SHORT COMMUNICATIONS

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A Painless and Direct Way from Integral to Discrete Fast Wavelet Transforms

Starting from the integral wavelet transform it is shown how the scaling function appears naturally in terms of bandpass filtering. Our arguments help to motivate the definition of multiscale analyses.

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1. Introduction

The importance of multiscale (multiresolution) analyses for the construction of orthogonal (semi-, biorthogonal) wavelets is beyond question, see, e.g., the textbooks of Chui [1], Daubechies [2], Louis, Maass, and Rieder [3], and Meyer [5].

In teaching wavelet courses one should therefore carefully introduce the crucial idea of multiscale analyses (MSA). A good motivation takes the foreknowledge of the audience into account. In this short note we start our course with the integral (continuous) wavelet transform and suggest an approach in which the scaling function, its relation to the wavelet, and the fast algorithms are derived from the wavelet in a natural (painless) way without touching MSAs.

At this point the audience will be convinced that scaling functions lie in the heart of constructing (discrete) wavelets. It will agree on the need for a general framework to study scaling equations. Now we can easily offer the concept of MSAs as such a framework.

The key to our considerations will be the well-known interpretation of the integral wavelet transform as a bandpass filter.

2. Point of departure: the integral wavelet transform as phase space representation

Let us start right away with the integral wavelet transform. We refer to the above cited textbooks for its detailed motivation and its advantages over the Fourier transform w.r.t. signal analysis.

A real-valued square-integrable function ψ of one real variable is called *wavelet* if

$$C_\psi := 2\pi \int_{\mathbb{R}} |\omega|^{-1} \, |\hat{\psi}(\omega)|^2 \, \mathrm{d}\omega < \infty \,. \tag{1}$$

Here, $\hat{\psi}$ denotes the Fourier transform $\hat{\psi}(\omega) := (2\pi)^{-1/2} \int_{\mathbb{R}} \psi(t) \, \mathrm{e}^{-it\omega} \, \mathrm{d}t$. Based on the two parameter family $\psi_{a,b}(t) := \psi((t-b)/a)/\sqrt{a}$ where $a > 0$ and $b \in \mathbb{R}$, we define the *integral wavelet transform* $L_\psi : L^2(\mathbb{R}) \to L^2(\mathbb{R}_{>0} \times \mathbb{R}, \mathrm{d}a \, \mathrm{d}b/a^2)$ by

$$L_\psi f(a, b) := \int_{\mathbb{R}} f(t) \, \psi_{a,b}(t) \, \mathrm{d}t \,, \qquad a > 0 \,, \qquad b \in \mathbb{R} \,.$$

The function f can be recovered from $L_\psi f$ by

$$f(t) = \frac{2}{C_\psi} \int_0^\infty \int_{\mathbb{R}} L_\psi f(a, b) \, \psi_{a,b}(t) \, a^{-2} \, \mathrm{d}b \, \mathrm{d}a \tag{2}$$

where the double integral has to be taken in the weak sense in general.

We now interpret the wavelet transform as a phase-space (time/frequency-space) representation. To this end we assume that the wavelet ψ is normalized by $\|\psi\|_{L^2(\mathbb{R})} = 1$ and localized in the time as well as the frequency domains. More precisely, let ψ be centered at $t_0 := \int_{\mathbb{R}} t |\psi(t)|^2 \, \mathrm{d}t$ with dispersion $\delta_t := \left(\int_{\mathbb{R}} (t - t_0)^2 \, |\psi(t)|^2 \, \mathrm{d}t \right)^{1/2}$. Without loss of generality we consider $t_0 = 0$ which can be achieved by translating the wavelet. In the positive frequency domain let $\hat{\psi}$ be localized at the center frequency $\omega_0 := \int_0^\infty \omega|\hat{\psi}(\omega)|^2 \, d\omega$ with dispersion $\delta_\omega := \left(\int_0^\infty (\omega - \omega_0)^2 \, |\hat{\psi}(\omega)|^2 \, d\omega\right)^{1/2}$. Hence, $\psi_{a,b}$ lies concentrated in the phase-space domain $D_{a,b}$ given by

$$D_{a,b} := [b - a\delta_t, \, b + a\delta_t] \times [(\omega_0 - \delta_\omega)/a, \, (\omega_0 + \delta_\omega)/a] \,. \tag{3}$$

Consequently, $L_\psi f(a, b) = \langle f, \psi_{a,b}\rangle_{L^2(\mathbb{R})} = \langle \hat{f}, \widehat{\psi_{a,b}}\rangle_{L^2(\mathbb{R})}$ represents the phase space contents of f at the time/frequency-point $(b, \omega_0/a)$. Please observe that the time resolution increases with the frequency. This effect is called the *zooming property* of the wavelet transform.

3. Semi-discrete wavelet transform: wavelet series

The step from the integral wavelet transform to wavelet series is straightforward. In view of the reconstruction formula (2) we are looking for wavelets ψ and discrete subsets $\{(a_j, b_k) \mid j, k \in \mathbb{Z}\}$ of $\mathbb{R}_{>0} \times \mathbb{R}$ such that

$$f(t) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} d_{j,k}\psi_{a_j, b_k}(t) \tag{4}$$

holds true for suitable coefficients $d_{j,k} = d_{j,k}(f)$. The function f is expressed as a superposition of wavelets. The magnitude of $d_{i,k}(f)$ measures the contribution of ψ_{a_j, b_k} to f or the phase-space contents of f in D_{a_j, b_k}, respectively. In this semi-discrete framework the expansion coefficients $d_{j,k}(f)$ play the role of $L_\psi f(a_j, b_k)$ but $d_{j,k}(f) \neq L_\psi f(a_j, b_k)$ in general.

We do not aim to handle expansions like (4) in full generality. The interested reader should consult the references [1, 2, 3, 5] and the literature cited therein.

For our purpose it is sufficient to consider dyadic expansions only, that is, $a_j := 2^{-j}$ and $b_{j,k} := ka_j$. The corresponding wavelets

$$\psi_{j,k}(t) := \psi_{a_j, b_{j,k}}(t) = 2^{j/2}\, \psi(2^j\, t - k)\,, \qquad j, k \in \mathbb{Z}\,,$$

are localized in phase-space at the points $(k/2^j, \, 2^j\,\omega_0)$ which lie more densely in time for increasing frequencies. Hence, small details of f are sampled at a higher resolution in the expansion (4), thereby reproducing the zooming property of the integral wavelet transform.

We further pose some restrictions on the wavelet. Basically, the set $\Psi = \{\psi_{j,k} \mid j, k \in \mathbb{Z}\}$ is required to be a stable basis of $L^2(\mathbb{R})$. That is, the linear span of Ψ lies dense in $L^2(\mathbb{R})$ and the norm equivalence

$$\Big\|\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} d_{j,k}\, \psi_{j,k}\Big\|^2_{L^2(\mathbb{R})} \sim \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |d_{j,k}|^2 \tag{5}$$

holds true for all square-summable sequences $\{d_{j,k}\}$. We use the notation $f \sim g$ to indicate the existence of two positive constants c_1 and c_2 such that $c_1 f \leq g \leq c_2 f$. We call a function ψ with the above properties a *discrete* wavelet.

Remark: *A discrete wavelet is indeed a wavelet in the more general sense of* (1).

Analyzing a function f by the semi-discrete wavelet transform consists in computing its expansion coefficients w.r.t. (4). How can this be done efficiently? The next two sections answer this question.

4. Discrete wavelets and scaling functions

From now on let ψ be a discrete wavelet. Any $f \in L^2(\mathbb{R})$ can be written as a series (4) which we split according to

$$f = P_l f + \sum_{j=l}^{\infty} Q_j f\,, \qquad l \in \mathbb{Z}\,, \tag{6}$$

where

$$Q_j f := \sum_{k\in\mathbb{Z}} d_{j,k}\, \psi_{j,k} \quad \text{and} \quad P_l f := \sum_{j=-\infty}^{l-1} Q_j f\,. \tag{7}$$

Lemma 4.1 provides some relations for later use.

Lemma 4.1: *Let ψ be a discrete wavelet and let P_l and Q_l be defined in* (7). *If $f \in L^2(\mathbb{R})$ then*
(i) $P_{l+1}P_l f = P_l f$ *and* $P_{l+1}Q_l f = Q_l f$,
(ii) $P_{l+1} f = P_l f + Q_l f$.

Proof: The assertion (ii) is trivial. To prove (i) one argues as follows. The operators $P_l, Q_l : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ are linear and (uniformly) bounded, see (5). Thus, $P_{l+1}P_l f = \sum_{j=-\infty}^{l-1} \sum_{k\in\mathbb{Z}} d_{j,k}(f)\, P_{l+1}\, \psi_{j,k}$. We may write $\psi_{j,k}$

$= \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} d_{m,n}(\psi_{j,k})\, \psi_{m,n}$ where $d_{m,n}(\psi_{j,k}) = 1$ for $(j, k) = (m, n)$ and $d_{m,n}(\psi_{j,k}) = 0$ otherwise. Since this representation of $\psi_{j,k}$ is unique, see (5), we have that $P_{l+1}\,\psi_{j,k} = \psi_{j,k}$ for $j \le l$. □

In Section 2 we associated the frequency band $B_j := [2^j(w_0 - \delta_\omega),\, 2^j(\omega_0 + \delta_\omega)]$ to $\psi_{j,k}$, see (3). Therefore, $P_l f$ is essentially a bandlimited version of f with frequencies in $[0,\, 2^{l-1}(\omega_0 + \delta_\omega)]$. Translated into the time domain this means that $P_l f$ contains all details of f being larger than $2^{1-l}\, 2\pi/(\omega_0 + \delta_\omega)$, see, e.g., [3] for the precise derivation. In other words, $P_l f$ is a lowpass filtered or smoothed version of f. Similarly, the frequency contents of $Q_j f$ is essentially restricted to B_j. Hence, $Q_j f$ is a bandpass filtered version of f representing all structures of f with sizes roughly between $2^{-j}\, 2\pi/(\omega_0 + \delta_\omega)$ and $2^{-j}\, 2\pi/(\omega_0 - \delta_\omega)$.

So, it is meaningful to call j the scale parameter where large j's correspond to small scales. The splitting (6) is accordingly called a *multiscale representation* as it decomposes a function into a smooth part and into a sum of finer details.

Let us reconsider the properties of P_l: it is (almost) a lowpass filter whose frequency band widens with l and $P_l f \to f$ as $l \to \infty$. In signal analysis filters with the properties of P_l are typically realized by convolutions where the kernels coincide with the corresponding impulse response, see, e.g., PAPOULIS [6]. For instance, let $h \in L^1(\mathbb{R})$ be a real-valued function being (essentially) bandlimited in $[-\Omega,\, \Omega]$. Further, let h be normalized by $\int_{\mathbb{R}} h(t)\, \mathrm{d}t = 1$. Then,

$$H_l f(t) := 2^l \int_{\mathbb{R}} h(2^l(t-\tau))\, f(\tau)\, \mathrm{d}\tau$$

is a lowpass filter with band $[-2^l\,\Omega,\, 2^l\,\Omega]$ and $H_l f \to f$ in $L^2(\mathbb{R})$ as $l \to \infty$.

Applying the trapezoidal rule with abscissae $b_{l,k} = k/2^l$, $k \in \mathbb{Z}$, to the above integral yields the approximation

$$H_l f(t) \approx \sum_{k\in\mathbb{Z}} 2^{-l/2}\, f(k/2^l)\, h_{l,k}(t)\,. \tag{8}$$

Therefore, we wish to know whether a real-valued function $\varphi \in L^2(\mathbb{R})$ exists with a normalized mean value, $\int_{\mathbb{R}} \varphi(t)\, \mathrm{d}t = 1$, such that

$$P_l f = \sum_{k\in\mathbb{Z}} c_{l,k}\varphi_{l,k} \quad \text{for any} \quad f \in L^2(\mathbb{R}) \tag{9}$$

where the coefficients $c_{l,k} = c_{l,k}(f)$ are uniquely determined. That is, if $P_l f = 0$ then $c_{l,k}(f) = 0$ for all $k \in \mathbb{Z}$.

A stronger ansatz (9) with $c_{l,k}(f) = 2^{-l/2} f(k/2^l)$ would not leave the necessary degrees of freedom to get from the approximation (8) to the exact representation (9). However, since $H_l \approx P_l$ and $H_l f \to f$ as $l \to \infty$, we expect that

$$|c_{l,k}(f) - 2^{-l/2}\, f(k/2^l)| \to 0\,, \quad \text{as} \quad l \to \infty\,, \tag{10}$$

for smooth f.

The statements of Lemma 4.1 impose necessary conditions on φ and ψ so that representation (9) may hold true.

Theorem 4.2: *Assume that P_l as defined in (7) can be expressed in the form (9). Then, there exist uniquely given real sequences $h = \{h_k\}_{k\in\mathbb{Z}}$, $g = \{g_k\}_{k\in\mathbb{Z}}$, $\tilde{h} = \{\tilde{h}_k\}_{k\in\mathbb{Z}}$, and $\tilde{g} = \{\tilde{g}_k\}_{k\in\mathbb{Z}}$ such that*

$$\varphi(x) = \sqrt{2} \sum_{k\in\mathbb{Z}} h_k \varphi(2x-k)\,, \tag{11}$$

$$\psi(x) = \sqrt{2} \sum_{k\in\mathbb{Z}} g_k \varphi(2x-k)\,, \tag{12}$$

and

$$\sqrt{2}\, \varphi(2x-n) = \sum_{k\in\mathbb{Z}} [\tilde{h}_{2k-n}\varphi(x-k) + \tilde{g}_{2k-n}\psi(x-k)]\,, \qquad n \in \mathbb{Z}\,. \tag{13}$$

Proof: The equalities (11) and (12) are nothing else than a rewriting of the equalities $P_1\varphi = \varphi$ and $P_1\psi = \psi$, respectively. Both latter expressions can easily be derived from Lemma 4.1 (i) when taking into account that $P_0\varphi = \varphi$ and $Q_0\psi = \psi$.

The verification of (13) is a little bit more involved. Here we apply Lemma 4.1 (ii) yielding $\varphi_{1,n} = P_0\varphi_{1,n} + Q_0\varphi_{1,n}$, $n \in \mathbb{Z}$, which we rewrite as

$$\varphi_{1,n} = \sum_{k\in\mathbb{Z}} c_{0,k}(\varphi_{1,n})\, \varphi_{0,k} + \sum_{k\in\mathbb{Z}} d_{0,k}(\varphi_{1,n})\, \psi_{0,k}\,.$$

Since $\varphi_{1,2r}(x) = \varphi_{1,0}(x-r)$ and $\varphi_{1,2r+1}(x) = \varphi_{1,1}(x-r)$, $r \in \mathbb{Z}$, we have that

$$\varphi_{1,2r} = \sum_{s\in\mathbb{Z}} [c_{0,s-r}(\varphi_{1,0})\, \varphi_{0,s} + d_{0,s-r}(\varphi_{1,0})\, \psi_{0,s}]\,,$$

$$\varphi_{1,2r+1} = \sum_{s\in\mathbb{Z}} [c_{0,s-r}(\varphi_{1,1})\, \varphi_{0,s} + d_{0,s-r}(\varphi_{1,1})\, \psi_{0,s}]\,.$$

Both relations from above imply (13) when setting $\tilde{h}_{2m} := c_{0,m}(\varphi_{1,0})$, $\tilde{h}_{2m+1} := c_{0,m}(\varphi_{1,1})$ and $\tilde{g}_{2m} := d_{0,m}(\varphi_{1,0})$, $\tilde{g}_{2m+1} := d_{0,m}(\varphi_{1,-1})$ for $m \in \mathbb{Z}$. □

Since φ satisfies the scaling or refinement equation (11) it is named *scaling function*. The scaling equation basically says that φ is a superposition of dilated and translated copies of itself. Relation (12) is most important: the wavelet is given as a sum of scaling functions. Starting with a scaling function there should be a good chance to find a corresponding discrete wavelet. So, in a first step one has to study non-trivial solutions of scaling equations.

Teaching a wavelet course we have now reached the perfect moment to introduce the concept of multiscale (or multiresolution) analysis originating in the work of Mallat and Meyer, see Mallat [4]. Roughly speaking, an MSA is a family $\{V_l\}_{l\in\mathbb{Z}}$ of subspaces of $X = L^2(\mathbb{R})$ satisfying $V_l \subset V_{l+1}$, $\cap V_l = \{0\}$, $\overline{\cup V_l} = X$, and $f(\cdot) \in V_l$ if and only if $f(2^{-l}\cdot) \in V_0$. The latter property is the special feature of MSAs: all spaces V_l are scaled versions of V_0.

The MSAs defined by $V_l = P_l X$, P_l as in (9), are a proven tool to study solutions of scaling equations. From here one can proceed with the detailed construction of orthogonal, semi- and biorthogonal wavelets as demonstrated, e.g., in the literature cited above.

If one gives an overview lecture to an audience which is merely interested in using (and not in constructing) wavelets as a tool like the Fourier transform, then one could immediately go from (11), (12), and (13) to the fast wavelet decomposition and reconstruction algorithms. Though these algorithms can be found in almost any book on wavelets we present them in the next section for the sake of completeness.

5. Fast discrete wavelet algorithms

In practice we do not deal with analytic but discrete signals. We require, then, that only the sampled values $s_n = s(hn)$, $n \in \mathbb{Z}$, of the signal s are available. Here, h is the sampling rate. For convenience let $h = 2^{-L}$ where $L \in \mathbb{N}$ (this is no principal restriction on h). We adapt the present situation to the semi-discrete framework of the former sections by carrying out a pre-processing step.

We consider the s_n's as scaled coefficients of a function f,

$$f(t) := \sum_{n\in\mathbb{Z}} 2^{-l/2}\, s_n \varphi_{L,n}(t)\,, \tag{14}$$

expanded in terms of the scaling function φ. Let us assume that φ belongs to the discrete wavelet ψ in the sense of representation (9). With other words, Theorem 4.2 may be applied.

In general, $f \neq P_L s$, that is, $2^{-L/2}\, s_n \neq c_{L,n}(s)$ for some $n \in \mathbb{Z}$. As already mentioned in the former section we expect, however, that $|2^{-L/2}\, s_n - c_{L,n}(s)|$ becomes smaller with increasing L whenever φ has a normalized mean value, see (10).

Remark: *In case of biorthogonal wavelets, for instance, the error $f - P_L s$ can be estimated analytically for smooth s thereby verifying* (10).

The highest frequency resolvable by the discrete samples s_n is limited by the sampling rate. In the language of the lowpass filters P_l this means that $P_L f = f$: the function f contains no details being smaller than $2^{-L}\, 2\pi/(\omega_0 + \delta_\omega)$. To obtain the wavelet expansion (4) of f only the coefficients $d_{j,n}(f)$ for $j \le L-1$ are required. Those can be computed easily from the samples $c_{L,n}(f) = 2^{-L/2}\, s_n$ as we demonstrate below.

We start by reformulating (13) into the slightly more general version $\varphi_{L,n} = \sum_k [\tilde{h}_{2k-n}\varphi_{L-1,k} + \tilde{g}_{2k-n}\psi_{L-1,k}]$. Now we plug the latter relation into (14) and change the order of summation to get the *discrete wavelet decomposition*

$$d_{L-1,k}(f) = \sum_{n\in\mathbb{Z}} \tilde{g}_{2k-n} c_{L,n}(f) \quad \text{and} \quad c_{L-1,k}(f) = \sum_{n\in\mathbb{Z}} \tilde{h}_{2k-n} c_{L,n}(f)\,. \tag{15}$$

Using both formulae recursively we are able to compute the $d_{j,k}(f)$'s for all scale parameters j smaller than L. Please note that the sequences $d^j = \{d_{j,k}\}_{k\in\mathbb{Z}}$ and $c^j = \{c_{j,k}\}_{n\in\mathbb{Z}}$ are obtained from c^{j+1} by discrete convolutions with the filters $\tilde{g}$ and $\tilde{h}$, respectively, followed by a downsampling by 2.

As easy as the decomposition is the reconstruction algorithm. Here, c^{j+1} is reconstructed from c^j and d^j by an upsampling by 2 followed by discrete convolutions. Specifically, the *discrete wavelet reconstruction* reads

$$c_{j+1,k} = \sum_{n\in\mathbb{Z}} (h_{k-2n} c_{j,n} + g_{k-2n} d_{j,n}) \tag{16}$$

which can be verified using (11) and (12).

In view of (15) and (16) one prefers discrete wavelets and scaling functions having finite and short filter sequences h, g, $\tilde{h}$, and $\tilde{g}$. In this situation the numerical effort for a complete decomposition and reconstruction of a finite set of samples $\{s_n\}_{n\in\mathscr{F}}$ is proportional to the number of samples, see [1, 2, 3, 5] for details. Then, one speaks of the *fast wavelet transform* which is easier to code and even faster than the fast Fourier transform.

A broad variety of discrete wavelets and their corresponding scaling functions are known in the literature which yield fast algorithms. The user has the freedom, which sometimes causes a lot of pain, to select the right pair for her/his specific application. An adequate choice requires a thorough knowledge of the properties of the different wavelets.

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Überabtastung und Rekonstruktion verlorener Werte bandbegrenzter Signale

A bandlimited signal sampled at a sampling rate higher than the Nyquist rate can be reconstructed uniquely from its samples even when a finite number of samples is being lost. This paper gives a solution to this problem which partially originates from Marks [1]. *The solution described here is available in a parallel and a sequential (iterative) form. The parallel implementation requires the solution of a single system of linear equations. Some investigation of the numerical properties of this system is done and a numerical example is given.*

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1. Einführung

In dieser Arbeit soll der Einfluß der Überabtastung auf die Möglichkeit der Rekonstruktion von äquidistant abgetasteten Funktionen bei Verlust endlich vieler Abtastwerte untersucht werden. Bei Abtastung mit der Nyquist-Rate, welcher bei gegebener Bandgrenze α eine Distanz von π/α entspricht, ist eine Rekonstruktion verlorengegangener Werte unmöglich, da die verbleibende Abtastmenge keine Eindeutigkeitsmenge für den zugrundeliegenden Funktionenraum mehr darstellt (siehe [5]). Ist die Abtastrate höher als die Nyquist-Rate, so bleibt die Darstellung der Funktion bei Auslassung endlich vieler Abtastpunkte eindeutig, so daß eine Rekonstruktion möglich wird. Dafür sollen hier ein Approximationsverfahren angegeben und dessen Konvergenzeigenschaften untersucht werden. Die Problemstellung geht auf [1] bzw. [2] zurück, wo allerdings die Frage nach der Lösbarkeit der entsprechenden Gleichungssysteme nicht beantwortet wurde.

2. Problemstellung und Lösung

Es sei f eine beliebige mit der Bandgrenze $n\pi$, $n > 0$ reell, bandbegrenzte Funktion mit endlicher Energie, gegeben durch ihre Abtastwerte auf der Punktmenge k/an, $k \in \mathbb{Z}$. Hierbei sei $a > 1$ eine beliebige reelle Zahl. Der lineare Raum dieser Funktionen sei mit $\mathbf{W}_{n\pi}$ bezeichnet. Die Norm $\|\cdot\|$ in $\mathbf{W}_{n\pi}$ sei die Energienorm, für eine Funktion f gegeben durch

$$\|f\| := \left(\int_{\mathbb{R}} |f(t)|^2 \, \mathrm{d}t\right)^{1/2} . \tag{1}$$