An Overview on Mathematical Methods in Tomography

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Preface

Focus: 2D X-ray computerized tomography

CT variants not addressed here:

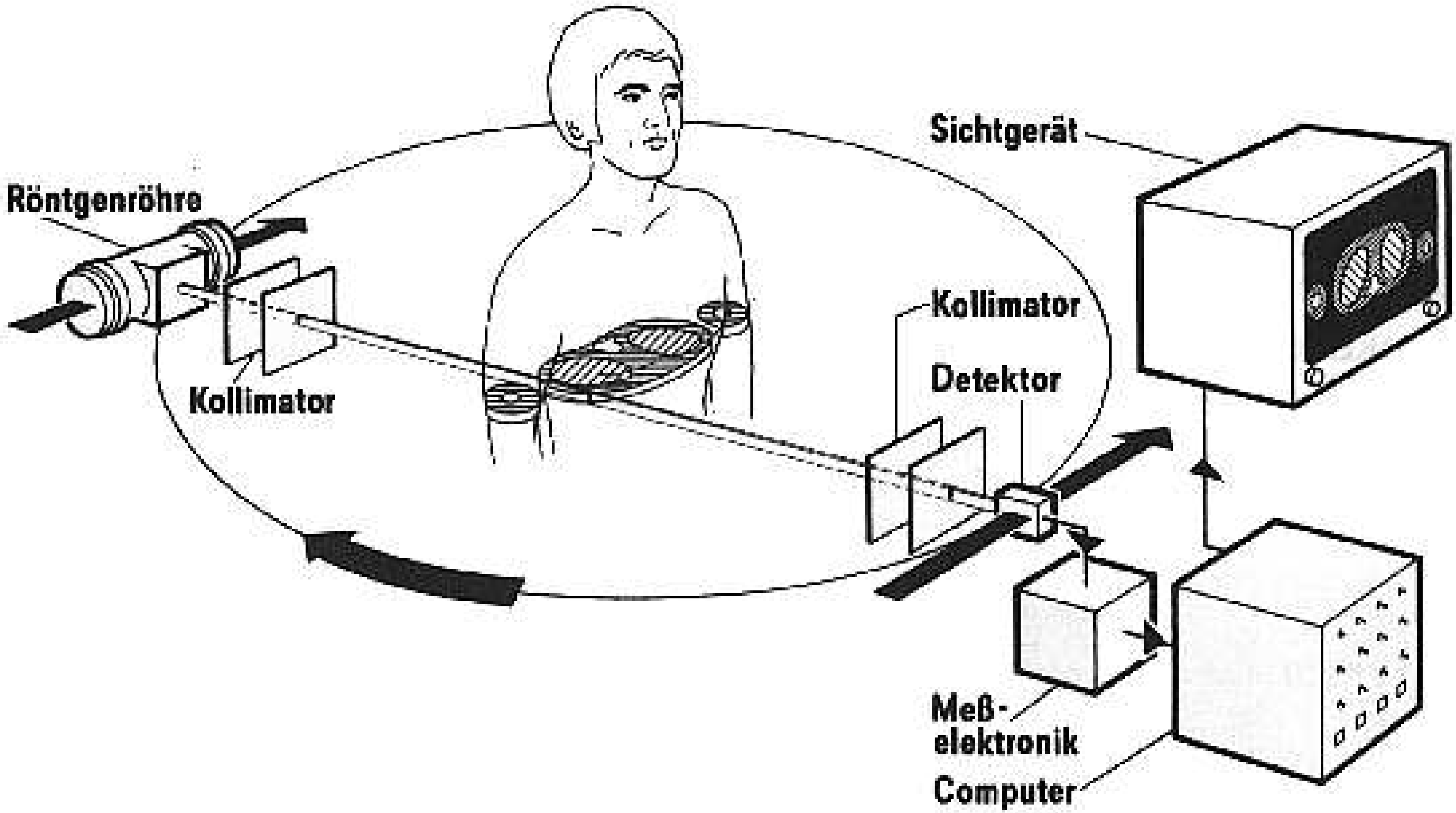
- 3D CT
- SPECT
- Doppler CT
- Diffusive (optical) CT
- MR imaging
- Impedance CT
- Ultrasound CT

F. Natterer, F. Wübbeling: *Mathematical Methods in Image Reconstruction*, SIAM 2001

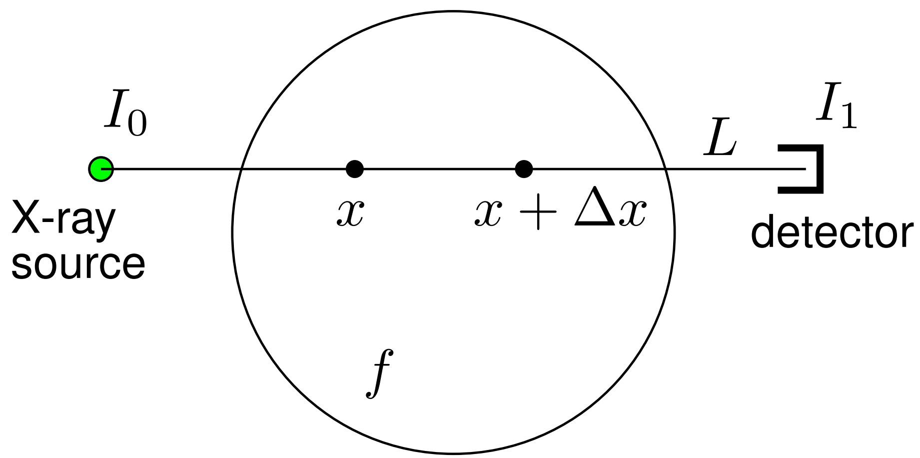
Contents

- Mathematical model for CT: the Radon transform
- Inversion formula: global and local tomography
- Non-uniqueness for discrete data
- Approximate inversion
- Filtered backprojection algorithm
- Computational examples

Principle of CT scanning device



The mathematical model



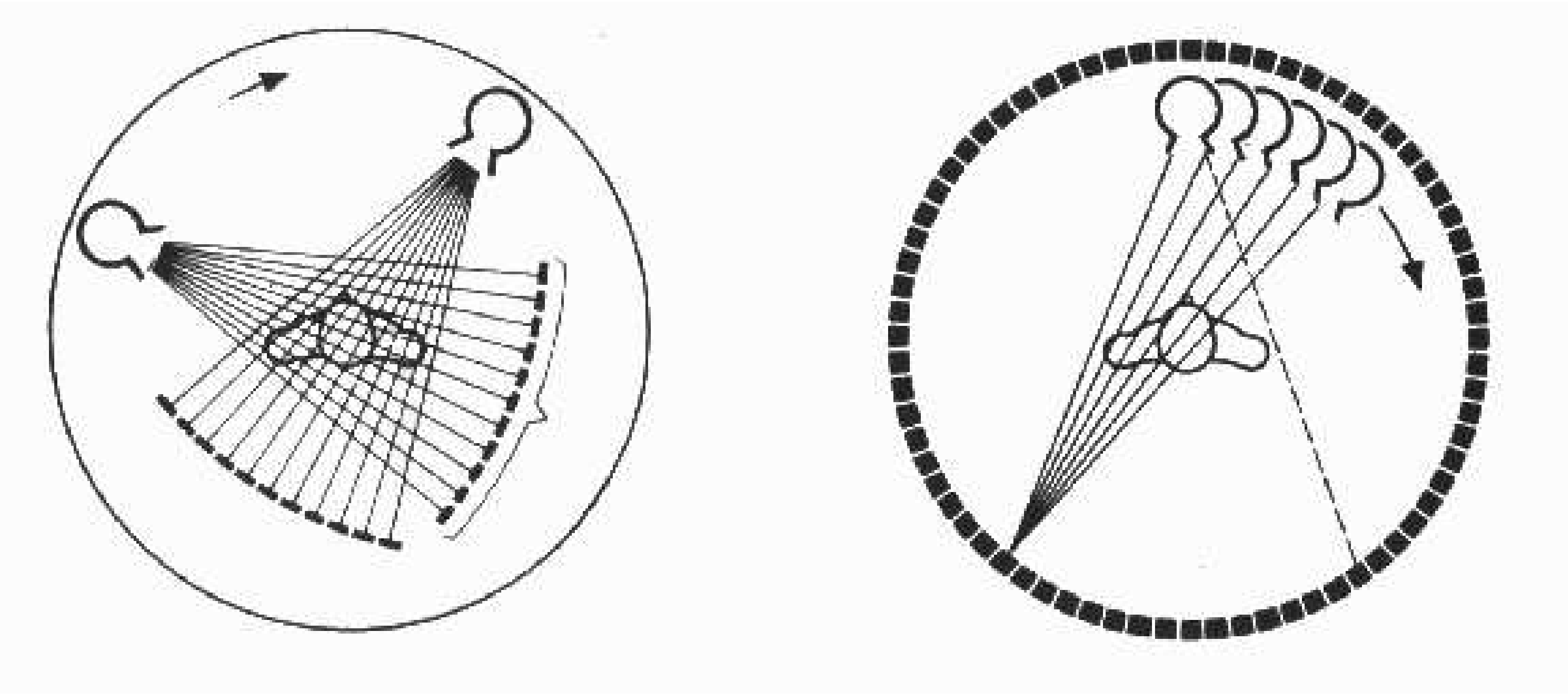
physical assump.: $I(x + \Delta x) - I(x) = -f(x)\,\|\Delta x\|\,I(x)$

$$\frac{I(x + \Delta x)\; - \;I(x)}{\|\Delta x\|} \;=\; -f(x)\;I(x)$$

$$\Delta x \to 0 \implies \partial_L \ln I(x) \;=\; -f(x)$$

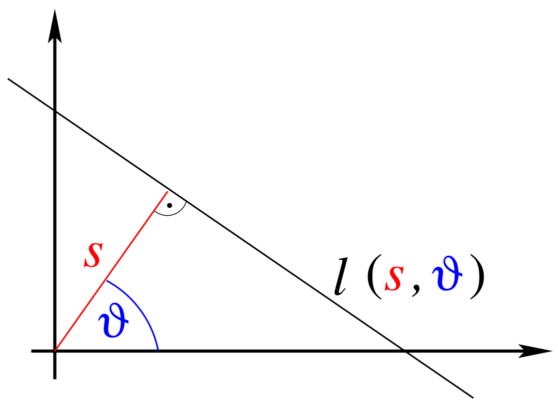
$$\boxed{\int_L f(x)\,\mathrm{d}\sigma(x) \;=\; \ln\bigl(I_0/I_1\bigr)}$$

CT scanning geometries



2D-Radon-Transform (parallel scanning geometry)

$$\mathbf{R}f(s,\vartheta) := \int_{l(s,\vartheta)\cap\Omega} f(x)\,\mathrm{d}\sigma(x)$$



tomographic inversion: $\mathbf{R}f(s,\vartheta) = g(s,\vartheta)$

$$\mathbf{R} : L^2(\Omega) \to L^2(Z), \quad Z = [-1,1] \times [0,\pi]$$

Johann Radon 1917, A. M. Cormack 1963, G. N. Houndsfield 1967

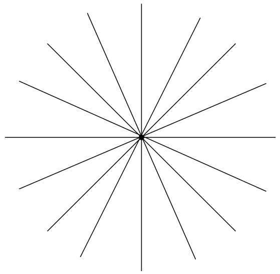
Inversion formula

Riesz potential $\Lambda^\alpha : H^t(\mathbb{R}^d) \to H^{t-\alpha}(\mathbb{R}^d)$, $\alpha > -d$

$$\widehat{\Lambda^\alpha f}(\xi) := \|\xi\|^\alpha \, \widehat{f}(\xi), \qquad \Lambda^\alpha = (-\Delta)^{\alpha/2}$$

backprojection $\mathbf{R}^* : L^2(Z) \to L^2(\Omega)$

$$\mathbf{R}^* g(x) = \int_0^\pi g(x^t\,\omega(\vartheta), \vartheta)\,\mathrm{d}\vartheta$$



$$\boxed{\Lambda^\alpha f = \frac{1}{2\pi}\,\mathbf{R}^*\Lambda_s^{1+\alpha}\,\mathbf{R}f, \quad f \in L^2(\Omega)}$$

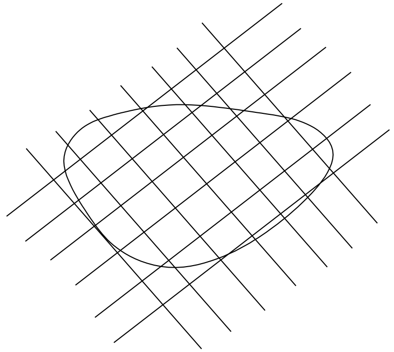
$\alpha = 0$: Radon 1917, general result: Smith, Solomon and Wagner 1977

Global and local tomography

$$\Lambda^\alpha f = \frac{1}{2\pi} \mathbf{R}^* \Lambda_s^{1+\alpha} \mathbf{R} f$$

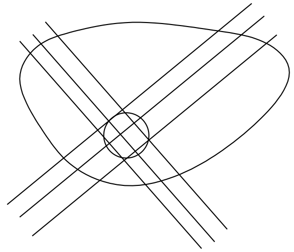
$\alpha = 0$: $\quad \Lambda_s = \mathcal{H} \frac{\mathrm{d}}{\mathrm{d}s}$, $\quad \mathcal{H}$ Hilbert transform

inversion formula for f is global



$\alpha = 1$: $\quad \Lambda_s^2 = -\frac{\mathrm{d}^2}{\mathrm{d}s^2}$, $\quad$ sing supp $\Lambda f \subset$ sing supp f

inversion formula for Λf is local



Local tomography

f

Λf

Non-Uniqueness for discrete data

Smith et al. 1977: s_i, $i = 1, \ldots, q$, ϑ_j, $j = 1, \ldots, p$

$$\exists f \neq 0 : \quad \mathbf{R}f(s_i, \vartheta_j) \;=\; 0 \quad \forall\, i, j$$

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Natterer 1980: (s_i, ϑ_j) rectangular grid with $h = 2/q = \pi/p$

$$f \text{ ghost} \quad \Longrightarrow \quad \|f\|_{L^2} \lesssim h^\beta \, \|f\|_{H_0^\beta}\,, \quad \beta > 1/2$$

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Louis 1984: $0 < \tau < 1$

$$f \text{ ghost} \quad \Longrightarrow \quad \int\limits_{|\xi|\leq\tau(p-1)} |\widehat{f}(\xi)|\, \mathrm{d}\xi \lesssim \mathrm{e}^{-\lambda(\tau)\,p}\, \|f\|_{L^1}$$

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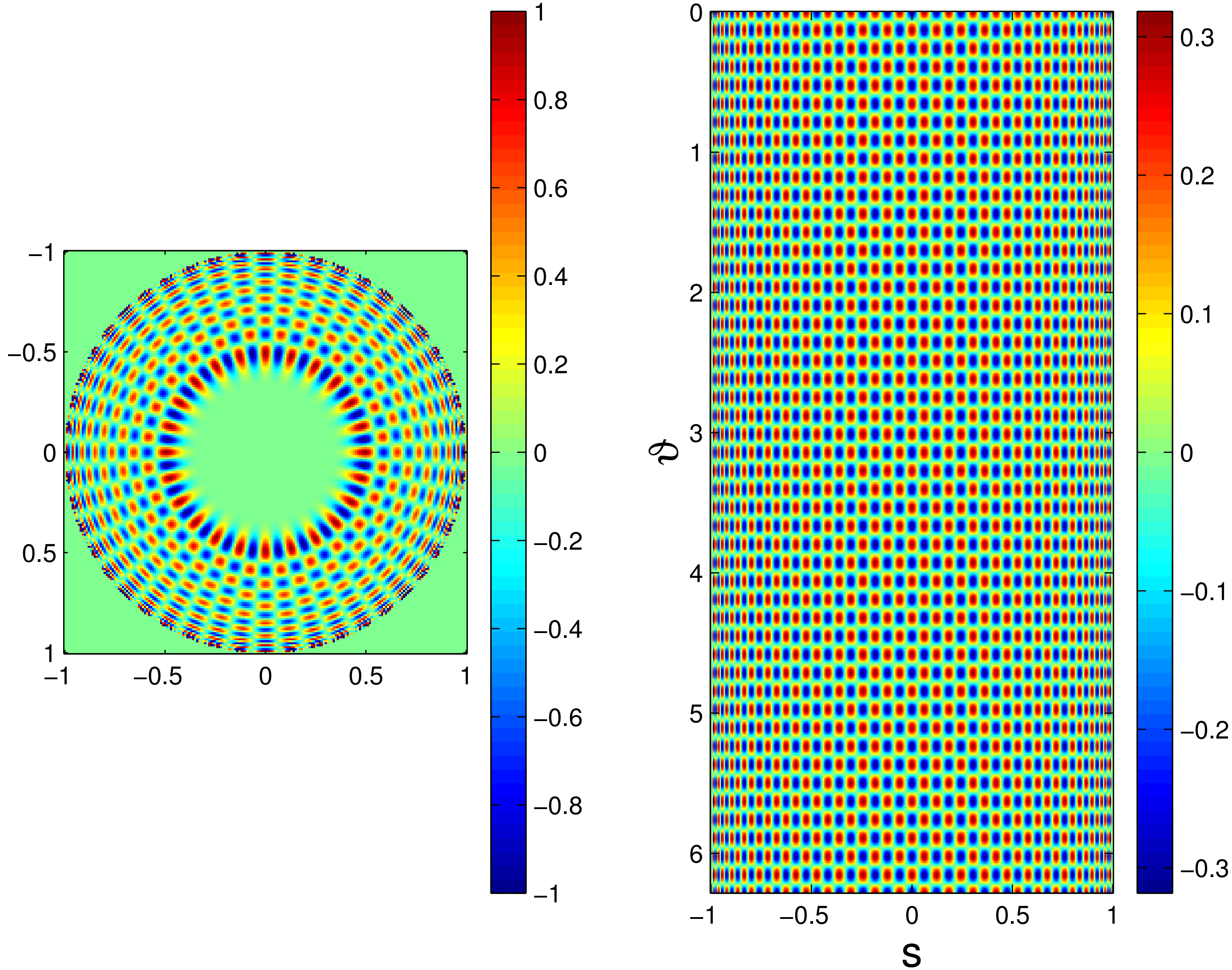
Further analytical aspects: stability, sampling and resolution

CT ghost



−1
−0.5
0
0.5
1
−1
−0.5
0
0.5
1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
0
1
2
3
4
5
6
ϑ
−1
−0.5
0
0.5
1
s
0.3
0.2
0.1
0
−0.1
−0.2
−0.3

Approximate inversion I

inversion formula: $f = \frac{1}{2\pi} \mathbf{R}^* \Lambda_s g \qquad g = \mathbf{R} f$

approx. inversion: $f \star e = \mathbf{R}^* (\upsilon \star_s \mathbf{R} f), \quad e = \mathbf{R}^* \upsilon$

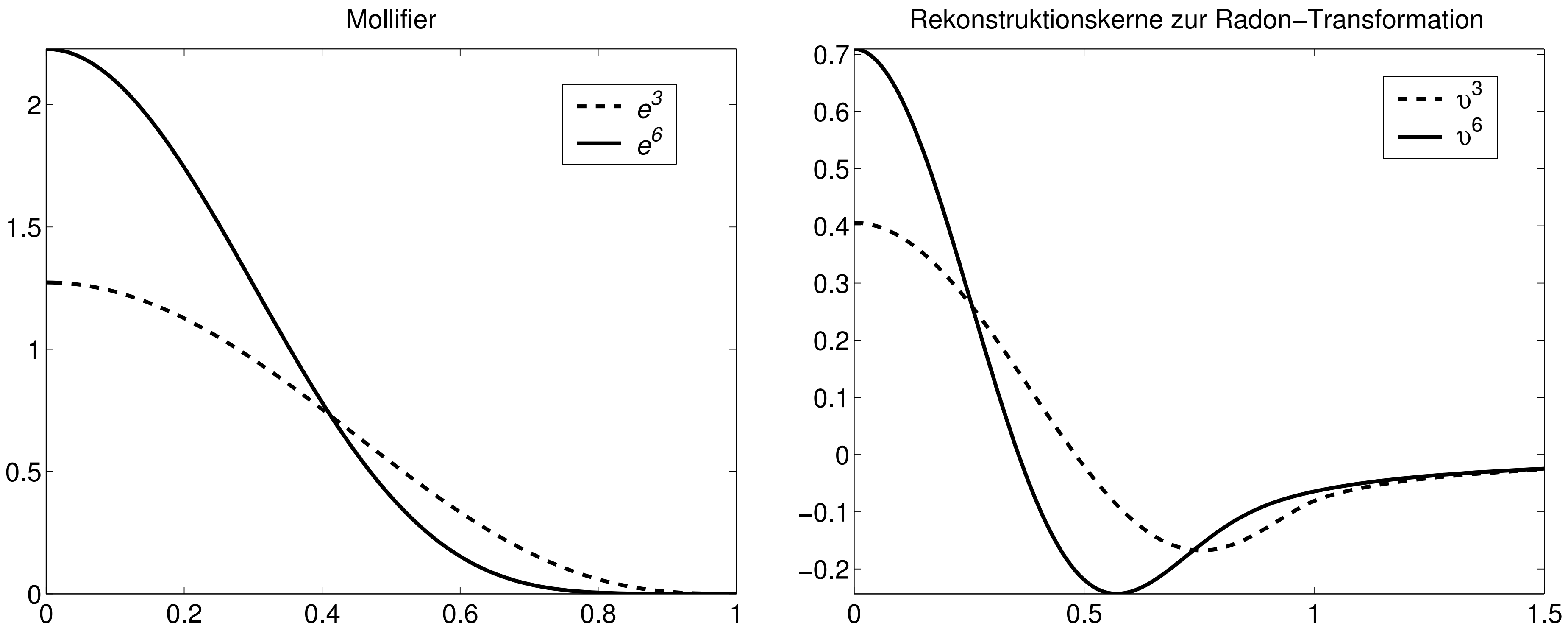
e mollifier ($e \approx \delta$, centered about 0 with mean value 1)
υ reconstruction filter/kernel

$$\upsilon = (2\pi)^{-1} \Lambda_s \mathbf{R} e \quad \Longrightarrow \quad e = \mathbf{R}^* \upsilon$$

$$e_\gamma(x) = \gamma^{-2} e(x/\gamma), \quad \upsilon_\gamma(s) = \gamma^{-2} \upsilon(s/\gamma), \quad \gamma > 0$$

$$f \star e_\gamma \to f \text{ as } \gamma \to 0$$

Approximate inversion II



Reconstruction algorithm I

approx. inversion: $f \star e_\gamma \;=\; \mathbf{R}^* \left(\upsilon_\gamma \star_s g\right)$

discrete data: $g_{\ell,j} = \mathbf{R} f(\ell/q,\, j\,\pi/p), \quad \ell = -q, \ldots, q, \quad j = 0, \ldots, p-1$

filtered backprojection: $f_R(x) = \mathbf{R}_p^* \big(\upsilon_\gamma \star_h g\big)(x)$

$$(\upsilon_\gamma \star_h g)_{k,j} = h \sum_{\ell=-q}^{q} \upsilon_\gamma\big(h(k-\ell)\big)\, g_{\ell,j}, \qquad h = 1/q$$

How to choose γ in relation to h?

Reconstruction algorithm II

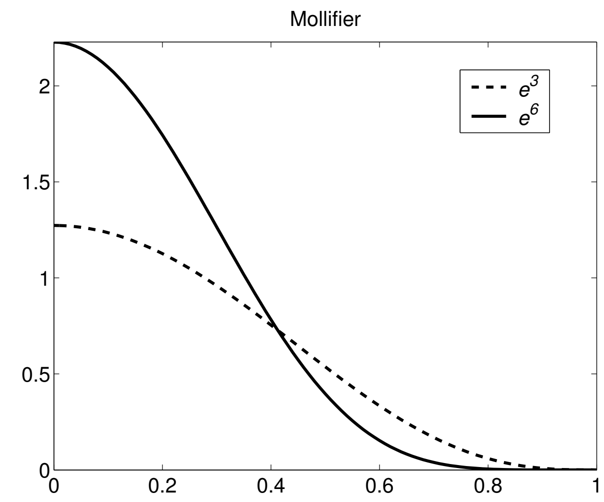
R. 2000: f essentially b-band-limited and $h \leq \pi/b$

$$f_R = f \star e_\gamma \; + \; \mathrm{m}(\gamma, h) \; \Lambda^{-1} f \; + \text{ discr. error}$$

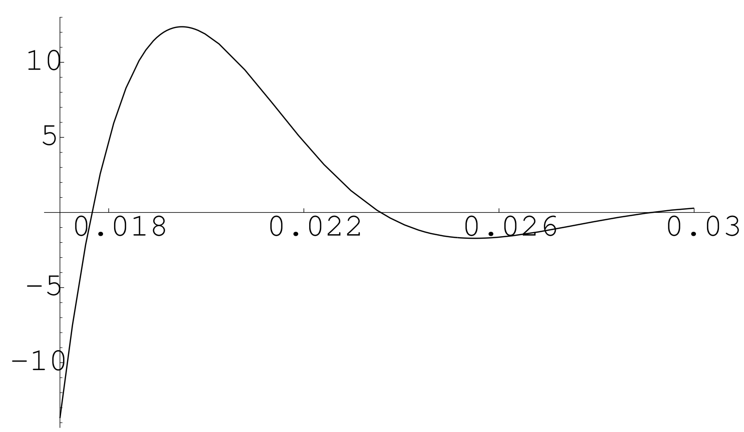
Strategy: Determine $\gamma = \gamma_h$ as a zero of $\mathrm{m}(\cdot, h)$, that is,

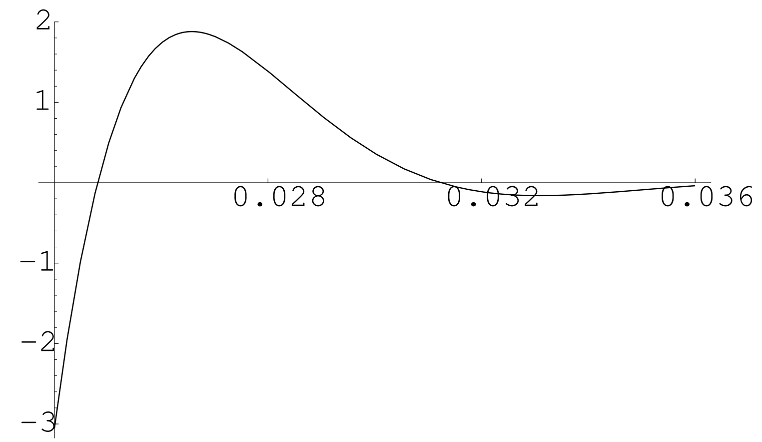
$$\mathrm{m}(\gamma_h, h) \; = \; 0$$

Reconstruction algorithm III

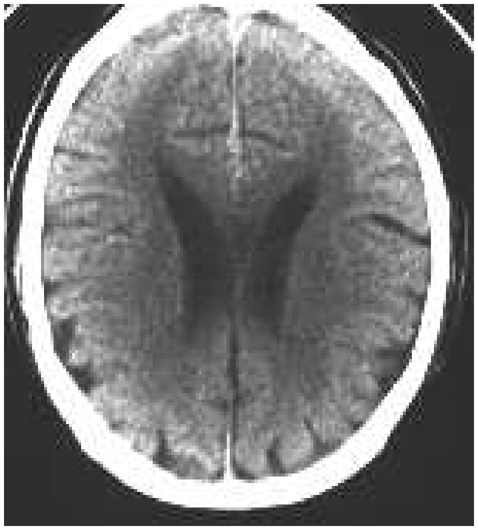


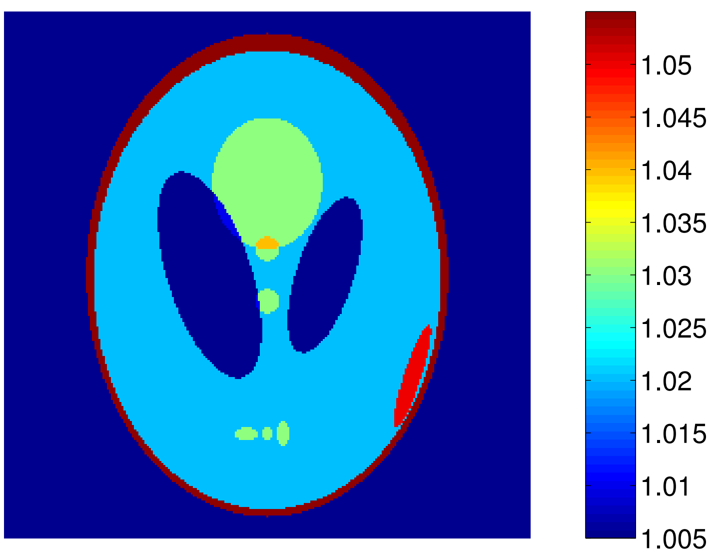
$$h = 0.01$$

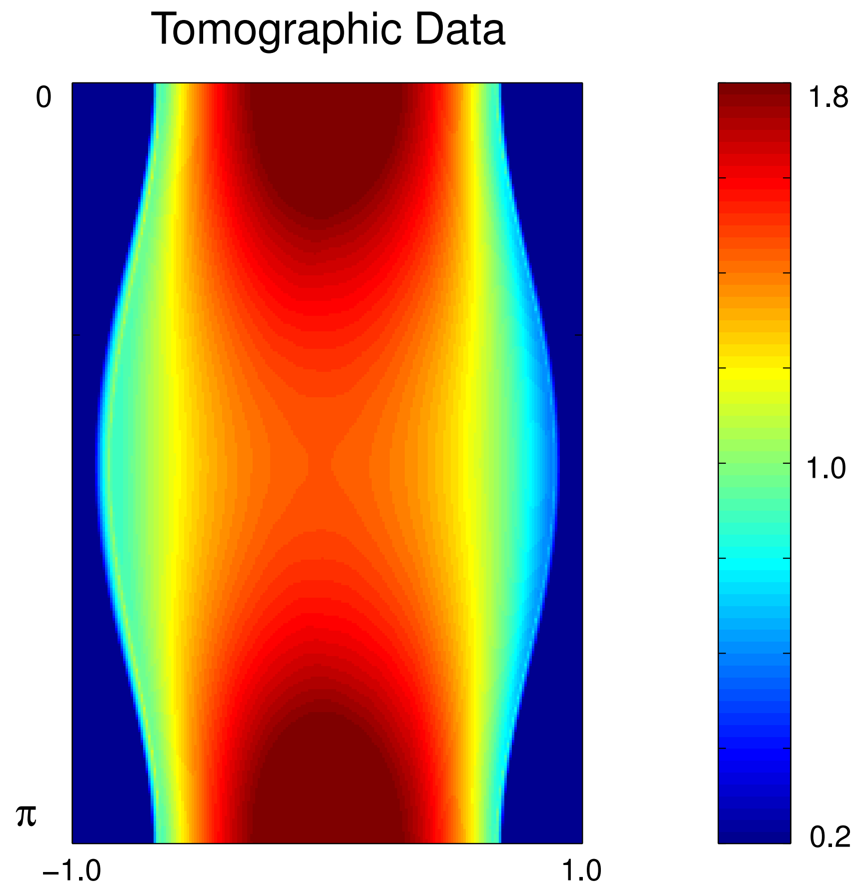




Shepp-Logan head phantom

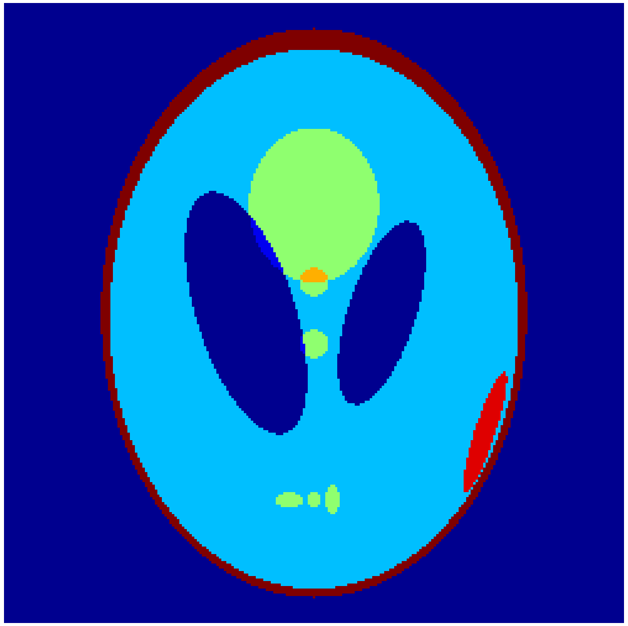




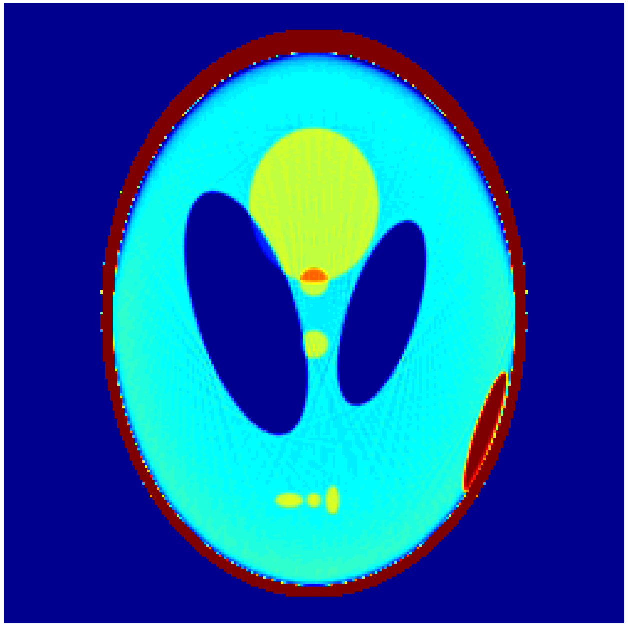


Reconstructions I

$$e(x) = \begin{cases} (1-\|x\|^2)^6 & : \quad \|x\| \le 1 \\ 0 & : \quad \text{otherwise} \end{cases}, \qquad h = 0.01$$

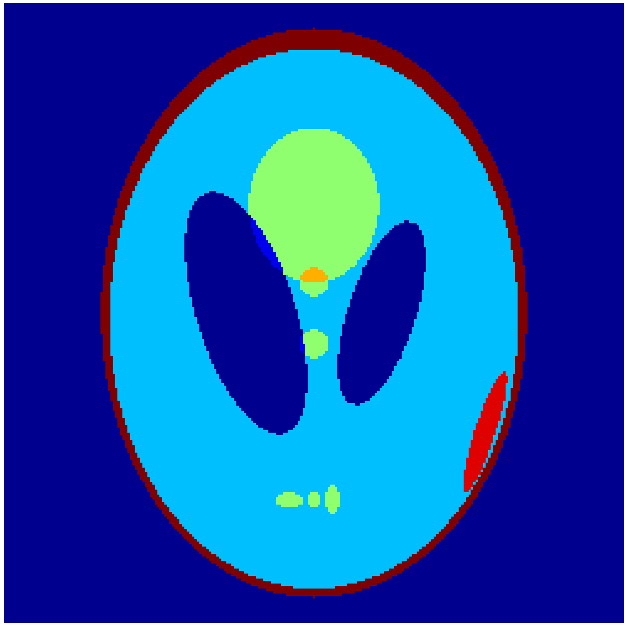


original

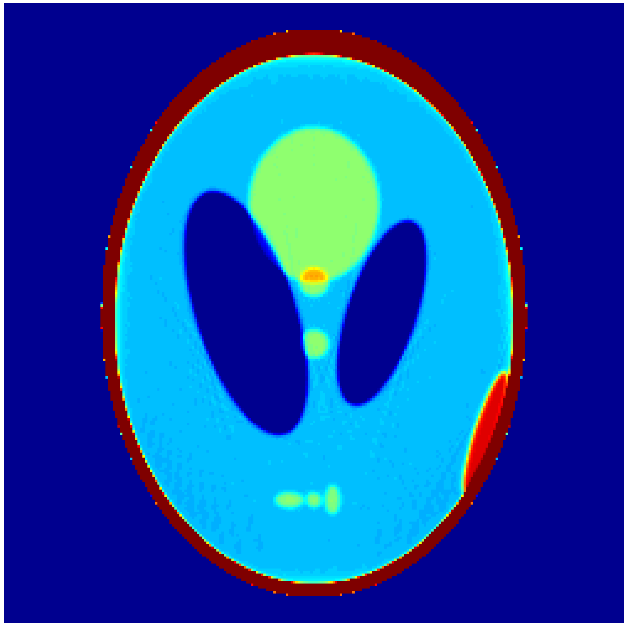


$\gamma = 0.01765299..$
(sm. zero of m),
rel. ℓ^2-error: 0.0816

Reconstructions II

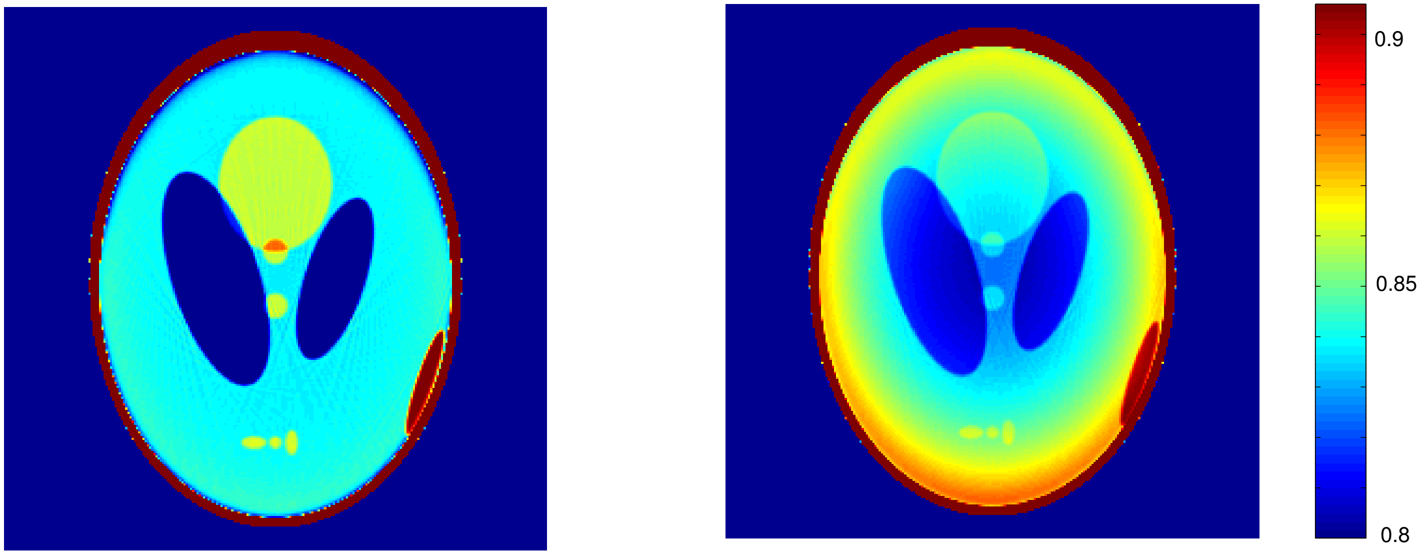


original



$\gamma = 0.02357177..$
(2nd sm. zero of m),
rel. ℓ^2-error: 0.1001

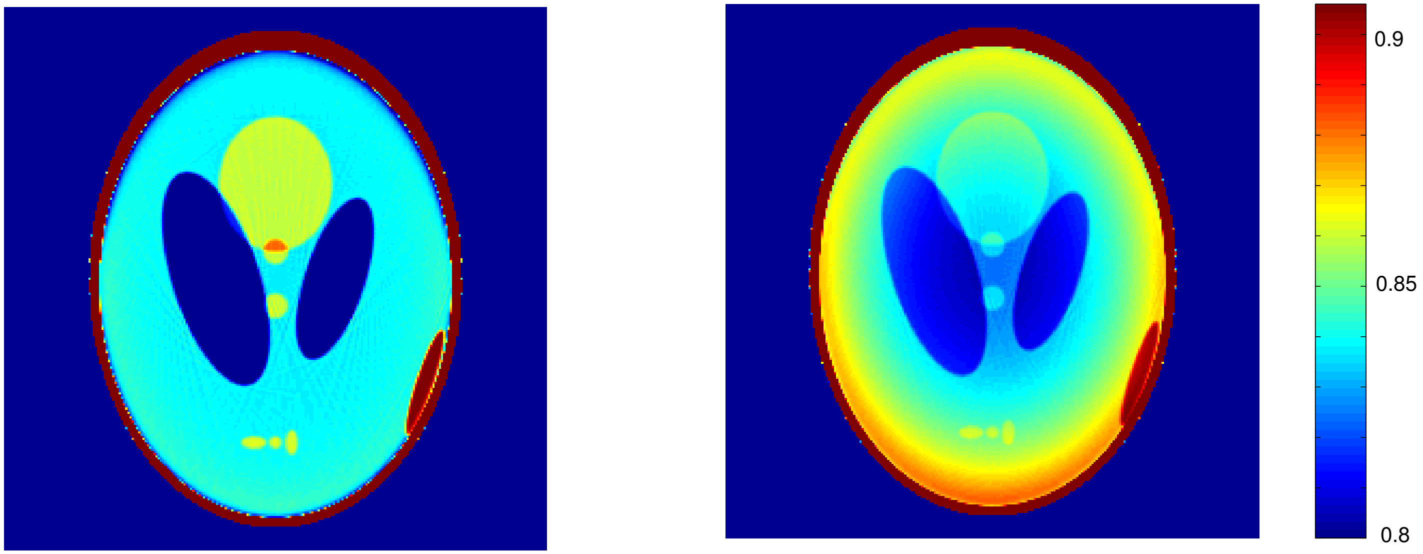
Violating the zero condition



$\gamma = 0.01765299..$
rel. ℓ^2-error: 0.0816

$\gamma = 0.0177..$
rel. ℓ^2-error: 0.1730

Violating the zero condition



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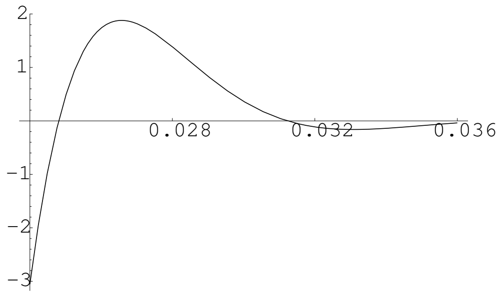

$$f_R = f \star e_\gamma + m(\gamma, h)\, \Lambda^{-1} f + \text{discr. error}$$