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Abstract

We consider the problem of minimizing the distance $\|f - \phi\|_{L^p(K)}$, where K is a subset of the complex unit circle $\partial\mathbb{D}$ and $\phi \in C(K)$, subject to the constraint that f lies in the Hardy space $H^p(\mathbb{D})$ and $|f| \leq g$ for some positive function g. This problem occurs in the context of filter design for causal LTI systems. We show that the optimization problem has a unique solution, which satisfies an extremal property similar to that for the Nehari problem. Moreover, we prove that the minimum of the optimization problem can be approximated by smooth functions. This makes the problem accessible for numerical solution, with which we deal in a follow-up paper.

1 Introduction

Let $\mathbb{D} = \{|z| < 1\}$ be the complex unit disk and $\partial\mathbb{D} = \{|z| = 1\}$ the complex unit circle. The Hardy space $H^\infty(\mathbb{D})$ is the space of bounded analytic functions on $\mathbb{D}$, see, e.g., [5, 9]. Via boundary values, $H^\infty(\mathbb{D})$ can be identified with a subspace of $L^\infty(\partial\mathbb{D})$. By $\mathcal{A}(\mathbb{D}) = H^\infty(\mathbb{D}) \cap C(\partial\mathbb{D})$ we denote the disk algebra.

We consider the optimization problem

$$\begin{array}{ll} \text{minimize} & \|f - \phi\|_{L^p(K)} \\ \text{subject to} & f \in E, \\ & |f| \leq g \quad \text{on } \partial\mathbb{D}, \end{array} \tag{OPT$_p$}$$

where $1 \leq p \leq \infty$. Here, E is either $H^\infty(\mathbb{D})$ or $\mathcal{A}(\mathbb{D})$. In the first case we denote the problem by (H-OPT$_p$), and in the second case we denote it by ($\mathcal{A}$-OPT$_p$). Further, $K \subset \partial\mathbb{D}$ is closed with positive measure, $g \in C(\partial\mathbb{D})$ with $g > 0$, and $\phi \in C(K)$ such that $|\phi| \leq g$ on K. Most of our theorems can actually be proved under weaker regularity conditions, for example for g continuous up to finitely many jump discontinuities. However, the corresponding proofs only become more cumbersome, but do not yield any insight. Therefore, we restrict our attention to continuous g and ϕ.

The problem (H-OPT$_p$) is a generalization or variation of various problems that have been studied before. The most classical of these problems is the Nehari problem (see, e.g., [5, 18])

$$\begin{array}{ll} \text{minimize} & \|f - \phi\|_{L^\infty(\partial\mathbb{D})} \\ \text{subject to} & f \in H^\infty(\mathbb{D}). \end{array} \tag{1}$$

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If $p=\infty$, $K=\partial\mathbb{D}$ and g is so large that the constraint $|f|\leq g$ is not active, then (H-OPT$_p$) is a Nehari problem. Other generalizations of the Nehari problem that have special cases in common with (H-OPT$_p$) have been considered. For example, if $p=\infty$, g is constant on $\partial\mathbb{D}\setminus K$ and g is so large on K that the constraint $|f|\leq g$ is not active on K, then (H-OPT$_p$) is a special case of a problem that has been studied by Baratchart, Leblond et al. in the context of system identification ([2], see also [1, 3]). Another related problem arising in H^∞ control theory has been studied by Helton et al. (see, e.g., [6, 7, 8, 10] and the references therein): Given a performance function $\Gamma:\partial\mathbb{D}\times\mathbb{C}\to[0,\infty)$ one is interested in minimizing $\|\Gamma(\cdot,f(\cdot))\|_{L^\infty(\partial\mathbb{D})}$ over $f\in H^\infty(\mathbb{D})$. If Γ were allowed to take the value ∞, then we could write (H-OPT$_p$) for $p=\infty$ as the minimization of $\|\Gamma(\cdot,f(\cdot))\|_{L^\infty(\partial\mathbb{D})}$ over $f\in H^\infty(\mathbb{D})$ with

$$\Gamma(\mathrm{e}^{i\theta},f(\mathrm{e}^{i\theta}))=\begin{cases}|f(\mathrm{e}^{i\theta})-\phi(\mathrm{e}^{i\theta})|, & |f(\mathrm{e}^{i\theta})|\leq g(\mathrm{e}^{i\theta}),\ \mathrm{e}^{i\theta}\in K,\\ 0, & |f(\mathrm{e}^{i\theta})|\leq g(\mathrm{e}^{i\theta}),\ \mathrm{e}^{i\theta}\in\partial\mathbb{D}\setminus K,\\ \infty, & |f(\mathrm{e}^{i\theta})|>g(\mathrm{e}^{i\theta}).\end{cases}$$

Our motivation for studying (OPT$_p$) comes from the design of causal linear time-invariant (LTI) systems. An LTI system L is a convolution operator, $Lf(t)=(h*f)(t)$, $t\in\mathbb{R}$, where the function $h:\mathbb{R}\to\mathbb{R}$ is called the impulse response of the system. The system is called causal or realizable, if for all $t_0\in\mathbb{R}$ it holds that $f(t)=0$ for $t<t_0$ implies $Lf(t)=0$ for $t<t_0$, or, equivalently, if $\operatorname{supp}h\subset[0,\infty)$. An alternative way to describe an LTI system is by its response to plane waves $\mathrm{e}^{i\omega_0\cdot}$. Taking the Fourier transform

$$\mathcal{F}f(\omega)=\widehat{f}(\omega)=\int_{\mathbb{R}}f(t)\mathrm{e}^{-i\omega t}\,\mathrm{d}t$$

of the system, one obtains $\widehat{Lf}(\omega)=T(\omega)\widehat{f}(\omega)$, where $T=\widehat{h}$ is the frequency response of the system. Especially, $L\mathrm{e}^{i\omega_0\cdot}=T(\omega_0)\mathrm{e}^{i\omega_0\cdot}$, that is, a plane wave $\mathrm{e}^{i\omega_0\cdot}$ is mapped to a plane wave multiplied by a (complex) scalar $T(\omega_0)$.

The space of frequency responses of causal LTI systems is rather restricted: Writing $h^\circ(t)=h(-t)$, we have $\widehat{T}=h^\circ$, or $\widehat{T^\circ}=h$. Therefore, $\operatorname{supp}\widehat{T}\subset(-\infty,0]$, or $\operatorname{supp}\widehat{T^\circ}\subset[0,\infty)$. If $T^\circ\in L^2(\mathbb{R})$, then by the Paley-Wiener Theorem [11, Theorem VI.7.2], $T^\circ\in H^2(\mathbb{C}^+)$. The space $H^p(\mathbb{C}^+)$, $1\leq p\leq\infty$, is the Hardy space on the complex upper half-plane $\mathbb{C}^+=\{\operatorname{Im}z>0\}$,

$$H^p(\mathbb{C}^+)=\left\{f:\mathbb{C}^+\to\mathbb{C}: f \text{ analytic on } \mathbb{C}^+,\ \sup_{y>0}\|f(\cdot+iy)\|_{L^p(\mathbb{R})}<\infty\right\}.$$

Via boundary values, $H^p(\mathbb{C}^+)$ is identified with a subspace of $L^p(\mathbb{R})$.

Additionally, the gain of energy of an LTI system is limited due to practical restrictions, i.e., in a given scenario one can expect that there exists a bounded function $G:\mathbb{R}\to[0,\infty)$ such that any frequency response T that can be physically realized satisfies $|T|\leq G$. For example, if the system that is to be designed is passive, then one can choose $G\equiv 1$. When T is bounded, especially $T^\circ\in H^\infty(\mathbb{C}^+)$.

In practice, LTI systems are only used in some frequency range $I\subset\mathbb{R}$. For a design problem, one then specifies a desired complex-valued frequency response $T_{\text{desired}}:I\to\mathbb{C}$ and tries to find a physically realizable system L such that the corresponding frequency response T_L is close to T_{desired} on I. In a situation where every transfer function T with $T^\circ \in H^\infty(\mathbb{C}^+)$ and $|T| \leq G$ is actually physically realizable, this is basically the problem (H-OPT$_p$), but posed on the complex upper half-plane. As usual, the problem can then be transported to the disk using a Möbius transformation and, if $p < \infty$, a scaling factor [9, Chapter 8].

Notice that, if $K = \overline{K}$ and g and ϕ are real symmetric, i.e., $\phi(\overline{z}) = \overline{\phi(z)}$, $z \in \partial\mathbb{D}$, and $g(\overline{z}) = g(z)$, $z \in K$, then a solution of (H-OPT$_p$) can always chosen to be real symmetric as well: If f^* is a solution of (H-OPT$_p$), then $(f^* + \overline{f^*(\overline{\cdot})})/2$ is a real symmetric solution. This is important, because the frequency response of an LTI system must be real symmetric, $H(-\omega) = \overline{H(\omega)}$, $\omega \in \mathbb{R}$.

The example that led us to the study of (H-OPT$_p$) is the design of dispersion compensating mirrors (DCMs) for the compression of ultra-short laser pulses [12, 15]. These mirrors consist of a stack of thin layers of typically two different dielectric materials with different refractive indices, which is deposited on some substrate, for example silica. A DCM constitutes a causal LTI system with frequency response R°, where R is called the reflection coefficient. For the mirror design problem, one specifies a frequency interval (or a collection of frequency intervals) I and a desired reflection coefficient R_{desired} on I. Usually, a high reflectance (HR) region is required, where $|R_{\text{desired}}| = 1$, and $\arg R_{\text{desired}}$ is determined by the phase shift that needs to be imposed on different frequencies. Additionally, it may be required to have a small frequency interval where the mirror is almost transparent, e.g., $|R| \leq 0.05$. This can be modelled by choosing $G = 0.05$ in the particular interval.

Unfortunately, the mapping which sends the layer structure n of the mirror to the corresponding reflection coefficient R_n has a rather complicated behavior. Usually, long optimization runs are necessary to find a mirror structure n such that R_n is close to R_{desired}. There is not even a useful characterization of the exact range of realizable reflection coefficients. The best thing we know is that R_n is real symmetric, $|R_n| \leq 1$ and $R_n \in H^\infty(\mathbb{C}^+)$ (see [16] for rigorous proofs of these facts). We can use this information to get an a priori bound on how small $\|R_n - R_{\text{desired}}\|_{L^p(\mathbb{R})}$ can in principle be made by solving

$$\begin{array}{ll} \text{minimize} & \|R - R_{\text{desired}}\|_{L^p(I)} \\ \text{subject to} & R \in H^\infty(\mathbb{C}^+),\ |R| \leq G,\ R \text{ real symmetric,} \end{array}$$

which, as mentioned before, can be transformed into (H-OPT$_p$).

The objective of this paper is to carry over results for the Nehari problem (1) to (H-OPT$_p$). For the Nehari problem, it is well-known that

(i) There is a solution.

(ii) If $\phi \in H^\infty(\mathbb{D}) + C(\partial\mathbb{D})$, then the solution f^* is unique and satisfies $|f^*(e^{i\theta}) - \phi(e^{i\theta})| = \|f^* - \phi\|_{L^\infty(\partial\mathbb{D})}$ for a.a. $e^{i\theta} \in \partial\mathbb{D}$.

(iii) If ϕ is Dini continuous, then the solution f^* is continuous.

The organization of the rest of this paper is as follows. In Section 2 we first show existence and uniqueness for (H-OPT$_p$). The main result of Section 2, Theorem 2, states that the solution of (H-OPT$_p$) satisfies an extremal property, which is similar to the extremal property (ii) for the Nehari problem. From this extremal property we deduce that in contrast to (iii) we cannot in general expect the solution of (H-OPT$_p$) to be continuous, even if ϕ is smooth. In Sections 3 and 4 we therefore deal with the relationship between ($\mathcal{A}$-OPT$_p$) and (H-OPT$_p$). In Section 3 we show that for $1 \le p < \infty$ the infimum of ($\mathcal{A}$-OPT$_p$) is equal to the minimum of (H-OPT$_p$), i.e., the minimum of (H-OPT$_p$) can be approximated with continuous feasible functions. In Section 4 we show that if K is the collection of finitely many intervals, then also for $p = \infty$ the infimum of ($\mathcal{A}$-OPT$_p$) is equal to the minimum of (H-OPT$_p$). We finish with some concluding remarks in Section 5.

2 Existence, Uniqueness and Extremal Properties

It is not hard to obtain a first result on existence and uniqueness for (H-OPT$_p$).

Theorem 1. (H-OPT$_p$) *has a solution,* $1 \le p \le \infty$*. If* $1 < p < \infty$*, then the solution is unique.*

Proof. Existence can be shown with a standard normal series argument as in [5, Chapter VI.1]. Uniqueness follows from the fact that the norm is strictly convex for $1 < p < \infty$. □

The main result of this section is

Theorem 2. *Let* f^* *be a solution of* (H-OPT$_p$) *and* $\tau^* = \|f^* - \phi\|_{L^p(K)} > 0$. *If* $1 \le p < \infty$*, then for almost all* $\mathrm{e}^{i\theta} \in \partial\mathbb{D} \setminus K$

$$|f^*(\mathrm{e}^{i\theta})| = g(\mathrm{e}^{i\theta}).$$

If $p = \infty$*, then, for almost all* $\mathrm{e}^{i\theta} \in \partial\mathbb{D}$*,* $f^*(\mathrm{e}^{i\theta})$ *is on the boundary of the set*

$$S(\theta, \tau^*) = \{z \in \mathbb{C} : |z| \le g(\mathrm{e}^{i\theta}),\ |z - \phi(\mathrm{e}^{i\theta})| \le \tau^* \text{ if } \mathrm{e}^{i\theta} \in K\}.$$

Moreover, for $1 < p \le \infty$*, the solution of* (H-OPT$_p$) *is unique. If* $K \ne \partial\mathbb{D}$*, then the solution of* (H-OPT$_p$) *is also unique for* $p = 1$.

Remark 3. *Similar extremal properties do not only hold true for the Nehari problem, but also for the related problems that we mentioned in Section 1, see Baratchart, Leblond, and Partington [2, Theorem 2], and Helton and Howe [6, Theorem 1]. In fact, the main ideas that we use to prove Theorem 2 come from [6].*

We are going to use the Hahn-Banach Theorem to prove Theorem 2. Before we can do this, we need an auxiliary result. Let

$$\mathscr{S}_p = \{f \in L^\infty(\partial\mathbb{D}) : |f| \le g,\ \|f - \phi\|_{L^p(K)} \le \tau^*\},$$

where τ^* is the minimum of (H-OPT$_p$).

Lemma 4. *Let* $1 \le p \le \infty$ *and assume that* $\tau^* > 0$.

(a) *The set* $\mathscr{S}_p$ *is convex.*

(b) *The interior of* $\mathscr{S}_p$ *is non-empty and disjoint from* $\mathcal{A}(\mathbb{D})$.

(c) *Every element of $\mathscr{S}_p$ is a pointwise limit of functions from $\mathscr{S}_p \cap C(\partial\mathbb{D})$.*

Proof. There is not much to show for (a), because it is immediate from the definition that $\mathscr{S}_p$ is convex. Moreover, it is clear that the interior of $\mathscr{S}_p$ cannot contain any function from $\mathcal{A}(\mathbb{D})$: If there were such a function f, we would have $\|f - \phi\|_{L^p(K)} < \tau^*$, contradicting the definition of τ^*. This is the second part of (b).

By Tietze's Extension Theorem [14, Theorem 20.4], ϕ can be extended to a function that is continuous on $\partial\mathbb{D}$. We also denote this extension by ϕ. We arrange it so that $|\phi| \le g$ on $\partial\mathbb{D}$. Now let $\epsilon = \tau^*/(2\|\mathbf{1}\|_{L^p(K)})$ and define

$$a(\mathrm{e}^{i\theta}) = \begin{cases} \phi(\mathrm{e}^{i\theta}), & |\phi(\mathrm{e}^{i\theta})| \le g(\mathrm{e}^{i\theta}) - \epsilon, \\ \frac{\phi(\mathrm{e}^{i\theta})}{|\phi(\mathrm{e}^{i\theta})|}(g(\mathrm{e}^{i\theta}) - \epsilon), & \text{otherwise.} \end{cases}$$

It is straightforward to prove that $\|a - \phi\|_{L^\infty(\partial\mathbb{D})} \le \epsilon$ and $|a| \le g - \epsilon$. Let $v \in L^\infty(\partial\mathbb{D})$ with $\|v\|_{L^\infty(\partial\mathbb{D})} \le \epsilon$. Then $\|(a + v) - \phi\|_{L^p(K)} \le \|2\epsilon\mathbf{1}\|_{L^p(K)} = \tau^*$ and $|a + v| \le g$, i.e., $a + v \in \mathscr{S}_p$. Because τ^* is positive by assumption, a lies in the interior of $\mathscr{S}_p$. This finishes the proof of (b).

The proof of (c) is not particularly hard, but a little more technical. We first consider the case $1 \le p < \infty$. Let $f \in \mathscr{S}_p$. Then there is a sequence $(\widetilde{f}_n) \subset C(\partial\mathbb{D})$ with $\|\widetilde{f}_n\|_{L^\infty(\partial\mathbb{D})} \le \|f\|_{L^\infty(\partial\mathbb{D})}$ such that $\widetilde{f}_n \to f$ a.e. (see, e.g., [14, Chapter 2]). By dominated convergence, $\|\widetilde{f}_n - \phi\|_{L^p(K)} \to \|f - \phi\|_{L^p(K)} \le \tau^*$. Now set

$$\widetilde{f}_n^1 = \phi + (\widetilde{f}_n - \phi)\frac{\|f - \phi\|_{L^p(K)}}{\|\widetilde{f}_n - \phi\|_{L^p(K)}}.$$

Then $\widetilde{f}_n^1 \in C(\partial\mathbb{D})$, $\widetilde{f}_n^1 \to f$ a.e., and moreover $\|\widetilde{f}_n^1 - \phi\|_{L^p(K)} = \|f - \phi\|_{L^p(K)} \le \tau^*$. However, it may not hold true that $|\widetilde{f}_n^1| \le g$. We therefore define functions f_n by

$$f_n(\mathrm{e}^{i\theta}) = \begin{cases} \widetilde{f}_n^1(\mathrm{e}^{i\theta}), & |\widetilde{f}_n^1(\mathrm{e}^{i\theta})| \le g(\mathrm{e}^{i\theta}), \\ \phi(\mathrm{e}^{i\theta}) + \mu_n(\mathrm{e}^{i\theta})(\widetilde{f}_n^1(\mathrm{e}^{i\theta}) - \phi(\mathrm{e}^{i\theta})), & \text{otherwise,} \end{cases}$$

where μ_n is a function on $\partial\mathbb{D}$ such that, if we are in the second case of the above definition, then $|f_n(\mathrm{e}^{i\theta})| = g(\mathrm{e}^{i\theta})$. Concretely, we set $\mu_n(\mathrm{e}^{i\theta}) = 1/p_\theta(\widetilde{f}_n^1(\mathrm{e}^{i\theta}))$, where p_θ is the Minkowski functional

$$p_\theta(z) = \inf\{t > 0 : |\phi(\mathrm{e}^{i\theta}) + t^{-1}(z - \phi(\mathrm{e}^{i\theta}))| \le g(\mathrm{e}^{i\theta})\}.$$

Then f_n is continuous, $|f_n| \le g$, and $f_n \to f$ pointwise a.e. Moreover, if $|\widetilde{f}_n^1(\mathrm{e}^{i\theta})| > g(\mathrm{e}^{i\theta})$, then $p_\theta(\widetilde{f}_n^1(\mathrm{e}^{i\theta})) > 1$, and therefore

$$|f_n(\mathrm{e}^{i\theta}) - \phi(\mathrm{e}^{i\theta})| = \left|\frac{\widetilde{f}_n^1(\mathrm{e}^{i\theta}) - \phi(\mathrm{e}^{i\theta})}{p_\theta(\widetilde{f}_n^1(\mathrm{e}^{i\theta}))}\right| \le |\widetilde{f}_n^1(\mathrm{e}^{i\theta}) - \phi(\mathrm{e}^{i\theta})|.$$

If follows that $\|f_n - \phi\|_{L^p(K)} \le \|\widetilde{f}_n^1 - \phi\|_{L^p(K)} \le \tau^*$, so $f_n \in \mathscr{S}_p \cap C(\partial\mathbb{D})$. This proves (c) for $1 \le p < \infty$.

It remains to consider the case $p = \infty$. Let $f \in \mathscr{S}_\infty$. Then we have $\|f - \phi\|_{L^\infty(K)} \le \tau^*$. As before, there is a sequence $(f_n^K) \subset C(K)$ such that $f_n^K \to f$ pointwise a.e. on K and $\|f_n^K - \phi\|_{L^\infty(K)} \le \tau^*$, and we can arrange it so that $|f_n^K| \leq g$ on K. By Tietze's Extension Theorem, every f_n^K can be extended to a function that is continuous on $\partial\mathbb{D}$. We also denote this extension by f_n^K, and we arrange it so that $|f_n^K| \leq g$ on $\partial\mathbb{D}$. Similarly, there is a sequence $(f_n^{\partial\mathbb{D}}) \subset C(\partial\mathbb{D})$ such that $f_n^{\partial\mathbb{D}} \to f$ pointwise a.e. on $\partial\mathbb{D}$, and we arrange it so that $|f_n^{\partial\mathbb{D}}| \leq g$ on $\partial\mathbb{D}$.

Now let $U_n \subset \partial\mathbb{D}$ be open with $K \subset U_n$ and $\operatorname{meas}(U_n \setminus K) \leq \frac{1}{2^n}$, where meas denotes Lebesgue measure on the circle. Regularity properties of Lebesgue measure ensure that this is always possible [14]. By Urysohn's Lemma there is $h_n \in C(\partial\mathbb{D})$ with $h_n \equiv 1$ on K, $h_n \geq 0$ and $\operatorname{supp} h_n \subset U_n$. Set

$$f_n = h_n f_n^K + (1 - h_n) f_n^{\partial\mathbb{D}}.$$

Then $f_n \in C(\partial\mathbb{D})$, $f_n \to f$ a.e., and $|f_n| \leq g$ on $\partial\mathbb{D}$. Moreover, $f_n = f_n^K$ on K, whence $\|f_n - \phi\|_{L^\infty(K)} = \|f_n^K - \phi\|_{L^\infty(K)} \leq \tau^*$. Therefore, $f_n \in \mathscr{S}_\infty$. This proves (c) for $p = \infty$. □

We are now ready to prove Theorem 2. Our proof goes along the lines of the proof of [6, Theorem 2].

Proof of Theorem 2. By the Hahn-Banach Theorem and properties (a) and (b) of Lemma 4, there is a nonzero $\lambda \in C(\partial\mathbb{D})^*$ such that

$$\operatorname{Re} \lambda(\mathscr{S}_p \cap C(\partial\mathbb{D})) \leq \operatorname{Re} \lambda(\mathcal{A}(\mathbb{D})). \tag{2}$$

Because $\mathcal{A}(\mathbb{D})$ is a linear space, it follows that either $\operatorname{Re} \lambda(\mathcal{A}(\mathbb{D})) = \mathbb{R}$ or $\operatorname{Re} \lambda(\mathcal{A}(\mathbb{D})) = 0$. But because of (2), $\operatorname{Re} \lambda(\mathcal{A}(\mathbb{D}))$ is bounded from below, whence

$$\operatorname{Re} \lambda(\mathcal{A}(\mathbb{D})) = 0. \tag{3}$$

Therefore, there is a nonzero $l \in H_0^1(\mathbb{D}) = \{f \in H^1(\mathbb{D}) : f(0) = 0\}$ such that

$$\lambda(f) = \int_{-\pi}^{\pi} f(\mathrm{e}^{i\theta}) l(\mathrm{e}^{i\theta}) \, \mathrm{d}\theta$$

for all $f \in C(\partial\mathbb{D})$, see, e.g., [5, Chapter IV]. Using the right hand side of the above equation, we can extend λ to all of $L^\infty(\partial\mathbb{D})$.

If $f \in \mathscr{S}_p$, then by property (c) of Lemma 4, there is a sequence $(f_n) \subset \mathscr{S}_p \cap C(\partial\mathbb{D})$ such that $f_n \to f$ pointwise a.e. Because (f_n) is bounded by g, dominated convergence yields $\lambda(f_n) \to \lambda(f)$. It follows from (2) and (3) that

$$\operatorname{Re} \lambda(f) \leq 0 \text{ for all } f \in \mathscr{S}_p. \tag{4}$$

Further, if $f \in H^\infty(\mathbb{D})$, then f is the pointwise limit of functions in $\mathcal{A}(\mathbb{D})$. (Take for example the Poisson integrals of f.) Dominated convergence and (3) imply

$$\operatorname{Re} \lambda(f) = 0 \text{ for all } f \in H^\infty(\mathbb{D}). \tag{5}$$

We now prove the assertion of the theorem for the case $1 \leq p < \infty$. Let f^* be a solution of (H-OPT$_p$). Then $f^* \in H^\infty(\mathbb{D}) \cap \mathscr{S}_p$. If $\partial\mathbb{D} \setminus K$ has zero measure, there is nothing to show, so we can assume that $\partial\mathbb{D} \setminus K$ has positive measure. Assume to the contrary that it is not true that $|f^*| = g$ a.e. on $\partial\mathbb{D} \setminus K$. Then there are a set $I \subset \partial\mathbb{D} \setminus K$ of positive measure and $\epsilon > 0$ such that $|f^*| + \epsilon \leq g$

on I. Let $h \in L^\infty(\partial\mathbb{D})$ be any function with $\|h\|_{L^\infty(\partial\mathbb{D})} \le \epsilon$ and $\operatorname{supp} h \subset I$. Then $f^* + h \in \mathscr{S}_p$, and

$$0 \overset{(4)}{\ge} \operatorname{Re} \int_{-\pi}^{\pi} \left(f^*(\mathrm{e}^{i\theta}) + h(\mathrm{e}^{i\theta})\right) l(\mathrm{e}^{i\theta})\, \mathrm{d}\theta \overset{(5)}{=} \operatorname{Re} \int_{-\pi}^{\pi} h(\mathrm{e}^{i\theta}) l(\mathrm{e}^{i\theta})\, \mathrm{d}\theta.$$

The same inequality follows for $-h$, ih and $-ih$, whence $\int_{-\pi}^{\pi} h(\mathrm{e}^{i\theta}) l(\mathrm{e}^{i\theta})\, \mathrm{d}\theta = 0$ for all $h \in L^\infty(\partial\mathbb{D})$. But then $l = 0$ on I, and therefore $l = 0$ on $\partial\mathbb{D}$. This is a contradiction to $l \ne 0$. Therefore, it must hold true that $|f^*| = g$ a.e. on $\partial\mathbb{D} \setminus K$.

The statement for the case $p = \infty$ follows with a similar argument.

From Theorem 1 we already know that the solution of (H-OPT$_p$) is unique for $1 < p < \infty$. For $p = \infty$, uniqueness follows from the fact that the sets $S(\theta, \tau^*)$ are strictly convex for all θ: If f_1^* and f_2^* are both solutions of (H-OPT$_\infty$), then $(f_1^* + f_2^*)/2$ is also a solution of (H-OPT$_\infty$), because the norm $\|\cdot\|_{L^\infty(K)}$ is convex. Because the sets $S(\theta, \tau^*)$ are strictly convex and $f_j^*(\mathrm{e}^{i\theta})$ is on the boundary of $S(\theta, \tau^*)$ for almost all $\mathrm{e}^{i\theta}$, $j = 1, 2$, it follows that $f_1^* = f_2^*$. If $p = 1$ and $K \ne \partial\mathbb{D}$, then $\partial\mathbb{D} \setminus K$ is nonempty and open and therefore has positive measure. Uniqueness then follows in the same way from the fact that the sets $\{z \in \mathbb{C} : |z| \le g(\mathrm{e}^{i\theta})\}$ are strictly convex for all $\mathrm{e}^{i\theta} \in \partial\mathbb{D} \setminus K$. □

A few remarks are in order.

Remark 5. *If $\tau^* = 0$, then ϕ is the restriction to K of some function in $H^\infty(\mathbb{D})$. Because K has positive measure, this implies uniqueness for* (H-OPT$_p$). *However, it is easy to see that the extremal property from Theorem 2 need not hold true in this case.*

Remark 6. *Theorem 2 still holds true if we admit more general g in Problem* (H-OPT$_p$)*, for example, if g is continuous up to finitely many jump discontinuities. We only used the continuity of g in the proof of Lemma 4(c). In order to prove Theorem 2 for this case, one has to adapt that proof. We leave out the details, because they are technical and do not add any insight.*

The following example demonstrates that Theorem 2 need not hold true if we drop the assumption that $|\phi| \le g$ on K.

Example 7. *Let $K = \partial\mathbb{D}$, $\phi(\mathrm{e}^{i\theta}) = 2$ and $g(\mathrm{e}^{i\theta}) = |2 + \mathrm{e}^{i\theta}|$. Then the solution of* (H-OPT$_\infty$) *is not unique, and also the extremal property from Theorem 2 is not satisfied. Indeed, because $g(\mathrm{e}^{i\pi}) = 1$, we have $\min_{f \in H^\infty(\mathbb{D}),\, |f| \le g} \|f - \phi\|_{L^\infty(K)} \ge 1$. On the other hand, let $f_0(\mathrm{e}^{i\theta}) = 1$, $f_1(\mathrm{e}^{i\theta}) = 2 + \mathrm{e}^{i\theta}$ and $f_\lambda(\mathrm{e}^{i\theta}) = \lambda f_0(\mathrm{e}^{i\theta}) + (1 - \lambda) f_1(\mathrm{e}^{i\theta})$. Then f_λ is feasible for* (H-OPT$_\infty$)*, $0 \le \lambda \le 1$, and $\|f_\lambda - \phi\|_{L^\infty(K)} = 1$. Thus, the solution of* (H-OPT$_\infty$) *is not unique. Further, f_λ does not satisfy the extremal property from Theorem 2 for $0 < \lambda < 1$.*

The reason why the proof of Theorem 2 fails if we drop the assumption $|\phi| \le g$ on K is that the set $\mathscr{S}_\infty$ may have empty interior, i.e., $\mathscr{S}_\infty$ may not satisfy property (b) from Lemma 4. Indeed, in Example 7 this is the case because of the singularity at $\theta = \pi$. However, one can show that $\mathscr{S}_\infty$ satisfies the conditions from Lemma 4 under additional assumptions, for example, if τ^* satisfies $\tau^* > \sup_{\mathrm{e}^{i\theta} \in \partial\mathbb{D}} |\phi(\mathrm{e}^{i\theta})| - g(\mathrm{e}^{i\theta})$.

Using Theorem 2, it is not hard to construct an example for which the solution of (H-OPT$_\infty$) is not continuous.

Example 8. *Let $K \subset \partial\mathbb{D}$ be closed with positive measure such that $\partial\mathbb{D} \setminus K$ has positive measure. Let $0 \neq \phi \in C(K)$ such that $\phi \notin H^\infty(\mathbb{D})|_K$, and let $g \equiv 3\|\phi\|_{L^\infty(K)}$. Let f^* be the solution of* (H-OPT$_\infty$) *with K, ϕ and g. Because 0 is feasible for* (H-OPT$_\infty$)*, $\|f^* - \phi\|_{L^\infty(K)} \leq \|\phi\|_{L^\infty(K)}$, and therefore $\|f\|_{L^\infty(K)} \leq 2\|\phi\|_{L^\infty(K)}$. On the other hand, because $\phi \notin H^\infty(\mathbb{D})|_K$, it follows that $\|f^* - \phi\|_{L^\infty(K)} > 0$. Then Theorem 2 yields $|f^*| = 3\|\phi\|_{L^\infty(K)}$ on $\partial\mathbb{D} \setminus K$. Therefore, f^* cannot be continuous.*

Remark 9. *Under some additional assumptions on g and ϕ we can obtain a weak continuity result. Recall that a function f defined on $\partial\mathbb{D}$ is called Dini continuous if for some $\epsilon > 0$ it holds that $\int_0^\epsilon \frac{\omega_f(t)}{t}\,\mathrm{d}t < \infty$, where $\omega_f(\delta) = \sup\{|f(\mathrm{e}^{i\theta}) - \mathrm{e}^{it}| : |\theta - t| < \delta\}$ is the modulus of continuity of f. Moreover, recall that the essential range of some $f \in L^\infty(\partial\mathbb{D})$ near $\mathrm{e}^{i\theta}$ is the set*

$$\operatorname{ess\,ran}(f, \mathrm{e}^{i\theta}) = \left\{ z \in \mathbb{C} : \begin{array}{l} f^{-1}(B_{\epsilon_1}(z)) \cap \mathrm{e}^{i[\theta-\epsilon_2, \theta+\epsilon_2]} \textit{ has positive} \\ \textit{Lebesgue measure for all } \epsilon_1, \epsilon_2 > 0 \end{array} \right\}.$$

Here, $B_\epsilon(z) = \{w \in \mathbb{C} : |z - w| < \epsilon\}$ denotes a ball in $\mathbb{C}$.

Assume that g and ϕ are Dini continuous. Let $\tau^* = \|f^* - \phi\|_{L^\infty(K)} > 0$ and let

$$\Gamma_1 = \{\mathrm{e}^{i\theta} \in \partial\mathbb{D} : \operatorname{ess\,ran}(f^*, \mathrm{e}^{i\theta}) \subset \partial B_{\tau^*}(\phi(\mathrm{e}^{i\theta}))\},$$
$$\Gamma_2 = \{\mathrm{e}^{i\theta} \in \partial\mathbb{D} : \operatorname{ess\,ran}(f^*, \mathrm{e}^{i\theta}) \subset B_{g(\mathrm{e}^{i\theta})}(0)\}.$$

By Theorem 2 we especially have $\partial\mathbb{D} \setminus K \subset \Gamma_2$. Using the techniques from Hui [10], one can show that f^* is continuous on Γ_1° and Γ_2°, where the little circle denotes the interior of a set. A result of Chirka [4, Theorem 33] then implies that if ϕ, $g \in C^k(\partial\mathbb{D})$, $k \geq 2$, then $f^* \in C^{k-1,1-\epsilon}(\Gamma_1^\circ \cup \Gamma_2^\circ \cup \mathbb{D})$ for any $\epsilon > 0$.

We do not know whether under the above assumptions f^* is also continuous on all of K°. The difficulty that arises when one tries to apply the techniques from [10] is that for some $\mathrm{e}^{i\theta} \in K$ the boundary of the set $S(\theta, \tau^*)$ is not an analytic curve.

3 Approximation by smooth functions, $1 \leq p < \infty$

We saw in Example 8 that we cannot in general expect the solution of (H-OPT$_p$) to be continuous, even if ϕ and g are smooth. For numerical computations, however, it is convenient to work in spaces of smooth functions. The following theorem shows that it is reasonable to work with smooth functions.

Theorem 10. *Let $1 \leq p < \infty$. Then the infimum of* ($\mathcal{A}$-OPT$_p$) *is equal to the minimum of* (H-OPT$_p$)*. In fact, let f^* be a solution of* (H-OPT$_p$)*. Then there is a sequence $(f_n) \subset \mathcal{A}(\mathbb{D})$ with $|f_n| \leq g$ on $\partial\mathbb{D}$ such that*

$$\|f_n - f^*\|_{L^p(\partial\mathbb{D})} \to 0 \qquad \textit{as } n \to \infty. \tag{6}$$

Furthermore, we may even arrange it for the f_n to be polynomials, that is, to be of the form

$$f_n(\mathrm{e}^{i\theta}) = \sum_{k=0}^{N_n - 1} \alpha_{N_n,k} \mathrm{e}^{ik\theta}. \tag{7}$$

If f^* is real symmetric, then we can arrange it for the f_n to be real symmetric, that is, to have real coefficients $\alpha_{N_n,k}$.

Proof. We first show that there is a sequence $(f_n) \subset \mathcal{A}(\mathbb{D})$ with $|f_n| \le g$ on $\partial\mathbb{D}$ that satisfies (6). For $0 < r < 1$ let f_r^* be the Poisson integral $f_r^*(\mathrm{e}^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^*(\mathrm{e}^{it})P_r(\theta - t)\,\mathrm{d}t$, where $P_r(\theta) = (1-r^2)/(1-2r\cos\theta + r^2)$ is the Poisson kernel for the disk. We then have $f_r^* \in \mathcal{A}(\mathbb{D})$ and $\|f_r^* - f^*\|_{L^p(\partial\mathbb{D})} \to 0$ as $r \nearrow 1$, but it might not be true that $|f_r^*| \le g$ on $\partial\mathbb{D}$. We are going to construct sequences (r_n) with $r_n \nearrow 1$ and (η_n) with $\eta_n \to 1$ such that $f_n = \eta_n f_{r_n}^*$ satisfies $|f_n| \le g$ on $\partial\mathbb{D}$. It then follows that

$$\begin{aligned}\|f_n - f^*\|_{L^p(\partial\mathbb{D})} &= \|\eta_n f_{r_n}^* - f^*\|_{L^p(\partial\mathbb{D})} \\ &\le \|f_{r_n}^* - f^*\|_{L^p(\partial\mathbb{D})} + |1-\eta_n|\|f_{r_n}^*\|_{L^p(\partial\mathbb{D})} \to 0\end{aligned}$$

as $n \to \infty$.

Fix $\epsilon > 0$. Because g is uniformly continuous on $\partial\mathbb{D}$, there is $\delta > 0$ such that

$$|\theta - t| \le \delta \quad \to \quad |g(\mathrm{e}^{i\theta}) - g(\mathrm{e}^{it})| \le \epsilon/2. \tag{8}$$

Furthermore, as $r \nearrow 1$, P_r becomes increasingly concentrated at 0 so that there is $\varrho = \varrho(\epsilon) < 1$ such that for all $r \in [\varrho, 1)$

$$\max_{t\in[-\pi,\pi]\setminus[-\delta,\delta]} |P_r(t)| < (\epsilon/2)\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(\mathrm{e}^{i\theta})|\,\mathrm{d}\theta\right)^{-1}. \tag{9}$$

Now for $r \in [\varrho, 1)$

$$\begin{aligned}|f_r^*(\mathrm{e}^{i\theta})| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f^*(\mathrm{e}^{i(\theta-t)})P_r(t)\,\mathrm{d}t\right| \\ &\le \frac{1}{2\pi}\left(\int_{[-\delta,\delta]} + \int_{[-\pi,\pi]\setminus[-\delta,\delta]} |f^*(\mathrm{e}^{i(\theta-t)})P_r(t)|\,\mathrm{d}t\right) \\ &\le \max_{t\in[\theta-\delta,\theta+\delta]} |f^*(\mathrm{e}^{it})| + \left(\max_{t\in[-\pi,\pi]\setminus[-\delta,\delta]} |P_r(t)|\right)\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f^*(\mathrm{e}^{it})|\,\mathrm{d}t\right).\end{aligned}$$

With $|f^*| \le g$ and (8) and (9),

$$|f_r^*(\mathrm{e}^{i\theta})| \le \max_{t\in[\theta-\delta,\theta+\delta]} |g(\mathrm{e}^{it})| + \epsilon/2 \le g(\mathrm{e}^{i\theta}) + \epsilon/2 + \epsilon/2 = g(\mathrm{e}^{i\theta}) + \epsilon.$$

To finish the proof of (6), let (ϵ_n) be a sequence of positive real numbers with $\epsilon_n \to 0$. Choose $r_n \ge \varrho(\epsilon_n)$ such that $r_n \nearrow 1$. Then

$$|f_{r_n}^*| \le g + \epsilon_n \le \left(1 + \frac{\epsilon_n}{\min_{\mathrm{e}^{i\theta}\in\partial\mathbb{D}} g(\mathrm{e}^{i\theta})}\right) g.$$

Set $\eta_n = \left(1 + \frac{\epsilon_n}{\min_{\mathrm{e}^{i\theta}\in\partial\mathbb{D}} g(\mathrm{e}^{i\theta})}\right)^{-1}$ and $f_n = \eta_n f_{r_n}^*$. Then $\eta_n \to 1$, and $|f_n| = |\eta_n f_{r_n}^*| \le g$, and of course f_n is in $\mathcal{A}(\mathbb{D})$. We have already seen that $\|f_n - f^*\|_{L^p(\partial\mathbb{D})} \to 0$ as $n \to \infty$.

Because the f_n are continuous, there is a sequence $(\widetilde{f}_n)$ of polynomials with $\|\widetilde{f}_n - f_n\|_{L^\infty(\partial\mathbb{D})} \le \epsilon_n$. As before, we can find $(\widetilde{\eta}_n)$ such that $|\widetilde{\eta}_n\widetilde{f}_n| \le g$ and $\|\widetilde{\eta}_n\widetilde{f}_n - f^*\| \to 0$.

Concerning the statement about symmetry, notice that if f^* is real symmetric, then due to the symmetry of the Poisson kernel the f_n are real symmetric. If the f_n are real symmetric, we can also choose the polynomials $\widetilde{f}_n$ to be real symmetric. □

4 Approximation by smooth functions, $p = \infty$

In the last section we saw that in the case $1 \leq p < \infty$, the infimum of ($\mathcal{A}$-OPT$_p$) is equal to the minimum of (H-OPT$_p$). In this section we show that under additional assumptions this is still true for $p = \infty$. The main part of this section is devoted to the proof of

Theorem 11. *Assume that K is the disjoint union of finitely many intervals of positive length, i.e., $K = \bigcup_{j=1}^{N} K_j$, where $K_j = \mathrm{e}^{i[\lambda_j, \rho_j]}$ for some $\lambda_j < \rho_j$. Then the infimum of* ($\mathcal{A}$-OPT$_\infty$) *is equal to the minimum of* (H-OPT$_\infty$)*.*

The proof of Theorem 11 is rather technical and lengthy. We divide it into several lemmas. Before we start with the proof, we try to give an idea of the structure.

- Lemmas 12 and 13 deal with the construction and properties of certain analytic functions ψ_δ mapping $\overline{\mathbb{D}}$ into $\mathbb{D}$. The important properties are that ψ_δ converges uniformly to the identity as $\delta \to 0$ and that $\psi_\delta(\mathrm{e}^{i\theta})$ converges *tangentially* to $\mathrm{e}^{i\theta}$ whenever $\mathrm{e}^{i\theta} \in \partial K$.

- The idea is to consider $f^* \circ \psi_\delta$, where f^* is the solution of (H-OPT$_\infty$), i.e., $\|f^* - \phi\|_{L^\infty(K)} = \min_{f \in H^\infty(\mathbb{D}), |f| \leq g} \|f - \phi\|_{L^\infty(K)} = \tau^*$. Importantly, $f^* \circ \psi_\delta \in \mathcal{A}(\mathbb{D})$. In Lemma 15 we prove that $\limsup_{\delta \to 0} \|f^* \circ \psi_\delta - \phi\|_{L^\infty(K)} \leq \tau^*$. Tangential convergence of ψ_δ on ∂K is a crucial ingredient of the proof.

- This does not prove Theorem 11 yet, since $f^* \circ \psi_\delta$ may not be feasible for ($\mathcal{A}$-OPT$_\infty$), i.e., we may not have $|f^* \circ \psi_\delta| \leq g$. However, we can multiply $f^* \circ \psi_\delta$ by some positive η such that $\eta(f^* \circ \psi_\delta)$ is feasible for ($\mathcal{A}$-OPT$_\infty$) and $\eta = \eta(\delta) \to 1$ as $\delta \to 0$. It will turn out that $\|\eta(\delta)(f^* \circ \psi_\delta) - \phi\|_{L^\infty(K)}$ converges to the minimum of (H-OPT$_\infty$) as $\delta \to 0$, which finishes the proof of Theorem 11.

In the following, we use the multivalued complex argument function that maps a complex number z with polar representation $z = r\mathrm{e}^{i\theta}$, $r > 0$, $\theta \in \mathbb{R}$, to the set $\theta + 2\pi\mathbb{Z}$. The advantage of using the multivalued argument is that rules like $\arg(zw) = \arg z + \arg w$, $z, w \in \mathbb{C} \setminus \{0\}$, hold, which are more tedious to write down if one restricts the argument, e.g., to $[-\pi, \pi)$. However, we do not make this explicit in our notation, i.e., we write $\arg z = \theta$ instead of $\arg z = \theta + 2\pi\mathbb{Z}$. We also write $\arg z \in I$ to express that there is some $\theta \in \arg z$ with $\theta \in I$.

We are now ready to start with the proof of Theorem 11.

Lemma 12. *Let $q : \partial\mathbb{D} \to [0, \infty)$ be Lipschitz continuous and let*

$$h(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\mathrm{e}^{i\theta} + z}{\mathrm{e}^{i\theta} - z} q(\mathrm{e}^{i\theta}) \, \mathrm{d}\theta, \qquad z \in \mathbb{D}.$$

For $\delta \in (0, 1)$ let $F_\delta(z) = z(1 - \delta h(z))$ and let

$$\psi_\delta(z) = \frac{F_\delta(z)}{\|F_\delta\|_{H^\infty(\mathbb{D})}} (1 - \delta^2).$$

Then the following statements hold true.

(a) *$\psi_\delta \in \mathcal{A}(\mathbb{D})$ and $\psi_\delta(\overline{\mathbb{D}}) \subset \mathbb{D}$.*

(b) *ψ_δ converges uniformly to the identity as $\delta \to 0$. More precisely, there is a constant $C > 0$ such that $\max_{z\in\overline{\mathbb{D}}} |\psi_\delta(z) - z| \le C\delta$, $\delta \in (0,1)$.*

(c) *If $\operatorname{Re} h(\mathrm{e}^{i\theta}) = 0$ and $\operatorname{Im} h(\mathrm{e}^{i\theta}) \neq 0$, then $\psi_\delta(\mathrm{e}^{i\theta}) \to \mathrm{e}^{i\theta}$ tangentially as $\delta \to 0$. More precisely,*

$$\arg \psi_\delta(\mathrm{e}^{i\theta}) = \theta + \arctan\left(-\delta \operatorname{Im} h(\mathrm{e}^{i\theta})\right),$$

and there are $\delta_0 > 0$ and $C > 0$ such that for $\delta \in (0, \delta_0)$ and for all $\mathrm{e}^{i\theta}$ with $\operatorname{Re} h(\mathrm{e}^{i\theta}) = 0$ and $\operatorname{Im} h(\mathrm{e}^{i\theta}) \neq 0$

$$\left|1 - |\psi_\delta(\mathrm{e}^{i\theta})|\right| \le C\delta^2.$$

Our definition of the function F_δ is inspired by a result of Nehari [13, Chapter V.11] concerning conformal mapping from the unit disk to nearly circular domains. Notice that the real part of h is the Poisson integral of q, and especially $\operatorname{Re} h = q$ on $\partial\mathbb{D}$. The important property in the following proof is that, because q and the Poisson kernel are non-negative, $\operatorname{Re} h \ge 0$ on $\overline{\mathbb{D}}$.

Proof. From the basic theory of Hardy spaces it is well-known that h is analytic on $\mathbb{D}$. Moreover, since q is Lipschitz continuous, $h \in \mathcal{A}(\mathbb{D})$, see, e.g., [5, Corollary III.1.4]. Therefore $\psi_\delta \in \mathcal{A}(\mathbb{D})$. From the definition of ψ_δ it is clear that $\psi_\delta(\overline{\mathbb{D}}) \subset \mathbb{D}$. This is (a).

In order to see (b), notice first that $(1-\delta|h(z)|)|z| \le |F_\delta(z)| \le (1+\delta|h(z)|)|z|$ implies

$$|1 - \|F_\delta\|_{H^\infty(\mathbb{D})}| \le \delta \|h\|_{H^\infty(\mathbb{D})}.$$

Then

$$\begin{aligned}
|\psi_\delta(z) - z| &\le |F_\delta(z) - z| + \left|\frac{1-\delta^2}{\|F_\delta\|_{H^\infty(\mathbb{D})}} - 1\right| |F_\delta(z)| \\
&\le \delta|zh(z)| + \left|1 - \delta^2 - \|F_\delta\|_{H^\infty(\mathbb{D})}\right| \frac{|F_\delta(z)|}{\|F_\delta\|_{H^\infty(\mathbb{D})}} \\
&\le \delta\|h\|_{H^\infty(\mathbb{D})} + \delta^2 + |1 - \|F_\delta\|_{H^\infty(\mathbb{D})}| \\
&\le (2\|h\|_{H^\infty(\mathbb{D})} + 1)\delta.
\end{aligned}$$

This is (b) with $C = 2\|h\|_{H^\infty(\mathbb{D})} + 1$.

It remains to prove (c). For any $\mathrm{e}^{i\theta} \in \partial\mathbb{D}$

$$\begin{aligned}
\arg \psi_\delta(\mathrm{e}^{i\theta}) &= \arg F_\delta(\mathrm{e}^{i\theta}) = \arg\left(\mathrm{e}^{i\theta}(1 - \delta h(\mathrm{e}^{i\theta}))\right) \\
&= \theta + \arg\left(1 - \delta h(\mathrm{e}^{i\theta})\right) = \theta + \arctan\left(\frac{-\delta \operatorname{Im} h(\mathrm{e}^{i\theta})}{1 - \delta \operatorname{Re} h(\mathrm{e}^{i\theta})}\right).
\end{aligned} \tag{10}$$

Assume especially that $\operatorname{Re} h(\mathrm{e}^{i\theta}) = 0$ and $\operatorname{Im} h(\mathrm{e}^{i\theta}) \neq 0$. Then

$$\left(\arg \psi_\delta(\mathrm{e}^{i\theta})\right) - \theta = \arctan\left(-\delta \operatorname{Im} h(\mathrm{e}^{i\theta})\right),$$

which converges linearly to zero as $\delta \to 0$. This is the first assertion of (c).

Since $\operatorname{Re} h$ is the Poisson integral of q and since $q \geq 0$ on $\partial\mathbb{D}$, we have $\operatorname{Re} h \geq 0$ on $\overline{\mathbb{D}}$, i.e., h only takes values in $\{\operatorname{Re} z \geq 0\}$. Then for $z \in \overline{\mathbb{D}}$ and $\delta \leq \frac{2}{\|\operatorname{Re} h\|_{L^\infty(\partial\mathbb{D})}}$

$$\begin{aligned} |F_\delta(z)| &= |z||1 - \delta h(z)| \leq \sqrt{|1 - \delta \operatorname{Re} h(z)|^2 + \delta^2 (\operatorname{Im} h(z))^2} \\ &\leq \sqrt{1 + \delta^2 (\operatorname{Im} h(z))^2} \leq 1 + \delta^2 \frac{(\operatorname{Im} h(z))^2}{2} \\ &\leq 1 + \delta^2 \frac{\|\operatorname{Im} h(z)\|^2_{L^\infty(\partial\mathbb{D})}}{2}. \end{aligned} \tag{11}$$

Both the condition for δ and the fact h only takes values in $\{\operatorname{Re} z \geq 0\}$ were needed for the second inequality. Moreover, if $\operatorname{Re} h(\mathrm{e}^{i\theta}) = 0$, then $|F_\delta(\mathrm{e}^{i\theta})| = \sqrt{1 + \delta^2 (\operatorname{Im} h(z))^2} \geq 1$, whence $\|F_\delta\|_{H^\infty(\mathbb{D})} \geq 1$. Therefore,

$$\begin{aligned} \left|1 - |\psi_\delta(\mathrm{e}^{i\theta})|\right| &= 1 - \frac{|F_\delta(\mathrm{e}^{i\theta})|}{\|F_\delta\|_{H^\infty(\mathbb{D})}}(1 - \delta^2) \\ &= \frac{\|F_\delta\|_{H^\infty(\mathbb{D})} - (1 - \delta^2)\sqrt{1 + \delta^2 (\operatorname{Im} h(\mathrm{e}^{i\theta}))^2}}{\|F_\delta\|_{H^\infty(\mathbb{D})}} \\ &\leq \|F_\delta\|_{H^\infty(\mathbb{D})} - (1 - \delta^2) \\ &\overset{(11)}{\leq} \delta^2 \left(\frac{\|\operatorname{Im} h(z)\|^2_{L^\infty(\partial\mathbb{D})}}{2} + 1 \right), \end{aligned}$$

which converges quadratically to zero as $\delta \to 0$. The second assertion of (c) therefore holds true with $C = \frac{\|\operatorname{Im} h(z)\|^2_{L^\infty(\partial\mathbb{D})}}{2} + 1$. To summarize, we have proved that the argument of $\psi_\delta(\mathrm{e}^{i\theta})$ converges linearly as $\delta \to 0$, while its modulus converges quadratically. This means that $\psi_\delta(\mathrm{e}^{i\theta}) \to \mathrm{e}^{i\theta}$ tangentially. □

We are going to apply Lemma 12 to a certain function q, which we construct in the following lemma.

Lemma 13. *There is a Lipschitz continuous function $q : \partial\mathbb{D} \to [0, \infty)$ such that*

$$h(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\mathrm{e}^{i\theta} + z}{\mathrm{e}^{i\theta} - z} q(\mathrm{e}^{i\theta})\, \mathrm{d}\theta$$

satisfies $\operatorname{Im} h(\mathrm{e}^{i\lambda_j}) < 0$, $\operatorname{Im} h(\mathrm{e}^{i\rho_j}) > 0$ and $\operatorname{Re} h(\mathrm{e}^{i\theta}) = 0$ for $\mathrm{e}^{i\theta}$ in some neighborhood of the points $\mathrm{e}^{i\lambda_1}, \ldots, \mathrm{e}^{i\lambda_N}$ and $\mathrm{e}^{i\rho_1}, \ldots, \mathrm{e}^{i\rho_N}$.

Proof. We begin with some simple estimates. First of all, recall from basic theory of Hardy spaces that for Lipschitz continuous q

$$\operatorname{Im} h(\mathrm{e}^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} q(\mathrm{e}^{i(\theta - t)}) \cot(t/2)\, \mathrm{d}t,$$

where the integral exists as a principal value integral (see, e.g., [9, Chapter 6]). Now assume that $q : \partial\mathbb{D} \to [0, \infty)$ is some Lipschitz continuous function with $0 \leq q \leq 1$, $\operatorname{supp} q \subset \mathrm{e}^{i[0,\sigma]}$ for some $0 < \sigma \leq \pi$, and $q(\mathrm{e}^{i\theta}) = 1$ for $\theta \in [\epsilon, \sigma - \epsilon]$

for some small $\epsilon > 0$. Then we have the estimate

$$\begin{aligned}\operatorname{Im} h(e^{i0}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} q(e^{-it})\cot(t/2)\,dt = \frac{1}{2\pi}\int_{-\sigma}^{0} q(e^{-it})\cot(t/2)\,dt \\ &\le \frac{1}{2\pi}\int_{-\sigma+\epsilon}^{-\epsilon}\cot(t/2)\,dt = -\frac{1}{\pi}\ln\left(\frac{\sin((\sigma-\epsilon)/2)}{\sin(\epsilon/2)}\right).\end{aligned} \tag{12}$$

Notice that we used $\sigma \le \pi$ for the inequality so that $\cot(t/2) \le 0$ for $t \in [-\sigma, 0]$. Similarly,

$$\operatorname{Im} h(e^{i\sigma}) \ge \frac{1}{\pi}\ln\left(\frac{\sin((\sigma-\epsilon)/2)}{\sin(\epsilon/2)}\right). \tag{13}$$

Moreover, if q is any Lipschitz continuous function on $\partial\mathbb{D}$ with $0 \le q \le 1$ and $q = 0$ on $e^{i[\theta-\eta,\theta+\eta]}$ for some $\eta > 0$ and $e^{i\theta} \in \partial\mathbb{D}$, then

$$\begin{aligned}|\operatorname{Im} h(e^{i\theta})| &= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} q(e^{i(\theta-t)})\cot(t/2)\,dt\right| \\ &\le \frac{1}{\pi}\int_{\eta}^{\pi}\cot(t/2)\,dt = -\frac{2}{\pi}\ln(\sin(\eta/2)).\end{aligned} \tag{14}$$

We are now going to construct a function q that satisfies all of the assertions of the lemma. Without loss of generality we can assume that $|K_j| \le \pi$ for all j. If this is not true, we can apply a Möbius transformation. Let $d_0 = \min_{j\ne l}\operatorname{dist}(K_j, K_l)$ and $M = -\frac{2}{\pi}\ln(\sin(d_0/2))$. Let $\epsilon > 0$ be so small that for all $j \in \{1, \dots, N\}$

$$\frac{1}{\pi}\ln\left(\frac{\sin((|K_j|-\epsilon)/2)}{\sin(\epsilon/2)}\right) \ge NM. \tag{15}$$

For each j let q_j be a Lipschitz continuous function on $\partial\mathbb{D}$ such that $\operatorname{supp} q_j \subset K_j = e^{i[\lambda_j,\rho_j]}$, $0 \le q_j \le 1$ and $q_j = 1$ on $e^{i[\lambda_j+\epsilon,\rho_j-\epsilon]}$. Then $q = \sum_{j=1}^{N} q_j$ satisfies all of the assertions of the lemma. Indeed, let

$$h_j(z) = \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}q_j(e^{i\theta})\,d\theta.$$

For $l \ne j$ we have $q_l = 0$ on $e^{i[\lambda_j-d_0,\lambda_j+d_0]}$ and $e^{i[\rho_j-d_0,\rho_j+d_0]}$. From (14) it follows that $|\operatorname{Im} h_j(e^{i\lambda_j})| \le M$ and $|\operatorname{Im} h_j(e^{i\lambda_j})| \le M$ for $l \ne j$, so

$$\left|\sum_{l\ne j}\operatorname{Im} h_l(e^{i\lambda_j})\right| \le (N-1)M \qquad \text{and} \qquad \left|\sum_{l\ne j}\operatorname{Im} h_l(e^{i\rho_j})\right| \le (N-1)M. \tag{16}$$

Further, by (12), (13) and (15) we have

$$\operatorname{Im} h_j(e^{i\lambda_j}) \le -NM \qquad \text{and} \qquad \operatorname{Im} h_j(e^{i\rho_j}) \ge NM. \tag{17}$$

Now (16) and (17) together give

$$\operatorname{Im} h(e^{i\lambda_j}) \le -M \qquad \text{and} \qquad \operatorname{Im} h(e^{i\rho_j}) \ge M.$$

Concerning the statement about the real part of h, we use that we have some freedom left in the construction. We can choose the q_j such that $q_j = 0$ for all j in some small neighborhood of the points $e^{i\lambda_1}, \dots, e^{i\lambda_N}$ and $e^{i\rho_1}, \dots, e^{i\rho_N}$. The statement then follows from the fact that $\operatorname{Re} h(e^{i\theta}) = q(e^{i\theta})$ for all $e^{i\theta} \in \partial\mathbb{D}$. $\square$

The following lemma seems quite obvious to us, but we are not aware of any reference. We therefore prove it for the convenience of the reader. Recall that for $f \in L^\infty(\partial\mathbb{D})$ the essential range of f on a measurable set $I \subset \partial\mathbb{D}$ is

$$\operatorname{ess\,ran}(f, I) = \left\{ z \in \mathbb{C} : \begin{array}{l} f^{-1}(B_\epsilon(z)) \cap I \text{ has positive Lebesgue} \\ \text{measure for all } \epsilon > 0 \end{array} \right\}.$$

Here, $B_\epsilon(z) = \{w \in \mathbb{C} : |z - w| < \epsilon\}$ denotes a ball in $\mathbb{C}$. When $\mathrm{e}^{i[\theta_1,\theta_2]}$ is an interval on $\partial\mathbb{D}$, we also write $\operatorname{ess\,ran}(f, [\theta_1, \theta_2])$ instead of $\operatorname{ess\,ran}(f, \mathrm{e}^{i[\theta_1,\theta_2]})$ for simplicity of notation.

Lemma 14. *Let $f \in H^\infty(\mathbb{D})$. Then for any $\epsilon > 0$ there is $\delta > 0$ such that if $z \in \mathbb{D}$ and $\mathrm{e}^{i\theta} \in \partial\mathbb{D}$ with $|z - \mathrm{e}^{i\theta}| \leq \delta$, then*

$$f(z) \in \operatorname{conv}(\operatorname{ess\,ran}(f, [\theta - \epsilon, \theta + \epsilon])) + B_\epsilon(0).$$

Proof. Fix some $\epsilon > 0$. Let $\delta > 0$ such that $\arg \mathrm{e}^{iB_\delta(0)} \subset [-\frac{\epsilon}{2}, \frac{\epsilon}{2}]$ and such that for all $r \in (1 - \delta, 1)$

$$\left| 1 - \frac{1}{\frac{1}{2\pi} \int_{[-\epsilon/2,\epsilon/2]} P_r(t)\,\mathrm{d}t} \right| \leq \frac{\epsilon}{2\|f\|_{H^\infty\mathbb{D})}} \tag{18}$$

and

$$\frac{1}{2\pi} \int_{[-\pi,\pi]\setminus[-\epsilon/2,\epsilon/2]} P_r(t)\,\mathrm{d}t \leq \frac{\epsilon}{2\|f\|_{H^\infty\mathbb{D})}}. \tag{19}$$

This is possible, because the Poisson kernel is an approximate identity.

Now let $z \in \mathbb{D}$ and $\mathrm{e}^{i\theta} \in \partial\mathbb{D}$ with $|z - \mathrm{e}^{i\theta}| \leq \delta$ and write $z = r\mathrm{e}^{i\tau}$ with $r \geq 0$ and $\tau \in \mathbb{R}$. Notice that $r \in (1 - \delta, 1)$. Then

$$f(z) = \frac{1}{2\pi} \left(\int_{[-\epsilon/2,\epsilon/2]} + \int_{[-\pi,\pi]\setminus[-\epsilon/2,\epsilon/2]} f(\mathrm{e}^{i(\tau-t)}) P_r(t)\,\mathrm{d}t \right) =: I_1 + I_2. \tag{20}$$

For the second integral it follows from (19) that $|I_2| \leq \frac{\epsilon}{2}$. For the first integral we have

$$\begin{aligned} I_1 = &\left(\frac{1}{\frac{1}{2\pi} \int_{[-\epsilon/2,\epsilon/2]} P_r(t)\,\mathrm{d}t} \right) \frac{1}{2\pi} \int_{[-\epsilon/2,\epsilon/2]} f(\mathrm{e}^{i(\tau-t)}) P_r(t)\,\mathrm{d}t \\ &+ \left(1 - \frac{1}{\frac{1}{2\pi} \int_{[-\epsilon/2,\epsilon/2]} P_r(t)\,\mathrm{d}t} \right) \frac{1}{2\pi} \int_{[-\epsilon/2,\epsilon/2]} f(\mathrm{e}^{i(\tau-t)}) P_r(t)\,\mathrm{d}t \\ =: &\, I_1' + I_1''. \end{aligned} \tag{21}$$

From (18) we get $|I_1''| \leq \frac{\epsilon}{2}$. Furthermore, the first term is a (weighted) average of f over $\mathrm{e}^{i[\tau-\epsilon/2,\tau+\epsilon/2]}$, so

$$I_1' \in \operatorname{conv}(\operatorname{ess\,ran}(f, [\tau - \epsilon/2, \tau + \epsilon/2])).$$

From $\arg \mathrm{e}^{iB_\delta(0)} \subset [-\frac{\epsilon}{2}, \frac{\epsilon}{2}]$ it follows that $\tau = \arg z \in [\theta - \frac{\epsilon}{2}, \theta + \frac{\epsilon}{2}]$, whence $[\tau - \epsilon/2, \tau + \epsilon/2] \subset [\theta - \epsilon, \theta + \epsilon]$. Therefore,

$$I_1' \in \operatorname{conv}(\operatorname{ess\,ran}(f, [\theta - \epsilon, \theta + \epsilon])). \tag{22}$$

Now (20)–(22) together with the estimates for $|I_1''|$ and $|I_2|$ yield

$$f(z) = I_1' + I_1'' + I_2 \in \operatorname{conv}(\operatorname{ess\,ran}(f, [\theta - \epsilon, \theta + \epsilon])) + B_\epsilon(0). \qquad \square$$

A big step towards the proof of Theorem 11 is

Lemma 15. *Let h be a function with the properties from Lemma 13. For $\delta > 0$ let ψ_δ be constructed from h as in Lemma 12. Let f^* be the solution of* (H-OPT$_\infty$) *and $\tau^* = \|f^* - \phi\|_{L^\infty(K)}$. Then*

$$\limsup_{\delta\to 0} \|f^* \circ \psi_\delta - \phi\|_{L^\infty(K)} \leq \tau^*.$$

Proof. Fix $\epsilon > 0$. Our proof consists of three steps. We first show that there is $\eta > 0$ such that for $\delta > 0$ small enough it holds that

$$|f^* \circ \psi_\delta - \phi| \leq \tau^* + \epsilon \tag{23}$$

on $\mathrm{e}^{i[\lambda_j,\lambda_j+\eta]}$ for all j. For this we need the tangential convergence property from Lemma 12. Similarly, we conclude that (23) holds on $\mathrm{e}^{i[\rho_j-\widetilde{\eta},\rho_j]}$ for small $\delta > 0$ and some $\widetilde{\eta} > 0$. Finally, we use Lemma 14 to show that (23) holds on $\mathrm{e}^{i[\lambda_j+\eta,\rho_j-\widetilde{\eta}]}$ for small $\delta > 0$.

Step 1: We show that there is $\eta > 0$ such that for $\delta > 0$ small enough it holds that $|f^* \circ \psi_\delta - \phi| \leq \tau^* + \epsilon$ on $\mathrm{e}^{i[\lambda_j,\lambda_j+\eta]}$ for all j. Write $h = u + iv$. Let $\delta_0 > 0$ be so small that

$$|t| \leq 2\delta_0\|v\|_{L^\infty(\partial\mathbb{D})} \quad \Rightarrow \quad |\phi(\mathrm{e}^{i(\theta+t)}) - \phi(\mathrm{e}^{i\theta})| \leq \frac{\epsilon}{2} \text{ if } \mathrm{e}^{i\theta},\ \mathrm{e}^{i(\theta+t)} \in K. \tag{24}$$

By the properties of h from Lemma 13 there is $\eta > 0$ such that for all $\theta \in \bigcup_{j=1}^N [\lambda_j, \lambda_j + \eta]$ we have $u(\mathrm{e}^{i\theta}) = 0$ and $v(\mathrm{e}^{i\theta}) \leq m < 0$ for some m. Since v is bounded away from zero, there is a constant $C_1 > 0$ such that for $0 < \delta \leq \delta_0$ and all $\theta \in \bigcup_{j=1}^N [\lambda_j, \lambda_j + \eta]$

$$C_1\delta \leq \arctan(-\delta v(\mathrm{e}^{i\theta})) \leq \|v\|_{L^\infty(\partial\mathbb{D})}\delta. \tag{25}$$

Now let $\theta \in [\lambda_j, \lambda_j + \eta]$ for some j and $0 < \delta \leq \delta_0$. Write $\psi_\delta(\mathrm{e}^{i\theta}) = r\mathrm{e}^{i\tau}$ with $r \geq 0$ and real τ. Then

$$\begin{aligned}|(f^* \circ \psi_\delta)&(\mathrm{e}^{i\theta}) - \phi(\mathrm{e}^{i\theta})| \\ &= |f^*(r\mathrm{e}^{i\tau}) - \phi(\mathrm{e}^{i\theta})| = \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} (f^*(\mathrm{e}^{i(\tau-t)}) - \phi(\mathrm{e}^{i\theta}))P_r(t)\,\mathrm{d}t\right| \\ &\leq \frac{1}{2\pi}\left(\int_{[-C_1\delta,C_1\delta]} + \int_{[-\pi,\pi]\setminus[-C_1\delta,C_1\delta]} |f^*(\mathrm{e}^{i(\tau-t)}) - \phi(\mathrm{e}^{i\theta})|P_r(t)\,\mathrm{d}t\right).\end{aligned} \tag{26}$$

We estimate the first integral. Let $|t| \leq \delta C_1$. From Lemma 12(c) we have $\tau = \theta + \arctan(-\delta v(\mathrm{e}^{i\theta}))$ (mod 2π), or

$$(\tau - t) - \theta = \arctan(-\delta v(\mathrm{e}^{i\theta})) - t \pmod{2\pi}. \tag{27}$$

Further,

$$|\arctan(-\delta v(\mathrm{e}^{i\theta})) - t| \overset{(25)}{\leq} (\|v\|_{L^\infty(\partial\mathbb{D})} + C_1)\delta \leq 2\delta_0\|v\|_{L^\infty(\partial\mathbb{D})},$$

so (24) implies $|\phi(\mathrm{e}^{i(\tau-t)}) - \phi(\mathrm{e}^{i\theta})| \leq \frac{\epsilon}{2}$. Then

$$\begin{aligned}|f^*(\mathrm{e}^{i(\tau-t)}) - \phi(\mathrm{e}^{i\theta})| &\leq |f^*(\mathrm{e}^{i(\tau-t)}) - \phi(\mathrm{e}^{i(\tau-t)})| + |\phi(\mathrm{e}^{i(\tau-t)}) - \phi(\mathrm{e}^{i\theta})| \\ &\leq |f^*(\mathrm{e}^{i(\tau-t)}) - \phi(\mathrm{e}^{i(\tau-t)})| + \frac{\epsilon}{2}.\end{aligned} \tag{28}$$

We now show that $\mathrm{e}^{i(\tau-t)} \in K_j = \mathrm{e}^{i[\lambda_j,\rho_j]}$ for δ small enough. From (27) we have $\tau - t = \theta + \arctan(-\delta v(\mathrm{e}^{i\theta})) - t \pmod{2\pi}$. Now let $\delta_1 > 0$ be so small that $\eta + (C_1 + \|v\|_{L^\infty(\partial\mathbb{D})})\delta_1 \leq |K_j|$ for all j. Notice that the choice of η made it necessary that $\eta < |K_j|$ for all j. Assume further that $0 < \delta \leq \delta_1$. Then by choice of θ, (25) and choice of t

$$\begin{aligned}\theta + \arctan(-\delta v(\mathrm{e}^{i\theta})) - t &\in [\lambda_j, \lambda_j + \eta] + [C_1\delta, \|v\|_{L^\infty(\partial\mathbb{D})}\delta] + [-C_1\delta, C_1\delta] \\ &= [\lambda_j, \lambda_j + \eta + (C_1 + \|v\|_{L^\infty(\partial\mathbb{D})})\delta] \\ &\subset [\lambda_j, \rho_j].\end{aligned}$$

It follows that $\mathrm{e}^{i(\tau-t)} \in K_j$. By assumption, $\|f^* - \phi\|_{L^\infty(K)} \leq \tau^*$, so

$$|f^*(\mathrm{e}^{i(\tau-t)}) - \phi(\mathrm{e}^{i(\tau-t)})| \leq \tau^*. \tag{29}$$

(28) and (29) together give

$$|f^*(\mathrm{e}^{i(\tau-t)}) - \phi(\mathrm{e}^{i\theta})| \leq \tau^* + \frac{\epsilon}{2},$$

and therefore the first integral in (26) can be estimated by

$$\frac{1}{2\pi}\int_{[-C_1\delta, C_1\delta]} |f^*(\mathrm{e}^{i(\tau-t)}) - \phi(\mathrm{e}^{i\theta})| P_r(t)\,\mathrm{d}t \leq \tau^* + \epsilon/2. \tag{30}$$

We estimate the second integral in (26) by

$$\begin{aligned}&\frac{1}{2\pi}\int_{[-\pi,\pi]\setminus[-C_1\delta, C_1\delta]} |f^*(\mathrm{e}^{i(\tau-t)}) - \phi(\mathrm{e}^{i\theta})| P_r(t)\,\mathrm{d}t \\ &\qquad \leq \left(\int_{[-\pi,\pi]\setminus[-C_1\delta, C_1\delta]} P_r(t)\,\mathrm{d}t\right)\left(\frac{1}{\pi}\|g\|_{L^\infty(\partial\mathbb{D})}\right).\end{aligned} \tag{31}$$

A straightforward calculation shows that

$$\begin{aligned}\int_{[-\pi,\pi]\setminus[-C_1\delta, C_1\delta]} P_r(t)\,\mathrm{d}t &= 2\pi - 4\arctan\left(\frac{1+r}{1-r}\tan\left(\frac{C_1\delta}{2}\right)\right) \\ &\leq 2\pi - 4\arctan\left(\frac{1}{C_2\delta^2}\tan\left(\frac{C_1\delta}{2}\right)\right),\end{aligned}$$

where we recall that $r = |\psi_\delta(\mathrm{e}^{i\theta})|$ and C_2 is the constant from Lemma 12(c). The last expression converges to zero as $\delta \to 0$. We want to emphasize that this is the point where tangential convergence is needed: In order for the expression inside of the arctan to converge to infinity, it is necessary that $1-r = 1-|\psi_\delta(\mathrm{e}^{i\theta})|$ converges to zero faster than linearly in δ. We conclude that there is $\delta_2 > 0$ such that $0 < \delta \leq \delta_2$ implies that the expression on the right hand side of (31) is smaller than $\frac{\epsilon}{2}$. Combining this with the estimates (26) and (30) we obtain that if $0 < \delta \leq \min\{\delta_0, \delta_1, \delta_2\}$, then for all $\theta \in \bigcup_{j=1}^N [\lambda_j, \lambda_j + \eta]$

$$|(f^* \circ \psi_\delta)(\mathrm{e}^{i\theta}) - \phi(\mathrm{e}^{i\theta})| \leq \tau^* + \epsilon. \tag{32}$$

Step 2: Similarly, one can show that there is $\delta_3 > 0$ so that this inequality holds for $0 < \delta \leq \delta_3$ and all $\theta \in \bigcup_{j=1}^N [\rho_j - \widetilde{\eta}, \rho_j]$ with some $\widetilde{\eta} > 0$.

Step 3: It remains to show that for small enough δ the inequality holds for $\theta \in \bigcup_{j=1}^{N}[\lambda_j+\eta, \rho_j-\widetilde{\eta}]$. This is an easy consequence of Lemma 14. By uniform continuity there is $\epsilon_1 > 0$ such that $\epsilon_1 \leq \frac{\epsilon}{2}$, $\epsilon_1 \leq \max\{\eta, \widetilde{\eta}\}$ and such that

$$|t| \leq \epsilon_1 \quad \Rightarrow \quad |\phi(\mathrm{e}^{i(\theta+t)}) - \phi(\mathrm{e}^{i\theta})| \leq \frac{\epsilon}{2} \text{ if } \mathrm{e}^{i\theta},\ \mathrm{e}^{i(\theta+t)} \in K. \tag{33}$$

By Lemma 14 there is $\epsilon_2 > 0$ such that

$$|\mathrm{e}^{i\theta} - z| < \epsilon_2 \quad \Rightarrow \quad f^*(z) \in \operatorname{conv}(\operatorname{ess\,ran}(f^*, [\theta-\epsilon_1, \theta+\epsilon_1])) + B_{\epsilon_1}(0). \tag{34}$$

Finally, by Lemma 12(b) there is $\delta_4 > 0$ such that for all $0 < \delta \leq \delta_4$ we have $\max_{z\in\mathbb{D}} |\psi_\delta(z) - z| \leq \epsilon_2$. Now let $\theta \in [\lambda_j+\eta, \rho_j-\widetilde{\eta}]$ for some j. Then for $0 < \delta \leq \delta_4$ we have $|\psi_\delta(\mathrm{e}^{i\theta}) - \mathrm{e}^{i\theta}| \leq \epsilon_2$, so

$$\begin{aligned} f^*(\psi_\delta(\mathrm{e}^{i\theta})) &\overset{(34)}{\in} \operatorname{conv}(\operatorname{ess\,ran}(f^*, [\theta-\epsilon_1, \theta+\epsilon_1])) + B_{\epsilon_1}(0) \\ &\subset \operatorname{conv}\left(\bigcup_{|t|\leq\epsilon_1} B_{\tau^*}(\phi(\mathrm{e}^{i(\theta+t)}))\right) + B_{\epsilon/2}(0) \\ &\qquad\qquad\qquad\qquad \text{since } \epsilon_1 \leq \max\{\eta, \widetilde{\eta}\} \\ &\overset{(33)}{\subset} B_{\tau^*+\epsilon/2}(\phi(\mathrm{e}^{i\theta})) + B_{\epsilon/2}(0) = B_{\tau^*+\epsilon}(\phi(\mathrm{e}^{i\theta})). \end{aligned}$$

This is just equation (32).

Summing up, we have shown that if $0 < \delta \leq \min\{\delta_0, \ldots, \delta_4\}$, then $\|f^* \circ \psi_\delta - \phi\|_{L^\infty(K)} \leq \tau^* + \epsilon$. Because $\epsilon > 0$ was arbitrary, this proves the lemma. □

Using the work we have done so far it is not hard any more to prove Theorem 11.

Proof of Theorem 11. Let f^* be the solution of (H-OPT$_\infty$) and $\tau^* = \|f^* - \phi\|_{L^\infty(K)}$. Fix $\epsilon > 0$. Let $\epsilon_1 > 0$ such that with

$$\eta = \left(1 + \frac{\epsilon_1}{\min_{\mathrm{e}^{i\theta}\in\partial\mathbb{D}} g(\mathrm{e}^{i\theta})}\right)^{-1}$$

we have $(1-\eta)\|f^*\|_{H^\infty(\mathbb{D})} < \epsilon/2$. Because ψ_δ converges uniformly to the identity as $\delta \to 0$ and since g is uniformly continuous, it follows from Lemma 14 as in the proof of Lemma 15 that for $\delta > 0$ small enough

$$|(f^* \circ \psi_\delta)(\mathrm{e}^{i\theta})| \leq g(\mathrm{e}^{i\theta}) + \epsilon_1.$$

By Lemma 15 we have for $\delta > 0$ small enough

$$\|f^* \circ \psi_\delta - \phi\|_{L^\infty(K)} \leq \tau^* + \frac{\epsilon}{2}.$$

From Lemma 12(a) it follows that $f^* \circ \psi_\delta \in \mathcal{A}(\mathbb{D})$. Moreover, for $\mathrm{e}^{i\theta} \in \partial\mathbb{D}$

$$|(f^* \circ \psi_\delta)(\mathrm{e}^{i\theta})| \leq g(\mathrm{e}^{i\theta}) + \epsilon_1 \leq \left(1 + \frac{\epsilon_1}{\min_{\mathrm{e}^{i\tau}\in\partial\mathbb{D}} g(\mathrm{e}^{i\tau})}\right) g(\mathrm{e}^{i\theta}),$$

whence $|\eta(f^* \circ \psi_\delta)| \leq g$. This means that $\eta(f^* \circ \psi_\delta)$ is feasible for ($\mathcal{A}$-OPT$_\infty$). Then

$$\begin{aligned}\tau^* &= \min_{f \in H^\infty(\mathbb{D}), |f| \leq g} \|f - \phi\|_{L^\infty(K)} \leq \inf_{f \in \mathcal{A}(\mathbb{D}), |f| \leq g} \|f - \phi\|_{L^\infty(K)} \\ &\leq \|\eta(f^* \circ \psi_\delta) - \phi\|_{L^\infty(K)} \leq \|f^* \circ \psi_\delta - \phi\|_{L^\infty(K)} + (1 - \eta)\|f^*\|_{H^\infty(\mathbb{D})} \\ &\leq \tau^* + \frac{\epsilon}{2} + \frac{\epsilon}{2} = \tau^* + \epsilon.\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, $\min_{f \in H^\infty(\mathbb{D}), |f| \leq g} \|f - \phi\|_{L^\infty(K)} = \inf_{f \in \mathcal{A}(\mathbb{D}), |f| \leq g} \|f - \phi\|_{L^\infty(K)}$. This is what we had to prove. □

It is easy to obtain a statement that is slightly stronger than Theorem 11.

Corollary 16. *Assume that K is the union of finitely many closed intervals. Let f^* be the solution of* (H-OPT$_\infty$). *Then there is a sequence $(f_n) \subset \mathcal{A}(\mathbb{D})$ with $|f_n| \leq g$ on $\partial\mathbb{D}$ such that $\|f_n - \phi\|_{L^\infty(K)} \to \|f^* - \phi\|_{L^\infty(K)}$ and such that (f_n) converges to f^* weakly* in $L^\infty(\partial\mathbb{D})$.*

Furthermore, we may even arrange it for the f_n to be polynomials. If f^* is real symmetric, then we can arrange it for the f_n to be real symmetric.

Proof. By Theorem 11 there is a sequence $(f_n) \subset \mathcal{A}(\mathbb{D})$ with $|f_n| \leq g$ and $\|f_n - \phi\|_{L^\infty(K)} \to \|f^* - \phi\|_{L^\infty(K)}$. Because (f_n) is bounded in $L^\infty(\partial\mathbb{D})$, there is a weakly* convergent subsequence (f_{n_l}). Denote its limit by $\widetilde{f}$. Then

$$\|\widetilde{f} - \phi\|_{L^\infty(K)} \leq \liminf_{l \to \infty} \|f_{n_l} - \phi\|_{L^\infty(K)} = \|f^* - \phi\|_{L^\infty(K)}.$$

Because the set $\{f \in H^\infty(\mathbb{D}) : |f| \leq g \text{ on } \partial\mathbb{D}\}$ is (sequentially) weakly* closed in $L^\infty(\partial\mathbb{D})$, we have $\widetilde{f} \in H^\infty(\mathbb{D})$ and $|\widetilde{f}| \leq g$. Therefore, $\widetilde{f}$ is also a solution of (H-OPT$_\infty$), and unique solvability implies $\widetilde{f} = f^*$. But then it follows that the whole sequence (f_n) converges weakly* to f^*: If there were infinitely many f_n outside of an arbitrary (weak* $L^\infty(\partial\mathbb{D})$-)neighborhood of f^*, we could use the preceding arguments to find a subsequence of these infinitely many f_n that converges to f^*, which is a contradiction.

That we can arrange it for the f_n to be polynomials and the statement about real symmetry can be shown as in the proof of Theorem 10. □

5 Concluding Remarks

Having established existence and uniqueness for (H-OPT$_p$) (Section 2), the next question is how to compute the solution. Because the solution of (H-OPT$_p$) can be approximated by polynomials (Sections 3 and 4), it seems suggestive to discretize the problem by replacing the space E in (OPT$_p$) by a finite dimensional space of polynomials. Moreover, one can replace the norm in the objective function by a quadrature approximation and check the constraint $|f| \leq g$ on a grid. In a follow-up paper [17], we will show that this approach is indeed reasonable: The solution of the thusly obtained discrete problem converges to the solution of (H-OPT$_p$) as the discretization becomes better. Moreover, the discrete problem can be cast in the form of a second-order cone program, which can be solved efficiently with modern numerical methods.

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