Assertions (a) and (b) result from Equation (3.2) in [Mah]. □

Proof of Lemma 1.1 Let $\pi = \pi_f \otimes \pi_\infty$ be an irreducible cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$. In the special case $\mu = 0$ we find

$$H^\bullet(\mathfrak{gl}_n, K_{n,\infty}; \pi_\infty)^{\mathrm{O}_n(\mathbb{R})} = H^\bullet(\mathfrak{sl}_n, \mathfrak{so}_n; \pi_\infty)^{\mathrm{O}_n(\mathbb{R})} \overset{\text{[BW], I.5}}{=} H^\bullet(\mathfrak{sl}_n, \mathrm{O}_n; \pi_\infty^{(\mathrm{O}_n)}).$$

The proof follows, since π occurs in cohomology in the sense of [KMS] exactly if the right hand side does not vanish (cf. p. 122 in [loc. cit.]), and π lies in $\mathrm{Coh}(\mathrm{GL}_n, 0)$ exactly if the left hand side does not vanish by Corollary 1.5. □

1.3. The Langlands parameter. We denote by $\mathfrak{L}_0^+(\mathrm{GL}_n)$ the set of all pairs $(\mathsf{w}, \mathbf{l})$, where $\mathsf{w} \in \mathbb{Z}$ and $\mathbf{l} = (l_1, \dots, l_n) \in \mathbb{Z}^n$ is a finite sequence satisfying $l_1 > \cdots > l_{\lfloor n/2 \rfloor} > 0$, $l_i + l_{n+1-i} = 0$ for all $i \in \{0, \dots, n\}$, and the purity condition

$$\mathsf{w} + \mathbf{l} \equiv \begin{cases} 1 \bmod (2), & \text{if } n \text{ is even,} \\ 0 \bmod (2), & \text{if } n \text{ is odd,} \end{cases} \tag{1.5}$$

where we identify w with $(\mathsf{w}, \dots, \mathsf{w})$. We note, that for n odd this immediately implies $\mathsf{w} \equiv \mathbf{l} \equiv 0 \bmod (2)$ since $l_{(n+1)/2} = 0$. If we let $\Phi_{\mathrm{GL}_n} = \Phi(\mathrm{GL}_n, T_n)$ denote the set of roots of T_n in GL_n and $\Phi^+_{\mathrm{GL}_n}$ the subset of positive roots determined by the choice of B_n, we see, that the sets $\mathfrak{L}_0^+(\mathrm{GL}_n)$ and $X_0^+(T_n)$ are in bijection:

$$\begin{array}{ccc} \mathfrak{L}_0^+(\mathrm{GL}_n) & \longleftrightarrow & X_0^+(T_n) \\ (\mathsf{w}, \mathbf{l}) & \mapsto & \mu = \dfrac{\mathsf{w} + \mathbf{l}}{2} - \varrho_n. \end{array} \tag{1.6}$$

Here,

$$\varrho_n = \frac{1}{2} \sum_{\alpha \in \Phi^+_{\mathrm{GL}_n}} \alpha = (\frac{n-1}{2}, \frac{n-3}{2}, \dots, -\frac{n-1}{2}) \quad \in X(\mathrm{GL}_n) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is the half-sum of positive roots of GL_n relative to T_n. Explicitly, we have

$$\mu = \begin{cases} \left(\frac{\mathsf{w}+l_1-(n-1)}{2}, \frac{\mathsf{w}+l_2-(n-3)}{2}, \dots, \frac{\mathsf{w}-l_1+(n-1)}{2}\right), & \text{if } n \text{ is even,} \\ \left(\frac{\mathsf{w}+l_1-(n-1)}{2}, \frac{\mathsf{w}+l_2-(n-3)}{2}, \dots, \frac{\mathsf{w}}{2}, \dots, \frac{\mathsf{w}-l_1+(n-1)}{2}\right), & \text{if } n \text{ is odd.} \end{cases}$$

In the inverse direction the parameter associated with a dominant, integral weight μ reads $(\mathsf{w}, \mathbf{l})$, where $\mathsf{w} = \mu_1 + \mu_n$ is the weight of μ and $\mathbf{l} = 2(\mu + \varrho_n) - \mathsf{w}$.

To any $(\mathsf{w}, \mathbf{l}) \in \mathfrak{L}_0^+(\mathrm{GL}_n)$ we attach an induced representation of Langlands type: we write D_l for the discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ of lowest weight $l+1$; we then set

$$J(\mathsf{w}, \mathbf{l}) := \begin{cases} \mathrm{Ind}_{Q(\mathbb{R})}^{\mathrm{GL}_n(\mathbb{R})}(|\cdot|_{\mathbb{R}}^{\mathsf{w}/2} \otimes D_{l_1}, \dots, |\cdot|_{\mathbb{R}}^{\mathsf{w}/2} \otimes D_{l_{n/2}}), & \text{if } n \text{ is even,} \\ \mathrm{Ind}_{Q(\mathbb{R})}^{\mathrm{GL}_n(\mathbb{R})}(|\cdot|_{\mathbb{R}}^{\mathsf{w}/2} \otimes D_{l_1}, \dots, |\cdot|_{\mathbb{R}}^{\mathsf{w}/2} \otimes D_{l_{(n-1)/2}}, |\cdot|_{\mathbb{R}}^{\mathsf{w}/2}), & \text{if } n \text{ is odd.} \end{cases}$$

Here, $Q \leq \mathrm{GL}_n$ is the parabolic subgroup of type $(2, \dots, 2)$ resp. $(2, \dots, 2, 1)$.

Let $(\mathsf{w}, \mathbf{l}) \in \mathfrak{L}_0^+(\mathrm{GL}_n)$ correspond to $\mu \in X_0^+(T_n)$ as in (1.6). By (3.6) of [Mah] any $\pi \in \mathrm{Coh}(\mathrm{GL}_n, \mu)$ has infinity component

$$\pi_\infty \cong \mathrm{sgn}^k \otimes J(-\mathsf{w}, \mathbf{l}), \quad k \in \mathbb{Z}/2\mathbb{Z}. \tag{1.7}$$

For later use (cf. Section 2) we remark that to each such representation π_∞ there is a corresponding representation π_∞^{W} of the Weil group $W_{\mathbb{R}}$ of $\mathbb{R}$ via Langlands correspondence. $W_{\mathbb{R}}$ is the non-split extension of $\mathbb{C}^\times$ by $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$ given by

$$W_{\mathbb{R}} = \mathbb{C}^\times \cup j\mathbb{C}^\times,$$

where $j^2 = -1$ and $jzj^{-1} = \bar{z}$ for all $z \in \mathbb{C}^\times$. Thus any representation of $W_{\mathbb{R}}$ is determined by how elements of the form $z = re^{i\theta}$ and j act. There are exactly three types of irreducible representations, which are given explicitly in Chapter 3 of [Kna1]:

- The one-dimensional representations $(+, t)$ with $t \in \mathbb{C}$, which act via φ are given by
$$\varphi(z) = |z|^t \text{ and } \varphi(j) = +1.$$
- The one-dimensional representations $(-, t)$ with $t \in \mathbb{C}$, which act via φ are given by
$$\varphi(z) = |z|^t \text{ and } \varphi(j) = -1.$$
- The two-dimensional representations (l, t), where $l \geq 1$ is an integer and $t \in \mathbb{C}$. In those we may always choose a basis $\{u, u'\}$ such that we have
$$\varphi(re^{i\theta})u = r^{2t}e^{il\theta}u,\ \varphi(re^{i\theta})u' = r^{2t}e^{-il\theta}u',\ \varphi(j)u = u',\ \varphi(j)u' = (-1)^l u,$$
where (l, t) acts via φ.

Using this notation we have

$$\pi_\infty^{\mathrm{W}} = \begin{cases} (l_1, \frac{-\mathsf{w}}{2}) \oplus (l_2, \frac{-\mathsf{w}}{2}) \oplus \cdots \oplus (l_{n/2}, \frac{-\mathsf{w}}{2}), & \text{if } n \text{ is even,} \\ (l_1, \frac{-\mathsf{w}}{2}) \oplus (l_2, \frac{-\mathsf{w}}{2}) \oplus \cdots \oplus (l_{(n-1)/2}, \frac{-\mathsf{w}}{2}) \oplus (\mathrm{sgn}^k, \frac{-\mathsf{w}}{2}), & \text{if } n \text{ is odd.} \end{cases}$$

We will need to determine the tensor product of two such Weil group representations. So let

$$\sigma_\infty \cong \mathrm{sgn}^{k'} \otimes J(-\mathsf{w}', \mathbf{l}'), \quad k' \in \mathbb{Z}/2\mathbb{Z}$$

be a representation of $\mathrm{GL}_m(\mathbb{R})$, notation being clear from the context. Analogously, we get

$$\sigma_\infty^{\mathrm{W}} = \begin{cases} (l'_1, \frac{-\mathsf{w}'}{2}) \oplus (l'_2, \frac{-\mathsf{w}'}{2}) \oplus \cdots \oplus (l'_{m/2}, \frac{-\mathsf{w}'}{2}) \oplus (\mathrm{sgn}^{k'}, \frac{-\mathsf{w}'}{2}), & \text{if } m \text{ is odd,} \\ (l'_1, \frac{-\mathsf{w}'}{2}) \oplus (l'_2, \frac{-\mathsf{w}'}{2}) \oplus \cdots \oplus (l'_{(m-1)/2}, \frac{-\mathsf{w}'}{2}), & \text{if } m \text{ is even.} \end{cases}$$

We want to calculate the tensor product of π_∞^{W} and $\sigma_\infty^{\mathrm{W}}$. Therefore we have to calculate the various tensor products of the building blocks. We distinguish three cases:

- Let σ, σ' be in $\{+, -\}$, and let t, t' be in $\mathbb{C}$. Obviously, we get
$$(\sigma, t) \otimes (\sigma', t') = \begin{cases} (+, t + t'), & \text{if } \sigma = \sigma', \\ (-, t + t'), & \text{if } \sigma \neq \sigma'. \end{cases}$$
- Let $l \geq 1$ be an integer, $\sigma \in \{+, -\}$ and t, t' in $\mathbb{C}$. Let further $\{u, u'\}$ be the special basis from the definition of (l, t) and v an arbitrary element of $(\pm, t')$. Then it is an easy calculation to show, that
$$(l, t) \otimes (\sigma, t') = (l, t + t'),$$
where an associated special basis is given by $\{u \otimes v, u' \otimes v\}$, if $\sigma = +$, and by $\{u \otimes v, -u' \otimes v\}$, if $\sigma = -$.
- Let $l, l' \geq 1$ be integers, and let t, t' be complex numbers. Let further be $\{u, u'\}$ and $\{v, v'\}$ the respective special bases of (l, t) and (l', t'). A quick calculation shows that $u \otimes v$ and $u' \otimes v'$ span a two-dimensional representation of the type $(l + l', t + t')$. Analogously,
$$(l - l', t + t') \text{ with special basis } \{(-1)^{l'} u \otimes v', u' \otimes v\} \text{ is well-defined for } l > l',$$
$$(l' - l, t + t') \text{ with special basis } \{(-1)^{l} u' \otimes v, u \otimes v'\} \text{ is well-defined for } l < l'.$$

In the case $l = l'$ the representation $(l,t)\otimes(l',t')$ is not irreducible any more, but splits into

$$\begin{aligned}&(+, t+t') \text{ spanned by } (-1)^l u\otimes v' + u'\otimes v,\\ &(-, t+t') \text{ spanned by } (-1)^{l-1} u\otimes v' + u'\otimes v.\end{aligned}$$

We may subsume the results of this case by

$$(l,t)\otimes(l',t') = \begin{cases}(l+l', t+t')\oplus(|l-l'|, t+t'), & \text{if } l\neq l',\\ (l+l', t+t')\oplus(+, t+t')\oplus(-, t+t'), & \text{if } l = l'.\end{cases}$$

We will only be interested in the case $m = n-1$. There we get

Proposition 1.6. *The tensor product $\pi_\infty^{\mathrm{W}}\otimes\sigma_\infty^{\mathrm{W}}$ takes the value*

$$\bigoplus_{i=1}^{\frac{n}{2}}(l_i, -\frac{\mathsf{w}+\mathsf{w}'}{2})\oplus\bigoplus_{i=1}^{\frac{n}{2}}\bigoplus_{j=1}^{\frac{n}{2}-1}\left[(l_i+l_j', -\frac{\mathsf{w}+\mathsf{w}'}{2})\oplus(|l_i-l_j'|, -\frac{\mathsf{w}+\mathsf{w}'}{2})\right],\quad \textit{if } n \textit{ is even},$$

$$\bigoplus_{j=1}^{\frac{n-1}{2}}(l_j', -\frac{\mathsf{w}+\mathsf{w}'}{2})\oplus\bigoplus_{i=1}^{\frac{n-1}{2}}\bigoplus_{j=1}^{\frac{n-1}{2}}\left[(l_i+l_j', -\frac{\mathsf{w}+\mathsf{w}'}{2})\oplus(|l_i-l_j'|, -\frac{\mathsf{w}+\mathsf{w}'}{2})\right],\quad \textit{if } n \textit{ is odd},$$

where by $(0, -\frac{\mathsf{w}+\mathsf{w}'}{2})$ we denote $(+, -\frac{\mathsf{w}+\mathsf{w}'}{2})\oplus(-, -\frac{\mathsf{w}+\mathsf{w}'}{2})$.[3]

1.4. Cohomology of locally symmetric spaces. Finally we need some notation for the cohomology of the orbifolds

$$S_n(K) := \mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})/KK_{n,\infty},$$

where K is a compact open subgroup of $\mathrm{GL}_n(\mathbb{A}_f)$. We set

$$\tilde{S}_n := \varprojlim_K S_n(K),$$

where K runs through all compact open subgroups of $\mathrm{GL}_n(\mathbb{A}_f)$. For any finite-dimensional representation (ϱ_μ, M_μ) we define the locally constant sheaf $\mathcal{M}_\mu = \mathcal{M}_{\mu,K}$ on $S_n(K)$ by setting $\mathcal{M}_{\mu,K}(U)$ for any open $U\subseteq S_n(K)$ to be the set of locally constant functions $f : \mathrm{pr}^{-1}(U)\to M_\mu$ satisfying

$$\forall\gamma\in\mathrm{GL}_n(\mathbb{Q}),\ z\in\mathrm{pr}^{-1}(U) : f(\gamma z) = \varrho_\mu(\gamma)(f(z)),$$

where $\mathrm{pr} : \mathrm{GL}_n(\mathbb{A})/KK_{n,\infty}\to S_n(K)$ is the natural projection. Analogously, we get a locally constant sheaf on $\tilde{S}_n$, noted $\mathcal{M}_\mu$ as well. Similarly, for any field extension $E/\mathbb{Q}$ we denote by $\mathcal{M}_{\mu,E} = \mathcal{M}_{\mu,E,K}$ the corresponding sheaf on $S_n(K)$. Analogously to the rational case, $\mathcal{M}_{\mu,E}$ denotes as well the respective sheaf on $\tilde{S}_n$.

We then define the cohomology groups with coefficients in $\mathcal{M}_{\mu,\mathbb{C},K}$ (cf. [Clo], p. 121):

$$H^\bullet_?(\tilde{S}_n, \mathcal{M}_{\mu,\mathbb{C}}) := \varinjlim_K H^\bullet_?(S_n(K), \mathcal{M}_{\mu,\mathbb{C},K}),\quad ?\in\{\text{ blank, c, cusp }\}.$$

These groups are modules under the canonical action of $\mathrm{GL}_n(\mathbb{A}_f)\times\mathrm{GL}_n(\mathbb{R})/\mathrm{GL}_n^+(\mathbb{R})$.

[3]Note, that since the integer l belonging to the representation (l,t) is at least 1, there is no conflict of notation.

2 The Rankin-Selberg convolution

We will now introduce the Rankin-Selberg L-series, whose critical values we want to study in Section 3. Starting from the Global Birch Lemma of [KMS] we will give a first description of those values in terms of integrals over certain Whittaker functions.

We fix a non-trivial character $\tau = \bigoplus_\ell \tau_\ell : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times$, such that for all finite places ℓ the conductor of τ_ℓ equals $\mathbb{Z}_\ell$. We also denote by

$$\tau(n) := \prod_{i=1}^{n-1} \tau(n_{i,i+1})$$

the induced (generic) character of $U_n(\mathbb{A})$. For any automorphic representation $\pi = \pi_f \otimes \pi_\infty$ we write $\mathscr{W}(\pi, \tau)$ for the Whittaker model of π with respect to τ. This Whittaker space can be described as the restricted tensor product of the local Whittaker spaces defined as in [JPS2] resp. [JPS1] in the infinite resp. finite case, that is

$$\mathscr{W}(\pi, \tau) = \mathscr{W}(\pi_\infty, \tau_\infty) \otimes \bigotimes_{\ell\nmid\infty}{}' \mathscr{W}(\pi_\ell, \tau_\ell).$$

Here, an element of $\mathscr{W}(\pi, \tau)$ is a tensor in $\bigotimes_\ell \mathscr{W}(\pi_\ell, \tau_\ell)$, where all but finitely many factors are given by the respective new vector v_ℓ^0 (cf. [JPS3]). This is possible, since π_ℓ is unramified for all but finitely many primes ℓ.

Now we fix a prime p, and two cuspidal automorphic representations $\pi = \pi_f \otimes \pi_\infty$ resp. $\sigma = \sigma_f \otimes \sigma_\infty$ of $\mathrm{GL}_n(\mathbb{A})$ resp. $\mathrm{GL}_{n-1}(\mathbb{A})$ both unramified at p. Following [JPS1] we now introduce the local Rankin-Selberg convolution for π and σ at some fixed prime number $\ell \neq p$: For each pair of Whittaker functions

$$(v_\ell, w_\ell) \in \mathscr{W}(\pi_\ell, \tau_\ell) \times \mathscr{W}(\sigma_\ell, \bar{\tau}_\ell)$$

the associated zeta integral

$$\psi_\ell(v_\ell, w_\ell; s) := \int\limits_{U_{n-1}(\mathbb{Q}_\ell)\backslash \mathrm{GL}_{n-1}(\mathbb{Q}_\ell)} v_\ell\begin{pmatrix} g & \\ & 1 \end{pmatrix} \cdot w_\ell(g) \cdot |\det(g)|^{s-\frac{1}{2}} dg$$

converges for $\mathrm{Re}(s)$ large enough. These zeta integrals span a fractional ideal L of the ring $\mathbb{C}[\ell^s, \ell^{-s}]$. In that way the local L-function $L(\pi_\ell, \sigma_\ell; s)$ is defined uniquely by fixing a polynomial $P(X) \in \mathbb{C}[X]$, such that $P(0) = 1$ and $P(\ell^{-s})^{-1}$ generates L, and by setting

$$P(\ell^{-s})^{-1} =: L(\pi_\ell, \sigma_\ell; s).$$

Obviously, we have a linear map on the tensor product $\mathscr{W}(\pi_\ell, \tau_\ell) \otimes \mathscr{W}(\sigma_\ell, \bar{\tau}_\ell)$ given by

$$\Psi_\ell : \begin{cases} \mathscr{W}(\pi_\ell, \tau_\ell) \otimes \mathscr{W}(\sigma_\ell, \bar{\tau}_\ell) \to \mathbb{C}(\ell^s), \\ v_\ell \otimes w_\ell \mapsto \Psi_\ell(v_\ell \otimes w_\ell; s) := \psi_\ell(v_\ell, w_\ell; s). \end{cases}$$

Moreover, if π_ℓ and σ_ℓ are both unramified, by §3.2 in [KMS] the zeta integral for the associated new vectors v_ℓ^0 and w_ℓ^0 represents the L-function

$$L(\pi_\ell, \sigma_\ell; s) = \Psi_\ell(v_\ell^0 \otimes w_\ell^0; s).$$

From now on we will write in short $t_\ell^0 := v_\ell^0 \otimes w_\ell^0$. Let S denote the set of primes ℓ, where π_ℓ or σ_ℓ is ramified. For any $\ell \in S$ there is a tensor $t_\ell^0 \in \mathscr{W}(\pi_\ell, \tau_\ell) \otimes \mathscr{W}(\sigma_\ell, \bar{\tau}_\ell)$ such that we have

$$L(\pi_\ell, \sigma_\ell; s) = \Psi_\ell(t_\ell^0; s).$$

Note, that for general n such a vector does not need to be pure. In the case $n = 3$ however, there is always a choice of a pure t_ℓ^0 (cf. [Rie]).

We will now consider pairs (v, w) of global Whittaker functions on $\mathrm{GL}_n(\mathbb{A})$ and $\mathrm{GL}_{n-1}(\mathbb{A})$ given as products of local Whittaker functions $v := \prod_\ell v_\ell$ and $w := \prod_\ell w_\ell$, where we choose $v_\ell = v_\ell^0$ and $w_\ell = w_\ell^0$ for ℓ not contained in $S \cup \{p\}$. For $\ell = p$ we let v_p and w_p vary among all Whittaker functions which are right invariant under the respective Iwahori subgroup I_n or I_{n-1}. Here, I_n consists of those matrices in $\mathrm{GL}_n(\mathbb{Z}_p)$ which are upper triangular modulo p. For $\ell \in S$ we will choose a tensor as described above.

For any choice of $v_\infty \in \mathscr{W}_0(\pi_\infty, \tau_\infty)$ and $w_\infty \in \mathscr{W}_0(\sigma_\infty, \bar{\tau}_\infty)$, for arbitrary v_ℓ, w_ℓ for $\ell \in S$, for $(v_\ell, w_\ell) = (v_\ell^0, w_\ell^0)$ for $\ell \notin S \cup \{p\}$, and for (v_p, w_p) like in the last paragraph we get global Whittaker functions (v, w) with associated automorphic forms (ϕ, φ). Here, the 0 in the index means, that we consider the space of $\mathrm{O}_n(\mathbb{R})$-finite resp. $\mathrm{O}_{n-1}(\mathbb{R})$-finite Whittaker functions (cf. [JPS2]). Like above we set

$$\mathscr{W}_0(\pi, \tau) = \mathscr{W}_0(\pi_\infty, \tau_\infty) \otimes \bigotimes_{\ell \nmid \infty}{}' \mathscr{W}(\pi_\ell, \tau_\ell).$$

The product of all local zeta integrals then becomes a Rankin-Selberg convolution (cf. [JS])

$$\prod_\ell \psi_\ell(v_\ell, w_\ell; s) = \int\limits_{\mathrm{GL}_{n-1}(\mathbb{Q})\backslash \mathrm{GL}_{n-1}(\mathbb{A})} \phi\begin{pmatrix} g & \\ & 1 \end{pmatrix} \cdot \varphi(g) \cdot |\det(g)|^{s-\frac{1}{2}} dg$$

for $\mathrm{Re}(s) \gg 0$, admitting an analytic continuation to an entire function in s (cf. [CP1], Prop. 6.1). This function only depends on the pure tensor $v \otimes w$ and can be extended linearly to the algebraic tensor product of Whittaker spaces $\mathscr{W}_0(\pi, \tau) \otimes \mathscr{W}_0(\sigma, \bar{\tau})$ by sending

$$\prod_\ell v_\ell \otimes \prod_\ell w_\ell \mapsto \prod_\ell \Psi_\ell(v_\ell \otimes w_\ell; s).$$

In particular we find (up to the infinity factor) the global L-function

$$L(\pi, \sigma; s) := \prod_\ell L(\pi_\ell, \sigma_\ell; s)$$

in the image of this map. For each choice of the pair (v_∞, w_∞) there is an entire function $P(s)$ such that

$$P(s) \cdot L(\pi_\infty, \sigma_\infty; s) = \Psi_\infty(v_\infty \otimes w_\infty; s),$$

and therefore

$$P(s) \cdot L(\pi, \sigma; s) = \Psi_\infty(v_\infty \otimes w_\infty; s) \cdot \prod_{\ell \nmid \infty} \Psi_\ell(t_\ell^0; s).$$

Recall from Section 1.3, that $L(\pi_\infty, \sigma_\infty; s)$ is given by the Weil group representation.

Writing each t_ℓ^0 for $\ell \in S$ as a sum of pure tensors leads to a finite sum of (global) pure tensors in $\mathscr{W}_0(\pi, \tau) \otimes \mathscr{W}_0(\sigma, \bar{\tau})$

$$\sum_j v_j \otimes w_j = (v_\infty \otimes w_\infty) \cdot \bigotimes_{\ell \nmid \infty} t_\ell^0. \tag{2.1}$$

We fix this explicit decomposition and in what follows our formulas will depend on it. Separating finite and infinite parts we will sometimes write $v_j = v_\infty \cdot v_{j,f}$ and $w_j = w_\infty \cdot w_{j,f}$. The associated automorphic forms ϕ_j and φ_j yield the integral representation

$$P(s) \cdot L(\pi, \sigma; s) = \sum_j \int \phi_j \begin{pmatrix} g & \\ & 1 \end{pmatrix} \varphi_j(g) |\det(g)|^{s-\frac{1}{2}} dg.$$

We will in particular consider modified (v_j, w_j)'s and (ϕ_j, φ_j)'s, where at $\ell = p$ the local component (v_p^0, w_p^0) is replaced by an arbitrary pair (v_p, w_p) of Whittaker functions invariant under the respective Iwahori subgroup.

We want to study the critical values of pairs (π,σ) of cohomological representations. Here, in analogy of Deligne's notion of critical values of motivic L-functions we say, that a half integer $s_0 \in \frac{1}{2}+\mathbb{Z}$ is *critical for* (π,σ) if neither $L(\pi_\infty^{\mathrm{W}}\otimes\sigma_\infty^{\mathrm{W}};s)$ nor $L((\pi_\infty^{\mathrm{W}}\otimes\sigma_\infty^{\mathrm{W}})\check{};1-s)$ has a pole in $s=s_0$. So let us study $L(\pi_\infty,\sigma_\infty;s)=L(\pi_\infty^{\mathrm{W}}\otimes\sigma_\infty^{\mathrm{W}};s)$ with

$$\begin{aligned}\pi_\infty &\cong \operatorname{sgn}^k\otimes J(-\mathsf{w},\mathbf{l}), \quad k\in\mathbb{Z}/2\mathbb{Z},\\ \sigma_\infty &\cong \operatorname{sgn}^{k'}\otimes J(-\mathsf{w}',\mathbf{l}'), \quad k'\in\mathbb{Z}/2\mathbb{Z}.\end{aligned}$$

We can give concrete formulæ for $L(\pi_\infty^{\mathrm{W}}\otimes\sigma_\infty^{\mathrm{W}};s)$ by (3.6) in [Kna1] and Proposition 1.6 by

$$\begin{aligned}&\prod_{i=1}^{\frac{n}{2}}\left[2(2\pi)^{-(s-\frac{\mathsf{w}+\mathsf{w}'}{2}+\frac{l_i}{2})}\cdot\Gamma\left(s-\frac{\mathsf{w}+\mathsf{w}'}{2}+\frac{l_i}{2}\right)\right]\cdot\prod_{i=1}^{\frac{n}{2}}\prod_{\substack{j=1\\ l_j'\neq l_i}}^{\frac{n}{2}-1}\Big[2(2\pi)^{-(s-\frac{\mathsf{w}+\mathsf{w}'}{2}+\frac{l_i+l_j'}{2})}\\ &\cdot\Gamma\left(s-\frac{\mathsf{w}+\mathsf{w}'}{2}+\frac{l_i+l_j'}{2}\right)\cdot 2(2\pi)^{-(s-\frac{\mathsf{w}+\mathsf{w}'}{2}+\frac{|l_i-l_j'|}{2})}\cdot\Gamma\left(s-\frac{\mathsf{w}+\mathsf{w}'}{2}+\frac{|l_i-l_j'|}{2}\right)\Big]\\ &\cdot\prod_{i=1}^{\frac{n}{2}}\prod_{\substack{j=1\\ l_j'=l_i}}^{\frac{n}{2}-1}\Big[2(2\pi)^{-(s-\frac{\mathsf{w}+\mathsf{w}'}{2}+l_i)}\Gamma\left(s-\frac{\mathsf{w}+\mathsf{w}'}{2}+l_i\right)\cdot\pi^{(s-\frac{\mathsf{w}+\mathsf{w}'}{2})/2}\cdot\Gamma\left(\frac{s}{2}-\frac{\mathsf{w}+\mathsf{w}'}{4}\right)\\ &\cdot\pi^{(s-\frac{\mathsf{w}+\mathsf{w}'}{2}+1)/2}\cdot\Gamma\left(\frac{s}{2}-\frac{\mathsf{w}+\mathsf{w}'}{4}+\frac{1}{2}\right)\Big],\end{aligned}$$

if n is even, and

$$\begin{aligned}&\prod_{j=1}^{\frac{n-1}{2}}\left[2(2\pi)^{-(s-\frac{\mathsf{w}+\mathsf{w}'}{2}+\frac{l_j'}{2})}\cdot\Gamma\left(s-\frac{\mathsf{w}+\mathsf{w}'}{2}+\frac{l_j'}{2}\right)\right]\cdot\prod_{i=1}^{\frac{n-1}{2}}\prod_{\substack{j=1\\ l_j'\neq l_i}}^{\frac{n-1}{2}}\Big[2(2\pi)^{-(s-\frac{\mathsf{w}+\mathsf{w}'}{2}+\frac{l_i+l_j'}{2})}\\ &\cdot\Gamma\left(s-\frac{\mathsf{w}+\mathsf{w}'}{2}+\frac{l_i+l_j'}{2}\right)\cdot 2(2\pi)^{-(s-\frac{\mathsf{w}+\mathsf{w}'}{2}+\frac{|l_i-l_j'|}{2})}\cdot\Gamma\left(s-\frac{\mathsf{w}+\mathsf{w}'}{2}+\frac{|l_i-l_j'|}{2}\right)\Big]\\ &\cdot\prod_{i=1}^{\frac{n-1}{2}}\prod_{\substack{j=1\\ l_j'=l_i}}^{\frac{n-1}{2}}\Big[2(2\pi)^{-(s-\frac{\mathsf{w}+\mathsf{w}'}{2}+l_i)}\Gamma\left(s-\frac{\mathsf{w}+\mathsf{w}'}{2}+l_i\right)\cdot\pi^{(s-\frac{\mathsf{w}+\mathsf{w}'}{2})/2}\cdot\Gamma\left(\frac{s}{2}-\frac{\mathsf{w}+\mathsf{w}'}{4}\right)\\ &\cdot\pi^{(s-\frac{\mathsf{w}+\mathsf{w}'}{2}+1)/2}\cdot\Gamma\left(\frac{s}{2}-\frac{\mathsf{w}+\mathsf{w}'}{4}+\frac{1}{2}\right)\Big],\end{aligned}$$

if n is odd. Now let s_0 be in $\frac{1}{2}+\mathbb{Z}$. We want to determine, if s_0 is critical for (π,σ). Using (1.2) we get: If for no pair (i,j) we have $l_i=l_j'$, then neither $L(\pi_\infty^{\mathrm{W}}\otimes\sigma_\infty^{\mathrm{W}};s)$ nor $L((\pi_\infty^{\mathrm{W}}\otimes\sigma_\infty^{\mathrm{W}})\check{};1-s)$ has a pole at $s=s_0$ exactly if the following inequalities in $\kappa:=\frac{1}{2}(\mathsf{w}+\mathsf{w}'+1)$ hold:

$$\begin{aligned}\left.\begin{matrix}\kappa-\frac{1+l_m}{2}\\ \kappa-\frac{1+l_m'}{2}\end{matrix}\right\}\quad &<\quad s_0\quad<\quad\begin{cases}\kappa+\frac{1+l_m}{2} & \text{if } n \text{ is even},\\ \kappa+\frac{1+l_m'}{2} & \text{if } n \text{ is odd},\end{cases}\\ \kappa-\frac{1+l_i+l_j'}{2}\quad &<\quad s_0\quad<\quad\kappa+\frac{1+l_i+l_j'}{2}\quad\text{for all } i,j,\\ \kappa-\frac{1+|l_i-l_j'|}{2}\quad &<\quad s_0\quad<\quad\kappa+\frac{1+|l_i-l_j'|}{2}\quad\text{for all } i,j \text{ fulfilling } l_i\neq l_j'.\end{aligned}$$

If there is a pair (i,j) with $l_i=l_j'$, then we get $\mathbf{l}'\equiv\mathbf{l}\equiv 0 \bmod (2)$, $\mathsf{w}+\mathsf{w}'\equiv 1 \bmod (2)$, and $\kappa\in\mathbb{Z}$. In this case neither $L(\pi_\infty^{\mathrm{W}}\otimes\sigma_\infty^{\mathrm{W}};s)$ nor $L((\pi_\infty^{\mathrm{W}}\otimes\sigma_\infty^{\mathrm{W}})\check{};1-s)$ has a pole at $s=s_0$, if and only if additionally to the inequalities above we have

$$s_0-\frac{1}{2}\not\equiv\kappa \bmod (2) \text{ or } \kappa-\frac{1}{2}<s_0<\kappa+\frac{1}{2}$$

and

$$s_0-\frac{1}{2}\equiv\kappa \bmod (2) \text{ or } \kappa-\frac{3}{2}<s_0<\kappa+\frac{3}{2}.$$

So if we have a pair (i,j) with $l_i=l_j'$, there is no critical s_0 with $s_0-\frac{1}{2}\equiv\kappa \bmod (2)$. It follows that only $s_0=\kappa-\frac{1}{2}$ is critical for (π,σ) in this case. On the other hand, if there is no such pair

(i,j), s_0 is critical for (π,σ), exactly if

$$\kappa - \frac{1+c_{\pi,\sigma}}{2} < s_0 < \kappa + \frac{1+c_{\pi,\sigma}}{2},$$

where by $c_{\pi,\sigma}$ we denote the minimum of all $|l_i - l'_j| \neq 0$ and of l_m resp. l'_m if n is even resp. odd. This set of critical elements is centered around κ. Since $c_{\pi,\sigma}$ is at least 1, we get

Proposition 2.1.

$$\kappa \text{ is critical for } (\pi,\sigma) \Longleftrightarrow \kappa \in \frac{1}{2} + \mathbb{Z} \Longleftrightarrow \mathsf{w} \equiv \mathsf{w}' \bmod (2).$$

This is why we will study L-values at κ in this paper. Note, that this is consistent with [KMS], where we have $\mathsf{w} = \mathsf{w}' = 0$ and thus $\kappa = \frac{1}{2}$.

We want to consider χ-twists of π for a finite idele class character $\chi = \prod_\ell \chi_\ell$ satisfying the properties

(a) $\chi_\infty = 1$,
(b) $\chi, \chi^2, \ldots, \chi^{n-1}$ have the same non-trivial conductor $f = p$-power.

The first assumption ensures that $P(s)$ will not change when varying χ in $\pi \otimes \chi$; obviously, the critical values do not change as well. The second assumption may possibly be omitted (cf. [Schm2], [Utz]).

Let $\tilde{\chi}_p$ denote the continuation of χ_p to $\mathbb{Z}_p$ by $\tilde{\chi}_p(px) = 0$ for all $x \in \mathbb{Z}_p$, and let $G(\chi_p)$ denote the Gauß sum of χ_p. Let further f be a non-trivial power of our fixed prime p and C_f the inverse image of the idele class group

$$\mathbb{Q}^\times \backslash \mathbb{Q}^\times \cdot \left(\mathbb{R}_{>0} \times \prod_{\ell \neq p,\infty} \mathbb{Z}_\ell^\times \times (1 + f^{2(n-1)})\mathbb{Z}_p \right) \subset \mathbb{Q}^\times \backslash \mathbb{A}^\times$$

under the determinant map

$$\det : \mathrm{GL}_{n-1}(\mathbb{Q}) \backslash \mathrm{GL}_{n-1}(\mathbb{A}) \to \mathbb{Q}^\times \backslash \mathbb{A}^\times.$$

With the same proof as for the corollary of the Global Birch Lemma in [KMS] we get

Lemma 2.2. *For any choice of (v_∞, w_∞) and any (v_p, w_p) right-invariant under the respective Iwahori subgroup the corresponding triples (P, ϕ_j, φ_j) for all j satisfy*

$$\begin{aligned} & v_p(1) \cdot w_p(1) \cdot P(\kappa) \cdot \prod_{i=1}^{n-1} \frac{G(\chi_p^i)(1-p^{-1})}{1-p^{-i}} \cdot L(\pi \otimes \chi, \sigma; \kappa) \\ = \; & \frac{p-1}{p} f^{2(n-1)} \sum_u \prod_{i=1}^{n-1} \tilde{\chi}_p(u_i^i) \sum_j \int_{C_f} \phi_j(\begin{pmatrix} g & \\ & 1 \end{pmatrix} \varphi^{-1} u \varphi) \varphi_j(g) |\det(g)|^{\kappa - \frac{1}{2}} dg, \end{aligned}$$

where $u = u_p$ (with $u_\ell = 1$ for all $\ell \neq p$) is taken from a representative system for $U_n(\mathbb{Z}_p)$ modulo $\varphi U_n(\mathbb{Z}_p)\varphi^{-1}$ with $\varphi = \mathrm{diag}(f^{-1}, \ldots, f^{-n})$, and the u_i run over the off-diagonal entries of u.

3 The algebraicity of the special L-value

From now on, let $\pi \in \mathrm{Coh}(\mathrm{GL}_n, \mu)$ and $\sigma \in \mathrm{Coh}(\mathrm{GL}_{n-1}, \nu)$ be two cohomological representations, where $\mu \in X_0^+(T_n)$ and $\nu \in X_0^+(T_{n-1})$. We will show that $L(\pi \otimes \chi, \sigma; \kappa)$ up to a constant factor independent of χ is an algebraic number. The idea is to make use of the non-vanishing of cohomology for π and σ. We thus will be able to construct a pairing on cohomology having certain integrals as values, that give a description of the L-values in question by Lemma 2.2. Since both representations are already defined over the algebraic numbers, and since this pairing respects algebraicity by construction, this will prove the assertion.

3.1. A map of differential forms. We will begin constructing a pairing on cohomology using a natural pairing on differential forms. However, this cannot be done straight forward, since belonging to π and σ we will get differentials on different symmetric spaces. So the first thing we will have to do is to describe a method that translates one type of differentials into the other. In this and the next two sections we will thus construct a chain map from the differential forms of the first type into those of the second one generalising the construction in [KMS], 3.3.

By [JPS3], Théorème (5.1) for $n \geq 3$ the global representations π and σ have finite parts π_f and σ_f with new vectors v_f resp. w_f right-invariant under some open compact subgroup $K \subseteq \mathrm{GL}_n(\hat{\mathbb{Z}})$ resp. $K' \subseteq \mathrm{GL}_{n-1}(\hat{\mathbb{Z}})$, such that the respective image under the determinant map is the full unit group $\hat{\mathbb{Z}}^\times$, i. e.

$$\det(K) = \det(K') = \hat{\mathbb{Z}}^\times.$$

We will assume $n \geq 3$ from now on. Moreover, the canonical embedding

$$j : \mathrm{GL}_{n-1} \to \mathrm{GL}_n,\ g \mapsto \begin{pmatrix} g & \\ & 1 \end{pmatrix}$$

sends K' into K, since by Théorème (4.1) of [loc.cit.] w_f is even right invariant under $j(\mathrm{GL}_{n-1}(\hat{\mathbb{Z}}))$, so we may choose K containing $j(K')$. Note, that by [JPS1] we are free to choose all additive characters τ_v needed in the definition of the respective Whittaker spaces $\mathscr{W}(\pi, \tau)$ and $\mathscr{W}(\sigma, \bar{\tau})$ to have exponent 0, what allows us to use those results.

Separating finite and infinite parts of adelic elements we write $g = (g_f, g_\infty)$ for $g \in \mathrm{GL}_n(\mathbb{A}) = \mathrm{GL}_n(\mathbb{A}_f) \times \mathrm{GL}_n(\mathbb{R})$. We put

$$\begin{aligned}
\mathscr{X}_n^1 &:= \mathrm{SL}_n(\mathbb{R}) / \mathrm{SO}_n(\mathbb{R}) = \mathrm{SL}_n^\pm(\mathbb{R}) / \mathrm{O}_n(\mathbb{R}),\\
\mathscr{X}_n &:= \mathrm{GL}_n(\mathbb{R}) / \mathrm{O}_n(\mathbb{R}) = \mathrm{GL}_n^+(\mathbb{R}) / \mathrm{SO}_n(\mathbb{R}) = \mathbb{R}_{>0} \times \mathscr{X}_n^1,\\
\Gamma &:= \{\gamma \in \mathrm{GL}_n^+(\mathbb{Q}) \mid \gamma_f \in K\} \subseteq \mathrm{SL}_n(\mathbb{Z}).
\end{aligned}$$

Then by the surjectivity of the determinant map, by strong approximation, we have the bijections

$$\Gamma \backslash \mathscr{X}_n \cong \mathrm{GL}_n(\mathbb{Q}) \backslash \mathrm{GL}_n(\mathbb{A}) / K \cdot \mathrm{O}_n(\mathbb{R}) \tag{3.1}$$

and

$$\Gamma \backslash \mathscr{X}_n^1 \cong \mathrm{GL}_n(\mathbb{Q}) \backslash \mathrm{GL}_n(\mathbb{A}) / K \cdot \mathrm{SO}_n(\mathbb{R}) Z_n^+(\mathbb{R}) \overset{1.4}{=} S_n(K). \tag{3.2}$$

The common dimension of $\mathscr{X}_n$ and $\Gamma \backslash \mathscr{X}_n$ is $d_n := \frac{n^2+n}{2}$. The same argument applies to GL_{n-1} with a discrete subgroup $\Gamma' \subseteq \mathrm{SL}_{n-1}(\mathbb{Z})$ attached to K'. For any element $h \in \mathrm{GL}_n(\mathbb{R})$ the embedding $j : \mathrm{GL}_{n-1} \to \mathrm{GL}_n$ induces an embedding of symmetric spaces

$$j_h : \mathscr{X}_{n-1} \to \mathscr{X}_n,\ g \cdot \mathrm{O}_{n-1}(\mathbb{R}) \mapsto h \cdot \begin{pmatrix} g & \\ & 1 \end{pmatrix} \cdot \mathrm{O}_n(\mathbb{R}).$$

We are in particular interested in those embeddings j_h which define maps of arithmetic quotients. For any $h \in \mathrm{GL}_n(\mathbb{Q})$ let

$$\Gamma'_h := \{\gamma \in \Gamma' \mid j(\gamma) \in h^{-1} \Gamma h\}.$$

Then j_h induces a proper mapping

$$\bar{j}_h : \Gamma'_h \backslash \mathscr{X}_{n-1} \to \Gamma \backslash \mathscr{X}_n,\ \Gamma'_h g \, \mathrm{O}_{n-1}(\mathbb{R}) \mapsto \Gamma h \begin{pmatrix} g & \\ & 1 \end{pmatrix} \mathrm{O}_n(\mathbb{R}).$$

We want to compose the maps j_h with the projections p_2 into the second component of $\mathscr{X}_n = \mathbb{R}_{>0} \times \mathscr{X}_n^1$, induced by the map

$$p_2 : \mathrm{GL}_n(\mathbb{R}) \to \mathrm{SL}_n^{\pm}(\mathbb{R}),\ g \mapsto g \cdot |\det(g)|^{-1/n}.$$

Recall that the passage to quotients only effects the second component, i. e.

$$\Gamma\backslash\mathscr{X}_n = \mathbb{R}_{>0} \times \Gamma\backslash\mathscr{X}_n^1.$$

On arithmetic quotients we have the homotopy equivalence

$$\bar{p}_2 : \Gamma\backslash\mathscr{X}_n \to \Gamma\backslash\mathscr{X}_n^1.$$

Of course the same arguments apply to $n-1$ instead of n.

For each $u \in U_n(\mathbb{Q})$ the map

$$J_u := \bar{p}_2 \circ \bar{j}_u : \begin{cases} \Gamma'_u\backslash\mathscr{X}_{n-1} & \to\ \Gamma\backslash\mathscr{X}_n^1 \\ \Gamma'_u g\,\mathrm{O}_{n-1} & \mapsto\ \Gamma u\left(\begin{smallmatrix} g & \\ & 1\end{smallmatrix}\right) \cdot |\det(g)|^{-1/n}\,\mathrm{O}_n \end{cases}$$

is proper by [KMS], p. 102. We want to keep track of the effect of these maps J_u on certain differential forms. We denote by l_u left translation by u and we decompose the map

$$p_2 \circ j_u : \mathrm{GL}_{n-1}(\mathbb{R}) \to \mathrm{SL}_n^{\pm}(\mathbb{R}),\ g \mapsto p_2(u \cdot j(g))$$

further into $p_2 \circ j_u = p_2 \circ l_u \circ j$. Since $\det(u) = 1$, the maps p_2 and l_u commute, hence we have

$$p_2 \circ j_u = l_u \circ p_2 \circ j.$$

We observe that $p_2 \circ j$ is an injective Lie group homomorphism and hence the induced map on invariant 1-forms is surjective. Specifically, letting $*$ denote dual vector space, this induced mapping

$$\delta(p_2 \circ j) : \mathfrak{sl}_n^* \to \mathfrak{gl}_{n-1}^*$$

is given by the formula

$$\delta(p_2 \circ j)(\omega)(X) := \omega(d(p_2 \circ j)(X)) \tag{3.3}$$

for $X \in \mathfrak{gl}_{n-1}$. Here $d(p_2 \circ j)$ denotes the Lie algebra homomorphism $\mathfrak{gl}_{n-1} \to \mathfrak{sl}_n$ induced by $p_2 \circ j$. Since the pullback l_u^* acts trivially on $\mathfrak{sl}_n^*$ we have

$$\delta(p_2 \circ j) = \delta(p_2 \circ j_u) = (d(p_2 \circ j_u))^*.$$

The map $\delta(p_2 \circ j)$ respects the Cartan decompositions

$$\mathfrak{sl}_n = \mathfrak{so}_n \oplus \tilde{\wp}_n \text{ and } \mathfrak{gl}_{n-1} = \mathfrak{so}_{n-1} \oplus \wp_{n-1}\,(= \mathfrak{k}_{n-1,\infty} \oplus \tilde{\wp}_{n-1}),$$

where $\mathfrak{so}_n$ denotes the set of skew symmetric $n \times n$ matrices and $\wp_n$ (resp. $\tilde{\wp}_n$) stands for the set of symmetric $n \times n$ matrices (resp. of trace equal to zero). In particular we have

$$\delta(p_2 \circ j)(\tilde{\wp}_n^*) = \wp_{n-1}^*.$$

We can now describe the map of differential forms

$$J_u^* : \Omega^\bullet(\Gamma\backslash\mathscr{X}_n^1, \mathcal{M}_{\mu,\mathbb{C}}) \to \Omega^\bullet(\Gamma'_u\backslash\mathscr{X}_{n-1}, \mathcal{M}_{\mu,\mathbb{C}})$$

in terms of the complex defining the Lie algebra cohomology. Note, that since $M_{\mu,\mathbb{C}}$ can be viewed as a $\mathrm{GL}_{n-1}(\mathbb{R})$-module via j, we can define a locally constant sheaf on $\tilde{S}_{n-1}$ just like in Section 1.4. We identify this sheaf with the one defined on $\tilde{S}_n$ and denote it by $\mathcal{M}_{\mu,\mathbb{C}}$ also. Since $M_{\mu,\mathbb{C}}$ can be viewed as a finite dimensional complex linear representation of $\mathrm{GL}_n(\mathbb{R})$ and therefore as one of any discrete subgroup Γ_n of $\mathrm{SL}_n^{\pm}(\mathbb{R})$, we may use Corollary VII.2.7 and VII.2.4 (5) of [BW] to get

$$\Omega^\bullet(\Gamma_n\backslash\mathscr{X}_n^1, \mathcal{M}_{\mu,\mathbb{C}}) \cong \left(\bigwedge^{\bullet} \tilde{\wp}_n^* \otimes C^\infty(\Gamma_n\backslash\,\mathrm{SL}_n^{\pm}(\mathbb{R}), M_{\mu,\mathbb{C}})\right)^{\mathrm{O}_n(\mathbb{R})}. \tag{3.4}$$

Here, we view the sheaf $\mathcal{M}_{\mu,\mathbb{C}}$ over $S_n(K_{\Gamma_n})$ as a sheaf over the arithmetic quotient $\Gamma_n\backslash\mathscr{X}_n^1$ via (3.2). Analogously we have

$$\Omega^\bullet(\Gamma_{n-1}\backslash\mathscr{X}_{n-1},\mathcal{M}_{\mu,\mathbb{C}}) \cong \left(\bigwedge^{\bullet}\wp_{n-1}^* \otimes C^\infty(\Gamma_{n-1}\backslash\operatorname{GL}_{n-1}(\mathbb{R}), M_{\mu,\mathbb{C}})\right)^{\operatorname{O}_{n-1}(\mathbb{R})} \tag{3.5}$$

for an arbitrary discrete subgroup Γ_{n-1} of $\operatorname{GL}_{n-1}(\mathbb{R})$, if we write $\mathcal{M}_{\mu,\mathbb{C}}$ as well for the locally constant sheaf of $M_{\mu,\mathbb{C}}$ over

$$\operatorname{GL}_{n-1}(\mathbb{Q})\backslash\operatorname{GL}_{n-1}(\mathbb{A})/K_{\Gamma_{n-1}}\operatorname{O}_{n-1}(\mathbb{R}) \overset{(3.1)}{\cong} \Gamma_{n-1}\backslash\mathscr{X}_{n-1}$$

that we get like in Section 1.4.

The dimension of $\wp_{n-1}^*$ is $d_{n-1}=\frac{n^2-n}{2}$, and the one of $\tilde{\wp}_n^*$ is $\tilde{d}_n := d_n - 1 = \frac{n^2+n}{2}-1$. We fix a basis $\{\omega_1,\dots,\omega_{\tilde{d}_n}\}$ of Maurer-Cartan forms in $\tilde{\wp}_n^*$ such that

$$\omega_i' := \delta(p_2\circ j)(\omega_i) \text{ for } i=1,\dots,d_{n-1}$$

is a basis of $\wp_{n-1}^*$ and $\omega_i'=0$ for $i>d_{n-1}$. Then the ω_i' for $1\le i\le d_{n-1}$ are Maurer-Cartan forms as well. For any set $I=\{i_1,\dots,i_r\}\subseteq\{1,\dots,\tilde{d}_n\}$ of r different elements $i_1,\dots,i_r$ we put $\omega_I := \omega_{i_1}\wedge\ldots\wedge\omega_{i_r}$ resp. $\omega_I' := \omega_{i_1}'\wedge\ldots\wedge\omega_{i_r}'$.

Lemma 3.1. *Let $r\in\mathbb{N}$. Given a differential form*

$$\eta = \sum_{|I|=r}\omega_I\phi_I \in \Omega^r(\Gamma\backslash\mathscr{X}_n^1,\mathcal{M}_{\mu,\mathbb{C}})$$

with $\phi_I\in C^\infty(\Gamma\backslash\operatorname{SL}_n^\pm(\mathbb{R}),M_{\mu,\mathbb{C}})$ we have

$$J_u^*(\eta) = \sum_{|I|=r}\omega_I'(\phi_I\circ p_2\circ j_u) \in \Omega^r(\Gamma_u'\backslash\mathscr{X}_{n-1},\mathcal{M}_{\mu,\mathbb{C}}).$$

Since J_u is proper we also get a map on differential forms with compact support

$$J_u^* : \Omega_c^\bullet(\Gamma\backslash\mathscr{X}_n^1,\mathcal{M}_{\mu,\mathbb{C}}) \to \Omega_c^\bullet(\Gamma_u'\backslash\mathscr{X}_{n-1},\mathcal{M}_{\mu,\mathbb{C}}),$$

just by replacing C^∞-functions by compactly supported C^∞-functions in our description above. We will later need a version of J_u^* on differential forms with certain growth conditions (which we get just the same).

3.2. Growth conditions. The next thing is to make precise those growth conditions. Let ϕ be a function in $C^\infty(\operatorname{SL}_n^\pm(\mathbb{R}),M_{\mu,\mathbb{C}})$, and $|\cdot| : M_{\mu,\mathbb{C}}\to\mathbb{R}$ an arbitrary norm of $M_{\mu,\mathbb{C}}$ as a $\mathbb{C}$-vector space. The function ϕ is of *moderate growth* or *slowly increasing*, if there is a constant C and a positive integer m such that for all $g\in\operatorname{SL}_n^\pm(\mathbb{R})$ we have

$$|\phi(g)| \le C\cdot||g||^m,$$

where $||g|| := \operatorname{tr}({}^tg\cdot g)^{1/2}$. The function ϕ is *fast decreasing*, if for each integer m there is a constant $C=C_m$ such that this inequality holds for all g. Those concepts are well-defined (i.e. independent of the norm $|\cdot|$) since all norms on $M_{\mu,\mathbb{C}}$ are equivalent, $M_{\mu,\mathbb{C}}$ being finite dimensional as a $\mathbb{C}$-vector space.

We will denote the compactly supported C^∞-functions by C_c^∞, the fast decreasing ones by C_{fd}^∞, and the ones of moderate growth by C_{mg}^∞. A differential form $\eta=\sum_I\omega_I\phi_I$ on $\Gamma\backslash\mathscr{X}_n^1$ is of *moderate growth* (resp. *fast decreasing*), if the ϕ_I have this property (cf. [Bor]). Following Borel we denote by $\Omega_{\mathrm{mg}}^\bullet(\Gamma\backslash\mathscr{X}_n^1,\mathcal{M}_{\mu,\mathbb{C}})$ (resp. $\Omega_{\mathrm{fd}}^\bullet(\Gamma\backslash\mathscr{X}_n^1,\mathcal{M}_{\mu,\mathbb{C}})$) the complex of forms $\eta\in\Omega^\bullet(\Gamma\backslash\mathscr{X}_n^1,\mathcal{M}_{\mu,\mathbb{C}})$ which together with their exterior de Rham differentials $d\eta$ are of moderate growth (resp. fast decreasing).

3.3. Integration along the fibre. In this section we want to find a map from the image of J_u^* to differentials on $\Gamma'_u\backslash\mathscr{X}^1_{n-1}$ such that the composition of this map and J_u^* is a chain map of the de Rham complex. In order to do this we will *integrate along the fibre*: We consider the canonical projection

$$\pi : \Gamma'_u\backslash\mathscr{X}_{n-1} = \Gamma'_u\backslash\mathscr{X}^1_{n-1} \times \mathbb{R}_{>0} \to \Gamma'_u\backslash\mathscr{X}^1_{n-1}$$

onto the first component and consider the push-forward π_* like in [BT], p. 37. We will show that for $n \geq 3$ the forms in

$$\Omega^\bullet(\Gamma'_u\backslash\mathscr{X}_{n-1}, \mathcal{M}_{\mu,\mathbb{C}}) = \Omega^\bullet(\Gamma'_u\backslash\mathscr{X}^1_{n-1} \times \mathbb{R}_{>0}, \mathcal{M}_{\mu,\mathbb{C}})$$

which are in the image $J_u^*(\Omega^\bullet_{\mathrm{fd}}(\Gamma\backslash\mathscr{X}^1_n, \mathcal{M}_{\mu,\mathbb{C}}))$ can be integrated along the fibre, i. e.

Lemma 3.2. *For $n \geq 3$ the push-forward π_* is a chain map lowering the degree of forms by one, more precisely*

$$\pi_* : J_u^*(\Omega^\bullet_{\mathrm{fd}}(\Gamma\backslash\mathscr{X}^1_n, \mathcal{M}_{\mu,\mathbb{C}})) \to \Omega^{\bullet-1}_{\mathrm{mg}}(\Gamma'_u\backslash\mathscr{X}^1_{n-1}, \mathcal{M}_{\mu,\mathbb{C}}).$$

Remark We need $n \geq 3$ only for the identifications (3.1) and (3.2). The lemma is true for $n = 2$ as well, if we view $\mathcal{M}_{\mu,\mathbb{C}}$ as a sheaf over the respective arithmetic quotients.

Proof. Let $\eta = \sum_{|I|=\bullet} \omega_I\phi_I$ be an arbitrary differential in $\Omega^\bullet_{\mathrm{fd}}(\Gamma\backslash\mathscr{X}^1_n, \mathcal{M}_{\mu,\mathbb{C}})$. Then we have $J_u^*(\eta) = \sum_{|I|=\bullet} \omega'_I \cdot (\phi_I \circ p_2 \circ j_u)$.

If $n \geq 2$, then like in the proof of Lemma 3.4 of [KMS] for each $N > 0$ there is a constant $C_{(N)}$ independent of g such that

$$\left|\phi_I\left(u\begin{pmatrix} g & \\ & 1\end{pmatrix} \cdot |\det(g)|^{-\frac{1}{n}}\right)\right| \leq C_{(N)} \cdot \min\{|\det(g)|^{-N}, |\det(g)|^{N}\}. \tag{3.6}$$

Let t denote the global parameter of the factor $\mathbb{R}_{>0}$ in $\Gamma'_u\backslash\mathscr{X}_{n-1}$. Integration along the fiber means that for each ω'_I having the invariant differential $\frac{dt}{t} =: \omega'_{d_{n-1}}$ as a wedge factor we must consider the integrals

$$\int_0^\infty \phi_I\left(u\begin{pmatrix} ht & \\ & 1\end{pmatrix} t^{\frac{1-n}{n}}\right)\frac{dt}{t} =: \check{\phi}_{I,u}(h)$$

for $h \in \mathrm{SL}^{\pm}_{n-1}(\mathbb{R})$. Those are absolutely convergent by (3.6). Moreover, the resulting functions $\check{\phi}_{I,u}$ are bounded, hence of moderate growth. For $\omega_{d_{n-1}} \notin I$ we set $\check{\phi}_{I,u} \equiv 0$. The same proof as for compact supports shows that integration along the fibre is a chain map lowering the degree of forms by 1, i. e.

$$\pi_* : J_u^*(\Omega^\bullet_{\mathrm{fd}}(\Gamma\backslash\mathscr{X}^1_n, \mathcal{M}_{\mu,\mathbb{C}})) \to \Omega^{\bullet-1}_{\mathrm{mg}}(\Gamma'_u\backslash\mathscr{X}^1_{n-1}, \mathcal{M}_{\mu,\mathbb{C}}).$$

If we write, in abuse of notation, $\omega'_{I\smallsetminus\{d_{n-1}\}}$ for the exterior product of the fitting $\omega'_i|_{\mathscr{X}^1_{n-1}}$, the image of π_* can be described by

$$\pi_* J_u^*(\eta) = \sum_{|I|=\bullet} \check{\phi}_{I,u}\omega'_{I\smallsetminus\{d_{n-1}\}}.$$

Note, that

$$d(\pi_* J_u^*(\eta)) = \pi_* J_u^*(d\eta)$$

has coefficient functions of moderate growth, since for $\eta \in \Omega^\bullet_{\mathrm{fd}}$ the coefficient functions of $d\eta$ are by definition also fast decreasing. So the proof of the lemma is complete. □

It follows that we have constructed a composed chain map

$$\Omega^\bullet_{\mathrm{fd}}(\Gamma\backslash\mathscr{X}^1_n, \mathcal{M}_{\mu,\mathbb{C}}) \xrightarrow{J_u^*} \mathrm{im}(J_u^*) \xrightarrow{\pi_*} \Omega^{\bullet-1}_{\mathrm{mg}}(\Gamma'_u\backslash\mathscr{X}^1_{n-1}, \mathcal{M}_{\mu,\mathbb{C}}). \tag{3.7}$$

Now let $\mathcal{M}_{\mu,\nu,\mathbb{C}}$ be the locally constant sheaf belonging to the tensor product $M_{\mu,\mathbb{C}} \otimes M_{\nu,\mathbb{C}}$. We want to construct a natural pairing

$$B_u : \Omega^{b_n}_{\rm fd}(\Gamma\backslash\mathscr{X}^1_n, \mathcal{M}_{\mu,\mathbb{C}}) \times \Omega^{b_{n-1}}_{\rm fd}(\Gamma'_u\backslash\mathscr{X}^1_{n-1}, \mathcal{M}_{\nu,\mathbb{C}}) \to \Omega^{\tilde{d}_{n-1}}_{\rm fd}(\Gamma'_u\backslash\mathscr{X}^1_{n-1}, \mathcal{M}_{\mu,\nu,\mathbb{C}}).$$

Note, that $b_n + b_{n-1} - 1 = \tilde{d}_{n-1} = \dim(\mathscr{X}^1_{n-1})$. By (3.7) it suffices to find a pairing

$$\tilde{B}_u : \Omega^{b_n-1}_{\rm mg}(\Gamma'_u\backslash\mathscr{X}^1_{n-1}, \mathcal{M}_{\mu,\mathbb{C}}) \times \Omega^{b_{n-1}}_{\rm fd}(\Gamma'_u\backslash\mathscr{X}^1_{n-1}, \mathcal{M}_{\nu,\mathbb{C}}) \to \Omega^{\tilde{d}_{n-1}}_{\rm fd}(\Gamma'_u\backslash\mathscr{X}^1_{n-1}, \mathcal{M}_{\mu,\nu,\mathbb{C}})$$

and to set $B_u(\eta, \eta') := \tilde{B}_u(\pi_* J_u^*(\eta), \eta')$.

3.4. A pairing on the differentials. We want to construct such a natural pairing $\tilde{B}_u$ now. In order to do this we need to write down elements $\check{\eta}$ of $\Omega^{b_n-1}_{\rm mg}(\Gamma'_u\backslash\mathscr{X}^1_{n-1}, \mathcal{M}_{\mu,\mathbb{C}})$ and η' of $\Omega^{b_{n-1}}_{\rm fd}(\Gamma'_u\backslash\mathscr{X}^1_{n-1}, \mathcal{M}_{\nu,\mathbb{C}})$ quite explicitly the way suggested by (3.4). In the proof of Lemma 3.2 we saw that the ω'_i with i running from 1 to $d_{n-1} - 1 = \tilde{d}_{n-1}$ form a basis of $\tilde{\wp}^*_{n-1}$. We let $\check{I}$ and I' run through the subsets of $\{1, \ldots, \tilde{d}_{n-1}\}$ like above Lemma 3.1, and let m resp. m' run through a basis of M_μ resp. M_ν.

By (3.4) we may write $\check{\eta} = \sum_{|\check{I}|=b_n-1} \omega'_{\check{I}}\check{\phi}_{\check{I}}$ with

$$\check{\phi}_{\check{I}} = \sum_m (\check{\phi}_{\check{I},m} \otimes m) \in C^\infty_{\rm mg}(\Gamma'_u\backslash\operatorname{SL}^\pm_{n-1}(\mathbb{R}), \mathbb{C}) \otimes M_{\mu,\mathbb{C}} = C^\infty_{\rm mg}(\Gamma'_u\backslash\operatorname{SL}^\pm_{n-1}(\mathbb{R}), M_{\mu,\mathbb{C}})$$

and $\eta' = \sum_{|I'|=b_{n-1}} \omega'_{I'}\varphi_{I'}$ with

$$\varphi_{I'} = \sum_{m'} (\varphi_{I',m'} \otimes m') \in C^\infty_{\rm fd}(\Gamma'_u\backslash\operatorname{SL}^\pm_{n-1}(\mathbb{R}), \mathbb{C}) \otimes M_{\nu,\mathbb{C}} = C^\infty_{\rm fd}(\Gamma'_u\backslash\operatorname{SL}^\pm_{n-1}(\mathbb{R}), M_{\nu,\mathbb{C}}).$$

Now consider the mapping given by

$$\tilde{B}_u(\check{\eta}, \eta') := \sum (\omega'_{\check{I}} \wedge \omega'_{I'}) \sum_{m,m'} \left(\check{\phi}_{\check{I},m} \cdot \varphi_{I',m'} \otimes (m \otimes m')\right),$$

where the first sum is over all pairs of subsets $\check{I}$ and I' of $\{1, \ldots, d_{n-1}\}$ fulfilling $|\check{I}| = b_n - 1$ and $|I'| = b_{n-1}$. It is well defined, since $\tilde{B}_u(\check{\eta}, \eta')$ is invariant under $\mathrm{O}_{n-1}(\mathbb{R})$, i. e. for all $g \in \mathrm{O}_{n-1}(\mathbb{R})$ we have

$$g.\tilde{B}_u(\check{\eta}, \eta') = \tilde{B}_u(g.\check{\eta}, g.\eta').$$

We use this formula for $\tilde{B}_u$ to describe B_u explicitly. So if we set $\check{\eta} := \pi_* J_u^*(\eta)$ with $\eta = \sum_I \phi_I\omega_I$ like before, and if we write $\check{\eta} = \sum_{|I|=b_n-1} \check{\phi}_{I,u}\omega'_{I\smallsetminus\{d_{n-1}\}}$ as in the proof of Lemma 3.2 with $\check{\phi}_{I,u} = \sum_m \check{\phi}_{I,u,m} \otimes m$, we get

$$B_u(\eta, \eta') = \sum_{\substack{|I|=b_n \\ |I'|=b_{n-1}}} \varepsilon_{I,I'} \sum_{m,m'} \left(\check{\phi}_{I,u,m} \cdot \varphi_{I',m'} \otimes (m \otimes m')\right) \omega'_1 \wedge \ldots \wedge \omega'_{\tilde{d}_{n-1}}, \tag{3.8}$$

where $\varepsilon_{I,I'} = \pm 1$ if $I \mathbin{\dot\cup} I' = \{1, \ldots, d_{n-1}\}$ and $\varepsilon_{I,I'} = 0$ otherwise, since for all $h \in \operatorname{SL}^\pm_{n-1}(\mathbb{R})$ we have

$$\begin{aligned}
\check{\phi}_{I,u}(h) &= \int_0^\infty \phi_I\left(u\begin{pmatrix} ht & \\ & 1\end{pmatrix} t^{\frac{1-n}{n}}\right)\frac{dt}{t} = \int_0^\infty \sum_m \phi_{I,m}\left(u\begin{pmatrix} ht & \\ & 1\end{pmatrix} t^{\frac{1-n}{n}}\right) \otimes m\,\frac{dt}{t} \\
&= \sum_m \int_0^\infty \phi_{I,m}\left(u\begin{pmatrix} ht & \\ & 1\end{pmatrix} t^{\frac{1-n}{n}}\right)\frac{dt}{t} \otimes m = \sum_m \left(\check{\phi}_{I,u,m}(h) \otimes m\right).
\end{aligned}$$

3.5. A cohomological pairing. By Theorem 5.2 of [Bor] the inclusion $\Omega_{\mathrm{c}} \hookrightarrow \Omega_{\mathrm{fd}}$ induces isomorphisms in cohomology. In particular, each fast decreasing cohomology class can be represented by a form with compact support. This allows us to integrate over the forms in the image of B_u. By §5 of [BT][4] we get an induced pairing

$$\mathscr{B}_u : H_{\mathrm{c}}^{b_n}(\Gamma\backslash\mathscr{X}_n^1, \mathcal{M}_{\mu,\mathbb{C}}) \times H_{\mathrm{c}}^{b_{n-1}}(\Gamma'\backslash\mathscr{X}_{n-1}^1, \mathcal{M}_{\nu,\mathbb{C}}) \to M_{\mu,\mathbb{C}} \otimes M_{\nu,\mathbb{C}}$$

on cohomology. It is given by

$$\mathscr{B}_u([\eta],[\eta']) := \int\limits_{\Gamma'_u\backslash\mathscr{X}_{n-1}^1} B_u(\eta, p_u^*(\eta')),$$

where $p_u : \Gamma'_u\backslash\mathscr{X}_{n-1}^1 \to \Gamma'\backslash\mathscr{X}_{n-1}^1$ is the natural projection. For simplicity we will write η' for $p_u^*(\eta')$.

By the previous section we can describe the values of $\mathscr{B}_u$ explicitly. Indeed we have

$$\begin{aligned}\int\limits_{\Gamma'_u\backslash\mathscr{X}_{n-1}^1} B_u(\eta,\eta') &\overset{(3.8)}{=} \int\limits_{\Gamma'_u\backslash\mathscr{X}_{n-1}^1} \sum_{I,I'} \varepsilon_{I,I'} \sum_{m,m'} \left(\check{\phi}_{I,u,m}\cdot\varphi_{I',m'}\otimes(m\otimes m')\right)\omega'_1\wedge\ldots\wedge\omega'_{\tilde{d}_{n-1}} \\ &= \int\limits_{\Gamma'_u\backslash\mathscr{X}_{n-1}^1} \sum_{I,I'} \varepsilon_{I,I'} \sum_{m,m'} \int_0^\infty \phi_{I,m}\left(u\begin{pmatrix} ht & \\ & 1\end{pmatrix} t^{\frac{1-n}{n}}\right)\frac{dt}{t}\cdot\varphi_{I',m'}(h)\otimes(m\otimes m')dh,\end{aligned}$$

recognising that a left-invariant $\tilde{d}_{n-1}$-form on $\mathscr{X}_{n-1}^1$ uniquely corresponds to a left-invariant measure dh on $\mathscr{X}_{n-1}^1 = \mathrm{SL}_{n-1}/\mathrm{SO}_{n-1}$ induced from a Haar measure on SL_{n-1}.

Because of the $\mathrm{O}_{n-1}(\mathbb{R})$-invariance of our differentials we can integrate over $\Gamma'_u\backslash\mathrm{SL}_{n-1}^{\pm}(\mathbb{R})$ instead of $\Gamma'_u\backslash\mathscr{X}_{n-1}^1 = \Gamma'_u\backslash\mathrm{SL}_{n-1}^{\pm}(\mathbb{R})/\mathrm{O}_{n-1}(\mathbb{R})$. The measure dh is the push forward of a Haar measure dg of $\mathrm{GL}_{n-1}(\mathbb{R})$ under the canonical projection. Let us extend the functions $\phi_{I,m}$ and $\varphi_{I',m'}$ in such a manner that they have actions of the respective centres via the central characters ω_π resp. ω_σ of our representations π resp. σ. Then we get

$$\int\limits_{\Gamma'_u\backslash\mathrm{GL}_{n-1}(\mathbb{R})} \sum_{I,I'}\varepsilon_{I,I'}\sum_{m,m'}\phi_{I,m}(u\begin{pmatrix} g & \\ & 1\end{pmatrix})\cdot\varphi_{I',m'}(g)\cdot\omega_\pi(|\det(g)|^{-\frac{1}{n}})\cdot\omega_\sigma(|\det(g)|^{-\frac{1}{n-1}})\otimes(m\otimes m')\,dg.$$

We know that the central character of $\pi_\infty\otimes\varrho_\mu$ is trivial, so that we have

$$\omega_\pi(|\det(g)|^{-\frac{1}{n}}) = \omega_{\check{\varrho}_\mu}(|\det(g)|^{-\frac{1}{n}}) = (|\det(g)|^{-\frac{1}{n}})^{-(\mu_1+\cdots+\mu_n)} = |\det(g)|^{\frac{\mathrm{w}}{2}}$$

and, analogously, $\omega_\sigma(|\det(g)|^{-\frac{1}{n-1}}) = |\det(g)|^{\frac{\mathrm{w}'}{2}}$. Recalling the definition of κ from Section 2 we can write

$$\mathscr{B}_u([\eta],[\eta']) = \int\limits_{\Gamma'_u\backslash\mathrm{GL}_{n-1}(\mathbb{R})} \sum_{I,I'}\varepsilon_{I,I'}\sum_{m,m'}\phi_{I,m}\left(u\begin{pmatrix} g & \\ & 1\end{pmatrix}\right)\cdot\varphi_{I',m'}(g)\cdot|\det(g)|^{\kappa-\frac{1}{2}}\otimes(m\otimes m')\,dg.$$

3.6. The Whittaker model. Choose a generator η_∞ of the one-dimensional $\mathbb{C}$-vector space $H^{b_n}(\mathfrak{gl}_n, K_{n,\infty}; \mathscr{W}_0(\pi_\infty,\tau_\infty)\otimes M_{\mu,\mathbb{C}})_\varepsilon$ in Corollary 1.5. Using the previous bases of $\tilde{\wp}_n^*$ and M_μ we can write

$$\eta_\infty = \sum_{|I|=b_n}\sum_m v_{\infty,I,m}\omega_I\otimes m$$

with Whittaker functions $v_{\infty,I,m}\in\mathscr{W}_0(\pi_\infty,\tau_\infty)$.

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[4] Bott and Tu work in the real case and with trivial coefficents, but the proof is the same in our situation. Note, that integration and tensoring with elements of $M_{\mu,\mathbb{C}}\otimes M_{\nu,\mathbb{C}}$ commutes.

The Fourier transform $\mathcal{F}(\pi) : \mathscr{W}(\pi,\tau) \xrightarrow{\sim} V_\pi \subset L^2_0(\mathrm{GL}_n(\mathbb{Q})\backslash \mathrm{GL}_n(\mathbb{A}))$ induces a mapping $\mathcal{F}(\pi)^{\mathrm{coh}}$ between the respective spaces of $(\mathfrak{gl}_n, K_{n,\infty})$-cohomology, which commutes with the action of $\mathrm{O}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$. Composing $\mathcal{F}(\pi)^{\mathrm{coh}}$ with the injection of $\mathscr{W}(\pi_f,\tau_f)$ into Lie algebra cohomology given by

$$\begin{array}{rcl}\mathscr{W}(\pi_f,\tau_f) & \hookrightarrow & H^{b_n}(\mathfrak{gl}_n, K_{n,\infty}; \mathscr{W}(\pi,\tau)\otimes M_{\mu,\mathbb{C}})_\varepsilon \\ v_f & \mapsto & v_f\cdot \eta_\infty\end{array}$$

we get

$$\tilde{\mathcal{F}}(\pi) = \mathcal{F}(\pi)^{\mathrm{coh}}\cdot\eta_\infty : \begin{cases}\mathscr{W}(\pi_f,\tau_f) & \to H^{b_n}(\mathfrak{gl}_n, K_{n,\infty}; V_\pi\otimes M_{\mu,\mathbb{C}})_\varepsilon, \\ v_f & \mapsto \eta := \sum\limits_{|I|=b_n}\sum\limits_m \phi_{I,m}\omega_I\otimes m,\end{cases}$$

where $\phi_{I,m}$ is the cusp form associated with $v_f v_{\infty,I,m}$ by $\mathcal{F}(\pi)$. Analogously, for a generator η'_∞ of the one-dimensional $\mathbb{C}$-vector space $H^{b_{n-1}}(\mathfrak{gl}_{n-1}, K_{n-1,\infty}; \mathscr{W}_0(\sigma_\infty,\bar{\tau}_\infty)\otimes M_{\nu,\mathbb{C}})_{\varepsilon'}$ we put

$$\tilde{\mathcal{F}}(\sigma) = \mathcal{F}(\sigma)^{\mathrm{coh}}\cdot\eta'_\infty : \begin{cases}\mathscr{W}(\sigma_f,\bar{\tau}_f) & \to H^{b_{n-1}}(\mathfrak{gl}_{n-1}, K_{n-1,\infty}; V_\sigma\otimes M_{\nu,\mathbb{C}})_{\varepsilon'}, \\ w_f & \mapsto \eta' := \sum\limits_{|I'|=b_{n-1}}\sum\limits_{m'} \varphi_{I',m'}\omega'_{I'}\otimes m',\end{cases}$$

where we use $\omega'_{I'}$ in the same sense as in the proof of Lemma 3.2. The rest of the notation should be clear.

By (2.1) we may decompose an element of $\mathscr{W}_0(\pi_f,\tau_f)\otimes\mathscr{W}_0(\sigma_f,\bar{\tau}_f)$ into a finite sum $\sum_j v_j\otimes w_j$ with pure tensors v_j and w_j in the respective restricted tensor product of local Whittaker spaces. Evaluating our pairing $\mathscr{B}_u$ at the corresponding η_j and η'_j we get

$$\mathscr{B}_u(\eta_j,\eta'_j) = \sum_{m,m'}\left(\sum_{I,I'}\varepsilon_{I,I'}\int\limits_{\Gamma'_u\backslash\mathrm{GL}_{n-1}(\mathbb{R})}\phi_{j,I,m}(uj(g))\varphi_{j,I',m'}(g)|\det(g)|^{\kappa-\frac{1}{2}}dg\right)\otimes(m\otimes m'),$$

where the cusp forms $\phi_{j,I,m}$ (belonging to $v_{j,f}v_{\infty,I,m}$) and $\varphi_{j,I',m'}$ (belonging to $w_{j,f}w_{\infty,I',m'}$) are restricted to the infinity component. Note, that we chose $v_{\infty,I,m}$ and $w_{\infty,I',m'}$ independent of j.

We are summing up terms like those on p. 123 of [KMS], and we can apply the same arguments. In order to do this we need to introduce some notation: From now on we denote conjugation by $\varphi := \mathrm{diag}(f^{-1},\dots,f^{-n})$ by the superscript φ, so that if $g=(g_{ij})\in\mathrm{GL}_n$, then $g^\varphi = \varphi g\varphi^{-1} = (f^{j-i}g_{ij})$. We will interpret $u\in U_n(\mathbb{Q})$ as an element of $U_n(\mathbb{Q}_p)$ and *not* embed $U_n(\mathbb{Q})$ diagonally into $U_n(\mathbb{A})$. From now on, we will only consider elements $u\in U_n(\mathbb{Q})$, that also lie in $U_n(\mathbb{Z}_p)^{\varphi^{-1}}\subset U_n(\mathbb{Q}_p)$. For those we write

$$K'_u := \{k\in K' \mid uj(k)u^{-1}\in K\}.$$

Like in [KMS] we get[5]

$$\mathscr{B}_u(\eta_j,\eta'_j) = \frac{\frac{p-1}{p}f^{2(n-1)}}{\mathrm{vol}(K'_u)}\cdot\sum_{m,m'}\left(\sum_{I,I'}\varepsilon_{I,I'}\int\limits_{C_f}\phi_{j,I,m}(j(g)u^{-1})\varphi_{j,I',m'}(g)|\det(g)|^{\kappa-\frac{1}{2}}dg\right)\otimes(m\otimes m').$$

3.7. Main Theorem. We will be interested in $\mathscr{B}_u(\eta_j,\eta'_j)$ from the last section as a function of u. So if λ is an (at first) arbitrary linear form on $M_{\mu,\mathbb{C}}\otimes M_{\nu,\mathbb{C}}$, we set

$$\mathscr{B}_\lambda(u) := \lambda\circ\sum_j\left(\mathscr{B}_u(\eta_j,\eta'_j)\right).$$

[5]Note, that in [KMS] the factor $\frac{p-1}{p}\cdot f^{2(n-1)}$ is actually missing in the cited formula on p. 123 and afterwards.

Using Lemma 2.2 we can express the value at $\kappa = \frac{1}{2} - \frac{\mathsf{w}+\mathsf{w}'}{2}$ of the Rankin-Selberg L-function in terms of the function $\mathscr{B}_\lambda(u)$ for a suitable choice of η_j's and η_j''s. In order to do this we define $P_{I,I',m,m'}(s)$ to be the entire function belonging to the pair $(v_{\infty,I,m}, w_{\infty,I',m'})$ (cf. 3.2 in [loc. cit.]) such that we have

$$\Psi(v_{\infty,I,m} \otimes w_{\infty,I',m'}; s) = P_{I,I',m,m'}(s) \cdot L(\pi_\infty, \sigma_\infty; s), \tag{3.9}$$

and

$$P_{\lambda,\infty}(s) := \sum_{I,I'} \varepsilon_{I,I'} \sum_{m,m'} \lambda(m \otimes m') P_{I,I',m,m'}(s). \tag{3.10}$$

This immediately leads to

Theorem A *Let $n \geq 3$. For all finite idele class character χ with trivial infinity part $\chi_\infty = 1$, and with $\chi, \chi^2, \ldots, \chi^{n-1}$ having the same non-trivial p-power conductor f we have the formula*

$$\begin{aligned} &v_p(1) w_p(1) P_{\lambda,\infty}(\kappa) \prod_{i=1}^{n-1} \frac{G(\chi_p^i)(1-p^{-1})}{1-p^{-i}} L(\pi \otimes \chi, \sigma; \kappa) \\ = &\sum_u \prod_{i=1}^{n-1} \tilde{\chi}_p(u_i^i) \operatorname{vol}(K'_{u^{\varphi^{-1}}}) \mathscr{B}_\lambda((u^{-1})^{\varphi^{-1}}), \end{aligned}$$

where $u = u_p$ (with $u_\ell = 1$ for all $\ell \neq p$) is taken from a representative system for $U_n(\mathbb{Z}_p)$ modulo $U_n(\mathbb{Z}_p)^\varphi$ with $\varphi = \operatorname{diag}(f^{-1}, \ldots, f^{-n})$, and the u_i run over the off-diagonal entries of u.

Unfortunately, we do not know if $P_{\lambda,\infty}$ is non-zero at $s = \kappa$ in general. However, in Section 4 we will show this in the case $n = 3$ for a suitable choice of λ.

3.8. Algebraicity. By [Sch], Satz 1.10, the cuspidal cohomology classes restrict to zero on the border of the Borel-Serre compactification of $S_n(K)$, so that we get an injection of cuspidal cohomology into cohomology with compact support:

$$H^\bullet_{\mathrm{cusp}}(\tilde{S}_n, \mathcal{M}_{\mu,\mathbb{C}}) \hookrightarrow H^\bullet_{\mathrm{c}}(\tilde{S}_n, \mathcal{M}_{\mu,\mathbb{C}}).$$

The latter is a module under $\operatorname{Aut}(\mathbb{C}/\mathbb{Q}) \times \mathrm{GL}_n(\mathbb{A}_f) \times \mathrm{GL}_n(\mathbb{R}) / \mathrm{GL}_n^+(\mathbb{R})$, where the actions of the factors commute and the (image of the) cuspidal cohomology even is defined over $\mathbb{Q}$ (cf. [Clo], Théorème 3.19). So this suggests that we try to choose the cuspidal cohomology classes $[\eta]$ and $[\eta']$ in such a way that the values of $\mathscr{B}_\lambda$ and therefore the L-values at $\frac{1}{2}$ are subject to good rationality conditions.

Let $\mathbb{Q}(\pi_f)$ denote the field of rationality of π_f in the notation of §3.1 in [Clo], that is the subfield of $\mathbb{C}$ fixed by the automorphisms $\alpha \in \operatorname{Aut}(\mathbb{C}/\mathbb{Q})$ fulfilling ${}^\alpha\pi_f \cong \pi_f$. It is a field of definition by Proposition 3.1 of [loc.cit.], and in our case in fact a number field by the Drinfel'd-Manin argument (cf. Proposition 3.16 in [loc.cit.]). For the field of rationality $\mathbb{Q}(\sigma_f)$ of σ_f the analogous statements hold.

If we denote by $F := \mathbb{Q}(\pi_f, \sigma_f)$ the smallest number field that contains $\mathbb{Q}(\pi_f)$ and $\mathbb{Q}(\sigma_f)$, the global (finite) Whittaker spaces $\mathscr{W}(\pi_f, \tau_f)$ and $\mathscr{W}(\sigma_f, \bar{\tau}_f)$ carry an F-structure, whose underlying F-spaces we denote by $\mathscr{W}_F(\pi_f, \tau_f)$ resp. $\mathscr{W}_F(\sigma_f, \bar{\tau}_f)$. Now since by Corollary 1.5 the cohomology spaces

$$H^{b_n}(\mathfrak{gl}_n, K_{n,\infty}; \mathscr{W}_0(\pi_\infty, \tau_\infty) \otimes M_{\mu,\mathbb{C}})_\varepsilon$$

and

$$H^{b_{n-1}}(\mathfrak{gl}_{n-1}, K_{n-1,\infty}; \mathscr{W}_0(\sigma_\infty, \bar{\tau}_\infty) \otimes M_{\nu,\mathbb{C}})_{\varepsilon'}$$

are one-dimensional, an immediate consequence is

Proposition 3.3. *We can normalise the ∞-part η_∞ by a non-trivial scalar factor such that for any Whittaker function $w_f \in \mathscr{W}_F(\pi_f, \tau_f)$ the cohomology class $[\eta]$ attached to $w_f \cdot \eta_\infty$ is F-rational, i. e.*

$$[\eta] \in H^{b_n}_{\text{cusp}}(\Gamma\backslash\mathscr{X}^1_n, \mathcal{M}_{\mu,F}) \subseteq H^{b_n}_{\text{c}}(\Gamma\backslash\mathscr{X}^1_n, \mathcal{M}_{\mu,\bar{\mathbb{Q}}}).$$

An analogous normalisation of η'_∞ yields

$$[\eta'] \in H^{b_{n-1}}_{\text{cusp}}(\Gamma'\backslash\mathscr{X}^1_{n-1}, \mathcal{M}_{\nu,F}) \subseteq H^{b_{n-1}}_{\text{c}}(\Gamma'\backslash\mathscr{X}^1_{n-1}, \mathcal{M}_{\nu,\bar{\mathbb{Q}}})$$

with the obvious notation.

The pairings $\mathscr{B}_u$ of cohomology spaces we considered in the sections before can be defined purely topologically and moreover with coefficients in an arbitrary subring of $\mathbb{C}$, in particular with coefficients in F. Furthermore we may choose the linear form λ to be induced from a linear form on the $\mathbb{Q}$-vector space M_μ or, slightly more general, from a linear form on the F-vector space $M_{\mu,F}$. By the definition of $\mathscr{B}_\lambda$ we then have

Corollary 3.4. *If the linear form λ is already defined over F, there is a choice of good local tensors t^0_ℓ of Whittaker functions for all $\ell \neq p$ such that for any "Iwahori fixed" pair*

$$(v_p, w_p) \in \mathscr{W}_F(\pi_p, \tau_p)^{I_n} \times \mathscr{W}_F(\sigma_p, \bar{\tau}_p)^{I_{n-1}}$$

the formula in Theorem 1 holds for the associated pairing $\mathscr{B}_\lambda$ with values $\mathscr{B}_\lambda(u)$ in the number field F.

4 The non-vanishing of the period

The algebraicity results of the last section have the one big flaw, that we can not guarantee the period $P_{\lambda,\infty}(\kappa)$ not to vanish. The second aim of this paper is to improve this situation. In this section we will study the case $n = 3$ and will show, that we have $P_{\lambda,\infty}(\kappa) \neq 0$ indeed (cf. Theorem B) for a suitable choice of λ. The general assumptions from the last section still hold.

The idea of proof is to construct a pairing on

$$\left(\bigwedge^2 \tilde{\wp}^*_3 \otimes \mathscr{W}_0(\pi_\infty, \tau_\infty) \otimes M_{\mu,\mathbb{C}}\right) \times \left(\tilde{\wp}^*_2 \otimes \mathscr{W}_0(\sigma_\infty, \bar{\tau}_\infty) \otimes M_{\nu,\mathbb{C}}\right),$$

whose image equals $P_{\lambda,\infty}(\kappa) \cdot \mathbb{C}$ if restricted to the one-dimensional cohomology modules

$$H^2(\mathfrak{gl}_3, K_{3,\infty}; \pi_\infty \otimes M_{\mu,\mathbb{C}})_\varepsilon \text{ and } H^1(\mathfrak{gl}_2, K_{2,\infty}; \sigma_\infty \otimes M_{\nu,\mathbb{C}})_{\varepsilon'}$$

for appropriate signs ε and ε'. This is done in Section 4.2. It remains to show, that the restricted pairing is not trivial, thus has an image isomorphic to $\mathbb{C}$. In order to do this, we split it up into a pairing $B_{\lambda,\infty}$ on the infinite Whittaker spaces times the coefficient modules and a pairing $B_\wedge$ on the exterior powers.

After proving some nice general properties of $B_{\lambda,\infty}$ in Section 4.3 we show in Section 4.4 for a particular λ (cf. Lemma 4.6), that $B_{\lambda,\infty}$ is not trivial restricted to the cohomological types. Finally, we show in Section 4.5, that $B_\wedge \otimes B_{\lambda,\infty}$ is not trivial restricted to cohomology, which proves Theorem B.

4.1. Notation.

In this section we want to get to know the modules we will be working with for the rest of this paper. Because of the small dimensions, everything is quite explicit.

$\mathfrak{so}_3$-modules. We may write $\mathfrak{so}_3(\mathbb{C}) := \mathfrak{so}_3 \otimes \mathbb{C} = \langle H, E_1, E_{-1}\rangle_{\mathbb{C}}$ with

$$H = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } E_{\pm 1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \pm i \\ -1 & \mp i & 0 \end{pmatrix},$$

where we have $[H, E_{\pm 1}] = \pm i E_{\pm 1}$ and $[E_1, E_{-1}] = 2iH$ for the Lie brackets. The standard torus is given by $\mathfrak{h}_3 = \langle H\rangle_{\mathbb{C}}$. We define e_1 to be the root given by $e_1(H) = 1$.

Now, for $k \in \mathbb{N}_0$ let $\mathscr{D}_k$ denote the irreducible $\mathfrak{so}_3$-module of highest weight ke_1. Let $v_{-k}^k, v_{1-k}^k, \dots, v_k^k$ be a $\mathfrak{so}_3$-basis of $\mathscr{D}_k$, where we put $v_i^k = 0$ for all $i \in \mathbb{Z} \smallsetminus \{-k, \dots, k\}$. We may choose this basis such that for all $i \in \mathbb{Z}$ we have

- $E_1 \,.\, v_i^k = v_{i+1}^k$,
- $E_{-1} \,.\, v_i^k = c_i^k v_{i-1}^k$ with $c_i^k = \begin{cases} -2\sum_{j=i}^k j & \text{if } -k \le i \le k, \\ 0 & \text{else.} \end{cases}$

From now on, we denote $\mathscr{W}_0(\pi_\infty, \tau_\infty) \otimes M_{\mu,\mathbb{C}}$ by V. By [Mah], Proposition 6.1.3, the $\mathrm{SO}_3(\mathbb{R})$-type of V supporting cohomology in $\bigwedge^2 \tilde{\wp}_3$ is $\mathscr{D}_3$. It occurs with multiplicity 1. Moreover the minimal $\mathrm{SO}_3(\mathbb{R})$-type of π_∞ is $\mathscr{D}_a$ for $a = \mu_1 - \mu_3 + 3$, and $\mathscr{D}_{a-3}$ is a maximal $\mathrm{SO}_3(\mathbb{R})$-type of M_μ as a $\mathfrak{so}_3$-module.

We will write $v_i := v_i^3$ and $c_i := c_i^3$. Furthermore, $\tilde{\wp}_3$ is isomorphic to $\mathscr{D}_2$. Here we will write $Z_i := v_i^2$ and $d_i := c_i^2$. We normalise those basis vectors by putting

$$Z_{-2} = \begin{pmatrix} 1 & -i & 0 \\ -i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ Z_{-1} = -2 \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ 1 & -i & 0 \end{pmatrix},\ Z_0 = -4 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

$$Z_1 = 12 \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ 1 & i & 0 \end{pmatrix},\ Z_2 = 24 \cdot \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$\mathfrak{so}_2$-modules. Analogously we may write $\mathfrak{so}_2(\mathbb{C}) = \mathfrak{so}_2 \otimes \mathbb{C} = \mathbb{C} \cdot H$, if we identify

$$H = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ with } \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By embedding $\wp_2$ into $\tilde{\wp}_3$ via

$$X \mapsto \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} - \frac{1}{3}\operatorname{tr}(X) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

we identify $\wp_2$ with $\langle Z_{-2}, Z_0, Z_2\rangle_{\mathbb{C}}$, and $\tilde{\wp}_2$ with $\langle Z_{-2}, Z_2\rangle_{\mathbb{C}}$. Here, the standard torus $\mathfrak{h}_2$ is all of $\mathfrak{so}_2(\mathbb{C})$. The root that sends H to 1 will be denoted with e_1 as well.

From now on, we denote $\mathscr{W}_0(\sigma_\infty, \bar{\tau}_\infty) \otimes M_{\nu,\mathbb{C}}$ by W. Like above, the $\mathrm{SO}_2(\mathbb{R})$-types supporting cohomology in $\tilde{\wp}_2$ are $\mathscr{D}'_2$ and $\mathscr{D}'_{-2}$, the irreducible representations of $\mathfrak{so}_2(\mathbb{C})$ with highest weight $2e_1$ resp. $-2e_1$. Again, the $\mathfrak{so}_2$-modules obtained by restriction are denoted the same. $\mathscr{D}'_2$ and $\mathscr{D}'_{-2}$ both occur with multiplicity 1, so that we have $W_{L'_2} \cong L'_2$ and $W_{L'_{-2}} \cong L'_{-2}$. Moreover, if $\mathscr{D}'_k$ denotes the irreducible $\mathfrak{so}_2$-module of weight ke_1 for $k \in \mathbb{Z}$, the minimal $\mathrm{SO}_2(\mathbb{R})$-types of σ_∞ are $\mathscr{D}'_b$ and $\mathscr{D}'_{-b}$ for $b = \nu_1 - \nu_2 + 2$, and M_ν as a $\mathfrak{so}_2$-module is the direct sum of $\mathscr{D}'_{2-b}, \mathscr{D}'_{4-b}, \dots, \mathscr{D}'_{b-2}$.

4.2. Pairings. For the moment let $n \geq 3$ be arbitrary again. In order to show that the value $P_{\lambda,\infty}(\kappa)$ in Theorem A does not vanish we study a pairing

$$B : \left(\bigwedge^{b_n} \tilde{\wp}_n^* \otimes \mathscr{W}_0(\pi, \tau) \otimes M_{\mu,\mathbb{C}} \right) \times \left(\bigwedge^{b_{n-1}} \tilde{\wp}_{n-1}^* \otimes \mathscr{W}_0(\sigma, \bar{\tau}) \otimes M_{\nu,\mathbb{C}} \right) \to \mathbb{C}$$

very similar to the pairing $\mathscr{B}_\lambda$ of Section 3 that takes the whole left side

$$v_p(1) w_p(1) P_{\lambda,\infty}(\kappa) \prod_{i=1}^{n-1} \frac{G(\chi_p^i)(1-p^{-1})}{1-p^{-i}} L(\pi \otimes \chi, \sigma; \kappa)$$

of the formula in the theorem as a value. We construct B as a tensor product of four pairings. In this way we can split up B and study our non-vanishing problem in the factors. Those are:

The pairing on the coefficient systems. We already know the pairing

$$B_\lambda = \lambda : \begin{cases} M_{\mu,\mathbb{C}} \times M_{\nu,\mathbb{C}} & \to \mathbb{C} \\ (m, m') & \mapsto \lambda(m \otimes m') \end{cases}$$

on the coefficient systems from Section 3.

The archimedean Rankin-Selberg pairing. We define a pairing

$$B_\infty : \begin{cases} \mathscr{W}_0(\pi_\infty, \tau_\infty) \times \mathscr{W}_0(\sigma_\infty, \bar{\tau}_\infty) & \to \mathbb{C} \\ (v_\infty, w_\infty) & \mapsto P_{v_\infty, w_\infty}(\kappa) \end{cases}$$

on the infinite parts of the Whittaker spaces. Here, it holds

$$\Psi(v_\infty, w_\infty; s) = P_{v_\infty, w_\infty}(s) \cdot L(\pi_\infty, \sigma_\infty; s)$$

like in Section 2. We will call B_∞, or in slight misuse of notation $B_{\lambda,\infty} := B_\lambda \otimes B_\infty$ as well, the *archimedean Rankin-Selberg pairing.*

The non-archimedean Rankin-Selberg pairing. On the finite parts of the Whittaker spaces we let

$$B_f : \begin{cases} \mathscr{W}_0(\pi_f, \tau_f) \times \mathscr{W}_0(\sigma_f, \bar{\tau}_f) & \to \mathbb{C}, \\ (v_f, w_f) & \mapsto \prod_{\ell \nmid \infty} \psi_\ell(v_\ell, w_\ell; \kappa). \end{cases}$$

We will call B_f the *non-archimedean Rankin-Selberg pairing.*

The pairing on the exterior powers. A problem in defining a pairing with values in $\mathbb{C}$ on the exterior powers of $\tilde{\wp}_n$ resp. $\tilde{\wp}_{n-1}$ is to make the arguments compatible. However, this is a problem we solved in Section 3: The differentials ω_I with $|I| = b_n$ generate $\bigwedge^{b_n} \tilde{\wp}_n^*$, and the differentials $\omega'_{I'}$ with $|I'| = b_{n-1}$ generate $\bigwedge^{b_{n-1}} \tilde{\wp}_{n-1}^*$. Thus a pairing of the sought-after type is given by

$$B_\wedge : \begin{cases} \bigwedge^{b_n} \tilde{\wp}_n^* \times \bigwedge^{b_{n-1}} \tilde{\wp}_{n-1}^* & \to \mathbb{C}, \\ (\omega_I, \omega'_{I'}) & \mapsto \varepsilon_{I,I'}. \end{cases}$$

We may now define B by putting

$$B(w, w') := \sum_{I,I'} \sum_{m,m'} B_\lambda(m, m') \cdot B_\infty(v_{I,m,\infty}, w_{I',m',\infty}) \cdot B_f(v_{I,f}, w_{I',f}) \cdot B_\wedge(\omega_I, \omega'_{I'}),$$

where we have $w = \sum_{|I|=b_n} \sum_m v_{I,m} \omega_I \otimes m$ and $w' = \sum_{|I'|=b_{n-1}} \sum_{m'} w_{I',m'} \omega'_{I'} \otimes m'$.

The next thing now is to determine the relation between B and $\mathscr{B}_\lambda$. In order to do this we compare the special L-values $L(\pi \otimes \chi, \sigma; \kappa)$ with zeta-integrals like they occur as values of B. Let $v_{j,I,m}$

resp. $w_{j,I',m'}$ be the Whittaker functions belonging to the automorphic forms $\phi_{j,I,m}$ resp. $\varphi_{j,I',m'}$ from the proof of Theorem A. From Section 2 we already know the "good tensors" t_ℓ^0 for (π_ℓ, σ_ℓ) at an arbitrary prime $\ell \neq p$ fulfilling

$$t_\ell^0 = \sum_j v_{j,\ell} \otimes w_{j,\ell},$$

where j runs through a finite sum independently of ℓ. Analogously, $\chi_\ell(\det) \cdot t_\ell^0$ is a "good tensor" for $(\pi_\ell \otimes \chi_\ell, \sigma_\ell)$. Like in the proof of the Global Birch Lemma (cf. [KMS]) it follows

$$L(\pi_\ell \otimes \chi_\ell, \sigma_\ell; s) = \Psi(\chi_\ell(\det) \cdot t_\ell^0; s) \text{ for } \ell \neq p, \infty.$$

At the place p we have $L(\pi_p \otimes \chi_p, \sigma_p; s) = 1$, since χ_p is ramified and π_p and σ_p are unramified. On the other hand, if we put

$$v_{j,p,\chi_p}(g) = \chi_p(\det(g)) \sum_u \prod_{i=1}^{n-1} \tilde{\chi}(u_i^i) v_{j,p}(g u^{\varphi^{-1}}),$$

where the summation is taken over a representative system for $U_n(\mathbb{Z}_p)$ modulo $U_n(\mathbb{Z}_p)^\varphi$, by Proposition 3.1 of [KMS] we get

$$\psi(v_{j,p,\chi_p}, w_{j,p}; s) = v_{j,p}(1) w_{j,p}(1) \prod_{i=1}^{n-1} \frac{G(\chi_p^i)(1-p^{-1})}{1-p^{-i}}.$$

Now that we know the respective values of the zeta integrals and the local Rankin-Selberg L-series at all places we use this information to find Whittaker functions such that $L(\pi \otimes \chi, \sigma; \kappa)$ occurs as a factor in the associated value of B. If we set

$$v_{j,f,\chi} = v_{p,\chi_p} \cdot \prod_{\ell \neq p,\infty} \chi_\ell(\det) v_{j,\ell}$$

we may define

$$v_{j,\chi} = \sum_{|I|=b_n} \sum_m v_{j,f,\chi} v_{\infty,I,m} \omega_I \otimes m \quad \text{and} \quad w_j = \sum_{|I'|=b_{n-1}} \sum_{m'} w_{j,f} w_{\infty,I',m'} \omega'_{I'} \otimes m'.$$

Compare with η and η' in Section 3.6. By Proposition 3.1 of [KMS] and (3.10) we get

$$B(v_{j,\chi}, w_j) = P_{\lambda,\infty}(\kappa) \cdot v_p(1) w_p(1) \prod_{i=1}^{n-1} \frac{G(\chi_p^i)(1-p^{-1})}{1-p^{-i}} \cdot \prod_{\ell \neq p,\infty} \psi(\chi_\ell(\det) v_{j,\ell}, w_{j,\ell}; \kappa).$$

All in all we have

$$\begin{aligned} \sum_j B(v_{j,\chi}, w_j) &= v_p(1) w_p(1) P_{\lambda,\infty}(\kappa) \prod_{i=1}^{n-1} \frac{G(\chi_p^i)(1-p^{-1})}{1-p^{-i}} \prod_{\ell \neq p,\infty} \Psi(\chi_\ell(\det) \cdot t_\ell^0; \kappa) \\ &= v_p(1) w_p(1) P_{\lambda,\infty}(\kappa) \prod_{i=1}^{n-1} \frac{G(\chi_p^i)(1-p^{-1})}{1-p^{-i}} \cdot L(\pi \otimes \chi, \sigma; \kappa). \end{aligned}$$

Like in the proof of the Global Birch Lemma we may express the values $B(v_{j,\chi}, w_j)$ as a sum of u-shifts, where u runs through a representative system of $U_n(\mathbb{Z}_p)$ modulo $U_n(\mathbb{Z}_p)^\varphi$.

Remark Because of our choice of Whittaker functions the image of $B_\lambda \otimes B_\infty \otimes B_\wedge$ restricted to the one-dimensional cohomology modules

$$H^{b_n}(\mathfrak{gl}_n, K_{n,\infty}; \mathscr{W}_0(\pi_\infty, \tau_\infty) \otimes M_{\mu,\mathbb{C}})_\varepsilon$$

and

$$H^{b_{n-1}}(\mathfrak{gl}_{n-1}, K_{n-1,\infty}; \mathscr{W}_0(\sigma_\infty, \bar{\tau}_\infty) \otimes M_{\nu,\mathbb{C}})_{\varepsilon'}$$

from Section 3 is generated by $P_{\lambda,\infty}(\kappa)$.

To prove, that $P_{\lambda,\infty}(\kappa)$ does not vanish, it would suffice to show that the restriction of $B_\lambda \otimes B_\infty \otimes B_\wedge$ to cohomology is not trivial. In the case $n = 3$ this will be done in the following sections for a suitable choice of λ.

4.3. The archimedean Rankin-Selberg pairing. In this section we want to study the archimedean Rankin-Selberg pairing we introduced in the last section.

Definition 4.1. *A pairing B of $(\mathfrak{gl}_n, \mathrm{O}_n(\mathbb{R}))$-modules R and S is called weakly equivariant, if B is $(\mathfrak{sl}_n, \mathrm{O}_n(\mathbb{R}))$-equivariant and for any $X \in \mathfrak{gl}_n$ there is a scalar $c(X) \in \mathbb{C}$ such that we have*

$$B(Xr, s) + B(r, Xs) = c(X) \cdot B(r, s) \quad \forall r \in R, s \in S.$$

We show the following

Proposition 4.2. *The archimedean Rankin-Selberg pairing B_∞ fulfils the following properties:*

(a) *B_∞ is weakly $(\mathfrak{gl}_{n-1}, \mathrm{O}_{n-1}(\mathbb{R}))$-equivariant,*
(b) *$B_{\lambda,\infty}(V, W) \not\equiv 0$ for any $\lambda \neq 0$.*

Proof. We first show that the zeta integral $\psi(v_\infty, w_\infty; \kappa)$ is $(\mathfrak{sl}_{n-1}, \mathrm{O}_{n-1}(\mathbb{R}))$-equivariant as a function on the Whittaker models. Therefor we have to show its $\mathrm{O}_{n-1}(\mathbb{R})$-equivariance[6], i. e.

$$\psi(\pi_\infty(h)v_\infty, \sigma_\infty(h)w_\infty; \kappa) = \psi(v_\infty, w_\infty; \kappa)$$

for all $h \in \mathrm{O}_{n-1}(\mathbb{R})$, and its $\mathfrak{sl}_{n-1}$-equivariance, i. e.

$$\psi(d\pi_\infty(X)v_\infty, w_\infty; \kappa) + \psi(v_\infty, d\sigma_\infty(X)w_\infty; \kappa) = 0$$

for all $X \in \mathfrak{sl}_{n-1}$. Here, $d\pi_\infty$ and $d\sigma_\infty$ are the *infinitesimal representations* belonging to the $\mathrm{GL}_{n-1}(\mathbb{R})$-representations π_∞ resp. σ_∞, that is for all $X \in \mathfrak{sl}_{n-1}$ we have

$$d\pi_\infty(X)(v_\infty) = \frac{d}{dt}\left(\pi_\infty(\exp(tX))v_\infty\right)|_{t=0}$$

and

$$d\sigma_\infty(X)(w_\infty) = \frac{d}{dt}\left(\sigma_\infty(\exp(tX))w_\infty\right)|_{t=0}.$$

We consider the Rankin-Selberg zeta integral

$$\psi(v_\infty, w_\infty; s) = \int_{U_{n-1}(\mathbb{R})\backslash \mathrm{GL}_{n-1}(\mathbb{R})} v_\infty\begin{pmatrix} g & \\ & 1\end{pmatrix} w_\infty(g)|\det(g)|^{s-\frac{1}{2}}dg$$

on $\mathscr{W}(\pi_\infty, \tau_\infty) \times \mathscr{W}(\sigma_\infty, \bar{\tau}_\infty)$. The group $\mathrm{GL}_{n-1}(\mathbb{R})$ acts on the tensor product of Whittaker spaces via right translation, so that we have

$$\psi(\pi_\infty(h)v_\infty, \sigma_\infty(h)w_\infty; s) = \int_{U_{n-1}(\mathbb{R})\backslash \mathrm{GL}_{n-1}(\mathbb{R})} v_\infty\begin{pmatrix} gh & \\ & 1\end{pmatrix} w_\infty(gh)|\det(g)|^{s-\frac{1}{2}}dg$$

for every $h \in \mathrm{GL}_{n-1}(\mathbb{R})$. Now we change the integration variable from g to gh^{-1}. Because of the transitivity of the action on the quotient $U_{n-1}(\mathbb{R})\backslash \mathrm{GL}_{n-1}(\mathbb{R})$ we get

$$\psi(\pi_\infty(h)v_\infty, \sigma_\infty(h)w_\infty; s) = |\det(h)|^{\frac{1}{2}-s} \cdot \psi(v_\infty, w_\infty; s).$$

Evidently $\psi(v_\infty, w_\infty; s)$ is $\mathrm{SL}^{\pm}_{n-1}(\mathbb{R})$-invariant, so that the $\mathrm{O}_{n-1}(\mathbb{R})$-invariance of ψ on the product $\mathscr{W}_0(\pi_\infty, \tau_\infty) \times \mathscr{W}_0(\sigma_\infty, \bar{\tau}_\infty)$ of the $\mathrm{O}_{n-1}(\mathbb{R})$-finite Whittaker spaces follows.

[6]Like in Section 3, we view π_∞ as a $\mathrm{GL}_{n-1}(\mathbb{R})$-module via j.

For the whole action of $\mathrm{GL}_{n-1}(\mathbb{R})$ we therefore have

$$\psi(\pi_\infty(h)v_\infty, \sigma_\infty(h)w_\infty; \kappa) = |\det(h)|^{\frac{1}{2}-\kappa} \cdot \psi(v_\infty, w_\infty; \kappa) \quad \forall h \in \mathrm{GL}_{n-1}(\mathbb{R}). \tag{4.1}$$

We use this fact to prove the weak equivariance of $\psi(v_\infty, w_\infty; \kappa)$. We have

$$\begin{aligned}
\psi(d\pi_\infty(X)v_\infty, w_\infty; \kappa) &= \int \frac{d}{dt}\, v_\infty\begin{pmatrix} g\cdot\exp(tX) & \\ & 1\end{pmatrix}\Big|_{t=0} w_\infty(g)\,|\det(g)|^{\frac{1}{2}-\kappa}dg \\
&= \frac{d}{dt}\int v_\infty\begin{pmatrix} g\cdot\exp(tX) & \\ & 1\end{pmatrix} w_\infty(g)\,|\det(g)|^{\frac{1}{2}-\kappa}dg\,\Big|_{t=0} \\
&\overset{(4.1)}{=} \frac{d}{dt}\int v_\infty\begin{pmatrix} g & \\ & 1\end{pmatrix} w_\infty(g\cdot\exp(-tX))\,|\det(g\cdot\exp(-tX))|^{\frac{1}{2}-\kappa}dg\,\Big|_{t=0} \\
&= \frac{d}{d(-t)}\int v_\infty\begin{pmatrix} g & \\ & 1\end{pmatrix} w_\infty(g\cdot\exp(tX))\,|\det(g)|^{\frac{1}{2}-\kappa}dg\,\Big|_{t=0} \\
&\quad + \frac{d}{dt}|\det(\exp(-tX))|^{\frac{1}{2}-\kappa}\Big|_{t=0} \cdot \int v_\infty\begin{pmatrix} g & \\ & 1\end{pmatrix} w_\infty(g)\,|\det(g)|^{\frac{1}{2}-\kappa}dg \\
&= -\psi(v_\infty, d\sigma_\infty(X)w_\infty; \kappa) - (\frac{1}{2}-\kappa)\,\mathrm{tr}(X)\cdot\psi(v_\infty, w_\infty, \kappa),
\end{aligned}$$

where integration is always over $U_{n-1}(\mathbb{R})\backslash \mathrm{GL}_{n-1}(\mathbb{R})$. All in all we find that $\psi(v_\infty, w_\infty; \kappa)$ is weakly $(\mathfrak{gl}_{n-1}, \mathrm{O}_{n-1}(\mathbb{R}))$-equivariant. Since we have

$$\psi(v_\infty, w_\infty; \kappa) = P_{v_\infty, w_\infty}(\kappa) \cdot L(\pi_\infty, \sigma_\infty; \kappa),$$

and $L(\pi_\infty, \sigma_\infty; \kappa)$ does not depend on the choice of our Whittaker functions, the same is true for $P_{v_\infty, w_\infty}(\kappa)$.

To show (b) we have to find elements of V and W for which $B_{\lambda,\infty}$ does not vanish. Recall that we fixed bases of $M_{\mu,\mathbb{C}}$ and $M_{\nu,\mathbb{C}}$ in Section 3. Let $m \in M_{\mu,\mathbb{C}}$ and $m' \in M_{\nu,\mathbb{C}}$ be such basis vectors fulfilling $\lambda(m \otimes m') \neq 0$. By Theorem 1.2 of [CP2] we are able to choose Whittaker functions $v_{m,\infty}$ in $\mathscr{W}_0(\pi_\infty, \tau_\infty)$ and $w_{m',\infty}$ in $\mathscr{W}_0(\sigma_\infty, \bar{\tau}_\infty)$ such that $P_{v_{m,\infty}, w_{m',\infty}}(\kappa)$ does not vanish. Hence, $v_{m,\infty} \otimes m$ and $w_{m',\infty} \otimes m'$ are suitable choices of elements of V and W such that $B_{\lambda,\infty}(V, W) \not\equiv 0$, which proves the proposition. □

4.4. Reduction to minimal K-types. We return to the case $n = 3$ now. Our aim in this section is to show

Theorem 4.3. *Let $\mathscr{D}_a$ be the minimal $\mathrm{SO}_3(\mathbb{R})$-type of $\mathscr{W}_0(\pi_\infty, \tau_\infty)$ with $3 \leq a$, and let $\mathscr{D}'_{\pm b}$ be the minimal $\mathrm{SO}_2(\mathbb{R})$-types of $\mathscr{W}_0(\sigma_\infty, \bar{\tau}_\infty)$ with $2 \leq b$. Assuming $b < a$ the pairing $B_{\lambda,\infty}$ remains non-trivial for a suitable choice of the linear form λ when restricted to $V_{\mathscr{D}_3}$ and $W_{\mathscr{D}'_{\pm 2}}$, i. e.*

$$B_{\lambda,\infty}(V_{\mathscr{D}_3}, W_{\mathscr{D}'_{\pm 2}}) = \mathbb{C}.$$

In a first step let

$$\langle\cdot,\cdot\rangle : \mathscr{W}_0(\pi_\infty, \tau_\infty) \times \mathscr{W}_0(\sigma_\infty, \bar{\tau}_\infty) \to \mathbb{C}$$

be an arbitrary non-trivial weakly $(\mathfrak{gl}_2, \mathrm{O}_2(\mathbb{R}))$-equivariant pairing of $\mathbb{C}$-vector spaces. We want to show the following

Theorem 4.4. *The pairing $\langle\cdot,\cdot\rangle$ is not trivial, if restricted to minimal K-types, that is*

$$\langle \mathscr{W}_0(\pi_\infty, \tau_\infty)_{\mathscr{D}_a}, \mathscr{W}_0(\sigma_\infty, \bar{\tau}_\infty)_{\mathscr{D}'_{\pm b}} \rangle \not\equiv 0.$$

Proof. Since $\mathscr{W}_0(\sigma_\infty, \bar\tau_\infty)$ is irreducible as $\mathfrak{gl}_2$-module, it is generated by a single element $w_{\pm b}$ of, say, weight $\pm be_1$. Because of the weak equivariance of $\langle\cdot,\cdot\rangle$ we get

$$\begin{aligned}\langle \mathscr{W}_0(\pi_\infty,\tau_\infty), \mathscr{W}_0(\sigma_\infty,\bar\tau_\infty)\rangle &= \langle \mathscr{W}_0(\pi_\infty,\tau_\infty), \mathscr{W}_0(\sigma_\infty,\bar\tau_\infty)_{\mathscr{D}'_{\pm b}}\rangle \\ &= \langle \mathscr{W}_0(\pi_\infty,\tau_\infty), w_{\pm b}\rangle\end{aligned}$$

But then we assumed $\langle \mathscr{W}_0(\pi_\infty,\tau_\infty), \mathscr{W}_0(\sigma_\infty,\bar\tau_\infty)\rangle \neq 0$, so that the Theorem follows from Proposition 4.5 below, which we could prove the same for $\mathscr{D}_b$ instead of $\mathscr{D}_{-b}$. □

Proposition 4.5. $\langle \mathscr{W}_0(\pi_\infty,\tau_\infty), \mathscr{W}_0(\sigma_\infty,\bar\tau_\infty)_{\mathscr{D}'_{-b}}\rangle = \langle \mathscr{W}_0(\pi_\infty,\tau_\infty)_{\mathscr{D}_a}, \mathscr{W}_0(\sigma_\infty,\bar\tau_\infty)_{\mathscr{D}'_{-b}}\rangle$.

Proof. Let $\mathfrak{U}(\mathfrak{sl}_3)$ be the universal envelopping algebra of $\mathfrak{sl}_3$, and let $p : \mathfrak{T}(\mathfrak{sl}_3) \to \mathfrak{U}(\mathfrak{sl}_3)$ be the belonging projection, where $\mathfrak{T}(\mathfrak{sl}_3)$ is the tensor algebra. Since each element of $\mathfrak{T}(\mathfrak{sl}_3)$ can be found as a representative of an element of $p(\bigoplus_{r=0}^{\infty}\bigotimes^r \tilde\wp_3)$, it holds

$$p\Big(\bigoplus_{r=0}^{\infty}\bigotimes^{r} \tilde\wp_3\Big) = \mathfrak{U}(\mathfrak{sl}_3). \tag{4.2}$$

This can be shown by proving that the left side contains a basis of $\mathfrak{sl}_3$. But then, $\tilde\wp_3$ is obviously contained, and we have

$$H = \frac{1}{48}i[Z_{-1},Z_1], E_1 = \frac{1}{48}[Z_0,Z_1], E_{-1} = \frac{1}{8}[Z_{-1},Z_0].$$

From now on we write $\tilde\wp_3^r$ for $\otimes^r\tilde\wp_3$. A direct implication of (4.2) is $\mathscr{W}_0(\pi_\infty,\tau_\infty) = \sum_{r\geq 0}\tilde\wp_3^r\,.\,\mathscr{W}_0(\pi_\infty,\tau_\infty)_{\mathscr{D}_a}$, where the dot denotes the action of $\mathfrak{U}(\mathfrak{sl}_3)$ on $\mathscr{W}_0(\pi_\infty,\tau_\infty)$. Thus the proof of Propostion 4.5 is reduced to showing that

$$\forall r\in\mathbb{N}_0 : \langle \tilde\wp_3^r\,.\,\mathscr{W}_0(\pi_\infty,\tau_\infty)_{\mathscr{D}_a}, \mathscr{W}_0(\sigma_\infty,\bar\tau_\infty)_{\mathscr{D}'_{-b}}\rangle \subseteq \langle \mathscr{W}_0(\pi_\infty,\tau_\infty)_{\mathscr{D}_a}, \mathscr{W}_0(\sigma_\infty,\bar\tau_\infty)_{\mathscr{D}'_{-b}}\rangle.$$

We will do this by induction on r. The case $r=0$ is trivial. The proof of the general step of the induction needs $r\geq 3$, so that we will show the cases $r=1$ and $r=2$ first.

Consider an arbitrary $v\in\mathscr{W}_0(\pi_\infty,\tau_\infty)_{\mathscr{D}_a}$. Since $\mathscr{W}_0(\pi_\infty,\tau_\infty)_{\mathscr{D}_a}$ is the direct sum of its weight spaces and because of the bilinearity of $\langle\cdot,\cdot\rangle$ we may assume v to have a weight $\mathsf{wt}(v)$ without loss of generality. But then we have

$$\mathsf{wt}(v)\langle v, w_{-b}\rangle = \langle H\,.\,v, w_{-b}\rangle = -\langle v, H\,.\,w_{-b}\rangle = b\langle v, w_{-b}\rangle,$$

so that $\langle v, w_{-b}\rangle = 0$ if $\mathsf{wt}(v)\neq be_1$. So it suffices to study the be_1 weight space of $\mathscr{W}_0(\pi_\infty,\tau_\infty)_{\mathscr{D}_a}$.

The case $r=1$. Remember $a\geq 3$. By the Clebsch-Gordan Formula for $\mathfrak{so}_3$ we have

$$\tilde\wp_3\otimes\mathscr{W}_0(\pi_\infty,\tau_\infty)_{\mathscr{D}_a} \cong \mathscr{D}_2\otimes\mathscr{D}_a \cong \bigoplus_{i=a-2}^{a+2}\mathscr{D}_i.$$

Recall $2\leq b\leq a$. So the be_1 weight space of $\tilde\wp_3\otimes\mathscr{W}_0(\pi_\infty,\tau_\infty)_{\mathscr{D}_a}$ is 3-dimensional for $b=a$, 4-dimensional for $b=a-1$, and 5-dimensional in all other cases. A set of generators is given by

$$\begin{aligned}&Z_0\otimes v_b^a,\\ &Z_2\otimes v_{b-2}^a,\\ &E_{-1}^{a-b-j}\,.\sum_{i=0}^{4}(-1)^i Z_{2-i}\otimes v_{a-2-j+i}^a \quad \in (\tilde\wp_3\otimes\mathscr{W}_0(\pi_\infty,\tau_\infty)_{\mathscr{D}_a})_{\mathscr{D}_{a-j}},\end{aligned}$$

where the allowed values in $\{0,1,2\}$ of j depend on $a-b$. $\mathscr{D}_a$ is the smallest $\mathrm{SO}_3(\mathbb{R})$-type in $\mathscr{W}_0(\pi_\infty,\tau_\infty)$, whence

$$E_{-1}^{a-b-j}\,.\sum_{i=0}^{4}(-1)^i Z_{2-i}\,.\,v_{a-2-j+i}^a = 0$$

for $j = 1, 2$. For $j = 0$ the term lies in the be_1 weight space of $(\tilde{\wp}_3 \otimes \mathscr{W}_0(\pi_\infty, \tau_\infty)_{\mathscr{D}_a})_{\mathscr{D}_a}$, the latter being isomorphic to $\mathscr{D}_a$, since the smallest K-type always occurs with multiplicity one, i. e.

$$E_{-1}^{a-b} \cdot \sum_{i=0}^{4} (-1)^i Z_{2-i} \,.\, v_{a-2+i}^a \in \mathbb{C} \cdot v_b^a.$$

Summing up, this means

$$(\tilde{\wp}_3 \otimes \mathscr{W}_0(\pi_\infty, \tau_\infty)_{\mathscr{D}_a})_{be_1} \subseteq \langle Z_0 \otimes v_b^a, Z_2 \otimes v_{b-2}^a \rangle_{\mathbb{C}} + \mathbb{C} \cdot v_b^a. \tag{4.3}$$

Since $Z_0 = \left(\begin{smallmatrix} -12 & 0 \\ 0 & -12 \end{smallmatrix}\right)$ lies in the centre of $\mathfrak{gl}_2$, it follows that $Z_0 \,.\, \mathscr{W}_0(\sigma_\infty, \bar{\tau}_\infty)_{\mathscr{D}'_{-b}}$ is a $\mathfrak{so}_2$-module. Because the action of Z_0 does not change the weight, and because of the multiplicity one of the smallest $\mathrm{SO}_2(\mathbb{R})$-type, we get $Z_0 \,.\, \mathscr{W}_0(\sigma_\infty, \bar{\tau}_\infty)_{\mathscr{D}'_{-b}} \subseteq \mathscr{W}_0(\sigma_\infty, \bar{\tau}_\infty)_{\mathscr{D}'_{-b}}$. Since for all $v^a = \sum_{i=-a}^{a} \alpha_i v_i^a \in \mathscr{W}_0(\pi_\infty, \tau_\infty)_{\mathscr{D}_a}$ we have

$$\langle Z_0 \,.\, v^a, w_{-b} \rangle = \langle Z_0 \,.\, \sum_{i=-a}^{a} \alpha_i v_i^a, w_{-b} \rangle = \sum_{\substack{i=-a \\ i \neq b}}^{a} \alpha_i \langle Z_0 \,.\, v_i^a, w_{-b} \rangle + \alpha_b \langle Z_0 \,.\, v_b^a, w_{-b} \rangle,$$

by the weak equivariance of $\langle \cdot, \cdot \rangle$ this means

$$Z_0 \,.\, v^a \in \ker\langle \cdot, w_{-b} \rangle + \mathbb{C} \cdot v_b^a =: \mathfrak{a}. \tag{4.4}$$

By a similar argument we even get

$$Z_0 \,.\, \mathfrak{a} \subseteq \mathfrak{a}. \tag{4.5}$$

Furthermore, by the $(\mathfrak{sl}_2, \mathrm{O}_2(\mathbb{R}))$-equivariance of $\langle \cdot, \cdot \rangle$, and since $\mathscr{D}'_{-b}$ is a minimal $\mathrm{SO}_2(\mathbb{R})$-type of $\mathscr{W}_0(\sigma_\infty, \bar{\tau}_\infty)$, we get

$$\langle Z_2 \,.\, v^a, w_{-b} \rangle = -\langle v^a, Z_2 \,.\, w_{-b} \rangle = -\langle v^a, 0 \rangle = 0. \tag{4.6}$$

for all $v^a \in \mathscr{W}_0(\pi_\infty, \tau_\infty)$. Together with (4.3) this proves the case $r = 1$.

The case $r = 2$. A set of generators of the be_1 weight space of $\tilde{\wp}_3^2 \,.\, \mathscr{W}_0(\pi_\infty, \tau_\infty)_{\mathscr{D}_a}$ is given by

$$\{Z_i Z_j \,.\, v_k^a \mid -2 \leq i, j \leq 2,\ -a \leq k \leq a,\ i + j + k = b\}.$$

Like in the case $r = 1$ we will show that all those generators lie in $\mathfrak{a}$. By (4.5) and (4.6) and by induction we already know this in the case $i \in \{0, 2\}$. Now, since $[Z_i, Z_j]$ lies in $\mathfrak{so}_3$ for all $i, j \in \{-2, \ldots, 2\}$, since $i + j + k = b$, and since v_k^a lies in $\mathscr{W}_0(\pi_\infty, \tau_\infty)_{\mathscr{D}_a}$, interchanging Z_i and Z_j only produces a summand in $\mathbb{C} \cdot v_b^a$, so that it suffices to show

$$\{Z_{-2}^2 \,.\, v_{b+4}, Z_{-2}Z_{-1} \,.\, v_{b+3}, Z_{-2}Z_1 \,.\, v_{b+1}, Z_{-1}^2 \,.\, v_{b+2}, Z_{-1}Z_1 \,.\, v_b, Z_1^2 \,.\, v_{b-2}\} \subseteq \mathfrak{a}. \tag{4.7}$$

Note, that for big values of b some of those terms vanish.

By the Lemma of Schur we have $\kappa(X, Y) = 6 \cdot \mathrm{tr}(XY)$ for the Killing form κ on $\mathfrak{so}_3$. From this we can calculate the Casimir operator $C_3 \in \mathfrak{U}(\mathfrak{sl}_3)$ explicitly. It holds

$$C_3 = -\frac{1}{12}\left(2i \cdot H + H^2 + E_1 E_{-1} - \frac{1}{48} \cdot Z_0^2 - \frac{1}{24} \cdot Z_{-2} Z_2 + \frac{1}{24} \cdot Z_{-1} Z_1\right).$$

Since $\mathscr{W}_0(\pi_\infty)$ is irreducible as an $\mathfrak{sl}_3$-module, C_3 acts like a scalar. By the anteceding we get

$$Z_{-1} Z_1 \,.\, \mathscr{W}_0(\pi_\infty)_{\mathscr{D}_a} \subseteq \mathfrak{a} \tag{4.8}$$

and even

$$Z_{-1} Z_1 = u_1 + u_2 \text{ with } u_1 \,.\, \mathfrak{a} \subseteq \mathfrak{a} \text{ and } u_2 \in \mathfrak{U}(\mathfrak{so}_3). \tag{4.9}$$

Like in the case $r = 1$ we want to make use of the fact that $\mathscr{D}_a$ is the smallest $\mathrm{SO}_3(\mathbb{R})$-type of $\mathscr{W}_0(\pi_\infty)$, i. e. of $(\tilde{\wp}_3 \,.\, \mathscr{W}_0(\pi_\infty)_{\mathscr{D}_a})_{\mathscr{D}_{a-j}} = 0$ for $j \in \{1, \dots, a\}$. For all those j we get

$$\sum_{i=0}^{4} (-1)^i Z_{2-i} \,.\, v^a_{a-2-j+i} = 0, \tag{4.10}$$

$$\sum_{i=0}^{4} (-1)^i (c^a_{a-2-j+i} Z_{2-i} \,.\, v^a_{a-3-j+i} + d_{2-i} Z_{1-i} \,.\, v^a_{a-2-j+i}) = 0. \tag{4.11}$$

Here, (4.10) just says that the maximal vectors in the respective $\mathscr{D}_{a-j}$ are zero, and (4.11) is (4.10) multiplied by E_{-1}.

Now consider the special case $j = (a-b)+1$ of (4.10). We multiply this equation by Z_1 and ignore summands containing Z_0 (by (4.5)), Z_2 (by (4.6)), or $Z_{-1}Z_1$ (by (4.8)) as lying in $\mathfrak{a}$. We find

$$Z_1^2 \,.\, v^a_{b-2} - Z_{-2}Z_1 \,.\, v^a_{b+1} \in \mathfrak{a}.$$

If $b = a$, then we have $Z_{-2}Z_1 \,.\, v^a_{b+1} = 0$ and thus $Z_1^2 \,.\, v^a_{b-2} \in \mathfrak{a}$. If $b \leq a-1$, we also multiply (4.11) for $j = a-b$ by Z_1, and analogously find

$$(d_2 - c^a_{b-1}) Z_1^2 \,.\, v^a_{b-2} - (d_{-1} - c^a_{b+2}) Z_{-2}Z_1 \,.\, v^a_{b+1} \in \mathfrak{a}.$$

But these terms are linearly independent in $Z_{-2}Z_1 \,.\, v^a_{b+1}$ and $Z_1^2 \,.\, v^a_{b-2}$, because of

$$\det \begin{pmatrix} 1 & -1 \\ d_2 - c^a_{b-1} & -d_{-1} + c^a_{b+2} \end{pmatrix} = 6b \geq 12 > 0,$$

so that for all pairs (a, b) we have

$$\{Z_{-2}Z_1 \,.\, v^a_{b+1}, Z_1^2 \,.\, v^a_{b-2}\} \subseteq \mathfrak{a}. \tag{4.12}$$

If $b > a-2$, we are done, since then all other terms in (4.7) are trivial. If $b \leq a-2$, we consider (4.10) in the case of $j = (a-b)-1$, this time multiplying the equation by Z_{-1}. This yields to $Z_{-1}^2 \,.\, v^a_{b+2} \in \mathfrak{a}$ for $b = a-2$. If $b \leq a-3$, we use (4.11) for $j = (a-b)-2$ and compare like above. Analogously, for all pairs (a, b) we get

$$\{Z_{-1}^2 \,.\, v^a_{b+2}, Z_{-1}Z_{-2} \,.\, v^a_{b+3}\} \subseteq \mathfrak{a}. \tag{4.13}$$

If $b > a-4$, we are done. For $b \leq a-4$ we multiply (4.10) for $j = (a-b)-2$ by Z_{-2} and find

$$Z_{-2}^2 \,.\, v^a_{b+4} \in \mathfrak{a} \tag{4.14}$$

for all pairs (a, b). (4.12), (4.13), and (4.14) together show the case $r = 2$.

The case $r \geq 3$. We consider terms of the form $Z_{i_1} \dots Z_{i_r} \,.\, v^a_k \in \tilde{\wp}_3^r \,.\, \mathscr{W}_0(\pi_\infty)_{\mathscr{D}_a}$ with $-2 \leq i_1, \dots, i_r \leq 2$, $-a \leq k \leq a$, and $i_1 + \dots + i_r + k = b$. We want to show that all of those lie in $\mathfrak{a}$. Like in the case $r = 2$ we may permute the Z_i at will to answer this question, so by (4.5) and (4.6) we only have to consider terms like

$$Z_{-2}^\alpha Z_{-1}^\beta Z_1^\gamma \,.\, v_k$$

with $\alpha + \beta + \gamma = r$ and $-2\alpha - \beta + \alpha + k = b$. Remember that this is a proof via induction over r. By the induction hypothesis $\tilde{\wp}_3^{r-2} \,.\, \mathscr{W}_0(\pi_\infty)_{\mathscr{D}_a}$ lies in $\mathfrak{a}$, so that by (4.9) and induction it is even enough to study those terms with $\beta = 0$ or $\gamma = 0$. We distinguish three cases:

At first let $\beta = \gamma = 0$. Then we only have to consider terms of the kind $Z_{-2}^r \,.\, v^a_k$ with $-2r + k = b$. Because of $r \geq 3$ it holds $k = b + 2r \geq 8$. On the other hand, multiplying (4.10) by Z_{-2}^{r-1} we find

$$-Z_{-2}^{r-1} Z_1 \,.\, v^a_{a-j-1} - Z_{-2}^{r-1} Z_{-1} \,.\, v^a_{a-j+1} - Z_{-2}^r \,.\, v^a_{a-j+2} \in \mathfrak{a}$$

for all $j \in \{1, \dots, a\}$. So if we assume the assertion for $\beta \neq 0$ and for $\gamma \neq 0$, we get $Z_{-2}^r \,.\, v^a_k \in \mathfrak{a}$ for all $k \geq 2$, and we are done in this case.

Now assume $\beta \neq 0$ and $\gamma = 0$. Set $u = Z_{-2}^{\tilde{\alpha}} Z_{-1}^{\tilde{\beta}}$ with $\tilde{\alpha} + \tilde{\beta} = r - 1$ and $\tilde{\beta} \neq 0$. We want to study the terms $uZ_{i_r} . v_k^a$ with $i_r \in \{-1, -2\}$. This is enough, since the factors in uZ_{i_r} commute modulo $\mathfrak{a}$, and since $\beta \neq 0$. Obviously, we have $k \geq 2\tilde{\alpha} + \tilde{\beta} + b - i_r \geq 5$. From (4.10) and (4.11) we get

$$-uZ_1 . v_{a-j-1}^a - uZ_{-1} . v_{a-j+1}^a + uZ_{-2} . v_{a-j+2}^a \in \mathfrak{a} \tag{4.15}$$

for all $j \in \{1, \dots, a\}$, and

$$-(4 + c_{a-j}^a) uZ_1 . v_{a-j-1}^a - (6 + c_{a-j+2}^a) uZ_{-1} . v_{a-j+1}^a + (4 + c_{a-j+3}^a) uZ_{-2} . v_{a-j+2}^a \in \mathfrak{a} \tag{4.16}$$

for all $j \in \{2, \dots, a+1\}$. Note, that by (4.9) and the induction hypothesis the respective first summands lie in $\mathfrak{a}$. So for $j = 1$ we directly get $uZ_{-1} . v_a^a \in \mathfrak{a}$. Since for the $j \in \{2, \dots, a\}$ we have

$$\det \begin{pmatrix} 1 & -1 \\ 6 + c_{a-j+2}^a & -4 - c_{a-j+3}^a \end{pmatrix} = (-2)(a - j + 1) \neq 0,$$

it follows

$$uZ_{-1} . v_k^a \in \mathfrak{a} \text{ for } 1 \leq k \leq a \quad \text{and} \quad uZ_{-2} . v_k^a \in \mathfrak{a} \text{ for } 2 \leq k \leq a,$$

which shows the assertion in this case by the anteceding.

The last case is $\beta = 0$ and $\gamma \neq 0$. Set $u = Z_{-2}^{\tilde{\alpha}} Z_1^{\tilde{\gamma}}$ with $\tilde{\alpha} + \tilde{\gamma} = r - 1$ and $\tilde{\gamma} \neq 0$. We want to study the terms $uZ_{i_r} . v_k^a$ with $i_r \in \{1, -2\}$. This is enough, since the factors in uZ_{i_r} commute modulo $\mathfrak{a}$, and since $\gamma \neq 0$. Keeping in mind that the factor u is different in this case, we can use (4.15) and (4.16) again, this time neglecting the respective second summands. Analogously this yields to

$$uZ_1 . v_k^a \in \mathfrak{a} \text{ for } -1 \leq k \leq a - 3 \quad \text{and} \quad uZ_{-2} . v_k^a \in \mathfrak{a} \text{ for } 2 \leq k \leq a.$$

Consider $uZ_1 . v_k^a$ for $k \geq a - 2$. Because of $b = -2\tilde{\alpha} + \tilde{\gamma} + 1 + k \geq a - 1 + \tilde{\gamma} - 2\tilde{\alpha}$ and $r \geq 3$ the exponent $\tilde{\alpha}$ has to be at least one. So modulo $\mathfrak{a}$ we may write $uZ_1 . v_k^a \equiv Z_{-2}^{\tilde{\alpha}-1} Z_1^{\tilde{\gamma}+1} Z_{-2} . v_k^a$, which lies in $\mathfrak{a}$ by the discussion above. We can apply the same trick for small values of k as well. Recall $\tilde{\gamma} > 0$. Thus $uZ_{-2} . v_k^a$ for $k \in \{-1, 0, 1\}$ lies in $\mathfrak{a}$, since we already know that $Z_{-2}^{\tilde{\alpha}+1} Z_1^{\tilde{\gamma}-1} Z_1 . v_k^a$ does. So up to now we have shown

$$uZ_1 . v_k^a \in \mathfrak{a} \text{ for } k \geq -1 \quad \text{and} \quad uZ_{-2} . v_k^a \in \mathfrak{a} \text{ for } k \geq -1. \tag{4.17}$$

We want to prove that the same is true for $-a \leq k < -1$. We do this inductively: Let $k_0 \in \{-1, \dots, 1 - a\}$. Assuming the assertion for $k \geq k_0$ we want to show that it is also true for $k_0 - 1$. We multiply (4.10) for $j = a + k_0$ with $uE_{-1}^{-2k_0}$ and get

$$u . \left(\sum_{i=0}^{4} (-1)^i \sum_{s=0}^{-2k_0} \binom{-2k_0}{s} \left(\prod_{t_1=1}^{s} d_{3-t_1-i} \prod_{t_2=1}^{-2k_0-s} c_{k_0-1+i-t_2}^a \right) Z_{2-s-i} . v_{k_0-2+s+i} \right) = 0. \tag{4.18}$$

Here the prefactor P_{k_0-1} of $uZ_1 . v_{k_0-1}$ is

$$\begin{aligned} P_{k_0-1} &= (-1)^0 \binom{-2k_0}{1} d_2 \prod_{t_2=1}^{-2k_0-1} c_{k_0-1-t_2}^a + (-1)^1 \binom{-2k_0}{0} \prod_{t_2=1}^{-2k_0} c_{k_0-2-t_2}^a \\ &= (8k_0 c_{-k_0-2}^a - c_{k_0-1}^a c_{k_0-2}^a) \cdot \prod_{t_2=2}^{-2k_0-1} c_{-k_0-1-t_2}^a \\ &= (-(c_{k_0-2}^a)^2 + (6k_0 + 4) c_{k_0-2}^a - 80k_0^2) \cdot \prod_{t_2=2}^{-2k_0-1} c_{-k_0-1-t_2}^a. \end{aligned}$$

P_{k_0-1} cannot be zero. Note, that the product in the last line is unequal to zero by the choice of k_0, and the term in parentheses could only vanish, if $c_{k_0-2}^a = 3k_0 + 2 \pm \sqrt{-71k_0^2 + 12k_0 + 4}$. But this cannot happen, since for $k_0 \leq -1$ the discriminant $-71k_0^2 + 12k_0 + 4$ is negative.

Now consider (4.18) modulo $\mathfrak{a}$. It reads

$$0 \equiv \sum_{i=0}^{4} P_{k_0-(2-i)} u Z_{2-i} \,.\, v^a_{k_0-(2-i)} \equiv P_{k_0-1} u Z_1 \,.\, v_{k_0-1} \bmod \mathfrak{a}$$

with the respective prefactors $P_{k_0-(2-i)}$. Note, that we may ignore the summands for $i = 2$ and $i = 0$ (by (4.5) and (4.6)), and for $i = 3$ (by (4.9), since $\tilde{\gamma} \neq 0$, and by induction over r). Finally, by induction over k_0 we may ignore the summand for $i = 4$.

Altogether we showed $uZ_1 \,.\, v_{k_0-1} \in \mathfrak{a}$. Like in the paragraph before (4.17) it follows that $uZ_{-2} \,.\, v^a_{k_0-1}$ lies in $\mathfrak{a}$ as well, which shows the step of our induction over k_0 and thereby the proposition. □

We now turn to the proof of Theorem 4.3. Since the minimal $\mathrm{SO}_3(\mathbb{R})$-type $\mathscr{D}_a$ has multiplicity 1 in π_∞, we find the $\mathscr{D}_3$-isotypical component $V_{\mathscr{D}_3}$ of V in

$$\mathscr{W}_0(\pi_\infty, \tau_\infty)_{\mathscr{D}_a} \otimes (M_\mu)_{\mathscr{D}_{a-3}} \cong \mathscr{D}_3 \oplus \mathscr{D}_4 \oplus \cdots \oplus \mathscr{D}_{2a-3}$$

by the Clebsch-Gordon formula. By the same argument for σ_∞ we find $W_{\mathscr{D}'_{-2}}$ in

$$\mathscr{W}_0(\sigma_\infty, \bar{\tau}_\infty)_{\mathscr{D}'_{-b}} \otimes (M_\nu)_{\mathscr{D}'_{b-2}} \cong \mathscr{D}'_{-2}$$

and $W_{\mathscr{D}'_2}$ in

$$\mathscr{W}_0(\sigma_\infty, \bar{\tau}_\infty)_{\mathscr{D}'_{b}} \otimes (M_\nu)_{\mathscr{D}'_{2-b}} \cong \mathscr{D}'_{2}.$$

We will now adjust our choice of the linear form $\lambda : M_\mu \otimes M_\nu \to \mathbb{C}$ to this situation.

Lemma 4.6. *There is a non-trivial $\mathfrak{so}_2$-invariant $\mathbb{Q}(i)$-rational linear form λ such that*

$$B_\lambda((M_\mu)_{\mathscr{D}_k}, (M_\nu)_{\mathscr{D}'_l}) \neq 0$$

if and only if $(k, l) \in \{(a-3, b-2), (a-3, 2-b)\}$.

Proof. Recall that a $\mathbb{Q}$-rational finite-dimensional representation ϱ of $\mathrm{SL}_n(\mathbb{R})$ always induces a $\mathbb{Q}$-rational representation of the Lie algebra $\mathfrak{sl}_n$ by restriction of the associated infinitesimal representation $d\varrho$ to $\mathfrak{sl}_n(\mathbb{Q})$. We may apply this to ϱ_μ and ϱ_ν for $n = 3, 2$ and furthermore restrict to $\mathrm{SO}_n(\mathbb{R})$. So in particular our $H \in \mathfrak{so}_2(\mathbb{Q})$ acts on $M_{\mu,\mathbb{Q}}$ and $M_{\nu,\mathbb{Q}}$. Once we want to pass to weight spaces we are obviously forced to enlarge the field of scalars to include $i = \sqrt{-1}$. Since we may choose $iH, E_1, E_{-1} \in \mathfrak{so}_3(\mathbb{Q}(i))$ as a Chevalley basis, M_μ decomposes as a direct sum of irreducible $\mathfrak{so}_3(\mathbb{C})$-modules

$$M_\mu \cong \mathscr{D}_{a-3} + \sum_{k<a-3} m(k)\mathscr{D}_k,$$

where in particular

$$M_{\mu,\mathbb{Q}(i)} \cap (M_\mu)_{\mathscr{D}_{a-3}} =: \mathscr{D}_{a-3,\mathbb{Q}(i)}$$

is an irreducible $\mathfrak{so}_3(\mathbb{Q}(i))$-module spanning $\mathscr{D}_{a-3}$ over $\mathbb{C}$. In a similar way $M_{\nu,\mathbb{Q}(i)}$ decomposes into weight spaces

$$M_{\nu,\mathbb{Q}(i)} = \mathscr{D}'_{2-b,\mathbb{Q}(i)} \oplus \cdots \oplus \mathscr{D}'_{b-2,\mathbb{Q}(i)},$$

which eventually allows us to define $\lambda : M_{\mu,\mathbb{Q}(i)} \otimes M_{\nu,\mathbb{Q}(i)} \to \mathbb{Q}(i)$ by setting

$$\lambda(m_{b-2} \otimes m'_{2-b}) = \lambda(m_{2-b} \otimes m'_{b-2}) = 1$$

for respective generators of the weight spaces $(\mathscr{D}_{a-3,\mathbb{Q}(i)})_{\pm(b-2)}$ and $\mathscr{D}'_{\pm(b-2),\mathbb{Q}(i)}$, and setting $\lambda = 0$ on the remaining part. □

Proof of Theorem 4.3. We now show that $B_{\lambda,\infty}(V_{\mathscr{D}_3}, W_{\mathscr{D}'_{-2}})$ is non-zero for B_λ like in Lemma 4.6. By Theorem 4.4 and by construction of λ the pairing $B_{\lambda,\infty}$ is non-trivial, when restricted to

$$\left(\mathscr{W}_0(\pi_\infty, \tau_\infty)_{\mathscr{D}_a} \otimes (M_\mu)_{\mathscr{D}_{a-3}}\right) \times \left(\mathscr{W}_0(\sigma_\infty, \bar{\tau}_\infty)_{\mathscr{D}'_{-b}} \otimes (M_\nu)_{\mathscr{D}'_{b-2}}\right).$$

In terms of the canonical bases $v^a_{-a},\dots,v^a_a$ of $\mathscr{W}_0(\pi_\infty,\tau_\infty)_{\mathscr{D}_a}$ and $m_{3-a},\dots,m_{a-3}$ of $(M_\mu)_{\mathscr{D}_{a-3}}$ as in Section 4.1 a highest weight vector of $V_{\mathscr{D}_3}$ is given by

$$v_3 := \sum_{k=-a+6}^{a} (-1)^k v^a_k \otimes m_{3-k}.$$

Thus the $2e_1$ weight space in $V_{\mathscr{D}_3} = \langle v_{-3},\dots,v_3\rangle_{\mathbb{C}}$ is generated by

$$E_{-1} \,.\, v_3 = \sum_{k=-a+6}^{a} (-1)^k \left(c^a_k v^a_{k-1} \otimes m_{3-k} + c^{a-3}_{3-k} v^a_k \otimes m_{2-k} \right).$$

So with generators w_{-b} of $\mathscr{W}_0(\sigma_\infty,\bar{\tau}_\infty)_{\mathscr{D}'_{-b}}$ and m'_{b-2} of $(M_\nu)_{\mathscr{D}'_{b-2}}$ we get by the $\mathfrak{so}_2$-equivariance of B_∞ and B_λ

$$B_{\lambda,\infty}(E_{-1}\,.\,v_3, w_{-b}\otimes m'_{b-2}) = (-1)^b (c^{a-3}_{3-b} - c^a_{b+1}) \cdot B_\infty(v^a_b, w_{-b}) \cdot B_\lambda(m_{2-b}, m'_{b-2}).$$

By Theorem 4.4 and the choice of λ as described in Lemma 4.6 it only remains to show that $c^a_{b+1} \neq c^{a-3}_{3-b}$ for $2 \leq b \leq a-1$, which is easily verified.

In the same manner we also get the non-vanishing of $B_{\lambda,\infty}(V_{\mathscr{D}_3}, W_{\mathscr{D}'_2})$, so the proof of Theorem 4.3 is complete. □

4.5. Reduction to cohomology. We still want to show that for $n = 3$ the value $P_{\lambda,\infty}(\kappa)$ in Theorem A does not vanish. Up to now we showed that $B_{\lambda,\infty}$ does not vanish on the cohomological K-types. Recalling the remark in Section 4.2 we want to prove, that $B_\wedge \otimes B_{\lambda,\infty}$ restricted to cohomology is still nontrivial. Like in Section 3.6 we may write

$$\left(\bigwedge^2 \tilde{\wp}_3^* \otimes V_{\mathscr{D}_3} \right)_{\varepsilon}^{\mathrm{SO}_3(\mathbb{R})} \qquad \text{resp.} \qquad \left(\tilde{\wp}_2^* \otimes (W_{\mathscr{D}'_{-2}} \oplus W_{\mathscr{D}'_2}) \right)_{\varepsilon'}^{\mathrm{SO}_2(\mathbb{R})}$$

for the respective cohomology spaces. This suggests to do the proof in two steps. At first we show

Proposition 4.7. $(B_\wedge \otimes B_\infty)\left(\left(\bigwedge^2 \tilde{\wp}_3^* \otimes V_{\mathscr{D}_3}\right)^{\mathfrak{so}_3}, \left(\tilde{\wp}_2^* \otimes (W_{\mathscr{D}'_{-2}} \oplus W_{\mathscr{D}'_2})\right)^{\mathfrak{so}_2} \right) \not\equiv 0.$

Proof. A basis of the $\mathfrak{so}_3$-module $\bigwedge^2 \tilde{\wp}_3$ is given by

$$\begin{array}{ll} 5Z_1 \wedge Z_2, & \\ 5Z_0 \wedge Z_2, & \\ 3Z_{-1}\wedge Z_2 + 2Z_0 \wedge Z_1, & Z_0 \wedge Z_1 - Z_{-1}\wedge Z_2, \\ 2Z_{-1}\wedge Z_1 + Z_{-2}\wedge Z_2, & Z_{-1}\wedge Z_1 - 2Z_{-2}\wedge Z_2, \\ Z_{-1}\wedge Z_0 + Z_{-2}\wedge Z_1, & 3Z_{-1}\wedge Z_0 - 2Z_{-2}\wedge Z_1, \\ Z_{-2}\wedge Z_0, & \\ Z_{-2}\wedge Z_{-1}, & \end{array} \tag{4.19}$$

whence $\bigwedge^2 \tilde{\wp}_3$ is isomorphic to $\mathscr{D}_1 \oplus \mathscr{D}_3$ as an $\mathfrak{so}_3$-module. The same is true for its dual $\bigwedge^2 \tilde{\wp}_3^*$, since $\tilde{\wp}_3 \cong \mathscr{D}_2$ is self-contragredient as an $\mathfrak{so}_3$-module. The cohomological $\mathrm{SO}_3(\mathbb{R})$-type $\mathscr{D}_3$ of V is isomorphic to $\mathscr{D}_3$. So by the Clebsch-Gordon formula we get

$$\begin{aligned} \bigwedge^2 \tilde{\wp}_3^* \otimes V_{\mathscr{D}_3} &\cong (\mathscr{D}_1 \oplus \mathscr{D}_3) \otimes \mathscr{D}_3 \\ &\cong (\mathscr{D}_2 \oplus \mathscr{D}_3 \oplus \mathscr{D}_4) \oplus (\mathscr{D}_0 \oplus \mathscr{D}_1 \oplus \mathscr{D}_2 \oplus \mathscr{D}_3 \oplus \mathscr{D}_4 \oplus \mathscr{D}_5 \oplus \mathscr{D}_6). \end{aligned}$$

The $\mathfrak{so}_3$-invariant vectors are just the $\mathscr{D}_0$-part by definition, so that it follows

$$\left(\bigwedge^2 \tilde{\wp}_3^* \otimes V_{\mathscr{D}_3} \right)^{\mathfrak{so}_3} = \left(\bigwedge^2 \tilde{\wp}_3^* \otimes V_{\mathscr{D}_3} \right)_{\mathscr{D}_0} \cong \mathscr{D}_0.$$

If we choose canonical basis vectors $v'_{-3}, v'_{-2}, v'_{-1}, v'_0, v'_1, v'_2, v'_3$ of $(\bigwedge^2 \tilde{\wp}_3^*)_{\mathscr{D}_3} \cong \mathscr{D}_3$ such that the weight of each v'_k is ke_1, a generator of the $\mathscr{D}_0$-component of $\bigwedge^2 \tilde{\wp}_3^* \otimes V_{\mathscr{D}_3}$ is given by $\sum_{k=-3}^{3}(-1)^k\, v'_{-k} \otimes v_k$.

Now consider the two cohomological $\mathrm{SO}_2(\mathbb{R})$-types $W_{\mathscr{D}'_{-2}} \cong \mathscr{D}'_{-2}$ and $W_{\mathscr{D}'_2} \cong \mathscr{D}'_2$. The $\mathfrak{so}_2$-types of $\tilde{\wp}_2$ are $\langle Z_2\rangle_{\mathbb{C}} \cong \mathscr{D}'_2$ and $\langle Z_{-2}\rangle_{\mathbb{C}} \cong \mathscr{D}'_{-2}$, so that we have $\tilde{\wp}_2 \cong \mathscr{D}'_{-2} \oplus \mathscr{D}'_2$. Since $\tilde{\wp}_2$ is self-contragredient with $H.Z^*_{-2} = 2iZ^*_{-2}$ and $H.Z^*_2 = -2iZ^*_2$, the same is true for $\tilde{\wp}_2^*$. We get

$$\tilde{\wp}_2^* \otimes W_{\mathscr{D}'_{-2}} \cong (\mathscr{D}'_{-2} \oplus \mathscr{D}'_2) \otimes \mathscr{D}'_{-2} \cong \mathscr{D}'_{-4} \oplus \mathscr{D}'_0$$

and

$$\tilde{\wp}_2^* \otimes W_{\mathscr{D}'_2} \cong (\mathscr{D}'_{-2} \oplus \mathscr{D}'_2) \otimes \mathscr{D}'_2 \cong \mathscr{D}'_0 \oplus \mathscr{D}'_4,$$

so that in both cases it follows

$$\left(\tilde{\wp}_2^* \otimes W_{\mathscr{D}'_{\pm 2}}\right)^{\mathfrak{so}_2} \cong \mathscr{D}'_0.$$

Now let w_{-2} and w_2 denote basis vectors of $W_{\mathscr{D}'_{-2}}$ resp. $W_{\mathscr{D}'_2}$, and choose a basis $\{w'_{-2}, w'_2\}$ of $\tilde{\wp}_2^*$ such that w'_{-2} has weight $-2e_1$ and w'_2 has weight $2e_1$. Then a basis of $(\tilde{\wp}_2^* \otimes W_{\mathscr{D}'_{-2}})^{\mathfrak{so}_2}$ resp. $(\tilde{\wp}_2^* \otimes W_{\mathscr{D}'_2})^{\mathfrak{so}_2}$ is given by $w'_2 \otimes w_{-2}$ resp. $w'_{-2} \otimes w_2$.

We still have to show that the restriction of $B_\wedge \otimes B_{\lambda,\infty}$ is not trivial. From Section 4.4 we know that $B_{\lambda,\infty}(v_k, w_{-2})$ vanishes for $k \neq 2$. But then by Theorem 4.4 we have $0 \not\equiv B_{\lambda,\infty}(V_{\mathscr{D}_3}, W_{\mathscr{D}'_2}) = B_{\lambda,\infty}(V_{\mathscr{D}_3}, w_{-2})$, so that $B_{\lambda,\infty}(v_2, w_{-2}) \neq 0$. Analogously, $B_{\lambda,\infty}(v_{-2}, w_2) \neq 0$ and $B_{\lambda,\infty}(v_a, w_2) = 0$ for $a \neq -2$. We showed for all $\alpha, \beta, \gamma \in \mathbb{C}$

$$\begin{aligned}(4.20)\qquad & (B_\wedge \otimes B_{\lambda,\infty})(\alpha \cdot \sum_{k=-3}^{3}(-1)^k\, v'_{-k} \otimes v_k, \beta \cdot w'_{-2} \otimes w_2 + \gamma \cdot w'_2 \otimes w_{-2}) \\ = \; & \alpha\gamma \cdot B_\wedge(v'_{-2}, w'_2) B_{\lambda,\infty}(v_2, w_{-2}) + \alpha\beta \cdot B_\wedge(v'_2, w'_{-2}) B_{\lambda,\infty}(v_{-2}, w_2),\end{aligned}$$

where the values of B_∞ do not vanish. So we reduced the proof to showing

$$B_\wedge(v'_2, w'_{-2}) \neq 0 \quad \text{and} \quad B_\wedge(v'_{-2}, w'_2) \neq 0.$$

In order to do this we choose bases of $\bigwedge^2 \tilde{\wp}_3^*$ and $\tilde{\wp}_2^*$ consisting of Maurer-Cartan forms like in Section 3.1. We set

$$\omega_1 := Z^*_{-2},\ \omega_2 := Z^*_2,\ \omega_3 := Z^*_0,\ \omega_4 := Z^*_{-1},\ \omega_5 := Z^*_1.$$

Recalling the embedding of $\wp_2$ into $\tilde{\wp}_3$ from Section 4.1 we also put

$$\omega'_1 := Z^*_{-2},\ \omega'_2 := Z^*_2,\ \omega'_3 := Z^*_0,$$

where we define Z^*_j by $Z^*_j(Z_i) = \delta_{ij}$ with $i, j \in \{-2, 0, 2\}$ in analogy to the above. It follows

$$\delta(p_2 \circ j)(\omega_i) = \begin{cases} \omega'_i & \text{if } i = 1,2,3, \\ 0 & \text{if } i = 4,5 \end{cases}$$

just like in Section 3.1.

Via (4.19) we can express v'_{-2}, v'_2 in terms of those Maurer-Cartan forms, and get

$$v'_{-2} = 5\, \omega_3 \wedge \omega_2 \text{ and } v'_2 = \omega_1 \wedge \omega_3.$$

Further we may set

$$w'_{-2} := \omega'_2 \text{ and } w'_2 := \omega'_1.$$

So, following the definition of $\varepsilon_{I,I'}$ in Section 3 and taking into account that ω'_3 corresponds to the differential $\frac{d}{dt}$ there, we find that $B_\wedge(v'_2, w'_{-2})$ and $B_\wedge(v'_{-2}, w'_2)$ do not vanish. $\square$

It remains to study the action of the groups of connected components of the respective orthogonal groups. The case of $\pi_0(\mathrm{O}_3)$ is already described by Corollary 1.5: Since $\varepsilon = \operatorname{sgn}(\omega_\pi(-1)(-1)^{\mathsf{wt}(\mu)/2}) = +$ we get

$$\left(\bigwedge^2 \tilde{\wp}_3^* \otimes V_{\mathscr{D}_3}\right)_+^{\mathrm{SO}_3(\mathbb{R})} = \left(\bigwedge^2 \tilde{\wp}_3^* \otimes V_{\mathscr{D}_3}\right)^{\mathfrak{so}_3}.$$

The case of $\pi_0(\mathrm{O}_2)$ is more interesting. If we set $\delta_2 = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ we may write

$$\mathrm{O}_2(\mathbb{R}) = \langle\delta_2\rangle \ltimes \mathrm{SO}_2(\mathbb{R}) \quad \text{resp.} \quad \mathrm{O}_2(\mathbb{C}) = \langle\delta_2\rangle \ltimes \mathrm{SO}_2(\mathbb{C}).$$

δ_2 acts on the weights τ of an arbitrary representation of $\mathfrak{so}_2$ by

$$\tau^{\delta_2}(H) = \tau(\delta_2^{-1}H\delta_2) = \tau(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}) = \tau(-H) = -\tau(H).$$

Thus δ_2 interchanges the two weights $-2e_1$ and $2e_1$ both in $\tilde{\wp}_2^*$ and W, whence it also interchanges the two $\mathfrak{so}_2$-modules in $(\tilde{\wp}_2^* \otimes (W_{\mathscr{D}'_{-2}} \oplus W_{\mathscr{D}'_2}))^{\mathfrak{so}_2}$ that are isomorphic to $\mathscr{D}'_0$. Without loss of generality we may assume that the basis vectors $w'_2 \otimes w_{-2}$ and $w'_{-2} \otimes w_2$ merge under the action of δ_2. Then we get

$$\left(\tilde{\wp}_2^* \otimes (W_{\mathscr{D}'_{-2}} \oplus W_{\mathscr{D}'_2})\right)_+^{\mathrm{SO}_2(\mathbb{R})} \cong \langle w'_2 \otimes w_{-2} + w'_{-2} \otimes w_2\rangle_{\mathbb{C}}.$$

and

$$\left(\tilde{\wp}_2^* \otimes (W_{\mathscr{D}'_{-2}} \oplus W_{\mathscr{D}'_2})\right)_-^{\mathrm{SO}_2(\mathbb{R})} \cong \langle w'_2 \otimes w_{-2} - w'_{-2} \otimes w_2\rangle_{\mathbb{C}}.$$

But since the $(B_\wedge \otimes B_{\lambda,\infty})$-value in (4.20) can not be zero in the cases $\alpha = \beta = \gamma = 1$ and $\alpha = \beta = -\gamma = 1$ simultaneously by the above-mentioned, it follows that there is a $\varepsilon' \in \{+,-\}$, such that

$$(B_\wedge \otimes B_{\lambda,\infty})\left(\left(\bigwedge^2 \tilde{\wp}_3^* \otimes V_{\mathscr{D}_3}\right)_+^{\mathrm{SO}_3(\mathbb{R})}, \left(\tilde{\wp}_2^* \otimes (W_{\mathscr{D}'_{-2}} \oplus W_{\mathscr{D}'_2})\right)_{\varepsilon'}^{\mathrm{SO}_2(\mathbb{R})}\right) \not\equiv 0.$$

Recalling the remark in Section 4.2 we find

Theorem B *For $n = 3$ let π_∞ have minimal K-type of highest weight $a \geq 3$ and σ_∞ minimal K-types of weight $\pm b$ such that $2 \leq b \leq a-1$. Then there is a $\mathbb{Q}(i)$-rational linear form λ such that $P_{\lambda,\infty}(\kappa)$ does not vanish.*

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Institut für Algebra und Geometrie, Universität Karlsruhe, 76128 Karlsruhe, Germany

E-mail address: Hendrik.Kasten@math.uni-karlsruhe.de

E-mail address: Claus.Schmidt@math.uni-karlsruhe.de