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Optimal Coating of Laser Mirrors for the Generation of Ultrashort Laser Pulses

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Ultrashort laser pulses play an important role in various scientific and industrial applications. For instance, they allow studying the dynamics in fast molecular processes or improve the power of imaging techniques in medical technology, as in optical coherence tomography.

The generation of such short pulses requires techniques for dispersion compensation over an enormous bandwidth. Therefore laser cavities with so-called double chirped mirrors are used. These mirrors consist of a series of alternating high- and low-index layers. The optical properties of the laser cavity depend on the material, the thickness and the number of the layers. So the generation of ultrashort laser pulses can be formulated as an inverse problem:

How have the mirrors to be coated to design laser cavities with prescribed spectral properties?

Mathematically, the coating of a mirror is described by a space (time) dependent refractive index n. The spectral properties of the mirror are modeled by its complex, frequency dependent reflection coefficient r. Now, the inverse problem reads: given r find n. We suggest to solve the inverse problem relying on its relation to an one-dimensional Schrödinger equation.

1 Introduction

The generation of ultrashort laser pulses requires dispersion compensation techniques over an enormous bandwidth. For their realization laser mirrors with coatings consisting of two alternating materials are used. In a simple-chirped mirror the thickness of one layer pair is chirped to higher values, such that longer wavelengths penetrate deeper into the mirror than shorter ones. This is necessary to compensate the dispersion caused by the laser cavity. Unfortunately large oscillations in the group delay occur. To avoid these oscillations a double-chirped mirror is used, where also the local coupling within one layer pair is chirped.

The optical properties of these coatings depend on material, thickness and number of the layers. So the generation of ultrashort laser pulses can be formulated as an inverse problem:

How do the mirrors have to be coated to design laser cavities with prescribed spectral properties?

2 Mathematical Modeling

Mathematically, the coating of a mirror is described by a space (travel time) dependent refractive index $n(z)$. The spectral properties of the mirror are modeled by its complex, frequency-dependent reflection coefficient $r(\omega) = R(\omega) \cdot e^{i \cdot \Phi(\omega)}$, where R is the (real) amplitude and Φ the phase shift depending on the frequency ω.

The optimal mirror coating is then equivalent to the problem finding a refractive index $n(z)$ which decribes the coating at position z to a given $r(\omega)$ which denotes the desired spectral properties.

In recent applications n is only a piecewise continuous function because every coating contains only a discrete number of layers. But in order to apply suitable mathematical theory the problem is solved continuously (in space). This is not a strong restriction from a practical point of view as in near future it will be possible to coat mirrors continuously.

Optical multilayer coatings like double-chirped mirrors (DCM) can be modeled by a reduced form of Maxwell's equations:

$$\begin{aligned} \frac{dE}{dz} &= -i\,\omega\,\mu_0\,H(z), \\ \frac{dH}{dz} &= -i\,\omega\,\varepsilon_0\,n^2(z)\,E(z), \end{aligned} \tag{1}$$

where $E(z)$ and $H(z)$ are the amplitude functions of the electric and the magnetic field at position z and $\mu_0,\ \varepsilon_0$ denote the permeability and permittivity of free space, respectively.

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Elimination of H and the change of variables $x(z) = \int_0^z \sqrt{\mu_0 \varepsilon_0}\, n(u)\, du$ from the physical depth z to the travel time x transform (1) into a Schrödinger equation

$$\frac{\partial^2 y}{\partial x^2}(\omega, x) + (\omega^2 - q(x))\, y(\omega, x) = 0, \quad \text{where} \tag{2}$$

$$\frac{1}{W} \cdot \frac{d^2 W}{dx^2} = q, \quad W^2(x) = \varepsilon_0 c_0 \cdot n(x), \ (c_0 = \text{vacuum light velocity}). \tag{3}$$

3 Design-Algorithm

Equations (2) and (3) propose the steps of a suitable design-algorithm. For given $r(\omega)$ the coefficient q in the Schrödinger equation (2) has to be determined; this is done in the next section. The potential q is the coefficient of a second order differential equation which has to be solved with suitable initial values. From the solution W the desired n can be computed by a simple transformation. Finally back change of variables from x to z has to be done in order to get a space dependent refractive index.

4 Solution of the Inverse Problem

The solution of the inverse problem for q is based on the following idea: Under certain conditions on r equation (2) can be transformed into an integral equation with an explicit solution K determining q uniquely.

Theorem 4.1 *(Gelfand, Levitan[1])*

For every reflection coefficient $r(\omega)$ without poles in the upper half complex plane and with a Fourier transform $F(x)$, continuous in $\mathbb{R}$, the potential q in the Schrödinger equation is given by

$$q(x) = 2\,\frac{dK(x,x)}{dx},$$

where the function K is the unique solution of the Gelfand-Levitan-Marchenko integral equation

$$F(x+y) + K(x,y) + \int_{-y}^{x} K(x,z) F(y+z)\, dz = 0. \tag{4}$$

Theorem 4.2 *(Kay[2])*

If r is a rational function $r(\omega) = r_0 \cdot \prod_{j=1}^{l} (\omega - \mu_j) \Big/ \prod_{k=1}^{m} (\omega - \lambda_k)$ with

- *poles and zeros such that the symmetry condition $r(\omega) = \overline{r(-\omega)}, \ \forall \omega \in \mathbb{R},$ holds (real filter),*
- *$Im(\lambda_k) < 0 \ \forall k \in \{1, \ldots, m\}$ (stable filter),*
- *$l \leq m - 2$,*

then the GLM-integral equation (4) has an explicit (real) solution $K(x,y) = \sum_{\mu=1}^{m} f_\mu(x)\, e^{-i\kappa_\mu y}$.

The m complex constants $\kappa_\mu, \ \mu = 1, \ldots, m$, are roots of the polynomial equation $1 - r(\omega) r(-\omega) = 0$. The functions $f_\mu(x)$ can be determined for every fixed x as the unique solutions of a linear system.

5 Numerical Realization

The goal to find an explicit solution for q in order to avoid early numerical approximations requires an input function r with a special structure, i.e. the modeling of r is a very important part of the algorithm.

Experimental data for $r(\omega)$ are given by engineers over a taylor series expansion for r about a centre frequency ω_0. It is now to find a rational function $\tilde{r}$ approximating this data. Therefore, in a first step a rational with symmetric poles and zeros is computed to get a real filter. Second, this filter is stabilized by reflecting the poles with positive imaginary part at the real axis to get a real and stable filter. The final result $\tilde{r}$ is the input for the design-algorithm. Our first numerical results are promising.

References

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