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On Bending-Torsional Flutter of a Cantilever with Tip Fluid Jet

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Introduction

From the 1950s to the 1970s, stability problems of structural members subjected to follower forces attracted very much attention. Meanwhile it seems to be clear (see [4], for instance) that many of such problems are academic and structural members subjected to fluid flow loading belong to the few representatives of non-conservative mechanical systems of practical relevance.

A special type of this problem class is a cantilevered bar subjected to a transverse follower force of fluid jet, which during the early 1970s (see [5,8], for example) was studied in many details. The investigations were based on linear models taking into consideration some pre-deformation due to the time-independent loading or on a simplified geometrically non-linear theory. In addition, the treatment was focused on a special arrangement producing the follower force of fluid jet. Since i) a consistent geometrically nonlinear theory of elasticity for such problems is available now (see [6,9] where the academic problem of a cantilever subjected to follower tip moment was treated) and ii) other specifications of tip jets are interesting to be discussed, the described problem is re-considered here.

In a first step, a sufficiently consistent formulation of the governing boundary value problem is presented, where two different mechanisms of fluid jet loads are discussed. The corresponding variational equations are derived next where a non-dimensional notation is introduced. This non-dimensional formulation is the key to recognizing the important parameters influencing the stability behavior and the flutter load, in particular qualitatively but also quantitatively.

Physical Model

Consider a slender beam of length L and mass per unit length $\mu_S = \rho A$ (ρ mass density, A cross-sectional area) with narrow rectangular cross-section (thickness $h \ll$ height H) so that the smaller of the bending stiffnesses EI_2 and the torsional stiffness GI_T are much smaller than the other bending stiffness EI_1, see Fig 1a. All data are assumed to be constant. One end of the beam is rigidly fixed and there is introduced a Cartesian reference frame $\{O\boldsymbol{e}_X\boldsymbol{e}_Y\boldsymbol{e}_Z\}$ with origin O coinciding with the centroid $S(Z=0)$ of the cross-sectional area, where the unit base vectors $\boldsymbol{e}_X, \boldsymbol{e}_Y$ correspond with the symmetry axes of that cross-section and $\boldsymbol{e}_Z$ is directed along the non-deformed bar axis. Due to the deformations (displacements u, v, w and a torsional angle φ), the centroid $S(Z)$ of a general cross-section located at the position $Z\boldsymbol{e}_Z$ displaces to s with a corresponding changed orientation denoted by the body-fixed reference frame $\{o\boldsymbol{e}_x\boldsymbol{e}_y\boldsymbol{e}_z\}$ with origin o coinciding with location s of the deformed cross-sectional centroid, where $\boldsymbol{e}_z$, $\boldsymbol{e}_y$ correspond with the symmetry axes of the actual cross-section (which itself remains undeformed) and $\boldsymbol{e}_z$ is the outside normal of that cross-section.

Two layouts of follower fluid jet load are considered, see Fig. 1b and c. The first one is that intro-

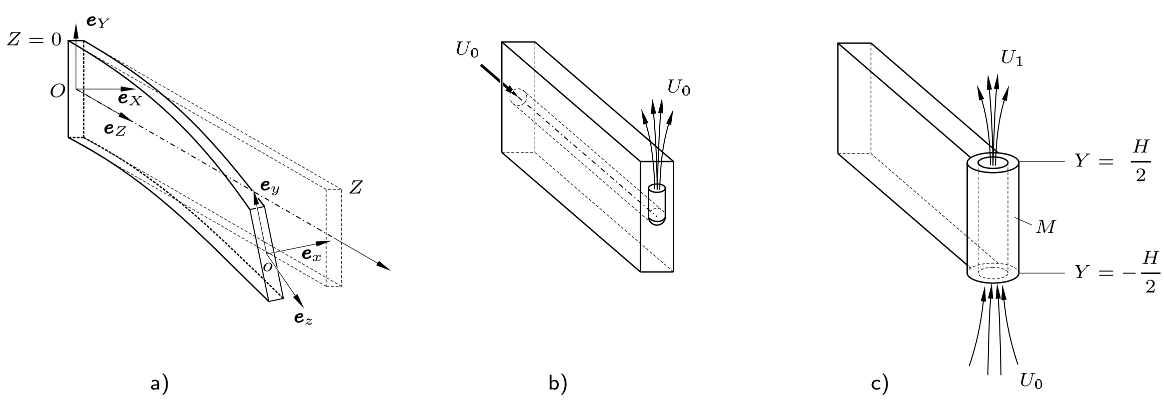

Figure 1. Bar Model, Coordinate Systems and Layouts of Fluid Jet Load.

duced by [5] and [8]. In this case, the bar has a uniform circular bore of certain suitable diameter along the Z-axis (in un-deformed state), through which an incompressible fluid (e. g., water) at constant speed U_0 is flowing. A nozzle is connected at the free end of the beam with the hole such that the fluid leaves the system in the deformed state at the centroid $s(Z = L)$ in the form of a jet following the end cross-section in its actual state shooting into the $\boldsymbol{e}_y(Z = L)$-direction.

The second specification is geared to that one suggested by Como [2] and Wohlhart [10] here also at the tip end. To make their academic follower force practicable, a slender rigid attachment of mass M and length H with a central hole as a model of a jet engine is appropriately fixed at the end cross-section along $\boldsymbol{e}_y(Z = L)$ where air is coming in with speed U_0 at $Y = -H/2$ and exhaust gas is leaving with larger speed $U_1 > U_0$ (neglecting the fuel mass rate against the air flow rate) at $Y = H/2$ so that a transverse follower propulsion in the $-\boldsymbol{e}_y$-direction results.

Formulation

The governing boundary value problems are derived based on the assumptions as follows: The fluid is inviscid. Rotary inertia and shear deformation of the beam are neglected as well as gravity effects. Then, Hamilton's principle

$$\delta \int_{t_0}^{t_1} (T - V)\,\mathrm{d}t + \int_{t_0}^{t_1} W_{\mathrm{virt}}\,\mathrm{d}t = 0$$

will be applied where T is the kinetic energy of the open system to be considered, V is the corresponding potential energy and W_{virt} contains the virtual work of all non-conservative forces and the contributions of mass transport over the open boundaries.

Independently from the two follower force realizations, there are the energy contributions

$$T_{\mathrm{S}} = \frac{1}{2}\int_0^L \mu_{\mathrm{S}}(u_{,t}^2 + v_{,t}^2 + k_{\mathrm{S}}^2\varphi_{,t}^2)\,\mathrm{d}Z, \quad k_{\mathrm{S}}^2 = \frac{I_1 + I_2}{A}$$

and

$$V_{\mathrm{S}} = \frac{1}{2}\int_0^L \left[EI_2 u_{,ZZ}^2 + EI_1(-v_{,ZZ} + \varphi u_{,ZZ})^2 + GI_T\varphi_{,Z}^2\right]\,\mathrm{d}Z$$

of the structural member together with its virtual work term

$$\begin{aligned} W_{\mathrm{virt,S}} &= -d_{\mathrm{e}}\int_0^L \mu_{\mathrm{S}}(u_{,t}\delta u + v_{,t}\delta v + k_S^2\varphi_{,t}\delta\varphi)\,\mathrm{d}Z \\ &\quad - d_{\mathrm{i}}\int_0^L (EI_2 u_{,ZZt}\delta u_{,ZZ} + EI_1 v_{,ZZt}\delta v_{,ZZ} + GI_T\varphi_{,Zt}\delta\varphi_{,Z})\,\mathrm{d}Z \end{aligned} \qquad (1)$$

characterizing external and internal damping in a linear formulation which is sufficient.

Two alternative fluid flow contributions to kinetic energy and virtual work representing the distinct follower force concepts have to be added. In case 1 where fluid flows through the central hole of the bar and leaves at the nozzle following the end cross-section into $\boldsymbol{e}_y(L)$-direction (here in such a form that the fluid speed within the nozzle remains unchanged), there is an additional kinetic energy

$$T_{\mathrm{F}}^{(1)} = \frac{1}{2}\int_0^L \mu_{\mathrm{F}}\left[u_{,t}^2 + v_{,t}^2 + 2U_0(u_{,t}u_{,Z} + v_{,t}v_{,Z})\right]\,\mathrm{d}Z$$

and a work term

$$W_{\mathrm{virt,F}}^{(1)} = -\mu_F U_0\Big[(u_{,t} - U_0\varphi)\delta u + (v_{,t} + U_0)\delta v\Big]_{Z=L}.$$

If the shortening of the bar were to be taken into consideration, a supplement would occur that would not significantly modify the stability analysis.

In case 2 (assuming that the fluid mass within the attached engine is negligible compared to M) the kinetic energy of the jet engine

$$T^{(2)} = \frac{1}{2}\Big[M(u_{,t}^2 + v_{,t}^2) + \frac{MH^2}{12}\varphi_{,t}^2\Big]_{Z=L}$$

and the contribution of the fluid jet

$$W_{\mathrm{virt,F}}^{(2)} = \mu_F\Big\{U_0\Big[(u_{,t} - U_0\varphi)\delta u + (v_{,t} + U_0)\delta v\Big] - U_1\Big[(u_{,t} - U_1\varphi)\delta u + (v_{,t} + U_1)\delta v\Big]\Big\}_{Z=L}$$

have to be added. Also in this case, the shortening of the bar end (together with the inclination angle $v_{,Z}(Z = L)$) can easily be taken into account.

It is straightforward now to evaluate Hamilton's principle for both cases to get the respective governing boundary value problems.

It will be mentioned that, for a consistent post buckling analysis or the examination of interacting oscillations in both lateral directions of the beam, a geometrically nonlinear theory of elasticity to the cubic order (see [1], for instance) or a nonlinear elastica theory (see [3,7], for instance) should be applied.

Evaluation and Results

In a first step, the steady deformation state is determined which is for both follower force realizations a pure time-independent bending about the stiffer of the two main inertia lateral axes without torsion:

$$u_0(Z,t), \varphi_0(Z,t) \equiv 0, \quad v_0(Z,t) = v_0(Z) \text{ where}$$
$$EI_1 v_{0,ZZ} = -\mu_{\mathrm{F}} U_0^2 (L - Z) \text{ (case 1) or } EI_1 v_{0,ZZ} = -\mu_{\mathrm{F}} (U_1^2 - U_0^2)(L - Z) \text{ (case 2)}.$$

Taking now solutions of the form

$$u(Z,t) = u_0 + \Delta u(Z,t), \quad v(Z,t) = v_0 + \Delta v(Z,t), \quad \varphi(Z,t) = \varphi_0 + \Delta\varphi(Z,t)$$

and substituting them into the governing nonlinear boundary value problems, we get – linearizing in the Δ-quantities – the variational equations as the starting point for the stability analysis. As expected the coupled boundary value problem in Δu and $\Delta\varphi$ constitute the lateral buckling problem while the other decoupled one in Δv describes simple damped oscillations.

Introducing non-dimensional variables and parameters, it becomes obvious that besides two characteristic damping coefficients D_{e} and D_{i} the stiffness ratio $GI_T/(EI_2)$ together with the slenderness $(k_{\mathrm{S}}/L)^2$ may drastically influence the eigenvalues as a function of the load parameter $\mu_{\mathrm{F}} U_0^2 L^2 / \sqrt{GI_T EI_2}$ (case 1) or $\mu_{\mathrm{F}}(U_1^2 - U_0^2) L^2 / \sqrt{GI_T EI_2}$ (case 2). The stability behavior will be discussed in detail where, in particular, some qualitative results not given in [5,8] will be presented.

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References

1. Botz, M., *Zur Dynamik von Mehrkörpersystemen mit elastischen Balken*, Diss. TU Darmstadt, Germany, 1992.
2. Como, M., Lateral Buckling of a Cantilever Subjected to a Transverse Follower Force, *Int. J. Solids Structures*, Vol. 2, 515–523, 1966.
3. Cusumano, J. P., and Moon, F. C., Chaotic Non-planar Vibrations of the Thin Elastica, Part I and II, *J. Sound Vibration*, Vol. 179, 185–208 and 209–226, 1995.
4. Elishakoff, I., Controversy Associated with the So-called "Follower Forces": Critical Overview, *Applied Mechanics Reviews*, Vol. 58, 117–142, 2005.
5. Kalmbach, C. F., Dowell, E. H., and Moon, F. C., The Suppression of a Dynamic Instability of an Elastic Body Using Feedback Control, *Int. J. Solids Structures*, Vol. 10, 361–381, 1974.
6. Meyer, A., Tan, H., and Wauer, J., Die Wirkung äußerer und innerer Dämpfung auf die Stabilität des Prandtlschen Kragträgers unter mitgehender Momentenbelastung, *Ing.-Archiv*, Vol. 54, 205–219, 1984.
7. Plaut, R. H., Postbuckling and Vibration of End-supported Elastica Pipes Conveying Fluid and Columns Under Follower Loads, *J. Sound Vibration*, Vol. 289, 264–277, 2006.
8. Prasad, S. N., and Herrmann, G., Stability of a Cantilevered Bar Subjected to a Transverse Follower Force of Fluid Jet, *Ing.-Archiv*, Vol. 39, 341–356, 1970.
9. Wauer, J., Der Prandtlsche Kragträger unter konservativer und nichtkonservativer Momentenbelastung, *ZAMP*, Vol. 29, 334–340, 1978.
10. Wohlhart, K., Dynamische Kippstabilität eines Plattenstreifens unter Folgelast, *Z. Flugwissenschaft*, Vol. 19, 291–298, 1971.