Pricing Policy for Selling Perishable Products under Demand Uncertainty and Substitution

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Abstract

In this thesis, we consider the problem of optimal pricing for selling multiple perishable products over a finite horizon of time under demand uncertainty and substitution. We suppose that the demand for a particular product in a store is the result of the consumers' arrival process and the consumers' choice behavior. We assume that the consumers arrive at the store according to a Poisson process with a gamma distributed rate and use the multinomial logit model to describe the consumers' choice behavior. We develop a periodic review of the pricing policy by which the firm, who sells a fixed amount of multiple perishable products, improves its total expected revenues. We consider two cases: First, we suppose that the firm knows the parameters of the gamma distribution of the consumers' arrival rate at the store and study how the optimal prices of the products change depending on the remaining time until the end of the selling season as well as the inventory levels of the products. In this case, our numerical study shows that (1) at a given inventory levels of the products, the optimal prices may decrease or increase as the time for selling them gets shorter (This result is different from that in the single-product case where the optimal price of the product is non-increasing as time progresses.), and (2) at a given time, the optimal price of each product increases as its inventory level decreases. We also present the simple path of the optimal prices to show how the optimal price patterns would be when both the inventory levels of the products and the remaining time to the end of the selling season vary. Second, we suppose that the firm does not know the parameters of the consumers' arrival rate distribution and examine the performance of our optimal pricing policy. For this case, we provide a Bayesian learning approach to capture uncertainty associated with the consumers' arrival rate at the store by using observation data from the early periods to estimate the parameters and forecast the future arrivals at the store, and we use the multinomial logit model under some specified assumptions to capture uncertainty associated with the consumers' choice behavior. We examine both the single- and the multi-update cases and present some numerical experiments to show how favorable the performance of our demand learning approach is, especially in comparison to those approaches that use other techniques to forecast future arrivals at the store such as moving average and exponential smoothing methods, that are widely used in revenue management applications.

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Abbreviations

AMR	*Advanced Market Research*
ALA	*Average-Rate Learning Approach*
BLA	*Bayesian Learning Approach*
CAB	*The U.S. Civil Aviation Board*
DP	*Dynamic Pricing*
EDLP	*Every Day Low Price*
ELA	*Exponential-Smoothing-Rate Learning Approaches*
IIA	Independence from Irrelevant Alternatives
IT	Information Technology
ML	*Maximum-Likelihood*
MNL	Moltinomial Logit
MSE	*Minimum Mean-Square Error*
M&S	*Marks & Spencer*
PIA	*Perfect Information Approach*
PROS	*Pricing & Revenue Optimization Science and Software*
RM	*Revenue Management*
RMS	*Revenue Management System*
YM	*Yield Management*

Chapter 1

Motivation & Overview

Motivation and Overview

Pricing is a critical issue in marketing activities because: (1) price creates a first impression for the consumers, (2) price is the most flexible marketing mix instrument, (3) price is an important marketing mix instrument that generates revenues, and (4) setting the right price for products is one of the most important marketing decisions which managers face. As setting the price for a product, one needs to take into account several internal and external aspects such as the marketing mix, the overall marketing strategy, costs, consumers' preferences, and market conditions.

The problem of setting the right price for a particular product becomes even more difficult for the case of selling perishable products such as apparel and seasonal goods because of high demand uncertainty and supply inflexibility. Demand uncertainty refers to the lack of information about how the product will be attractive to the consumers in the market. In the context of selling perishable products, the inventory decisions (ordering) often have to be made in advance while demand for the product is not yet realized. Furthermore, there are many cases where the firm does not have any opportunity of replenishment over the selling season, because the lead times of replenishment are relatively longer than the selling season. These problems are considered as supply inflexibility. In practice, the firms who are selling perishable products often face substantial losses due to the mismatch between demand and supply. For example, M&S lost £150 million due to failures in matching supply with demand in 1998-99 (Christopher and Towill, 2002). In such cases, pricing decisions play an important role in managing demand, i.e., balancing demand and inventory, as well as improving revenues.

Revenue management applications have successfully been used in many industries such as airline, hotel, and retailing where the firms sell perishable products under demand uncertainty and supply inflexibility. Revenue management (RM) is concerned with demand uncertainty and mainly focuses on resolving it (e.g., Talluri and van Ryzin, 2004). For example, revenue management aims at helping the firm to predict how demand for a product will react to changes of price over time. The most familiar example probably comes from the airline industry, where the tickets for the same flight may be sold at many different fares throughout the booking horizon depending on product restrictions as well as the remaining time until departure and the number of unsold seats. During the last decades, the use of such strategies has been taken over from transportation industry to other areas and is increasingly important in retailing, telecommunications, entertainment, financial services, health care, and manufacturing, etc. (see Chiang et al., 2007).

Dynamic pricing (DP) is perhaps the best well-known application of revenue management by which firms are able to manage demand for the available inventory of products by changing prices over time in order to maximize the total expected revenues. In the last few decades, there has been a growing literature on dynamic pricing that considers the problem of selling a fixed amount of perishable products over a finite horizon of time. This growing interest is influenced by three main aspects: (1) the increased availability of demand data, (2) the ease of changing prices, and (3) the availability of decision-support tools to analyze demand data (e.g., Elmaghraby and Keskinocak, 2003).

In the dynamic pricing context, it is assumed that demand for a perishable product is price-sensitive while consumer's valuations change over time, and pricing decisions play as the most important role in managing demand and consequently improving revenues. Consider a firm who owns a fixed amount of a single perishable product. At the beginning of the selling season, he faces two main questions with respect to his price decisions:

1. At which price should the product be sold, i.e., the determination of the initial price?
2. How should the price change over time?

While setting high prices yields larger expected revenues but may result in high numbers of unsold units, low prices lead to the smaller expected revenues and the higher risk of stockout situation before the end of the selling season. Hence, the seller should attempt to dynamically balance the expected current and future revenues by setting different prices over time.

In terms of literature, there are two types of dynamic pricing models: (1) deterministic demand models and (2) stochastic demand models. While deterministic demand models assume that the seller has perfect information about demand process (e.g., Gallego and van Ryzin, 1994, Smith and Achabal, 1998), stochastic demand models consider some uncertainty in association with demand process (e.g., Gallego and van Ryzin, 1994, Zhao and Zhang, 2000). Most of the existing research consider the single perishable product case under Markovian assumptions with respect to the demand process in which consumers arrive at a store according to a Poisson process with rate λ . An arriving consumer will purchase one unit of the product if the posted price is not greater than his reservation price. As the common result in the revenue management literature, Gallego and van Ryzin (1994) have shown, for the problem of selling a fixed amount of a single perishable product over a finite horizon of time:

1. At a given point in time, the optimal price decreases as the inventory level of the product increases.
2. For a given inventory of the product, the optimal price increases if there is more time to sell inventory.
3. More amount of inventory leads to higher expected revenues.
4. More time left to the end of the selling season leads to a higher expected revenue.

Under Markovian assumptions the determination of optimal prices requires the solution of the Hamilton-Jacobi-Bellman equation that is in general a complex differential equation for which closed-form solutions are reported only for the case of exponential demand rate (e.g., Kincaid and Darling, 1963, Gallego and van Ryzin, 1994). Furthermore, because of the memoryless property of Markovian processes such as the Poisson process, the knowledge of the number of arriving consumers up to time t , i.e., $[0,t]$, provides nothing about the consumers' arrivals in the future, i.e., $(t,T]$ where T is the length of the selling season, thus, *demand learning* is not possible.

Demand learning is an effective approach to improve revenue by resolving demand uncertainty and providing a better demand forecasting. In a few last years, we have witnessed a growing interest in considering demand learning for the problem of selling a single perishable product where the demand for a product is assumed to be the result of the consumers' arrivals and choice behavior. For instance, Aviv and Pazgal (2005b) assumed that the consumers arrive at the store according to a Poisson process with a rate that depends on general market conditions rather than on the product's price and that an arriving consumer purchases one unit of the product if the posted price is not greater than his reservation price, i.e., the maximum price that he is willing to pay for the product. This approach enables the seller to learn about demand patterns during the selling season and improve his total expected revenues. Still, the authors assumed that the seller observes only completed sales, so that arrivals of the consumers which may provide information about market conditions are not considered if such consumers do not purchase at all.

In contrast to the single perishable product case, the problem of determining an optimal pricing policy for multiple perishable products has received considerably less attention in the revenue management literature (e.g., Elmaghraby and Keskinocak, 2003, Bitran and Caldentey, 2003). While, in the single perishable product case, it is assumed that demand for the product depends on price and the seller determines the optimal price policy based on the remaining time until the end of the selling season and inventory level of the product. In the multiple perishable products case, the demand for a particular product not only depends on its own price, but also on prices of all other products. Hence, the seller has to take into consideration both the inventory level of the interest product and the inventory levels of all other products as well as the time until the end of the selling season to determine the optimal prices. In fact, in the multiple perishable products case, the demand for a particular product should be modeled as a choice problem facing a set of (substitutable) products determining the right prices for the products will be quite more complicated than the case of pricing a single perishable product.

In this thesis, we are going to provide an **optimal pricing policy** to optimize the revenue of selling a given inventory of **multiple perishable products** under **demand uncertainty** and **substitution**. We assume that the demand for each product can be modeled with the help of two elements: (1) the consumers' arrival rate and (2) the consumers' choice behavior.

In terms of the consumers' arrival rate, we assume that the consumers arrive at the store according to a Poisson process ($N_\tau, \tau \in [0,t]$) with a gamma distributed rate λ that depends on the regular purchase patterns rather than on the prices of the products. Under this assumption we provide a Bayesian learning approach to capture the demand uncertainty associated with the consumers' arrival rate. We divide the selling season into T equal periods where T does not depend on the consumers' arrival rate and use the number of arriving consumers at the store during the t first periods to estimate the parameters of the gamma distribution and compute the probability distribution function of the consumers' arrivals in the next periods $t+1,\ldots,T$.

In terms of the consumers' choice behavior, we assume that each arriving consumer chooses his favorite product based on the multinomial logit model that is used to capture the demand uncertainty associated with the consumers' choice behavior. In our demand learning model, we provide an optimal pricing policy based on products' *qualities*, *prices*, and *inventory levels* as well as the remaining *time* to the end of the selling season.

The structure of this thesis is as follows: Chapter **2** deals with pricing decisions and revenue optimization. In this chapter, we address the importance of pricing decisions in managing demand and optimizing revenue for selling perishable products over a short period of time. Then, we review pricing models in the literature on revenue management. After a short introduction to static pricing models, we consider the problem of selling a fixed amount of a perishable product over a finite horizon of time for both deterministic and stochastic demand models in dynamic pricing applications. At the end of this chapter, we also present a review of related works and highlight the main results. Chapter **3** addresses demand uncertainty as one of the most important problems faced by managers to achieve success in business, especially for those firms who are selling perishable products over a finite horizon of time. Demand forecasting and demand learning are in the centre of our focus in this chapter. We consider the consumers' arrival process and point out some forecasting methods that are widely used in revenue management applications. We also discuss our Bayesian learning approach by which the firm learns about the consumers' arrival rate from the observed demand data in the earlier periods. Chapter **4** is devoted to the consumers' choice behavior as the second element of demand learning model. The reservation price and the multinomial choice models will be discussed in this chapter. The reservation price model is widely used in the problem of selling a single perishable product, while the multinomial choice model is preferred in solving the problem of selling multiple perishable products. In chapter **5** we provide an optimal pricing policy to manage demand of selling multiple perishable products under demand uncertainty and substitution based on products' *qualities*, *prices*, and *inventory levels* as well as the remaining *time* to the end of the selling season. The overall aim of our optimal pricing policy is to maximize the expected revenues gathered from a given inventory levels of the products over a finite selling horizon. We will also present a review of the literature on the problem of selling multiple perishable products. A numerical study will be provided in chapter **6**. In this chapter, we will consider several cases and present some numerical experiments to illustrate how our optimal pricing policy works and show the optimal price patterns with respect to the remaining time and the available inventory of the products. Then, we will consider the determination of an optimal pricing policy by using our demand learning model for both the single- and the multi-update cases and compare the performance of the Bayesian learning approach to the other demand learning approaches. At the end of this chapter, we will also point out some selected extensions of our demand learning model that address some real situations in the market. Finally, in chapter **7**, we conclude our work with a summary of our main results and future directions.

Chapter 2

Pricing
&
Revenue Optimization

2.1 Introduction

In this chapter, we look at the role of pricing decisions for the success of marketing activities, especially in the case of managing demand for selling perishable products over a short period of time in order to maximize revenues.

As was already mentioned in the motivation part of chapter 1, price is of fundamental importance as: (1) Price creates a first impression for consumers with respect to the underlying product. Often times consumers’ perceptions of a product are formed as soon as they learn the price. It is important for firms to know if consumers are likely to make the decision to buy a product when all they know is its price. If so, pricing may become the most important of all marketing activities if it can be shown that consumers tend to avoid learning more about the product than the price; (2) Price is the most flexible marketing mix instrument. Unlike product and distribution decisions, which can take months or years to change, price can be adapted very rapidly. The flexibility of pricing decisions is particularly important in times when the firm seeks to quickly stimulate demand or respond to changing market conditions; (3) Price is an important marketing mix variable that generates revenues; all others, i.e., product, promotion, and distribution, involve expenditures of funds (e.g., Rao, 1984); (4) Setting the right price for products is one of the most important marketing decisions which managers face (e.g., Monroe and Cox, 2001). Prices set too low may mean that the firm will miss additional profits that could be earned if the consumers are willing to spend more to acquire the product. Additionally, attempts to raise an initially low priced product to a higher price may meet by consumer resistance when people feel that the firm is attempting to take advantage of their consumers. Prices set too high can also impact revenue as it prevents interested consumers from purchasing the product. Setting the right price level often requires considerable market knowledge and, especially with new products, testing of different pricing options.

From a firm point of view, profits are determined by the difference between revenues and costs; Revenues that a firm obtains from selling a product are determined by multiplying price per unit sold by the number of units sold. On the side of consumers, prices influence what and how much of each product to buy. Thus, pricing decisions affect both firms’ profits (revenues) and the consumers’ purchase behavior (e.g., Monroe, 1990).

According to studies, pricing is one of the fastest and the most effective ways for a firm to maximize its profit (e.g., Marn et al., 2003). For example, Marn and Rosiello (1992) compared the profit implications of a *1* percent improvement in different control variables and showed that based on *2,263* companies' average economics in Compustat[1] aggregate data, the leverage of improved pricing is high in comparison to other control variables, e.g., the profit implications of *1*% increase in unit sales volume- assuming no decrease in price- yields a *3.3*% increase in profit while *1*% improvement in price without changing the unit sales volume would improve the company's profit by *11.1*%. That is, improvements in price have three to four times the effect on profitability as proportional increases in sales volume. Another study produced by A.T. Kearney (2000) also showed that improvements in profit associated with changes in four main control variables as price, sales volume, variable and fixed costs, have similar relationships as those reported by Mc Kinsey (1992), see table *2.1*.

1% Change in **Control Variables**	*Improvement of Profit* **(McKinsey, 1992)**	*Improvement of Profit* **(A.T. Kearney, 2000)**
Price↑	**11.1%**	**8.2%**
Variable Costs↓	**7.8%**	**5.1%**
Sales Volume↑	**3.3%**	**3.0%**
Fixed Costs↓	**2.3%**	**2.0%**

Table 2.1 *Profit improvement per 1% change in control variables.*
Sources: McKinsey (1992), A.T. Kearney (2000) studies; Phillips (2005).

Although pricing is the strongest determinant of revenues/profits in marketing activities, setting the right price for the right consumer at the right time is one of the most complex and challenging decisions that managers have to face in order to improve sales profitability.

[1] Standard & Poor's Compustat data has delivered high-quality, standardized fundamental market data to investment professionals around the world (www.compustatresources.com).

In practice, when determining pricing strategies and setting the right prices for products, one needs to take into account several internal and external aspects such as marketing mix, marketing strategy, production costs, consumers' preferences, elasticity of demand, and market conditions. For instance, while the cost-driven pricing strategy considers the cost of producing and selling activities, the value-driven pricing strategy uses consumer's willingness to pay, i.e., the maximum value that a consumer accepts to pay for a product (see § *4.2* for more details). Market-driven pricing is another pricing strategy in which the price is determined by competitive conditions in the market. Readers, who are interested in pricing decisions and pricing strategies, are referred to Nagle and Hogan (2006) and Armstrong and Kotler (2007) for comprehensive reviews.

In the context of pricing decisions, pricing is considered as a process by which revenue is optimized by maximizing the number of buyers (demand) together with a minimum reduction in price, or maximizing price together with a minimal loss of buyers (e.g., Joseph, 2007). In other words, the goal of pricing optimization is to improve the total revenues of a business by determining the optimal prices based on the relationship between demand and price.

In the past few years, price optimization applications such as every-day-low-price (EDLP) pricing (e.g., Sin et al., 2007), promotional pricing (e.g., Kwong, 2003, Aydin and Ziya, 2008), and markdown pricing (e.g., Lazear, 1986, Pashigian, 1988, Pashigan and Bowen, 1991, Federgruen and Heching, 1997, Kwon et al., 2008) are increasingly used by retailers and consumer products companies. Many firms, especially in apparel and seasonal goods, are also using price optimization software within such offers as SAP, Oracle, PROS, etc. in order to improve revenues. Advanced Market Research (AMR) has estimated that the price management applications market was about $*348* million in 2007 and will grow to approximately $*1.1* billion in 2010.

Revenue management- also known as yield management (YM) - has gained attention recently as one of the most successful application areas of operations research. Revenue management refers to the strategies and tactics used to manage the allocation of selling capacity to different demand classes over time in order to maximize revenues (e.g., Phillips, 2005). Revenue management can also be considered as the art of maximizing revenues generated from a limited capacity of a product over a finite horizon of time by selling each product to the right consumer at the right time for the right price (e.g., Pak and

Piersma, 2003). More broadly, revenue management is concerned with demand-management decisions and the methodology and systems required making them (e.g., Talluri and Ryzin, 2004).

In terms of history, revenue management could be traced back to the seventies of the last century when the U.S. Civil Aviation Board (CAB) loosened control of airline prices (e.g., Belobaba, 1987, McGill and van Ryzin, 1999). Nowadays, revenue management applications have become crucial with respect to business success in many other industries such as hotel (e.g., Bitran and Mondschein, 1995), car rental agency (e.g., Geraghty and Johnson, 1997), and retailing (e.g., Elmaghraby and Keskinocak, 2003) where firms sell *perishable products* to the price-sensitive consumers in the market. For a comprehensive review of recent developments of revenue management in different industries, the interested readers are referred to Chiang et al. (2007) and the references therein (see also Talluri and Ryzin, 2004, chapter *10*). Experiences show that revenue management makes it possible to yield *3*% to *7*% increasing of revenues and consequently *50*% to *100*% increasing of profit[2] (e.g., Cross, 1997) which is a significant effect on the profitability of operations. For instance, according to Economagic (www.economagic.com), the US retail sales in 2007 were estimated to be more than $*46* trillion. That is, the application of revenue management could result in more than $*1.3* trillion potential increasing in revenues.

Perishable products such as apparel, holiday merchandise, events' tickets, hotel rooms, etc. have three major characteristics:

1. There is a well defined finite selling horizon;
2. Consumer valuations change over time;
3. The marginal cost of selling one more item is little.

Coupling these characteristics of the perishable products with supply side limitations creates a big challenge for manager to make the right price decisions for the products because:

1. Demand uncertainty is high due to uniqueness of the products and time varying valuations of the consumers.

[2] Profit = Revenue – Cost.

2. Replenishment lead-times are relatively long in comparison to the length of the selling horizon which limits replenishment of the inventory during the selling season.
3. At the end of the selling season the salvage value (price) will be very low.

In revenue management applications, setting the price for a product depends on the product itself, the consumer's valuation (willingness to pay), and the time of purchase, see Figure *2.1*.

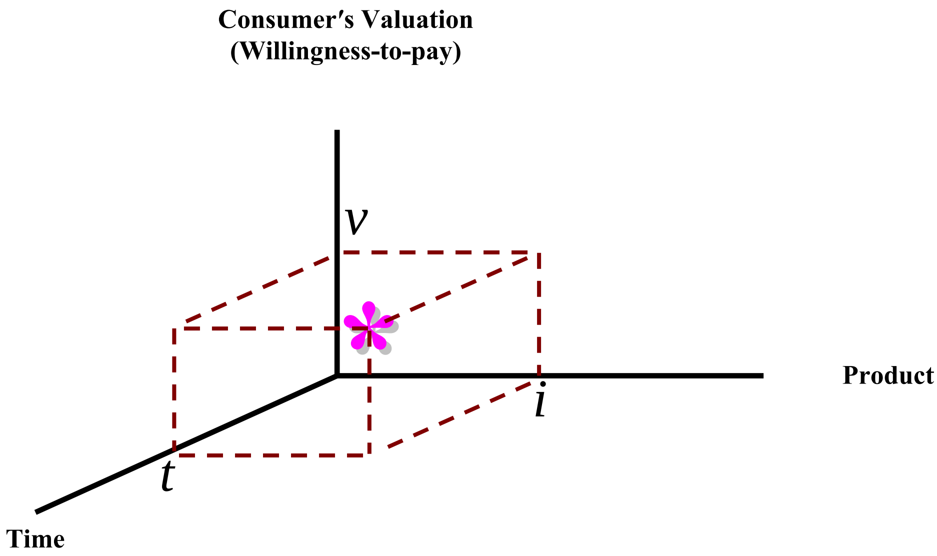

Figure 2.1 *Different aspects affecting price decisions.*

In this perspective, revenue management could be considered as a form of price discrimination. Thus, price discrimination is the topic in pricing strategies that we are going to discuss next.

2.2 Price Discrimination

Price discrimination is a pricing strategy in which different prices are charged to different consumers or segments for the same product. The aim of price discrimination is to extract as much consumer surplus[3] as possible without loss of consumers. There are three types of price discrimination:

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[3] Consumer surplus refers to the consumer's benefit by purchasing a product for a price that is less than he would be willing to pay. In fact, the individual consumer surplus is the difference between the maximum price which a consumer would be willing to pay (or reservation price) for the amount that he buys and the actual price.

2.2.1 First Degree of Price Discrimination

With respect to first degree price discrimination, price for a product varies by consumer. It is applied when the seller knows the maximum willingness to pay of each consumer in the market. In fact, the seller charges each consumer the maximum price that he is willing to pay. To illustrate how the seller gains more revenues/profits by applying first degree price discrimination we consider the following example. Suppose that demand for a particular product is a linear function of price p as $d(p) = 20 - 4p$. Then the revenue function is $r(p) = pd(p)$. Considering unit (order) costs of c, the profit function will be $B(p) = (p-c)d(p)$. Then

$$B(p) = 20(p-c) - 4(p-c)p = -4p^2 - (20-4c)p - 20c$$

and

$$\frac{d}{dp}B(p) = 20 - 4(p-c) - 4p = -8p + 4c + 20.$$

By considering $c = 1$, the optimal price with respect to the maximum profit is given by

$$\frac{d}{dp}B(p) = 0 \Rightarrow 24 - 8p = 0 \Rightarrow p = 3.$$

That is, $p = 3$ is the price at which profit is maximized, $B(3) = 16$, and $r(3) = 24$. At this price, $d(3) = 8$ units will be sold, see Figure *2.2* on the left. But, revenue can achieve the value $r(.) = 30$, i.e., *25*% more than in the single-price case, if we use four different prices, $P = \{4.5, 4, 3.5, 3\}$ and first serve those two consumers who are willing to pay the (highest) price of $p = 4.5$, then those two consumers with restriction price of $p = 4$ and so on. Then, if such a price strategy is possible, the seller initially sets the price at $p = 4.5$ and sells *2* units to those consumers, whose reservation price is equal or greater than $p = 4.5$, and gains $r(4.5) = 9$ and $B(4.5) = 7$. Then, he lowers the price to $p = 4$ and sells *2* more units more to those consumers, whose reservation price is equal or greater than $p = 4$ and gains again $2 \times (4) = 8$ units of revenue and $2 \times (4-1) = 6$ units of profit, and so on. As it shown in Figure *2.2* on the right, profit from selling eight units will be *22*, i.e., 37.5% more than that in the single-price case. Therefore, the revenue/profit obtained from selling product sequentially at different prices is significantly greater than when the seller uses a uniform price.

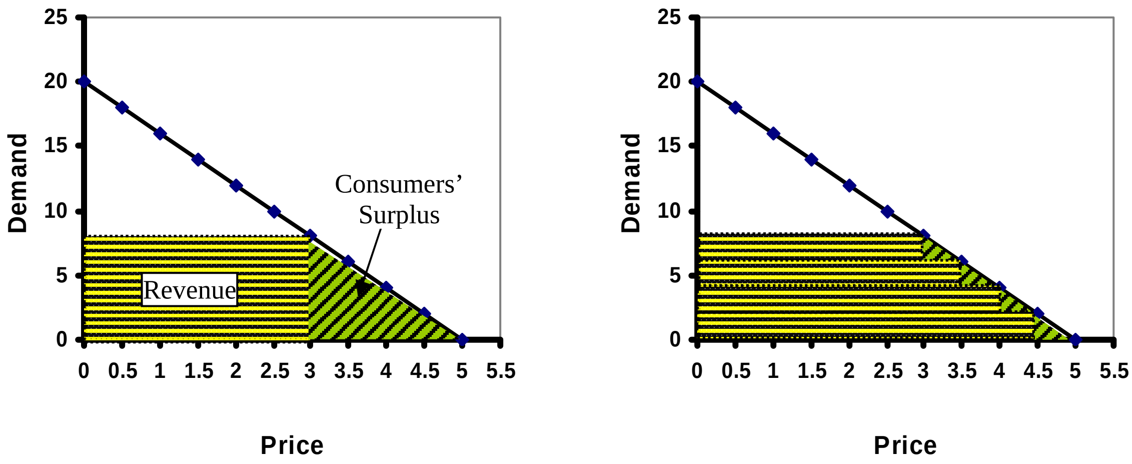

Figure 2.2 *First degree of price discrimination.*

Intuitively, if the seller uses prices with smaller differences, he will even gain more revenue/profit, of course, depending on the costs and the demand function. As the number of different price values tends to infinity, we will have perfect first-degree price discrimination in which each consumer pays his reservation price (according to his willingness-to-pay) for the product. In our example, if the seller is able to use perfect first-degree price discrimination, $p \in (1,5)$, his total revenue tends to $\frac{1}{2}16 \times (5+1) = 48$. Although it is difficult to find perfect price discrimination in real situations, we can point out bargaining in the used-product-markets and the negotiation price in B2B where the seller can find customers' willingness to pay as examples.

2.2.2 Second Degree of Price Discrimination

With respect to second degree price discrimination, price varies according to different purchase behavior. It is applied when the seller is not able to distinguish the different types of consumers and uses consumer preference information. Indeed, consumers are grouped into different types (segments) through their purchase behavior. In apparel and high-tech industries, for example, firms, who are usually uncertain about how much the consumers are willing to pay for the product, set a high initial price to capture revenues from the high-valuation consumers, who are willing to pay higher for the product, and mark it down before the end of the selling season to capture revenues from the low-valuation consumers, who are willing to pay lower for the product in order to clear inventory, i.e., the firm uses markdown pricing as a segmentation mechanism. In fact, the firms suppose that there are, for example, two main types of consumers in the market, the high-valuation consumers, who prefer to purchase the product earlier by paying higher than the low-valuation consumers, who are willing to wait for markdowns. However, the firms do not know whether an arriving consumer is a high- or low-valuation consumer, so that they set different prices at different times, i.e., using a time-based price discrimination. Each consumer shows to which segment, the high-valuation or the low-valuation, he belongs to in a self-selection mechanism. There are also some other types of second degree price discrimination such as quantity-based and channel-based price strategies where firms use other variants of second degree price discrimination. In quantity-based price discrimination, consumers pay different prices based on the amount of product that they buy while in channel-based price discrimination different prices are charged to the consumers who buy product from different sale channels such as Internet, TV-Shop, etc.

2.2.3 Third Degree of Price Discrimination

With respect to third degree price discrimination, price varies according to consumer segments. In contrast to second degree price discrimination, in this case, the firm is able to divide consumers into several segments and set different prices based on the segments. Different prices associated with the location of the stores and demographic variables, e.g., discounts for students, could be considered as third degree price discrimination examples. In such cases, the firm determines the price for each segment according to its demand function. Figure 2.3 shows an example of third degree price discrimination.

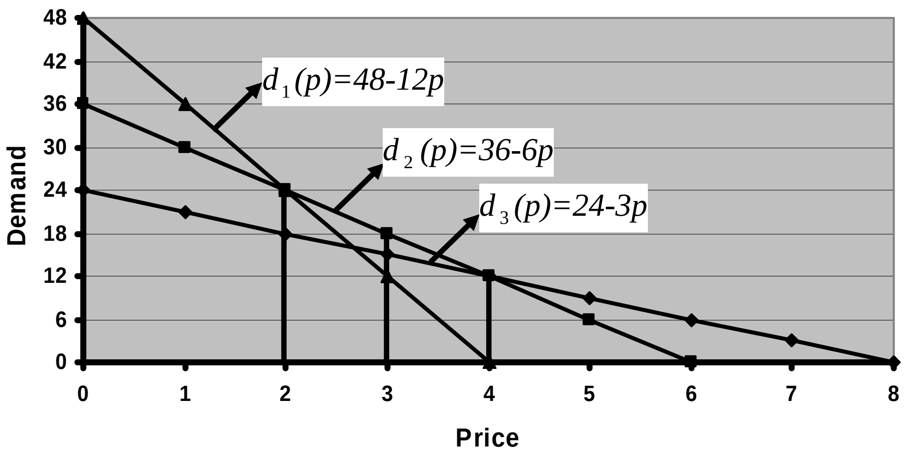

Figure 2.3 *Third degree of price discrimination.*

In this example, a firm sells a particular product to a market medeled by three segments. Each segment is characterized by a specified demand function. As shown in Figure 2.3, the firm charges different prices for the segments in such a way that revenues is maximized, i.e., he sets the price at which price elasticity of demand[4] is equal to $\varepsilon_p = -1$. Optimal prices, sales and revenues with respect to each segment are provided in table *2.2*.

Segment	*Demand Function*	*Price*	*Sales*	*Revenue*
1	$d_1(p)=48-12p$	***2***	*24*	*48*
2	$d_2(p)=36-6p$	***3***	*18*	*54*
3	$d_3(p)=24-3p$	***4***	*12*	*48*
				Sum = 150

Table 2.2 *Third degree of price discrimination.*

2.3 Pricing Models in Revenue Management

In this section, we review pricing models that are used the revenue management literature on the problem of selling a single perishable product over a finite horizon of time. There are two types of pricing policies that are widely used in revenue management applications (e.g., Elmaghraby and Keskinocak, 2003):

1. *Posted-price policy* in which products are sold at take-it-leave-it prices determined by the seller;
2. *Price-discovery policy* in which prices are determined via a bidding process such as auctions.

In this work, we will focus on the posted-price policy and its applications in pricing and revenue optimization and we refer the interested readers to Klemperer (2004) for an excellent study on both the theory and practice in auctions.

The posted-price policy itself has two forms, *static* and *dynamic pricing models*. While in static pricing models the seller sets a fixed price for the product over a relatively long-

[4] Price elasticity of demand is defined as the measure of responsiveness in the quantity demanded for a commodity as a result of change in price of the same commodity. It is a measure of how consumers react to a change in price

term period, in dynamic pricing models the seller dynamically changes the price over time based on factors such as time of sale, demand information, and the inventory level.

In what follows we look at static and dynamic pricing models in the revenue management literature. We begin with static pricing models by introducing the *newsvendor* problem. Then, we describe dynamic pricing models as our main subject and address the problem of selling perishable product over a finite horizon of time under both deterministic and stochastic demand assumptions. Finally, we present a review of pricing models in the revenue management literature and highlight main results.

2.3.1 Static Pricing Models

Static pricing models are appropriate for the products having the following characteristics:

1. Short selling period.
2. High costs of changing price.
3. Legal regulations that force the price to be fixed.

The newsvendor[5] problem is perhaps the best known static (single-period) pricing model. In the newsvendor formulation, a decision maker, who faces an uncertain price-dependent demand for the product, decides how much of the product to stock for a single selling period before demand can be observed (e.g., Porteus, 1990). The problem is particularly important for products with significant demand uncertainty and large overstock and stockout costs. Considering random demand *D* with the cumulative distribution function *F(.)*, unit order costs *c*, selling price *p*, and salvage value per unit υ, the optimal inventory level q^*, will be as follows:

$$P(D \le q^*) = \frac{p-c}{p-\upsilon} \qquad \textit{2.1.a}$$

or

$$q^* = F^{-1}\left(\frac{p-c}{p-\upsilon}\right). \qquad \textit{2.1.b}$$

The proof is provided in appendix **B**.

In the literature, there are also studies that used the newsvendor framework to analyze the problem of determining the optimal price based on the demand function and inventory

[5] The newsvendor (or newsboy) model is a mathematical model in operations management and applied economics used to determine optimal inventory levels.

for a single period. For example, Petruzzi and Dada (1999) examined both the additive and the multiplicative demand cases and presented an optimal pricing policy with inventory consideration. In the additive case, demand is assumed to be a continuous random variable of the form:

$$D(p,\omega) = d(p) + \omega \tag{2.2}$$

where ω is a zero-mean random variable that does not depend on the price. The authors considered $d(p) = a - bp$ ($a>0$, $b>0$) in the additive case and showed that the optimal price in the stochastic demand case p_{ad}^{*st}, is not greater than the optimal price in the deterministic demand case $p_{ad}^{*\det}$, i.e., $p_{ad}^{*st} \le p_{ad}^{*\det}$.

In the multiplicative case, demand is assumed to be a continuous random variable of the form:

$$D(p,\omega) = \omega d(p) \tag{2.3}$$

where ω is a non-negative random variable with mean one that does not depend on the price. In the multiplicative demand case, the authors considered $d(p) = ap^{-b}$ ($a>0$, $b>0$) and showed that the optimal price in the stochastic demand case p_{mu}^{*st}, is not smaller than the optimal price in the deterministic demand case $p_{mu}^{*\det}$, i.e., $p_{mu}^{*st} \ge p_{mu}^{*\det}$.

2.3.2 Dynamic Pricing Models

Dynamic pricing models are concerned with how demand responses to price changes over time with respect to consumers' preferences and market conditions. In terms of applications, dynamic pricing models are particularly useful for those industries having:

1. High start-up costs;
2. Perishable capacity;
3. Short selling horizon;
4. A demand that is stochastic and price-sensitive.

Pricing policies that take into account dynamic pricing are today the key elements of success in business, because price is one of the most effective marketing mix variables by which managers can manipulate demand for the product over time. In recent years, the rapid developments of information technology (IT), the Internet, and e-commerce have had strong influences on the development of dynamic pricing policies in different industries. As Elmaghraby and Keskinocak (2003) state, there are three main aspects that contribute to the growing attention to apply dynamic pricing policies in practice:

1. The increased availability of demand data.
2. The ease of changing prices due to new technologies.
3. The availability of decision-support tools for analyzing demand data.

In practice, a variety of mathematical models has been used in computing optimal prices over time. Narahari et al. (2005) provided a list of five categories of models that are used in dynamic pricing:

1. *Inventory-based models*: These are models where pricing decisions are primarily based on the inventory levels.
2. *Data-driven models*: These models use statistical or similar techniques for utilizing the available data of consumers' preferences and buying patterns to compute the optimal dynamic prices.
3. *Game theory models*: In a multi-seller scenario, the sellers may compete for the same pool of consumers and this induces a dynamic pricing game among the sellers. Game theoretic models lead to interesting ways of computing optimal dynamic prices in such situations.
4. *Machine learning models*: An e-business market provides a rich playground for online learning by buyers and sellers. The sellers can potentially learn buyer preferences and buying patterns and use algorithms to dynamically price their offerings in order to maximize revenues or profits.
5. *Simulation models*: It is well known that simulation can always be used in any decision making problem. A simulation model for dynamic pricing may use any of the four models stated above or use a prototype system or any other way of mimicking the dynamics of the system.

In terms of modeling dynamic pricing applications, there are some key assumptions that determine which type of model should be used (e.g., Talluri and Ryzin 2004.)

1. *Deterministic vs. stochastic demand*: It depends on whether the seller has perfect information about demand or whether there is uncertainty about it.

2. *Myopic- vs. strategic-consumers*: The myopic-consumer models assume that a consumer makes a purchase once the posted price is below his valuation (willingness-to-pay) without considering future path of prices. But, the strategic-consumer models suppose that the consumers in the market take into account future path of prices when making purchase decisions (Aviv and Pazgal, 2008). Although the strategic-consumer model is more realistic, the myopic-consumer model is more tractable and widely used.
3. *Monopoly, oligopoly, and perfect competition*: Under assumption of perfect competition a firm can not influence the market price, thus, dynamic pricing will be not applicable. In the non-perfect competition cases, the assumption of oligopoly competition seems to be more realistic than the monopoly case. But, because of the complexity of the analysis and the difficulty of collecting competitors' data, monopoly models are more commonly used in dynamic pricing models.
4. *Dependent vs. independent demand over time*: Dependence or independence of demand is concerned with whether or not future demand for the product is affected by observed demand. The demand for a particular product is dependent on many aspects, e.g.: (i) the product is durable, (ii) the size of market population or the firm's fraction is very small, or (iii) consumers' knowledge about the product and consumers' willingness-to-pay for the product play important roles in making decisions.
5. *Replenishment vs. no replenishment of inventory*: Whether or not inventory replenishment is possible during the planned horizon affects whether as a seller needs to make the inventory decisions up front, before the selling season starts, or whether he will have access to additional units during the selling season.

In the next two sections, we will study deterministic and stochastic demand models in revenue management applications. As the basic problem of dynamic pricing in revenue management, we consider that a seller, who owns a fixed amount of a perishable product, aims to maximize his total revenues by selling inventory to a price-sensitive market over a finite horizon of time [$0,T$]. We assume that the consumers in the market are myopic, the seller is a monopolist, and there is not any opportunity to replenish inventory during the sales season. This situation arises in many industries such as apparel and retailing where the seller has to order in advance and replenishment usually is not possible because the lengths of lead-times are relatively larger than the selling season. Therefore, demand for the product will be a function of *time* and *price*[6].

2.3.2.1 Deterministic Demand Models

Deterministic models assume that the seller has perfect information about the demand process. Although this assumption is not suitable for applications where demand is hardly predictable at the beginning of the selling season, e.g., for new products and seasonal goods, deterministic models are widely used in practice because (i) deterministic models are easy to analyze, and (ii) they provide an upper bound with respect to the expected revenues of stochastic demand models as a good first-order approximation (e.g., Gallego and van Ryzin 1994, 1997).

We consider a retailer who wants to sell a fixed amount of a seasonal product q_0 over the time interval [*0,T*] divided into *T* periods. Let p_t and $d(p_t)$ denote, respectively, the price and the demand rate in period *t=1,…,T*. We assume that demand for the product in each period can be described by a regular demand function that is characterized by the following properties:

Assumption 2.1 Regular Demand Function:

1. *The demand function is continuously differentiable on* Ω_p.
2. *The demand function is strictly decreasing,* $\frac{\partial d(p_t)}{\partial p_t} = d_p(p_t) < 0$, *on* Ω_p.
3. *The demand function is bounded above and below:* $0 \le d(p_t) < \infty$, $\forall p_t \in \Omega_p$.
4. *The demand tends to zero for sufficiently high prices,* $\inf_{p_t \in \Omega_p} d(p_t) = 0$.
5. *The revenue function* $r(p_t) = p_t d(p_t)$ *is finite for all* $p_t \in \Omega_p$ *and has a finite maximizing price* $p^0 \in \Omega_p$.

6 There are some studies which assume that the availability of inventory also affects demand for the product. According to Soysal (2007) when inventory is limited, a consumer can not wait for a sale without taking into account the stockout risk. The author showed that although a diminishing of the availability of the product has a negative effect on the total quantity sold, it can improve sellers' profit.

Under this assumption the demand function is continuous, decreasing, bounded and tends to zero for sufficiently high prices. In this setting, the revenue management problem can be written as follows:

$$\begin{aligned} R &= \max \sum_{t=1}^{T} r(p_t) \\ s.t. \quad & \sum_{t=1}^{T} d(p_t) \le q_0 \\ & d(p_t) \ge 0 \\ & p_t \ge 0. \end{aligned} \tag{2.4}$$

Let $J(p) = \frac{\partial}{\partial p_t} r(p_t)$ be the marginal revenue such that:

Assumption 2.2 Monotonic Marginal Revenue: *The marginal revenue* $\frac{\partial}{\partial p_t} r(p_t)$ *is strictly decreasing in price.*

This assumption guarantees that the revenue function $r(p_t)$ is a concave function of the price that guarantees that the first-order conditions are sufficient for determining an optimal price (e.g., Talluri and van Ryzin, 2004). With $H(p_t) = (p_t - l)d(p_t)$ as the corresponding Hamiltonian function where $l \ge 0$ is the Lagrangian multiplier on the inventory constraint $\sum_{t=1}^{T} d(p_t) \le q_0$ and $l(q_0 - \sum_{t=1}^{T} d(p_t)) = 0$ the optimal price is given by

$$p_t^* = l - \frac{d(p_t^*)}{d_p(p_t^*)}, \tag{2.5}$$

where $d_p(p_t^*)$ is the partial derivation of $d(p_t^*)$ with respect to the optimal price and the Lagrange multiplier l can be considered as the marginal opportunity cost of inventory. In this setting, the price elasticity of demand with respect to the optimal price $\varepsilon_{p_t^*}$ is given by

$$\varepsilon_{p_t^*} = \frac{p_t^*}{d(p_t^*)} d_p(p_t^*). \tag{2.6}$$

From (*2.5*)

$$\frac{d_p(p_t^*)}{d(p_t^*)} = \frac{1}{l - p_t^*}. \tag{2.7}$$

Therefore

$$\varepsilon_{p_t^*} = \frac{p_t^*}{d(p_t^*)} d_p(p_t^*) = \frac{p_t^*}{l - p_t^*}. \qquad 2.8$$

If $l = 0$, i.e., the inventory constraint is not active, then the optimal policy is equal to

$$p_t^* = -\frac{d(p_t^*)}{d_p(p_t^*)}. \qquad 2.9$$

And then

$$\varepsilon_{p_t^*} = \frac{p_t^*}{d(p_t^*)} d_p(p_t^*) = -1. \qquad 2.10$$

On the other hand, if the inventory constraint is active, i.e., $q_0 - \sum_{t=1}^{T} d(p_t^*) = 0$, then the optimal policy is equal or greater than $-\frac{d(p_t^*)}{d_p(p_t^*)}$, because $l \geq 0$. That is, $p_t^* = -\frac{d(p_t^*)}{d_p(p_t^*)}$, is a lower bound on the optimal policy in deterministic demand models.

In the special case of time-homogeneous demand, i.e., $d(p_t) = d(p)$, a fixed-price solution can be optimal over the selling season (e.g., Gallego and van Ryzin, 1994). In this case, the optimal fixed-price will either be the price at which inventory will be sold out exactly at the end of the horizon that is called the *inventory clearing price* or the price at which revenue would be maximized that is called the *revenue maximizing price*.

Let p^0 denotes the revenue maximizing price. According to the assumptions of *2.1* and *2.2* at this price the marginal revenue is equal to zero, i.e., $J(p^0) = \frac{\partial}{\partial p^0} r(p^0) = 0$, and the price elasticity $\varepsilon_{p^0} = -1$, then $p^0 = -\frac{d(p^0)}{d_p(p^0)}$. Let $\overline{p}$ to be the inventory clearing price, i.e., $\overline{p}$ is chosen in such a way that $d(\overline{p}) = \frac{q_0}{T}$. Then

$$p^* = \max\{p^0, \overline{p}\}. \qquad 2.11$$

In what follows we present an example to illustrate the fixed-price case. Suppose that the demand function for the product is as $d(p) = 20 - 4p$, then we have $r(p) = 20p - 4p^2$, $J(p) = \frac{\partial}{\partial p} r(p) = 20 - 8p$, $p^0 = 2.5$, and $d(p^0) = 10$, see Figure *2.4*. Keeping the length of the selling season T constant when $\frac{q_0}{T} < d(p^0)$, i.e., $d(\overline{p}) < d(p^0)$, then $\overline{p} > p^0$. Thus the optimal fixed-price will be equal to $\overline{p}$, i.e., $p^* = \overline{p}$. In this case, the optimal revenue increases as the initial inventory increases (until $\frac{q_0}{T} = d(p^0)$). On the other hand, when $\frac{q_0}{T} \geq d(p^0)$, $d(\overline{p}) \geq d(p^0)$, then $\overline{p} \leq p^0$. Thus the optimal fixed-price is equal to p^0, i.e., $p^* = p^0$. Therefore in this case, the optimal price and the optimal revenue will remain constant as the initial inventory increases.

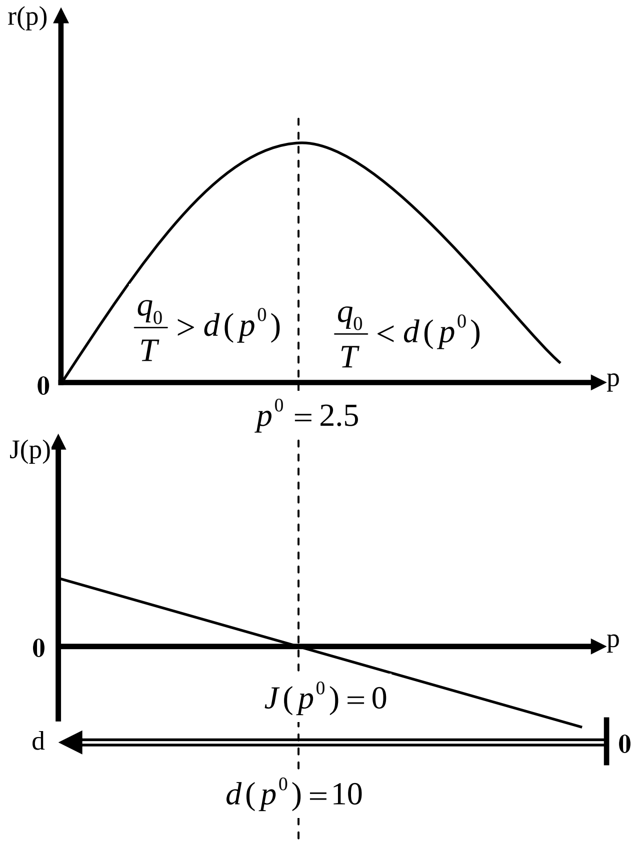

Figure 2.4 *Revenue and marginal revenue curves for the case of the linear-demand function as* $d(p) = 20 - 4p$.

Gallego and van Ryzin (1994) argued that the fixed-price heuristic of deterministic models for the entire of the horizon $p^* = \max\{p^0, \overline{p}\}$ will be asymptotically optimal in two limiting cases: (1) The number of items is large and there is enough time to sell them; and (2) There is the potential for a large number of sales at the revenue maximizing price. The quality of the deterministic approximation in this case also depends on the coefficient of variation $c.v. = \sigma / \mu$ rather than the variance σ^2 itself. For instance, if the length of the selling season increases, we should expect that the variance of the cumulative demand will also increase but the coefficient of variation will probably decrease. Similarly, products facing a high volume demand are likely to have a small coefficient of variation (e.g., Bitran and Caldentey, 2003).As a result, for the case of time-homogeneous demand the optimal price will be:

1. Non-increasing in the initial inventory;
2. Non-decreasing in the length of the selling horizon.

To show how the optimal price and the optimal revenue change with respect to the initial inventory and the length of the selling season, we consider a deterministic log-linear demand case $d(p) = 30e^{-p}$ as an example.

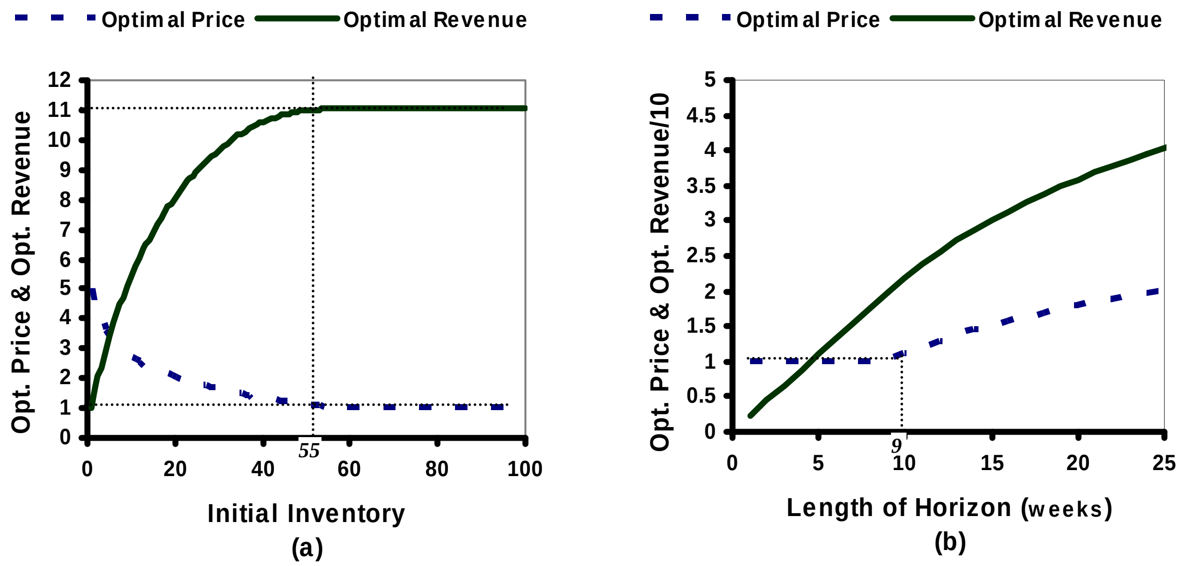

Figure 2.5 *Optimal price & optimal revenue for the deterministic (log-linear) demand case with rate* $d(p) = 30e^{-p}$.

Figure *2.5.a* shows- keeping constant the length of the selling season, in this example *T=5* weeks- that the optimal revenue is non-decreasing as the initial inventory increases while the optimal price is non-increasing as the initial inventory increases. Moreover, there is an optimal inventory level, in this example $q^* = 55$, that maximizes the optimal revenue and above this threshold, additional units of inventory do not change both the optimal price and the optimal revenue. According to Figure *2.5.b*, in contrast, if the initial inventory remains unchanged, in this example $q_0 = 100$, the optimal revenue is strictly increasing in the length of the selling season while the optimal price is non-decreasing. There is also a threshold with respect to the length of horizon, in this example $T_{threshold} \cong 9$, such that the optimal price increases by increasing the length of horizon to a value larger than the threshold.

2.3.2.2 Stochastic Demand Models

Dynamic pricing with stochastic demand are more complex to compute in comparison with deterministic demand cases. Stochastic models are clearly used more appropriately to describe real situations where the paths of demand and inventory are unpredictable over time and managers are forced to react dynamically to the market by adjusting prices as uncertainty reveals itself. In terms of literature, there are two types of stochastic demand models:

1. Continuous-time models;
2. Discrete-time models.

2.3.2.2.1 Continuous time models: In the case of continuous time models, Kincaid and Darling (1963) addressed the general issue of how to dynamically price a perishable product where the demand for the product follows a Poisson process with rate $d(p_t)$ and each arriving consumer at time *t* has a reservation price for the product that is a random variable with the distribution function for not buying $F(p_t) = P(v_t < p_t)$ where v_t is the reservation price in period *t*. The authors considered two cases: In the first case, the seller does not post prices, but receives offers from potential incoming buyers, which he either accepts or rejects. It is assumed that each arriving consumer offers his reservation price, i.e., consumers do not act as strategic players; In the second case, the seller posts the price at p_t and each arriving consumer purchases the product only if his reservation price is not less than p_t. Then, the demand process in the second case will be Poisson with rate $D(p_t) = A_t(1 - F(p_t))$ where A_t is the number of arriving consumers at time *t* and

$1-F(p_t)$ is the fraction of the arriving consumers, who are willing to pay p_t or more for the product.

Considering the available inventory at time *t* to be q_t, the optimality conditions for the value function $V_t(q_t)$ as follows:

$$V_t(q_t) = \max_{p_t \geq 0} E[p_t E[\min\{D(p_t), q_t\}] + V_{t+1}(q_t - D(p_t))]. \qquad 2.12$$

The optimal expected revenue gathered from time *t* to the end of the selling season by selling the available inventory and the optimal price can be determined for both cases, and closed-form solutions are reported for the special case $F(p_t) = 1 - e^{-p_t}$. As price is determined, the optimality condition (Hamilton-Jacobi-Bellman equation) is given by (e.g., Bitran and Caldentey, 2003)

$$-\frac{\partial V_t(q_t)}{\partial t} = \max_{p_t \geq 0}\{A_t(1-F(p_t)) \times (p_t - \Delta V_t(q_t))\}. \qquad 2.13$$

where $\Delta V_t(q_t) = V_t(q_t) - V_t(q_t - 1)$ is the expected marginal revenue of the q_t^{th} unit of inventory at time *t* (the opportunity cost of selling a unit of inventory at time *t* when the available inventory is q_t). According to this condition, it is easy to see that the optimal price satisfies $p_t \geq \Delta V(q_t)$. Under some restrictions on $F(p_t)$ and its density function $f(p_t)$, the first-order condition characterizes the optimal price as follows:

$$p^*_{t,q_t} = \frac{1-F(p^*_{t,q_t})}{f(p^*_{t,q_t})} + \Delta V_t(q_t). \qquad 2.14$$

Thus, the problem of computing an optimal price strategy reduces to the computation of the opportunity cost $\Delta V(q_t)$.

Similar to Kincadic and Darling (1963), Gallego and van Ryzin (1994) proved that the value function $V_t(q_t)$ is strictly increasing and strictly concave in both the remaining time until the end of the selling season, i.e., *T-t*, as well as the inventory level q_t. Furthermore, they showed that the optimal price p^*_{t,q_t} is increasing in the remaining time and decreasing in inventory level. That is:

1. At a given point in time, the optimal price decreases as inventory increases;

2. For a given level of inventory, the optimal price increases if there is more time to sell.

As an important limitation, Kincadic and Darling (1963) discussed there are no exact closed-form solutions for the optimal price strategy in equation (*2.13*) for the other demand functions in which the consumers' reservation price is not exponentially distributed. To avoid this limitation, some studies considered the discrete time models that we will discuss next.

2.3.2.2.2 Discrete time models: In the case of discrete time models, the length of the selling season is assumed to be divided into *T* periods such that there is only one consumer per period who is willing to pay ν_t for the product in period *t*. The reservation price in period *t* assumed to be a random variable with distribution $F(\nu_t) = P(\nu_t \le \nu)$. Therefore, if the seller sets the price at p_t in period *t*, he will sell exactly one unit of the product if $\nu_t > p_t$, with probability $1-F(p_t)$. Let $V_t(q_t)$ be the optimal expected revenues of the available inventory q_t at the beginning of period *t*, *t=1,…,T*, $d(p_t) = 1-F(p_t)$ the (average) demand rate, and $r(p_t) = p_t d(p_t)$ the revenue function, so that the problem can be formulated as follows (e.g., Talluri and van Ryzin, 2004):

$$\begin{aligned} V_t(q_t) &= \max_{d(p_t)\ge 0}\{d(p_t)(p_t + V_{t+1}(q_t - 1)) + (1-d(p_t))V_{t+1}(q_t)\} \\ &= \max_{d(p_t)\ge 0}\{r(p_t) - d(p_t)\Delta V_{t+1}(q_t)\} + V_{t+1}(q_t) \end{aligned} \qquad 2.15$$

where $V_{T+1}(q_t) = 0$ and $V_t(0) = 0$. Under assumption of monotonic marginal revenue, the necessary and sufficient conditions for the optimal rate $d(p_t^*)$ are $J(d(p_t^*)) = \Delta V_{t+1}(q_t)$.

Proposition 2.1

If assumption *2.2* holds, then the expected marginal value of inventory $\Delta V_{t+1}(q_t)$ is decreasing in *t* and q_t (e.g., Talluri and van Ryzin, 2004). That is:

1. $\Delta V_{t+1}(q_t) \le \Delta V_t(q_t)$;
2. $\Delta V_t(q_t + 1) \le \Delta V_t(q_t)$.

Therefore, the higher marginal values correspond to the lower optimal demand rates and hence the optimal higher prices. In fact, $\Delta V_{t+1}(q_t) \le \Delta V_t(q_t)$ states that the expected marginal value of inventory is greater as there are more time until the end of the selling season and $\Delta V_t(q_t+1) \le \Delta V_t(q_t)$ states that the expected marginal value of inventory increases as decreasing inventory.

To illustrate how the optimal price changes in stochastic demand models, we provide a numerical example. According to Gallego and van Ryzin (1994) the optimal solution for $d(p) = ae^{-p}$, $a > 0$, when there are q_t units of the product at the beginning of the given period *t*, *t=1,…,T*, will be as follows:

$$p_t^*(q_t) = V_t(q_t) - V_t(q_t - 1) + 1$$

where $V_t(q_t)$, the optimal expected revenues generated over (*t*,*T*], is given by

$$V_t(q_t) = \ln\left(\sum_{i=0}^{q_t} (d(p_t^0)t)^i \frac{1}{i!}\right) \quad , \quad d(p_t^0) = a/e$$ [7].

Suppose that the entire of the selling season is divided into *T=50* periods and the seller has $q_0 = 20$ units of the product at time zero. The seller updates the price at the start of each period based on the remaining time and the inventory on hand. Figure *2.6* shows a sample path of the optimal price when the demand function has the form $d(p) = 100e^{-p}$. Note that the x-axis in Figure *2.6* represents the number of the passed periods. As shown in Figure *2.6* after setting the initial price at time zero, the optimal price decreases as the time progresses without any sale and jumps up after each sale. Moreover, the slope at which the price decreases over time tends to increase as the selling horizon gets shorter.

In our example where $d(p) = 100e^{-p}$, the initial inventory $q_0 = 20$, and *T=1*[8], then, the revenue maximizing price and the inventory clearing price are given by

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[7] Under the assumption of $d(p) = ae^{-p}$, $a > 0$, the revenue maximizing price is given by

$$r(p) = pd(p) = ape^{-p} \rightarrow \frac{\partial \mathrm{r(p)}}{\partial \mathrm{p}} = ae^{-p} - ape^{-p}$$

$$\frac{\partial \mathrm{r(p)}}{\partial \mathrm{p}} = 0 \rightarrow ae^{-p}(1-p) = 0 \rightarrow p = 1$$

Then,

$$d(p_t^0) = ae^{-1} = a/e.$$

[8] For the single fixed price, we consider the entire of the selling season as one period.

$$r(p) = pd(p) = 100pe^{-p} \rightarrow \frac{\partial r}{\partial p} = 100e^{-p}(1-p) \rightarrow \frac{\partial r}{\partial p} = 0 \rightarrow p^0 = 1$$

$$d(\bar{p}) = q_0 \rightarrow 100e^{-\bar{p}} = 20 \rightarrow \bar{p} = 1.61\,.$$

Therefore, the optimal fixed-price will be $p^* = \max\{p^0, \bar{p}\} = 1.61$, see Figure *2.6*.

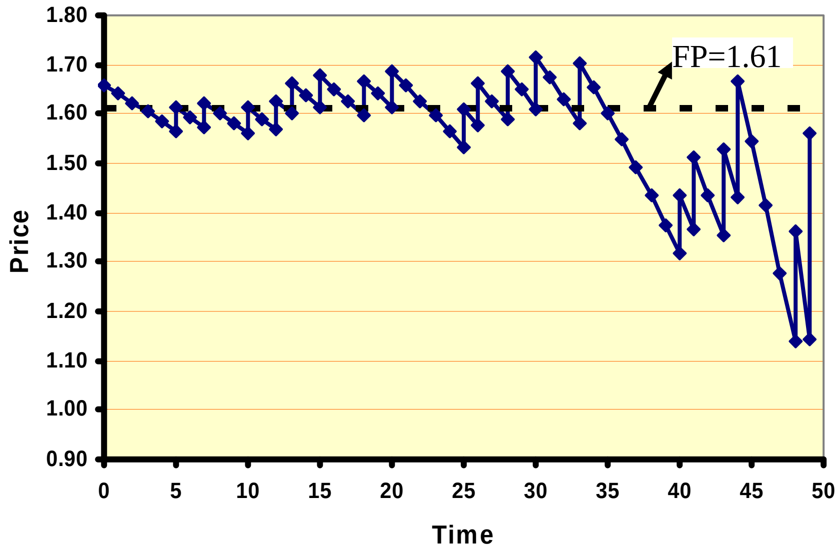

Figure 2.6 *A sample path of the optimal price and the optimal fixed-price of a stochastic demand* $D(t, d(p_t), \xi_t)$ *where* $d(p_t) = 100e^{-p_t}, q_0 = 20$.

In practice, implementing and controlling the optimal prices conduced from discrete time models is difficult (e.g., Bitran and Mondschein, 1997). As an alternative, the single fixed-price policy for whole of the selling season could be used if the firm has a large number of units to sell. The fixed-price model is simple and easy to implement and control. Hence, even if price changes are possible, managers often choose to use the fixed-price policy instead of the continuous time policies.

In what follows we review some other studies that considered some special cases of the problem of pricing a single perishable product over a finite horizon of time and highlight the main results. There are some studies in the revenue management literature that focused on the time of changing the price during the selling season. Feng and Gallego (1995) studied the problem of selling a given inventory over a finite horizon where there is a unique price change during the sales season and the seller has is to decide the optimal timing of changing the price. They showed that it is optimal to decrease (respectively increase) the initial price as soon as the remaining time falls below (respectively above) a time threshold that depends on the number of the remaining units. Feng and Xiao (2000) presented the problem of pricing the same product at different levels. They improved the model considered in Feng and Gallego (1995) and showed that the optimal policy is monotone.

Zhao and Zheng (2000) studied a dynamic pricing model for selling a given stock of a perishable product over a finite time horizon. In this setting, consumers arrive according to a non-homogeneous Poisson process. The author assumed that consumers' reservation price distribution changes over time, derived an optimality condition equivalent to (*2.13*), and showed that the value function $V_t(q_t)$ is concave on both the level of inventory and the duration of the selling season. They also showed that at any given time, the optimal price decreases with inventory and found a sufficient condition on the reservation price distribution that guarantees that the optimal price is non-decreasing on the duration of the selling horizon.

Bitran and Mondschein (1997) considered a continuous time problem of pricing a seasonal product where a seller faces a stochastic arrival of the consumers with different valuations of the product. They modeled optimal pricing policies as functions of time and inventory and showed that demand uncertainty leads to higher prices, larger discounts, and more unsold inventory. The authors also addressed the practical problem of implementing continuous time models and pointed out some reasons such as coordinating and management costs associated with changing prices and destruction of product's value to explain why the managers prefer to revise their pricing policies periodically instead of continuously. They extended the basic model to incorporate periodic reviews where prices are allowed to change at discrete time intervals and found that the loss experienced by implementing periodic pricing reviews instead of the continuous time pricing policies is very small if the seller selects an appropriate number of reviews.

In the retail chain context, Bitran et al. (1998) extended the single perishable product periodic review formulation of Bitran and Mondschein (1997) to the case of a retail chain. The authors assumed that a certain amount of a product is sold at different stores and each store has its own demand pattern. Under the constraint that at every given time the price must be the same at all the locations, the authors derived optimality conditions and a set of heuristics for two cases when inventory transfers among stores are allowed and not allowed and proposed a methodology to set prices of seasonal products where the retail chain changes prices periodically according to the inventory level as well as the time left to the end of the selling season. The authors described the consumers' arrival process by a Poisson distribution with a time dependent arrival rate (non-homogeneous Process) and supposed a probability distribution for the consumers' reservation price to capture the heterogeneity of the consumers' purchasing behavior. The heuristics are constructed using a rolling horizon approach, whereby at every decision point the price is computed assuming that this is the last time that the price will be revised. Computational experiments showed that this type of heuristic performs quite well with an average error of *2%–3%*. The paper also includes a set of numerical experiments that were conducted using real data collected from a retail chain store; see also Federgruen and Heching (2002).

Smith and Achabal (1998) also studied clearance pricing policies in retail chain. They assumed that demand for the product is a function of price, the inventory on-hand, and seasonal effects and found that prices should be set higher before the clearance period starts, and then reduced more sharply during the clearance period. Mantralla and Rao (2001) also considered the problem of optimal ordering and markdown decisions in retailing. The authors provided a decision-support system by which retailers are able to determine the optimal initial inventory and then markdown pricing decisions. Gupta et al. (2006) studied the problem of setting prices for clearing retail inventories of fashion goods, presented a discrete time model for both deterministic and stochastic demand, and showed that the penalty for choosing the markdown price once and then keeping it unchanged is small either when the mean reservation prices do not change over time or when the mean of the reservation prices drops significantly after the first period.

Chapter 3

Demand Forecasting
&
Demand Learning

3.1 Introduction

Demand forecasting is essential to make decisions of business activities such as production (ordering) and marketing (pricing, distribution, etc.). Most of firms cannot simply wait for demand to emerge and then react to it. Instead, they must anticipate and plan for future demand so that they can react immediately to consumer orders as they occur. For instance, in the case of selling perishable products over a finite horizon of time, the firm has to order in advance while demand is not yet realized. In addition, the replenishment of inventory usually is not possible within the selling period (supply inflexibility) and firms have to determine the initial inventory and price according to primary forecasts of demand. In such cases, the seller usually sets an initial price for the product according to his belief (prediction) about demand for the product and changes it over time based on the realized demand, the remaining amount of stocked items, and the time until the end of the selling season. The ultimate aim of the seller is to maximize the total revenues of selling inventory by setting the right price for the product at the right time. In order to set the right price for the product one needs to know how demand reacts to change of prices over time.

In practice, however, there are many situations where the firm does not have full knowledge of the demand patterns when he determines the prices. Moreover, the demand patterns may be changing over time in ways that are not necessarily predictable. From the firm's point of view, the more uncertainty associated with future demand, the more difficulty to make the effective pricing decisions occurs and the more possibilities to make potentially bad decisions as well. Thus, resolving demand uncertainty plays an important role for managing demand and the success of the firm's operations in the market.

This chapter is devoted to study demand uncertainty as one of the most important problems faced by managers when determining the optimal prices for perishable products over time. There are several demand forecasting methods used to resolve demand uncertainty. We begin with a short introduction to revenue management systems that demand forecasting is one of the most important parts of it, and then focus on demand forecasting methods that are used commonly in revenue management applications. After that, we will consider the consumers' arrival rate as one of the sources of demand uncertainty and provide a Bayesian learning approach to compute the distribution of the consumers' arrivals during the remaining time until the end of the selling season.

Additionally, we present a review of demand learning literature and highlight the main results.

As mentioned in the preceding chapter, revenue management is concerned with demand-management decisions and the methodology and *systems* required making them. A revenue management system (RMS) is a tool that supports decision making with respect to the revenue problems such as pricing decisions, distribution decisions, and promotion decisions. As shown in Figure *3.1*, a RMS includes four main steps:

1. *Data collection*: Collecting relevant data about the product such as prices, demand, competition, and other factors that affect sales.
2. *Estimation & forecasting*: Determining a demand model, estimating the parameters of the demand model, and forecasting demand based on these parameters.
3. *Optimization*: Finding an optimal set of controls such as prices, allocations, markdowns, etc., to maximize expected revenues.
4. *Control*: Controlling the sales and inventory.

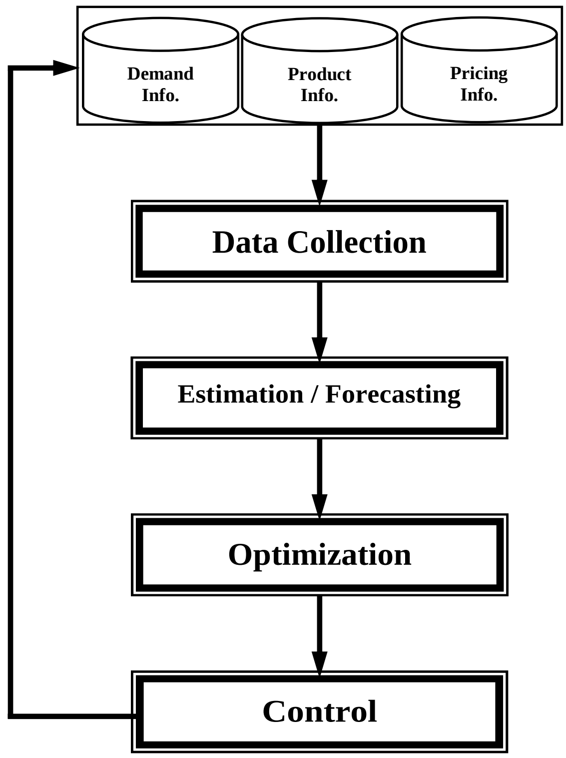

Figure 3.1 *Revenue management systems.*
Source: Talluri and van Ryzin (2004).

3.2 Demand Forecasting

Forecasting is the key tool from which all other revenue management subjects originate. As forecasts are looking into the unknown future, some level of error between demand forecasts and actual demand has to be expected. Thus, the goal of forecasting is to minimize the error between the forecast values of demand and the actual demand. A good demand forecasting will help a firm to increase its revenues by minimizing the lost sales.

Since most of the revenue management applications such as dynamic pricing use stochastic models of demand, the optimization models require good estimates of the probability distribution of future demand or at least parameter estimates for the assumed demand distribution. Such estimates can be made in two ways:

1. *Parametric estimation:* A specific functional form of demand is assumed and the estimates of the parameters of this function have to be calculated.
2. *Nonparametric estimation:* Distributions or functions have to be directly estimated based on observed data.

While nonparametric estimations seem to be more general, they have two main flaws (e.g., Talluri and van Ryzin, 2004): First, they often require much more information than is available in many revenue management applications to obtain reasonable estimates of future demand, especially for the case of selling new products or of fashion or seasonal goods; Second, they may not provide good estimates of future demand even if they fit the observed data well. In contrast to nonparametric estimations, parametric estimations are more widely used in RM applications because they are better able to smooth out the noise within demand data. Further, they have the advantage of providing estimates of demand that extend beyond the range the observed demand data, and are generally more robust to errors and noise in data. However, parametric estimations can suffer from specification errors, i.e., they assumed that demand distribution is significantly different from the real demand distribution (which can be tested).

In terms of estimation, there are two well-known approaches to find estimators: (1) Minimum mean-square error (MSE) and (2) Maximum-likelihood (ML). Minimum mean-square error estimators which have been mostly studied in the case of linear regression are those values of parameters that minimize the sum of squared differences between the observed and expected values of the observations. Maximum-likelihood estimators, in contrast, are the values of the parameters for which the observed sample is most likely to have occurred.

While estimation is concerned with the problem of finding the parameters of the demand model to describe the observed data, forecasting involves predicting future demand that is unobserved yet. Demand forecasting is our next topic in this chapter in which we review some methods that will be used in this work.

3.3 Forecasting Methods

There are various forecasting methods which can be divided into two main categories: (1) *Qualitative methods* that are based on opinions and intuition and (2) *Quantitative methods* that use mathematical and historical data to make forecasts. In what follows we briefly review some forecasting methods that are mostly used in demand forecasting and refer the reader to Makridakis et al. (1998), Talluri and van Ryzin (2004), and Armstrong and Green (2005) for a review on forecasting methods.

3.3.1 Qualitative Forecasting Methods

Qualitative forecasting methods are most appropriate when there is little historical data available. In respect to our work, these methods can be used to determine the initial inventory, i.e., how many units should be ordered, and the initial price.

3.3.1.1 Unaided Judgment

It is common practice to ask experts what will happen. This is a good method to use when

- Experts are unbiased.
- Large changes are unlikely.
- Relationships are well understood by experts.
- Experts possess privileged information.
- Experts receive accurate and well-summarized feedback about their forecasts.

3.3.1.2 Delphi

In Delphi forecasting method, the administrator should recruit between five and twenty suitable experts and poll them for their forecasts and reasons. The administrator then provides the experts with anonymous summary statistics on the forecasts, and experts' reasons for their forecasts. The process is repeated until there is little change in forecasts between rounds – two or three rounds are usually sufficient. The Delphi forecast is the median or mode of the experts' final forecasts.

3.3.1.3 Structured Analogies

The outcomes of similar situations from the past (analogies) may help a marketer to forecast the outcome of a new (target) situation. For example, the introduction of new products in the interest market can provide analogies for the outcomes of the subsequent release of similar products in other markets. To use the structured analogies method, an administrator prepares a description of the target situation and selects experts who have knowledge of analogous situations; preferably direct experience. The experts identify and describe analogous situations, rate their similarity to the target situation, and match the outcomes of their analogies with potential outcomes in the target situation. The administrator then derives forecasts from the information the experts provided on their most similar analogies.

3.3.1.4 Judgmental Decomposition

The basic idea behind judgemental decomposition is to divide the forecasting problem into parts that are easier to forecast than the whole. A decision maker forecasts the parts individually, using methods appropriate to each part. Finally, the parts are combined to obtain a forecast.

3.3.1.5 Judgmental Bootstrapping

Judgmental bootstrapping converts subjective judgments into structured procedures. Experts are asked what information they use to make predictions about a class of situations. They are then asked to make predictions for diverse cases, which can be real or hypothetical. The resulting data are then converted to a model by estimating a regression equation relating the judgmental forecasts to the information used by the forecasters. Judgemental bootstrapping models are most useful for repetitive complex forecasting problems where data on the dependent variable are not available (e.g. demand for a new product) or data does not vary sufficiently for the estimation of an econometric model.

Judgmental bootstrapping also allows experts to see how they are weighting various factors. This knowledge can help to improve judgmental forecasting. Bootstrapping also allows for estimating effects of changing key variables when historical data are not sufficient to allow for estimates.

3.3.1.6 Expert Systems

Expert systems are structured representations of the rules experts use to make predictions or diagnoses. Expert systems forecasting involves identifying forecasting rules used by experts and rules learned from empirical research. One should aim for simplicity and completeness in the resulting system, and the system should explain forecasts to users.

3.3.1.7 Conjoint Analysis

By surveying consumers about their preferences for alternative product designs in a structured way, it is possible to infer how different features will influence demand. The potential customer is thus forced to make trade-offs among various features by choosing one of each pair of offerings in a way that is representative of how they would choose in the marketplace. The resulting data can be analysed by regressing respondents' choices against the product features.

3.3.2 Quantitative Forecasting Methods

In quantitative forecasting methods, historical data is needed and mathematical methods are used for forecasting.

3.3.2.1 Time Series Forecasting Methods

Time series forecasting methods are perhaps the most frequently used class of techniques among all the quantitative forecasting methods. They are based on assumption that future is an extension of the past. In these methods, historical data is used to predict future demand. Among the different time series forecasting methods, we use the special cases of moving average forecasting and exponential smoothing forecasting methods as two alternatives of our Bayesian forecasting approach to predict the number of arriving consumers at the store. These forecasting approaches are widely used in the revenue management applications.

3.3.2.1.1 Moving average methods: In these simple forecasting methods, we assume that the values of the forecasts corresponding to the next periods are equal to the average of the k last observations. Let T be the number of periods and $n_1,...,n_t$ be the observed values in the periods $1,...,t$. The forecast of the next period, i.e., period $t+1$, is given by

$$\hat{N}_{t+1} = \frac{n_t + n_{t-1} + ... + n_{t-k+1}}{k}. \qquad 3.1$$

As the value of period $t+1$ unfolds, i.e., n_{t+1}, the forecast of period $t+2$ is given by

$$\hat{N}_{t+2} = \frac{n_{t+1} + n_t + ... + n_{t-k+2}}{k} = \hat{N}_{t+1} + \frac{n_{t+1} - n_{t-k+1}}{k}. \qquad 3.2$$

As a special case, if $k=t$ the forecasts of the next periods would be simply equal to the average of all observations that we are using it in our work.

3.3.2.1.2 Exponential smoothing methods: Exponential smoothing methods are commonly used in revenue management practice because they are simple and robust (Talluri and van Ryzin, 2004). The simplest version of exponential smoothing methods is defined by a single parameter, $0 < \alpha < 1$. Let T be the number of periods and $n_1,...,n_t$ be the observed values in the periods $1,...,t$. The forecast of the next period, i.e., period $t+1$, is given by

$$\hat{N}_{t+1} = \alpha \sum_{k=0}^{t-1} (1-\alpha)^k n_{t-k}. \qquad 3.3$$

In exponential smoothing methods, smaller values of α smooth the forecasts more and spread the weights over a longer course, while larger values of α make the forecasts more responsive to the recent observations. Interested readers are referred to Silver and Peterson (1985) for a review of moving average and exponential smoothing forecasting methods in more details.

3.3.2.2 Bayesian Forecasting Methods

Bayesian forecasting is a natural product of the Bayesian approach to inference. The Bayesian approach in general requires explicit formulation of a model and conditioning on known quantities, in order to draw inferences about unknown ones. Bayesian forecasting methods use the Bayes formula to merge a prior belief about forecast values with information obtained from observed data. The methods are especially useful when there is no historical data such as introducing new products, fashion/style and seasonal goods.

Consider a model with parameter δ. The seller has some initial believes about the value of this parameter and collects data to improve this understanding. Under Bayesian analysis the seller's believes about the parameter is represented by a probability distribution over all possible values that the parameter can take, where the probability represents how likely the seller thinks it is for the parameter to take a particular value. Prior to collecting data, the seller's believes are based on logic, intuition, or past analyses. These believes are represented by a density on δ, called the prior distribution and denoted $f(\delta)$. The seller collects data in order to improve his ideas about the value of δ. Suppose the seller observes a sample as $Y = \{y_1,..., y_n\}$. Based on this sample information, the seller changes (updates) his ideas about δ. The updated believes are represented by a new density on δ as $g(\delta \mid Y)$ called the posterior distribution. This posterior distribution depends on *Y*, since it incorporates the information that is contained in the observed sample. The question that arises is how exactly do the seller's believe about δ change from observing *Y*? That is, how does the posterior distribution $g(\delta \mid Y)$ differ from the prior distribution $f(\delta)$? There is a precise relationship between the prior and posterior distribution, established by Bayes' rule. Let $P(y_n \mid \delta)$ be the probability that y_n is the n^{th} value. The probability of observing the sample outcomes *Y* is

$$L(Y \mid \delta) = \prod_{k=1}^{n} P(y_k \mid \delta). \qquad 3.4$$

This is the likelihood function of the observed choices. Note that it is a function of the parameters δ.

Bayes' rule provides the mechanism by which the seller improves his ideas about δ (e.g., Train, 2003). By the rules of conditioning,

$$g(\delta \mid Y)L(Y) = L(Y \mid \delta)f(\delta), \qquad 3.5$$

where *L(Y)* is the marginal probability of *Y*, marginal over δ:

$$L(Y) = \int L(Y \mid \delta)f(\delta)d\delta. \qquad 3.6$$

Both sides of equation (*3.6*) represent the joint probability of *Y* and δ, with the conditioning in opposite directions. The left-hand side is the probability of *Y* times the probability of δ given *Y*, while the right-hand side is the probability of δ times the probability of *Y* given δ. Rearranging, we have

$$g(\delta \mid Y) = \frac{L(Y \mid \delta) f(\delta)}{L(Y)}. \qquad 3.7$$

This equation is Bayes' rule applied to the prior and posterior distributions. In general, Bayes rule links conditional and unconditional probabilities in any setting and does not imply a Bayesian perspective on statistics. Bayesian statistics arises when the unconditional probability is the prior distribution and the conditional probability is the posterior distribution. We can express equation (*3.7*) in a more compact and convenient form. The marginal probability of *Y,* i.e., *L(Y),* is constant with respect to δ and, more specifically, is the integral of the numerator of (*3.7*). As such, *L(Y)* is simply the normalizing constant that assures that the posterior distribution integrates to *1*, as required for any proper density. Using this fact, equation (*3.7*) can be stated more short by saying simply that the posterior distribution is proportional to the prior distribution times the likelihood function:

$$g(\delta \mid Y) \propto L(Y \mid \delta) f(\delta). \qquad 3.8$$

Intuitively, the probability that the seller assigns to a given value for the parameters *after* seeing the sample is the probability that he assigns *before* seeing the sample times the probability (i.e., likelihood) that those parameter values would result in the observed choices.

Regarding to our work, we assume that values, for instance, $N_1, N_2, ..., N_t, ..., N_T$ describe a sequence of independent and identically distributed random variables and that N_t has a probability density function $f(n \mid \delta)$ that is a function of an unknown parameter δ. The parameter δ itself is assumed to be a random variable with a probability density function $g(\delta)$, called the *prior distribution*. Let $g_0(\delta)$ presents our prior distribution and n_1 denotes the first observed value. The posterior distribution of δ is given by

$$g_1(\delta) = \frac{g_0(\delta) f(n_1 | \delta)}{\int_0^\infty g_0(\delta) f(n_1 | \delta) d\delta}. \qquad 3.9$$

Then the Bayes estimator δ^* is the expected value of δ based on the posterior distribution:

$$\delta^* = E[\delta] = \int_0^\infty \delta \, g_1(\delta) d\delta \qquad 3.10$$

The estimator δ^* has several useful theoretical properties. As an important advantage, it minimizes the variance of the forecast error. The value δ^* is used in forecasting by setting $\hat{N}_t = E[N_t | \delta^*]$. The readers are referred to West and Harrison (1989) as an excellent review of Bayesian learning and forecasting.

In Bayesian probability theory, a class of the prior probability distributions $P(\delta)$ is said to be conjugate to a class of likelihood functions $P(x | \delta)$ if the resulting posterior distributions $P(\delta | x)$ are in the same family as $P(\delta)$. That is, the posterior distribution of the parameter δ has the same distribution as the prior. We point out here some pairs of conjugate families of prior distributions (see for more details DeGroot and Schervish, 2002):

Bernoulli-beta: Let $N_1, N_2, ..., N_t$ be 0-1 random variables having a Bernoulli distribution with $P(N_k = 1) = \delta$, and δ has a beta distribution with parameters (α, β). After observing $n_1, n_2, ..., n_t$, δ has a beta distribution with parameters and ($\alpha + \sum_{k=1}^{t} n_k$, $\beta + t - \sum_{k=1}^{t} n_k$).

Poisson-gamma: Let $N_1, N_2, ..., N_t$ have a Poisson distribution with mean λ, and λ has a gamma distribution with parameters (α, β). After observing $n_1, n_2, ..., n_t$, λ has a gamma distribution with parameters ($\alpha + \sum_{k=1}^{t} n_k$, $\beta + t$).

Exponential-gamma: Suppose that $N_1, N_2, ..., N_t$ have an exponential distribution with rate λ, and λ has a gamma distribution with parameters (α, β). After observing $n_1, n_2, ..., n_t$, λ has a gamma distribution with parameters ($\alpha + t$, $\beta + \sum_{k=1}^{t} n_k$).

Normal-inverse gamma: Suppose that $N_1, N_2, ..., N_t$ have a normal distribution with mean μ and unknown variance σ^2, and σ^2 has an inverse gamma distribution with parameters (α, β). After observing $n_1, n_2, ..., n_t$, σ^2 has an inverse gamma distribution with parameters ($\alpha + t/2$, $\beta + 1/2 \sum_{k=1}^{t} (n_k - \mu)^2$).

In this work, we suppose that the demand for each product is the result of an arrival process and a consumer choice behavior. In the following section, we focus on consumer arrival processes. The consumer choice behavior will be discussed in the next chapter.

3.4 Consumer Arrival Process

In what follows we consider consumer arrival processes. Suppose there is a consumer population of size M in the market for the product, where each consumer has probability ζ of arriving at the store during the sales season. Then the number of arriving consumers N has a binomial distribution:

$$P(N=n)=\binom{M}{n}\zeta^{n}(1-\zeta)^{M-n} \qquad 3.11$$

with the expected number of arriving consumers $E[N]=\sum_{n=1}^{M} nP(N=n)=\zeta M$. Supposing that we increase *M*, while keeping the expected number of arriving consumers constant, i.e., $E[N]=\zeta M=\lambda$. Thus $\zeta=\lambda/M$.

$$\begin{aligned}\lim_{M\to\infty} P(N=n) &= \lim_{M\to\infty}\binom{M}{n}\zeta^{n}(1-\zeta)^{M-n}\\ &= \lim_{M\to\infty}\binom{M}{n}(\frac{\lambda}{M})^{n}(1-\frac{\lambda}{M})^{M-n}\end{aligned}$$

Then

$$\lim_{M\to\infty} P(N=n)=\frac{\lambda^{n}}{n!}e^{-\lambda}. \qquad 3.12$$

That is, the number of arriving consumers at the store *N* has a Poisson distribution with parameter λ where the parameter λ represents the expected number of the consumers' arrivals. As a result, the Poisson distribution can be used to model the number of arriving consumers from a large population of potential consumers.

We consider the case in which the consumers' arrivals at the store occur over the time interval [*0,T*], where *T* is the length of the selling season. We assume that for each consumer, his arrival time is generated according to a probability distribution on [*0,T*], independent from any other consumer. We split [*0,T*] in two intervals [*0,t*] and (*t,T*] and assume that each consumer determines his arrival time according to a uniform distribution on [*0,T*]. Under these assumptions every instant in [*0,T*] is equally likely and then a consumer arrival occurs in [*0,t*] with probability *t/T*. Let $N(t)$ denotes the number of the consumers' arrivals in [*0,t*], for $0 \le t \le T$ and $E[N(T)] = \lambda T$, thus on average λ consumer arrivals occur per time unit (period). That is, *N(t)* and *N(t,T)=N(T)-N(t)* have independent Poisson distributions with parameters λt and $\lambda(T - t)$. Therefore

$$\lim_{M \to \infty} P(N(t) = n) = \frac{(\lambda t)^n}{n!} e^{-(\lambda t)} \tag{3.13}$$

and

$$\lim_{M \to \infty} P(N(t,T) = n) = \frac{(\lambda(T-t))^n}{n!} e^{-(\lambda(T-t))} \tag{3.14}$$

3.5 Poisson Process

The Poisson process is a very well-known stochastic process for modeling purposes in numerous practical applications, e.g., to model consumer arrival process (e.g., Tijms, 1986, Ross, 2002). A Poisson process *N* is a stochastic process in which arrivals (events) occur continuously and independently of one another. More specifically, Poisson processes are counting processes.

Definition 3.1 Poisson Random Variable: *A random variable N* with outcomes *0,1,2,…, and mean* Λ *is a Poisson random variable if* $P(N = n) = \dfrac{e^{-\Lambda}\Lambda^n}{n!}$.

According to the definition $E[N] = Var[N] = \Lambda$. That is, variance-to-mean ratio for a Poisson random variable is equal to *1*.

Definition 3.2 Poisson Process: *A Poisson process N with the arrival rate function (intensity)* λ *is a stochastic process that has the following properties:*

1. *N is a counting process- N(0)=0, and for t>0, N(t) is non-decreasing and takes on only non-negative integer values;*
2. *Increments on non-overlapping time intervals are independent of one another as random variables, i.e., any set of increments* $N(t_l + \tau_l) - N(t_l)$ *for* $l = 1,2,\dots,n$ *is independent; and*

3. *For all* $t \geq 0$ *and* $\tau > 0$, *the increment* $N(t+\tau) - N(t)$ *is a Poisson random variable with mean* $\Lambda = \int_t^{t+\tau} \lambda(x)dx$.

Let *N(t)* be the number of arrivals that occur after time *0* up through and including time *t>0*, *N(t)* is a random variable for each value of *t*.

Homogeneous vs. non-homogeneous Poisson process: A homogeneous Poisson process is characterized by a rate function λ such that the number of arrivals over the time interval $(t, t+\tau]$ follows a Poisson distribution with parameter $\lambda\tau$. That is

$$P[(N(t+\tau) - N(t)) = n] = \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!}, \qquad n = 0,1,... \qquad 3.15$$

where $N(t+\tau) - N(t)$ is the number of arrivals within the time interval $(t, t+\tau]$. While the rate function λ is constant in homogeneous Poisson process, in the non-homogeneous Poisson process it is given by $\lambda(t)$. In this case, the expected number of arrivals during the time interval $(t, t+\tau]$ is

$$\lambda_{t\in(t,t+\tau]} = \int_t^{t+\tau} \lambda(t)dt. \qquad 3.16$$

Therefore, the number of arrivals follows a Poisson process that is given by

$$P[(N(t) - N(t+\tau) = n] = \frac{\mathrm{e}^{-\int_t^{t+\tau}\lambda(t)dt}(\int_t^{t+\tau}\lambda(t)dt)^n}{n!}, \qquad n = 0,1,... \qquad 3.17$$

Memoryless property of Poisson process: A Poisson process is generally characterized by the property of memoryless that means the number of arrivals occurring in any interval of time after time *t* is independent of the number of arrivals occurring before time *t*.

$$P\{X > t+\tau \mid X > t\} = P\{X > \tau\}, \qquad \tau \geq 0 \qquad 3.18$$

Sum of Poisson processes: Suppose that the Poisson processes N_1 with rate λ_1 and N_2 with rate λ_2 are independent. Then the counting process N defined by $N(t) = N_1(t) + N_2(t)$ is a Poisson process with rate λ given by

$$\lambda(t) = \lambda_1(t) + \lambda_2(t). \qquad 3.19$$

Parameter estimation: Suppose that we observe the number of arrivals periodically. Given a sample of *t* observation $n_1,...,n_t$, we interested in estimating the value of the rate λ of the corresponding Poisson process. To calculate the maximum likelihood value of λ, we can use the log-likelihood function as follows:

$$\begin{aligned} L(\lambda) &= \log \prod_{k=1}^{t} f(n_k \mid \lambda) \\ &= \sum_{k=1}^{t} \log(\frac{e^{-\lambda}\lambda^{n_k}}{n_k!}) \\ &= -t\lambda + (\sum_{k=1}^{t} n_k)\log(\lambda) - \sum_{k=1}^{t}\log(n_k!) \end{aligned} \quad 3.20$$

Take the derivative of $L(\lambda)$ with respect to λ and equate it to zero:

$$\frac{d}{d\lambda}L(\lambda) = 0 \Rightarrow -n + \frac{1}{\lambda}(\sum_{k=1}^{t} n_k) = 0 \Rightarrow \hat{\lambda} = \frac{\sum_{k=1}^{t} n_k}{t} \quad 3.21$$

Consider $N(t) = \sum_{k=1}^{t} n_k$, then *N(t)/t* is an unbiased estimator of λ. We also have $N(t) = \sum_{s=1}^{t} N(s-1,s)$, with all *N(s−1,s)* independent and identically distributed. Thus the law of large numbers applies, and $N(t)/t \to \lambda$.

To check whether $n_1,...,n_t$ are likely from a homogeneous Poisson process we can calculate the sample variance as follows:

$$s_n^2 = \frac{1}{t-1}\sum_{k=1}^{t}(n_k - \hat{\lambda})^2. \quad 3.22$$

If $s_n^2 \approx \hat{\lambda}$, then it is well possible that the numbers of arrivals have the same distribution. However, it often occurs that $s_n^2 > \hat{\lambda}$ that is called over-dispersion with respect to the Poisson process that arises from either the arrivals are not Poisson, or the arrivals are Poisson but with different parameters.

Bayesian inference: In Bayesian inference, the conjugate prior distribution for the rate of the Poisson process λ is assumed to be a gamma distribution with parameters (α, β). Thus

$$f(\lambda;\alpha,\beta)=\frac{\beta(\beta\lambda)^{\alpha-1}e^{-\beta\lambda}}{\Gamma(\alpha)}, \qquad \lambda>0\,. \qquad 3.23$$

Then, given the sample of *t* observations $n_1,...,n_t$ and a prior of a gamma distribution with parameters (α,β), the posterior distribution of the rate λ will be again a gamma distribution with parameters $(\alpha+\sum_{k=1}^{t}n_k\,,\beta+t)$.

3.6 Literature Review of Demand Learning

Demand learning is an effective approach to resolve demand uncertainty in which a decision maker learns from the observed demand information after the selling season unfolds by updating the forecasts of future demand.

Demand learning models have been studied in both the inventory control and revenue management literature to better forecast future demand. The inventory control problem is mainly concerned with determining the optimal inventory for the next periods and most of the studies in this context use periodic review models in which demand in each period is considered to be a random variable that follows a distribution function with an unknown parameter. At the beginning of each period, the observed demand data is used to revise the unknown parameter of the demand distribution by using Bayes' rule in order to determine the optimal inventory for the coming periods. As more observations become available, demand uncertainty is resolved and the parameter of the demand distribution approaches its true value (e.g., Scarf, 1959, Murray and Silver, 1966, Azoury, 1985, Popovic, 1987, Bradford and Sugrue, 1990, Hill, 1999, Lariviere and Porteus, 1999, Berk et al., 2007).

Lazear (1986) considered a two-period problem for the situation of selling a product. In this model, it is assumed that a firm sells a single unit of the product to a price-sensitive market of *M* consumers. He assumes that each arriving consumer is willing to pay the amount *v* for the product, i.e., *v* is the consumer's reservation price. The firm does not know *v* with certainty, but has some prior belief of the density of *v*. In each period, there are two types of consumers, (1) *buyers* whose reservation price is not less than the posted price and who will buy the product and (2) *shoppers* whose reservation price is less than the posted price and who will leave the store empty-handed. Hence, the firm has information about the probability that an arriving consumer is a buyer or a shopper. Based on the prior belief of the consumers' reservation price, the author used a Bayesian approach to update the consumers' reservation price and set the price for the second period. The author showed that the price is monotonically decreasing in time.

Fisher and Raman (1996) documented a dramatic improvement in forecast accuracy that can be achieved after having observed initial sales for fashion items. They modeled and analyzed the decisions required under Quick Response where firms can quickly react to demand by improving of ordering (production) lead times and gave a method for estimating the demand probability distributions. Eppen and Iyer (1997) proposed an inventory control model in which demand during the first phase of the sales season is correlated with the demand during the remainder of the selling season. This correlation is due mainly to the uncertainty about the exact parameters of the underlying demand process.

Petruzzi and Dada (2001) considered a two-period model where the firm faces uncertain demand which depends on both price and the inventory level. The model is an extension of the newsvendor problem by considering price as a decision variable. The seller determines (1) how much to buy from a supplier for delivery in the periods *1* and *2* before the beginning of the season and (2) the price at which the product should be sold. After having observed data of the sales in the first period, the seller determines the stocking quantity and the selling price for the second period. The authors showed that all decisions can be determined uniquely as a function of the first period stocking factor. They also found that the cost of learning is a consequence of censored information and shared with the consumer in the form of a higher selling price when demand uncertainty is additive, see also Dada et al. (2007).

While most research in inventory control assumes that the price is exogenous and the firm decides how much inventory has to be replenished in each time period, there are some studies that consider both pricing and inventory decisions (e.g., Subramanian and Shoemaker, 1996, Chan et al., 2004).

Petruzzi and Dada (2002) studied the problem of determining optimal stocking and pricing policies over time when a given market parameter of the demand process, though fixed, is initially unknown. In this setting, because of the initially unknown market parameter, the decision maker begins with a subjective probability distribution associated with demand. Learning occurs as the firm monitors the market's response to its decisions and then updates its characterization of the demand function. The authors found that the first-period optimal selling price increases with the length of the problem horizon. However, for a given problem horizon, prices can rise or fall over time depending on how the scale parameter influences demand.

Bitran and Wadhwa (1996) described a modeling approach for using observed sales data to update demand information over time, and showed how this can be applied in an optimal dynamic pricing model for seasonal products. They used the demand model developed by Bitran and Mondschein (1993). In their methodology the entire sales season is divided into T discrete periods and it is assumed that for each period there is a reservation price distribution associated with the consumer population. Consumers in each period arrive at the store in the form of a Poisson process with an arrival rate that is known to the planner, and that is independent of price. Each store arrival rate is drawn randomly from the reservation price distribution. Each arriving consumer is assumed to purchase one unit of the product if price of the product is not larger than his reservation price. The authors showed that a firm who incorporates a demand learning approach will be able to correct the price early in the season and improve its revenues.

Smith and Achabal (1998) considered clearance (markdown) pricing policies for retail chains. They assumed that the demand (sales) rate to be a function of price, seasonal effects and the remaining units of the product. The authors used a combination of regression analysis and subjective inputs in order to estimate the parameters of the model.

There are also some studies that incorporate both pricing and inventory decisions to improve the profit of the operation. Jorgensen et al. (1999) considered a monopolist firm that plans its production, inventory, and pricing policy over a fixed and finite horizon. They developed a dynamic model of pricing, production and inventory management, which allows for learning effects on both the demand and the production side. The author assumed that the demand rate at the current period depends on price and the cumulative sales in the earlier periods. Burnetas and Smith (2001) examined the combined problem of pricing and ordering for a perishable product with an unknown demand distribution and censored demand observations resulting from lost sales, faced by a monopolistic retailer. They**Fehler! Textmarke nicht definiert.** developed an adaptive pricing and ordering policy with the asymptotic property that the average realized profit per period converges with probability one to the optimal value under complete information on the distribution.

In the dynamic pricing literature, most of the studies assume that consumers arrive according to a Poisson process with a given rate (e.g., Kincaid and darling, 1963, Gallego and van Ryzin, 1994). That is, increments time intervals are independent of one another as random variables. Thus, knowing the number of consumers that arrived up to time t, provides no information about the number of arriving consumers in the remaining time to the end of the sales course. As a result, the problem of optimal pricing is formulated as a Markov decision process (e.g., Subramanian et al., 1999, Feng and Gallego, 2000).

In the last few years, researchers focused on learning from sales information during the selling season. Aviv and Pazgal (2005b) considered a retailer of a fashionable good during a short sales season that is uncertain about how successful the product will be in the market and studied a problem where the arrival rate of potential consumers at the store is constant and independent of the prevailing prices. Each consumer has a reservation price for the product and will buy a single unit of the product as long as the price is below his reservation price. Consumers are assumed to be heterogeneous in their reservation prices. The retailer knows the distribution of the reservation prices in the population, but not their values at the individual consumer level. The authors proposed a Bayesian updating model that incorporates three types of uncertainty: (1) the uncertainty about the number and timing of the consumers' arrivals to the store, (2) the uncertainty about the reservation price of each consumer, which affects the individual purchasing decision, and (3) the uncertainty about how successful the product is in the market. In the learning process, the authors assumed that the seller observes only completed sales, so that an arrival of consumer which may provide information about the market condition is not recorded if the consumer did not purchase the product. They showed that (1) the optimal prices tend to be higher with higher levels of uncertainty about the market condition, (2) over the sales season prices fall down continuously, but jump upwards at the points of sale, and (3) the optimal expected revenues increase as a function of the length of the sales season, but in a diminishing rate. Similarly, Aviv and Pazgal (2005a) proposed a partially observed Markov decision process framework to compute an upper bound on the seller's revenues and derive some heuristics to approximate an optimal pricing policy. Xu and Hopp (2005) proposed a piecewise linear demand model with unknown parameters and used Bayes updating to investigate some properties of the optimal price process. Lobo and Boyd (2003) studied the monopolistic problem with uncertain demand and considered a linear demand model with an additive random noise that follows a normal distribution and developed approximate solutions to an optimal pricing policy by exploiting convex optimization. Bertsimas and Perakis (2006) considered a discrete time model in which demand is a linear function of the price with unknown coefficient and studied both the monopolistic and oligopolistic cases. Instead of Bayesian learning, the authors used a least squares estimation embedded in a dynamic program with incomplete state information. Some approximations and heuristics are proposed to reduce the dimensionality of the problem. In the infinite horizon setting, Cope (2006) analyzed the problem of dynamic pricing through a nonparametric Bayesian demand model; pricing decisions in his model, however, are independent of the inventory levels.

Lin (2005) presented a dynamic pricing model where consumers arrive in accordance with a conditional Poisson process, whose rate is not known to the seller in advance. As the sale moves forward, the seller uses sales data from the realized demand to fine-tune the arrival rate estimation, and then uses the fine-tuned arrival rate estimation to better understand the demand curve in the future. Consequently, the seller updates the future demand distribution periodically sets the product's price to maximize the expected total revenues.

Araman and Caldentey (2005) studied the problem faced by a retailer who sells non-perishable products to a price sensitive market in which the uncertainty in the demand rate is modeled by a single parameter to capture the unknown size of the market. The distribution of the unknown parameter is updated using Bayesian learning. The authors considered two cases. In the first case, the retailer is constrained to sell the entire initial stock of the non-perishable product before a different assortment is considered. In the second case, the retailer is able to stop selling the non-perishable product at any time to switch to a different menu of products.

Chapter 4

Consumer Choice Models

4.1 Introduction

In the present chapter, we look at the second element of our demand model; the consumers' choice process. In this context, we are particularly interested in the choice behavior of a large number of consumers expressed as aggregated demand for a certain product among a finite set of products. Since this aggregated demand is itself the result of each consumer's choice behavior in the market, we need a consumer choice model to describe demand for the product. In terms of literature, there are two kinds of the consumer choice models: (1) *Reservation price models* where consumer choice behavior only depends on the consumer's reservation price for the product. Reservation price models are mostly used in the case of the single perishable product problem (e.g., Lazear, 1986, Gallego and van Ryzin, 1994, Aviv and Pazgal, 2005b) and (2) *Discrete choice models* where a consumer facing a finite set of alternatives has to choose one product from the set.

4.2 Reservation Price Models

As mentioned in chapter *2*, price creates a first impression for the consumers and is the only marketing mix variable that generates revenues. By definition, price is the amount of money that a consumer pays for the product. Reservation price models assume that price is the only factor influencing the consumer purchase behaviour and consequently that price forms the demand for an individual product.

Since the term of the reservation price can be confused with the term of the maximum price, we introduce the term of a maximum price before going on to discuss reservation price models.

Definition 4.1 ***Maximum Price (Total Economic Value)***: *The maximum price of a product is the price of the consumer's best alternative that is called the reference price plus (minus) the value of whatever differentiates the underlying product from the best alternative that is called differentiation value*

$$p_{\max} = p_{ref} + v_{dif} \qquad 4.1$$

where $p_{\max}$ denotes the maximum price, p_{ref} denotes the reference price, and v_{dif} is the differentiation value (e.g., Nagle and Holden, 2002).

In this definition, the reference price for the underlying product is the price of the best alternative product from the consumer perspective. The differentiation value is the value of any differences between the reference product (the best alternative) and the underlying product. Determining the maximum price insists of *4* necessary steps (e.g., Nagle and Holden, 2002):

1. Identifying the cost of the competitive product that the consumer views as the best alternative.
2. Identifying all factors that differentiate the underlying product from the best competitive product such as superior performance, additional features, etc.
3. Determining the value to the consumer of these differentiating factors. The positive and negative values associated with the product's differentiating attributes comprise the differentiation value.
4. Summing the reference price (value) and the differentiation value to determine the maximum price.

Definition 4.2 ***Reservation Price (Willingness-to-Pay)****: A reservation price is the (maximum) price that a consumer is willing to pay for the underlying product.*

Reservation price choice models assume that an arriving consumer will purchase one unit of the product if his reservation price v equals or exceeds the posted price p; otherwise, he will not purchase the product. Unlike the maximum price the reservation price does not depend on a reference product. The reservation price, indeed, can be considered as utility of the product (e.g., Aydin and Ryan, 2000). In reservation price models, the reservation price of a consumer determines whether or not he purchases the product. If the posted-price is equal to the consumer's reservation price, the consumer gains the same surplus from purchasing and not purchasing the product (e.g., Breidert, 2005).

Although, there are many approaches such as analyzing market data, consumer surveys, and conjoint analyses for measuring consumers' reservation prices (see Breidert et al., 2006 for a review), the reservation prices of consumers are often unknown to the seller, especially in the cases of new products and fashion goods. In the revenue management literature, most of the studies assume the reservation prices across the potential consumers to be independent identically distributed variables following a certain distribution such as exponential and Weibull distributions and the seller attempts to estimate the parameters of the distribution (e.g., Gallego and van Ryzin, 1994, Bitran and Wadhwa, 1996, Gaul and Darzian Azizi, 2009). If the reservation prices of the consumers in the market follows a distribution with a probability density function (p.d.f.) as $f(v)$ and the seller sets the price at p then an arriving consumer will purchase the product with probability $1-F(p)$ where $F(v)$ is the cumulative distribution function (c.d.f.).

For example, if the reservation prices of the consumers in the market follows an exponential distribution with a probability density function as:

$$f(x;\theta)=\theta\, e^{-\theta x}, \quad x \geq 0 \tag{4.2}$$

where $\theta>0$ is the parameter of the distribution, often called the *rate parameter*, then the cumulative distribution function is given by

$$F(x;\theta)=1-e^{-\theta x}, \quad x \geq 0 \tag{4.3}$$

That is, if the seller sets the price at p the probability that a consumer purchases the product is $1-F(x;\theta)=e^{-\theta x}$.

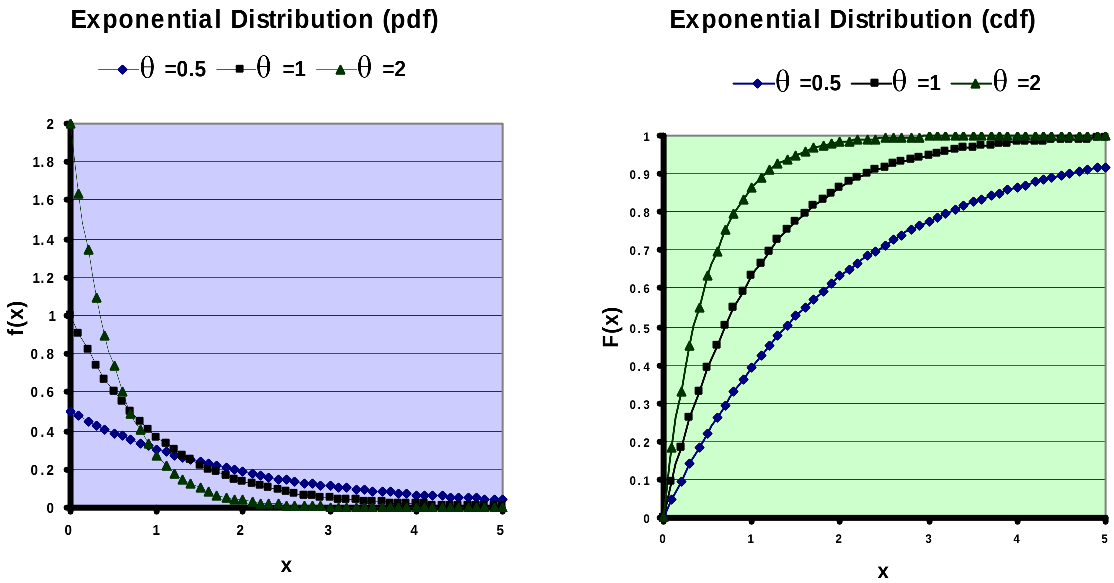

Figure 4.1 *Exponential distribution's p.d.f & c.d.f.*

4.3 Discrete Choice Models

Discrete choice models describe decision makers' choices among alternatives (e.g., Train, 2003). A decision maker can be a consumer and the alternatives might represent competing products. Discrete choice models are usually used in the case of the multiple perishable products problem (e.g., McFadden, 1986, Mahajan and van Ryzin, 2001, Dong et al., 2008). Consider a consumer facing a finite set of alternatives, *S*, including *m* products[9], i.e., $|S| = m$, has to choose one product from the set. In this situation, we are interested in how the consumer makes decisions to buy a certain product or leave the store empty-handed. This is, the basic problem of discrete choice models which is our focus in this chapter.

Discrete choice problems have been studied for many years to describe how an individual chooses a certain alternative among several alternatives (e.g., Thurstone, 1927, Luce, 1959, Manski and McFadden, 1981, McFadden, 1982 and 2001). In the theory of the consumer choice behavior, there are two approaches to treat discrete choice problems:

1. *Deterministic approach*: In this approach, we assume that each consumer has a utility function by which he is able to rank the alternatives in a rational manner and chooses either product i, $i=1,\ldots,m-1$ that is the best in his ranking to buy or decides not to buy, i.e., choosing alternative m.
2. *Probabilistic approach*: In this approach, we assume that different consumers are sensitive to different attributes of the products in different situations that are not necessarily known to the seller. Therefore, the seller has to use a probabilistic choice model to explain the consumers' choice behavior.

4.3.1 Constant vs. Random Utility Models

Suppose that a set of m products S is presented to a population of consumers in the market. We are interested in determining the fraction of the population choosing product i, $i=1,2,\ldots,m$. Since there is uncertainty about consumers' preferences and the consumers' purchase behavior, we use a probabilistic mechanism to capture this phenomenon. The

[9] Since there is always an option for each consumer to purchase nothing, we consider the m^{th} product as the no-purchase option such that the consumer who prefers it leaves the store empty-handed. It can also be considered as a product presented by other firms in the market.

probabilistic mechanism is used to determine the probability that a consumer in the market will choose product *i* and consequently the fraction of the population choosing product *i*.

Let U_i denote the utility that a consumer in the population has for buying product *i*, $i \in S$. In consumer theory, product *i* is chosen if and only if $U_i = \max\{U_j : j \in S\}$. In choice theory, there are two basic models of discrete choice that are used to determine the probability of choosing a certain product (e.g., Anderson et al., 1992):

(1) *Constant utility models* such as the Luce model and the Tversky model in which the decision is stochastic, while the utility of different products is deterministic;

(2) *Random utility models* such as the McFadden model and the Thurstone model in which the decision rule is deterministic, while the utility is stochastic.

The constant utility models assume that the utility of the products is constant and known to the seller and the choice probabilities for each consumer are functions parameterized by the utilities of the products. In these models the consumer does not necessarily choose the product that yields the highest utility but he has a probability of choosing each product. The consumer is assumed to behave according to the choice probabilities defined by a probability distribution function over the set of alternatives that includes the utilities as parameters.

The random (stochastic) utility models, in contrast, assume that the consumer chooses the alternative with the highest utility. However, the seller has not perfect knowledge of the utility functions of the U_i. Instead the probability of choosing product *i*, $P_S(i)$, is equal to the probability that the utility of product *i*, U_i, is greater than or equal to the utilities of all products in the choice set. That is

$$P_S(i) = P(U_i = \max\{U_j : j \in S\}) \qquad 4.4$$

There is a variety of reasons that makes the random utility models are useful in practice. First, the random utility models can represent unobserved taste variations of the consumers. Second, they can model unobserved attributes affecting the consumers' choice. Third, they can model the variations of the consumer behavior. Finally, the random utility models can capture the unpredictable behavior of the consumers.

In random utility models, the utility is divided into two additive parts, deterministic and random as $U_i = u_i + \varepsilon_i$ where u_i is the representative (systematic) component and ε_i

is a mean-zero random (disturbance) component. The representative component could be defined as a function of various observable attributes of the product.

The random utility models can be categorized in two main groups: the binomial (binary) and the multinomial choice models. The binomial choice models are used for the cases in which there are only two alternatives to choose from and the multinomial choice models are used when there are more than two alternatives faced by the consumers in the market. In what follows we provide a brief introduction to the binary choice models and then focus on the multinomial choice models.

4.3.2 Binary Choice Models

We consider the case in which there are only two alternatives to choose, i.e., $m=2$ and $S=\{1,2\}$. This case arises when there is only one product and each consumer in the market makes the decision to buy or not to buy it. Then, a consumer will decide to buy one unit of the product *1* with the probability $P_S(1) = P(U_1 \geq U_2)$, or leaves the store empty-handed with probability $P_S(2) = P(U_2 \geq U_1) = 1 - P_S(1)$. Considering that $U_1 = u_1 + \varepsilon_1$ and $U_2 = u_2 + \varepsilon_2$,

$$\begin{aligned} P_S(1) &= P(U_1 \geq U_2) \\ &= P(u_1 + \varepsilon_1 \geq u_2 + \varepsilon_2) \end{aligned}$$

Then,

$$P_S(1) = P(\varepsilon_2 - \varepsilon_1 \leq u_1 - u_2)\,. \qquad 4.5$$

4.3.2.1 Binary Probit

The probit model is derived under assumption of normal distributed unobserved utility components (e.g., Train, 2003). Assume that ε_1 and ε_2 are both normally distributed variables with mean zero and variances σ_1^2 and σ_2^2, i.e., $\varepsilon_i \sim N(0,\sigma_i^2)$ for $i=1,2$, then the term $\varepsilon = \varepsilon_2 - \varepsilon_1$ is also normally distributed with mean zero but with variance $\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\sigma_{12}$ where σ_{12} is the covariance of the random components. Hence, the probability of choosing the product to buy, $P_S(1)$ is given by

$$P_S(1) = P(\varepsilon_2 - \varepsilon_1 \leq u_1 - u_2)$$

Then

$$P_S(1) = \int_{-\infty}^{u_1-u_2} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2}(\frac{\varepsilon}{\sigma})^2} d\varepsilon, \qquad \sigma > 0,$$

or

$$P_S(1) = \Phi(\frac{u_1 - u_2}{\sqrt{2\sigma}}), \qquad 4.6$$

where $\Phi(.)$ denotes the standard normal distribution. This model is called the *binary probit*.

4.3.2.2 Binary Logit

Although binary probit is both intuitively reasonable and there is at least some theoretical grounds for its assumptions about the distribution of ε_i, *i=1,2*, it has the unfortunate property of not having a closed form (e.g., Ben-Akiva and Lerman, 1985, Cramer, 1991). Instead, we must express the choice probability as an integral. The binary logit that is "probitlike" and analytically more convenient satisfies this goal. The binary logit model arises from assumption of logistically distributed random variables. That is

$$F(\varepsilon) = \frac{1}{1+e^{-\varsigma\varepsilon}}, \qquad \varsigma > 0, \ -\infty < \varepsilon < +\infty \qquad 4.7$$

where ς is the scale parameter. In this formulation, ε has a mean zero and variance $\frac{\pi^2}{3\varsigma^2}$ (π = 3.14...), so that the probability of choosing the product to buy, $P_S(1)$ is given by

$$P_S(1) = \frac{e^{\varsigma u_1}}{e^{\varsigma u_1} + e^{\varsigma u_2}}. \qquad 4.8$$

There are also some other binary choice models such as the linear probability model and the arctan probability model which are used to model consumer choice behavior (e.g., Ben-Akiva and Lerman, 1985). In this work, we are particularly interested to model consumer choice behavior in which consumers are facing more than two alternatives. Therefore, we will focus on the multinomial logit model in the next section.

4.3.3 Multinomial Logit Model

The multinomial logit (MNL) model is a generalization of the binary logit model to m products, $m > 2$. The multinomial logit (MNL) model is the best known utility-based probabilistic discrete choice model that has been extensively used in econometric (e.g., McFadden, 1980) and marketing (e.g., McFadden, 1986) studies to describe the demand of individuals facing discrete choices.

The multinomial logit model can be derived by assuming that all random components $\varepsilon_i, \forall i \in S$ are i.i.d. random variables with a Gumbel distribution. That is

$$f(\varepsilon_i) = \varsigma\, e^{-\varsigma(\varepsilon_i - \eta)} e^{-e^{-\varsigma(\varepsilon_i - \eta)}} \qquad 4.9$$

and

$$F(\varepsilon_i) = e^{-e^{-\varsigma(\varepsilon_i - \eta)}} \qquad 4.10$$

where $\varsigma > 0$ is a scale parameter and η is a location parameter. The mean of the Gumbel distribution is $\eta + \gamma / \varsigma$ where γ is Euler's constant (=0.5772…), and the variance is $\frac{\pi^2}{6\varsigma^2}$.

To compute the probability that a consumer facing a set of alternatives chooses a certain alternative, firstly we point out some basic properties of the Gumbel distribution.

4.3.3.1 Gumbel Distribution

The Gumbel distribution has some useful analytical properties (Ben-Akiva and Lerman, 1985):

1. If ε is Gumbel distributed with parameters (η, ς), then $\alpha\varepsilon + V$, $\forall V > 0, \forall \alpha > 0$ is Gumbel distributed with parameters ($\alpha\eta + V, \varsigma / \alpha$).
2. If ε_1 and ε_2 are independent Gumbel distributed variables with parameters (η_1, ς) and (η_2, ς), then $\varepsilon^* = \varepsilon_1 + \varepsilon_2$ is a logistically distributed variable:
$$F(\varepsilon^*) = \frac{1}{1 + e^{\varsigma(\eta_2 - \eta_1 - \varepsilon^*)}}.$$
3. If ε_1 and ε_2 are independent Gumbel distributed variables with parameters (η_1, ς) and (η_2, ς) respectively, then $\max(\varepsilon_1, \varepsilon_2)$ is Gumbel distributed with parameters $\left(\frac{1}{\varsigma}\ln(e^{\varsigma\eta_1} + e^{\varsigma\eta_2}), \varsigma\right)$.

4. As the most important property, if $\varepsilon_1, \varepsilon_2, ..., \varepsilon_m$ are m independent Gumbel distributed random variables with parameters $(\eta_1, \varsigma), (\eta_2, \varsigma), \ldots, (\eta_m, \varsigma)$, respectively, then $\max(\varepsilon_1, \varepsilon_2, ..., \varepsilon_m)$ is Gumbel distributed with parameters $(\frac{1}{\varsigma}\ln\sum_{i=1}^{m} e^{\varsigma\eta_i}, \varsigma)$, i.e., the distribution of the maximum of m independent Gumbel random variables with the same scale parameter ς is also a Gumbel random variable.

Using the properties of the Gumbel distribution the multinomial logit model of choosing alternative *1* is derived as follows (e.g., Ben-Akiva and Lerman, 1985):
Let $\eta = 0$ for all the random variables.

$$\begin{aligned} P_S(1) &= P(U_1 \geq U_i, \forall i \in S) \\ &= P(U_1 = \max U_i, \forall i \in S) \\ &= P[u_1 + \varepsilon_1 = \max(u_i + \varepsilon_i), \forall i \in S] \end{aligned}$$

Define $U^* \geq \max_{i=2,3,...,m}(u_i + \varepsilon_i)$. From property *4*, U^* is Gumbel distributed with parameters $(\frac{1}{\varsigma}\ln\sum_{i=2}^{m} e^{\varsigma u_i}, \varsigma)$. Using property *1*, we can write $U^* = u^* + \varepsilon^*$ where $u^* = \frac{1}{\varsigma}\ln\sum_{i=2}^{m} e^{\varsigma u_i}$ and ε^* is Gumbel distributed with parameters $(0, \varsigma)$. Since

$$\begin{aligned} P_S(1) &= P(u_1 + \varepsilon_1 \geq u^* + \varepsilon^*) \\ &= P[(u^* + \varepsilon^*) - (u_1 + \varepsilon_1) \leq 0], \end{aligned}$$

by property *2* we will have

$$\begin{aligned} P_S(1) &= \frac{1}{1 + e^{\varsigma(u^* - u_1)}} \\ &= \frac{e^{\varsigma u_1}}{e^{\varsigma u_1} + e^{\varsigma u^*}} = \frac{e^{\varsigma u_1}}{e^{\varsigma u_1} + \exp(\ln\sum_{i=2}^{m} e^{\varsigma u_i})} \\ &= \frac{e^{\varsigma u_1}}{\sum_{i=1}^{m} e^{\varsigma u_i}}. \end{aligned} \qquad \text{4.11}$$

That is, the probability that the consumer chooses a particular alternative depends on the utilities of each alternative, u_i, *i=1,…,m* , and ς that represents the degree of heterogeneity in consumer tastes.

The relation of the logit probability to the representative utility is sigmoid, or S-shaped, as shown in Figure *4.2*. This shape has implications for the impact of changes in explanatory variables. If the representative utility of an alternative is very low compared with other alternatives, a small increase in the utility of this alternative has only a little effect on the probability of its being chosen: the other alternatives are still sufficiently better so that this small improvement does not help much. Similarly, if one alternative is far superior to the others in observed attributes, a further increase in its representative utility has little effect on the choice probability. The point at which the increase in representative utility has the greatest effect on the probability of being chosen is when the probability is close to *0.5*, meaning a *50–50* chance for the alternative being chosen. In this case, a small improvement tips the balance in consumers' choices, inducing a large change in probability.

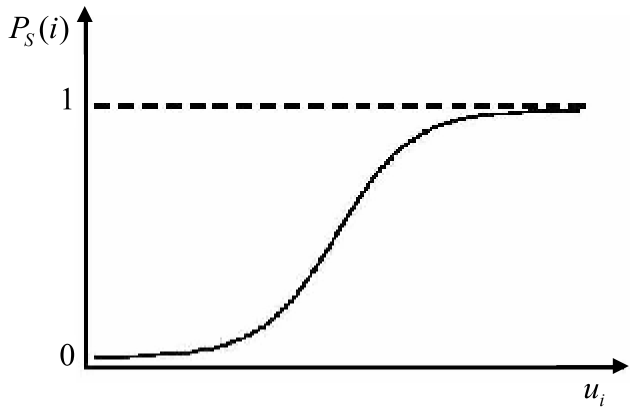

Figure 4.2 *Graph of logit choice probability curve.*

4.3.3.2 Properties of the Multinomial Logit Model

1. If heterogeneity in consumer tastes is very large, $\varsigma \to \infty$, then the variance of the ε_i, $i = 1,...,m$, tends to zero as well and the MNL model reverts to a deterministic model as follows:

$$\lim_{\varsigma\to\infty} P_S(i) = \begin{cases} 1 & \textit{if} \quad u_i = \max\limits_{j=1,...,m} u_j \\ 0 & \textit{otherwise.} \end{cases}$$

That is, all the consumers will choose the product with the greatest value of the utilities $u_i = \max\limits_{j=1,2,...,m} u_j$. In this case, if there are *k>1* alternatives with maximum utility $\lim\limits_{\varsigma\to\infty} P_S(i) = 1/k$ for the alternatives with maximum utility, and zero for the others.

2. If ς tends to zero, then the variance of the ε_i, $i = 1,..,m$, tends to infinity. Thus the deterministic part of the utility function u_i has no effect on the consumers' purchase behavior and consumers will choose random among the products (e.g., Anderson and de Palma, 1992).

That is,

$$\lim_{\varsigma \to 0} P_S(i) = \frac{1}{m}.$$

Figure *4.3* illustrates the choice probabilities of the (binomial) logit model for different values of ς.

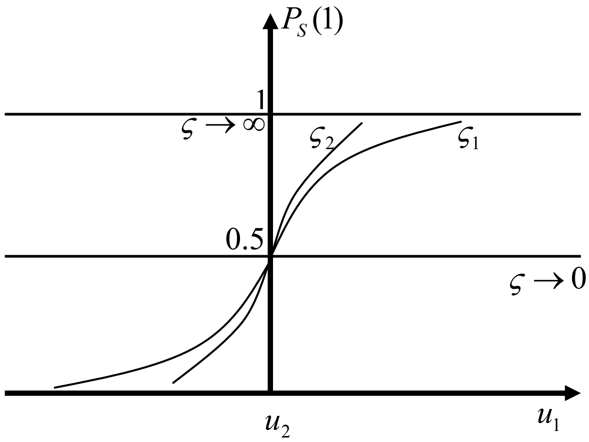

Figure 4.3 *Choice probabilities for the binomial logit model, i.e., m=2, for different values of* ς, $0 < \varsigma_1 < \varsigma_2 < \infty$.
Source: Anderson et al. (1992)

3. $\dfrac{\partial P_S(i)}{\partial u_i} \propto +P_S(i)(1 - P_S(i))$; That is, the purchase probability of a product is a monotonically increasing function of the product's utility.
4. $\dfrac{\partial P_S(i)}{\partial u_j} \propto -P_S(i)P_S(j)$; That is, the purchase probability of a product is a monotonically decreasing function of the utilities of the other products.
5. $e_{u_i}^{P_S(i)} = (1 - P_S(i))/\varsigma$; That is, the own (utility) elasticity is positive and the purchase probability of product *i* decreases as its own utility decreases.

6. $e_{u_j}^{P_S(i)} = -P_S(j)u_j / \varsigma$; That is, the cross (utility) elasticity is negative and the purchase probability of product i decreases as the utility of another product increases.
7. If the alternative j , $j=1,\ldots,m$ is dropped from (respectively added to) the choice set, then the increase (respectively decrease) in probability of choosing i, $i=1,\ldots,m$ and $i \neq j$, is proportional to the probability when j was (respectively was not) in the choice set. That is

$$P_{S-\{j\}}(i) = \alpha P_S(i), \quad i = 1,\ldots,m \;\; and \;\; i \neq j$$

where

$$\alpha = \frac{\sum_k e^{\varsigma u_k}}{\sum_{k \neq j} e^{\varsigma u_k}}$$

8. If the utility of the alternative i, u_i, changes to $\overline{u}_i$, the new choice probability is

$$\overline{w}_i = \frac{e^{\varsigma \overline{u}_i}}{\sum_{j=1}^{m} w_j e^{\varsigma \overline{u}_j}}.$$

4.3.3.3 Limitation of the Multinomial Logit Model

The MNL model has the limitation of the so-called independence from irrelevant alternatives (IIA) property: For all $S \subseteq A$, $T \subseteq A$ such that $S \subseteq T$ and for all $i \in S$ and $j \in S$,

$$\frac{P_S(i)}{P_S(j)} = \frac{P_T(i)}{P_T(j)}. \qquad 4.12$$

That is, the ratio of choice properties for alternatives i and j is independent of the choice set that contains the alternatives. Therefore, the MNL model should be restricted to choice sets containing alternatives that are equally dissimilar.

Despite this limitation, the MNL models are widely used in researches because of its useful properties. In the next chapter, we will use the MNL model as a part of our demand model.

Chapter 5

Pricing
of
Multiple Perishable Products

5.1 Introduction

In this chapter, we focus on the problem of determining an optimal pricing policy for selling a set of multiple perishable products under demand uncertainty and substitution. As mentioned, in the case of selling multiple perishable products, the demand for a particular product does not only depend on its own price and inventory level but also on the price and inventory of the other products that could be considered as a substitution product. Furthermore, in the case of perishable products, the demand for each product depends on the remaining time until the end of the selling season as well. That is, the demand for a particular product depends on: (1) its own price and inventory level, (2) the prices and inventory levels of all other products, and (3) the remaining time of the selling season. Therefore, determining the right prices for multiple perishable products over a finite horizon of time is considerably more complicated than the case of selling a single perishable product where demand for the product depends on time and inventory.

In this work, we consider a firm who presents a set of perishable products, e.g., different kinds of a summer T-shirt, different kinds of tickets for a special event, etc. to a price-sensitive population of consumers over a finite horizon of time. In such cases, there are some factors such as quality, price, and inventory (availability) of the products as well as time that affect demand for the products. This situation arises in many industries such as apparel and retailing where the firms have to order in advance and then have limited control over the quality of the products[10]. Furthermore, there often is not any opportunity of replenishment during the selling season because of the long supply lead-times, i.e., supply inflexibility. In practice, the firms have to choose from a set of the available products in the market and order a certain amount of them before the selling season starts based on some expectations of market conditions by applying a qualitative forecasting method. In such cases, price as a decision variable plays a very important role in matching demand and supply and consequently improving (optimizing) the total revenues.

In this thesis, we are going to provide a demand learning approach to determine an optimal pricing policy for selling multiple perishable products under demand uncertainty and substitution based on the products' *qualities*, *prices*, and *inventory* as well as the remaining *time* to the end of the selling season. The overall aim of our optimal price policy

[10] The firms themselves have to choose from the presented products and then they are not completely able to determine the quality of the products which they want to sell.

is to maximize the expected revenues obtained from selling a given inventory of the products over a finite selling horizon.

The problem of managing demand and inventory decisions of multiple perishable products under demand substitution is considered in both the operations research literature on inventory and revenue management (e.g., dynamic pricing) and the economics and marketing literature on consumer choice behaviour.

In the inventory management literature, many papers have studied the problem of assortment decisions while prices were not decision variables. For example, van Ryzin and Mahajan (1999) considered the retail optimal assortment problem in which a retailer decides which subset of the products should be offered and how much inventory of each product should be stocked. They assumed that prices are exogenously determined and applied the multinomial logit (MNL) model to a stochastic single-period assortment planning problem under static demand substitution, i.e., each potential consumer considers the subset *S* of products offered by the retailer and may choose a product to buy or may decide not to buy at all. Under static demand substitution they assumed that consumers make their choice based only on knowledge of the set *S*, and have no knowledge of the inventory status of the products. If a consumer's favorite product is unavailable he does not undertake a substitute choice and the sale is lost. As a result, demand is independent of the availability of the products, though it depends on the initial set of products in the assortment. That is, demand substitution for the products is determined only by the initial assortment and the predetermined price. As the authors have pointed out, static demand substitution is not realistic in practice. However, they studied some retail environment where this assumption reasonably approximates certain types of consumer behaviour and showed that the optimal assortment consists of a certain number of the most popular products.

Similarly, Smith and Agrawal (2000) studied the problem of multi-product inventory management with demand substitution in retailing by considering general random substitution patterns within the set of products and a fixed cycle for replenishment with no lead time needed for quantity adjustment. An inventory policy specifies both the products to be stocked and the initial inventory level for each product. Therefore, the optimal policy is determined by maximizing the expected profit per cycle, subject to floor space, assortment size, etc. In terms of consumer demand models, the authors assumed that the demand per cycle can be deduced from the random number of arriving consumers that follows a binomial distribution. In this setting, the number of consumers per cycle who initially prefer a particular product is negative binomial whose distribution parameters are known to the retailer. In terms of demand substitution, the authors assumed that if product i from the initial set of the products is unavailable, the consumer who prefers product i may choose a second product, say j, as substitute and if product j is unavailable as well, a lost sale will be the result. That is, there is only one substitution attempt if the consumer's first choice is not available. The authors developed an inventory management methodology that jointly optimizes the stock levels and the choice of the products to stock.

Mahajan and van Ryzin (2001) developed a solution for the retail optimal assortment problem but under dynamic demand substitution in which the inventory availability affects demand substitution as well as prices. They considered a single-period inventory model of multiple perishable products, where each product has a unit selling price but the prices can differ and are again assumed to be exogenously determined. The authors used a model formulated as the newsvendor problem where the retailer decides about the initial inventory level of each product before demand is realized. They assumed that the consumer's choice is based on a simple utility maximization mechanism in which the utility of buying the product represents the consumer surplus and used several choice models such as the multinomial logit model, Markovian second choice, universal backup, and Lancaster demand. During the sales period, each arriving consumer chooses his favorite product from the available products (not the initial assortment) based on a stochastic utility maximization criterion. The authors showed that the retailer should stock relatively more units of popular products and relatively less units of not so popular products.

Bassok et al. (1999) also studied a single-period multi-product inventory problem with downward demand substitution where demand for product i can be satisfied by product j for $i \geq j$. Here, products are ordered with respect to their prices, i.e., $p_1 \geq p_2 \geq \dots$. The authors provided a greedy algorithm that solves the allocation problem. Kök and Fisher (2007) considered the assortment optimization problem in retailing under both static and dynamic substitution. They formulated a general assortment optimization problem and solved it by providing methodologies for estimating the parameters of the model for different sets of available data. The authors used an optimization algorithm by which the retailer can add products at stores that carry less than the full assortment and delete products from stores with full assortment. This study showed that products with higher demand and higher margin should be included first in the assortment and should receive a larger amount of the overall inventories.

Unlike the studies above, Aydin and Ryan (2000) examined joint inventory and pricing decisions problems in inventory (assortment) management. They considered the problem of selecting and pricing a product line in the context of retailing and assumed that the consumers' purchase process consists of two stages:

1. The consumers' arrivals at the store.
2. The consumers' decision of buying a particular product.

The authors assumed that the consumers' arrivals at the store follow a Poisson process such that the arrival rate does not depend on the variety offered by the retailer. Moreover, they assumed that the reservation price of each consumer for product *i* can be explained by the utility U_i which is a function of a fixed component, u_i, plus a mean-zero random variable ε_i that follows a Gumbel distribution, i.e., $U_i = u_i + \varepsilon_i$, and use the MNL model to determine the purchase probabilities. In the general case, they considered a retailer, who offers a set of products decides how much of each product should be ordered and at what price it should be sold.

Recently, researchers focus on dynamic pricing of substitutable perishable products and adjust prices dynamically to affect consumers' substitution behaviour where replenishment is not possible during the selling horizon (e.g., Zhang and Cooper, 2006). Bitran et al. (2005) studied the problem of finding an optimal pricing policy to maximize the total expected revenues over a finite horizon of time. They assumed that consumers arrive at the store according to an exogenous stochastic process and provided asymptotic approximations for both the unlimited and limited supply cases. Liu und Milner (2006) considered the problem of dynamically pricing multiple items over a finite horizon of time under a common pricing constraint. Such cases arise for example in apparel where retailers have to set the same price for different size/color of a certain product. The authors presented a model of continuous time pricing with stochastic demand. Unlike the expectation that prices increase as inventory decreases they showed that a decrease in inventory may also lead to a reduction in price because of differences in the demand rates for alternative items.

According to our knowledge, Dong and Kouvelis (2008) is the latest work that considers dynamic pricing and inventory control of substitutable products. The authors assume that the consumers' arrivals at a store follow a Poisson process and divide the entire of the selling season into T time periods in such a way that the probability of more than one arrival within a period can be ignored (see *§2.3.2.2*). They use the MNL model to describe consumers' choice behavior and develop a stochastic dynamic programming formulation to determine the optimal prices of the products over time. As an interesting result, they show that in contrast to the single perishable product case the optimal prices of the products may not be decreasing in time. They also show that the optimal dynamic pricing converges to the static pricing when inventory of all products are not scare.

Dividing the entire of the selling season into T time periods such that the probability of more than one arrival within a period can be ignored and using a dynamic programming formulation to solve the optimality equation of the dynamic pricing problem results in exhaustive computational efforts. Moreover, the optimal prices will have to be updated after each period, i.e., a huge number of price changes may occur, which would result in problems concerning coordinating and managing operations in practice (e.g., Bitran and Mondschein, 1997).

To avoid such problems, we present a periodic pricing policy for the problem of selling a set of multiple perishable products over a finite horizon of time in which the retailer who aims to maximize his total expected revenues, sets the prices based on observation, the inventory levels of the products, and the remaining time to the end of the selling season. We divide the entire of the selling season into T equal periods, e.g., days, weeks, months, etc. such that T does not depend on the consumers' arrivals, i.e., more than one arrival per period is allowed. In our framework, we assume that the consumers' arrivals at the store follow a Poisson process with rate λ that does not depend on the prices and is also unknown to the retailer but follows a gamma distribution. Under this assumption we use a Bayesian learning approach that makes it possible to capture demand uncertainty associated with the consumers' arrival rate. We also use the MNL model to capture demand uncertainty associated with the consumers' purchase behavior.

In what follows we describe our demand learning model that will be used to determine an optimal pricing policy for selling multiple perishable products under demand uncertainty and substitution that included a Bayesian learning approach to learn about the consumers' arrival rate and a demand substitution model to describe the consumers' choice behavior. Then, we formulate the optimal pricing problem faced by a retailer who sells a limited inventory of a set of multiple perishable products over a finite horizon of time.

5.2 Model Description

We consider a retailer who is selling a set of I perishable products, i.e., $S = \{1,2,..,I\} \cup \{0\}$ [11]. The products have to be ordered before the selling season starts and there is no possibility to replenish inventory during the selling season. Let $Q^0 = (q_1^0,...,q_I^0)$ be the vector of the initial inventory of the products and $P^0 = (p_1^0,..., p_I^0)$ the vector of the initial price of the products. We divide the entire of the selling season into T equal periods. The problem of the retailer is to determine the optimal prices $P^t = (p_1^t,..., p_I^t)$ at the end of a given period t by taking into account the remaining inventory $Q^t = (q_1^t,...,q_I^t)$ in order to maximize the expected total revenues. Figure *5.1* describes this problem.

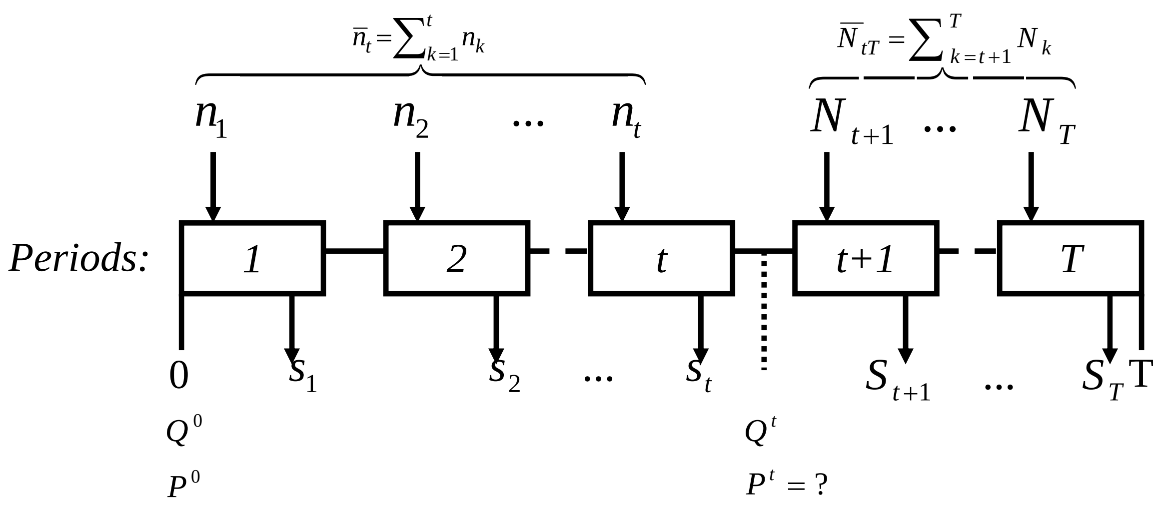

Figure 5.1 *A* description of the problem.

[11] {0} is assumed to be the no-purchase option. In fact, {0} represents a product that is not in the current set of products.

where

Q^0 : Vector of the initial inventory of the products.

P^0 : Vector of the initial prices of the products.

n_t : Number of arrivals at the store in period *t*.

$\overline{n}_t$: Number of arrivals in the *t* first periods.

s_t : Vector of sales in period *t*, $s_t = (s_{1t},...,s_{It})$.

N_{t+1}: Forecast of arrivals in period *t+1*.

$\overline{N}_{tT}$: Expected number of arrivals during the periods *t+1,…,T*.

S_{t+1} : Vector of the expected sales in periods *t+1*.

Q^t : Vector of the available inventory of the products at the end of period *t*.

P^{t*} : Vector of the optimal prices of the products for the next periods *t+1,…,T*.

In this framework, we assume that the demand for each product is the result of two elements:

1. The consumers' arrival rate;
2. The consumers' choice behavior.

5.2.1. Consumers' Arrival Rate

In terms of the consumers' arrival rate, we assume that the arrival at the store is an independent stochastic process that is not influenced by the products' prices and inventory levels. Bitran and Mondschein (1997) argued that the arrival rate of potential consumers to a store follows a regular purchasing pattern during the selling season rather than a function of individual prices is reasonable in the case of selling seasonal products and operating with a strategy of periodic pricing reviews until the products are sold.

Following Gaul and Darzian Azizi (2009), we assume that consumers arrive at the store according to a homogeneous Poisson process. That is, the number of arriving consumers at the store during the time interval [*0,t*], $\overline{N}_t$, has a Poisson distribution with mean $E[\overline{N}_t] = \int_0^t \lambda dt = \lambda t$. Thus, the conditional probability of observing n consumers arriving at the store over the time interval [*0,t*] is given by

$$P(\overline{N}_t = n \mid \lambda) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \qquad n = 0,1,2,... \qquad 5.1$$

Suppose that λ is unknown to the retailer but follows a gamma distribution with a prbability function as:

$$f(\lambda;a,b)=\frac{be^{-b\lambda}(b\lambda)^{a-1}}{\Gamma(a)}, \quad \lambda\geq 0. \qquad 5.2$$

Therefore, the expected number of arriving consumers at the store within the time interval [*0,t*] is $E[\overline{N}_t]=a/b$ and the variance is $Var(\overline{N}_t)=a/b^2$. In practice, using a gamma distribution for the rate of consumers' arrival is preferred to the application of normal and exponential distributions (e.g., Burgin, 1975):

1. It is defined only for non-negative values.
2. It ranges from a monotonic decreasing function (exponential distribution) through unimodal distributions skewed to the right to a normal distribution.
3. It is mathematically tractable in applications.
4. Considering the gamma distribution as a prior distribution results consequently in a posterior gamma distribution.

Moreover, the probability distribution function of the total number of arriving consumers at the store during an arbitrary time-interval $[0,t]$ by using Bayes' rule is:

$$\begin{aligned}P(\overline{N}_t=n)&=\int_0^\infty P(\overline{N}_t=n|\lambda)f(\lambda;a,b)d\lambda\\&=\frac{\Gamma(n+a)}{n!\Gamma(a)}\left(\frac{b}{b+t}\right)^a\left(\frac{t}{b+t}\right)^n, \quad n=0,1,2,... \qquad 5.3\end{aligned}$$

where $\overline{N}_t$ is the number of arriving consumers in the time interval [*0,t*] and $\Gamma(a)=(a-1)!=\int_0^\infty x^{a-1}e^{-x}dx, \forall a\in Z_+$ is the gamma function. That is, the probability distribution function of observing $\overline{N}_t$ arrivals at a store is given by a *negative binomial distribution* with parameters (a , $\frac{b}{b+t}$). As a special case when variance of the consumers' arrivals of the *t* first periods tends to zero (equivalently, *a* tends to infinity) by substituting $b=a/\mu$ in (*5.3*), we can see that $\overline{N}_t$ converges to a Poisson distribution with mean equal to λt (e.g, Lin, 2005). This case is the most frequently used in the literature on revenue management (e.g., Bitran and Mondschein, 1997, Bitran et al., 1998).

There are many studies on the negative binomial distribution for modeling demand and its advantages in the marketing science literature, see for instance Lilien et al. (1992). One of the most important advantages of the negative binomial distribution for modeling demand is that the variance generally exceeds its mean (e.g., Agrawal and Smith, 1996). Therefore, the coefficient of variation will be larger than that on the Poisson distribution with the same expected value; in the revenue management literature, as McGill and van Ryzin (1999) pointed out, both homogenous and non-homogeneous Poisson processes lead to Poisson cumulative arrival distributions whose expected value and variation are equal to λ, i.e. , the coefficient of variation is $1/\sqrt{\lambda}$ that is much lower than what is encountered in practice.

To compute the distribution of arrivals at a store during the time (*t*,*T*], we use another important advantage of assuming that the consumers' arrival rate follows a gamma distribution; If the number of accurate arrivals at the store from the beginning of the selling season up to and including time t , i.e., the t first periods, is equal to $\overline{n}_t = \sum_{k=1}^{t} n_k$, the posterior distribution of the consumers' arrival rate will be again a gamma distribution whose parameters will be (a_{tT}, b_{tT}) where $a_{tT} = a_t + \overline{n}_t$ and $b_{tT} = b_t + t$, see §*3.3.2.2*. That is, the posterior probability distribution function of the consumers' arrival rate to the store $g(\lambda)$ will be as follows:

$$g(\lambda) = \frac{(b+t)e^{-(b+t)\lambda}((b+t)\lambda)^{a+\overline{n}_t-1}}{\Gamma(a+\overline{n}_t)} \qquad 5.4$$

Using equation (*5.4*) the retailer will be able to calculate the probability distribution of the number of consumers that will arrive at the store during the time (*t*,*T*], i.e., in the periods *t+1,…,T*, according to

$$P(\overline{N}_{tT} = n) = \int_0^\infty P(\overline{N}_{tT} = n|\lambda)g(\lambda)d\lambda$$

$$= \int_0^\infty \frac{e^{-(T-t)\lambda}((T-t)\lambda)^n}{n!} \frac{(b_t+t)e^{-(b_t+t)\lambda}((b+t)\lambda)^{a_t+\overline{n}_t-1}}{(a+\overline{n}_t-1)!}$$

$$= \frac{\Gamma(n+a_t+\overline{n}_t)}{n!\Gamma(a_t+\overline{n}_t)}\left(\frac{b_t+t}{b_t+T}\right)^{a_t+\overline{n}_t}\left(\frac{T-t}{b_t+T}\right)^n \qquad 5.5$$

(where $\overline{N}_{tT}$ is the number of consumers who arrive at the store during the remaining periods) that follows a negative binomial distribution with parameters ($a_t + \overline{n}_t$,(b_t +*t*)/(b_t + *T*)).

5.2.2. Consumers' Choice Behavior

In terms of the second element of our demand model, we use the multinomial logit model to describe the consumers' choice behavior. The MNL model is a special case of stochastic utility model for a statistically homogeneous population. As discussed in chapter *4*, the MNL model is a discrete choice model in which an arriving consumer facing a set of products with corresponding prices chooses at most one unit of a product to buy or decides to buy nothing, in order to maximize his utility. We assume that the consumer's utility from purchasing product *i* is given by $U_i = u_i + \varepsilon_i$, where the representative component u_i is a deterministic utility value and the random components ε_i's are independent identically distributed variables from a Gumbel distribution with mean zero and variance $\pi^2 / 6\varsigma^2$, $\varsigma > 0$. That is,

$$f(x) = \varsigma\, e^{-\varsigma(\varepsilon-\eta)}(e^{-e^{-\varsigma(\varepsilon-\eta)}}), \quad \varsigma > 0 \text{ and } -\infty < \varepsilon < +\infty., \qquad 5.6$$

The probability that an arriving consumer selects product *i* to buy is given by $w_i = P[U_i = \max U_j, j \in S]$, i.e., the probability that product *i* has the highest utility among the products.

In our demand model, we consider both assortment-based (static) and stockout-based (dynamic) demand substitution. The following assumptions are required for our demand model:

A1. *Each consumer has a favorite product within the assortment that he will buy if it is available (i.e., assortment-based substitution), and if the favorite product is unavailable he may choose another product to purchase, or leaves the store empty-handed (i.e., stockout-based substitution).*

A2. *A consumer who prefers product i may choose the second product, say j, if product i is not available and if product j is unavailable as well, he leaves the store empty-handed, i.e., only one substitute attempt is considered (see Smith and Agrawal, 2000).*

A3. *The probability of choosing product i by an arriving consumer as the first choice is independent from the number of arrivals, inventory, and time.*

A4. *Each consumer will purchase at most one unit of his favorite product.*

As discussed in chapter *4*, if the ε_i 's are i.i.d according to a Gumbel distribution, then the choice probabilities are given by

$$w_i = \frac{e^{\varsigma u_i}}{\sum_{j \in S} e^{\varsigma u_j}}, \quad \forall i \in S \qquad 5.7$$

5.3 Optimal Pricing Policy

In this section, we formulate an optimal pricing policy for multiple substitutable products. We assume that the representative component u_i is a linear function of quality and price as $u_i = z_i - p_i$ where z_i is the quality index of product *i*, measured in dollars, and p_i is the price of product *i*. Furthermore, $u_0 + \varepsilon_0$ is assumed to be the utility of the no-purchase action. For simplicity, we assume that $\varsigma = 1$, i.e., there is not any heterogeneity in consumer tastes, and $u_0 = 0$, i.e., the consumer does not gain any utility when he leaves the store empty-handed, then $e^{u_0} = 1$.

As a reasonable assumption, we suppose that there is a finite predetermined set of prices for the products denoted as $\mathrm{P} = (P_1, P_2, ..., P_I)$ where $P_i = \{p_{i0}, p_{i1}, ..., p_{ik_i}\}$, $k_i \in \mathbb{N}$, in which $p_{i0} > p_{i1} > ... > p_{ik_i}$. According to static demand substitution, an arriving consumer selects product *i* with probability w_i or leaves the store empty-handed with probability w_0 that are given from (*5.7*) as follows:

$$w_i = \frac{e^{z_i - p_i}}{1 + \sum_{j=1,2,...,I} e^{z_j - p_j}}. \qquad 5.8$$

$$w_0 = \frac{1}{1 + \sum_{j=1,2,...,I} e^{z_j - p_j}}. \qquad 5.9$$

We consider $p_{i0} = \infty$ as the null price, as defined in Gallego and van Ryzin (1997), which is used to “turn off” demand for product *i*, i.e., $w_i(p_{i0} = \infty) = 0$. In fact, the retailer sets the price of product *i* at $p_{i0} = \infty$ once product *i* runs out of the stock, i.e., setting the price for product *i* at $p_{i0} = \infty$ is equal to the situation that it is not longer available.

Suppose that after passing period *t*, *t=1,…,T,* the retailer wants to set the optimal price for the available product *i*, *i=1,…,I* from the corresponding predetermined set of prices $P_i = \{p_{i0}, p_{i1},..., p_{ik_i}\}$. We define the following notation for our demand model:

T is the length of the selling season, i.e., the number of periods.

$\mathrm{P} = (P_1, P_2,..., P_I)$ is the vector of the set of allowable prices which $P_i = \{p_{i0}, p_{i1},..., p_{ik_i}\}$, $k_i \in \mathbb{N}$.

$Z = (z_1, z_2,..., z_I)$ is the quality vector of the products in which $z_1 \geq z_2 \geq ... \geq z_I$.

$\overline{N}_{tT}$ is the expected number of arrivals during the periods *t+1,…,T*, i.e., (*t*,*T*].

$g(n) = P(\overline{N}_{tT} = n)$ is the probability that *n* consumers arrive at the store during the periods *t+1,…,T*.

w_i^t is the probability that an arriving consumer after period *t* choose product *i* as the first choice.

w_0^t is the probability that an arriving consumer after period *t* leaves the store empty-handed.

$\overline{\rho}_{ji}(X^t)$ is the probability that an arriving consumer after period *t*, who prefers unavailable product *j* as the first choice, will switch to product *i* (stockout-based demand substitution with considering the availability of all products).

$X^t = (X_1^t,..., X_I^t)$ is the vector of the availability of the products after period *t*.

X_i^t is a binary variable after period *t* that depends on the availability of product *i*.

ρ_{ji}^t is the probability that an arriving consumer after period *t*, who prefers unavailable product *j* as the first choice, will switch to product *i* (stockout-based demand substitution without considering the availability of all products).

$Q^t = (q_1^t,..., q_I^t)$ is the inventory vector of the products at the end of period *t*, *t=1,…,T*.

$Q^0 = (q_1^0,..., q_I^0)$ is the vector of the initial inventory of the products.

$\mathrm{P}^t = (p_1^t,..., p_I^t)$ is the vector of prices of the products for the periods *t+1,…,T*.

$\mathrm{P}^0 = (p_1^0,..., p_I^0)$ is the vector of the initial prices of the products.

$\mathrm{P}^{t*} = (p_1^{t*}, p_2^{t*},..., p_I^{t*})$ is the vector of the optimal prices of the products for the periods *t+1,…,T*, i.e., (*t,T*].

$D_i^t(n, \mathrm{P}^t, Q^t, X^t, Z)$ is the expected demand for product *i* if *n* consumers arrive at the store during the periods *t+1,…,T*.

$R_i^t(n, \mathrm{P}^t, Q^t, X^t, Z)$ is the expected revenues gathered from selling the available inventory of product *i*, q_i^t, at prices described by P^t if *n* consumers arrive at the store during the periods *t+1,…,T*.

$R^t(\mathrm{P}^t, Q^t, X^t, Z)$ is the total expected revenues gathered from selling the available inventory of the products Q^t at prices described by P^t during the periods *t+1,…,T*.

Based on our assumptions, the demand for product *i* will be either from the consumers who prefer it as the first choice or from the consumers, who prefer an unavailable product, say $j \neq i$, and choose product *i* as the second choice. As $\overline{N}_{tT} = n$ consumers arrive at the store during the periods *t+1,…,T*, the demand for product *i* is given by

$$D_i^t(n, \mathrm{P}^t, Q^t, X^t, Z) = nw_i^t + \sum_{j \neq i} (nw_j^t - q_j^t)^+ \overline{\rho}_{ji}(X^t) \qquad 5.10$$

where nw_i^t, *i=1,…,I*, is the expected demand for product *i* from the consumers who select product *i* as the first choice (i.e., the assortment-based substitution) and $\sum_{j \neq i} (nw_j^t - q_j^t)^+ \overline{\rho}_{ji}(X^t)$ is demand for product *i* from the consumers who prefer unavailable product $j \neq i$ and consider product *i* as the second choice (i.e., the stockout-based substitution[12]). In (*5.10*), we define

$$(nw_j^t - q_j^t)^+ = \begin{cases} nw_j^t - q_j^t & if \quad nw_j^t > q_j^t \\ 0 & otherwise \end{cases}. \qquad 5.11$$

And $\overline{\rho}_{ji}(X^t)$ is given by

[12] Note that if an arriving consumer's first and second choices are not available, then he will leave the store empty-handed.

$$\bar{\rho}_{ji}(X^t) = \frac{X_i^t e^{z_i - p_i^t}}{1 + \sum_{r \neq j} X_r^t e^{z_r - p_r^t}}, \qquad 5.12$$

where $X^t = (X_1^t, ..., X_I^t)$ is the vector of the availability of the products and for $i=1,\ldots,I$, X_i^t is a binary variable defined as follows:

$$X_i^t = \begin{cases} 1 & \text{if } nw_i^t < q_i^t \\ 0 & \textit{otherwise.} \end{cases}$$

If $X_i^t = 1, \forall\, i=1,\ldots,I$, then we will have only the assortment-based substitution situation. As shown in Figure *5.2*, if *n* consumers arrive at the store during the next periods, nw_i^t consumers choose product *i* as the first choice. If $nw_i^t > q_i^t$, i.e., the demand for product *i* is greater than its inventory level, then $nw_i^t - q_i^t$ consumers, whose first choice is product i, may switch to another product *j* with probability $\bar{\rho}_{ij}(X^t)$ or may leave the store empty-handed with probability.

$$\bar{\rho}_{i0}(X^t) = \frac{1}{1 + \sum_{r=1}^{I} X_r^t e^{z_r - p_r^t}} = 1 - \sum_{k \neq i} \bar{\rho}_{ik}(X^t) \qquad 5.13$$

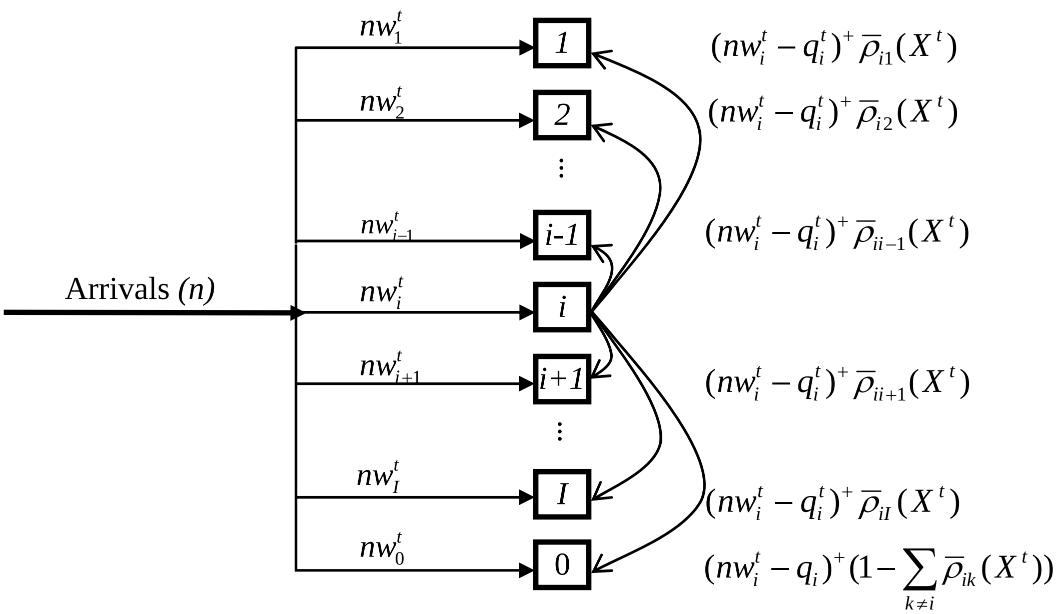

Figure 5.2 *A view of demand for the product under substitution.*

Therefore, the total expected revenues collected from selling the inventory of product *i*, q_i^t, at price p_i^t during the next periods will be as follows:

$$R_i^t(n,\mathrm{P}^t,Q^t,X^t,Z)=\min(D_i^t(n,\mathrm{P}^t,Q^t,X^t,Z),q_i^t)\,p_i^t\,. \qquad 5.14$$

That is, at the end of period *t*, the expected revenues collected from each product depends on the number of arriving consumers, the posted prices and the inventory levels of the products as well as the quality vector. Then the optimal expected revenues gathered from selling the available inventory of products during the periods *t+1,…,T* , R^{t^*}, is given by

$$R^{t^*}(\mathrm{P}^{t^*},Q^t,X^t,Z)=\max_{\mathrm{P}^{t^*}\in\mathrm{P}}\sum_{n=1}^{\infty}\left(ng(n)\sum_{i=1}^{I}R_i^{t^*}(n,\mathrm{P}^{t^*},Q^t,X^t,Z)\right) \qquad 5.15$$

where P^{t^*} is the set of the optimal prices and for *n=0,1,2,…*

$$g(n)=\frac{\Gamma(n+a_t+\bar{n}_t)}{n!\,\Gamma(a_t+\bar{n}_t)}\left(\frac{b_t+t}{b_t+T}\right)^{a_t+\bar{n}_t}\left(\frac{T-t}{b_t+T}\right)^{n} \quad if\ \ \sigma^2\neq 0 \qquad 5.16$$

$$g(n)=\frac{e^{-\lambda(T-t)}(\lambda(T-t))^n}{n!} \qquad otherwise. \qquad 5.17$$

Example 5.1

To explain how to compute the expected revenues by using (*5.8*)-(*5.17*), we provide a simple example. Consider a retailer selling three substitutable products, i.e., *|S|=4*. Let *Z=(14,12,8)* be the quality vector and *Q=(20,30,20)* denote the available inventory of product at the end of period *t*. We are going to compute the total expected revenues if *n=100* consumers arrive at the store while the price vector is as *P=(15,10.5,7.5)*. The probability that each consumer considers product *i*, *i=1,2,3* as the first choice is given by (*5.8*) and (*5.9*) (see table *5.1*, *i=0* denotes the no-buying option).

i	q_i	u_i	p_i	w_i
1	20	14	15	0.05
2	30	12	10.5	0.60
3	20	8	7.5	0.22
0	-	0	0	0.13
				1.00

Table 5.1 *Inventory, utility, price, and the probability of buying the products.*

We consider two different cases. In the first case, we assume that an arriving consumer, who prefers unavailable product *j* as the first choice, will switch to another product, say $i \neq j$, with considering the availability of all products. Then, the demand for each product is given by (*5.10*) as follows:

$$\begin{aligned} D_1(n,P,Q,X,Z) &= nw_1 + (nw_2 - q_2)^+ \bar{\rho}_{21}(X) + (nw_3 - q_3)^+ \bar{\rho}_{31}(X) \\ &= 5 + 30\bar{\rho}_{21}(X) + 2\bar{\rho}_{31}(X) \\ D_2(n,P,Q,X,Z) &= nw_2 + (nw_1 - q_1)^+ \bar{\rho}_{12}(X) + (nw_3 - q_3)^+ \bar{\rho}_{32}(X) \\ &= 60 + (0)\bar{\rho}_{12}(X) + 2\bar{\rho}_{32}(X) \\ D_3(n,P,Q,X,Z) &= nw_3 + (nw_1 - q_1)^+ \bar{\rho}_{13}(X) + (nw_2 - q_2)^+ \bar{\rho}_{23}(X) \\ &= 22 + (0)\bar{\rho}_{13}(X) + 30\bar{\rho}_{23}(X) \end{aligned}$$

Table *5.2* provides the probability of switching from unavailable product *j* to product $i \neq j$ in this case, i.e., $\bar{\rho}_{ji}(X)$. Then, the demand for each product is given by

$$\begin{aligned} D_1(n,P,Q,X,Z) &= 13.6 \\ D_2(n,P,Q,X,Z) &= 60 \\ D_3(n,P,Q,X,Z) &= 22 \end{aligned}$$

X_i	i	1	2	3
1	1	**-**	**0**	**0**
0	2	**0.27**	**-**	**0**
0	3	**0.27**	**0**	**-**

Table 5.2 *The probability of stockout-based substitution with considering the availability of all products.*

And the expected revenues collected from selling the inventory of product *i*, *i=1,…,3* is given by (*5.14*) as follows:

$$\begin{aligned} R_1(n,P,Q,X,Z) &= \min(D_1,q_1)\,p_1 = \min(13.6,20)15 = 204 \\ R_2(n,P,Q,X,Z) &= \min(D_2,q_2)\,p_2 = \min(60,30)10.5 = 315 \\ R_3(n,P,Q,X,Z) &= \min(D_3,q_3)\,p_3 = \min(22,20)7.5 = 150 \end{aligned}$$

Therefore, the total expected revenues gathered from selling inventory is given by (*5.15*)

$$R(n,P,Q,X,Z) = \sum_{i=1}^{I} R_i(n,P,Q,X,Z) = 679$$

In the second case, we assume that an arriving consumer, who prefers unavailable product *j* as the first choice, will switch to another product, say $i \neq j$, without considering the availability of all products with the probability denoted as ρ_{ji}. To compute ρ_{ji}, one needs to substitute $X_j = 0$ and $X_i = 1$, $i \in \{1,2,3\} - \{j\}$ in equation (*5.12*). That is

$$\rho_{ji} = \frac{e^{z_i - p_i}}{1 + \sum_{r \neq j} e^{z_r - p_r}}. \qquad 5.18$$

Table *5.3* provides these probabilities.

i	1	2	3
1	**-**	**0.63**	**0.23**
2	**0.12**	**-**	**0.55**
3	**0.06**	**0.77**	**-**

Table 5.3 *The probability of stockout-based substitution without considering the availability of all products.*

Then, the expected demand for each product is given by

$$\begin{aligned} D_1(n,P,Q,X,Z) &= nw_1 + (nw_2 - q_2)^+ \rho_{21} + (nw_3 - q_3)^+ \rho_{31} \\ &= 5 + 30\rho_{21} + 2\rho_{31} \\ &= 5 + (30)0.12 + (2)0.06 \\ &= 8.7 \\ D_2(n,P,Q,X,Z) &= nw_2 + (nw_1 - q_1)^+ \rho_{12} + (nw_3 - q_3)^+ \rho_{32} \\ &= 60 + (0)\rho_{12} + 2\rho_{32} \\ &= 60 + (0)0.63 + (2)0.77 \\ &= 61.5 \\ D_3(n,P,Q,X,Z) &= nw_3 + (nw_1 - q_1)^+ \rho_{13} + (nw_2 - q_2)^+ \rho_{23} \\ &= 22 + (0)\rho_{13} + 30\rho_{23} \\ &= 22 + (0)0.23 + (30)0.55 \\ &= 38.5 \end{aligned}$$

In this case, the total expected revenues collected from selling the inventory of product *i*, *i=1,…,3* is given by

$$\begin{aligned} R_1(n,P,Q,X,Z) &= \min(D_1, q_1)\,p_1 = \min(8.7,20)15 = 130.5 \\ R_2(n,P,Q,X,Z) &= \min(D_2, q_2)\,p_2 = \min(61.5,30)10.5 = 315 \\ R_3(n,P,Q,X,Z) &= \min(D_3, q_3)\,p_3 = \min(38.5,20)7.5 = 150 \end{aligned}$$

Therefore, the total expected revenues gathered from inventory is given by

$$R(n, P, Q, X, Z) = \sum_{i=1}^{I} R_i(n, P, Q, X, Z) = 595.5$$

That is 12% less than the total expected revenues in the first case. This simple example shows that in the case of the stockout-based substitution with considering the availability of all products performs significantly better.

Chapter 6

Numerical Study

In this chapter, we will consider several cases in terms of determining the optimal prices and present some numerical experiments to illustrate how our demand leaning approach works and highlight its performance. In §6.1, we determine the optimal pricing policy by using our approach to show how the optimal prices of the products change dependent on the remaining time and the inventory levels. In this case, we ignore learning from the earlier periods and suppose that the retailer has perfect information about the parameters of the consumers' arrival rate distribution. In §6.2, we bring demand learning into play to determine the optimal prices and compare the performance of our demand learning approach to exponential smoothing and m-period moving average approaches. Finally, in §6.3 we present some extensions of our demand learning model to address selected situations in the market.

In what follows we consider the problem of setting the optimal prices for the case in which a retailer is going to sell three versions of a product differentiated in quality, say the high- (***H***), the medium- (***M***) and the low-quality (***L***) versions of a perishable product, i.e., $I=3$. The selling season is divided into $T=10$ periods, say days. Based on a pre-sale market research, the retailer believes that the mean of the consumers' arrival rate will be $\mu = 5$ per period[13]. Also, the retailer believes that the potential consumers in the market evaluate the products with the quality vector denoted by $Z=(14,12,8)$, measured in dollar. Of course, one can consider many different specific situations with respect to the number of products I, the length of the selling season T, the mean and variance of the consumers' arrival rate, inventory levels, and the quality vector.

6.1 Optimal Pricing Policy (without demand learning)

In this section, we are particularly interested in studying how our demand learning approach works. We assume that the consumers' arrivals at the store follow a Poisson process with rate λ that has a gamma distribution whose parameters (a,b) are known to the retailer in advance. That is, we suppose that at the beginning of each period the retailer knows the mean μ and the variance σ^2 of the consumers' arrival rate λ. Then the parameters of the corresponding gamma distribution will be given by $a = \mu^2 / \sigma^2, b = \mu / \sigma^2$.

[13] We consider different values of variance for different experiments.

6.1.1 Optimal Prices in Time

First, we study how the optimal prices change with respect to the time left to the end of the sales season. As mentioned in chapter *2*, in the case of selling a single perishable product, the optimal price decreases in the remaining time until the end of the horizon. We will examine whether or not this property holds in the multi-product case.

Consider the following situation: At the beginning of each period the retailer has *21* units; *4* units of the high-quality version, *10* units of the medium-quality version, and *7* units of the low-quality version. That is, the inventory levels remain unchanged as time passes. In this case, we suppose that the variance of the consumers' arrival rate tends to zero, i.e., the consumers' arrivals follow a Poisson distribution with $\mu = \sigma^2 = 5$. First, we consider the single perishable product problem case based on our model in which we suppose that there is only one product to sell and each arriving consumer either buys the product with probability P_{buy} or leaves the store empty-handed with probability $1 - P_{buy}$. Figure *6.1* shows the optimal prices as a function of the remaining time. We can see that as discussed in chapter *2*, the optimal price is non-decreasing in time. Figure *6.1* on the left shows the optimal price as there are just *4* units of the high-quality version, or just *10* units of the medium-quality version, or just *7* units of the low-quality version at the beginning of each period. But, the figure on the right shows the optimal price when there are *21* units of one a particular version.

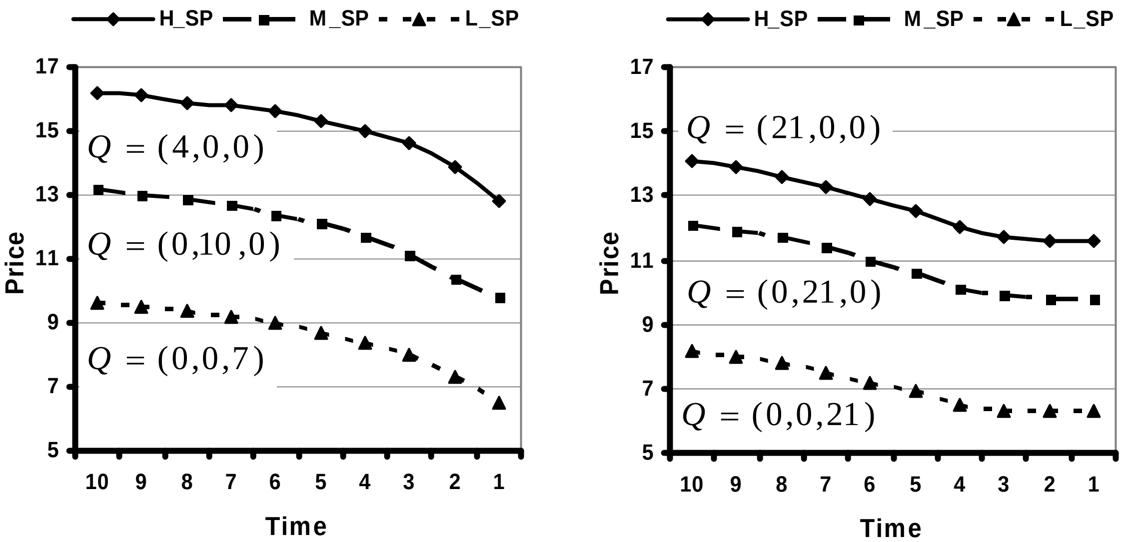

Figure 6.1 *The optimal price as a function of the remaining time for the single perishable*

Now, we consider the multi-product case. Figure *6.2* shows the optimal prices as a function of the remaining time when all three versions of the product, $Q = (4,10,7)$, are considered. Interestingly, unlike in the single perishable product case the optimal price may even increase in the case of the multiple perishable products as shorting the length of horizon.

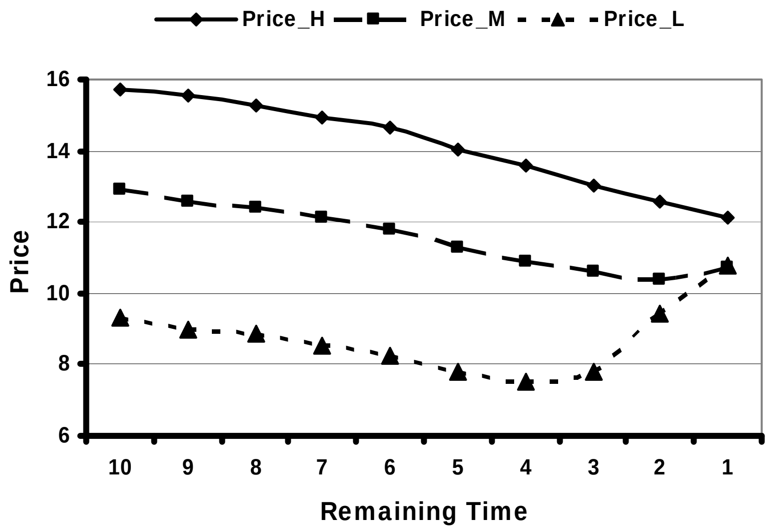

Figure 6.2 *The optimal price as a function of the remaining time for the multiple products case.*

Figure *6.2* shows that the optimal price for each product decreases initially as time passes, but the optimal prices of the medium-quality version and especially the low-quality version increase as there is a shorter time until the end of the selling season. To explain this behavior of the optimal prices in the multi-product case, we refer to lemma *5.1* (§ *5.2.2*).

Lemma 6.1

a. *The purchase probability of product i increases when its price decreases.*
b. *The purchase probability of product i increases when the price for a product* $j \neq i$ *increases.*

The proof is provided in appendix **B**.

Lemma *6.1.a* states that the purchase probability of a product increases when the product's price decreases. Therefore, the retailer can increase the probability of selling a certain product by decreasing its own price if there is enough time to sell inventory. That is, over the earlier periods during which the inventory levels of the products remained unchanged, the optimal price of each product decreases as the time to the end of the selling season gets shorter. Hence, decreasing price increases the purchase probability of the corresponding product. As time passes, in contrast, there will likely be fewer consumers to buy the products, therefore the retailer can increase his total expected revenues by selling more units of the product whose revenue margin, i.e., price, is greater than others. To do so, the retailer should not only decrease price of the high-quality version, but he should also increase the price of the other products, i.e., the medium- and the low-quality versions, because the purchase probability of a product increases as prices of the other products increase (lemma *6.1.b*). Moreover, we observe a larger price increasing for the low-quality version, i.e., the product with the least revenue margin, in comparison with the medium-quality version.

6.1.2 Optimal Price in Inventory

In the next experiment, we will show how the optimal price of each product varies when only its own inventory level is changed. We consider the retailer, who determines the initial prices of the products according to different inventory levels of the interest product as the length of the selling season is *T=10*. In Figure *6.3*, we can see that similar to the single perishable product case the optimal price of each product will increase, as its inventory decreases and the inventory levels of all other products remains constant.

For the case of the high-quality version, we suppose that the consumers' arrival rate has $\mu = 5$ with variance $\sigma^2{}_H = 1$ and the inventory levels of the medium- and low-quality versions stay unchanged, $q_M = 10$ and $q_L = 7$. Figure *6.3.a* shows that the optimal price for the high-quality version increases as its inventory level decreases.

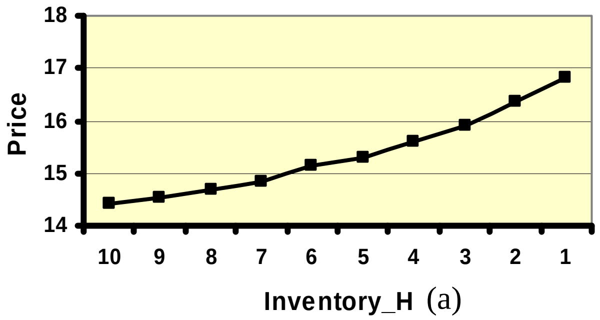

Figure 6.3.a *The optimal price as a function of the inventory level (high-quality version).*

For the case of the medium-quality version, we suppose that the consumers' arrival rate has $\mu = 5$ with variance $\sigma^2{}_M = 4$ and the inventory levels of both the high- and low-quality versions stay unchanged, $q_H = 4$ and $q_L = 7$. Figure *6.3.b* shows that the optimal price for the medium-quality version increases as its inventory level decreases.

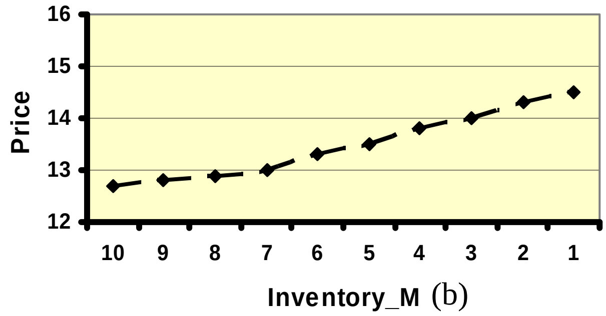

Figure 6.3.b *The optimal price as a function of the inventory level (medium-quality version).*

For the case of the low-quality version, we suppose that the consumers' arrival rate has $\mu = 5$ with variance $\sigma^2{}_L = 9$ and the inventory levels of the high- and medium-quality versions stay unchanged, $q_H = 4$ and $q_M = 10$. Figure *6.3.c* shows that the optimal price for the low-quality version increases as its inventory level decreases.

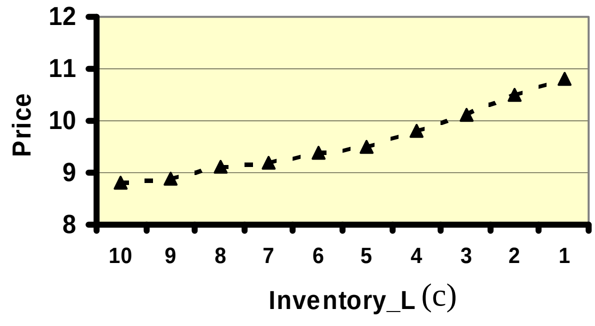

Figure 6.3.c *The optimal price as a function of the inventory level (low-quality version).*

6.1.3 Sample Path of the Optimal Prices

In the next experiment, we consider the effects of both the remaining time and the inventory levels of the products on the optimal prices. Figure *6.4* shows a sample path of the optimal prices when both the inventory level of the products and the length of time horizon change. For this case, we suppose again that the retailer has perfect information of the consumers' arrivals at the store, i.e., the mean of the arrival rate $\mu = 5$ and the variance of the consumers' arrival rate tends to zero.

This figure shows how the optimal price patterns can be in the multi-product problem. As mentioned in chapter 2, in the case of selling a single perishable product: (1) the optimal price decreases as the time passes that we consider it as time-effect and (2) the optimal price increases (decreases) as decreasing (increasing) inventory that we consider it as the inventory-effect. In this example, after passing *4* periods the optimal price of the high-quality version decreases while its inventory did not change, it agrees with lemma *5.1.a*. But the optimal prices for both of the medium- and the low-quality versions also decrease while their inventory levels decreases. That is, the time-effect dominates the inventory-effect.

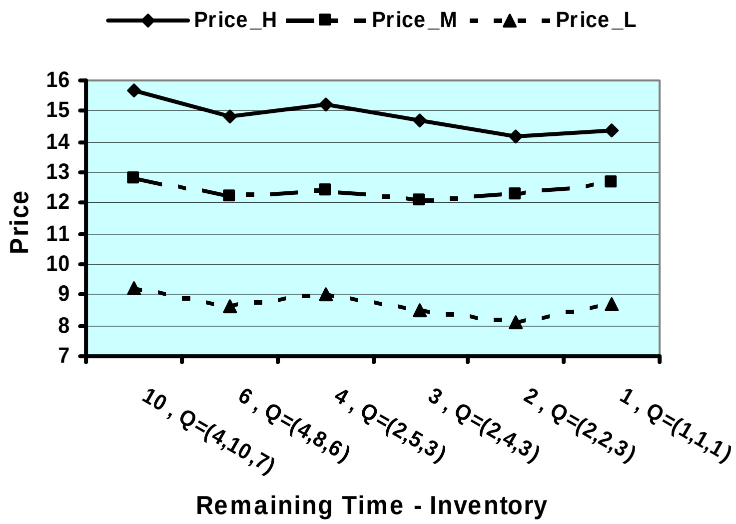

Figure 6.4 *The sample path of the optimal prices.*

In contrast, the optimal price for the medium-quality version goes down, i.e., the time-effect dominates the inventory-effect, and then up, i.e., the inventory-effect dominates the time-effect, during the periods 4^{th}, 3^{rd} and 2^{nd} while the inventory of two other products stays constant. Also, the optimal prices increase at the beginning of the last period in comparison to the second period that means the inventory-effect is greater than the time effect for all product versions.

6.2 Optimal Pricing Policy (with demand learning)

In this section, we consider the cases in which the retailer uses our optimal pricing policy to maximize the expected revenues from selling inventory of the products. At every decision point, the retailer updates the prices assuming that this is the last time of revising prices, i.e., a two-period problem. Hereafter, we suppose that the retailer presents only two versions of the product[14], the high- and the low-quality versions.

As mentioned in chapter *3*, there are many forecasting methods used in revenue management applications to make pricing decisions. In the following experiments, we apply our demand learning approach to determine the optimal prices. As mentioned in § *5.2.1*, if the number of accurate arrivals of consumers at the store from the beginning of the selling season up through and including time t, i.e., the t earlier periods, is equal to $\bar{n}_t = \sum_{k=1}^{t} n_k$, the probability distribution of the consumers' arrivals at the store that will show up at the store during the time-interval $(t,T]$, i.e., the next periods, is given by a negative binomial distribution, see equation (*5.5*). Then, at the beginning of period $t+1$, the retailer is able to update the prices under assumption that the number of the consumers' arrivals in the next periods follow a gamma distribution with the parameters $a = \mu_t^2 / \sigma_t^2$ and $b = \mu_t / \sigma_t^2$ where μ_t and σ_t^2 are computed from observations during the t first periods.

To show the performance of our Bayesian learning approach (*BLA*), we conduct a simulation to generate the random number of arrivals at the store during the sales season. Then, we determine an optimal pricing policy for the products based on time and inventory and compare the total revenues gathered by applying the following approaches:

1. *Perfect information approach (PIA):* In this approach we assume that the retailer knows the real number of arriving consumers during the next periods, so that he can determine the best optimal pricing policy and consequently gather the maximum potential revenues.
2. *Average-rate learning approach (ALA):* In this approach the retailer determines the optimal prices by assuming that the consumers' arrival rate at the store during the

[14] It is just because of reducing the time of simulation.

next periods is the average of the consumers' arrivals during the passed periods (see (*3.1*))

$$\hat{N}_{t+1} = \frac{n_1 + n_2 + ... + n_t}{t}$$

As in our work we assume that every decision point to be the last time at which we revise our knowledge about future demand, at the end of period t the forecast for the periods $t+1$, $\ldots, T$ are given by

$$\begin{aligned} \hat{N}_{t+k} &= \hat{N}_{t+1} \quad , \quad k = 2,...,T-t \\ \overline{N}_{tT} &= (T-t)\hat{N}_{t+1} \end{aligned} \qquad 6.1$$

3. *Exponential-smoothing-rate learning approaches (ELA):* In this approach the retailer uses the exponential smoothing method with different parameters to forecast the number of arriving consumers in the next periods by considering observations during the earlier periods (see (*3.3*))

$$\hat{N}_{t+1} = \alpha \sum_{k=0}^{t-1} (1-\alpha)^k n_{t-k}$$

And again, the forecast for the periods $t+1, \ldots, T$ are given by

$$\begin{aligned} \hat{N}_{t+k} &= \hat{N}_{t+1} \quad , \quad k = 2,...,T-t \\ \overline{N}_{tT} &= (T-t)\hat{N}_{t+1} \end{aligned} \qquad 6.2$$

In the followings, we present several numerical examples to show the performance of our Bayesian learning approach in comparison with others.

6.2.1 Single-Update Case

First, we consider the single-update case. At time zero, the retailer sets the initial prices from the predetermined set of prices based on his initial forecasts of the gamma distribution parameters, i.e., the distribution of the consumers' arrivals, the inventory levels, and the length of the selling season. As the sales season unfolds, the retailer collects sales information periodically to revise the parameters of the consumers' arrivals distribution (see Figure *5.1*).

Figure *6.5* shows the performance of the different learning approaches when the retailer updates the prices after passing the 3^{rd}, 5^{th}, or 7^{th} period. For instance, if it is supposed that the retailer sets the prices after the 3^{rd} period, we will do as follows:

In each simulation run[15], we generate the number of arriving consumers at the store for all *10* periods according to the parameters of the gamma distribution that are not known to the retailer. For this example, we consider the special case in which the mean of the arrival rate is $\mu = 5$ and the variance of the arrival rate tends to zero. That is, the number of arriving consumers follows a Poisson distribution with mean and variance $\mu = \sigma^2 = 5$. In this experiment, based on the initial prices and the initial inventory, the real number of consumers' arrivals, and our demand distribution we compute:

1. The revenues gathered in the *3* earlier periods.
2. The available inventory of the products at the end of period 3, Q^3.

After that, we use different learning approaches to determine the optimal prices for the remaining periods. Then, we apply the optimal prices from different approaches to compute the expected revenues with respect to each approach based on the real data. Finally, we compare the total revenues gathered from each approach to the case of the perfect information approach. This procedure also is repeated at the end of *5th* and *7th* period.

Figure 6.*5* shows that for all approaches the ratio of the expected revenues improves as the retailer uses more demand information. As the most important result, our Bayesian learning approach performs better than others, especially when the retailer updates the prices in the earlier periods. This is important because in the context of selling perishable products under high demand uncertainty, the retailer should effectively response to the demand pattern of the products in the market as soon as possible. As the length of horizon gets shorter, the market becomes less sensitive to the prices. According to Soysal (2007) an early price update improves revenues significantly more than a late price updates, because as time passes the price-sensitivity of the consumers dramatically decreases. Also, the figure shows that the *ALA* relatively performs better than the *ELA*.

[15] In this work, we use the statistic software of *R* to generate data and computations.

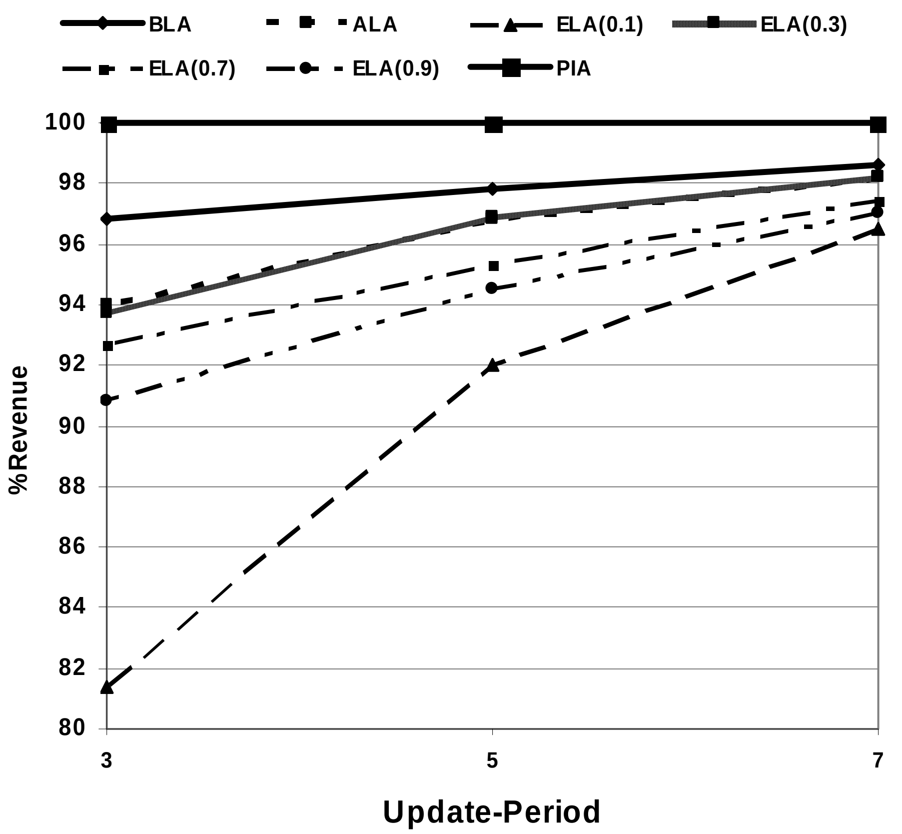

Figure 6.5 P*erformance of the single-update case for different learning approaches.*

Now, we take into consideration the effect of the variance of the consumers' arrival rate on the performance of the *BLA* and the *ALA*[16]. As we expected, Figures *6.6.a-c* show that the performances of both the *ALA* and the *BLA* decrease as the variance of the consumers' arrival rate increases.

These figures also show that the gap between the *ALA* and the *BLA* increases with increasing uncertainty about the consumers' arrival rate, e.g., for the case in which the retailer updates the prices after passing *3* periods the gap between the performances of two approaches increases from *0.8*% ($\sigma^2 = 1$) to *2.8*% ($\sigma^2 = 9$). But, using more information improves the performance of both the *BLA* and the *ALA* and decreases the gap between them.

[16] Our study shows that the *ALA* generally performs better than the *ELA*'s. Then, we go on with the *BLA* and the *ALA*.

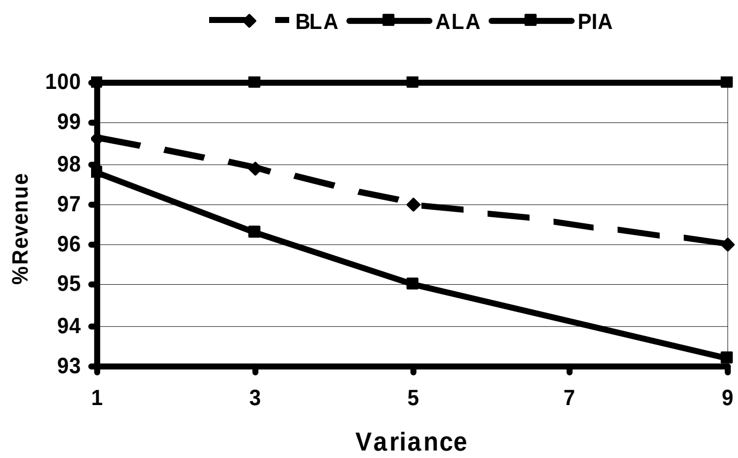

Figure 6.6.a *Performance of the single-update at the end of 3rd period for the BLA and the ALA.*

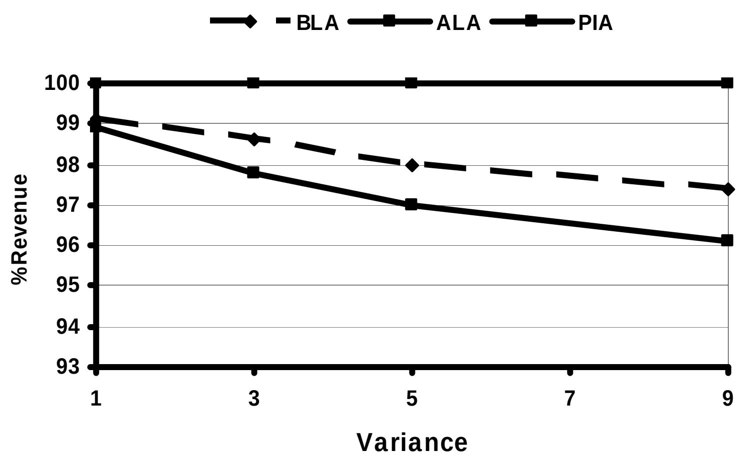

Figure 6.6.b *Performance of the single-update at the end of 5th period for the BLA and the ALA.*

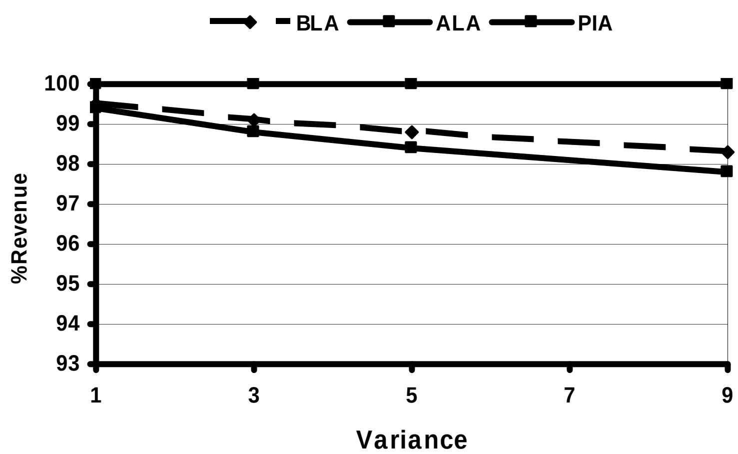

Figure 6.6.c *Performance of the single-update at the end of 7th period for the BLA and the ALA.*

6.2.2 Multi-Update Case

In this case, we examine the performances of the *BLA* and the *ALA* where there is more than one opportunity to refine the prices. The retailer determines the initial prices from an allowable price set according to his initial forecasts and the initial inventory, $Q^0 = (4,10,7)$. The results from two approaches are compared to the *PIA* in which the retailer knows the real values of the consumers' arrivals in all periods. We study different levels of uncertainty regarding to the consumers' arrival rate. We also compare the result of the single-update in which the retailer updates the prices only one time at the end of the *3rd* period with the results of the multi-update cases in which the retailer updates the prices at the end of the periods 3rd, 4th, 5th, 6th, and 7th. Figure *6.7* shows the result of our simulation:

1. Applying the multi-update method improves the performance of both the *BLA* and the *ALA*.
2. The gap between the revenues obtained from the multi-update case and the revenues obtained from the single-update case increases as the variance of the consumers' arrival rate increases, e.g., from *1.4*% ($\sigma^2 = 1$) to *2.7*% ($\sigma^2 = 9$) for the *ALA* and from *0.3*% ($\sigma^2 = 1$) to *1.7*% ($\sigma^2 = 9$) for the *BLA*.

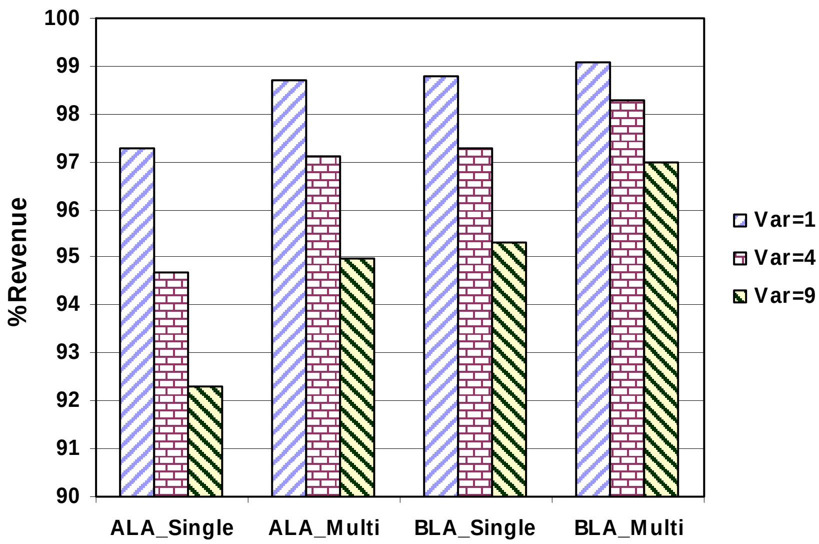

Figure 6.7 *Performance of the single- and the multi-update cases for the BLA and the ALA.*

3. In the multi-update case, the gap between the total revenues obtained by the *ALA* and the *BLA* increases as the variance of the consumers' arrival rate increases, from *0.4*% ($\sigma^2 = 1$) to *2.0*% ($\sigma^2 = 9$). Similarly, in the single-update case, the gap between the revenues obtained by the *ALA* and the *BLA* increases as the variance of the consumers' arrival rate increases, from *1.5*% ($\sigma^2 = 1$) to *3.0*% ($\sigma^2 = 9$).
4. As an interesting result, using the *BLA* in the single-update case yields more revenues than using the *ALA* in the multi-update case. It shows the robustness of the Bayesian learning approach.

6.3 Extensions

In this section, we consider two extensions of our basic demand learning model in order to support some selected situations in the market. In the first extension, we relax assumption that the consumers' arrival rate is constant in the whole of the selling season, i.e., non-homogeneous arrival process, but such that the retailer knows how it changes over the time. In the second one, we consider the case in which the retailer is not sure about the z_i's, i.e., the representative component of utility and uses observations to updates them.

6.3.1 Non-homogenous Consumers' Arrival Rate

An interesting extension to our work is the case of non-homogenous consumers' arrival rate in which the consumers' arrival rate into the store changes over time. It could be due to some unpredictable factors such as weather, especially for apparel and energy distributors, and market conditions that may affect overall demand for the products during the remaining periods. For instance, consider a retailer who sells different kinds of a summer T-shirt. He orders in advance a certain amount of different kinds of the product based on some predictions. After passing a few periods of the sales season, say *3-4* weeks, the retailer receives new information showing that during the remaining periods weather will change and not be as warm as he expected at the beginning of the selling season, i.e., the arrival rate will reduce in the next periods in comparison with the earlier periods. Since we assume that the consumers' arrival rate follows a gamma distribution, we can use the scaling property of the gamma distribution to response new situation.

***Lemma 6.2**: If X is a random variable that has a gamma distribution with parameters (a,b), then Y=kX, $k\in\mathbb{R}$, has a gamma distribution with parameters (a,b/k).*

Suppose that $\lambda_{t|t\in(t,T]} = k\lambda_{t|t\in[0,t]}$. If $\lambda_{t|t\in[0,t]}$ has a gamma distribution with parameters (*a*,*b*), $\lambda_{t|t\in(t,T]}$ will be a gamma-distributed random variable with parameters (*a,b/k*) because of lemma *6.2*. That is, the probability distribution of the number of arriving consumers during the next periods is given by

$$P(\overline{N}_{tT} - \overline{n}_t = n) = \frac{\Gamma(n + a_t + \overline{n}_t)}{n!\Gamma(a_t + \overline{n}_t)}\left(\frac{k(b_t + t)}{b_t + t + k(T - t)}\right)^{a_t + \overline{n}_t}\left(\frac{k(T - t)}{b_t + t + k(T - t)}\right)^n . \qquad 6.3$$

Then, the optimal expected revenues gathered from selling the available inventory of products during the periods *t+1,…,T* , is given by (*5.15*)

$$R^{t*}(P^{t*}, Q^t, X^t, Z) = \max_{P^{t*}\in \mathrm{P}} \sum_{n=1}^{\infty}\left(ng(n)\sum_{i=1}^{I} R_i^{t*}(n, \mathrm{P}^{t*}, Q^t, X^t, Z)\right)$$

where P^{t*} is the set of the optimal prices and for *n=0,1,2,…*

$$g(n) = \frac{\Gamma(n + a_t + \overline{n}_t)}{n!\Gamma(a_t + \overline{n}_t)}\left(\frac{k(b_t + t)}{b_t + t + k(T - t)}\right)^{a_t + \overline{n}_t}\left(\frac{k(T - t)}{b_t + t + k(T - t)}\right)^n \quad \text{if } \sigma^2 \neq 0 \qquad 6.4$$

$$g(n) = \frac{e^{-\lambda(T-t)}(\lambda(T - t))^n}{n!} \qquad \text{otherwise} . \qquad 6.5$$

6.3.2 Revising the Quality Vector

In this case, we suppose that the retailer is also interested in revising the quality vector of the products during the sales season. Because the retailer collects sales information- the number of arriving consumers and the number of each product sold- periodically, if the set of the available products remains unchanged during some earlier periods, we will be able to compute the quality vector of the products as well.

Suppose that the retailer wants to update the quality vector as well as the prices after passing *t* period for the first time. Let $\overline{n}_t$ be the number of arriving consumer during the *t* earlier periods and at the end of the period *t* all products are still available. Then, for product *i* , *i=1,…,I*

$$\bar{s}_{it} = \bar{n}_t \frac{e^{z_i - p_i}}{1 + \sum_{j=1}^{I} e^{z_j - p_j}}, \qquad 6.6$$

where $\bar{s}_{it}$ is the amount of product *i* that are sold during the *t* earlier periods.

Defining $Y = 1 + \sum_{j=1}^{I} e^{z_j - p_j}$, then the quality index for product *i* is given by

$$z_i = \ln(\frac{Y\bar{s}_{it}}{\bar{n}_t}) + p_i \qquad 6.7$$

The proof is provided in appendix **B**.

Chapter 7

Conclusions
&
Further Directions

7.1 Conclusions

In this thesis, we considered the problem of selling substitutable perishable products over a finite horizon of time. We provided a Bayesian learning approach to determine optimal pricing policies based on the products' *qualities*, *prices*, and *inventory* as well as the remaining *time* to the end of the selling season. We assumed that the demand for each product at the store is the result of two elements: (1) the consumers' arrival rate, and (2) the consumer choice behavior.

In terms of the consumers' arrival rate, we assumed that the consumers arrive at the store according to a Poisson process with a gamma distributed rate λ. Under this assumption and using Bayes' rule, the number of arriving consumers during a given time interval [*0*,*t*], follows a negative binomial distribution whose parameters depend on time *t* and the parameters of the gamma distribution. In our Bayesian learning approach, the selling season is divided into *T* equal time periods and the seller uses demand observations of the earlier periods to revise the parameters of the consumers' arrival rate. Therefore, the seller knows the parameters of the distribution based on the number of arriving consumers at the end of a given period, *t*.

In terms of the consumers' purchase behavior, we used the multinomial logit model to compute the probability that an arriving consumer facing a set of products chooses his favorite product to buy. We assumed that the consumer's utility from purchasing product *i* is given by $U_i = u_i + \varepsilon_i$, where the representative component u_i is a deterministic utility value and the random component ε_i is an independent identically distributed variable from a Gumbel distribution with mean zero. We assumed that the deterministic utility u_i is a linear function of quality and price as $u_i = z_i - p_i$ where z_i is the quality index and p_i is the price of product *i*. We considered both assortment- and stockout-based demand substitution and assumed that:

A1. Each consumer has a favorite product within the assortment, i.e., the initial set of products, and will buy it if it is available (assortment-based substitution), and if the favorite product is unavailable he may choose other products to purchase (stockout-based substitution), or leaves the store empty-handed.

A2. A consumer who prefers product i may choose a second product, say j, as substitute if product i is not available, and if product j is unavailable too, he leaves the store empty-handed (restriction to only one substitute attempt).

A3. The probability of product i being chosen by an arriving consumer as the first choice is independent from the number of arriving consumers.

A4. Each consumer will purchase at most one unit of his favorite product.

In this framework, demand for product i is either through consumers who prefer it as the first choice or through the consumers who prefer the unavailable product $j \neq i$ and choose product i as the second favorite product. Therefore, the demand for a particular product depends on the number of arriving consumers, the product's price, inventory, and quality index, as well as the prices, inventories, and quality indexes of all other products. Using the number of arriving consumers' distribution and the choice probabilities given by the multinomial logit model, we presented an optimal pricing policy by which the seller is able to improve his total expected revenues.

We have done a numerical study to show how our optimal pricing policy works and presented its performance. We considered two cases. First, we assumed that the seller knows the parameters of the consumers' arrival rate at the store and studied how the optimal prices of the products change with respect to the remaining time to the end of the selling season and the available inventory of the products. Our numerical experiment showed that (1) at a given inventory level, the optimal price of the products may decrease or increase as time progresses which is different from the known single product case, and (2) at a given time, the optimal price of each product increases as its inventory level decreases. We also presented an example of the simple path of the optimal prices to show how the optimal price patterns would be when both the inventory levels and the remaining time to the end of the selling season are changed. Second, we considered the cases where the seller does not know the parameters of the consumers' arrival rate but uses demand information of the earlier periods to update the parameters of the demand distributions in the stores. We studied both the single- and the multi-update cases and computed the total expected revenues gathered from our Bayesian learning approach (*BLA*) and two others; (1) The seller uses the average of the number of the consumers' arrivals in the earlier periods to forecast the number of arriving consumers in the next periods (*ALA*), and (2) the seller uses the exponential smoothing approaches with different parameters to forecast the number of arriving consumers in the next periods (*ELA*). Then, we compared the performances each approach by using the real data (deterministic case).

We showed that for all three forecasting approaches the results (revenues) improve as the seller uses more demand information. As the most important result, our Bayesian learning approach performs better than two others, especially when the seller updates the prices in the earlier periods. In the context of selling perishable products under demand uncertainty, the seller should effectively respond to demand pattern in the market as soon as possible because the market becomes less sensitive to the price as time passes. Our numerical study also showed that the *ALA* performs relatively better than the *ELA* approach. In other experiment, we examined the effect of increasing the variance of the consumers' arrival rate, i.e., increasing demand uncertainty on the performance of the *BLA* and the *ALA*. We observed that the performances of both approaches decline as uncertainty about the consumers' arrival rate increases. Moreover, the gap between the performance of the *ALA* and the *BLA* improves as the variance of the consumers' arrival rate increases, e.g., for the case in which the retailer updates the prices after *3* periods the gap between the performances of two approaches improves from *0.8*% ($\sigma^2 = 1$) to *2.8*% ($\sigma^2 = 9$). Still, using more information improves the performance of both the *BLA* and the *ALA* and decreases this gap. In our last experiment, we compared the performances of the single- and the multi-update and observed:

- Using the multi-update approach improves the performance of both the *BLA* and the *ALA*.
- The gap between the revenues obtained from the multi-update case and the revenues obtained from the single-update case increases as the variance of the consumers' arrival rate increases, e.g., from *1.4*% ($\sigma^2 = 1$) to *2.7*% ($\sigma^2 = 9$) for the *ALA* and from *0.3*% ($\sigma^2 = 1$) to *1.7*% ($\sigma^2 = 9$) for the *BLA*.
- In the multi-update case, the gap between the revenues obtained by the *ALA* and the *BLA* increases as the variance of the consumers' arrival rate increases, from *0.4*% ($\sigma^2 = 1$) to *2.0*% ($\sigma^2 = 9$).

- In the single-update case, the gap between the revenues obtained by the *ALA* and the *BLA* increases as the variance of the consumers' arrival rate increases, from *1.5*% ($\sigma^2 = 1$) to *3.0*% ($\sigma^2 = 9$).
- As an interesting result, using the *BLA* approach in the single-update case gives more revenues than using the *ALA* in the multi-update case. This shows the robustness of the *BLA*.

Finally, we considered two extensions to our optimal pricing policy problem in order to support some real situations in the market. In the first extension, we relaxed assumption that the mean of the consumers' arrival rate is constant over the selling season and modified our demand model by using the scaling property of the gamma distribution. In the second extension, we considered the situation where the seller is interested in learning about the quality vector as well as the consumers' arrival rate.

7.2 Further Directions

In this thesis, we assumed that there is not any opportunity to replenish inventory during the selling season. One can relax this assumption and develop an optimal pricing policy combined with an optimal ordering policy. That is, at a given time the seller determines which level of inventory of the products would be optimal as well as the prices. This case could be considered as an extension of our work.

In this work, we supposed that the quality vector of the products is the same for all consumers in the market. One can consider different segments of the consumers in the market with different quality vectors as well as different arrival rates.

Considering several stores (sales channels) is another direction for future research. In this case, both the consumers' arrival rate and the quality vector could be different among the stores.

Finally, we assumed that there is only one substitution attempt if the consumer's first choice is not available. Considering all the possible substitution attempts could be an interesting subject for future directions.

Some Useful Distributions

In this appendix, we consider some useful distributions which are used in our work. First, we point out Bernoulli, Poisson, binomial, negative binomial, and exponential distributions, and then consider the gamma distribution in more detail.

1. Bernoulli (Alternative) Distribution

A random variable X on $\{0,1\}$ has a Bernoulli distribution with parameter $p \in (0,1)$ if $P(X=1) = 1 - P(X=0) = p$. Letting $q = 1 - p$, the probability function of X can be written as follows:

$$f(x \mid p) = \begin{cases} p^x q^{1-x} & \text{for } x = 0,1, \\ 0 & \textit{otherwise.} \end{cases}$$

If X has a Bernoulli distribution with parameter p, then

$$E[X] = p\,,\; Var(X) = p(1-p)\,.$$

2. Binomial Distribution

A binomial random variable X on $\{0,1,\ldots\}$ with parameters n and p has the following probability function:

$$f(x \mid n, p) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & \text{for } x = 0,1,\ldots,n, \\ 0 & \textit{otherwise.} \end{cases}$$

If X has a Binomial distribution with parameters n and p, then

$$E[X] = np\,,$$

and

$$Var(X) = np(1-p)\,.$$

3. Negative Binomial Distribution

A negative binomial random variable X with parameters r and p has the following probability function:

$$f(x \mid r, p) = \begin{cases} \binom{r+x-1}{x} p^r (1-p)^x & \text{for } x = 0,1,2,\ldots, \\ 0 & \textit{otherwise.} \end{cases}$$

If X has a Binomial distribution with parameters n and p, then

$$E[X] = \frac{r(1-p)}{p}, \quad Var(X) = \frac{r(1-p)}{p^2}.$$

4. Poisson distribution

Let X be a random variable with a discrete distribution, and suppose that the values of X must be non-negative integers. The Poisson distribution is a discrete distribution on N_0. For a Poisson distribution N with parameter λ holds:

$$P(N = n) = \frac{\lambda^n}{n!} e^{-\lambda}.$$

We have $E[N] = \lambda$ and $\sigma^2(N) = \lambda$.

5. Exponential Distribution

It is said that a random variable X has an exponential distribution with parameter θ ($\theta > 0$) if X has a continuous distribution for which the probability distribution function is specified as follows:

$$f(x \mid \theta) = \begin{cases} \theta e^{-\theta x} & \text{for } x > 0, \\ 0 & \text{for } x \le 0. \end{cases}$$

If X has an exponential distribution with parameter θ, then $P(X \le s) = 1 - e^{-\theta s}$ and

$$E[X] = \int_0^\infty t\theta\, e^{-\theta t} dt = \frac{1}{\theta}, \quad Var[X] = \frac{1}{\theta^2}.$$

As an extremely important property of the exponential distribution, it is the only continuous distribution with the memoryless property:

$$P(X \le t + s \mid X > t) = \frac{P(X \le t + s, X > t)}{P(X > t)} = \frac{P(X \le t + s) - P(X \le t)}{e^{-\theta t}},$$

$$P(X \le t + s \mid X > t) = \frac{e^{-\theta t} - e^{-\theta(t+s)}}{e^{-\theta t}} = 1 - e^{-\theta s} = P(X \le s)$$

The exponential distribution is a special case of the Erlang distribution where the shape parameter *k=1*.

6. Erlang Distribution

If X is a continuous random variable that has an Erlang distribution, the probability function will be as follows:

$$f(x) = \frac{\theta\, e^{-\theta x} (\theta\, x)^{k-1}}{(k-1)!}, \quad 0 \le x < \infty, \quad \theta > 0, \quad k > 0$$

If X has a Erlang distribution with parameters k and θ, then

$$E[X] = k/\theta \;,\; Var[X] = k/\theta^2$$

The Erlang distribution is a special case of the gamma distribution where the shape parameter k is integer.

7. Gamma Distribution

The sum of k independent exponentially distributed random variables with parameter $\theta > 0$ has a gamma distribution with parameters (k, θ) where k is called the shape parameter and θ is called the scale parameter. If X is a continuous random variable that has a gamma distribution, the probability function will be as follows:

$$f(x) = \frac{\theta\, e^{-\theta x} (\theta\, x)^{k-1}}{\Gamma(k)}, \quad 0 \le x < \infty, \quad \theta > 0, \quad k > 0$$

where is $\Gamma(k)$ is the gamma function, defined by $\Gamma(k) = \int_0^\infty \theta^k x^{k-1} e^{-\theta x} dx$ or

$\Gamma(k) = \int_0^\infty x^{k-1} e^{-z} dx$.

- The mean μ of the distribution is

$$\mu = \int_0^\infty x f(x) dx = k/\theta .$$

- The variance σ^2 of the distribution is

$$\sigma^2 = \int_0^\infty (x-\mu)^2 f(x) dx = k/\theta^2 .$$

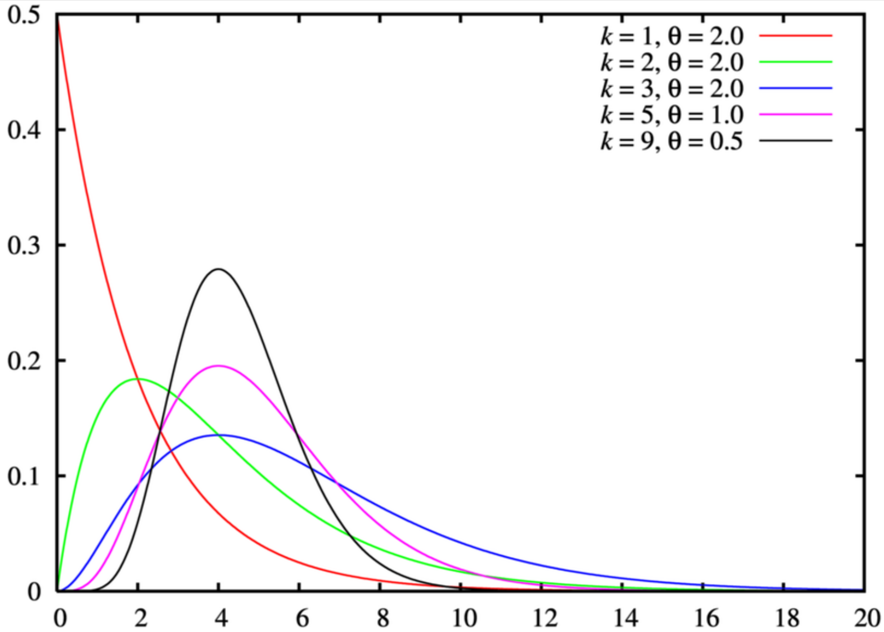

Figure A.1 *Probability density function of the gamma distribution.*

- If k is an integer, then the cumulative density function will be as follows:

$$F(x) = 1 - e^{-x/\theta} \sum_{n=0}^{k-1} \frac{(x/\theta)^n}{n!}.$$

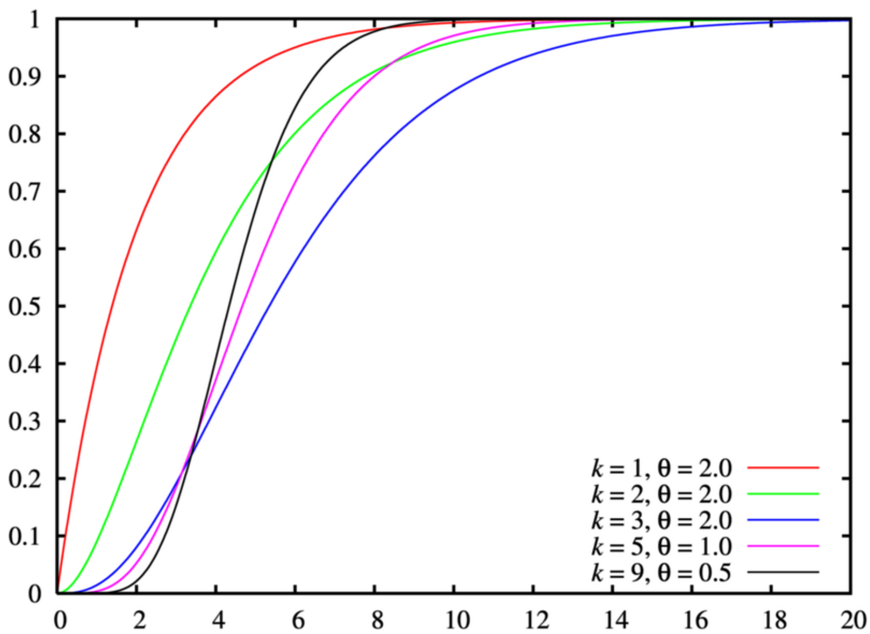

Figure A.2 *Cumulative density function of the gamma distribution.*

- If k is an integer, then $\Gamma(k) = (k-1)!$
- One of the most important properties of the gamma function is $\Gamma(k) = (k-1)\Gamma(k)$.
- If $X \sim gamma(1,\theta)$, then X has an exponential distribution with parameter θ. That is $f(x) = \theta e^{-\theta x}$.
- The chi-square distribution with k degrees of freedom is $X \sim gamma(k/2,2)$.
- If k is an integer, then the gamma distribution with parameters (k,θ) is called the k-Erlang(θ) distribution.
- If $X \sim gamma(k,2)$, then X has a chi-square distribution with $2k$ degrees of freedoms.
- If $X_1, X_2,..., X_m$ are independent random variables with $X_i(k_i,\theta)$, then $(X_1 + X_2 + ... + X_m) \sim gamma(k_1 + k_2 + ... + k_m, \theta)$

Fitting the Gamma Distribution to the Observed Data

To fit the gamma distribution to the observed data one can use the mean and the variance of the observed data as follows:

$$k = \mu^2 / \sigma^2 \quad , \quad \theta = \mu / \sigma^2 .$$

Alternatively, the parameters may be determined using the maximum likelihood (Burgin, 1975). The maximum likelihood estimates of (k,θ) are given by

$$\theta = k/\mu , \quad \ln k - (d/dk)\ln\Gamma(k) = \ln\mu - \ln G$$

where $G = (\prod_{i=1}^{n} x_i)^{1/n}$ is the geometric mean of the data. Then k is approximately

$$k \approx \frac{1}{4\alpha}[1 + \sqrt{(1 + \frac{4\alpha}{3})}]$$

where $\alpha \approx \frac{1}{2k} + \frac{1}{12k^2}$.

- **Proof of Eq. (2.1.a) and (2.1.b).**

Let D denote the one period random demand, with mean $\mu = E[D]$ and variance $\sigma^2 = Var[D]$. Let c be the unit cost, $p > c$ the selling price and $v < c$ the salvage value. If q units are ordered, then $\min(D,q)$ units are sold and $(q-D)^+ = \max(q-D,0)$ units are salvaged. The profit is given by $p\min(q,D) + \upsilon(q-D)^+ - cq$. The expected profit is well defined and given by:

$$B(q) = pE\min(q,D) + \upsilon E(q-D)^+ - cq$$

Using the fact that $\min(q,D) = D-(D-q)^+$ we can write the expected profit as

$$B(q) = (p-c)\mu - G(q).$$

where $G(q) = (c-\upsilon)E(q-D)^+ + (p-c)E(D-q)^+ \geq 0$.

Let $h = c-\upsilon$ and $b = p-c$. It is convenient to think of h the per unit overage cost and of b as per unit underage cost.

The problem of maximizing $B(q)$ can be considered as that of minimizing the expected overage and underage cost *G(q)*.

Let $G^{\det}(q) = h(\mu-q)^+ + b(q-\mu)^+$. This represents the cost when D is deterministic, i.e., $P(D=\mu)=1$. So that $q=\mu$ minimizes $G^{\det}(q)$ and $G^{\det}(\mu)=0$, so $B^{\det}(\mu) = (p-c)\mu$.

Let $g(x) = hx^+ + bx^-$, then *G(q)* can be written as $G(q) = E[g(q-D)]$. Since g is convex and convexity is preserved by linear transformations and by the expectation operator it follows that G is also convex. By Jensen's inequality $G(q) \geq G^{\det}(q)$. As a result, $B(q) \leq B^{\det}(q) \leq B^{\det}(\mu) = (p-c)\mu$. Thus, the expected profit is lower than it would be in the case of deterministic demand.

If the distribution of D is continuous, we can find an optimal solution by taking the derivative of G and setting it to zero. Since we can interchange the derivative and the expectation operators, it follows that

$$G'(q) = h\delta(D-q)$$

where

$$\delta(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise}. \end{cases}$$

Since $E\delta(q-D)=P(q-D>0)$ and $E\delta(D-q)=P(D-q>0)$, it follows that

$$G'(q)=hP(q-D>0)-bP(D-q>0).$$

Setting the derivative to zero reveals that

$$F(q)\equiv P(D\le q)=\frac{b}{b+h}=\frac{p-c}{p-v}$$

If F is strictly increasing then F has an inverse and there is a unique optimal solution given by

$$P(D\le q^*)=\frac{p-c}{p-\upsilon}$$

or

$$q^*=F^{-1}(\frac{p-c}{p-\upsilon}).\square$$

- **Proof of Assumption 2.2**

Under assumption 2.1, we will have

$$J(p_t)=\frac{\partial}{\partial p_t}r(p_t)=\frac{\partial(p_t d(p_t))}{\partial p_t}=d(p_t)+p_t\frac{\partial d(p_t)}{\partial p_t}.$$

As the price in period t increases, both $d(p_t)$ and $p_t\frac{\partial d(p_t)}{\partial p_t}$ decrease. Therefore $d(p_t)+p_t\frac{\partial d(p_t)}{\partial p_t}$ decreases as p_t increases. $\square$

- **Proof of Lemma 6.1**

a)

$$\frac{\partial w_i}{\partial p_i}=\frac{-e^{z_i-p_i}(1+\sum_{j=1,2,\dots,I}e^{z_j-p_j})+e^{z_i-p_i}e^{z_i-p_i}}{(1+\sum_{j=1,2,\dots,I}e^{z_j-p_j})^2}$$

$$=\frac{-e^{z_i-p_i}[(1+\sum_{j=1,2,\dots,I}e^{z_j-p_j})-e^{z_i-p_i}]}{(1+\sum_{j=1,2,\dots,I}e^{z_j-p_j})^2}$$

$$=\frac{-e^{z_i-p_i}}{(1+\sum_{j=1,2,\dots,I}e^{z_j-p_j})}\frac{(1+\sum_{j=1,2,\dots,I}e^{z_j-p_j})-e^{z_i-p_i}}{(1+\sum_{j=1,2,\dots,I}e^{z_j-p_j})}$$

$$\frac{\partial w_i}{\partial p_i}=-w_i(1-w_i).\square$$

b)

$$\frac{\partial w_i}{\partial p_j} = \frac{0.(1+\sum_{j=1,2,...,I} e^{z_j - p_j}) + e^{z_j - p_j} e^{z_i - p_i}}{(1+\sum_{j=1,2,...,I} e^{z_j - p_j})^2}$$

$$= \frac{e^{z_j - p_j}}{(1+\sum_{j=1,2,...,I} e^{z_j - p_j})} \frac{e^{z_i - p_i}}{(1+\sum_{j=1,2,...,I} e^{z_j - p_j})}$$

$$\frac{\partial w_i}{\partial p_j} = w_j w_i \ . \square$$

- **Proof of Eq. (6.7)**

$$\begin{cases} \bar{s}_{1t} = \bar{n}_t \dfrac{e^{z_1 - p_1}}{1+\sum_{j=1}^{I} e^{z_j - p_j}} \\ \bar{s}_{2t} = \bar{n}_t \dfrac{e^{z_2 - p_2}}{1+\sum_{j=1}^{I} e^{z_j - p_j}} \\ . \\ . \\ . \\ \bar{s}_{It} = \bar{n}_t \dfrac{e^{z_I - p_I}}{1+\sum_{j=1}^{I} e^{z_j - p_j}} \end{cases}$$

Defining $Y = 1 + \sum_{j=1}^{I} e^{z_j - p_j}$

$$\begin{cases} \bar{n}_t e^{z_1 - p_1} = Y\bar{s}_{1t} \\ \bar{n}_t e^{z_2 - p_2} = Y\bar{s}_{2t} \\ . \\ . \\ . \\ \bar{n}_t e^{z_I - p_I} = Y\bar{s}_{It} \end{cases} \Rightarrow \bar{n}_t (Y-1) = Y \sum_{i=1}^{I} \bar{s}_{it} \Rightarrow Y = \frac{\bar{n}_t}{\bar{n}_t - \sum_{i=1}^{I} \bar{s}_{it}}$$

$$e^{z_i - p_i} = \frac{Y\bar{s}_{it}}{\bar{n}_t} \Rightarrow z_i - p_i = \log(\frac{Y\bar{s}_{it}}{\bar{n}_t}) \Rightarrow z_i = \log(\frac{Y\bar{s}_{it}}{\bar{n}_t}) + p_i \ . \square$$

A_t	Number of arrivals in period *t*.
$B(p)$	Profit as a function of price *p*.
$B(q)$	Profit as a function of inventory *q*.
c	Unit order cost.
c.v.	Coefficient of variation.
$D_i^t(.)$	Demand for product *i* during the periods $t+1,\ldots,T$.
D, $D(.)$	Random demand.
$d(p)$	Demand rate or function of price *p*.
$d(p^*)$	Demand rate at the optimal price p^*.
$d(p_t)$	Demand rate or function of price p_t in period *t*.
$d(p_t^*)$	Demand rate at the optimal price p_t^* in period *t*.
$d(\overline{p})$	Demand rate at the inventory clearing price $\overline{p}$.
$d(p^0)$	Demand rate at the revenue maximizing price p^0.
$e_{u_i}^{P_S(i)}$	Own (utility) elasticity.
$e_{u_j}^{P_S(i)}$	Cross (utility) elasticity.
$f(.)$, $g(.)$	Probability density function.
$F(.)$	Cumulative distribution function.
$H(.)$	The Hamilton function.
$J(.)$	Marginal revenue.
l	Lagrangian multiplier.
M	Size of consumer population in the market.
m	Number of options (alternatives).
$N(t)$	Number of arrivals in the time interval $[0,t]$.
$N(T)$	Number of arrivals in the time interval $[0,T]$.
$N(t,T)$	Number of arrivals in the time interval $(t,T]$.
N	Random variable that count the number of arrivals.
N_t	Value of random variable in period *t*.
n_t	Number of arrivals in period *t*. Also t^{th} observed value.
$\hat{N}_t$	Forecast of value for period *t*.
$\overline{N}_t$	Number of arrivals in the time interval $[0,t]$.
$\overline{N}_{tT}$	Number of arrivals in the time interval $(t,T]$.
$\overline{n}_t$	Number of arrivals during the *t* earlier periods.
$r(p)$	Revenue as a function of price.
$r(p^*)$	Revenue at the optimal price.
$r(p_t)$	Revenue as a function of price in period *t*.
$r(p_t^*)$	Revenue at the optimal price in period *t*.
P	Set of allowable prices for the products.
P_i	Set of allowable prices for product *i*, $i=1,\ldots,I$.

P^0	Vector of the initial prices of the products.
P^t	Vector of prices for the periods $t+1,\ldots,T$.
P^{t*}	Vector of the optimal prices for the periods $t+1,\ldots,T$.
p_{0i}	Null price at which demand for product i will be equal zero.
p^*	Optimal price.
p_t	Price in period t.
p_t^*	Optimal price in period t.
p_{t,q_t}^*	Optimal price in period t as the inventory level is q_t.
p^0	Revenue maximizing price.
$\overline{p}$	Inventory clearing price.
$p_{mu}^{*\det}$	Optimal price in the deterministic demand case for the multiplicative demand.
p_{mu}^{*st}	Optimal price in the stochastic demand case for the multiplicative demand.
$p_{ad}^{*\det}$	Optimal price in the deterministic demand case for the additive demand.
p_{ad}^{*st}	Optimal price in the stochastic demand case for the additive demand.
$p_{\max}$	Maximum price that consumer will pay for the product.
p_{ref}	Reference price.
Q^t	Vector of the available inventory of the products at the end of period t.
Q^0	Vector of the initial inventory of the products.
q_0	Initial inventory.
q_t	Inventory at time t.
q^*	Optimal inventory.
q_i^t	Inventory of product i at the end of period t.
$R_i^t(.)$	Total expected revenues with respect to product i during the periods $t+1,\ldots,T$.
$R^t(.)$	Total expected revenues during the periods $t+1,\ldots,T$.
S	Set of alternatives.
$\overline{s}_{it}$	Number of units of product i sold in the t earlier periods.
T	Number of periods; Length of the horizon.
t	Time; index of period.
U_i	Random utility (random variable).
u_i	Representative component of random utility.
$V_t(q_t)$	Value function.
$\Delta V_t(q_t)$	Expected marginal revenue of the q_t^{th} unit of inventory in period t.
v_{dif}	Differentiation value.
w_i	Probability that an arriving consumer chooses product i as the first choice.

w_0	Probability that an arriving consumer leaves the store empty-handed.
X	Vector of the availability of the products.
X_i	A binary variable depends on the availability of product *i*.
Z	Quality vector of the products.
ε_i	Mean-zero random component of random utility for product *i*.
ε_p	Price elasticity of demand.
ζ	Probability that a consumer will arrive at the store during the sales season.
Λ	Parameter of Poisson distribution.
λ	Arrival rate.
$\lambda(t)$	Arrival rate function.
$\hat{\lambda}$	The maximum likelihood estimate of λ.
μ	Mean of the consumers' arrival rate at the store.
ν	Reservation price.
ν_t	Reservation price in period *t*.
$\overline{\rho}_{ji}(.)$	Probability that an arriving consumer, who prefers unavailable product *j* as the first choice, will switch to product *i* (stockout-based demand substitution with considering the availability of all products).
ρ_{ji}	Probability that an arriving consumer, who prefers unavailable product *j* as the first choice, will switch to product *i* (stockout-based substitution without considering the availability of all products).
σ^2	Variance of the customers' arrival rate.
υ	Salvage value per unit.
$\Phi(.)$	Standard normal distribution.
Ω_p	Constraint set of prices *p* in the single-product problem.

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