Modern random measures: Palm theory and related models

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This paper aims at developing the theory of Palm measures of stationary random measures on locally compact second countable groups. The focus will be on recent developments concerning invariant transport-kernels and related invariance properties of Palm measures, shift-coupling, and mass-stationarity. Stationary partitions and (invariant) matchings will serve as extensive examples. Many recent results will be extended from the Abelian (or $\mathbb{R}^d$) case to general locally compact groups.

1 Motivation

Palm probabilities are a very important concept in theory and application of point processes and random measures, see e.g. Matthes, Kerstan and Mecke [22], Kallenberg [15], Stoyan, Kendall and Mecke [30], Daley and Vere-Jones [5], Thorisson [32], and Kallenberg [16]. The Palm distribution of a stationary random measure M on an locally compact group G is describing the statistical behaviour of M as seen from a *typical* point in the mass of M. Actually it is mathematically more fruitful to consider a general stationary probability measure on an abstract sample space equipped with a flow. This measure is then itself a Palm probability measure, obtained when M is given as the Haar measure on G. This general approach is also supported by many applications that require to consider a random measure together with other (jointly stationary) random measures and fields. In stochastic geometry, for instance, already the basic notions (e.g. typical cell, typical face, rose of directions) require the use of Palm probability measure in such a setting. And the *refined Campbell theorem* is a simple, yet powerful tool for handling them.

In his seminal paper [23] Mecke has introduced and studied Palm measures of stationary random measures on Abelian groups. Although Palm distributions were defined in Tortrat [34] in case of a group and in Rother and Zähle [28] in case of a homogeneous space, these more general cases have found little attention in the literature. A very general approach to invariance properties of Palm measures as well as historical comments and further references can be found in Kallenberg [17].

Recent years have seen some remarkable progress in understanding invariance properties of Palm probability measures and associated transport and coupling questions.

Liggett [21] found a way to remove a head from i.i.d. sequence of coin tosses so that the rest of the the coin tosses are again i.i.d.! This is an explicit *shift-coupling* of an i.i.d. sequence and its Palm version: they are the same up to a shift of the origin. Holroyd and Peres [13] and Hoffman, Holroyd and Peres [11] have used a *stable marriage algorithm* for transporting a given fraction of Lebesgue measure on $\mathbb{R}^d$ to an ergodic point process. Again this yields an explicit shift-coupling of a point process with its Palm version. In dimension $d \geq 3$, Chatterjee, Peled, Peres and Romik [3] found an (invariant) *gravitational allocation rule* transporting Lebesgue measure to a homogeneous Poisson process exhibiting surprisingly nice moment properties. Motivated by [13] and [11], Last [18] has studied general stationary partitions of $\mathbb{R}^d$. Last and Thorisson [20] have developed a general theory for invariant (weighted) *transport-kernels* on Abelian groups.

Another related line of research can be traced back to Mecke's [23] intrinsic characterization of Palm measures via an integral equation. Thorisson [32] called a simple point processes on $\mathbb{R}^d$ *point-stationary* if its distribution is invariant under *bijective point-shifts* against any independent stationary background. He proved that point-stationarity is a characterizing property of Palm versions of stationary point processes. Heveling and Last [9], [10] showed that the independent stationary background could be removed from the definition of point-stationarity. Bijective point shifts are closely related to invariant graphs and trees on point processes, see Holroyd and Peres [12], Ferrari, Landim and Thorisson [6] and Timar [33]. The extension of point-stationarity to general random measures is not straightforward. Last and Thorisson [20] call a random measure on an Abelian group *mass-stationary* if, informally speaking, the neutral element is a typical location in the mass. They could prove that mass-stationarity is indeed a characterizing property of Palm measures.

In this paper we will develop Palm theory for stationary random measures on locally compact second countable groups. In doing so we will extend many recent results from the Abelian (or $\mathbb{R}^d$) case to general groups. After having introduced stationary random measures and their Palm measures we will focus on invariant weighted transport-kernels balancing stationary random measures, invariance and transport properties of Palm measures and mass-stationarity. In the final two sections we will apply parts of the theory to *stationary partitions* and (invariant) *matchings* of point processes.

2 Stationary random measures

We consider a topologial (multiplicative) group G that is assumed to be a locally compact, second countable Hausdorff space with unit element e and Borel σ-field $\mathcal{G}$. A classical source on such groups is [25], see also chapter 2 of [16]. A measure η on G is *locally finite* if it is finite on compact sets. There exists a left-invariant Haar measure on G , i.e. a locally finite measure satisfying

$$\int f(hg)\lambda(dg) = \int f(g)\lambda(dg), \quad h \in G, \tag{2.1}$$

for all measurable $f : G \to \mathbb{R}_+$, where $\mathbb{R}_+ := [0, \infty)$. This measure is unique up to normalization. The *modular function* is a continuous homomorphism $\Delta : G \to (0, \infty)$

satisfying

$$\int f(gh)\lambda(dg) = \Delta(h^{-1}) \int f(g)\lambda(dg), \quad h \in G, \tag{2.2}$$

for all f as above. This modular function has the property

$$\int f(g^{-1})\lambda(dg) = \int \Delta(g^{-1}) f(g)\lambda(dg). \tag{2.3}$$

The group G is called *unimodular*, if $\Delta(g) = 1$ for all $g \in G$.

We denote by $\mathbf{M}$ the set of all locally finite measures on G, and by $\mathcal{M}$ the cylindrical σ-field on $\mathbf{M}$ which is generated by the evaluation functionals $\eta \mapsto \eta(B)$, $B \in \mathcal{G}$. The *support* $\operatorname{supp} \eta$ of a measure $\eta \in \mathbf{M}$ is the smallest closed set $F \subset G$ such that $\eta(G \setminus F) = 0$. By $\mathbf{N} \subset \mathbf{M}$ (resp. $\mathbf{N}_s \subset \mathbf{M}$) we denote the measurable set of all (resp. *simple*) *counting measures* on G, i.e. the set of all those $\eta \in \mathbf{M}$ with discrete support and $\eta\{g\} := \eta(\{g\}) \in \mathbb{N}_0$ (resp. $\eta\{g\} \in \{0, 1\}$) for all $g \in G$. We can and will identify $\mathbf{N}_s$ with the class of all *locally finite* subsets of G, where a set is called locally finite if its intersection with any compact set is finite.

We will mostly work on a σ-finite measure space $(\Omega, \mathcal{A}, \mathbb{P})$ (see Remark 2.8). Although $\mathbb{P}$ need not be a probability measure, we are still using a probabilistic language. Moreover, we would like to point out already at this early stage, that we will consider several measures on $(\Omega, \mathcal{A})$. A *random measure* on G is a measurable mapping $M : \Omega \to \mathbf{M}$. A random measure is a (simple) *point process* on G if $M(\omega) \in \mathbf{N}$ (resp. $M(\omega) \in \mathbf{N}_s$) for all $\omega \in \Omega$. A random measure M can also be regarded as a *kernel* from Ω to G. Accordingly we write $M(\omega, B)$ instead of $M(\omega)(B)$. If M is a random measure, then the mapping $(\omega, g) \mapsto \mathbf{1}\{g \in \operatorname{supp} M(\omega)\}$ is measurable.

We assume that $(\Omega, \mathcal{A})$ is equipped with a *measurable flow* $\theta_g : \Omega \to \Omega$, $g \in G$. This is a family of measurable mappings such that $(\omega, g) \mapsto \theta_g \omega$ is measurable, θ_e is the identity on Ω and

$$\theta_g \circ \theta_h = \theta_{gh}, \quad g, h \in G, \tag{2.4}$$

where $\circ$ denotes composition. This implies that θ_g is a bijection with inverse $\theta_g^{-1} = \theta_{g^{-1}}$. A random measure M on G is called *invariant* (or *flow-adapted*) if

$$M(\theta_g \omega, gB) = M(\omega, B), \quad \omega \in \Omega,\ g \in G,\ B \in \mathcal{G}, \tag{2.5}$$

where $gB := \{gh : h \in B\}$. This means that

$$\int f(h) M(\theta_g \omega, dh) = \int f(gh) M(\omega, dh) \tag{2.6}$$

for all measurable $f : G \to \mathbb{R}_+$. We will often skip the ω in such relations, i.e. we write (2.6) as $\int f(h)M(\theta_g, dh) = \int f(gh)M(\theta_e, dh)$ or $\int f(h)M(\theta_g, dh) = \int f(gh)M(dh)$. Recall that θ_e is the identity on Ω. Still another way of expressing (2.5) is

$$M \circ \theta_g = gM, \quad g \in G, \tag{2.7}$$

where for $\eta \in \mathbf{M}$ and $g \in G$ the measure $g\eta$ is defined by $g\eta(\cdot) := \int \mathbf{1}\{gh \in \cdot\}\eta(dh)$.

A measure $\mathbb{P}$ on $(\Omega, \mathcal{A})$ is called *stationary* if it is invariant under the flow, i.e.

$$\mathbb{P} \circ \theta_g = \mathbb{P}, \quad g \in G, \tag{2.8}$$

where θ_g is interpreted as a mapping from $\mathcal{A}$ to $\mathcal{A}$ in the usual way:

$$\theta_g A := \{\theta_g \omega : \omega \in A\}, \quad A \in \mathcal{A},\, g \in G.$$

Example 2.1. Consider the measurable space $(\mathbf{M}, \mathcal{M})$ and define for $\eta \in \mathbf{M}$ and $g \in G$ the measure $\theta_g \eta$ by $\theta_g \eta := g\eta$, i.e. $\theta_g \eta(B) := \eta(g^{-1}B)$, $B \in \mathcal{G}$. Then $\{\theta_g : g \in G\}$ is a measurable flow and the identity M on $\mathbf{M}$ is an invariant random measure. A stationary probability measure on $(\mathbf{M}, \mathcal{M})$ can be interpreted as the distribution of a *stationary random measure*.

Example 2.2. Let $(E, \mathcal{E})$ be a Polish space and assume that Ω is the space of all measures ω on $G \times E \times G \times E$ such that $\omega(B \times E \times G \times E)$ and $\omega(G \times E \times B \times E)$ are finite for compact $B \subset G$. The σ-field $\mathcal{A}$ is defined as the cylindrical σ-field on Ω. It is stated in [27] (and can be proved as in [22]) that $(\Omega, \mathcal{A})$ is a Polish space. For $g \in G$ and $\omega \in \Omega$ we let $\theta_g \omega$ denote the measure satisfying

$$\theta_g \omega(B \times C \times B' \times C') = \omega(g^{-1}B \times C \times g^{-1}B' \times C')$$

for all $B, B' \in \mathcal{G}$ and $C, C' \in \mathcal{E}$. The random measures M and N defined by $M(\omega, \cdot) := \omega(\cdot \times E \times G \times E)$ and $N(\omega, \cdot) := \omega(G \times E \times \cdot \times E)$ are invariant. Port and Stone [27] (see also [8]) call a stationary probability measure on $(\Omega, \mathcal{A})$ concentrated on the set of integer-valued $\omega \in \Omega$ a (translation invariant) *marked motion process*. The idea is that the (marked) points of M move to the points of N in one unit of time.

Example 2.3. Assume that G is an additive Abelian group and consider a flow $\{\tilde{\theta}_g : g \in G\}$ as in [20] (see also [26]). In our current setting this amounts to define $gh := h + g$ and $\theta_g := \tilde{\theta}_g^{-1}$. It is somewhat unfortunate that in the point process literature it is common to define the shift of a measure $\eta \in \mathbf{M}$ by $g \in G$ by $g^{-1}\eta$ and not (as it would be more natural) by $g\eta$. Here we follow the terminology of [17].

Remark 2.4. Our setting accomodates stationary marked point processes (see [5], [22], Remark 3) as well as stochastic processes (fields) jointly stationary with a random measure M (see [32]). The use of an abstract flow $\{\theta_g : g \in G\}$ acting directly on the underlying sample space is making the notation quite efficient. A more general framework is to work on an abstract measure space and to replace (2.8) by a distributional invariance, see [17].

We now fix a σ-finite stationary measure $\mathbb{P}$ on $(\Omega, \mathcal{A})$ and an invariant random measure M on G. Let $w : G \to \mathbb{R}_+$ be a measurable function having $\int w(g)\lambda(dg) = 1$. The measure

$$\mathbb{Q}_M(A) := \iint \mathbf{1}\{\theta_g^{-1}\omega \in A\} w(g) M(\omega, dg) \mathbb{P}(d\omega), \quad A \in \mathcal{A}, \tag{2.9}$$

is called the *Palm measure* of M (with respect to $\mathbb{P}$). A more succinct way of writing (2.9) is

$$\mathbb{Q}_M(A) = \mathbb{E} \int \mathbf{1}\{\theta_g^{-1} \in A\} w(g) M(dg), \quad A \in \mathcal{A}, \tag{2.10}$$

where $\mathbb{E}$ denotes integration with respect to $\mathbb{P}$. The following lemma shows that $\mathbb{Q}_M$ is concentrated on $\{e \in \operatorname{supp} M\}$. An interpretation of the Palm measure will be given in Remark 2.10.

Proposition 2.5. *We have* $\mathbb{Q}_M(e \notin \operatorname{supp} M) = 0$.

Proof. From the definition (2.10) and (2.7) we have

$$\begin{aligned}\mathbb{Q}_M(e \notin \operatorname{supp} M) &= \mathbb{E} \int \mathbf{1}\{e \notin \operatorname{supp} M \circ \theta_g^{-1}\} w(g) M(dg) \\ &= \mathbb{E} \int \mathbf{1}\{e \notin \operatorname{supp} g^{-1} M\} w(g) M(dg) \\ &= \mathbb{E} \int \mathbf{1}\{g \notin \operatorname{supp} M\} w(g) M(dg) = 0,\end{aligned}$$

proving the assertion. □

Our first result is the *refined Campbell theorem* connecting $\mathbb{P}$ and $\mathbb{Q}_M$. In the canonical case of Example 2.1 the result was derived in [34]. In particular, the result will show that the definition (2.10) is independent of the choice of the function w.

Theorem 2.6. *For any measurable* $f : \Omega \times G \to \mathbb{R}_+$,

$$\mathbb{E} \int f(\theta_g^{-1}, g) M(dg) = \mathbb{E}_{\mathbb{Q}_M} \int f(\theta_e, g) \lambda(dg). \tag{2.11}$$

Proof. From definition (2.10) and Fubini's theorem we obtain that

$$\mathbb{E}_{\mathbb{Q}_M} \int f(\theta_e, g) \lambda(dg) = \mathbb{E} \iint f(\theta_h^{-1}, g) w(h) \lambda(dg) M(dh),$$

Using Fubini's theorem again, as well as (2.3), we get

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}_M} \int f(\theta_e, g) \lambda(dg) &= \mathbb{E} \iint f(\theta_h^{-1}, g^{-1}) \Delta(g^{-1}) w(h) \lambda(dg) M(dh) \\ &= \mathbb{E} \iint f(\theta_h^{-1}, g^{-1}h) \Delta(g^{-1}h) w(h) \lambda(dg) M(dh),\end{aligned}$$

where we have used left-invariance of λ for the second equality. From Fubini's theorem we obtain

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}_M} \int f(\theta_e, g) \lambda(dg) &= \mathbb{E} \iint f(\theta_{gh}^{-1}, h) \Delta(h) w(gh) M \circ \theta_g^{-1}(dh) \lambda(dg) \\ &= \mathbb{E} \iint f(\theta_h^{-1} \circ \theta_g^{-1}, h) \Delta(h) w(gh) M \circ \theta_g^{-1}(dh) \lambda(dg) \\ &= \mathbb{E} \iint f(\theta_h^{-1}, h) \Delta(h) w(gh) M(dh) \lambda(dg),\end{aligned}$$

where we have used the flow property (2.4) and (2.7) for the second and stationarity of $\mathbb{P}$ for the final equation. The result now follows from

$$\Delta(h) \int w(gh) \lambda(dg) = \Delta(h) \Delta(h^{-1}) \int w(g) \lambda(dg) = 1.$$

□

Remark 2.7. Establish the setting of Example 2.3. Then (2.11) means that

$$\mathbb{E}\int f(\tilde{\theta}_g, g)M(dg) = \mathbb{E}_{\mathbb{Q}_M}\int f(\tilde{\theta}_e, g)\lambda(dg). \tag{2.12}$$

Equation (2.11) or rather its equivalent version

$$\mathbb{E}\int \mathbf{1}\{(\theta_e, g) \in \cdot\}M(dg) = \mathbb{E}_{\mathbb{Q}_M}\int \mathbf{1}\{(\theta_g, g) \in \cdot\}\lambda(dg),$$

is known as *skew factorization* of the *Campbell measure* of M. A general discussion of this technique can be found in [17].

Let $w' : G \to \mathbb{R}_+$ be another measurable function having $\int w'(g)\lambda(dg) = 1$ and take $A \in \mathcal{A}$. Then (2.11) implies that

$$\begin{aligned}\mathbb{E}\int \mathbf{1}\{\theta_g^{-1} \in A\}w'(g)M(dg) &= \mathbb{E}\int \mathbf{1}\{\theta_g^{-1} \in A\}w'(g)M(dg)\\ &= \mathbb{E}_{\mathbb{Q}_M}\int \mathbf{1}\{\theta_e \in A\}w'(g)\lambda(dg) = \mathbb{Q}_M(A).\end{aligned}$$

Hence the definition (2.9) is indeed independent of the choice of w.

The *intensity* γ_M of M is defined by

$$\gamma_M := \mathbb{Q}_M(\Omega) = \mathbb{E}\int w(g)M(dg). \tag{2.13}$$

We have $\gamma_M = \mathbb{E}M(B)$ for any $B \in \mathcal{G}$ with $\lambda(B) = 1$. The refined Campbell theorem implies the ordinary Campbell theorem

$$\mathbb{E}\int f(g)M(dg) = \gamma_M \int f(g)\lambda(dg), \tag{2.14}$$

for all measurable $f : G \to \mathbb{R}_+$. In case $0 < \gamma_M < \infty$ we can define the *Palm probability measure* of M by $\mathbb{Q}_M^0 := \gamma_M^{-1}\mathbb{Q}_M$.

Remark 2.8. Working with a (stationary) σ-finite measure $\mathbb{P}$ rather than with probability measure doesn't make the theory more complicated. In fact, some of the fundamental results can even be more easily stated this way. An example is the one-to-one correspondence between $\mathbb{P}$ and the Palm measure $\mathbb{P}_M$, see e.g. Theorem 7.1. Otherwise, extra technical integrability assumptions are required (see Theorem 11.4 in [16]). Another advantage is that in some applications it is the Palm probability measure that has a probabilistic interpretation. This measure can be well-defined also in case where $\mathbb{P}$ is not a finite measure.

To derive another corollary of the refined Campbell theorem, we take a measurable function $\tilde{w} : \mathbf{M} \times G \to \mathbb{R}_+$ satisfying

$$\int \tilde{w}(\eta, g)\eta(dg) = 1, \tag{2.15}$$

whenever $\eta \in \mathbf{M}$ is not the null measure. For one example of such a function we refer to [23]. We then have the *inversion formula*

$$\mathbb{E}\mathbf{1}\{M(G) > 0\}f = \mathbb{E}_{\mathbb{Q}_M} \int \tilde{w}(M \circ \theta_g, g) f \circ \theta_g \lambda(dg) \tag{2.16}$$

for all measurable $f : \Omega \to \mathbb{R}_+$. This is a direct consequence of the refined Campbell theorem (2.11). A first useful consequence of the inversion formula is the following.

Proposition 2.9. *The Palm measure $\mathbb{Q}_M$ is σ-finite.*

Proof. As $\mathbb{P}$ is assumed σ-finite, there is a positive measurable function f on Ω such that $\mathbb{E}f < \infty$. The function $\int \tilde{w}(M \circ \theta_g, g) f(\theta_g) \lambda(dg)$ is positive on $\{M(G) > 0\}$ and has by (2.16) a finite integral with respect to $\mathbb{Q}_M$. Since $\mathbb{Q}_M(M(G) = 0) = 0$ (see Lemma 2.5) we obtain that $\mathbb{Q}_M$ is σ-finite. □

The *invariant* σ-field $\mathcal{I} \subset \mathcal{A}$ is the class of all sets $A \in \mathcal{A}$ satisfying $\theta_g A = A$ for all $g \in G$. Let M be an invariant random measure with finite intensity and define

$$\hat{M} := \mathbb{E}\Big[\int w(g) M(dg) \Big| \mathcal{I}\Big], \tag{2.17}$$

where the conditional expectation is defined as for probability measures. Since $\hat{M} \circ \theta_g = \hat{M}$, $g \in G$, the refined Campbell theorem (2.11) implies that $\mathbb{E}\mathbf{1}_A \int w(g) M(dg) = \mathbb{Q}_M(A)$ for all $A \in \mathcal{I}$. Therefore definition (2.17) is independent of the choice of w. If $\mathbb{P}$ is a probability measure and $G = \mathbb{R}^d$, then $\hat{M}$ is called *sample intensity* of M, see [22] and [16]. Assuming that $\mathbb{P}(\hat{M} = 0) = 0$, we define the *modified Palm measure* $\mathbb{Q}^*_M$ (see [22], [32], [18]) by

$$\mathbb{Q}^*_M(A) = \mathbb{E}\hat{M}^{-1} \int \mathbf{1}\{\theta_g^{-1} \in A\} w(g) M(dg). \tag{2.18}$$

Conditioning shows that

$$\mathbb{Q}^*_M(A) = \mathbb{P}(A), \quad A \in \mathcal{I}. \tag{2.19}$$

Comparing (2.18) and (2.9) yields

$$d\mathbb{Q}^*_M = \hat{M}^{-1} d\mathbb{Q}_M. \tag{2.20}$$

The refined Campbell theorem (2.11) takes the form

$$\mathbb{E}\hat{M}^{-1} \int f(\theta_g^{-1}, g) M(dg) = \mathbb{E}_{\mathbb{Q}^*_M} \int f(\theta_e, g) \lambda(dg). \tag{2.21}$$

Remark 2.10. If $\mathbb{P}$ is a probability measure and M is a simple point process with a positive and finite intensity, then the Palm probability measure $\mathbb{Q}^0_M$ can be interpreted as a conditional probability measure given that M has a point in e, see [17]. The modified version describes the underlying stochastic experiment as seen from a randomly chosen point of M, see [22], [32]. Both measures agree iff $\hat{M}$ is $\mathbb{P}$-a.e. constant and in particular if $\mathbb{P}$ is *ergodic*, i.e. $\mathbb{P}(A) = 0$ or $\mathbb{P}(\Omega \setminus A) = 0$ for all $A \in \mathcal{I}$.

3 Stationary marked random measures

Let $(\mathrm{S}, \mathcal{S})$ be some measurable space and M' a kernel from Ω to $G \times \mathrm{S}$. We call M' *marked random measure* (on G with mark space S) if $M := M'(\cdot \times \mathrm{S})$ is a random measure on G. Daley and Vere-Jones [5] call M the *ground process* of M'.

A marked random measure M' is *invariant* if $M'(\cdot \times B)$ is invariant for all $B \in \mathcal{S}$. In many applications M' is of the form

$$M' = \int \mathbf{1}\{(g, \delta(g)) \in \cdot\} M(dg), \tag{3.22}$$

where $\delta : \Omega \times G \to \mathrm{S}$ is mesurable. If $g \in \operatorname{supp} M$ we think of $\delta(g)$ as *mark* of g. If M is invariant and δ is invariant in the sense that

$$\delta(\omega, g) = \delta(\theta_g^{-1}\omega, e), \quad \omega \in \Omega,\ g \in G, \tag{3.23}$$

then it is easy to check that M' is invariant.

Let M' be an invariant marked random measure and $\mathbb{P}$ a σ-finite stationary measure on $(\Omega, \mathcal{A})$. The *Palm measure* of M' is the measure $\mathbb{Q}_{M'}$ on $\Omega \times \mathrm{S}$ defined by

$$\mathbb{Q}_{M'}(A) = \mathbb{E} \int \mathbf{1}\{(\theta_g^{-1}, z) \in A\} w(g) M'(d(g, z)), \quad A \in \mathcal{A} \otimes \mathcal{S}. \tag{3.24}$$

Note that $\mathbb{Q}_{M'}(\cdot \times B)$ is the Palm measure of $M'(\cdot \times B)$. The refined Campbell theorem (2.11) takes the form

$$\mathbb{E} \int f(\theta_g^{-1}, g, z) M'(d(g, z)) = \iint f(\omega, g, z) \lambda(dg) \mathbb{Q}_{M'}(d(\omega, z)) \tag{3.25}$$

for all measurable $f : \Omega \times G \times \mathrm{S} \to \mathbb{R}_+$.

Assume that $(\Omega, \mathcal{A})$ is a Borel space. (This is e.g. the case in Example 2.1.) If $\mathbb{Q}_{M'}(\Omega \times \cdot)$ is a σ-finite measure, then we may *disintegrate* $\mathbb{Q}_{M'}$, to get another form of (3.25). For simplicity we even assume that the intensity γ_M of $M := M'(\cdot \times \mathrm{S})$ is finite. Assuming also $\gamma_M > 0$ we can define the *mark distribution* $\mathbb{W}$ of M' by $\mathbb{W} := \gamma_M^{-1} \mathbb{Q}_{M'}(\Omega \times \cdot)$. There exists a stochastic kernel $(z, A) \mapsto \mathbb{Q}_{M'}^z(A)$ from S to Ω satisfying

$$\mathbb{Q}_{M'}(d(\omega, z)) = \gamma_M \mathbb{Q}_{M'}^z(d\omega) \mathbb{W}(dz).$$

Therefore (3.25) can be written as

$$\mathbb{E} \int f(\theta_g^{-1}, g, z) M'(d(g, z)) = \gamma_M \iiint f(\omega, g, z) \lambda(dg) \mathbb{Q}_{M'}^z(d\omega) \mathbb{W}(dz). \tag{3.26}$$

Our next lemma provides an elegant way for handling stationary marked random measures. A kernel κ from $\Omega \times G$ to S is called *invariant* if

$$\kappa(\omega, g, \cdot) = \kappa(\theta_g^{-1}\omega, e, \cdot), \quad \omega \in \Omega,\ g \in G. \tag{3.27}$$

Lemma 3.1. *Let M' be an invariant marked random measure and assume that $(S, \mathcal{S})$ is a Borel space. Then there this an invariant stochastic kernel κ from $\Omega \times G$ to* S *such that*

$$M' = \iint \mathbf{1}\{(g, z) \in \cdot\} \kappa(g, dz) M(dg) \quad \mathbb{P} - a.e. \tag{3.28}$$

Proof. Define a measure C' on $\Omega \times G \times \mathrm{S}$ by

$$C'(\cdot) := \mathbb{E} \int \mathbf{1}\{(\theta_e, g, z) \in \cdot\} M'(d(g, z)).$$

From stationarity of $\mathbb{P}$ and invariance of M' we easily obtain that

$$\int \mathbf{1}\{(\theta_h \omega, hg, z) \in \cdot\} C'(d(\omega, g, z)) = C', \quad h \in G.$$

Moreover, the measure $C := C'(\cdot \times S)$ is σ-finite. We can now apply Theorem 3.5 in [17] to obtain an invariant kernel κ satisfying

$$C' = \iint \mathbf{1}\{(\omega, g, z) \in \cdot\} \kappa(\omega, g, dz) C(d(\omega, g)). \tag{3.29}$$

(In fact the theorem yields an invariant kernel κ', satisfying this equation. But in our specific situation we have $\kappa'(\omega, g, \mathrm{S}) = 1$ for C-a.e. (ω, g), so that κ' can be modified in an obvious way to yield the desired κ.) Equation (3.29) implies that

$$\mathbb{E}\mathbf{1}_A M'(B) = \mathbb{E}\mathbf{1}_A \iint \mathbf{1}\{(g, z) \in B\} \kappa(g, dz) M(dg), \quad B \in \mathcal{G} \times \mathcal{S},$$

for all $A \in \mathcal{A}$. Therefore $M'(B) = \iint \mathbf{1}\{(g, z) \in B\} \kappa(g, dz) M(dg)$ holds $\mathbb{P}$-a.e. Since $\mathcal{G} \times \mathcal{S}$ is countably generated, (3.28) follows. $\square$

We now assume that (3.28) holds for an invariant kernel κ. Invariance of κ implies that the Palm measure of M' is given by

$$\mathbb{Q}_{M'} = \mathbb{E}_{\mathbb{Q}_M} \int \mathbf{1}\{(\theta_e, z) \in \cdot\} \kappa(e, dz). \tag{3.30}$$

The refined Campbell theorem (3.26) reads

$$\mathbb{E} \int f(\theta_g^{-1}, g, z) M'(d(g, z)) = \mathbb{E}_{\mathbb{Q}_M} \iint f(\omega, g, z) \kappa(e, dz) \lambda(dg). \tag{3.31}$$

The mark distribution is given by $W = \gamma_M^{-1} \mathbb{E}_{\mathbb{Q}_M} \kappa(e, \cdot)$. In the special case (3.22) the refined Campbell theorem (3.31) says that

$$\mathbb{E} \int f(\theta_g^{-1}, g, z) M'(d(g, z)) = \mathbb{E}_{\mathbb{Q}_M} \int f(\omega, g, \delta(e)) \lambda(dg). \tag{3.32}$$

The mark distribution is given by $W = \gamma_M^{-1} \mathbb{Q}_M(\delta(e) \in \cdot)$.

4 Invariant transport-kernels

We first adapt the terminology from [20] to our present more general setting. A *transport-kernel* (on G) is a Markovian kernel T from $\Omega \times G$ to G. We think of $T(\omega, g, B)$ as proportion of mass transported from location g to the set B, when ω is given. A *weighted transport-kernel* (on G) is a kernel from $\Omega \times G$ to G such that $T(\omega, g, \cdot)$ is locally finite for all $(\omega, g) \in \Omega \times G$. If T is finite then the mass at g is weighted by $T(\omega, g, G)$ before being transported by the normalized T. A weighted transport kernel T is called *invariant* if

$$T(\theta_g \omega, gh, gB) = T(\omega, h, B), \quad g, h \in G,\ \omega \in \Omega,\ B \in \mathcal{G}. \tag{4.1}$$

Quite often we use the short-hand notation $T(g, \cdot) := T(\theta_e, g, \cdot)$. If M is an invariant random measure on G and $N := \int T(\omega, g, \cdot) M(\omega, dg)$ is locally finite for each $\omega \in \Omega$, then N is again an invariant random measure. Our interpretation is, that T transports M to N in an invariant way.

Example 4.1. Consider a measurable function $t : \Omega \times G \times G \to \mathbb{R}_+$ and assume that t is invariant, i.e.

$$t(\theta_k \omega, kg, kh) = t(\omega, g, h), \quad \omega \in \Omega,\ g, h, k \in G. \tag{4.2}$$

Let M be an invariant random measure on G and define

$$T(g, B) := \int_B t(g, h) M(dh).$$

Then (4.1) holds. Such functions t occur in the *mass-transport principle*, see [2] and Remark 4.6 below. The number $t(\omega, g, h)$ is then interpreted as the mass sent from g to h when the configuration ω is given.

Example 4.2. Consider a measurable mapping $\tau : \Omega \times G \to G$. The transport kernel T defined by $T(g, \cdot) := \delta_{\tau(g)}$ is invariant if and only τ is *covariant* in the sense that

$$\tau(\theta_g \omega, gh) = g\tau(\omega, h), \quad \omega \in \Omega,\ g, h \in G. \tag{4.3}$$

In this case we call τ *allocation rule*. This terminology is taken from [13].

Example 4.3. Consider the setting of Example 2.2 and let $\mathbb{P}$ be a σ-finite stationary measure on $(\Omega, \mathcal{A})$ concentrated on the set Ω' of all integer-valued $\omega \in \Omega$. Define an invariant transport-kernel T by

$$T(\omega, g, \cdot) = \frac{1}{M(\omega, \{g\})} \sum_{h : \omega(g,h) > 0} \omega(g, h) \delta_h, \quad \omega \in \Omega',\ g \in G, \tag{4.4}$$

if $M(\omega, \{g\}) > 0$ and $T(\omega, g, \cdot) := \delta_g$ otherwise, where $\omega(g, h) := \omega(\{g\} \times E \times \{h\} \times E)$. It can be easily checked that T is $\mathbb{P}$-a.e. (M, N)-*balancing*. A general criterion for the existence of balancing transport-kernels is given in Section 6.

Let M and N be two invariant random measures on G. A weighted transport-kernel T on G is called (M, N)-*balancing* if

$$\int T(\omega, g, \cdot) M(\omega, dg) = N(\omega, \cdot) \tag{4.5}$$

holds for all $\omega \in \Omega$. In case $M = N$ we also say that T is M-*preserving*. If $\mathbb{Q}$ is a measure on $(\Omega, \mathcal{A})$ such that (4.5) holds for $\mathbb{Q}$-a.e. $\omega \in \Omega$ then we say that T is $\mathbb{Q}$-a.e. (M, N)-balancing.

The next result is a fundamental transport property of Palm measures. It generalizes Theorem 4.2 in [20].

Theorem 4.4. *Let $\mathbb{P}$ be a σ-finite stationary measure on $(\Omega, \mathcal{A})$. Consider two invariant random measures M and N on G and let T and T^* be invariant weighted transport-kernels satisfying*

$$\iint \mathbf{1}\{(g,h) \in \cdot\} T(\omega, g, dh) M(\omega, dg) = \iint \mathbf{1}\{(g,h) \in \cdot\} T^*(\omega, h, dg) N(\omega, dh) \tag{4.6}$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$. Then we have for any measurable function $f : \Omega \times G \to \mathbb{R}_+$ that

$$\mathbb{E}_{\mathbb{Q}_M} \int f(\theta_g^{-1}, g^{-1}) \Delta(g^{-1}) T(e, dg) = \mathbb{E}_{\mathbb{Q}_N} \int f(\theta_e, g) T^*(e, dg). \tag{4.7}$$

Proof. Since $\int w(g^{-1}h)\lambda(dh) = 1$, $g \in G$, we have from Fubini's theorem that

$$\begin{aligned} I &:= \mathbb{E}_{\mathbb{Q}_N} \int f(\theta_e, g) T^*(c, dg) = \mathbb{E}_{\mathbb{Q}_N} \iint f(\theta_e, g) w(g^{-1}h) \lambda(dh) T^*(c, dg) \\ &= \mathbb{E}_{\mathbb{Q}_N} \iint f(\theta_e, g) w(g^{-1}h) T^*(c, dg) \lambda(dh). \end{aligned}$$

Next we use the refined Campbell theorem (2.11) and (4.1) (applied to T^*) to get

$$\begin{aligned} I &= \mathbb{E} \iint f(\theta_h^{-1}, g) w(g^{-1}h) T^*(\theta_h^{-1}, e, dg) N(dh) \\ &= \mathbb{E} \iint f(\theta_h^{-1}, h^{-1}g) w(g^{-1}hh) T^*(\theta_e, h, dg) N(dh) \\ &= \mathbb{E} \iint f(\theta_h^{-1}, h^{-1}g) w(g^{-1}hh) T(\theta_e, g, dh) M(dg), \end{aligned}$$

where the last equality is due to assumption (4.6). We now make the above steps in the reversed direction. Again by (4.1),

$$I = \mathbb{E} \iint f(\theta_{gh}^{-1}, h^{-1}) w(hgh) T(\theta_g^{-1}, e, dh) M(dg).$$

Since $\theta_{gh}^{-1} = \theta_h^{-1} \circ \theta_g^{-1}$ we can use the refined Campbell theorem, to obtain

$$I = \mathbb{E}_{\mathbb{Q}_M} \iint f(\theta_h^{-1}, h^{-1}) w(hgh) T(\theta_e, e, dh) \lambda(dg).$$

The result now follows from

$$\int w(hgh)\lambda(dg) = \int w(gh)\lambda(dg) = \Delta(h^{-1}), \quad h \in G. \qquad \square$$

Theorem 4.4 will play a key role in establishing an invariance property of Palm measures, see Theorem 5.1 below. One interesting special case is the following generalization of Neveu's [26] exchange formula from Abelian to locally compact groups. Another special case is Theorem 9.1 below.

Corollary 4.5. *Let $\mathbb{P}$ be a σ-finite stationary measure on $(\Omega, \mathcal{A})$ and M, N invariant random measure on G. Then we have for any measurable function $f : \Omega \times G \to \mathbb{R}_+$ that*

$$\mathbb{E}_{\mathbb{Q}_M} \int f(\theta_g^{-1}, g^{-1}) \Delta(g^{-1}) N(dg) = \mathbb{E}_{\mathbb{Q}_N} \int f(\theta_e, g) M(dg). \tag{4.8}$$

Proof. Apply Theorem 4.4 with $T(g, \cdot) := M$ and $T^*(g, \cdot) := N$. □

Remark 4.6. Let $B \in \mathcal{G}$ have positive and finite Haar measure. Using the definition (2.10) of Palm measures (applied to $w := \lambda(B)^{-1}\mathbf{1}_B$) as well as invariance of M and N, we can rewrite the exchange formula (4.8) as

$$\begin{aligned} \mathbb{E} \iint & \mathbf{1}\{h \in B\} f(\theta_g^{-1}, g^{-1}h) \Delta(g^{-1}) \Delta(h) N(dg) M(dh) \\ &= \mathbb{E} \iint \mathbf{1}\{h \in B\} f(\theta_h^{-1}, h^{-1}g) M(dg) N(dh). \end{aligned} \tag{4.9}$$

The function $t(\omega, g, h) := f(\theta_g^{-1}\omega, g^{-1}h)$ is invariant in the sense of (4.2). Equation (4.9) implies

$$\mathbb{E} \iint \mathbf{1}\{g \in B\} t(h, g) \Delta(g^{-1}) \Delta(h) N(dh) M(dg) = \mathbb{E} \iint \mathbf{1}\{g \in B\} t(g, h) N(dg) M(dh) \tag{4.10}$$

for all invariant t. In case $M = N$ this gives a version of the *mass-transport principle* (see [2]) for stationary random measures on groups. It will be shown in [19] that Neveu's exchange formula (4.8) can be generalized to jointly stationary random measures on a homogeneous space. In fact, the papers [2] and [1] show that the mass-transport principle can be extended beyond this setting.

Corollary 4.7. *Let the assumptions of Theorem 4.4 be satisfied. Assume moreover that M and N have finite intensities and that $\mathbb{P}(\hat{M} = 0) = \mathbb{P}(\hat{N} = 0) = 0$. Then we have for any measurable function $f : \Omega \times G \to \mathbb{R}_+$ that*

$$\hat{M} \mathbb{E}_{\mathbb{Q}_M^*} \Big[\int f(\theta_g^{-1}, g^{-1}) \Delta(g^{-1}) T(c, dg) \Big| \mathcal{I} \Big] = \hat{N} \mathbb{E}_{\mathbb{Q}_N^*} \Big[\int f(\theta_e, g) T^*(e, dg) \Big| \mathcal{I} \Big],$$

$\mathbb{P}$-a.e. for any choice of the conditional expectations.

Proof. Define the random variables X and X' by $X := \int f(\theta_g^{-1}, g^{-1}) \Delta(g^{-1}) T(e, dg)$ and $X' := \int f(\theta_e, g) T^*(c, dg)$ and let $A \in \mathcal{I}$. Due to (2.19) we have $\mathbb{Q}_M^* = \mathbb{Q}_N^*$ on $\mathcal{I}$. Hence we have to show that

$$\mathbb{E}_{\mathbb{Q}_M^*} \mathbf{1}_A \hat{M} X = \mathbb{E}_{\mathbb{Q}_N^*} \mathbf{1}_A \hat{N} X'.$$

By (2.20) this amounts to $\mathbb{E}_{\mathbb{Q}_M} \mathbf{1}_A X = \mathbb{E}_{\mathbb{Q}_N} \mathbf{1}_A X'$, i.e. to a consequence of (4.7). □

5 Invariance properties of Palm measures

In this section we fix a stationary σ-finite measure $\mathbb{P}$ on $(\Omega, \mathcal{A})$. In the special case of an Abelian group the following fundamental invariance property of Palm measures has recently been established in [20].

Theorem 5.1. *Consider two invariant random measures M and N on G and an invariant weighted transport-kernel T. Then T is $\mathbb{P}$-a.e. (M,N)-balancing iff*

$$\mathbb{E}_{\mathbb{Q}_M} \int f(\theta_g^{-1}) \Delta(g^{-1}) T(e, dg) = \mathbb{E}_{\mathbb{Q}_N} f \tag{5.1}$$

holds for all measurable $f : \Omega \to \mathbb{R}_+$.

Proof. Assume first that T is $\mathbb{P}$-a.e. (M,N)-balancing. Lemma 5.2 below shows that there exists an invariant transport-kernel T^* satisfying (4.6) for $\mathbb{P}$-a.e. $\omega \in \Omega$. Applying (4.7) to a function not depending on the second argument, yields (5.1).

Let us now assume that (5.1) holds. Take a measurable function $f : \Omega \times G \to \mathbb{R}_+$. By the refined Campbell theorem and (5.1),

$$\mathbb{E} \int f(\theta_g^{-1}, g) N(dg) = \mathbb{E}_{\mathbb{Q}_M} \iint f(\theta_h^{-1}, g) \Delta(h^{-1}) T(e, dh) \lambda(dg).$$

By Fubini's theorem and (2.2) this equals

$$\mathbb{E}_{\mathbb{Q}_M} \iint f(\theta_h^{-1}, gh) \Delta(h) T(e, dh) \lambda(dg) = \mathbb{E} \iint f(\theta_h^{-1} \circ \theta_g^{-1}, gh) T(\theta_g^{-1}, e, dh) M(dg),$$

where the equality is again due to the refined Campbell theorem, this time applied to M. By (4.1),

$$\mathbb{E} \int f(\theta_g^{-1}, g) N(dg) = \mathbb{E} \iint f(\theta_h^{-1}, h) T(g, dh) M(dg).$$

A straightforward substitution yields

$$\mathbb{E} \int \tilde{f}(\theta_e, g) N(dg) = \mathbb{E} \iint \tilde{f}(\theta_e, g) T(h, dg) M(dh),$$

for all measurable functions $\tilde{f} : \Omega \times G \to \mathbb{R}_+$. From this we obtain by a standard procedure that $N(B) = \int T(\theta_e, g, B) M(dg)$ $\mathbb{P}$-a.e. for all $B \in \mathcal{G}$. Since $\mathcal{G}$ is countably generated, this concludes the proof of the theorem. $\square$

The above proof has used the following lemma:

Lemma 5.2. *Assume that T is a $\mathbb{P}$-a.e. (M,N)-balancing invariant weighted transport-kernel. Then there exists an invariant transport-kernel T^* on G such that (4.6) holds for $\mathbb{P}$-a.e. $\omega \in \Omega$.*

Proof. Consider the following measure W on $\Omega \times G \times G$:

$$W := \iiint \mathbf{1}\{(\omega, g, h) \in \cdot\} T(\omega, g, dh) M(\omega, dg) \mathbb{P}(d\omega).$$

Stationarity of $\mathbb{P}$, (2.5), and (4.1) easily imply that

$$\int \mathbf{1}\{(\theta_g \omega, kg, kh) \in \cdot\} W(d(\omega, g, h)) = W, \quad k \in G.$$

Moreover, as T is a $\mathbb{P}$-a.e. (M, N)-balancing, we have

$$W' := \int \mathbf{1}\{(\omega, h) \in \cdot\} W(d(\omega, g, h)) = \iint \mathbf{1}\{(\omega, h) \in \cdot\} N(\omega, dh) \mathbb{P}(d\omega). \tag{5.2}$$

This is a σ-finite measure on $\Omega \times G$. As in the proof of Lemma 3.1 we can now apply Theorem 3.5 in [17] to obtain an invariant transport-kernel T^* satisfying

$$W = \iint \mathbf{1}\{(\omega, g, h) \in \cdot\} T^*(\omega, h, dg) W'(d(\omega, h)).$$

Recalling the definition of W and the second equation in (5.2) we get that

$$\iiint \mathbf{1}\{(\omega, g, h) \in \cdot\} T(\omega, g, dh) M(\omega, dg) \mathbb{P}(d\omega)$$
$$= \iiint \mathbf{1}\{(\omega, g, h) \in \cdot\} T^*(\omega, h, dg) N(\omega, dh) \mathbb{P}(d\omega)$$

and hence the assertion of the lemma. ◻

Example 5.3. Consider the setting of Example 2.2 and let $\mathbb{P}$ be a σ-finite stationary measure on $(\Omega, \mathcal{A})$ concentrated on the set Ω' of all integer-valued $\omega \in \Omega$. Applying Theorem 5.1 with T given by (4.4) yields

$$\int \frac{1}{M(\omega, \{e\})} \sum_{h:\omega(e,h)>0} f(\theta_h \omega) \omega(e, h) \Delta(g^{-1}) \mathbb{Q}_M(d\omega) = \mathbb{E}_{\mathbb{Q}_N}[f] \tag{5.3}$$

for any measurable $f : \Omega \to \mathbb{R}_+$. Specializing to the case of a function f depending only on $N(\omega)$ (and to an Abelian subgroup of $\mathbb{R}^d$) yields Theorem 6.5 in Port and Stone [27]. In the special case $G = \mathbb{R}$ (and under further restrictions on the support of $\mathbb{P}$), (5.3) is Theorem (6.5) in [8].

In fact, the paper [27] is using a specific form of the Palm measure $\mathbb{Q}_M$. To explain this, we assume that $\mathbb{P}(M' \in \cdot)$ is σ-finite, where M' is the marked random measure defined by $M'(\omega) := \omega(\cdot \times G \times E)$. Then there is a Markov kernel K from $\mathbf{M}_E$ to Ω satisfying

$$\mathbb{P} = \int K(\eta, \cdot) \mathbb{P}(M' \in d\eta).$$

Here $\mathbf{M}_E$ denotes the space of all measures η on $G \times E$ such that $\eta(\cdot \times E) \in \mathbf{M}$. (Similarly as in Example 2.2 this space can be equipped with a σ-field and a flow.) By Theorem 3.5 in [17] we can assume that K is invariant. Of course, if $\mathbb{P}$ is a probability measure, then

$K(M', A)$ is a version of the conditional probability of $A \in \mathcal{A}$ given M. Using invariance of K, it is straightforward to check that

$$\mathbb{Q}_M = \int K(\eta, \cdot)\mathbb{Q}_M(M' \in d\eta). \tag{5.4}$$

Let M and N be random measures on G. We call an allocation rule τ (see Example 4.2) $\mathbb{P}$-a.e. (M, N)-*balancing*, if

$$\int \mathbf{1}\{\tau(g) \in \cdot\}M(dg) = N \tag{5.5}$$

holds $\mathbb{P}$-a.e., i.e. if the transport T defined by $T(g, \cdot) := \delta_{\tau(g)}$ is $\mathbb{P}$-a.e. (M, N)-balancing. For an allocation rule τ it is convenient to introduce the measurable mapping $\theta_\tau : \Omega \to \Omega$ by

$$\theta_\tau(\omega) := \theta_{\tau(\omega,e)}\omega. \tag{5.6}$$

Similarly we define $\theta_\tau^{-1} : \Omega \to \Omega$ by $\theta_\tau^{-1}(\omega) := \theta_{\tau(\omega,e)}^{-1}\omega$.

Corollary 5.4. *Consider two invariant random measures M and N and let τ be an allocation rule. Then τ is $\mathbb{P}$-a.e. (M, N)-balancing iff*

$$\mathbb{E}_{\mathbb{Q}_M} f(\theta_\tau^{-1})\Delta(\tau(e)^{-1}) = \mathbb{E}_{\mathbb{Q}_N} f \tag{5.7}$$

holds for all measurable $f : \Omega \to \mathbb{R}_+$.

In the Abelian case Corollary 5.4 can be found in [20]. If in addition $M = N$, then one implication is (essentially) a consequence of Satz 4.3 in [24]. The special case $M = \lambda$ was treated in [7].

6 Existence of balancing weighted transport-kernels

We fix a stationary σ-finite measure $\mathbb{P}$ on $(\Omega, \mathcal{A})$ and consider two invariant random measures M, N on G. Our aim is to establish a necessary and sufficient condition for the existence of (M, N)-balancing invariant weighted transport-kernels T satisfying

$$\int \Delta(g^{-1})T(e, dg) = 1. \tag{6.1}$$

Theorem 6.1. *Assume that M and N have positive and finite intensities. Then there exists a $\mathbb{P}$-a.e. (M, N)-balancing invariant weighted transport-kernel satisfying* (6.1) *iff*

$$\mathbb{E}[M(B)|\mathcal{I}] = \mathbb{E}[N(B)|\mathcal{I}] \quad \mathbb{P} - a.e. \tag{6.2}$$

for some $B \in \mathcal{G}$ satisfying $0 < \lambda(B) < \infty$.

Proof. Let $B \in \mathcal{G}$ satisfy $0 < \lambda(B) < \infty$. For any $A \in \mathcal{I}$ we have from the refined Campbell theorem (2.11) that

$$\lambda(B)\mathbb{Q}_M(A) = \mathbb{E}[\mathbf{1}_A M(B)], \quad \lambda(B)\mathbb{Q}_N(A) = \mathbb{E}[\mathbf{1}_A N(B)]. \tag{6.3}$$

Assume now that T is a $\mathbb{P}$-a.e. (M, N)-balancing invariant weighted transport-kernel satisfying (6.1). Then Theorem 5.1 implies for all $A \in \mathcal{I}$ the equality $\mathbb{Q}_M(A) = \mathbb{Q}_N(A)$. Thus (6.3) implies $\mathbb{E}\mathbf{1}_A M(B) = \mathbb{E}\mathbf{1}_A N(B)$ and hence (6.2).

Let us now assume that (6.2) holds for some $B \in \mathcal{G}$ satisfying $0 < \lambda(B) < \infty$. Since $\mathbb{E}M(\cdot)$ and $\mathbb{E}N(\cdot)$ are multiples of λ, M and N have the same intensities. We assume without loss of generality that these intensities are equal to 1. From (6.3) and conditioning we obtain that $\mathbb{Q}_M = \mathbb{Q}_N$ on $\mathcal{I}$. Using the group-coupling result in Thorisson [31] as in Last and Thorisson [20], we obtain a stochastic kernel $\tilde{T}$ from Ω to G satisfying

$$\mathbb{Q}_N(A) = \mathbb{E}_{\mathbb{Q}_M} \int \mathbf{1}\{\theta_g^{-1} \in A\}\tilde{T}(\theta_e, dg), \quad A \in \mathcal{A}. \tag{6.4}$$

Let T' be the kernel from Ω to G defined by

$$T'(\omega, \cdot) := \int \mathbf{1}\{g \in \cdot\}\Delta(g)\tilde{T}(\omega, dg).$$

Since Δ is continuous, $T'(\omega, B)$ is finite for all $\omega \in \Omega$ and all compact $B \subset G$. The invariant weighted transport-kernel T defined by $T(\omega, g, B) := T'(\theta_g^{-1}\omega, g^{-1}B)$ satisfies (6.1) by definition. And because of (6.4) it does also satisfy (5.1). Theorem 5.1 implies that T is $\mathbb{P}$-a.e. (M, N)-balancing. $\square$

7 Mecke's characterization of Palm measures

Let M be an invariant random measure on G. In contrast to the previous sections we do not fix a stationary measure on $(\Omega, \mathcal{A})$. Instead we consider here a measure $\mathbb{Q}$ on $(\Omega, \mathcal{A})$ as a candidate for a Palm measure $\mathbb{Q}_M$ of M w.r.t. some stationary measure $\mathbb{P}$ on $(\Omega, \mathcal{A})$. In case G is an Abelian group (and within a canonical framework) the following fundamental characterization theorem was proved in [23]. In a canonical framework (and for finite intensities) the present extension has been established in [28], even in the more general case of a random measure on an homogeneous space.

Theorem 7.1. *The measure $\mathbb{Q}$ is a Palm measure of M with respect to some σ-finite stationary measure iff $\mathbb{Q}$ is σ-finite, $\mathbb{Q}(M(G) = 0) = 0$, and*

$$\mathbb{E}_{\mathbb{Q}} \int f(\theta_g^{-1}, g^{-1})\Delta(g^{-1})M(dg) = \mathbb{E}_{\mathbb{Q}} \int f(\theta_e, g)M(dg) \tag{7.1}$$

holds for all measurable $f : \Omega \times G \to \mathbb{R}_+$.

Proof. If $\mathbb{Q}$ is a Palm measure of M, then $\mathbb{Q}$ is σ-finite by Proposition 2.9. Equation $\mathbb{Q}(M(G) = 0) = 0$ holds by Proposition 2.5, while the Mecke equation (7.1) is a special case of (4.7).

Let us now conversely assume the stated conditions. Using the function $\tilde{w}$ occuring in (2.16), we define a measure $\mathbb{P}$ on Ω by

$$\mathbb{P}(A) := \mathbb{E}_{\mathbb{Q}} \int \mathbf{1}\{\theta_g \in A\} \tilde{w}(gM, g) \lambda(dg), \quad A \in \mathcal{A}. \tag{7.2}$$

Take a measurable $f : \Omega \times G \to \mathbb{R}_+$. By (2.15),

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}} \int f(\theta_g, g) \lambda(dg) &= \mathbb{E}_{\mathbb{Q}} \iint f(\theta_g, g) \tilde{w}(gM, h) gM(dh) \lambda(dg) \\ &= \mathbb{E}_{\mathbb{Q}} \iint f(\theta_g, g) \tilde{w}(M \circ \theta_g, gh) \lambda(dg) M(dh). \end{aligned}$$

Using assumption (7.1), we get

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}} \int f(\theta_g, g) \lambda(dg) \\ &= \mathbb{E}_{\mathbb{Q}} \iint f(\theta_g \circ \theta_{h^{-1}}, g) \tilde{w}(M \circ \theta_g \circ \theta_{h^{-1}}, gh^{-1}) \Delta(h^{-1}) \lambda(dg) M(dh) \\ &= \mathbb{E}_{\mathbb{Q}} \iint f(\theta_{g^{-1}h^{-1}}, g^{-1}) \tilde{w}(M \circ \theta_{g^{-1}h^{-1}}, g^{-1}h^{-1}) \Delta(h^{-1}) \Delta(g^{-1}) \lambda(dg) M(dh) \\ &= \mathbb{E}_{\mathbb{Q}} \iint f(\theta_{g^{-1}}, g^{-1}h) \tilde{w}(M \circ \theta_{g^{-1}}, g^{-1}) \Delta(g^{-1}) \lambda(dg) M(dh) \\ &= \mathbb{E}_{\mathbb{Q}} \iint f(\theta_g, gh) \tilde{w}(M \circ \theta_g, g) \lambda(dg) M(dh), \end{aligned}$$

where we have used properties of the Haar measure λ for the second, third, and fourth equality. This implies

$$\mathbb{E}_{\mathbb{Q}} \int f(\theta_g, g) \lambda(dg) = \mathbb{E}_{\mathbb{Q}} \iint f(\theta_g, h) \tilde{w}(M \circ \theta_g, g) M \circ \theta_g(dh) \lambda(dg),$$

so that by definition (7.2)

$$\mathbb{E}_{\mathbb{Q}} \int f(\theta_g, g) \lambda(dg) = \mathbb{E} \int f(\theta_e, h) M(dh). \tag{7.3}$$

Since $\mathbb{Q} \otimes \lambda$ is σ-finite, there is a measurable function $\tilde{f} : \Omega \times G \to (0, \infty)$ having $\mathbb{E}_{\mathbb{Q}} \int \tilde{f}(\theta_e, g) \lambda(dg) < \infty$. Defining a positive measurable function $f : \Omega \times G \to \mathbb{R}_+$ by $f(\omega, h) := \tilde{f}(\theta_{h^{-1}}\omega, h)$, we obtain

$$\mathbb{E}_{\mathbb{Q}} \int f(\theta_g, g) \lambda(dg) = \mathbb{E}_{\mathbb{Q}} \int \tilde{f}(\theta_e, g) \lambda(dg) < \infty.$$

Hence (7.3) implies that the $\mathbb{P}$-a.e. positive measurable function $\int f(\theta_e, g) M(dg)$ has a finite integral with respect to $\mathbb{P}$. Therefore $\mathbb{P}$ is σ-finite.

Next we show that $\mathbb{P}$ is stationary. By invariance of λ we have for $A \in \mathcal{A}$ and $h \in G$

$$\mathbb{P}(A) = \mathbb{E}_{\mathbb{Q}} \int \mathbf{1}\{\theta_{h^{-1}g} \in A\} \tilde{w}(M \circ \theta_{h^{-1}g}, h^{-1}g)\lambda(dg)$$

$$= \mathbb{E}_{\mathbb{Q}} \int \mathbf{1}\{\theta_g \in \theta_h A\} \tilde{w}(M \circ \theta_{h^{-1}} \circ \theta_g, h^{-1}g)\lambda(dg).$$

Applying (7.3), yields

$$\mathbb{P}(A) = \mathbb{E} \int \mathbf{1}\{\theta_e \in \theta_h A\} \tilde{w}(M \circ \theta_{h^{-1}}, h^{-1}g) M(dg) = \mathbb{P}(\theta_h A),$$

since $\int \tilde{w}(M \circ \theta_{h^{-1}}, h^{-1}g) M(dg) = 1$ $\mathbb{P}$-a.e.

It remains to show that $\mathbb{Q}$ is the Palm measure $\mathbb{Q}_M$ of M with respect to $\mathbb{P}$. By (2.10) and (7.3),

$$\mathbb{Q}_M(A) = \mathbb{E} \int \mathbf{1}\{\theta_h^{-1} \in A\} w(h) M(dh) = \mathbb{E}_{\mathbb{Q}} \int \mathbf{1}\{\theta_g^{-1} \circ \theta_g \in A\} w(g)\lambda(dg).$$

Hence $\mathbb{Q}_M(A) = \mathbb{Q}(A)$, as desired. $\square$

8 Mass-stationarity

We fix an invariant random measure M on G and a σ-finite measure $\mathbb{Q}$ on $(\Omega, \mathcal{A})$. Our aim is to establish *mass-stationarity* as another characterizing property of Palm measures of M. This is generalizing one of the main results in Last and Thorisson [20] from Abelian to arbitrary groups satisfying the general assumptions.

Let $C \in \mathcal{G}$ be relatively compact and define a Markovian transport kernel T_C by

$$T_C(g, B) := M(gC)^{-1} M(B \cap gC), \quad g \in G,\, B \in \mathcal{G}, \tag{8.1}$$

if $M(gC) > 0$, and by letting $T_C(g, \cdot)$ equal some fixed probability measure, otherwise. In the former case $T_C(g, \cdot)$ is just governing a G-valued stochastic experiment that picks a point uniformly in the mass of M in gC. Since M is invariant, it is immediate that T_C is invariant too. If $0 < \lambda(C) < \infty$ we also define the uniform distribution λ_C on C by $\lambda_C(B) := \lambda(B \cap C)/\lambda(C)$. The interior (resp. boundary) of a set $C \subset G$ is denoted by $\operatorname{int} C$ (resp. ∂C).

A σ-finite measure $\mathbb{Q}$ on $(\Omega, \mathcal{A})$ is called *mass-stationary for M* if $\mathbb{Q}(M(G) = 0) = 0$ and

$$\mathbb{E}_{\mathbb{Q}} \iint \mathbf{1}\{(\theta_g^{-1}, hg) \in A\} T_C(h^{-1}, dg)\lambda_C(dh) = \mathbb{Q} \otimes \lambda_C(A), \quad A \in \mathcal{A} \otimes \mathcal{G}, \tag{8.2}$$

holds for all relatively compact sets $C \in \mathcal{G}$ with $\lambda(C) > 0$ and $\lambda(\partial C) = 0$.

Remark 8.1. Assume that $\mathbb{Q}$ is a probability measure. Let C be as assumed in (8.2). Extend the space $(\Omega, \mathcal{A}, \mathbb{Q})$, so as to carry random elements U, V in G such that θ_e and U are independent, U has distribution λ_C, and the conditional distribution of V given

(θ_e, U) is uniform in the mass of M on $U^{-1}C$. (The mappings θ_g, $g \in G$, are extended, so that they still take values in the original space Ω.) Then (8.2) can be written as

$$(\theta_V^{-1}, UV) \overset{d}{=} (\theta_e, U). \tag{8.3}$$

Mass-stationarity of $\mathbb{Q}$ requires that this holds for all such pairs (U, V).

Theorem 8.2. *There exists a σ-finite stationary measure $\mathbb{P}$ on $(\Omega, \mathcal{A})$ such that $\mathbb{Q} = \mathbb{Q}_M$ iff $\mathbb{Q}$ is mass-stationary for M.*

Proof. Assume first that $\mathbb{Q} = \mathbb{Q}_M$ is the Palm measure of M with respect to a σ-finite stationary measure $\mathbb{P}$. Let C be as in (8.2) and D a measurable subset of C. Define a weighted transport-kernel T by

$$T(g, B) := \iint \mathbf{1}\{k \in B, h^{-1}k \in D\}\mathbf{1}\{h^{-1}g \in C\}T_C(h, dk)\lambda(dh).$$

By invariance of T_C and left-invariance of λ,

$$T(\theta_g, g, gB) = \iint \mathbf{1}\{k \in B, h^{-1}gk \in D\}\mathbf{1}\{h^{-1}g \in C\}T_C(g^{-1}h, dk)\lambda(dh) = T(\theta_e, e, B).$$

Hence T is invariant. Let $g \in G$. From the properties of the set C and continuity of group mulitplication we have for λ-a.e. h such that $h^{-1}g \in C$ the relationship $h^{-1}g \in \operatorname{int} C$. If $h^{-1}g \in \operatorname{int} C$ and $g \in \operatorname{supp} M$, then $g \in \operatorname{int} hC$ and $M(hC) > 0$. Using this and the definition of T_C, we obtain

$$\begin{aligned}\int T(g, B)M(dg) &= \iint \mathbf{1}\{k \in B, h^{-1}k \in D\}\lambda(dh)M(dk)\\ &= \iint \mathbf{1}\{k \in B, h^{-1} \in D\}\lambda(dh)M(dk) = M(B)\lambda^*(D),\end{aligned}$$

where $\lambda^*(D) := \int \mathbf{1}\{h^{-1} \in D\}\lambda(dh)$. Therefore we obtain from Theorem 5.1 that

$$\mathbb{E}_{\mathbb{Q}} \int f(\theta_g^{-1})\Delta(g^{-1})T(e, dg) = \lambda^*(D)\mathbb{E}_{\mathbb{Q}} f \tag{8.4}$$

for all measurable $f : \Omega \to \mathbb{R}_+$. The above left-hand side equals

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}} \iint f(\theta_g^{-1})\Delta(g^{-1})\mathbf{1}\{g \in hD, h^{-1} \in C\}T(e, dg)\lambda(dh)\\ &\qquad = \mathbb{E}_{\mathbb{Q}} \iint f(\theta_g^{-1})\Delta(g^{-1})\mathbf{1}\{hg \in D, h \in C\}T(e, dg)\lambda^*(dh).\end{aligned}$$

Hence we obtain from (8.4) and a monotone class argument

$$\mathbb{E}_{\mathbb{Q}} \iint f(\theta_g^{-1}, hg)\Delta(g^{-1})\mathbf{1}\{h \in C\}T(e, dg)\lambda^*(dh) = \mathbb{E}_{\mathbb{Q}} \int f(\theta_e, h)\mathbf{1}\{h \in C\}\lambda^*(dh)$$

for all measurable $f : \Omega \times G \to \mathbb{R}_+$. By (2.3) this means

$$\mathbb{E}_{\mathbb{Q}} \iint f(\theta_g^{-1}, hg)\Delta(g^{-1})\Delta(h^{-1})\mathbf{1}\{h \in C\}T(e, dg)\lambda(dh)$$
$$= \mathbb{E}_{\mathbb{Q}} \int f(\theta_e, h)\Delta(h^{-1})\mathbf{1}\{h \in C\}\lambda(dh).$$

Applying this with $f(\omega, k)$ replaced by $f(\omega, k)\Delta(k)$ gives

$$\mathbb{E}_{\mathbb{Q}} \iint f(\theta_g^{-1}, hg)\mathbf{1}\{h \in C\}T(e, dg)\lambda(dh) = \mathbb{E}_{\mathbb{Q}} \int f(\theta_e, h)\mathbf{1}\{h \in C\}\lambda(dh).$$

This is equivalent to (8.2).

Now we assume conversely, that $\mathbb{Q}$ is mass-stationary. Our strategy is to derive the Mecke equation (7.1) and to apply Theorem 7.1. Let $f : \Omega \to \mathbb{R}_+$ be measurable, C be as in (8.2) and D a measurable subset of C. By (8.2)

$$\mathbb{E}_{\mathbb{Q}} \iint f(\theta_g^{-1})M(h^{-1}C)^{-1}\mathbf{1}\{hg \in D, h \in C\}M(dg)\lambda(dh) = \lambda(D)\mathbb{E}_{\mathbb{Q}}f,$$

where $1/0 := 0$. Therefore,

$$\mathbb{E}_{\mathbb{Q}} \iint f(\theta_g^{-1})M(gh^{-1}C)^{-1}\mathbf{1}\{h \in D, hg^{-1} \in C\}\Delta(g^{-1})M(dg)\lambda(dh) = \lambda(D)\mathbb{E}_{\mathbb{Q}}f.$$

Since D can be any measurable subset of C, we get

$$\mathbb{E}_{\mathbb{Q}} \int f(\theta_g^{-1})M(gh^{-1}C)^{-1}\mathbf{1}\{hg^{-1} \in C\}\Delta(g^{-1})M(dg) = \mathbb{E}_{\mathbb{Q}}f$$

for λ-a.e. $h \in C$. In particular we also have for λ-a.e. $h \in C$ that

$$\mathbb{E}_{\mathbb{Q}} \int f(\theta_g^{-1})\tilde{f}(M \circ \theta_g^{-1})M(gh^{-1}C)^{-1}\mathbf{1}\{hg^{-1} \in C\}\Delta(g^{-1})M(dg) = \mathbb{E}_{\mathbb{Q}}f\tilde{f}(M),$$

for any measurable $\tilde{f} : \mathbf{M} \to \mathbb{R}_+$. Since $\mathcal{M}$ is countably generated, we may choose the corresponding λ-null set independent of $\tilde{f}$. In particular we may take for any h outside a λ-null set, $\tilde{f}(\eta) := \eta(h^{-1}C)$. Then $\tilde{f}(M \circ \theta_g^{-1}) = M \circ \theta_g^{-1}(h^{-1}C) = M(gh^{-1}C)$, so that

$$\mathbb{E}_{\mathbb{Q}} \int f(\theta_g^{-1})\mathbf{1}\{hg^{-1} \in C\}\Delta(g^{-1})M(dg) = \mathbb{E}_{\mathbb{Q}}fM(h^{-1}C)$$

for λ-a.e. $h \in C$. From here we can proceed as in Last and Thorisson [20], to get the full Mecke equation (7.1). $\square$

9 Stationary partitions

The topic of this section are (extended) stationary partitions as introduced (in case $G = \mathbb{R}^d$) and studied in [18]. The motivation of [18] was to connect stationary tessellations of classical stochastic geometry (see e.g. [30], [29]) with recent work in [13] and [11] on allocation rules transporting Lebesgue measure to a simple point process. Our terminology here is closer to stochastic geometry.

Let N be an invariant simple point process on G. As mentioned earlier, we identify N with its support. A *stationary partition* (based on N) is a pair (Z,τ) consisting of a *measurable* set $Z:\Omega\to\mathcal{G}$ and an allocation rule τ such that $\tau(g)\in N$ whenever $g\in Z$. We also assume that $\{Z=\emptyset\}=\{N=\emptyset\}$. Measurability of Z just means that $(\omega,g)\mapsto \mathbf{1}\{g\in Z(\omega)\}$ is measurable, while covariance of Z means that

$$Z(\theta_g\omega)=gZ(\omega),\quad \omega\in\Omega,\, g\in G. \tag{9.1}$$

For convenience we also assume that $\tau(g)=g$, $g\in G$, whenever $N=\emptyset$. Define

$$C(\omega,g):=\{h\in Z(\omega):\tau(\omega,h)=g\},\quad \omega\in\Omega,\, g\in G. \tag{9.2}$$

Note that $C(g)=\emptyset$ whenever $g\notin N\neq\emptyset$. The system $\{C(g):g\in N\}$ forms a partition of Z into measurable sets, provided that $N\neq\emptyset$. Equations (9.1) and (4.3) imply the following covariance property:

$$C(\theta_h\omega,hg)=hC(\omega,g),\quad \omega\in\Omega,\, g,h\in G. \tag{9.3}$$

Although we do not make any topological or geometrical assumptions, we refer to $C(g)$ as *cell* with (generalized) *centre* $g\in N$. We do not assume that $g\in C(g)$ and some of the cells might be empty.

We now fix a σ-finite stationary measure $\mathbb{P}$ on $(\Omega,\mathcal{A})$. The following theorem generalizes Theorem 7.1 in [18] from the case $G=\mathbb{R}^d$ to general groups. The former case is also touched by Lemma 16 in [11].

Theorem 9.1. *Let (Z,τ) be a stationary partition. Then we have for any measurable $f,\tilde{f}:\Omega\to\mathbb{R}_+$,*

$$\mathbb{E}\mathbf{1}\{e\in Z\}\Delta(\tau^*(e))\tilde{f}f(\theta_{\tau^*})=\mathbb{E}_{\mathbb{Q}_N}f\int_{C(e)}\tilde{f}(\theta_g^{-1})\lambda(dg), \tag{9.4}$$

where $\tau^*(g):=\tau(g)^{-1}$, $g\in G$, and θ_{τ^*} is defined by (5.6).

Proof. Consider the random measure $M:=\lambda(Z\cap\cdot)$. By covariance (9.1) of Z and invariance of λ we have $M(\theta_g,gB)=\int\mathbf{1}\{h\in gB\cap gZ\}\lambda(dh)=M(B)$ for all $g\in G$ and $B\in\mathcal{S}$. Hence M is invariant. An equally simple calculation shows that the Palm measure of M is given by

$$\mathbb{Q}_M(A)=\mathbb{P}(A\cap\{e\in Z\}),\quad A\in\mathcal{A}. \tag{9.5}$$

Define transport-kernels T and T^* by $T(g,\cdot):=\delta_{\tau(g)}$ and $T^*(g,\cdot):=\lambda(C(g)\cap\cdot)$. Since $\tau(g)\in N$ whenever $g\in Z$, it is straigthforward to check that (4.6) holds, even for all $\omega\in\Omega$. By (4.3), T is invariant. Invariance of T^* follows from (9.3) and invariance of λ:

$$T^*(\theta_g,gh,gB)=\lambda(C(\theta_g,gh)\cap gB)=\lambda(gC(h)\cap gB)=T^*(h,B).$$

Theorem 4.4 implies that (4.7) holds. Applying this formula to the measurable function $(\omega,g)\mapsto f(\omega)\tilde{f}(\theta_g^{-1}\omega)$ and taking into account (9.5) as well as the definitions of T and T^*, yields the assertion (9.4). $\square$

The special choice $\tilde{f}\equiv 1$ in (9.4) yields the following relationship between the measure $\mathbb{E}\mathbf{1}\{e\in Z\}\Delta(\tau^*(e))\mathbf{1}\{\theta_{\tau^*}\in\cdot\}$ and a *volume-weighted* version of the Palm measure $\mathbb{Q}_N$. In case of $G=\mathbb{R}^d$ we refer to Section 4 of [18].

Proposition 9.2. *Let (Z, τ) be a stationary partition. Then we have for any measurable $f : \Omega \to \mathbb{R}_+$,*

$$\mathbb{E}\mathbf{1}\{e \in Z\}\Delta(\tau^*(e))f(\theta_{\tau^*}) = \mathbb{E}_{\mathbb{Q}_N}\lambda(C(e))f. \tag{9.6}$$

Remark 9.3. The special case $f \equiv 1$ of (9.6) gives

$$\mathbb{E}_{\mathbb{Q}_N}\lambda(C(e)) = \mathbb{E}\mathbf{1}\{e \in Z\}\Delta(\tau^*(e)) \tag{9.7}$$

If N has a positive and finite intensity γ_N and G is unimodular, this yields the intuitively obvious formula

$$\mathbb{E}_{\mathbb{Q}_N^0}\lambda(C(e)) = \mathbb{P}(e \in Z)\gamma_N^{-1}. \tag{9.8}$$

Define

$$V(g) := \{h \in G : \tau(h) = \tau(g)\}$$

as the cell containing $g \in G$.

Corollary 9.4. *Let (Z, τ) be a stationary partition. Then we have for any $\beta \geq 0$ that*

$$\mathbb{E}\mathbf{1}\{e \in Z\}\Delta(\tau^*(e))\lambda(V(e))^\beta = \mathbb{E}_{\mathbb{Q}_N}\lambda(C(e))^{\beta+1}. \tag{9.9}$$

Proof. For $h \in N$ we get from (9.3) that $C(\theta_{h^{-1}}, e) = h^{-1}C(h)$. Assuming $e \in Z$, we can apply this fact to $h := \tau(e) \in N$, to obtain

$$C(\theta_{\tau^*}, e) = \tau^*(e)V(e). \tag{9.10}$$

We now apply (9.6) to $f := \lambda(C(e))^\beta$. Using (9.10) together with invariance of λ, yields (9.9). ☐

A stationary partition (Z, τ) is called *proper*, if

$$\mathbb{Q}_N(\lambda(C(e) = 0) = \mathbb{Q}_N(\lambda(C(e) = \infty) = 0. \tag{9.11}$$

In the unimodal case the second equation is implied by (9.7). The following two results can be proved as in Section 5 of [18]. The details are left to the reader.

Proposition 9.5. *Let (Z, τ) be a stationary and proper partition. Then we have for any measurable $f : \Omega \to \mathbb{R}_+$,*

$$\mathbb{E}\mathbf{1}\{e \in Z\}\Delta(\tau^*(e))\lambda(V(e))^{-1}f(\theta_{\tau^*}) = \mathbb{E}_{\mathbb{Q}_N}f. \tag{9.12}$$

Corollary 9.6. *Let (Z, τ) be a stationary and proper partition. Then* (9.9) *holds for all $\beta \in \mathbb{R}$. If the intensity γ_N of N is finite, then we have in particular,*

$$\gamma_N = \mathbb{E}\mathbf{1}\{e \in Z\}\Delta(\tau^*(e))\lambda(V(e))^{-1}. \tag{9.13}$$

From now on we assume that N has a finite intensity and $\mathbb{P}(\hat{N} = 0) = 0$. Just for simplicity we also assume that $\mathbb{P}$ (and hence also $\mathbb{Q}_N^*$) is a probability measure. We first note the following consequence of the proof of Theorem 9.7 and Corollary 4.7:

Corollary 9.7. *Let (Z,τ) be a stationary partition. We have for any measurable $f,\tilde{f}:\Omega\to\mathbb{R}_+$,*

$$\mathbb{E}[\mathbf{1}\{e\in Z\}\tilde{f}f(\theta_{\tau^*})\Delta(\tau^*(e))|\mathcal{I}]=\hat{N}\mathbb{E}_{\mathbb{Q}_N^*}\Big[f\int_{C(e)}\tilde{f}(\theta_g^{-1})\lambda(dg)\Big|\mathcal{I}\Big], \tag{9.14}$$

$\mathbb{P}$-a.e. for any choice of the conditional expectations. In particular,

$$\hat{N}^{-1}\mathbb{E}[\mathbf{1}\{e\in Z\}f(\theta_{\tau^*}\Delta(\tau^*(e))|\mathcal{I}]=\mathbb{E}_{\mathbb{Q}_N^*}[f\lambda(C(e))|\mathcal{I}]. \tag{9.15}$$

Let $\alpha>0$. Essentially following [11] (dealing with the case $G=\mathbb{R}^d$), we call a stationary partition (Z,τ) (based on N) *α-balanced*, if

$$\mathbb{P}(\lambda(C(g))=\alpha\hat{N}^{-1}\text{ for all }g\in N)=1. \tag{9.16}$$

The significance of α-balanced stationary partitions is due to the following theorem. The result extends Theorem 13 in [13] and Theorem 9.1 in [18] (both dealing with $\alpha=1$) from $\mathbb{R}^d$ to general groups.

Theorem 9.8. *Let $\alpha>0$. A stationary partition (Z,τ) is α-balanced iff*

$$\mathbb{Q}_N^*=\alpha^{-1}\mathbb{E}\mathbf{1}\{e\in Z\}\Delta(\tau^*(e))\mathbf{1}\{\theta_{\tau^*}\in\cdot\}. \tag{9.17}$$

Proof. If (Z,τ) is a α-balanced stationary partition, then (9.17) follows from (9.15).

Assume now that (9.17) holds. Since $\mathbb{Q}_N^*$ has the invariant density $\hat{N}$ with respect to $\mathbb{Q}_N$, (9.17) implies that

$$\mathbb{Q}_{\alpha N}=\mathbb{E}\mathbf{1}\{e\in Z\}\hat{N}\mathbf{1}\{\theta_{\tau^*}\in\cdot\}\Delta(\tau^*(e)), \tag{9.18}$$

where we have also used that $\mathbb{Q}_{\alpha N}=\alpha\mathbb{Q}_N$. Using the invariant weighted transport-kernels $T(g,\cdot):=\hat{N}\mathbf{1}\{g\in Z\}\delta_{\tau(g)}$, this reads

$$\mathbb{Q}_{\alpha N}=\mathbb{E}\int\mathbf{1}\{\theta_g^{-1}\in\cdot\}\Delta(g^{-1})T(e,dg).$$

Since $\mathbb{P}$ is the Palm measure of λ, we get from Theorem 5.1 that T is $\mathbb{P}$-a.e. $(\lambda,\alpha N)$-balancing. Therefore we have $\mathbb{P}$-a.e. that

$$\int\mathbf{1}\{g\in Z,\tau(g)\in\cdot\}\lambda(dg)=\alpha\hat{N}^{-1}N(\cdot).$$

This is just saying that (Z,τ) is α-balanced. $\square$

Remark 9.9. If a α-balanced stationary partition (Z,τ) is given, then (9.17) provides an explicit method for constructing the modified Palm probability measure $\mathbb{Q}_N^*$ by a *shift-coupling* with the stationary measure $\mathbb{P}$. In case G is a unimodal group, (9.17) simplifies to

$$\mathbb{Q}_N^*=\alpha^{-1}\mathbb{E}\mathbf{1}\{e\in Z\}\mathbf{1}\{\theta_{\tau^*}\in\cdot\}. \tag{9.19}$$

Since $\alpha=\mathbb{P}(e\in Z)$, this means that $\mathbb{Q}_N^*=\mathbb{P}(\theta_{\tau^*}\in\cdot|e\in Z)$.

The actual construction of α-balanced partitions is an interesting topic in its own right. Triggered by [21], the case $G = \mathbb{R}^d$ was discussed in [13] and [11]. Among many other things it was shown there that α-balanced partitions do actually exist for any $\alpha \le 1$. The occurence of the sample intensity $\hat{N}$ in (9.16) is explained by the spatial ergodic theorem, see Proposition 9.1 in [18], at least in case $G = \mathbb{R}^d$. The paper [13] has also results on discrete groups in case $\alpha = 1$. It might be conjectured that α-balanced partitions exist for all $\alpha \le 1$, provided that the Haar measure λ is diffuse.

10 Matchings and point stationarity

We consider an invariant simple point process N on G. A *point-allocation* for N is an allocation rule τ having $\tau(g) \in N$ whenever $g \in N$. (Recall that we identify N with its support.) We also assume that

$$\tau(g) = g, \quad g \notin N. \tag{10.1}$$

A point-allocation for N is called *bijective* if $g \mapsto \tau(g)$ is a bijection on N whenever $N(G) > 0$. In the Abelian (or $\mathbb{R}^d$) case the following special case of Corollary 5.4 is discussed in [32], [9], and [10]. Recall the notation introduced at (5.6).

Corollary 10.1. *Let $\mathbb{P}$ be a σ-finite stationary measure $\mathbb{P}$ on $(\Omega, \mathcal{A})$ and τ be a point-allocation for N. Then τ is $\mathbb{P}$-a.e. bijective iff*

$$\mathbb{E}_{\mathbb{Q}_N} f(\theta_\tau^{-1})\Delta(\tau(e)^{-1}) = \mathbb{E}_{\mathbb{Q}_N} f \tag{10.2}$$

holds for all measurable $f : \Omega \to \mathbb{R}_+$.

A *N-matching* is a point-allocation τ such that $\tau(\tau(g)) = g$ for all $g \in N$. (We don't require that $\tau(g) \ne g$ for $g \in N$.) In the canonical case $\Omega = \mathbf{N}_s$ with N being the identity on Ω (and with the flow given as in Example 2.1) we just say that τ is a matching. Our next result is generalizing the point process case of Theorem 1.1 in [10] (dealing with an Abelian group). We assume that there exist a measurable and injective function $I : \Omega \to [0, 1]$. As any Borel space has this property, this is no serious restriction of generality.

Theorem 10.2. *A measure $\mathbb{Q}$ on $(\Omega, \mathcal{A})$ is a Palm measure of N with respect to some σ-finite stationary measure iff $\mathbb{Q}$ is σ-finite, $\mathbb{Q}(e \notin N) = 0$, and*

$$\mathbb{E}_{\mathbb{Q}} f(\theta_\tau^{-1})\Delta(\tau(e)^{-1}) = \mathbb{E}_{\mathbb{Q}} f \tag{10.3}$$

holds for all N-matchings τ and all measurable $f : \Omega \to \mathbb{R}_+$.

Our proof of Theorem 10.2 requires the following generalization of a result in [10], that is of interest in its own right.

Proposition 10.3. *There exist N-matchings τ_k, $k \in \mathbb{N}$, such that for $e \in N$*

$$\{e\} \cup \{g \in N : \theta_g \ne \theta_e\} \subset \{\tau_k(e) : k \in \mathbb{N}\}. \tag{10.4}$$

The proof of Proposition 10.3 is based on several lemmas. A *subprocess* of N is an invariant simple point process S such that $S \subset N$.

Lemma 10.4. *Let S be a subprocess of N and τ a matching. Then $\tau'(\omega, g) := \tau(S(\omega), g)$ is a N-matching.*

Proof. Covariance of τ' is a direct consequence of the invariance of N and covariance of τ. We have to show that $\tau'(g) \in N$ and $\tau'(\tau'(g))$ if $g \in N$. If $g \notin S$ then (10.1) (for τ') implies $\tau(g) = g \in N$ and in particular $\tau'(\tau'(g)) = g$. If $g \in S$, then $\tau'(g) = \tau(S, g) \in S \subset N$. Moreover, since τ is a matching we have $\tau'(\tau'(g)) = \tau(S, \tau(S, g)) = g$.

To check the asserted matching property we take $g \in N$. If $g \notin S$ then (10.1) (for τ') implies $\tau(g) = g$ and in particular $\tau(\tau(g)) = g$. $\square$

Lemma 10.5. *There exists a countable family of subprocesses $\{S_n : n \in \mathbb{N}\}$ of N such that, for any $B \in \mathcal{G}$ with compact closure, $\omega \in \Omega$ and $g, h \in N(\omega)$, there exists $n \in \mathbb{N}$ with*

$$I(\theta_k^{-1}\omega) \in \{I(\theta_g^{-1}\omega), I(\theta_h^{-1}\omega)\}, \quad k \in S_n(\omega) \cap B. \tag{10.5}$$

Proof. Let $\{(q_n, r_n, s_n) : n \in \mathbb{N}\}$ be dense in $[0,1]^3$. Define the simple point processes S_n, $n \in \mathbb{N}$, by

$$S_n(\omega) := \{k \in N(\omega) : \min\{|I(\theta_k^{-1}\omega) - q_n|, |I(\theta_k^{-1}\omega) - r_n|\} \le s_n\}.$$

The measurability of S_n follows from the measurability of I. Invariance of S_n follows from invariance of N and the flow property (2.4).

Fix $B \in \mathcal{G}$ with compact closure, $\omega \in \Omega$ and $g, h \in N(\omega)$. From the local finiteness of $N(\omega)$ we deduce that the set

$$\{I(\theta_k^{-1}\omega) : k \in N(\omega) \cap B\}$$

is a finite subset of $[0,1]$. Hence, there exists $\varepsilon > 0$ such that

$$\{|I(\theta_k^{-1}\omega) - I(\theta_g^{-1}\omega)|, |I(\theta_k^{-1}\omega) - I(\theta_h^{-1}\omega)| : k \in N(\omega) \cap B\} \cap (0, \varepsilon) = \emptyset.$$

Moreover, there exists $n \in \mathbb{N}$ such that $0 < s_n < \varepsilon/2$ and both $|I(\theta_g^{-1}\omega) - q_n| < s_n$ and $|I(\theta_h^{-1}\omega) - r_n| < s_n$ hold, finishing the proof of the proposition. $\square$

Let $\eta \in \mathbf{N}_s$ and take a Borel set $B \in \mathcal{G}$. Then call $h \in \eta$ a *B-neighbour* of $g \in \eta$ if $h \in gB$. Clearly, h is a B-neighbour of g if and only if g is a B^*-neighbour of h, where the inverted set B^* is defined by $B^* := \{g^{-1} : g \in B\}$. We say that h is the *unique B-neighbour* of g if $gB \cap \eta = \{h\}$. A point h can be the unique B-neighbour of g, and g a non-unique B^*-neighbour of h. We then define an allocation rule $\pi_B : \mathbf{N}_s \times G \to G$ by

$$\pi_B(\eta, g) := \begin{cases} h, & \text{if } g \text{ and } h \text{ are mutual unique } B \cup B^*\text{-neighbours,} \\ g, & \text{otherwise.} \end{cases} \tag{10.6}$$

Lemma 10.6. *There exists a countable family of N-matchings (σ_n) such that*

$$\{e\} \cup \{g \in N(\omega) : I(\theta_g^{-1}\omega) \ne I(\omega) \text{ and } I(\theta_g^{-1}\omega) \ne I(\theta_g\omega)\} \subset \{\sigma_n(\omega, e) : n \in \mathbb{N}\} \tag{10.7}$$

for all $\omega \in \Omega$ such that $e \in N(\omega)$.

Proof. Let $\{B_m\}$ be a countable base of the topology of G. We consider the matchings $\tau_m := \pi_{B_m}, m \in \mathbb{N}$, defined in (10.6) and the family of subprocesses $\{S_n : n \in \mathbb{N}\}$ from Lemma 10.5.

Let $\omega \in \Omega$ such that $e \in N(\omega)$. Let $g \in N(\omega)$ such that $I(\theta_g^{-1}\omega) \neq I(\omega)$ and $I(\theta_g^{-1}\omega) \neq I(\theta_g\omega)$. There exists an open set U with compact closure in G such that $e, g, g^{-1}, g^2 \in U$. By Proposition 10.5 there exists $n \in \mathbb{N}$ such that, for all $k \in S_n(\omega) \cap U$, we have $I(\theta_k^{-1}\omega) \in \{I(\omega), I(\theta_g^{-1}\omega)\}$. From $I(\theta_g^{-1}\omega) \notin \{I(\omega), I(\theta_g\omega)\}$ and injectivity of I we obtain $\theta_g^{-1}\omega \neq \omega$ and $\theta_g^{-1}\omega \neq \theta_g\omega$. This gives $\theta_{g^{-2}}\omega \neq \theta_{g^{-1}}\omega$ and $\theta_{g^{-2}} \neq \omega$. We deduce that $I(\theta_{g^2}^{-1}\omega) \notin \{I(\omega), I(\theta_g^{-1}\omega)\}$, so $g^2 \notin S_n(\omega)$. In the same (even simpler) way we get $g^{-1} \notin S_n(\omega)$

Since $\{B_m\}$ is a base of the topology of G there exists $m \in \mathbb{N}$ such that $B_m \cap N(\omega) = \{g\}$, $gB_m^* \cap N(\omega) = \{e\}$, $B_m^* \cap N(\omega) \subset \{g^{-1}\}$ and $gB_m \cap N(\omega) \subset \{g^2\}$. We conclude that $\tau_m(S_n(\omega), e) = g$.

Reenumeration of the countable family of N-matchings $(\omega, h) \mapsto \tau_m(S_n(\omega), h)$ (see Lemma 10.4) yields a family $\{\sigma_n : n \in \mathbb{N}\}$ that satisfies (10.7). $\square$

Lemma 10.7. *There exists a countable family of N-matchings (σ_n) such that*

$$\{g \in N(\omega) : I(\theta_g^{-1}\omega) \neq I(\omega) \text{ and } I(\theta_g^{-1}\omega) = I(\theta_g\omega)\} \subset \{\sigma_n(\omega, e) : n \in \mathbb{N}\} \tag{10.8}$$

for all $\omega \in \Omega$ with $e \in N(\omega)$.

Proof. For $n \in \mathbb{N}$, define a covariant mapping $\sigma_n : \Omega \times G \to G$ by

$$\sigma_n(e) := \begin{cases} g, & \text{if } N \cap B_n = \{g\}, N \cap B_n^* = \{g^{-1}\}, N \cap gB_n^* = \{e\}, \\ & N \cap gB_n = \{g^2\} \text{ and } I(\theta_g) = I(\theta_g^{-1}) > I, \\ g, & \text{if } N \cap B_n^* = \{g\}, N \cap B_n = \{g^{-1}\}, N \cap gB_n = \{e\}, \\ & N \cap gB_n^* = \{g^2\} \text{ and } I(\theta_g) = I(\theta_g^{-1}) < I, \\ e, & \text{otherwise,} \end{cases} \tag{10.9}$$

and $\sigma_n(\omega, g) := g\sigma_n(\theta_g^{-1}\omega, e)$. Take $\omega \in \Omega$ with $e \in N(\omega)$ and $g \in N(\omega)$ such that $I(\theta_g^{-1}\omega) = I(\theta_g\omega)$. Then $g^{-1}, g^2 \in N(\omega)$. Moreover, there exists $n \in \mathbb{N}$ such that g is the unique B_n-neighbour of e in $N(\omega)$, g^{-1} the unique B_n^*-neighbour of e, e the unique B_n^*-neighbour of g in $N(\omega)$ and g^2 the unique B_n-neighbour of g. Hence, $\sigma_n(\omega, e) = g$ and the inclusion (10.8) holds.

It remains to show that σ_n is for each $n \in \mathbb{N}$ a N-matching. It is clearly sufficient to assume $e \in N$ and to prove that $\sigma_n(g) = e$ if $\sigma_n(e) = g \neq e$. Let $g \in N$. By definition of σ_n and invariance of N we have for $h \in G$ that

$$\sigma_n(g) = \begin{cases} gh, & \text{if } g^{-1}N \cap B_n = \{h\}, g^{-1}N \cap B_n^* = \{h^{-1}\}, g^{-1}N \cap hB_n^* = \{e\}, \\ & g^{-1}N \cap gB_n = \{h^2\} \text{ and } I(\theta_{hg}) = I(\theta_{hg}^{-1}) > I(\theta_g), \\ gh, & \text{if } g^{-1}N \cap B_n^* = \{h\}, g^{-1}N \cap B_n = \{h^{-1}\}, g^{-1}N \cap gB_n = \{e\}, \\ & g^{-1}N \cap gB_n^* = \{h^2\} \text{ and } I(\theta_{hg}) = I(\theta_{hg}^{-1}) < I(\theta_g), \\ g, & \text{otherwise.} \end{cases}$$

Putting $h := g^{-1}$ and using $\sigma_n(e) = g$ (i.e. the conditions on the right-hand side of (10.9)) we see that indeed $\sigma_n(g) = e$. $\square$

Combining Lemma 10.6 with Lemma 10.7 yields Proposition 10.3.

Proof of Theorem 10.2: One implication follows from Corollary 10.1. So we assume that $\mathbb{Q}$ is σ-finite, $\mathbb{Q}(e \notin N) = 0$, and that (10.2) holds for all τ and f as stated there. We will check the Mecke equation (7.1).

Without loss of generality we assume that $e \in N$ everywhere on Ω. Let τ be a N-matching. Since $\tau(e) \in N$ we get from the matching property and covriance (writing $g := \tau(e)$) that $\tau(\theta_g^{-1}, e) = g^{-1}\tau(\theta_e, g) = g^{-1}$, i.e.

$$\tau(\theta_\tau^{-1}, e) = \tau(\theta_e, e)^{-1}. \tag{10.10}$$

Therefore we obtain for all measurable $f : \Omega \times G \to \mathbb{R}_+$ from (10.2) that

$$\mathbb{E}_{\mathbb{Q}}\mathbf{1}\{\theta_\tau \neq \theta_e\}f(\theta_e, \tau(e)) = \mathbb{E}_{\mathbb{Q}}\mathbf{1}\{\theta_\tau^{-1} \neq \theta_e\}f(\theta_e, \tau(e)^{-1})\Delta(\tau(e)^{-1}). \tag{10.11}$$

Applying Proposition 10.3 yields

$$\mathbb{E}_{\mathbb{Q}}\int \mathbf{1}\{\theta_g \neq \theta_e\}f(\theta_e, g)N(dg) = \sum_{k=1}^{\infty} \mathbb{E}_{\mathbb{Q}}f_k(\theta_e, \tau_k(e))f(\theta_e, \tau_k(e)), \tag{10.12}$$

where

$$f_k(g) := \mathbf{1}\{\theta_g \neq \theta_e\}\mathbf{1}\{\tau_m(e) \neq g \text{ for } 1 \leq m \leq k-1\}.$$

We claim that

$$f_k(\theta_{\tau_k}^{-1}, \tau_k(e)^{-1}) = f_k(\theta_e, \tau_k(e)). \tag{10.13}$$

To see this, we take $\omega \in \Omega$ and set $g := \tau_k(\omega, e)$. We have to show that

$$f_k(\theta_g^{-1}\omega, g^{-1}) = f_k(\omega, g). \tag{10.14}$$

For $k \leq 2$ and $m \leq k-1$ we have from covariance that $\tau_m(\theta_g^{-1}\omega, e) \neq g^{-1}$ is equivalent to $g^{-1}\tau_m(\omega, g) \neq g^{-1}$, i.e. to $\tau_m(\omega, g) \neq e$. Since τ_m is matching the latter is also equivalent to $\tau_m(\omega, e) \neq g$. Hence (10.14) follows. Using this fact, we obtain from (10.12) and (10.11)

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\int \mathbf{1}\{\theta_g \neq \theta_e\}f(\theta_e, g)N(dg) &= \sum_{k=1}^{\infty} \mathbb{E}_{\mathbb{Q}}f_k(\theta_e, \tau_k(e))f(\theta_{\tau_k}^{-1}, \tau_k(e)^{-1})\Delta(\tau_k(e)^{-1}) \\ &= \mathbb{E}_{\mathbb{Q}}\int \mathbf{1}\{\theta_g \neq \theta_e\}f(\theta_g^{-1}, g^{-1})\Delta(g^{-1})N(dg).\end{aligned}$$

In order to show (7.1) it remains to verify

$$\mathbb{E}_{\mathbb{Q}}\int \mathbf{1}\{\theta_g = \theta_e\}f(\theta_e, g)N(dg) = \mathbb{E}_{\mathbb{Q}}\int \mathbf{1}\{\theta_g = \theta_e\}f(\theta_g^{-1}, g^{-1})\Delta(g^{-1})N(dg). \tag{10.15}$$

For any fixed $\omega \in \Omega$ the closed set $H_\omega := \{g \in G : \theta_g\omega = \omega\}$ is a subgroup of G. Indeed, if $g, h \in H_\omega$ then $\theta_{gh}\omega = \theta_g \circ \theta_h\omega = \omega$, while $g \in H_\omega$ is equivalent to $g^{-1} \in H_\omega$. Since $e \in N(\omega)$ we have, moreover, for all $g \in H_\omega$ that $g \in gN(\omega) = N(\theta_g\omega) = N(\omega)$. Hence H_ω is a subset of $N(\omega)$ and in particular unimodular. On the other hand we clearly have that $h^{-1}gh \in H_\omega$ for all $h \in G$ and $g \in H_\omega$, so that H_ω is a normal subgroup of G. Therefore we obtain from Proposition II.21 in [25] that $\Delta(g) = 1$ for all $g \in H_\omega$. Hence (10.15) is equivalent to

$$\mathbb{E}_{\mathbb{Q}} \int \mathbf{1}\{\theta_g = \theta_e\} f(\theta_e, g) N(dg) = \mathbb{E}_{\mathbb{Q}} \int \mathbf{1}\{\theta_g = \theta_e\} f(\theta_e, g^{-1}) N(dg). \tag{10.16}$$

This follows from $\{g \in N : \theta_g = \theta_e\} = \{g^{-1} : \theta_g = \theta_e, g \in N\}$. □

One might wonder whether the N-matchings in Theorem 10.2 may be chosen N-*measurable*, i.e. measurable with respect to $\sigma(N) \otimes \mathcal{G}$. In the canonical case $\Omega = \mathbf{N}_s$ this is trivially true. Otherwise, applying Lemma 10.6 to the canonical case and composing with N, yields the following result without any assumptions on $(\Omega, \mathcal{A})$.

Proposition 10.8. *There exist N-matchings τ_k, $k \in \mathbb{N}$, such that for $e \in N$*

$$\{e\} \cup \{g \in N : N \circ \theta_g \neq N\} \subset \{\tau_k(e) : k \in \mathbb{N}\}. \tag{10.17}$$

It is easy to see (cf. [10]) that the (random) set $\{g \in N : N \circ \theta_g^{-1} = N\}$ cannot be exhausted by matchings as in Proposition 10.8. In case $G = \mathbb{R}^d$, however, the set can be exhausted by bijective point-allocations, see [9]. Proposition 10.8 together with a suitable modification of the arguments of the proof of Theorem 10.2 then leads to the following Theorem 4.1 in [9]. We skip further details.

Theorem 10.9. *Let N be an invariant simple point process on $\mathbb{R}^d$ and $\mathbb{Q}$ be a measure on $(\Omega, \mathcal{A})$. Then $\mathbb{Q}$ is a Palm measure of N with respect to some σ-finite stationary measure iff $\mathbb{Q}$ is σ-finite, $\mathbb{Q}(0 \notin N) = 0$, and*

$$\mathbb{E}_{\mathbb{Q}} f(\theta_\tau^{-1}) = \mathbb{E}_{\mathbb{Q}} f \tag{10.18}$$

holds for all N-measurable bijective point-allocations τ and all measurable $f : \Omega \to \mathbb{R}_+$.

Property (10.18) has been called *point-stationarity* of $\mathbb{Q}$ (with respect to N), see [32]. We conjecture that Theorem 10.9 holds for general groups G, i.e. that the validity of (10.3) for all N-measurable bijective point-allocations τ is characterizing Palm measures.

The matchings considered in this section are examples of what the authors of [14] call a *one-color matching scheme*. *Mutual nearest neighbour matching* (see [6], [12], [4]) is another example of such matchings. The *two-color matching* schemes studied in [14] are an interesting example of an allocation rule balancing different jointly stationary (simple) point processes.

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