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Non-Smooth Motion of Rotor Systems with Frictional Contacts

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ABSTRACT

Many structural imperfections such as cracks or delaminations of plies in composites can be typically modeled as unilateral constraints, in particular if breathing of the gaps due to these defects is involved. Imperfections of such a kind are also of interest in rotor dynamics. The described problems basically have in common that the mathematical model results in systems of Differential-Algebraic-Equations that are valid only almost everywhere due to switching conditions. Especially for the usage in parameter estimation an efficient simulation of such systems is imperative and hence modal model reduction techniques are often applied. Model reduction with underlying continuous systems introduces artificial rigidity. In conjunction with unilateral contact problems this requires– as known for systems of rigid bodies– a consistent calculation of the non-smooth transition from one contact configuration to the upcoming one. This paper presents an efficient method to deal with this problem. Upon applying Hamilton's Principle and spatial discretization, the equations of motion are derived comprising both normal and tangentially frictional contact regions.

KEY WORDS

Delaminated Rotors, unilateral constraint, Hamilton's Principle, Frictional Contacts.

1 INTRODUCTION

Systems with friction and normal contact play an important role in machinery dynamics. As publications give evidence, many issues in rotor dynamics are associated with contact problems:

- problems of rotor systems with unilateral constraints due to normal contact such as rotors with cracks, cp. review [17], or delamination, e.g. [19, 18];
- problems with frictional contacts due to rotor-stator rubbing, e.g. [20, 5] or due to loose parts impacting on rotating parts, e.g. [4];
- stability problems due to internal friction, however, often modeled as viscous damping, e.g. [7, 6].

Most of the publications contribute to this class of problems in rotor dynamics with simplifying assumptions on the contact situation, which are not necessarily inaccurate, but could be difficult to modify as to dealing with more complex situations. A modeling method of such a kind is to employ stiff elastic structural members like unilateral spring-dashpot elements to model normal contact, i.e., a force can only be transmitted if it is not a tensile one. Apart from the fact that the ideal contact with $\textit{stiffness} \rightarrow \infty$ is impossible to simulate in finite time, there arise other problems even for finite parameters. It is reported in [12] (and references therein) that parameter tuning of spring-dashpot contact elements to fit simulations to measurements of harmonically forced bending vibrations of mutually contacting beams was infeasible in general. A set of parameters turned out to be very sensitive to the excitation frequency and hence cannot be interpreted as a set of pure material constants. Such modeling techniques tend to be tuning of parameters rather than deriving parameters from physical models. This is common practice and not meant to be criticized in general. However, it might be recommendable to use more abstract tuning parameters that are more convenient to apply. This benefit costs the lack of a direct physical interpretation. In conjunction with vibro-impact problems it is often proposed, as for instance in [8] and eventually in [12], to use

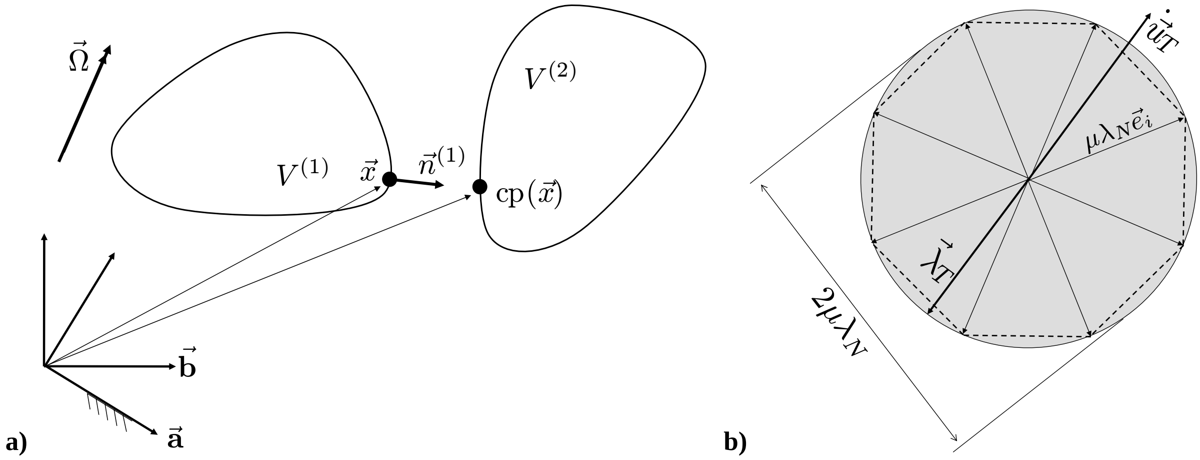

Figure 1: **a)** Sketch of potential contact. **b)** Coulomb friction disc and polygonal approximation (dashed).

restitution coefficients. One obvious benefit compared to spring-dashpot elements is the reduction of the number of parameters to be tuned from 2 to 1.

The drawback of restitution methods are the difficulties that arise with multiple contacts. The local application of restitution methods, i.e., only at that place where the contact happens, in general yields erroneous results in rigid-body dynamics, since the results– in some cases– can only be explained by supposing tensile contact forces, which is an inconsistence. It is also known for a long time that the local application of unilateral impenetrability conditions is another source of inconsistency with rigid bodies: The conclusion that contact no longer persists if the required contact force becomes a tensile one is in general not correct, compare [3]. Theoretical aspects of existence and uniqueness of consistent solutions of unilaterally constrained systems are considered in [11, 15, 10], for the application of consistent restitution methods in rigid-body dynamics see [14], for instance.

As will be seen, the same problems arise with continuous structures after discretization. Many investigations in dynamics of continuous rotor systems base on discretized models, be it modally discretized structures, FE-structures, etc.. It seems therefore worthwhile to reproduce the theory of non-smooth systems with multiple frictional contacts particularly with regard to continuous rotor systems.

The paper is organized as follows:

- Derivation of the discrete equations of motion of continuous structures with unilateral frictional contacts in a rotating reference frame. This is done upon application of Hamilton's Principle and discretization of the variational formulation according to the Ritz method.
- Investigation of stationary solutions. It turns out that centrifugal effects significantly affect the stationary solutions. The formulation is extended so that it can take some spacial cases of contact friction into account even for stationary solutions.
- Derivation of consistent transition conditions for the non-smooth motion.
- Presentation of a modeling example: Rotating shaft with through-width delamination.

2 EQUATIONS OF MOTION

In what follows, the theory of frictionless unilateral constraints for rotor systems in [16] is briefly reproduced, however, now with friction taken into account.

2.1 Hamilton's Principle for rotating systems with frictional unilateral contacts

Figure 1a depicts an inertial system $\vec{\mathbf{a}} = (\vec{a}_1\,\vec{a}_2\,\vec{a}_3)^\intercal$ and a rotating (constant angular velocity $\vec{\Omega}$) frame $\vec{\mathbf{b}} = (\vec{b}_1\,\vec{b}_2\,\vec{b}_3)^\intercal$, each with orthonormal basisvectors $\vec{a}_k, \vec{b}_k$, so that $\vec{\mathbf{a}}\vec{\mathbf{a}}^\intercal = \vec{\mathbf{b}}\vec{\mathbf{b}}^\intercal = \mathbf{I} \in \mathbb{R}^{3\times 3}$ equals the identity matrix. It is thereby assumed that the scalar products between the vectors are evaluated according to the rules of matrix multiplication.

Given two elastic bodies $V^{(k)}, k = 1, 2$, or two separate parts of one body, whose boundaries $\partial V^{(k)}$ are subdivided into disjoint sets $\Gamma_\sigma^{(k)}$, $\Gamma_u^{(k)}$,$\Gamma_c^{(k)}$, where stress-, displacement- or impenetrability boundary conditions are applied, respectively. The potential contact point is formally denoted by the *closest-point-operator* $\mathrm{cp}(\cdot)$. The impenetrability condition can thus be expressed by a gap function g as

$$g(\vec{x}, \vec{u}(\vec{x})) \geq 0 \text{ on } \Gamma_c^{(1)} \tag{1}$$

with the related definition and variation

$$\text{cp}(\vec{x}) = \arg\min_{\vec{y}\in V^{(2)}} |\vec{x} - \vec{y}|\,, \tag{2}$$

$$g\left(\vec{x}, \vec{u}\left(\vec{x}\right)\right) = -\left(\vec{x} + \vec{u}^{(1)}\left(\vec{x}\right) - \text{cp}(\vec{x}) - \vec{u}^{(2)}\left(\text{cp}(\vec{x})\right)\right) \cdot \vec{n}^{(1)}, \tag{3}$$

$$\delta g = n^{(1)} \cdot \left(-\delta\vec{u}^{(1)} + \delta\vec{u}^{(2)}\right) \tag{4}$$

wherein $\vec{u}^{(k)}, \vec{n}^{(1)}$ are displacement vectors and normal unit vectors, respectively. Omitted superscribting is meant to be read: $\vec{u}(\vec{x}) = \vec{u}^{(k)}$ if $\vec{x} \in V^{(k)}$.

Hamilton's Principle reads

$$\int_{t_0}^{t_1} (\delta(T - U) + \overbrace{(W_0 + W_c)}^{W_{\text{virt}}})\, dt = 0, \tag{5}$$

with kinetic energy T and elastic potential energy U, the latter one as usual expandable to comprise other forces that have a potential and that are not taken into account in the virtual work W_{virt}.

Upon partial integration in the time domain and exploiting vanishing variations at the time boundaries t_0, t_1, the variation of the kinetic energy can also be written as

$$\delta T = -\int_V \rho\delta\vec{r} \cdot \ddot{\vec{r}}\, dV \tag{6}$$

with density ρ and $V = V^{(1)} \cup V^{(2)}, \vec{r} = \vec{x} + \vec{u}$, Lagrangian reference point $\vec{x}$. Expressing the displacement with respect to the rotating frame and thereafter calculating acceleration and the variation yields

$$\vec{u} = \vec{\mathbf{b}}^{\mathsf{T}}\mathbf{u};\ \mathbf{u} = (u_1\, u_2\, u_3)^{\mathsf{T}}, \tag{7}$$

$$\ddot{\vec{r}} = \vec{\mathbf{b}}^{\mathsf{T}}\ddot{\mathbf{u}} + 2\vec{\Omega} \times \vec{\mathbf{b}}^{\mathsf{T}}\dot{\mathbf{u}} + \vec{\Omega} \times (\vec{\Omega} \times (\vec{x} + \vec{\mathbf{b}}^{\mathsf{T}}\mathbf{u})), \tag{8}$$

$$\delta\vec{r} = \delta\mathbf{u}^{\mathsf{T}}\vec{\mathbf{b}}. \tag{9}$$

The virtual work in Eq. (5) is the sum of the usual part W_0, which does not take the contact into account, and the part W_c that includes traction on the contact boundary:

$$W_0 = \sum_{k=1,2}\left(\int_V \vec{f} \cdot \delta\vec{u}\, dV + \int_{\Gamma_\sigma} \vec{t}^{\,*} \cdot \delta\vec{u}\, d\Gamma\right)^{(k)}, \tag{10}$$

$$W_c = \sum_{i=1,2}\int_{\Gamma_c^{(i)}} \vec{t}^{\,(i)} \cdot \delta\vec{u}^{(i)}\, d\Gamma \tag{11}$$

$$= \int_{\Gamma_c^{(1)}} (\lambda_N\delta g + \vec{\lambda}_T \cdot \delta\vec{u}_T)\, d\Gamma. \tag{12}$$

Herein is $\vec{f}$ a given volume force in V, is $\vec{t}^{\,*}$ the prescribed stress vector on Γ_σ and is $\vec{t}$ the unknown stress vector in the still unknown contact zone Γ_c. Equation (12) follows from $\vec{t}^{\,(2)} = -\vec{t}^{\,(1)}$, i.e., *Actio=Reactio* and from the decomposition of the stress vector into normal and tangential part. Eventually, Eq. (12) follows from Eq. (4).

The still unknown contact zone can be made constant by expanding it to $\Gamma_c^{(1)} = \partial V^{(1)} \setminus \Gamma_\sigma^{(1)} \setminus \Gamma_u^{(1)}$ and thereby demanding

$$\lambda_N \geq 0, g \geq 0, \lambda_N g = 0, \tag{13}$$
$$\dot{\vec{u}}_T \cdot \vec{\lambda}_T \rightarrow \text{Min}, \tag{14}$$
$$\mu\lambda_N - |\vec{\lambda}_T| \geq 0 \tag{15}$$

for all $\vec{x} \in \Gamma_c^{(1)}$. The optimization problem (14) with auxiliary condition (15) is an alternative expression for the well known Coulomb friction law. Either one states, see **Fig. 1b**, that the friction force $\vec{\lambda}_T$ is antiparallel to the relative sliding velocity $\dot{\vec{u}}_T$ in case of sliding, whereas in case of sticking($|\dot{\vec{u}}_T| = 0$) the magnitude of the constraint force is bounded by $\mu\lambda_N > 0$ with the friction coefficient μ. The formulation as optimization problem (14, 15) is known as the *Maximum Dissipation Principle* [15].

Equations (13-15) ensure that the integrant in Eq. (12) does not contribute to the integral value if the contact is open, i.e. $g > 0$, since the third condition in Eq. (13) ensures in this case that $\lambda_N = 0$ and thus, with Eq. (15), that $|\vec{\lambda}_T| \leq 0$, which is only possible for $\vec{\lambda}_T = \vec{0}$. Equations (13) are called the *Signorini* complementarity conditions. Together with Eqs. (14, 15) they define the frictional contact problem according to Coulomb's friction law.

2.2 Discretization

The discretization process consists of two tasks:
1.) Discretization of the displacements $\mathbf{u}$ and referring variations in Eq. (7):

$$\mathbf{u} = \mathbf{\Phi}(\vec{x})^\mathsf{T}\mathbf{q}(t),\ \delta\mathbf{u}^\mathsf{T} = \delta\mathbf{q}^\mathsf{T}\mathbf{\Phi}(\vec{x}), \tag{16}$$

with appropriately chosen admissible ansatz functions $\mathbf{\Phi}_k(\vec{x}) \in \mathbb{R}^3$, $k = 1, \ldots, n$, which are concatenated in $\mathbf{\Phi}(\vec{x}) = (\mathbf{\Phi}_1(\vec{x}), \ldots, \mathbf{\Phi}_n(\vec{x}))^\mathsf{T} \in \mathbb{R}^{3\times n}$ and n generalized coordinates $\mathbf{q} = (q_1, \ldots, q_n)^\mathsf{T}$.
2.) Discretization of the unknown contact interaction forces in normal and tangential direction and the referring contact laws (13-15). The gap function is thereby assumed to be given in a linear formulation after disretization, which is a reasonable assumption on little gap breathing. The referring normal contact force is discretized by appropriate ansatz functions as well, whereby the column matrix $\boldsymbol{\lambda}_N$ contains their weighting factors:

$$g = \mathbf{g}_N(\vec{x})^\mathsf{T}\mathbf{q} + g_N(\vec{x}),\ \delta g = \delta\mathbf{q}^\mathsf{T}\mathbf{g}_N(\vec{x}), \tag{17}$$
$$\lambda_N(\vec{x}, t) = \mathbf{\Lambda}_N(\vec{x})^\mathsf{T}\boldsymbol{\lambda}_N(t),\ \mathbf{\Lambda}_N(\vec{x}) = (\Lambda_{N_1}(\vec{x}), \ldots, \Lambda_{N_m}(\vec{x}))^\mathsf{T} \in \mathbb{R}^m,\ \boldsymbol{\lambda}_N = (\lambda_{N_1} \ldots \lambda_{N_m}). \tag{18}$$

The SIGNORINI inequality conditions (13) are approximated by discretization through weighted averaging in $\Gamma_c^{(1)}$ to ensure that the inequalities are at least satisfied on an average:

$$\underbrace{\int_{\Gamma_c^{(1)}} \mathbf{\Lambda}_N \mathbf{g}_N^\mathsf{T}\, d\Gamma}_{\mathbf{J}_N^\mathsf{T}}\ \mathbf{q} + \underbrace{\int_{\Gamma_c^{(1)}} \mathbf{\Lambda}_N g_N\, d\Gamma}_{\mathbf{j}_N} \geq \mathbf{0},\ \ \boldsymbol{\lambda}_N \geq \mathbf{0},\ \boldsymbol{\lambda}_N^\mathsf{T}\left(\int_{\Gamma_c^{(1)}} \mathbf{\Lambda}_N \mathbf{g}_N^\mathsf{T}\, d\Gamma\, \mathbf{q} + \int_{\Gamma_c^{(1)}} \mathbf{\Lambda}_N g_N\, d\Gamma\right) = 0. \tag{19}$$

In order to preserve meaningful inequalities, the weighting functions must be non-negative in $\Gamma_c^{(1)}$. As to the discretisation of the contact laws (13-15) it is assumed for the sake of brevity that they have to be satisfied only at a number s of discrete points $\vec{x}_k \in \Gamma_c$, which can be accomplished by employing Dirac delta functions $\Lambda_{N_k} = \delta(\vec{x} - \vec{x}_k)$ as force ansatz functions in Eqs. (18,19). This means that Eqs. (14,15) simultaneously hold at each of these points:

The auxiliary condition (15) of the optimization problem is non-differentiable. By approximating the friction disc, see **Fig 1b**, by a polygon that is spanned by an even number of r unit vectors $\vec{e}_i$ so that each vector is part of a pair $\vec{e}_i = -\vec{e}_j$, one gets for each contact point $\vec{x}_k$ a discretized and differentiable optimization problem of the form

$$\dot{\mathbf{q}}^\mathsf{T}\mathbf{g}_T \sum_{i=1}^{r} \beta_i \mathbf{e}_i \rightarrow \text{Min},\ \beta_i \geq 0,\ \mu\lambda_N - \sum_{i=1}^{r} \beta_i \geq 0, \tag{20}$$

whereby $\mathbf{g}_T^\mathsf{T}\dot{\mathbf{q}}$ and $\mathbf{e}_i$ are coordinate column matrices of the tangential velocity and spanning vectors $\vec{e}_i$, respectively, with respect to a local tangential coordinate system and $\vec{\lambda}_T = \sum \beta_i \vec{e}_i$.

The auxiliary conditions of the optimization problem (20) are differentiable and thus the referring *Kuhn-Tucker conditions* [1] for the optimum exist and read here

$$\mathbf{0} \leq \mathbf{B}_k^\mathsf{T}\mathbf{g}_{T_k}^\mathsf{T}\dot{\mathbf{q}} + \mathbf{1}_k z_k \perp \boldsymbol{\beta}_k \geq \mathbf{0}; \;\; 0 \leq \mu_k \lambda_{N_k} - \mathbf{1}_k^\mathsf{T}\boldsymbol{\beta}_k \perp z_k \geq 0, \tag{21}$$

wherein $\mathbf{B}_k = [\mathbf{e}_1 \ldots \mathbf{e}_r]_k$, $\mathbf{1}_k = (1\,1\,\ldots\,1)_k^\mathsf{T}$, $\boldsymbol{\beta}_k = (\beta_1 \ldots \beta_r)_k^\mathsf{T}$ and z_k is a slack variable for the friction force auxiliary condition of the optimization problem. The property $\mathbf{a} \perp \mathbf{b}$ for a pair of fitting column matrices is an expression for the orthogonality $\mathbf{a}^\mathsf{T}\mathbf{b} = 0$. The subscripts k in Eq. (21) indicate that parameters may depend on the contact position $\vec{x}_k$.

Upon discretization of the Hamilton functional in Eqn. (5) – here with the simplifying assumption of negligible geometrical non-linear effects – by means of the displacement and variation approximations in Eq. (16) and application of the reformulation of the kinetic energy variation in Eqs. (6-9) one eventually ends up with

$$\mathbf{M}\ddot{\mathbf{q}} + (\mathbf{G}+\mathbf{D})\dot{\mathbf{q}} + \overbrace{(\mathbf{K}+\mathbf{P})}^{\mathbf{Q}}\mathbf{q} = \mathbf{f} + \mathbf{J}_N\boldsymbol{\lambda}_N + \mathbf{J}_T\boldsymbol{\lambda}_T, \tag{22}$$

$$\mathbf{0} \leq \mathbf{J}_N^\mathsf{T}\mathbf{q} + \mathbf{j}_N \perp \boldsymbol{\lambda}_N \geq \mathbf{0}, \tag{23}$$

$$\mathbf{0} \leq \mathbf{J}_T^\mathsf{T}\dot{\mathbf{q}} + \mathbf{E}\mathbf{z} \perp \boldsymbol{\lambda}_T \geq \mathbf{0}; \;\; 0 \leq \boldsymbol{\mu}\boldsymbol{\lambda}_N - \mathbf{E}^\mathsf{T}\boldsymbol{\lambda}_T \perp \mathbf{z} \geq 0\;. \tag{24}$$

Equations (24) are obtained from Eqs. (21) by taking all contact points into account with $\boldsymbol{\lambda}_T = (\boldsymbol{\beta}_1^\mathsf{T} \ldots \boldsymbol{\beta}_s^\mathsf{T})^\mathsf{T}$, $\mathbf{z} = (z_1 \ldots z_s)^\mathsf{T}$ and all parameters $\mathbf{g}_{T_k}\mathbf{B}_k, \mathbf{1}_k, \mu_k$ appropriately concatenated in the matrices $\mathbf{J}_T, \mathbf{E}, \boldsymbol{\mu}$, respectively. In Eq. (22) $\mathbf{M}$ denotes the *pd* (*positive definite*) and symmetric mass matrix, $\mathbf{G}$ the skew-symmetric gyroscopic matrix, $\mathbf{D}$ the *pd* and symmetric damping matrix, comprising internal damping effects, for instance modeled as *Rayleigh*-damping and external damping effects, $\mathbf{P}$ is the skew-symmetric damping matrix due to external damping in the rotating reference frame. The overall stiffness matrix $\mathbf{K} = \mathbf{K}_0 + \mathbf{K}_c$ comprises elastic effects in the symmetric and here presumedly *pd* matrix $\mathbf{K}_0$ and centrifugal effects by

$$\mathbf{K}_c = -\Omega^2 \int_V \rho \boldsymbol{\Phi}\mathbf{W}_k\boldsymbol{\Phi}^\mathsf{T}\, dV, \tag{25}$$

where moreover is assumed that the angular velocity vector with magnitude Ω is parallel to the k-th coordinate axis in the rotating frame $\vec{\mathbf{b}}$, i.e. $\vec{\Omega} = \Omega\vec{b}_k$. This implies $\mathbf{W}_k = \mathbf{diag}(1-\delta_{ik}; i=1,2,3)$ and hence that $\mathbf{K}_c$ is *nsd* (*negative semi-definite*). For more details on the derivation of the system matrices see [16].

Equations (22-24) define a system of *Differential Algebraic Equations* (*DAE*) that describes the motion as long as no zero crossings in the inequalities occur. If this happens, additional assumptions – transition laws – have to be made, which are discussed in section 4.

3 STATIONARY SOLUTIONS

Systems with friction are in general hysteretic and thus dependant on their evolution history. Any investigation of stationary solutions therefore requires additional restrictive presumptions to define special cases that can be considered non-hysteretic.

3.1 Formulation as Linear Complementarity Problem

In this section it is assumed that the systems allow the application of a contact force law similar to (13-15) obtained by replacing $\dot{\vec{u}}_T$ in the objective function by $\vec{u}_T$, i.e.

$$\lambda_N \geq 0, g \geq 0, \lambda_N g = 0; \quad \vec{u}_T \cdot \vec{\lambda}_T \rightarrow \text{Min}, \quad \mu\lambda_N - |\vec{\lambda}_T| \geq 0 \tag{26}$$

$$\mathbf{u}_T = \mathbf{g}_T^\mathsf{T}\mathbf{q} + \mathbf{g_0}_T \tag{27}$$

and hence also presuming that the tangential displacements initially vanish and that in case of a non-sticking contact the displacement history were such that $\vec{u}_T$ evolved rectilinearly and unidirectionally and therefore the friction force be again anti-parallel to $\vec{u}_T$, see **Fig. 1b** ($\dot{\vec{u}}_T$ replaced by $\vec{u}_T$). Furthermore, it is assumed that the tangential displacement is linear affine in the generalized coordinates $\mathbf{q}$ after discretization, whereby Eq. (27) already shows the decomposition in a local tangential reference frame.

The discretisation of the contact laws is analogous to Eqs. (20-24), however, now with $\vec{u}_T$ replacing $\dot{\vec{u}}_T$, i.e., Eq. (27) replacing $\mathbf{g}_T^\intercal \dot{\mathbf{q}}$. Since stationary solutions are sought, all time derivatives and time-dependant excitations are omitted in Eq. (22). Elimination of the generalized coordinates $\mathbf{q}$ reduces Eqs. (22-24) to

$$\mathbf{0} \leq \underbrace{\begin{bmatrix} \mathbf{J}_N^\intercal \mathbf{Q}^{-1}\mathbf{J}_N & \mathbf{J}_N^\intercal \mathbf{Q}^{-1}\mathbf{J}_T & \mathbf{0} \\ \mathbf{J}_T^\intercal \mathbf{Q}^{-1}\mathbf{J}_N^\intercal & \mathbf{J}_T^\intercal \mathbf{Q}^{-1}\mathbf{J}_T & \mathbf{E} \\ \boldsymbol{\mu} & -\mathbf{E}^\intercal & \mathbf{0} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} \boldsymbol{\lambda}_N \\ \boldsymbol{\lambda}_T \\ \mathbf{z} \end{pmatrix}}_{\mathbf{x}} + \underbrace{\begin{pmatrix} \mathbf{j}_N + \mathbf{J}_N^\intercal \mathbf{Q}^{-1}\mathbf{f} \\ \mathbf{j}_T + \mathbf{J}_T^\intercal \mathbf{Q}^{-1}\mathbf{f} \\ \mathbf{0} \end{pmatrix}}_{\mathbf{b}} \perp \begin{pmatrix} \boldsymbol{\lambda}_N \\ \boldsymbol{\lambda}_T \\ \mathbf{z} \end{pmatrix} \geq \mathbf{0}. \tag{28}$$

This algebraic problem is known as *Linear Complementarity Problem* – abbreviated LCP($\mathbf{A}, \mathbf{b}$) – where for given matrices $\mathbf{A}, \mathbf{b}$ any solution $\mathbf{x}$ is sought. To the authors' knowledge there is no general existence condition for solutions of a LCP, but only for special cases: The LCP (28) is guaranteed to have a – not necessarily unique – solution for a symmetric and *pd* matrix $\mathbf{Q}$ [2], i.e. vanishing external damping and subcritical rotational speed. It also has a solution for vanishing friction and *pd* – but not necessarily symmetric – matrix $\mathbf{Q}$ [16].

4 TRANSIENT SOLUTION

The DAEs (22-24) are solved for a constant constraint configuration as long as the inequalities show no zero crossing from above. If this happens, consistent transitions to the new constraint configuration have to be calculated [14].

4.1 Frictional Constraints Transition

For the following considerations only these constraints are relevant that are active as equality in Eqs. (23,24) or are just about to become active or inactive since a zero crossing was just detected at time ${}^{-}t$ solely within the frictional constraints (24) and the normal forces $\boldsymbol{\lambda}_N$ in the conditions (23). It now hast to be decided, which constraints remain active at ${}^{+}t$ and thus further hold as equality. Since the relative tangential velocity ${}^{+}\dot{\vec{u}}_T$ at ${}^{+}t$ for constraints that just started sliding has the same direction as the tangential acceleration ${}^{+}\ddot{\vec{u}}_T$, the friction law (14) holds also on acceleration level and therefore $\ddot{\vec{u}}_T$ replaces $\dot{\vec{u}}_T$. An analogous reasoning as in section 3.1 yields a complementarity formulation for the accelerations ${}^{+}\ddot{\mathbf{q}}$, however, now with a matrix $\mathbf{g}_T^\intercal {}^{+}\ddot{\mathbf{q}} + \mathbf{g_1}_T$ expressing the coordinates of $\ddot{\vec{u}}_T$ in a local tangential coordinate frame. With ${}^{+}\mathbf{q} = {}^{-}\mathbf{q}$ and ${}^{+}\dot{\mathbf{q}} = {}^{-}\dot{\mathbf{q}}$ only jumps in the accelerations can occur and therefore eliminating ${}^{+}\ddot{\mathbf{q}}$ yields a LCP

$$\mathbf{0} \leq \underbrace{\begin{bmatrix} \mathbf{J}_N^\intercal \mathbf{M}^{-1}\mathbf{J}_N & \mathbf{J}_N^\intercal \mathbf{M}^{-1}\mathbf{J}_T & \mathbf{0} \\ \mathbf{J}_T^\intercal \mathbf{M}^{-1}\mathbf{J}_N^\intercal & \mathbf{J}_T^\intercal \mathbf{M}^{-1}\mathbf{J}_T & \mathbf{E} \\ \boldsymbol{\mu} & -\mathbf{E}^\intercal & \mathbf{0} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} \boldsymbol{\lambda}_N \\ \boldsymbol{\lambda}_T \\ \mathbf{z} \end{pmatrix}}_{\mathbf{x}} + \underbrace{\begin{pmatrix} \mathbf{j}_N + \mathbf{J}_N^\intercal \mathbf{M}^{-1}\mathbf{f}^* \\ \mathbf{j}_T + \mathbf{J}_T^\intercal \mathbf{M}^{-1}\mathbf{f}^* \\ \mathbf{0} \end{pmatrix}}_{\mathbf{b}} \perp \begin{pmatrix} \boldsymbol{\lambda}_N \\ \boldsymbol{\lambda}_T \\ \mathbf{z} \end{pmatrix} \geq \mathbf{0} \tag{29}$$

with $\mathbf{f}^* = \mathbf{f} - (\mathbf{G}+\mathbf{D})^{-}\dot{\mathbf{q}} + (\mathbf{K}+\mathbf{P})^{-}\mathbf{q}$. Since $\mathbf{M}$ is *pd* and symmetric, it can be shown that the LCP($\mathbf{A}, \mathbf{b}$) here always has a solution. It is again noted that the matrices $\mathbf{J}_N, \mathbf{J}_T, \ldots$ in the LCP (28) are not identical with those in LCP (29) since the latter ones only comprise the active constraints.

The LCP (29) is for vanishing ${}^{-}\dot{\mathbf{q}}, {}^{-}\ddot{\mathbf{q}}$ also suitable to judge preloaded systems wether or not given stationary initial conditions ${}^{-}\mathbf{q}$ can be beard without forcing sticking frictional contacts to glide.

4.2 Impact Laws

When displacement constraints are about to be violated, i.e., a zero crossing within the Signorini Conditions (23) is detected, a percussive impact occurs, which will be classically treated by means of a restitutional impact law as usually done in rigid-body dynamics.

Immediately before impact, the set of active constraints be expanded by the constraints that just have been touched and reduced by those constraints that are about to become inactive, i.e. whose referring entries in $\mathbf{J}_N^\intercal {}^{-}\dot{\mathbf{q}}$ are positive. The problem now is to decide which of those constraints remain active immediately after the impact at ${}^{+}t$. This decision requires additional assumptions. One has to expect velocity jumps from ${}^{-}\dot{\mathbf{q}}$ to ${}^{+}\dot{\mathbf{q}}$, which are

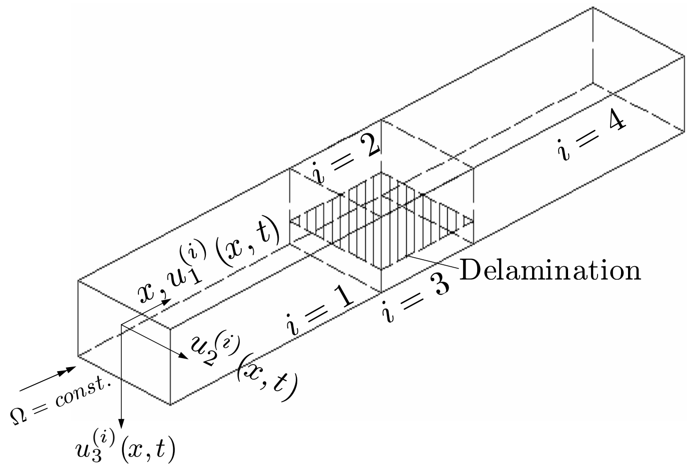

Figure 2: Rotating beam with delaminated region.

the new initial velocities for the upcoming constraint configuration. As usual it is assumed that during the impact the positions do not change, i.e. ${}^{+}\mathbf{q} = {}^{-}\mathbf{q}$. Upon assuming infinitely short impact duration and neglecting non impulsive forces $\mathbf{f}$ in Eq. (22), integration of the referring equation yields

$$\mathbf{M}{}^{+}\dot{\mathbf{q}} = \mathbf{M}{}^{-}\dot{\mathbf{q}} + \mathbf{J}_N\hat{\boldsymbol{\lambda}}_N + \mathbf{J}_T\hat{\boldsymbol{\lambda}}_T, \tag{30}$$

$$\mathbf{0} \le \mathbf{J}_N^{\mathsf{T}}\,{}^{+}\dot{\mathbf{q}} + \boldsymbol{\varepsilon}\mathbf{J}_N^{\mathsf{T}}\,{}^{-}\dot{\mathbf{q}} \perp \hat{\boldsymbol{\lambda}}_N \ge \mathbf{0}, \tag{31}$$

$$\mathbf{0} \le \mathbf{J}_T^{\mathsf{T}}\,{}^{+}\dot{\mathbf{q}} + \dot{\mathbf{g}}_{\mathbf{0}T} + \mathbf{E}\mathbf{z} \perp \hat{\boldsymbol{\lambda}}_T \ge \mathbf{0};\;\; 0 \le \boldsymbol{\mu}\hat{\boldsymbol{\lambda}}_N - \mathbf{E}^{\mathsf{T}}\hat{\boldsymbol{\lambda}}_T \perp \mathbf{z} \ge 0\;. \tag{32}$$

The complementarity conditions (31) express a restitution law that permits opening contacts without normal impulses $\hat{\boldsymbol{\lambda}}_N$ transmitted. The matrix $\boldsymbol{\varepsilon} = \mathbf{diag}(\varepsilon_k)$ carries the restitution coefficients for each contact. The complementarities (32) express the Kuhn-Tucker conditions for the polygonally approximated, compare section 2.2, tangential impact law

$${}^{+}\dot{\vec{u}}_T \cdot \hat{\vec{\lambda}}_T \to \text{Min},\;\; \mu\hat{\lambda}_N - |\hat{\vec{\lambda}}_T| \ge 0, \tag{33}$$

whereby $\hat{(\cdot)}$ indicates the referring integral over the infinitely short impact duration. As done above, Eqs. (30-32) can be transformed to a LCP, which can be shown to always have a not necessarily unique solution since $\mathbf{M}$ is *pd*.

5 EXAMPLE: DELAMINATED ROTATING EULER-BERNOULLI BEAM

The model was already presented in [18]. It consists of Euler-Bernoulli beams $i = 1, \ldots, 4$ (beam cross section area A, second order area moments I_2, I_3 , elastic modulus E, length L), each one allowing longitudinal and lateral displacements $\mathbf{u}^{(i)} = (u_1^{(i)}, u_2^{(i)}, u_3^{(i)})^{\mathsf{T}}$ with appropriate matching and boundary conditions, see **Fig. 2**.

5.1 Derivation of System Matrices

Here, the continuous system is discretized by employing eigenfunctions $\boldsymbol{\varphi}_\ell(\xi)$, $0 \le \xi \le 1$, $\ell = 1, \ldots, n$ of the self-adjoint non-rotating problem as Ritz ansatz functions for the Hamilton functional in Eq. (5).

With $\mathbf{u} = \left(\mathbf{u}^{(1)\intercal} \dots \mathbf{u}^{(4)\intercal}\right)^\intercal$, one obtains from the variation of the elastic potential

$$\delta U = \int_{\xi=0}^{1} \mathcal{D}[\delta \mathbf{u}]^\intercal \mathcal{D}[\mathbf{u}] \, d\xi = \sum_k \sum_\ell \delta q_k \underbrace{\int_{\xi=0}^{1} \mathcal{D}[\boldsymbol{\varphi}_k]^\intercal \mathcal{D}[\boldsymbol{\varphi}_\ell] \, d\xi}_{K_{0\,kl}} \; q_\ell \tag{34}$$

with the operator

$$\mathcal{D}[\cdot] = \textbf{diag}\left(\mathcal{D}^{(i)}[\cdot]\right), \mathcal{D}^{(i)}[\cdot] = \begin{pmatrix} \sqrt{EA^{(i)}/L^{(i)}}\frac{\partial(\cdot)}{\partial\xi} & 0 & 0 \\ 0 & \sqrt{EI_2{}^{(i)}/L^{(i)3}}\frac{\partial^2(\cdot)}{\partial\xi^2} & 0 \\ 0 & 0 & \sqrt{EI_3{}^{(i)}/L^{(i)3}}\frac{\partial^2(\cdot)}{\partial\xi^2} \end{pmatrix} \tag{35}$$

the elastic stiffness matrix $\mathbf{K}_0 = [K_{0\,kl}]$, for instance, which is diagonal due to self-adjointness (orthogonality of eigenfunctions).

The gap function, see Eq. (3), is here simply the opening of the delamination, i.e. $g = u_3^{(3)} - u_3^{(2)} \geq 0$ and hence Eq. (17) corresponds to

$$g = \underbrace{(0\dots0\ -1\,0\dots0\,1\,0\dots0)}_{\mathbf{g}_N(\xi)^\intercal} \overbrace{\left[\,\boldsymbol{\varphi}_1\,\dots\,\boldsymbol{\varphi}_n\,\right]}^{\boldsymbol{\Phi}(\xi)^\intercal} \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} \tag{36}$$

with appropriately distributed zeros and ones in the first matrix. Accordingly, the tangential relative displacement, see Eqs. (20, 27), is given by

$$\mathbf{u}_T = \begin{pmatrix} u_1^{(2)} - u_1^{(3)} \\ u_2^{(2)} - u_2^{(3)} \end{pmatrix} = \underbrace{\begin{pmatrix} 0\dots0\,1\,0\dots0\ -1\,0\dots0 \\ 0\dots0\,1\dots0\ -1\,0\dots0 \end{pmatrix} \left[\,\boldsymbol{\varphi}_1\,\dots\,\boldsymbol{\varphi}_n\,\right]}_{\mathbf{g}_T(\xi)^\intercal} \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}. \tag{37}$$

The derivation of the other system matrices is similar, as shown in [16], but omitted for the sake of brevity.

5.2 Simulation Results

The required solutions of the LCPs (28,29) are obtained by means of a modified version of *Lemke*'s algorithm, see [9].

The beam depicted in **Fig. 2** (length 1m, width=height=0.02m, density 7500kg/m^3, delaminated region from x=0.1m to x=0.9m, beam heights at delamination 0.011m and 0.009m) rotates in the gravitational field (9.81 m/s^2) with rotational speed near the first gravitational resonance at 21 Hz. Various simulations with varying rotational speeds indicate that significant gap vibrations with partially closed gaps only occur in the vicinity of the first gravitational resonance. Already at the second gravitational resonance dominant centrifugal forces prevent the gap from closing. The influence of interfacial friction is negligible since the relative tangential displacements, see Eq. (37), turns out to be very small at the actual rotational speed.

The discretized system has 16 degrees of freedom (dof) corresponding to mass-normalized eigenshapes of the non-rotating model. Nonlinear effects can be retrieved in the phase plots. **Fig. 3a** shows scaled plots for the first three modes. However, the nonlinear effects are small. Even for such unrealistically long delamination regions as assumed in this example, the nonlinearity due to changing contact configurations is hardly to recover in the power spectrum density. Here, the power spectrum density in **Fig. 3c** is obtained from displacements in the rotating frame at a position in the undamaged region of the beam. There is virtually no significant peak beside the one stemming from the gravitational excitation. One must therefore conclude that a failure identification procedure based upon the determination of the harmonic distortion is not suitable with gravitationally harmonically excited and delaminated rotating beam structures like depicted in **Fig. 2**.

As to the convergence of the numerical method it was observed that the solution of the discussed exact method with domain stitching could also be approximately obtained with a penalty method. To this end, the penetration g_i

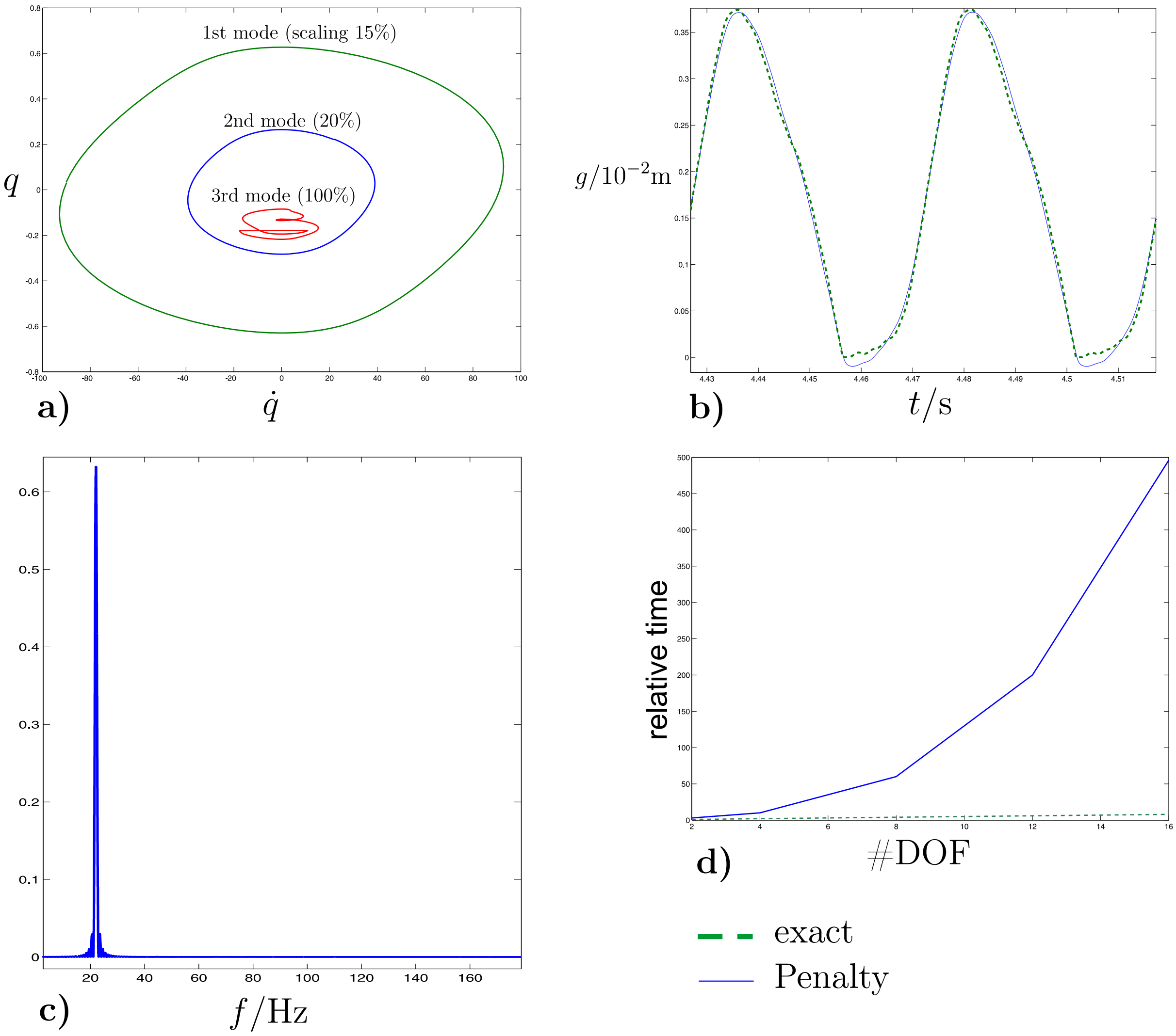

Figure 3: **a)** Phase plots of first 3 modal coordinates at steady state vibration near first gravitational resonance at 21 Hz. **b)** Gap opening at middle of gap – Penalty solution for time comparison only, see Fig. d). **c)** Power spectrum of displacement in a undamaged region of the beam. **d)** Time comparison exact method vs. moderately stiff penalty approach.

of the i-th contact, see Eq. 17, is penalized with a force $\lambda_{N_i} = -\max(0, c_p g_i + d_p \dot{g}_i)$ and friction is neglected. The penalty parameters stiffness c_p and damping d_p had to be appropriately tuned to fit the penalty solution to the exact solution for different restitution coefficients e_i. **Fig. 3d** shows the computation time comparison of MATLAB implementations for reduced models with an increasing number of dof. Penalty methods require the usage of special solvers for stiff differential equations (here with MATLAB ode15s) whereas the exact model still can be treated with ordinary Runge-Kutta solvers (here ode45), which are usually much faster than the stiff counterparts. This confirms the major drawback that is often asserted for penalty methods with vibro-impact systems, see [13]. The penalty solution depicted in **Fig. 3b** for the case of ideally plastic impacts ($e_i = 0$) is obtained by employing penalty spring-dashpot elements with greatly reduced stiffness and damping and serves only as the penalty counterpart for fair computation time comparisons. The computation time for less compliant an thus more realistic elements should be even higher.

6 CONCLUSION

A modeling process was presented for continuous rotor systems with frictional contacts. It bases on the derivation of the equations of motion as DAEs for structural variant systems due to changing contact configurations. Under some kinematical linearity assumptions (i.e. small and not explicitly time-dependant gap functions) it could be shown that stationary solutions necessarily exist at subcritical rotational speeds for frictionless systems and for frictional systems without external damping. At present, no theory can decide in general wether solutions exist at supercritical rotational speed and indeed there are cases in which they do not.

Moreover, consistent contact transition conditions were formulated since solely locally applied restitution-, impenetrability- and friction-laws show inconsistent results. Hence, the application of the derived more complex transition conditions is in general also required with coarsely discretized continuous structures and not only in rigid body dynamics.

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