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Constrained Hardy Space Approximation II: Numerics*

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Abstract

In a previous paper [*Constrained Hardy Space Approximation*, preprint available at `http://www.mathematik.uni-karlsruhe.de/iwrmm/seite/preprints/`] we considered the problem of minimizing the distance $\|f - \phi\|_{L^p(K)}$, where K is a subset of the complex unit circle $\partial\mathbb{D}$ and $\phi \in C(K)$, subject to the constraint that f lies in the Hardy space $H^\infty(\mathbb{D})$ and $|f| \leq g$ for some positive function g. This problem occurs in the context of filter design for causal LTI systems. In this paper we devise a general discretization scheme for this problem and show convergence as the discretization becomes better. We derive several concrete discretizations and cast them in the form of second-order cone programs, which can be solved efficiently. We demonstrate this practically with a problem from the design of dispersion compensating mirrors for the generation of ultrashort laser pulses. A MATLAB implementation of our method is available at `http://www.mathematik.uni-karlsruhe.de/grk1294/~schneck/`.

1 Introduction and Motivation

Let $\mathbb{D} = \{|z| < 1\}$ be the complex unit disk and $\partial\mathbb{D} = \{|z| = 1\}$ the complex unit circle. By $H^\infty(\mathbb{D})$ we denote the Hardy space $H^\infty(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C} : f \text{ analytic and bounded}\}$, see, e.g., [6, 11]. Functions in $H^\infty(\mathbb{D})$ have boundary values on $\partial\mathbb{D}$, and so $H^\infty(\mathbb{D})$ can be identified with a subspace of $L^\infty(\partial\mathbb{D})$. By $\mathcal{A}(\mathbb{D}) = H^\infty(\mathbb{D}) \cap C(\partial\mathbb{D})$ we denote the disk algebra, i.e., the subspace of $H^\infty(\mathbb{D})$ of functions with continuous boundary values.

In this paper we are concerned with the numerical solution of the optimization problem

$$\begin{array}{ll} \text{minimize} & \|f - \phi\|_{L^p(K)} \\ \text{subject to} & f \in H^\infty(\mathbb{D}), \\ & |f| \leq g \quad \text{on } \partial\mathbb{D}, \end{array} \tag{H-OPT$_p$}$$

where $1 \leq p \leq \infty$. Here, $K \subset \partial\mathbb{D}$ is closed with positive measure, $g \in C(\partial\mathbb{D})$ with $g > 0$, and $\phi \in C(K)$ such that $|\phi| \leq g$ on K.

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1.1 Motivation

Problems of the form (H-OPT$_p$) occur in the context of the design of causal linear time-invariant (LTI) systems with a desired frequency response. An LTI system L is a convolution operator, $Lf(t) = (h * f)(t)$, $t \in \mathbb{R}$, where the function $h : \mathbb{R} \to \mathbb{R}$ is called the impulse response of the system. The system is called causal or realizable, if for all $t_0 \in \mathbb{R}$ it holds that $f(t) = 0$ for $t < t_0$ implies $Lf(t) = 0$ for $t < t_0$, or, equivalently, if $\operatorname{supp} h \subset [0, \infty)$. An alternative way to describe an LTI system is by its response to plane waves $\mathrm{e}^{i\omega_0 \cdot}$. Taking the Fourier transform

$$\mathcal{F}f(\omega) = \widehat{f}(\omega) = \int_{\mathbb{R}} f(t)\mathrm{e}^{-i\omega t}\,\mathrm{d}t$$

of the system, one obtains $\widehat{Lf}(\omega) = T(\omega)\widehat{f}(\omega)$, where $T = \widehat{h}$ is the frequency response of the system. Especially, $L\mathrm{e}^{i\omega_0 \cdot} = T(\omega_0)\mathrm{e}^{i\omega_0 \cdot}$, that is, a plane wave $\mathrm{e}^{i\omega_0 \cdot}$ is mapped to a plane wave multiplied by a (complex) scalar $T(\omega_0)$.

The space of frequency responses of causal LTI systems is rather restricted: Writing $h^\circ(t) = h(-t)$, we have $\widehat{T} = h^\circ$, or $\widehat{T^\circ} = h$. Therefore, $\operatorname{supp} \widehat{T} \subset (-\infty, 0]$, or $\operatorname{supp} \widehat{T^\circ} \subset [0, \infty)$. The connection to Hardy spaces is as follows: The space $H^p(\mathbb{C}^+)$, $1 \le p \le \infty$, is the Hardy space on the complex upper half-plane $\mathbb{C}^+ = \{\Im z > 0\}$,

$$H^p(\mathbb{C}^+) = \left\{ f : \mathbb{C}^+ \to \mathbb{C} : f \text{ analytic on } \mathbb{C}^+,\ \sup_{y>0} \|f(\cdot + iy)\|_{L^p(\mathbb{R})} < \infty \right\}.$$

Functions from $H^p(\mathbb{C}^+)$ have boundary values on $\mathbb{R}$, and so $H^p(\mathbb{C}^+)$ can be identified with a subspace of $L^p(\mathbb{R})$. The Paley-Wiener Theorem [13, Theorem VI.7.2] gives a characterization of this subspace in the case $p = 2$: $H^2(\mathbb{C}^+) = \{f \in L^2(\mathbb{R}) : \operatorname{supp} \widehat{f} \subset [0, \infty)\}$. Thus, if $T^\circ \in L^2(\mathbb{R})$, then $T^\circ \in H^2(\mathbb{C}^+)$.

Additionally, the gain of energy of an LTI system is limited due to practical restrictions, i.e., in a given scenario one can expect that there exists a bounded function $G : \mathbb{R} \to [0, \infty)$ such that any frequency response T that can be physically realized satisfies $|T| \le G$. For example, if the system that is to be designed is passive, then one can choose $G \equiv 1$. When T is bounded, especially $T^\circ \in H^\infty(\mathbb{C}^+)$.

In practice, LTI systems are only used in some frequency range $I \subset \mathbb{R}$. For a design problem, one then specifies a desired complex-valued frequency response $T_{\text{desired}} : I \to \mathbb{C}$ and tries to find a physically realizable system L such that the corresponding frequency response T_L is close to T_{desired} on I. In a situation where every frequency response T with $T^\circ \in H^\infty(\mathbb{C}^+)$ and $|T| \le G$ is actually physically realizable, and when one measures the distance between T_{desired} and T in the $L^p(I)$-norm, this is basically the problem (H-OPT$_p$), but posed in $H^p(\mathbb{C}^+)$ instead of $H^p(\mathbb{D})$. Using the Möbius transformation $w \mapsto \frac{iw+1}{-iw+1}$, which maps $\mathbb{C}^+$ conformally to $\mathbb{D}$, the problem can then be transported to the disk to obtain a problem of the form (H-OPT$_p$), see also, e.g., [10].

The particular example which led us to the study of (H-OPT$_p$) is the design of dispersion compensating mirrors (DCMs) for the generation of ultra-short laser pulses [15, 17]. We explain this in detail in the last section of this paper.

1.2 Related problems

The problem (H-OPT$_p$) can be seen as a generalization of the classical Nehari problem (see, e.g., [6, 24])

$$\begin{array}{ll} \text{minimize} & \|f-\phi\|_{L^\infty(\partial\mathbb{D})} \\ \text{subject to} & f \in H^\infty(\mathbb{D}), \end{array}$$

where $\phi \in L^\infty(\partial\mathbb{D})$. If $p = \infty$, $K = \partial\mathbb{D}$ and g is so large that the constraint $|f| \leq g$ is not active, then (H-OPT$_p$) is a Nehari problem. Several other generalizations of the Nehari problem that have special cases in common with (H-OPT$_p$) have been considered, see, e.g., Baratchart, Leblond et al. [3, 4, 5]) or Helton et al. [8, 9, 10, 12].

For the Nehari problem, there is an explicit representation of the solution [24], which can be turned into an algorithm. For the problems studied by Baratchart, Leblond et al. and Helton et al., this is not the case, but there are implicit representations of the solution or optimality conditions which can be used to obtain algorithms that converge to the solution (see the already cited references).

1.3 Previous and new results

We have dealt with the theory of (H-OPT$_p$) in [19]. Let us briefly recall the most important results which we will need in this paper.

Theorem 1 (Theorem 2 in [19])**.** *Let f^* be a solution of* (H-OPT$_p$) *and* $\tau^* = \|f^*-\phi\|_{L^p(K)} > 0$*. If $1 \leq p < \infty$, then for almost all* $\mathrm{e}^{i\theta} \in \partial\mathbb{D} \setminus K$

$$|f^*(\mathrm{e}^{i\theta})| = g(\mathrm{e}^{i\theta}).$$

If $p = \infty$, then, for almost all $\mathrm{e}^{i\theta} \in \partial\mathbb{D}$, $f^*(\mathrm{e}^{i\theta})$ is on the boundary of the set

$$S(\theta,\tau^*) = \{z \in \mathbb{C} : |z| \leq g(\mathrm{e}^{i\theta}),\ |z-\phi(\mathrm{e}^{i\theta})| \leq \tau^* \text{ if } \mathrm{e}^{i\theta} \in K\}.$$

Moreover, for $1 < p \leq \infty$, the solution of (H-OPT$_p$) *is unique. If $K \neq \partial\mathbb{D}$, then the solution of* (H-OPT$_p$) *is also unique for $p = 1$.*

Theorem 2 (Theorems 10, 11 and Corollary 16 in [19])**.** *Let f^* be a solution of* (H-OPT$_p$).

1. *If $1 \leq p < \infty$, then there is a sequence $(f_n) \subset \mathcal{A}(\mathbb{D})$ with $|f_n| \leq g$ on $\partial\mathbb{D}$ that converges to f^* in $L^p(\partial\mathbb{D})$.*
2. *If $p = \infty$ and additionally K is the union of finitely many closed intervals, then there is a sequence $(f_n) \subset \mathcal{A}(\mathbb{D})$ with $|f_n| \leq g$ on $\partial\mathbb{D}$ such that $\|f_n-\phi\|_{L^\infty(K)} \to \|f^*-\phi\|_{L^\infty(K)}$ and such that (f_n) converges to f^* weakly* in $L^\infty(\partial\mathbb{D})$.*

Furthermore, we may even arrange it for the f_n to be polynomials. If f^* is real symmetric, i.e., $f(\mathrm{e}^{i\omega}) = \overline{f(\mathrm{e}^{-i\omega})}$, $\mathrm{e}^{i\omega} \in \partial\mathbb{D}$, then we can arrange it for the f_n to be real symmetric.

Unlike for the problems in Section 1.2, we have no representation or characterization of the solution of (H-OPT$_p$) which could be turned into an algorithm. We therefore take the more straightforward approach of discretizing (H-OPT$_p$) directly. This method is actually quite efficient, because the discretization can be done in a way that results in discrete problems which can be handled well with existing algorithms. Our theoretical results from [19], Theorems 1 and 2, will then be essential to show convergence.

The organization of the rest of this paper is as follows. We start in Section 2 by stating some assumptions and fixing notation. In Sections 3 and 4 we devise a general discretization scheme for (H-OPT$_p$). Theorem 2 suggests that it is reasonable to replace the space $H^\infty(\mathbb{D})$ by a finite dimensional subspace of polynomials. This yields a semi-discrete problem, which we consider in Section 3. In Section 4 we then obtain a fully discrete problem by replacing the norm in the objective function by a quadrature approximation and checking the constraint $|f| \le g$ on a grid. We show that the minimum of the discrete problem converges to the minimum of (H-OPT$_p$), and that the solution of the discrete problem converges to the solution of (H-OPT$_p$), in the cases $p = 1$ and $p = \infty$ under the additional hypothesis on K from Theorems 1 and 2. In Section 5 we consider several concrete discretizations. In Section 6 we show how to recast these discrete problems as second-order cone programs, for which there already exist very efficient solvers [20, 21]. We finish in Section 7 by applying our results to a design problem for a special LTI system, a dispersion compensating mirror for the generation of ultra-short laser pulses.

2 Assumptions and notation

Notice that because the impulse response h of an LTI system is real-valued (see Section 1.1), its frequency response $T = \widehat{h}$ is real symmetric, i.e., $T(-\omega) = \overline{T(\omega)}$, $\omega \in \mathbb{R}$. Therefore, we assume throughout this paper that in the problem (H-OPT$_p$), K, g and ϕ are also symmetric, i.e., $K = \overline{K}$, $\phi(\mathrm{e}^{-i\theta}) = \overline{\phi(\mathrm{e}^{i\theta})}$, $\mathrm{e}^{i\theta} \in \partial\mathbb{D}$, and $g(\mathrm{e}^{-i\theta}) = g(\mathrm{e}^{i\theta})$, $\mathrm{e}^{i\theta} \in K$. Because our motivation is the design of LTI systems, we are only interested in real symmetric solutions of (H-OPT$_p$). Notice that there is always such a solution: If f^* is a solution of (H-OPT$_p$), then $(f^* + \overline{f^*(\bar{\cdot})})/2$ is a real symmetric solution. Optimization therefore takes place in the space

$$\begin{aligned}\mathcal{H}^\infty(\mathbb{D}) &= \left\{ f \in H^\infty(\mathbb{D}) : f(\mathrm{e}^{i\theta}) = \overline{f(\mathrm{e}^{-i\theta})},\ \mathrm{e}^{i\theta} \in \partial\mathbb{D} \right\} \\ &= \left\{ f \in L^\infty(\partial\mathbb{D}) : \begin{array}{l} \widehat{f}_k = 0 \text{ for integers } k < 0, \\ \widehat{f}_k \in \mathbb{R} \text{ for integers } k \ge 0 \end{array} \right\}\end{aligned}$$

instead of $H^\infty(\mathbb{D})$. Here, $\widehat{f}_k = \int_{-\pi}^{\pi} f(\theta)\mathrm{e}^{-ik\theta}\,\mathrm{d}\theta$, $k \in \mathbb{Z}$, are the Fourier coefficients of f.

Let $N \in \mathbb{N}$. For computations we are going to use the finite dimensional subspaces

$$\mathcal{H}^\infty_N(\mathbb{D}) = \left\{ f_\alpha : f_\alpha(\mathrm{e}^{i\theta}) = \sum_{k=0}^{N-1} \alpha_k \mathrm{e}^{ik\theta},\ \alpha = (\alpha_0, \dots, \alpha_{N-1})^\top \in \mathbb{R}^N \right\}$$

and

$$\mathcal{L}_N^\infty(\partial\mathbb{D}) = \left\{ f_\beta : f_\beta(\mathrm{e}^{i\theta}) = \sum_{k=-N}^{N-1} \beta_k \mathrm{e}^{ik\theta},\ \beta = (\beta_{-N}, \ldots, \beta_{N-1})^\top \in \mathbb{R}^{2N} \right\}.$$

By $\mathcal{X}$ we usually denote a grid on $\partial\mathbb{D}$, i.e., a set of finitely many points from $\partial\mathbb{D}$. Given two points $\mathrm{e}^{i\theta}$ and $\mathrm{e}^{i\tau}$ on $\partial\mathbb{D}$, we define the *distance between* $\mathrm{e}^{i\theta}$ *and* $\mathrm{e}^{i\tau}$ to be

$$\operatorname{dist}\left(\mathrm{e}^{i\theta}, \mathrm{e}^{i\tau}\right) = \min_{\mathrm{e}^{i(\theta+\mu)}=\mathrm{e}^{i\tau}} |\mu|.$$

Clearly, we always have $\operatorname{dist}\left(\mathrm{e}^{i\theta}, \mathrm{e}^{i\tau}\right) \le \pi$. The *fineness of the grid* $\mathcal{X}$, i.e., the maximal distance between two neighboring points, is

$$h_{\max}(\mathcal{X}) = \max_{\mathrm{e}^{i\theta}\in\mathcal{X}} \min_{\mathrm{e}^{i\tau}\in\mathcal{X}\setminus\{\mathrm{e}^{i\theta}\}} \operatorname{dist}\left(\mathrm{e}^{i\theta}, \mathrm{e}^{i\tau}\right).$$

3 Semi-discrete problem

In a first step we replace the space $H^\infty(\mathbb{D})$ in (H-OPT$_p$) by the discrete space $\mathcal{H}_N^\infty(\mathbb{D})$. We therefore consider the semi-discrete problem

$$\begin{array}{ll} \text{minimize} & \|f-\phi\|_{L^p(K)} \\ \text{subject to} & |f| \le g \quad \text{on } \partial\mathbb{D}, \\ & f \in \mathcal{H}_N^\infty(\mathbb{D}). \end{array} \tag{SDP$_p$}$$

We also write (SDP$_p(N)$) in order to denote the above problem with a specific N. For $1 < p < \infty$, (SDP$_p$) has a unique solution since the objective function is strictly convex and we are minimizing over a compact and convex set.

We have the following convergence result.

Theorem 3. *If $1 \le p < \infty$, then the minimum of* (SDP$_p(N)$) *converges to the minimum of* (H-OPT$_p$) *as $N \to \infty$, that is, if f_N^* is a solution of* (SDP$_p(N)$), *and f^* is a solution of* (H-OPT$_p$), *then*

$$\|f_N^* - \phi\|_{L^p(K)} \to \|f^* - \phi\|_{L^p(K)} \qquad \text{as } N \to \infty.$$

If K is the union of finitely many closed intervals, the same holds true for $p = \infty$.

Proof. Fix $\epsilon > 0$. By Theorem 2 there is a polynomial $\widetilde{f}$ with $|\widetilde{f}| \le g$ such that

$$\|\widetilde{f} - \phi\|_{L^p(K)} \le \min_{f\in H^\infty(\mathbb{D}),\ |f|\le g} \|f-\phi\|_{L^p(K)} + \epsilon.$$

Then for $N \ge \deg \widetilde{f} - 1$

$$\begin{aligned} \|f_N^* - \phi\|_{L^p(K)} &= \min_{f\in\mathcal{H}_N^\infty(\mathbb{D}),\ |f|\le g} \|f-\phi\|_{L^p(K)} \le \|\widetilde{f}-\phi\|_{L^p(K)} \\ &\le \min_{f\in H^\infty(\mathbb{D}),\ |f|\le g} \|f-\phi\|_{L^p(K)} + \epsilon = \|f^*-\phi\|_{L^p(K)} + \epsilon. \end{aligned}$$

On the other hand, since f_N^* is feasible for (H-OPT$_p$),

$$\|f^* - \phi\|_{L^p(K)} \le \|f_N^* - \phi\|_{L^p(K)}.$$

Together we have for $N \geq \deg \widetilde{f} - 1$

$$\|f^* - \phi\|_{L^p(K)} \leq \|f_N^* - \phi\|_{L^p(K)} \leq \|f^* - \phi\|_{L^p(K)} + \epsilon.$$

Therefore, $\|f_N^* - \phi\|_{L^p(K)} \to \|f^* - \phi\|_{L^p(K)}$ as $N \to \infty$. $\square$

Corollary 4. *Let f_N^* be a solution of $(\mathrm{SDP}_p(N))$, and let f^* be a solution of (H-OPT_p). If $1 < p < \infty$, then (f_N^*) converges to f^* strongly in $L^p(\partial\mathbb{D})$. If $p = \infty$ and K is the union of finitely many intervals, then (f_N^*) converges to f^* weakly* in $L^\infty(\partial\mathbb{D})$. If $p = 1$ and $K \neq \partial\mathbb{D}$, then (f_N^*) converges to f^* weakly in $L^1(\partial\mathbb{D})$ and strongly in $L^1(\partial\mathbb{D} \setminus K)$.*

Proof. The sequence (f_N^*) is bounded in $L^\infty(\partial\mathbb{D})$. If $p = \infty$, then there is a weakly* convergent subsequence $(f_{N_l}^*)$. If $1 \leq p < \infty$, then there is a subsequence $(f_{N_l}^*)$ which converges weakly in $L^p(\partial\mathbb{D})$: In the case $1 < p < \infty$ this is due to the fact that $(f_{N_l}^*)$ is especially bounded in $L^p(\partial\mathbb{D})$ and that the unit ball in reflexive spaces in weakly sequentially compact. In the case $p = 1$ we notice that, since (f_N^*) is especially bounded in $L^2(\partial\mathbb{D})$, we can extract a subsequence $(f_{N_l}^*)$ which converges weakly in $L^2(\partial\mathbb{D})$. But weak convergence in $L^2(\partial\mathbb{D})$ implies weak convergence $L^1(\partial\mathbb{D})$. Denote the limit in any case by $\widetilde{f}$.

Because the norm is sequentially lower semicontinuous with respect to the weak and weak* topologies,

$$\|\widetilde{f} - \phi\|_{L^p(K)} \leq \liminf_{l\to\infty} \|f_{N_l}^* - \phi\|_{L^p(K)} = \|f^* - \phi\|_{L^p(K)}.$$

Equality on the right hand side follows from Theorem 3. The set of functions that is feasible for (H-OPT_p), $\{f \in H^\infty(\mathbb{D}) : |f| \leq g \text{ on } \partial\mathbb{D}\}$, is weakly closed in $L^p(\partial\mathbb{D})$ for $1 \leq p < \infty$, and (sequentially) weakly* closed in $L^\infty(\partial\mathbb{D})$. Therefore, the weak (or weak*) limit $\widetilde{f}$ is also feasible for (H-OPT_p), whence $\|\widetilde{f} - \phi\|_{L^p(K)} = \|f^* - \phi\|_{L^p(K)}$.

Uniqueness of the solution of (H-OPT_p) now implies $\widetilde{f} = f^*$. (In the case $p = 1$ we need $K \neq \partial\mathbb{D}$ for uniqueness.) But then it follows that the whole sequence (f_N^*) converges weakly (or weakly*) to f^*: If there were infinitely many f_N outside of an arbitrary (weak $L^p(\partial\mathbb{D})$- or weak* $L^\infty(\partial\mathbb{D})$-)neighborhood of f^*, we could use the preceding arguments to find a subsequence of these infinitely many f_N that converges to f^*, which is a contradiction.

It remains to show the statements about strong convergence in the cases $1 < p < \infty$ and $p = 1$. If $1 < p < \infty$, then weak convergence, $f_N^* - \phi \rightharpoonup f^* - \phi$ in $L^p(K)$, and convergence of the norm, $\|f_N^* - \phi\|_{L^p(K)} \to \|f^* - \phi\|_{L^p(K)}$, imply that $f_N^* - \phi \to f^* - \phi$ strongly in $L^p(K)$ (see [2, 6.6]), and therefore $f_N^* \to f^*$ strongly in $L^p(K)$. Further, by Theorem 1, $|f^*| = g$ a.e. on $\partial\mathbb{D} \setminus K$. Therefore,

$$\|g\|_{L^p(\partial\mathbb{D}\setminus K)} = \|f^*\|_{L^p(\partial\mathbb{D}\setminus K)} \leq \liminf_{l\to\infty} \|f_N^*\|_{L^p(\partial\mathbb{D}\setminus K)} \leq \|g\|_{L^p(\partial\mathbb{D}\setminus K)},$$

from which it follows that $\|f_N^*\|_{L^p(\partial\mathbb{D}\setminus K)} \to \|g\|_{L^p(\partial\mathbb{D}\setminus K)}$. As before, weak convergence and convergence of the norm imply that $f_N^* \to f^*$ strongly in $L^p(\partial\mathbb{D} \setminus K)$. Together, $f_N^* \to f^*$ strongly in $L^p(\partial\mathbb{D})$ for $1 < p < \infty$.

Finally, because $|f^*| = g$ a.e. on $\partial\mathbb{D} \setminus K$ by Theorem 1 and because $|f_N^*| \leq g$ for all N, weak convergence $f_N^* \rightharpoonup f^*$ in $L^1(\partial\mathbb{D} \setminus K)$ implies strong convergence $f_N^* \to f^*$ in $L^1(\partial\mathbb{D} \setminus K)$, see, e.g., [22, Theorem 1]. $\square$

4 Fully discrete problem

Unfortunately, we are not aware of any "nice" method to check the constraint $|f| \le g$ on the complete circle. For a complete discretization we only check the constraint on some grid $\mathcal{X} \subset \partial\mathbb{D}$. Moreover, it may not be possible to compute the objective function exactly. We therefore replace $\|f-\phi\|_{L^p(K)}$ by some quadrature approximation $T^p(f-\phi)$ and obtain the fully discrete problem

$$\begin{array}{ll} \text{minimize} & T^p(f-\phi) \\ \text{subject to} & |f(\mathrm{e}^{i\theta})| \le g(\mathrm{e}^{i\theta}), \quad \theta \in \mathcal{X}, \\ & f \in \mathcal{H}_N^\infty(\mathbb{D}). \end{array} \tag{FDP$_p$}$$

If we want to denote the above problem with, e.g., a specific grid $\mathcal{X}$, a specific approximation T^p, or a specific N, we write (FDP$_p(\mathcal{X})$), (FDP$_p(\mathcal{X}, T^p)$), (FDP$_p(\mathcal{X}, T^p, N)$) and so on. The set of feasible functions $\{f \in \mathcal{H}_N^\infty(\mathbb{D}) : |f(\mathrm{e}^{i\theta})| \le g(\mathrm{e}^{i\theta}),\ \theta \in \mathcal{X}\}$ is convex and closed in the finite dimensional space $\mathcal{H}_N^\infty(\mathbb{D})$. From Lemma 7 below it follows that if the grid $\mathcal{X}$ is fine enough, then the set of feasible functions is also bounded and therefore, because $\mathcal{H}_N^\infty(\mathbb{D})$ is finite dimensional, compact. Thus, (FDP$_p$) has a solution. If additionally the quadrature approximation is strictly convex, then the solution is unique.

We assume that we are given a sequence (T_n^p) of quadrature approximations that converges locally uniformly for functions $f \in \mathcal{H}_N^\infty(\mathbb{D})$, i.e.,

$$\sup_{f \in \mathcal{H}_N^\infty(\mathbb{D}),\ \|f\|_{L^\infty(\partial\mathbb{D})} \le 1} \left| T_n^p(f-\phi) - \|f-\phi\|_{L^p(K)} \right| \to 0 \qquad \text{as } n \to \infty. \tag{1}$$

We show in this section that as the grid $\mathcal{X}$ on which we check the constraint becomes finer and as the approximation T^p becomes better, the minimum of the fully discrete problem (FDP$_p$) converges to the minimum of the semi-discrete problem (SDP$_p$):

Theorem 5. *Let $1 \le p \le \infty$ and fix $N \in \mathbb{N}$. Let $(\mathcal{X}_n)$ be a sequence of grids on $\partial\mathbb{D}$ with $h_{\max}(\mathcal{X}_n) \to 0$ as $n \to \infty$, and let (T_n^p) be a sequence of quadrature approximations to $\|\cdot\|_{L^p(K)}$ such that* (1) *holds. Then the minimum of the fully discrete problem* (FDP$_p(\mathcal{X}_n, T_n^p, N)$) *converges to the minimum of the semi-discrete problem* (SDP$_p(N)$) *as $n \to \infty$, i.e., if $f_{N,n}^*$ is a solution of* (FDP$_p(\mathcal{X}_n, T_n^p, N)$)*, and f_N^* is a solution of* (SDP$_p(N)$)*, then*

$$T_n^p(f_{N,n}^* - \phi) \to \|f_N^* - \phi\|_{L^p(K)} \qquad \text{as } n \to \infty.$$

Before we can prove Theorem 5 we need a lemma which gives a bound on the derivative of functions that are feasible for (FDP$_p(\mathcal{X}, N)$).

Lemma 6. *Fix $N \in \mathbb{N}$. There are $h_0 > 0$ and $C > 0$ such that for any grid $\mathcal{X}$ with $h_{\max}(\mathcal{X}) \le h_0$ and any f which is feasible for* (FDP$_p(\mathcal{X}, N)$) *the estimate*

$$\|f'\|_{L^\infty(\partial\mathbb{D})} \le C\|g\|_{L^\infty(\partial\mathbb{D})}.$$

holds true. Here, $f'(\mathrm{e}^{i\theta}) = \frac{\mathrm{d}}{\mathrm{d}\theta} f(\mathrm{e}^{i\theta})$.

Proof. Let $\mathcal{X} = \{\mathrm{e}^{i\theta_1}, \dots, \mathrm{e}^{i\theta_n}\} \subset \partial\mathbb{D}$ be some grid. Let $f \in \mathcal{H}_N^\infty(\mathbb{D})$. Then f has the form $f(\mathrm{e}^{i\theta}) = \sum_{k=-N}^{N-1} \beta_k \mathrm{e}^{ik\theta}$ for some $\beta = (\beta_k)_{k=-N}^{N-1} \in \mathbb{R}^{2N}$. We write the relations

$$f_\beta(\mathrm{e}^{i\theta_j}) = \sum_{k=-N}^{N-1} \beta_k \mathrm{e}^{ik\theta_j}, \qquad j = 1, \dots, n,$$

in matrix form

$$B(\mathcal{X})\beta = f(\mathcal{X}). \tag{2}$$

Here, $f(\mathcal{X})$ is the vector $f(\mathcal{X}) = (f(\mathrm{e}^{i\theta_j}))_{j=1}^n \in \mathbb{C}^n$ and $B(\mathcal{X})$ is the matrix $B(\mathcal{X}) = (b_{jk}) \in \mathbb{C}^{n\times 2N}$, where $b_{jk} = \mathrm{e}^{ik\theta_j}$, $j = 1,\ldots,n$, $k = -N,\ldots,N-1$.

Now let $\theta_j = j\pi/N$ and take the grid $\mathcal{X}^N = \{\mathrm{e}^{i\theta_j} : j = 1,\ldots,2N\}$ with $n = 2N$ points. Then $\frac{1}{\sqrt{2N}}B(\mathcal{X}^N)$ is unitary. Since the set of invertible matrices is open, and since matrix inversion is a continuous function on this set, there is $h_1 > 0$ such that if the grid $\widetilde{\mathcal{X}} = \{\mathrm{e}^{i\widetilde{\theta}_1},\ldots,\mathrm{e}^{i\widetilde{\theta}_{2N}}\}$ satisfies

$$\max_{j=1,\ldots,2N} |\mathrm{e}^{i\theta_j} - \mathrm{e}^{i\widetilde{\theta}_j}| \le h_1, \tag{3}$$

then $B(\widetilde{\mathcal{X}})$ is still invertible and

$$\|B(\widetilde{\mathcal{X}})^{-1}\|_{\infty\to 1} \le 2\|B(\mathcal{X}^N)^{-1}\|_{\infty\to 1}. \tag{4}$$

Here, $\|\cdot\|_{\infty\to 1}$ is the operator norm $\|A\|_{\infty\to 1} = \sup_{\|x\|_\infty\le 1}\|Ax\|_1$.

To finish the proof, choose $h_0 > 0$ so small that any grid $\mathcal{X} \subset \partial\mathbb{D}$ with $h_{\max}(\mathcal{X}) \le h_0$ has a subgrid $\widetilde{\mathcal{X}} \subset \mathcal{X}$, consisting of $2N$ points, that satisfies (3). Let $\mathcal{X}$ be such a grid and $\widetilde{\mathcal{X}}$ a subgrid with (3). Let f be feasible for $(\mathrm{FDP}_p(\mathcal{X}, N))$, i.e., $f \in \mathcal{H}^\infty_N(\mathbb{D})$ with $|f(\mathrm{e}^{i\theta})| \le g(\mathrm{e}^{i\theta})$ for all $\mathrm{e}^{i\theta} \in \mathcal{X}$. Then f has the form $f(\mathrm{e}^{i\theta}) = \sum_{k=0}^{N-1}\alpha_k\mathrm{e}^{ik\theta}$ for some $\alpha = (\alpha_k)_{k=0}^{N-1} \in \mathbb{R}^N$. By (2) we have

$$B(\widetilde{\mathcal{X}})\begin{pmatrix}0_{\mathbb{R}^N}\\ \alpha\end{pmatrix} = f(\widetilde{\mathcal{X}}),$$

whence

$$\begin{aligned}\|\alpha\|_1 &\le \|B(\widetilde{\mathcal{X}})^{-1}\|_{\infty\to 1}\|f(\widetilde{\mathcal{X}})\|_\infty = \|B(\widetilde{\mathcal{X}})^{-1}\|_{\infty\to 1}\max_{\mathrm{e}^{i\widetilde{\theta}}\in\widetilde{\mathcal{X}}}|f(\mathrm{e}^{i\widetilde{\theta}})|\\ &\overset{(4)}{\le} 2\|B(\mathcal{X}^N)^{-1}\|_{\infty\to 1}\|g\|_{L^\infty(\partial\mathbb{D})}.\end{aligned}$$

Therefore,

$$\|f'\|_{L^\infty(\partial\mathbb{D})} = \sup_{\mathrm{e}^{i\theta}\in\partial\mathbb{D}}\left|\sum_{k=1}^{N-1} ik\alpha_k\mathrm{e}^{ik\theta}\right| \le N\|\alpha\|_1 \le 2N\|B(\mathcal{X}^N)^{-1}\|_{\infty\to 1}\|g\|_{L^\infty(\partial\mathbb{D})}.$$

Thus, the lemma holds with $C = 2N\|B(\mathcal{X}^N)^{-1}\|_{\infty\to 1}$. □

Next, we show that if the grid $\mathcal{X}_n$ is fine enough, then functions that are feasible for the fully discrete problem $(\mathrm{FDP}_p(\mathcal{X}_n, N))$ are almost feasible for the semi-discrete problem $(\mathrm{SDP}_p(N))$.

Lemma 7. *Fix $N \in \mathbb{N}$. Then for any $\epsilon > 0$ there is $h_1 > 0$ such that if $\mathcal{X}$ is a grid with $h_{\max}(\mathcal{X}) \le h_1$ and f is feasible for* $(\mathrm{FDP}_p(\mathcal{X}, N))$*, then*

$$|f| \le g + \epsilon.$$

Proof. Since g is uniformly continuous, there is $h_2 > 0$ such that for all $|\mu| \le h_2$ and all $\mathrm{e}^{i\theta} \in \partial\mathbb{D}$ we have

$$|g(\mathrm{e}^{i\theta}) - g(\mathrm{e}^{i(\theta+\mu)})| \le \epsilon/2. \tag{5}$$

Suppose that $\mathcal{X}$ is a grid with

$$h_{\max}(\mathcal{X}) \leq \min\{h_0, h_2, \epsilon(C\|g\|_{L^\infty(\partial\mathbb{D})})^{-1}\}, \tag{6}$$

where h_0 and C are the constants from Lemma 6. Let f be feasible for the fully discrete problem $(\mathrm{FDP}_p(\mathcal{X}, N))$ and let $\mathrm{e}^{i\theta} \in \partial\mathbb{D}$. Choose $\mathrm{e}^{it} \in \mathcal{X}$ such that $\operatorname{dist}\left(\mathrm{e}^{i\theta}, \mathrm{e}^{it}\right) \leq h_{\max}(\mathcal{X})/2$. Then

$$\begin{aligned}
|f(\mathrm{e}^{i\theta})| &\leq |f(\mathrm{e}^{it})| + \frac{h_{\max}(\mathcal{X})}{2}\|f'\|_{L^\infty(\partial\mathbb{D})} \\
&\leq g(\mathrm{e}^{it}) + \frac{h_{\max}(\mathcal{X})}{2}C\|g\|_{L^\infty(\partial\mathbb{D})} \qquad \text{by Lemma 6} \\
&\leq g(\mathrm{e}^{i\theta}) + \frac{\epsilon}{2} + \frac{\epsilon}{2} \qquad \text{by (5) and (6)} \\
&= g(\mathrm{e}^{i\theta}) + \epsilon.
\end{aligned} \tag{7}$$

Thus, the lemma holds true with $h_1 = \min\{h_0, h_2, \epsilon(C\|g\|_{L^\infty(\partial\mathbb{D})})^{-1}\}$. ☐

We are now ready to prove Theorem 5.

Proof of Theorem 5. Fix an arbitrary $\epsilon > 0$. Let $\epsilon_1 > 0$ be so small that for

$$\eta = \left(1 + \frac{\epsilon_1}{\min_{\mathrm{e}^{i\theta}\in\partial\mathbb{D}} g(\mathrm{e}^{i\theta})}\right)^{-1} \tag{8}$$

it holds true that $(1-\eta)\|\phi\|_{L^p(K)} \leq \epsilon$. (This is possible since $\eta \to 1$ as $\epsilon_1 \to 0$.) By Lemma 7 there is $n_0 \in \mathbb{N}$ such that functions that are feasible for $(\mathrm{FDP}_p(\mathcal{X}_n, N))$, $n \geq n_0$, satisfy

$$|f| \leq g + \epsilon_1. \tag{9}$$

Because of (1) we can possibly increase n_0 such that for all $n \geq n_0$ and all $f \in \mathcal{H}_N^\infty(\mathbb{D})$ with $\|f\|_{L^\infty(\partial\mathbb{D})} \leq \|g\|_{L^\infty(\partial\mathbb{D})} + \epsilon_1$

$$\left| T_n^p(f-\phi) - \|f-\phi\|_{L^p(K)} \right| \leq \epsilon. \tag{10}$$

So (10) especially holds true for all f that are feasible for $(\mathrm{FDP}_p(\mathcal{X}_n, N))$, $n \geq n_0$.

Now fix $n \geq n_0$. Let $f_{N,n}^*$ be a solution of $(\mathrm{FDP}_p(\mathcal{X}_n, T_n^p, N))$, and let f_N^* be a solution of $(\mathrm{SDP}_p(N))$. Then

$$T_n^p(f_{N,n}^* - \phi) \leq T_n^p(f_N^* - \phi) \leq \|f_N^* - \phi\|_{L^p(K)} + \epsilon. \tag{11}$$

The first inequality holds true because $f_{N,n}^*$ is a solution of $(\mathrm{FDP}_p(\mathcal{X}_n, T_n^p, N))$ and f_N^* is feasible for $(\mathrm{FDP}_p(\mathcal{X}_n, T_n^p, N))$, and the second inequality is due to (10).

On the other hand, take η from (8) and set $f_{N,n} = \eta f_{N,n}^*$. Because $f_{N,n}^*$ is feasible for $(\mathrm{FDP}_p(\mathcal{X}_n, N))$, we have $|f_{N,n}^*| \leq g + \epsilon_1$ by (9), and therefore

$$|f_{N,n}^*| \leq \left(1 + \frac{\epsilon_1}{\min_{\mathrm{e}^{i\theta}\in\partial\mathbb{D}} g(\mathrm{e}^{i\theta})}\right) g.$$

So $f_{N,n} = \eta f^*_{N,n} \le g$, that is, $f_{N,n}$ is feasible for the semi-discrete problem. Therefore,

$$\|f^*_N - \phi\|_{L^p(K)} \le \|f_{N,n} - \phi\|_{L^p(K)}.$$

Further, since $\eta \le 1$ and $(1-\eta)\|\phi\|_{L^p(K)} \le \epsilon$,

$$\begin{aligned}\|f_{N,n} - \phi\|_{L^p(K)} &= \|\eta f^*_{N,n} - \phi\|_{L^p(K)} \le \eta\|f^*_{N,n} - \phi\|_{L^p(K)} + (1-\eta)\|\phi\|_{L^p(K)} \\ &\le \|f^*_{N,n} - \phi\|_{L^p(K)} + \epsilon.\end{aligned}$$

Because (10) holds true for $f^*_{N,n}$, we end up with

$$\|f^*_N - \phi\|_{L^p(K)} \le T^p_n(f^*_{N,n} - \phi) + 2\epsilon \tag{12}$$

for $n \ge n_0$.

(11) and (12) imply that $T^p_n(f^*_{N,n} - \phi) \to \|f^*_N - \phi\|_{L^p(K)}$ as $n \to \infty$. □

Theorem 3 and Theorem 5 together imply

Corollary 8. *Let $(\mathcal{X}_n)$ be a sequence of grids on $\partial\mathbb{D}$ with $h_{\max}(\mathcal{X}_n) \to 0$ as $n \to \infty$, and let (T^p_n) be a sequence of quadrature approximations to $\|\cdot\|_{L^p(K)}$ such that* (1) *holds.*

If $1 \le p < \infty$, then for each $N \in \mathbb{N}$ we can choose $n(N)$ such that the minimum of the fully discrete problem $(\mathrm{FDP}_p(\mathcal{X}_{n(N)}, T^p_{n(N)}, N))$ *converges to the minimum of* (H-OPT_p) *as $N \to \infty$. If K is the union of finitely many closed intervals of positive measure, this also holds true for $p = \infty$.*

Similar to Corollary 4 we can prove

Corollary 9. *Let $(\mathcal{X}_n)$ be a sequence of grids on $\partial\mathbb{D}$ with $h_{\max}(\mathcal{X}_n) \to 0$ as $n \to \infty$, and let (T^p_n) be a sequence of quadrature approximations to $\|\cdot\|_{L^p(K)}$ such that* (1) *holds. Let f^* be a solution of* (H-OPT_p) *and let $f^*_{N,n}$ be a solution of* $(\mathrm{FDP}_p(\mathcal{X}_n, T^p_n, N))$.

1. *If $1 < p < \infty$, then for each $N \in \mathbb{N}$ we can choose $n(N)$ such that the minimum of* $(\mathrm{FDP}_p(\mathcal{X}_{n(N)}, T^p_{n(N)}, N))$ *converges to the minimum of* (H-OPT_p) *and such that $f^*_{N,n(N)} \to f^*$ in $L^p(\partial\mathbb{D})$ as $N \to \infty$.*

2. *If $p = \infty$ and K is the union of finitely many closed intervals, then for each $N \in \mathbb{N}$ we can choose $n(N)$ such that the minimum of the discrete problem* $(\mathrm{FDP}_p(\mathcal{X}_{n(N)}, T^p_{n(N)}, N))$ *converges to the minimum of* (H-OPT_p) *and such that $f^*_{N,n(N)} \xrightharpoonup{*} f^*$ in $L^\infty(\partial\mathbb{D})$ as $N \to \infty$.*

3. *If $p = 1$, then for each $N \in \mathbb{N}$ we can choose $n(N)$ such that the minimum of the discrete problem* $(\mathrm{FDP}_p(\mathcal{X}_{n(N)}, T^p_{n(N)}, N))$ *converges to the minimum of* (H-OPT_p)*, and, if $K \ne \partial\mathbb{D}$, such that $f^*_{N,n(N)} \rightharpoonup f^*$ in $L^1(\partial\mathbb{D})$ and $f^*_{N,n(N)} \to f^*$ in $L^1(\partial\mathbb{D} \setminus K)$ as $N \to \infty$.*

Proof. By Corollary 8, for any $1 \le p \le \infty$ we can choose $n(N)$ such that the minimum of the fully discrete problem $(\mathrm{FDP}_p(\mathcal{X}_n, T^p_n, N))$ converges to the minimum of (H-OPT_p). As we saw in the proof of Theorem 5, we can additionally achieve that

$$\left| T^p_n(f^*_{N,n(N)} - \phi) - \|f^*_{N,n(N)} - \phi\|_{L^p(K)} \right| \le \epsilon_N \tag{13}$$

and $|f^*_{N,n(N)}| \leq g+\epsilon_N$ with $\epsilon_N \to 0$ as $N \to \infty$. Especially, $(f^*_{N,n(N)})$ is bounded in $L^\infty(\partial\mathbb{D})$. As in the proof of Corollary 4 we can extract a subsequence which converges weakly in the case $1 \leq p < \infty$ and weakly* in the case $p = \infty$. Denote the limit by $\widetilde{f}$. As before, lower semicontinuity of the norm implies $\|\widetilde{f} - \phi\|_{L^p(K)} \leq \|f^* - \phi\|_{L^p(K)}$.

Now for any $\epsilon > 0$ all but possibly finitely many $f^*_{N,n(N)}$ lie in the set $\{f \in H^\infty(\mathbb{D}) : |f| \leq g+\epsilon \text{ on } \partial\mathbb{D}\}$. This set is weakly closed in $L^p(\mathbb{D})$, $1 \leq p < \infty$, and (sequentially) weakly* closed in $L^\infty(\mathbb{D})$. Therefore, $|\widetilde{f}| \leq g + \epsilon$ for any $\epsilon > 0$, i.e., $|\widetilde{f}| \leq g$. But this means that $\widetilde{f}$ is feasible for (H-OPT$_p$). If follows that $\widetilde{f}$ is a solution of (H-OPT$_p$). Unique solvability then implies $\widetilde{f} = f^*$. (In the case $p = 1$ we need $K \neq \partial\mathbb{D}$.) As before, by uniqueness of the limit the whole sequence $(f^*_{N,n(N)})$ converges weakly (or weakly* if $p = \infty$) to f^* in $L^p(\partial\mathbb{D})$ as $N \to \infty$.

Because of (13) and because $T^p_n(f^*_{N,n(N)} - \phi)$ converges to $\|f^* - \phi\|_{L^p(K)}$, $\|f^*_{N,n(N)} - \phi\|_{L^p(K)} \to \|f^* - \phi\|_{L^p(K)}$. Also, it follows as in the proof of Corollary 4 that $\|f^*_{N,n(N)}\|_{L^p(\partial\mathbb{D}\setminus K)} \to \|f^*\|_{L^p(\partial\mathbb{D}\setminus K)}$. As before, we obtain that $f^*_{N,n(N)} \to f^*$ strongly in $L^p(\partial\mathbb{D})$ for $1 < p < \infty$. In the case $p = 1$, weak convergence $f^*_{N,n(N)} \rightharpoonup f^*$ in $L^1(\partial\mathbb{D} \setminus K)$, together with the properties $|f^*_{N,n(N)}| \leq g + \epsilon_N$ with $\epsilon_N \to 0$ and $|f^*| = g$ a.e. on $\partial\mathbb{D} \setminus K$, implies strong convergence $f^*_{N,n(N)} \to f^*$ in $L^1(\partial\mathbb{D} \setminus K)$, see, e.g., [22, Lemma 2]. $\square$

We finish this section with examples of quadrature approximations that satisfy (1).

Example 10 (Rectangle rule, case $1 \leq p < \infty$)**.** *Assume that K is the union of finitely many closed intervals of positive measure and that ϕ is smooth. Let $(\mathcal{X}_n)$ be a sequence of grids with $h_{\max}(\mathcal{X}_n) \to 0$ as $n \to \infty$. Then (1) is fulfilled for the rectangle rule*

$$T^p_n(f - \phi) = \left(\sum_{\mathrm{e}^{i\theta} \in K \cap \mathcal{X}_n} \left| f(\mathrm{e}^{i\theta}) - \phi(\mathrm{e}^{i\theta}) \right|^p h_\theta \right)^{1/p},$$

where

$$h_\theta = \frac{\min\{\mu > 0 : \mathrm{e}^{i(\theta-\mu)} \in \mathcal{X}\} + \min\{\mu > 0 : \mathrm{e}^{i(\theta+\mu)} \in \mathcal{X}\}}{2}.$$

The reason why (1) holds for the rectangle rule is Lemma 6 and the fact that the error of the rectangle rule can be estimated by the derivative of the integrand.

Example 11 (Exact quadrature, case $p = 2$)**.** *We will see in the next section that for $p = 2$, $\phi \in \mathcal{L}^\infty_{N_\phi}(\partial\mathbb{D})$ for some $N_\phi \in \mathbb{N}$ and $f \in \mathcal{H}^\infty_N(\mathbb{D})$ it is in principle possible to compute $\|f - \phi\|_{L^p(K)}$ exactly. Assume that (ϕ_n) is a sequence with $\phi_n \in \mathcal{L}^\infty_n(\partial\mathbb{D})$ and $\|\phi_n - \phi\|_{L^2(K)} \to \infty$ as $n \to \infty$ and suppose that*

$$T^2_n(f - \phi) = \|f - \phi_n\|_{L^2(K)}.$$

Then

$$\begin{aligned}\left|T^2_n(f - \phi) - \|f - \phi\|_{L^2(K)}\right| &= \left|\|f - \phi_n\|_{L^2(K)} - \|f - \phi\|_{L^2(K)}\right| \\ &\leq \|\phi - \phi_n\|_{L^2(K)},\end{aligned}$$

so (T^2_n) satisfies (1).

Example 12 (Case $p = \infty$)**.** *Assume that K is the union of finitely many closed intervals of positive measure and that ϕ is smooth. Let $(\mathcal{X}_n)$ be a sequence of grids with $h_{\max}(\mathcal{X}_n) \to 0$ as $n \to \infty$. Then* (1) *is fulfilled for*

$$T_n^p(f - \phi) = \max_{\mathrm{e}^{i\theta} \in K \cap \mathcal{X}_n} |f(\mathrm{e}^{i\theta}) - \phi(\mathrm{e}^{i\theta})|.$$

The reason why (1) *holds is again Lemma 6.*

5 Discretization: Examples

In this section we explicitly write down the different discretizations of (H-OPT$_p$) corresponding to the quadrature approximations from Examples 10–12.

5.1 $1 \le p < \infty$, rectangle rule

Let $\mathcal{X} = \{\mathrm{e}^{i\theta_1}, \dots, \mathrm{e}^{i\theta_d}\}$ and let $N \in \mathbb{N}$. With the rectangle rule to approximate $\|f - \phi\|_{L^2(K)}$, see Example 10, we get the fully discrete problem

$$\begin{array}{ll} \text{minimize} & \left(\sum_{j \in K \cap \mathcal{X}} \left| f_\alpha(\mathrm{e}^{i\theta_j}) - \phi_j \right|^p h_{\theta_j} \right)^{1/p} \\ \text{subject to} & \left| f_\alpha(\mathrm{e}^{i\theta_j}) \right| \le g_j, \qquad j \in \mathcal{X}, \end{array}$$

in the optimization variable $\alpha = (\alpha_k)_{k=0}^{N-1} \in \mathbb{R}^N$. Here, $f_\alpha(\mathrm{e}^{i\theta}) = \sum_{k=0}^{N-1} \alpha_k \mathrm{e}^{ik\theta}$. By the sloppy notation $j \in \mathcal{X}$ we mean $\mathrm{e}^{i\theta_j} \in \mathcal{X}$, similarly for $j \in K \cap \mathcal{X}$. Also, we write $\phi_j = \phi(\mathrm{e}^{i\theta_j})$ and $g_j = g(\mathrm{e}^{i\theta_j})$, $j = 1, \dots, d$.

Let us assume that $\mathcal{X}$ is symmetric, i.e., $\mathcal{X} = \overline{\mathcal{X}}$, and write

$$\mathcal{X}^+ = \mathcal{X} \cap \{\mathrm{e}^{i\theta} : \theta \in [0, \pi]\}.$$

Then due to the symmetry of f_α and g it suffices to consider

$$\begin{array}{lll} \text{minimize} & F_p(\alpha) = \left(\sum_{j \in K \cap \mathcal{X}} \left| f_\alpha(\mathrm{e}^{i\theta_j}) - \phi_j \right|^p h_{\theta_j} \right)^{1/p} \\ \text{subject to} & \left| f_\alpha(\mathrm{e}^{i\theta_j}) \right| - g_j \le 0, \qquad j \in \mathcal{X}^+. \end{array} \tag{D-OPT$_p$}$$

5.2 $p = 2$, exact quadrature

We come back to Example 11. Suppose that ϕ is nice enough and can be written (or well approximated) in the form

$$\phi(\mathrm{e}^{i\theta}) = \sum_{k=-N_\phi}^{N_\phi - 1} \beta_k \mathrm{e}^{ik\theta}$$

with some $N_\phi \in \mathbb{N}$ and $\beta = (\beta_k)_{k=-N_\phi}^{N_\phi-1} \in \mathbb{R}^{2N_\phi}$. (The β_k must be real due to the real symmetry of ϕ.) Then it is possible to compute $\|f_\alpha - \phi\|_{L^2(K)}$ exactly. Without loss of generality we may assume that $N_\phi \ge N$ with possibly some of the β_k equal to zero. To simplify notation in the following, we write $f_\alpha(\mathrm{e}^{i\theta}) = \sum_{k=-N_\phi}^{N_\phi-1} \alpha_k \mathrm{e}^{ik\theta}$ with $\alpha_k = 0$ for $k \notin \{0, 1, \ldots, N-1\}$. We have

$$\begin{aligned}
\|f_\alpha - \phi\|_{L^2(K)}^2 = \int_K |f_\alpha(\mathrm{e}^{i\theta}) - \phi(\mathrm{e}^{i\theta})|^2 \,\mathrm{d}\theta &= \int_K \left| \sum_{k=-N_\phi}^{N_\phi-1} (\alpha_k - \beta_k)\mathrm{e}^{ik\theta} \right|^2 \mathrm{d}\theta \\
&= \int_K \sum_{k,l=-N_\phi}^{N_\phi-1} (\alpha_k - \beta_k)(\alpha_l - \beta_l)\mathrm{e}^{i(k-l)\theta} \,\mathrm{d}\theta \\
&= \sum_{k,l=-N_\phi}^{N_\phi-1} (\alpha_k - \beta_k)(\alpha_l - \beta_l) \int_K \mathrm{e}^{i(k-l)\theta} \,\mathrm{d}\theta.
\end{aligned}$$

With

$$m_j = \int_K \mathrm{e}^{ij\theta} \,\mathrm{d}\theta \tag{14}$$

we get the discrete problem

$$\begin{array}{ll}
\text{minimize} & \widetilde{F}_2(\alpha) = \left(\displaystyle\sum_{k,l=-N_\phi}^{N_\phi-1} m_{k-l}(\alpha_k - \beta_k)(\alpha_l - \beta_l) \right)^{1/2} \\
\text{subject to} & \left| f_\alpha(\mathrm{e}^{i\theta_j}) \right| - g_j \le 0, \qquad j \in \mathcal{X}^+.
\end{array} \tag{D-OPT$_{\widetilde{2}}$}$$

5.3 $p = \infty$

In the case $p = \infty$ we use the approximation from Example 12 and obtain

$$\begin{array}{ll}
\text{minimize} & F_\infty(\alpha) = \max\limits_{j \in K \cap \mathcal{X}^+} \left| f_\alpha(\mathrm{e}^{i\theta_j}) - \phi_j \right| \\
\text{subject to} & \left| f_\alpha(\mathrm{e}^{i\theta_j}) \right| - g_j \le 0, \qquad j \in \mathcal{X}^+.
\end{array} \tag{D-OPT$_\infty$}$$

Notice that due to symmetry we only take the maximum over $j \in K \cap \mathcal{X}^+$.

6 SOCP formulations

In order to solve the discrete problems from the previous section numerically, it is advisable to reformulate them as second-order cone programs (SOCPs). A (dual) SOCP is a problem of the form

$$\begin{array}{ll}
\text{maximize} & b^\top y \\
\text{subject to} & A_j^\top y + z^j = c_j, \quad j = 1, \ldots, \nu, \\
& z^j \in \mathcal{Q}_{n_j}, \quad j = 1, \ldots, \nu.
\end{array}$$

in the optimization variables y and $z = (z^1; \ldots; z^\nu)^*$. Here, $b \in \mathbb{R}^m$ for some positive integer m, $c^j \in \mathbb{R}^{n_j}$ for some positive integers n_j, $j = 1, \ldots, \nu$, and $A^j \in \mathbb{R}^{m \times n_j}$. Moreover, $\mathcal{Q}_n$ is the standard second-order cone of dimension $n \in \mathbb{N}$, $\mathcal{Q}_n = \left\{ (\xi_0; \xi) \in \mathbb{R} \times \mathbb{R}^{n-1} : \|\xi\|_2 \le \xi_0 \right\}$. For $n = 1$, this has to be read

*The semicolon is used to denote row-wise concatenation of vectors or matrices, i.e., $(z^1; z^2) = \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}$.

as $\mathcal{Q}_1 = \{\xi_0 \in \mathbb{R} : 0 \leq \xi_0\}$. SOCPs can be solved very efficiently via interior-point methods, see, e.g., [1, 14, 16]. Also, there are a number of free software packages available that can solve SOCPs [20, 21].

SOCPs include as special cases quadratically constrained quadratic programs (QCQPs) and p-norm minimization for rational p. A general strategy how to cast these problems into SOCP form is described in [1]. For example, in order to obtain SOCP formulations of (D-OPT$_2$), (D-OPT$_{\widetilde{2}}$) and (D-OPT$_\infty$), one can first write these problems as QCQPs, which can then be recast as SOCPs. For rational p, one can use the method from [1] to convert p-norm minimization problems to SOCPs.

A detailed derivation of the SOCP formulations of (D-OPT$_2$), (D-OPT$_{\widetilde{2}}$) and (D-OPT$_\infty$) is in [18]. Because the derivations are mechanical and do not yield any particular insights, we merely state the results for (D-OPT$_{\widetilde{2}}$) and (D-OPT$_\infty$) in the following subsections.

6.1 $p = 2$, exact quadrature

Recall that for exact quadrature we need to be in the situation that $\phi(\mathrm{e}^{i\theta}) = \sum_{k=-N_\phi}^{N_\phi-1} \beta_k \mathrm{e}^{ik\theta}$ with some $N_\phi \in \mathbb{N}$ and $\beta = (\beta_k)_{k=-N_\phi}^{N_\phi-1} \in \mathbb{R}^{2N_\phi}$. Let

$$\widetilde{M} = (m_{k-l})_{k,l=0}^{N-1} \in \mathbb{R}^{N \times N},$$

where the m_j are as in (14). Moreover, let

$$\widetilde{q} = (\widetilde{q}_k)_{k=0,\dots,N-1}, \qquad \widetilde{q}_k = -\sum_{l=-N_\phi}^{N_\phi-1} m_{k-l}\beta_l.$$

Further, let $\widetilde{S}$ be a matrix such that $\widetilde{S}^\top \widetilde{S} = \widetilde{M}$. We can choose $\widetilde{S} \in \mathbb{R}^{m \times N}$ for some $m \leq N$. Finally, for $j \in \mathcal{X}^+$ let

$$\gamma_j = (\cos(k\theta_j))_{k=0,\dots,N-1} \quad \text{and} \quad \sigma_j = (\sin(k\theta_j))_{k=0,\dots,N-1}.$$

With these definitions the SOCP formulation of (D-OPT$_{\widetilde{2}}$) is

$$\begin{aligned}
\text{max.} \quad & \begin{pmatrix} -1 \\ 0_{\mathbb{R}^{N\times 1}} \end{pmatrix}^\top \begin{pmatrix} t \\ \alpha \end{pmatrix} \\
\text{s.t.} \quad & \begin{pmatrix} -\frac{1}{2} & \widetilde{q}^\top \\ \frac{1}{2} & -\widetilde{q}^\top \\ 0_{\mathbb{R}^{m\times 1}} & -\widetilde{S} \end{pmatrix} \begin{pmatrix} t \\ \alpha \end{pmatrix} + z^K = \begin{pmatrix} \frac{1}{2}\left(1 - \left(\sum_{k,l=-N_\phi}^{N_\phi-1} m_{k-l}\beta_k\beta_l\right)\right) \\ \frac{1}{2}\left(1 + \left(\sum_{k,l=-N_\phi}^{N_\phi-1} m_{k-l}\beta_k\beta_l\right)\right) \\ 0_{\mathbb{R}^{m\times 1}} \end{pmatrix}, \\
& z^K \in \mathcal{Q}_{m+2}, \\
& \begin{pmatrix} 0 & 0_{\mathbb{R}^{1\times N}} \\ 0 & -\gamma_j^\top \\ 0 & -\sigma_j^\top \end{pmatrix} \begin{pmatrix} t \\ \alpha \end{pmatrix} + z^j = \begin{pmatrix} g_j \\ 0 \\ 0 \end{pmatrix}, \quad j \in \mathcal{X}^+, \\
& z^j \in \mathcal{Q}_3, \quad j \in \mathcal{X}^+.
\end{aligned} \tag{SOCP$_{\widetilde{2}}$}$$

If α is a minimizer of (D-OPT$_{\widetilde{2}}$), then $(\widetilde{F_2}(\alpha)^2; \alpha)$ is a maximizer of (SOCP$_{\widetilde{2}}$). Conversely, if $(t; \alpha)$ is a maximizer of (SOCP$_{\widetilde{2}}$), then α is a minimizer of (D-OPT$_{\widetilde{2}}$), and $\widetilde{F_2}(\alpha) = t^{1/2}$.

6.2 $p = \infty$

For $j \in K \cap \mathcal{X}^+$ let

$$q^{(j)} = (q_k^{(j)})_{k=0,\ldots,N-1}, \qquad q_k^{(j)} = -\Re\left(\mathrm{e}^{-ik\theta_j}\phi_j\right).$$

The SOCP formulation of (D-OPT$_\infty$) is

$$\begin{array}{ll}
\text{max.} & \begin{pmatrix} -1 \\ 0_{\mathbb{R}^{N\times 1}} \end{pmatrix}^\top \begin{pmatrix} t \\ \alpha \end{pmatrix} \\
\text{s.t.} & \begin{pmatrix} -\frac{1}{2} & {q^{(j)}}^\top \\ \frac{1}{2} & -{q^{(j)}}^\top \\ 0 & -\gamma_j^\top \\ 0 & -\sigma_j^\top \end{pmatrix} \begin{pmatrix} t \\ \alpha \end{pmatrix} + z^{K,j} = \begin{pmatrix} \frac{1}{2}(1-|\phi_j|^2) \\ \frac{1}{2}(1+|\phi_j|^2) \\ 0 \\ 0 \end{pmatrix}, \quad j \in K \cap \mathcal{X}^+, \\
& z^{K,j} \in \mathcal{Q}_4, \quad j \in K \cap \mathcal{X}^+, \\
& \begin{pmatrix} 0 & 0_{\mathbb{R}^{1\times N}} \\ 0 & -\gamma_j^\top \\ 0 & -\sigma_j^\top \end{pmatrix} \begin{pmatrix} t \\ \alpha \end{pmatrix} + \begin{pmatrix} \zeta_j^0 \\ \zeta_j^1 \\ \zeta_j^2 \end{pmatrix} = \begin{pmatrix} g_j \\ 0 \\ 0 \end{pmatrix}, \quad j \in \mathcal{X}^+, \\
& z^j \in \mathcal{Q}_3, \quad j \in \mathcal{X}^+.
\end{array} \tag{SOCP$_\infty$}$$

If α is a minimizer of (D-OPT$_\infty$), then $(F_\infty(\alpha)^2; \alpha)$ is a maximizer of (SOCP$_\infty$). Conversely, if $(t;\alpha)$ is a maximizer of (SOCP$_\infty$), then α is a minimizer of (D-OPT$_\infty$), and $F_\infty(\alpha) = t^{1/2}$.

7 Numerical experiments

In order to solve the SOCPs from the previous section practically, we decided to use a slightly modified version of the SDPT3 software package [21]. In its original form, SDPT3 works with sparse matrices. However, the matrices in (SOCP$_{\tilde{2}}$) and (SOCP$_\infty$) are dense. By instead using a dense format in some critical places and by using a highly optimized implementation of the basic linear algebra subprograms [7], we could achieve a speed-up of SDPT3 for our problems of around 100 on a system with two Quad-Core Opteron 2352 processors. For example, the computation time for Example 1 below with $p = \infty$, $N = 2^{12}$ and $d = 2^{12}$ was 578 seconds.

7.1 Example 1: Test problem

As a test problem we took the function ϕ in the top row of Figure 1. It is defined by $\phi(\mathrm{e}^{i\theta}) = |\phi(\mathrm{e}^{i\theta})|\mathrm{e}^{-i\theta}$, where $|\phi(\mathrm{e}^{i\theta})|$ is as in the top left of Figure 1. Moreover, we took $g \equiv 1$ and $K = \{\mathrm{e}^{i\theta} : \theta \in [\frac{\pi}{4}, \frac{3\pi}{4}] \cup [\frac{-3\pi}{4}, \frac{-\pi}{4}]\}$. We solved a series of problems for $p = 2$ with exact quadrature ($N_\phi = 2^{14}$) and $p = \infty$. We varied the dimension of the space $\mathcal{H}_N^\infty(\mathbb{D})$ by taking $N \in \{2^4, 2^5, \ldots, 2^{12}\}$. For each N we solved the problem for several grids of the form

$$\mathcal{X}_d = \left\{\mathrm{e}^{i0\pi/d}, \mathrm{e}^{i1\pi/d}, \ldots, \mathrm{e}^{i(2d-1)\pi/d}\right\}.$$

We took $d \in \{2^{\log_2 N}, \ldots, 2^{14}\}$.

The solutions for $N = 2^{12}$ and $d = 2^{14}$ are shown in the middle row ($p = 2$) and bottom row ($p = \infty$) of Figure 1. One can see nicely that the solutions

$f^*_{2,N,d}$ and $f^*_{\infty,N,d}$ show the properties of Theorem 1: The absolute value is (almost) equal to 1 on $\partial\mathbb{D} \setminus K$. Moreover, in the case $p = \infty$, $|\phi - f^*_{\infty,N,d}|$ is (almost) constant on K.

We also investigated numerically how the minimum of the discrete problem behaves when N is fixed and d is increased. Let us denote by $\tau^p_{N,d}$ the minimum of the optimization problem with certain p, N and d. Let

$$\delta^p_{N,d} = \left|\tau^p_{N,d} - \tau^p_{N,d/2}\right| \tag{15}$$

be the difference between two minima when the number of grid points is doubled. In Figure 2 we show how $\delta^p_{N,d}$ behaves for fixed N when we vary d. We observe that both for $p = 2$ and for $p = \infty$, $\delta^p_{N,d}$ behaves like d^{-2}. However, by looking only at the left starting points of the curves in Figure 2 (marked by circles) we observe that in the case $p = 2$, also $\delta^p_{d,d}$ behaves like d^{-2}, while in the case $p = \infty$ the decay of $\delta^p_{d,d}$ is significantly slower. This indicates that for $p = 2$ it should suffice to choose $d = N$, while for $p = \infty$ it might be advisable to choose d larger than N.

7.2 Example 2: Dispersion compensating mirror

We consider an example which comes from the design of dielectric mirrors that are used for dispersion compensation inside a laser cavity in order to generate ultra-short light pulses [15, 17]. Such mirrors consist of a stack of thin layers of typically two different dielectric materials with different refractive indices, which is deposited on some substrate, for example silica. The mirror constitutes a causal LTI system with frequency response R°, where $R^\circ(\omega) = R(-\omega)$, $\omega \in \mathbb{R}$. The function R is called reflection coefficient. A plane wave $\mathrm{e}^{i\omega\cdot}$ that is incident on the mirror gives rise to a reflected wave $R(\omega)\mathrm{e}^{-i\omega\cdot}$ propagating in the opposite direction. The modulus $|R(\omega)|$ describes the amplitude of the reflected wave, and the argument $\arg R(\omega)$ describes the phase shift.

We consider the example of a mirror like the one that is used in [23]. The mirror should have the following properties:

1. There is a frequency interval I_1, containing most of the spectrum of the pulses generated by the laser, where the mirror is highly reflective and imposes a specified frequency-dependent phase shift. The desired phase shift φ is usually given as a polynomial around some center frequency $\omega_0 \in I_1$,
$$\varphi(\omega) = \sum_{\nu=0}^{l} \frac{1}{\nu!} D_\nu (\omega - \omega_0)^\nu. \tag{16}$$
The numbers D_ν are called dispersion coefficients. The shape of the pulse after the phase shift does not depend on D_0 and D_1 [18]. Therefore, in practice only the group delay dispersion $GDD = \varphi''$ is relevant. However, the coefficient D_1 causes a shift of the pulse, so D_1 is connected to the thickness of the mirror. Therefore, there are practical restrictions to how large or small D_1 can be chosen.

2. There is a frequency interval I_2 where the mirror is almost transparent, which can be used for optical pumping.

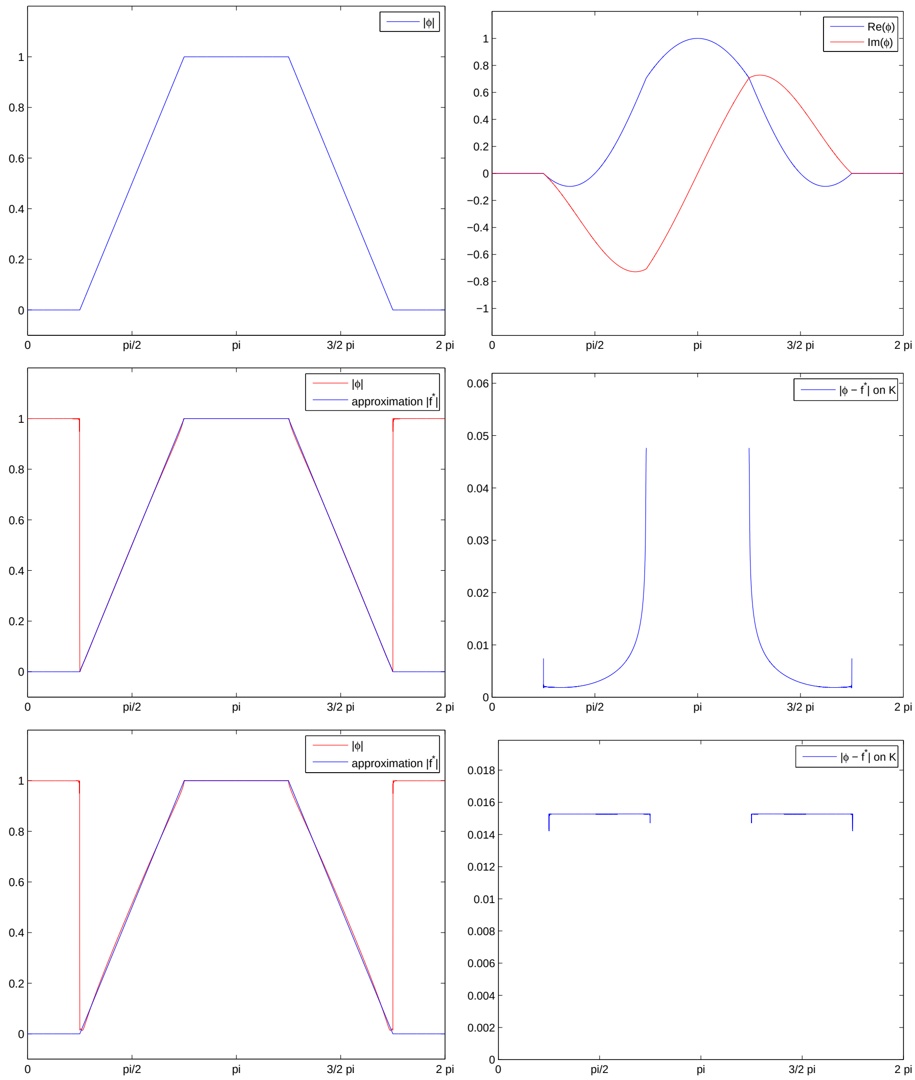

Figure 1: Results from Example 1 with $N = 4096$, $d = 16384$. *Top left:* Absolute value of some function ϕ. *Top right:* Real and imaginary part of ϕ. *Middle left:* Solution f^* for the case $p = 2$, exact quadrature. *Middle right:* Difference between ϕ and f^* on K for $p = 2$, exact quadrature. *Bottom left:* Solution f^* for the case $p = \infty$. *Bottom right:* Difference between ϕ and f^* on K for $p = \infty$.

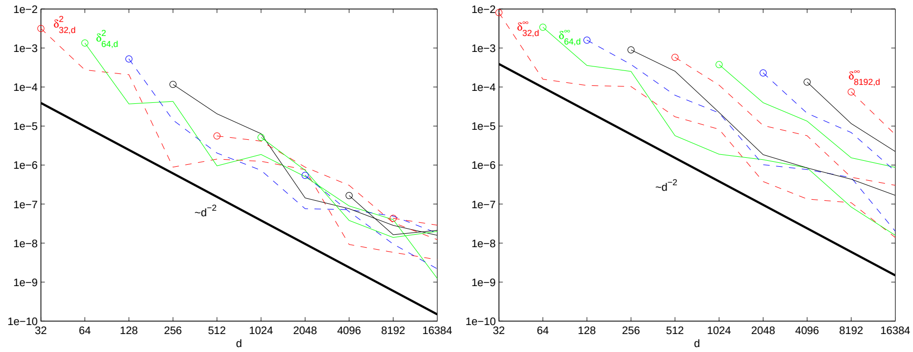

Figure 2: Example 1. *Left:* Case $p = 2$. Each curve shows how $\delta^2_{N,d}$ (see equation (15)) from Example 1 varies when N is fixed and d is varied. The starting point of each curve is marked by a circle. The curve starting at $d = 32$ is $\delta^2_{32,d}$, the curve starting at $d = 64$ is $\delta^2_{64,d}$, and so on. The solid black line indicates a decay of d^{-2}. *Right:* Case $p = \infty$. Each curve shows how $\delta^\infty_{N,d}$ varies when N is fixed and d is varied.

In our example we take $l = 5$ and $D_5 = -9.6928\,\mathrm{fs}^5$, $D_4 = 19.9324\,\mathrm{fs}^4$, $D_4 = -38.1923\,\mathrm{fs}^3$ $D_2 = -53.5991\,\mathrm{fs}^2$ at a center wavelength of 798.7735 nm. Moreover, we choose $D_1 = -27$ fs and $D_0 = 0$. The frequency interval of interest is $[729\,\mathrm{nm}, 995\,\mathrm{nm}]$, or, with $\omega = 2\pi c_0/\lambda$, where c_0 is the speed of light in vacuum, λ is wavelength and ω is angular frequency, $I_1 = [1.8931\,\mathrm{fs}^{-1}, 2.5839\,\mathrm{fs}^{-1}]$. For the desired reflection coefficient on I_1 we have $R_{\mathrm{desired}}(\omega) = \mathrm{e}^{i\varphi(\omega)}$, where φ is as in (16), see the top left of Figure 3. Moreover, the pump window is $[670\,\mathrm{nm}, 690\,\mathrm{nm}]$, or $I_2 = [2.7299\,\mathrm{fs}^{-1}, 2.8114\,\mathrm{fs}^{-1}]$. The amplitude reflectivity in the pump window should be smaller or equal to 0.05, see the top right of Figure 3.

The mirror design problem then consists of finding a layer structure n such that the corresponding reflection coefficient R_n is close to R_{desired} on I_1 in a suitable sense, and such that the constraint $|R_n| \le 0.05$ on I_2 is satisfied. Unfortunately, the mapping $n \mapsto R_n$ has a rather complicated behavior. Usually, long optimization runs are necessary to find a mirror structure n such that R_n is close to R_{desired}. There is not even a useful characterization of the exact range of realizable reflection coefficients. The best thing we know is that R_n is real symmetric (i.e., $R_n(-\omega) = \overline{R_n(\omega)}$, $\omega \in \mathbb{R}$), $|R_n| \le 1$ and $R_n \in H^\infty(\mathbb{C}^+)$ (see [18] for rigorous proofs of these facts). We can use this information to get an a priori bound on how small $\|R_n - R_{\mathrm{desired}}\|_{L^p(I_1)}$ can in principle be made by solving

$$\begin{array}{ll} \text{minimize} & \|R - R_{\mathrm{desired}}\|_{L^p(I_1)} \\ \text{subject to} & R \in H^\infty(\mathbb{C}^+),\ |R| \le G,\ R \text{ real symmetric}, \end{array} \tag{R-OPT$_p$}$$

where $G \equiv 0.05$ on I_2 and $G \equiv 1$ otherwise as in the top right of Figure 3.

Let us consider the case $p = \infty$. We can solve the problem (R-OPT$_\infty$) by transporting it from the real line to the unit circle, i.e., by casting it in the form

(H-OPT$_\infty$). To this end, we use the isometry

$$\left\{ \begin{array}{rcl} T_\infty : H^\infty(\mathbb{D}) & \longrightarrow & H^\infty(\mathbb{C}^+) \\ f & \longmapsto & \widetilde{f},\ \widetilde{f}(w) = f\left(\frac{iw+1}{-iw+1}\right), \end{array} \right.$$

We obtain a problem of the form (H-OPT$_\infty$) with ϕ and g as in the bottom row of Figure 3 and $K = \exp(i([-2.4031, -2.1696] \cup [2.1696, 2.4031]))$. ($K$ consists of two intervals due to real symmetry.) We should point out that g is not continuous in this example, whereas our general assumption in this paper and in [19] was that $g \in C(\partial\mathbb{D})$. However, our theorems can be generalized to include, for example, piecewise continuous g. We did not do this, because it makes the proofs more tedious, especially in [19], but does not add any insight.

We then solved the discrete problem with $N = 2^{12}$ on a grid $\mathcal{X}$ which contained the points $\left\{e^{i0\pi/d}, e^{i1\pi/d}, \ldots, e^{i(2d-1)\pi/d}\right\}$, $d = 2^{12}$. The solution of the discrete problem has a large derivative at the boundary points of K. (The solution of the (H-OPT$_\infty$) is not continuous on ∂K in this case, compare Example 8 in [19].) Therefore, we additionally refined the grid around K, so for the final grid we had $|\mathcal{X}| = 16348$.

The solution of the discrete problem, transported back to the real line, is shown in Figure 4. The minimal reflectivity in the HR region is 0.9978. One can see that the GDD of the solution oscillates rather wildly around the desired GDD, especially towards the boundary of K. The magnitude of the oscillations is just about tolerable for practical use. Notice that the "spikes" in the GDD are at the boundary of the points of the set where the solution has absolute value (approximately) 1. This is in accordance with Remark 9 in [19].

Concerning practice, we come to the following conclusions. In order for a DCM to work inside a laser cavity, it needs to be highly reflective. In our example, a minimal amplitude reflectivity of 0.999 in the HR region is required. Our results show that this design goal cannot be accomplished with the parameters that we used. For the actual mirror design process one then has two options:

- First of all, one can vary the desired parameter D_1. By doing this, one can shift the effective turning points of incident waves in the HR region deeper into the mirror. Further numerical experiments show that the minimum of (H-OPT$_\infty$) can be made smaller by varying D_1. On the other hand, the minimum of (H-OPT$_\infty$) varied only very slightly in our numerical experiments when we varied the desired D_0.

- The second option is to tweak the objective function that measures the distance between R_n and R_{desired}. In the optimization runs of the actual mirror design process, one usually does not use the L^p-distance $\|R_n - R_{\text{desired}}\|_{L^p(I)}$, but one measures the distance of the amplitude reflectivity, $|R_n| - |R_{\text{desired}}|$, and the distance of the phase (or GDD) separately, multiplies them with weights and adds them up. One can then use a larger weight for the distance of the amplitude reflectivity. In this way one can achieve that in a good local minimum of the objective function, the amplitude reflectivity is larger than 0.999 in the HR region, but on the flip side one then has to accept deviations from the desired phase (or GDD) that are larger than those in Figure 4.

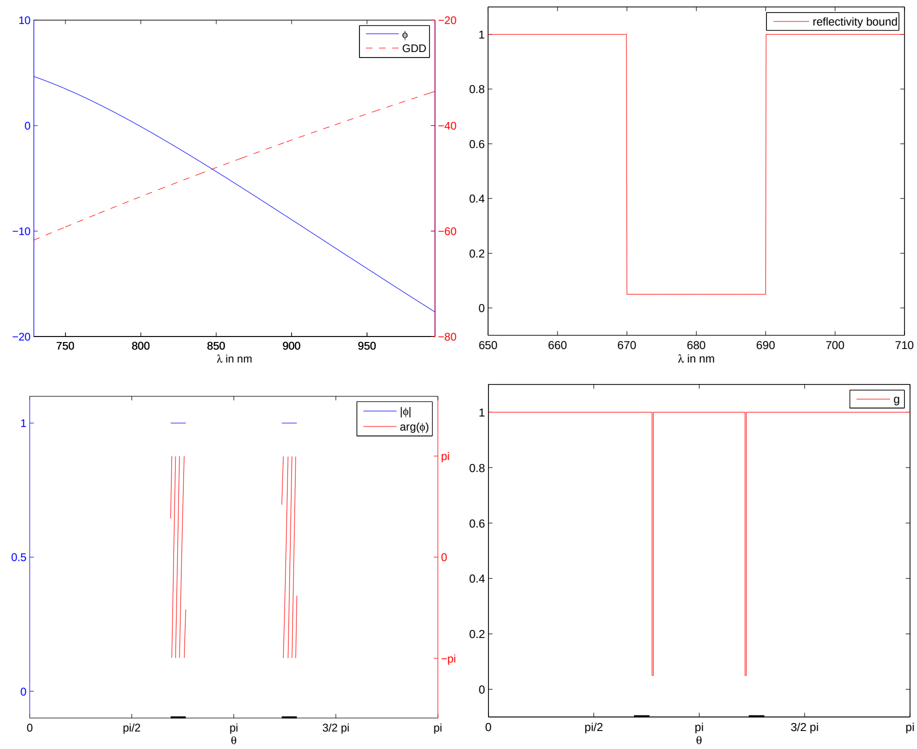

Figure 3: Example 2, dispersion compensating mirror with pump window. *Top left:* Desired phase shift and GDD in HR region [729 nm, 995 nm]. *Top right:* Reflectivity bound in pump window [670 nm, 690 nm]. *Bottom left:* $\arg\phi$ and $|\phi|$ (desired reflection coefficient transported to the circle). The set K is indicated on the x-axis. *Bottom right:* g (reflectivity bound transported to the circle).

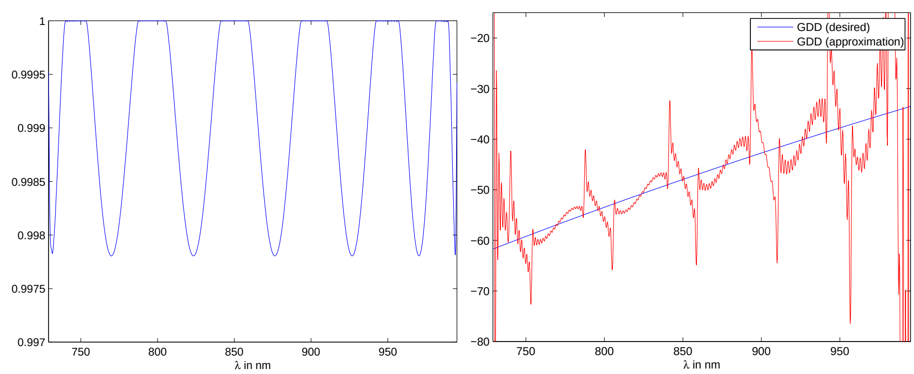

Figure 4: Example 2, results. *Left:* Absolute value of solution in HR region, transported back to the line. *Right:* Desired GDD and GDD of the solution.

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