Stationary random measures on homogeneous spaces

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Abstract

The subject of this paper are stationary random measures on a homogeneous space and their Palm measures. After the discussion of some fundamental properties as the refined Campbell theorem we will study invariant transports, invariance properties of Palm measures and stationary partitions. A key tool will be simple transformation of stationary random measures that will allow to extend recent results for stationary random measures on a group to the more general case of a homogeneous state space.

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1 Introduction

Palm measure and Palm probabilities are a fundamental and important concept in theory and application of point processes and random measures, see e.g. Matthes, Kerstan and Mecke [11], Kallenberg [4], Stoyan, Kendall and Mecke [17], Daley and Vere-Jones [1], Thorisson [18], Kallenberg [5] and Schneider and Weil [16]. In his seminal paper [12] Mecke has introduced and studied Palm measures of stationary random measures on Abelian groups. Later Tortrat [19] and Rother and Zähle [15] introduced Palm measures of stationary random measure on a group and on a homogeneous space, respectively. However, these more general cases have found little attention in the literature. The most general approach to invariance properties of Palm measures can be found in Kallenberg [6].

In this paper we will advance Palm theory for stationary random measures on a homogeneous space S. A simple but crucial tool will be a deterministic transformation of a stationary random measure on S into a stationary random measure on the group G. As this transformation is preserving the Palm measure, it can be used for extending recent results from the group case to general homogeneous spaces. This approach is not only short and elegant but has also the advantage of highlighting the important special case of a stationary random measure on a group. Stationary random measures and their Palm

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measures are introduced in Section 3. Sections 4-6 are extending some of the results in [9, 8] on invariance and transport properties of Palm measures. In Section 7 we will establish Mecke's [12] intrinsic characterization of Palm measure in our setting, see also [15]. In the two final sections we will apply the theory to stationary partitions (see [3, 7, 2] for the case $S = G = \mathbb{R}^d$) of (a random subset of) S.

2 Homogeneous spaces

We consider a topologial (multiplicative) group G that is assumed to be a locally compact, second countable Hausdorff space with unit element e and Borel σ-field $\mathcal{G}$. A classical source on such groups is [13], see also chapter 2 of [5]. A measure ν on G is *locally finite* if it is finite on compact sets. There exists a left-invariant Haar measure on G, i.e. a locally finite measure satisfying

$$\int f(hg)\lambda(dg) = \int f(g)\lambda(dg), \quad h \in G, \tag{2.1}$$

for all measurable $f : G \to \mathbb{R}_+$, where $\mathbb{R}_+ := [0, \infty)$. This measure is unique up to normalization. The *modular function* is a continuous homomorphism $\Delta : G \to (0, \infty)$ satisfying

$$\int f(gh)\lambda(dg) = \Delta(h^{-1}) \int f(g)\lambda(dg), \quad h \in G, \tag{2.2}$$

for all f as above. This modular function has the property

$$\int f(g^{-1})\lambda(dg) = \int \Delta(g^{-1})f(g)\lambda(dg). \tag{2.3}$$

The group G is called *unimodular*, if $\Delta(g) = 1$ for all $g \in G$.

Next we consider another locally compact second countable Hausdorff space S with Borel σ-field $\mathcal{S}$. We assume that the group G is *operating continuously* on S. This means that there is a continuous mapping $(g, x) \mapsto gx$ from $G \times S$ to S having $ex = x$ and $h(gx) = (hg)x$ for all $h, g \in G$ and $x \in S$. We assume that G is operating *transitively*, i.e. that the *projection* $\pi_x : G \to S$, $\pi_x(g) := gx$, is surjective for one (and hence for all) $x \in S$. We also assume that G operates *properly*, i.e. that $\pi_x^{-1}K$ is compact in G for any compact $K \subset S$. Then S is called *homogeneous space*.

A measure μ on S is *invariant* (or G-invariant) if

$$\int f(gx)\mu(dx) = \int f(x)\mu(dx), \quad g \in G, \tag{2.4}$$

for all measurable $f : S \to \mathbb{R}_+$. Up to a factor there is only one such invariant measure which is locally finite. In fact, we can and will choose

$$\mu := \lambda \circ \pi_c^{-1}, \tag{2.5}$$

where c is some fixed element of S. For any $x \in S$ we let $G_x := \{g \in G : gc = x\}$. Since G_c is a compact subgroup of G, there is a unique left- and right-invariant probability measure λ_c on G_c. It is convenient to extend λ_c to G by setting $\lambda_c(G \setminus G_c) := 0$. We have

$$\Delta(g) = 1, \quad g \in G_c. \tag{2.6}$$

Indeed, if $w : S \to \mathbb{R}_+$ is measurable with $\int w d\mu = 1$, then (2.5) and (2.2) imply for $g \in G_c$ that

$$1 = \int w(hc)\lambda(dh) = \int w(hgc)\lambda(dh) = \Delta(g^{-1}) \int w(hc)\lambda(dh) = \Delta(g^{-1}).$$

For any $x \in S$ we choose some $g_x \in G_x$ and define a probability measure λ_x on G by

$$\lambda_x(B) := \int \mathbf{1}\{g_x g \in B\}\lambda_c(dg), \quad B \in \mathcal{G}. \tag{2.7}$$

By G_c-invariance of λ_c, this definition is independent of the choice of g_x. This does also imply that

$$\int \mathbf{1}\{g \in \cdot\}\lambda_{hx}(dg) = \int \mathbf{1}\{hg \in \cdot\}\lambda_x(dg), \quad h \in G,\, x \in S. \tag{2.8}$$

Note that λ_x is supported by G_x. Moreover, the proof of Theorem 2.29 in [5] shows that $(x, B) \mapsto \lambda_x(B)$ is a kernel disintegrating the Haar measure λ as follows:

$$\int f(g)\lambda(dg) = \iint f(g)\lambda_x(dg)\mu(dx). \tag{2.9}$$

In fact, (2.9) is a straightforward consequence of (2.5), Fubini's theorem and (2.6).

Important examples of groups (e.g. in stochastic geometry) are the translation group, the rotation group and the group of rigid motions, all defined as subgroups of the group of bijective affine maps. The linear (resp. affine) Grassmannian of k-dimensional linear (resp. affine) subspaces of $\mathbb{R}^d$ is a homogeneous space under the group of rotations (resp. rigid motions). More details on these (and related spaces) can be found in the Appendix of [16].

For further reference we state the important special case of a group acting on itself explicitly:

Example 2.1. Assume that $S := G$ and that $(g, x) \mapsto gx$ is just the multiplication in the group. Take $c := e$. Then $\mu = \lambda$. Moreover, $G_h = \{h\}$ and $\lambda_h(\{h\}) = 1$ for all $h \in G$.

3 Stationary random measures

We denote by $\mathbf{M}(S)$ the set of all locally finite measures on S, and by $\mathcal{M}(S)$ the cylindrical σ-field on $\mathbf{M}(S)$ which is generated by the evaluation functionals $\nu \mapsto \nu(B)$, $B \in \mathcal{S}$. We often write $\mathbf{M}$ and $\mathcal{M}$ instead of $\mathbf{M}(S)$ and $\mathcal{M}(S)$, respectively. The *support* $\operatorname{supp} \nu$ of a measure $\nu \in \mathbf{M}$ is the smallest closed set $F \subset G$ such that $\nu(G \setminus F) = 0$. By $\mathbf{N} \subset \mathbf{M}$ (resp. $\mathbf{N}_s \subset \mathbf{M}$) we denote the measurable set of all (resp. *simple*) *counting measures* on S, i.e. the set of all those $\nu \in \mathbf{M}$ with discrete support and $\nu\{x\} := \nu(\{x\}) \in \mathbb{N}_0$ (resp. $\nu\{x\} \in \{0, 1\}$) for all $x \in S$. We can and will identify $\mathbf{N}_s$ with the class of all *locally finite* subsets of S, where a set is called locally finite if its intersection with any compact set is finite.

Although we mostly work on a σ-finite measure space $(\Omega, \mathcal{F}, \mathbb{P})$ ($\mathbb{P}$ need not be a probability measure), we are still using a probabilistic language. Together with $\mathbb{P}$ we will consider several other measures on $(\Omega, \mathcal{F})$. A *random measure* on S is a measurable mapping $\xi : \Omega \to \mathbf{M}$. A random measure is a (simple) *point process* on S if $\mathbb{P}(\xi \notin \mathbf{N}) = 0$ (resp. $\mathbb{P}(\xi \notin \mathbf{N}_s) = 0$). A random measure ξ can also be regarded as a *kernel* from Ω to S. Accordingly we write $\xi(\omega, B)$ instead of $\xi(\omega)(B)$. If ξ is a random measure, then the mapping $(\omega, x) \mapsto \mathbf{1}\{x \in \operatorname{supp} \xi(\omega)\}$ is measurable.

We assume that $(\Omega, \mathcal{F})$ is equipped with a *measurable flow* $\theta_g : \Omega \to \Omega$, $g \in G$. This is a family of measurable mappings such that $(\omega, g) \mapsto \theta_g \omega$ is measurable, θ_e is the identity on Ω and

$$\theta_g \circ \theta_h = \theta_{gh}, \quad g, h \in G, \tag{3.1}$$

where $\circ$ denotes composition. This implies that θ_g is a bijection with inverse $\theta_g^{-1} = \theta_{g^{-1}}$. A random measure ξ on S is called *invariant* (or *flow-adapted*) if

$$\xi(\theta_g \omega, gB) = \xi(\omega, B), \quad \omega \in \Omega,\ g \in G,\ B \in \mathcal{S}, \tag{3.2}$$

where $gB := \{gx : x \in B\}$. This means that

$$\int f(x)\xi(\theta_g \omega, dx) = \int f(gx)\xi(\omega, dx) \tag{3.3}$$

for all measurable $f : S \to \mathbb{R}_+$. We will often skip the ω in such relations, i.e. we write (3.3) as $\int f(x)\xi(\theta_g, dx) = \int f(gx)\xi(\theta_e, dx)$ or $\int f(x)\xi(\theta_g, dx) = \int f(gx)\xi(dx)$. Recall that θ_e is the identity on Ω. Still another way of expressing (3.2) is

$$\xi \circ \theta_g = g\xi, \quad g \in G, \tag{3.4}$$

where for $\nu \in \mathbf{M}$ and $g \in G$ the measure $g\nu$ is defined by $g\nu(\cdot) := \int \mathbf{1}\{gx \in \cdot\}\nu(dx)$.

In order to formulate a close relationship between invariant random measures on S and G we need the following Lemma.

Lemma 3.1. *If $\nu \in \mathbf{M}(S)$ then $\nu' := \int \lambda_x(\cdot)\nu(dx) \in \mathbf{M}(G)$.*

Proof. Let $B \subset G$ be compact. We first prove that

$$K := \{x \in S : G_x \cap B \neq \emptyset\}$$

is compact. Let $\mathcal{U} \subset \mathcal{G}$ be an open cover of K. Then $\{\pi_c^{-1}U : U \in \mathcal{U}\}$ is an open cover of B. Indeed, if $g \in B$ then there is a $x \in S$ with $g \in G_x$, i.e. $x \in K$. This means that $g \in \pi_c^{-1}U$ for some $U \in \mathcal{U}$. Since B is compact, there is a finite subset $\mathcal{U}'$ of $\mathcal{U}$ such that $\{\pi_c^{-1}U : U \in \mathcal{U}'\}$ is an open cover of B. But then $\mathcal{U}'$ is an open cover of K. Indeed, if $x \in K$, then we may take some $g \in G_x \cap B$. In particular, there is some $U \in \mathcal{U}'$ such that $g \in \pi_c^{-1}U$, i.e. $x = \pi_c g \in U$. We conclude that K is compact. It follows that

$$\nu'(B) = \int \lambda_x(B)\nu(dx) \leq \int \mathbf{1}\{G_x \cap B \neq \emptyset\}\nu(dx) = \nu(K) < \infty.$$

Therefore, ν' is locally finite. $\square$

The following lemma is a strengthening of Theorem 7.3 in [6] in our more specific situation. It will play a crucial role in the sequel. Invariance of a random measure on G is defined in the setting of Example 2.1.

Lemma 3.2. *If ξ is an invariant random measure on S, then*

$$\xi' := \int \lambda_x(\cdot)\xi(dx) \tag{3.5}$$

is an invariant random measure on G.

Proof. Let $B \in \mathcal{G}$. Since $x \mapsto \lambda_x(B)$ is measurable, $\xi'(B)$ is measurable as well. If B is compact, then $\xi'(B)$ is finite by Lemma 3.1. Using (2.8) and (3.2) we get for any $g \in G$ and $B \in \mathcal{S}$ that

$$\xi'(\theta_g, gB) = \int \lambda_x(gB)\xi(\theta_g, dx) = \int \lambda_{gx}(gB)\xi(dx) = \xi'(B).$$

Therefore, ξ' is invariant. ☐

Remark 3.3. If ξ' is given as in (3.5), then

$$\xi = \int \mathbf{1}\{gc \in \cdot\}\xi'(dg). \tag{3.6}$$

Conversely, if ξ' is an invariant random meansure on G, then (3.6) defines an invariant random measure on S.

A measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is called *stationary* if it is invariant under the flow, i.e.

$$\mathbb{P} \circ \theta_g = \mathbb{P}, \quad g \in G, \tag{3.7}$$

where θ_g is interpreted as a mapping from $\mathcal{F}$ to $\mathcal{F}$ in the usual way:

$$\theta_g A := \{\theta_g \omega : \omega \in A\}, \quad A \in \mathcal{F},\, g \in G.$$

Example 3.4. Consider the measurable space $(\mathbf{M}, \mathcal{M})$ and define for $\nu \in \mathbf{M}$ and $g \in G$ the measure $\theta_g \nu$ by $\theta_g \nu := g\nu$, i.e. $\theta_g \nu(B) := \nu(g^{-1}B)$, $B \in \mathcal{S}$. Then $\{\theta_g : g \in G\}$ is a measurable flow and the identity ξ on $\mathbf{M}$ is an invariant random measure. A stationary probability measure on $(\mathbf{M}, \mathcal{M})$ can be interpreted as the distribution of a *stationary random measure.*

Example 3.5. Assume that G is an additive Abelian group and that $S = G$. Consider a flow $\{\tilde{\theta}_g : g \in G\}$ as in [9] (see also [14]). In our current setting this amounts to define $gx := x + g$ and $\theta_g := \tilde{\theta}_g^{-1}$. It is somewhat unfortunate that in the point process literature it is common to define the shift of a measure $\nu \in \mathbf{M}$ by $g \in G$ by $g^{-1}\nu$ and not (as it would be more natural) by $g\nu$. Here we follow the terminology of [6].

Remark 3.6. Our setting accomodates stationary marked point processes (see [1, 11]) as well as stochastic processes (fields) jointly stationary with a random measure ξ (see [18]). The use of an abstract flow $\{\theta_g : g \in G\}$ acting directly on the underlying sample space is making the notation quite efficient. An even more general framework would be to let the group operate on the appropriate state spaces and to replace (3.7) by a distributional invariance, see [6]. Such an approach might be better suited for the case of a group operating in a non-transitive way. A specific non-transitive case is treated in Remark 3.12.

We now fix a σ-finite stationary measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ and an invariant random measure ξ on S. Let $w : S \to \mathbb{R}_+$ be a measurable function having $\int w(x)\mu(dx) = 1$. The measure

$$\mathbb{P}_\xi(A) := \iiint \mathbf{1}\{\theta_g^{-1}\omega \in A\} w(x)\lambda_x(dg)\xi(\omega, dx)\mathbb{P}(d\omega), \quad A \in \mathcal{F}, \tag{3.8}$$

is called the *Palm measure* of ξ (with respect to $\mathbb{P}$). A more succinct way of writing (3.8) is

$$\mathbb{P}_\xi(A) = \mathbb{E}_\mathbb{P} \iint \mathbf{1}\{\theta_g^{-1} \in A\} w(x)\lambda_x(dg)\xi(dx), \quad A \in \mathcal{F}, \tag{3.9}$$

where $\mathbb{E}_\mathbb{P}$ denotes integration with respect to $\mathbb{P}$. It is easy to see that $\mathbb{P}_\xi$ is concentrated on $\{c \in \operatorname{supp} \xi\}$, i.e.

$$\mathbb{P}_\xi(c \notin \operatorname{supp} \xi) = 0. \tag{3.10}$$

If ξ' is an invariant random measure on G, then (3.9) simplifies to

$$\mathbb{P}_{\xi'}(A) = \mathbb{E}_\mathbb{P} \int \mathbf{1}\{\theta_g^{-1} \in A\} w'(g)\xi'(dg), \quad A \in \mathcal{F}, \tag{3.11}$$

where $w' : G \to \mathbb{R}_+$ is a measurable function having $\int w'(g)\lambda(dg) = 1$. The following *refined Campbell theorem* connects $\mathbb{P}$ and $\mathbb{P}_{\xi'}$. For a proof we refer to [19], [6] and [8].

Proposition 3.7. *Let ξ' be an invariant random measure on G. Then $\mathbb{P}_{\xi'}$ is σ-finite and we have fo any measurable $f : \Omega \times G \to \mathbb{R}_+$,*

$$\mathbb{E}_\mathbb{P} \int f(\theta_g^{-1}, g)\xi'(dg) = \mathbb{E}_{\mathbb{P}_{\xi'}} \int f(\theta_e, g)\lambda(dg). \tag{3.12}$$

Equation (3.12) or rather its equivalent version

$$\mathbb{E}_\mathbb{P} \int \mathbf{1}\{(\theta_e, g) \in \cdot\}\xi(dg) = \mathbb{E}_{\mathbb{P}_\xi} \int \mathbf{1}\{(\theta_g, g) \in \cdot\}\lambda(dg),$$

is known as *skew factorization* of the *Campbell measure* of ξ'. A general discussion of this technique can be found in [6].

Remark 3.8. For additive groups (3.12) is mostly written as

$$\mathbb{E}_\mathbb{P} \int f(\tilde{\theta}_g, g)\xi(dg) = \mathbb{E}_{\mathbb{P}_\xi} \int f(\theta_e, g)\lambda(dg), \tag{3.13}$$

see Example 3.5.

Equation (3.12) shows in particular that definition (3.11) is independent of the choice of w', see the argument below. For instance we may choose $w'(g) := w(gc)$, where w is as in (3.8). This yields:

Lemma 3.9. *Let ξ be an invariant random measure on S, and define ξ' by* (3.5)*. Then* $\mathbb{P}_\xi = \mathbb{P}_{\xi'}$.

The previous lemma is a convenient tool for extending results for Palm measures of invariant random measures on groups to the case of a homogeneous state space. We start with the basic refined Campbell theorem. In particular, this theorem will show that the definition (3.9) is independent of the choice of the function w. In the canonical case of Example 3.4 and for finite intensities the result was derived in Rother and Zähle [15].

Theorem 3.10. *Let ξ be an invariant random measure on S. Then $\mathbb{P}_\xi$ is σ-finite and we have for any measurable $f : \Omega \times G \to \mathbb{R}_+$,*

$$\mathbb{E}_{\mathbb{P}} \iint f(\theta_g^{-1}, g)\lambda_x(dg)\xi(dx) = \mathbb{E}_{\mathbb{P}_\xi} \int f(\theta_e, g)\lambda(dg). \tag{3.14}$$

Proof. Define ξ' by (3.5) and apply Proposition 3.7 together with Lemma 3.9. $\square$

Let $w' : S \to \mathbb{R}_+$ be another measurable function having $\int w'(x)\mu(dx) = 1$ and take $A \in \mathcal{F}$. Then (3.14) implies that

$$\begin{aligned}\mathbb{E}_{\mathbb{P}} \iint \mathbf{1}\{\theta_g^{-1} \in A\}w'(x)\lambda_x(dg)\xi(dx) &= \mathbb{E}_{\mathbb{P}} \iint \mathbf{1}\{\theta_g^{-1} \in A\}w'(gc)\lambda_x(dg)\xi(dx) \\ &= \mathbb{E}_{\mathbb{P}_\xi} \int \mathbf{1}\{\theta_e \in A\}w'(gc)\lambda(dg) = \mathbb{P}_\xi(A).\end{aligned}$$

Hence the definition (3.8) is indeed independent of the choice of w.

The *intensity* γ_ξ of ξ is defined by

$$\gamma_\xi := \mathbb{P}_\xi(\Omega) = \mathbb{E}_{\mathbb{P}} \int w(x)\xi(dx). \tag{3.15}$$

We have $\gamma_\xi = \mathbb{E}_{\mathbb{P}}\xi(B)$ for any $B \in \mathcal{S}$ with $\mu(B) = 1$. The refined Campbell theorem implies the ordinary Campbell theorem

$$\mathbb{E}_{\mathbb{P}} \int f(x)\xi(dx) = \gamma_\xi \int f(x)\mu(dx), \tag{3.16}$$

for all measurable $f : S \to \mathbb{R}_+$. We just have to use that $f(x) = \int f(gc)\lambda_x(dg)$. In case $0 < \gamma_\xi < \infty$ we can define the *Palm probability measure* of ξ by $\mathbb{P}_\xi^0 := \gamma_\xi^{-1}\mathbb{P}_\xi$.

To derive another corollary of the refined Campbell theorem, we take a measurable function $\tilde{w} : \mathbf{M} \times S \to \mathbb{R}_+$ satisfying

$$\int \tilde{w}(\nu, x)\nu(dx) = 1, \tag{3.17}$$

whenever $\nu \in \mathbf{M}$ is not the null measure. For one example of such a function we refer to [12]. We then have the *inversion formula*

$$\mathbb{E}_{\mathbb{P}}\mathbf{1}\{\xi(S) > 0\}f = \mathbb{E}_{\mathbb{P}_\xi} \int \tilde{w}(\xi \circ \theta_g, gc)f(\theta_g)\lambda(dg) \tag{3.18}$$

for all measurable $f : \Omega \to \mathbb{R}_+$. This is a direct consequence of the refined Campbell theorem (3.14).

The *invariant* σ-field $\mathcal{I} \subset \mathcal{F}$ is the class of all sets $A \in \mathcal{F}$ satisfying $\theta_g A = A$ for all $g \in G$. Let ξ be an invariant random measure with finite intensity and define

$$\hat{\xi} := \mathbb{E}_{\mathbb{P}}\Big[\int w(x)\xi(dx)\Big|\mathcal{I}\Big], \tag{3.19}$$

where the conditional expectation is defined as for probability measures. Since $\hat{\xi} \circ \theta_g = \hat{\xi}$, $g \in G$, the refined Campbell theorem (3.14) implies that $\mathbb{E}_{\mathbb{P}} \mathbf{1}_A \int w(x)\xi(dx) = \mathbb{P}_\xi(A)$ for all $A \in \mathcal{I}$. Therfore definition (3.19) is independent of the choice of w. If $\mathbb{P}$ is a probability measure and $S = G = \mathbb{R}^d$, then $\hat{\xi}$ is called *sample intensity* of ξ, see [11] and [5]. Assuming that $\mathbb{P}(\hat{\xi} = 0) = 0$, we define the *modified Palm measure* $\mathbb{P}^*_\xi$ (see [11, 18, 7]) by

$$\mathbb{P}^*_\xi(A) = \mathbb{E}_{\mathbb{P}} \hat{\xi}^{-1} \iint \mathbf{1}\{\theta_g^{-1} \in A\} w(x) \lambda_x(dg) \xi(dx). \tag{3.20}$$

Conditioning shows that

$$\mathbb{P}^*_\xi(A) = \mathbb{P}(A), \quad A \in \mathcal{I}. \tag{3.21}$$

Comparing (3.20) and (3.8) yields

$$d\mathbb{P}^*_\xi = \hat{\xi}^{-1} d\mathbb{P}_\xi. \tag{3.22}$$

The refined Campbell theorem (3.14) takes the form

$$\mathbb{E}_{\mathbb{P}} \hat{\xi}^{-1} \iint f(\theta_g^{-1}, g) \lambda_x(dg) \xi(dx) = \mathbb{E}_{\mathbb{P}^*_\xi} \int f(\theta_e, g) \lambda(dg). \tag{3.23}$$

Remark 3.11. If $\mathbb{P}$ is a probability measure and ξ is a simple point process with a positive and finite intensity, then the Palm probability measure $\mathbb{P}^0_\xi$ can be interpreted as a conditional probability measure given that ξ has a point in c. This could be justified by Theorem 12.8 in [4]. The modified version describes the underlying stochastic experiment as seen from a randomly chosen point of ξ, see [11], [18]. Both measures agree iff $\hat{\xi}$ is $\mathbb{P}$-a.e. constant and in particular if $\mathbb{P}$ is *ergodic*, i.e. $\mathbb{P}(A) = 0$ or $\mathbb{P}(\Omega \setminus A) = 0$ for all $A \in \mathcal{I}$.

The above theory can be extended to certain non-transitive situations:

Remark 3.12. Let $(S', \mathcal{S}')$ be some measurable space and ζ a kernel from Ω to $S \times S'$. We call ζ *marked random measure* (on S with mark space S') if $\zeta(\cdot \times S')$ is a random measure on S. Such a marked random measure is *invariant* if $\zeta(\cdot \times B')$ is invariant for all $B' \in \mathcal{S}'$. In this case the *Palm measure* $\mathbb{P}_\zeta$ of ζ is the measure on $\Omega \times S'$ defined by

$$\mathbb{P}_\zeta(A) = \mathbb{E}_{\mathbb{P}} \iint \mathbf{1}\{(\theta_g^{-1}, z) \in A\} w(x) \lambda_x(dg) M(d(x,z)), \quad A \in \mathcal{F} \otimes \mathcal{S}'.$$

Note that $\mathbb{P}_\zeta(\cdot \times B')$ is the Palm measure of $\zeta(\cdot \times B')$. The refined Campbell theorem (3.14) takes the form

$$\mathbb{E}_{\mathbb{P}} \iint f(\theta_g^{-1}, g, z) \lambda_x(dg) M(d(x,z)) = \iint f(\omega, g, z) \lambda(dg) \mathbb{P}_\zeta(d(\omega, z)) \tag{3.24}$$

for all measurable $f : \Omega \times G \times S' \to \mathbb{R}_+$.

Assume that $(\Omega, \mathcal{F})$ is a Borel space. (This is e.g. the case in Example 3.4.) If $\mathbb{P}_\zeta(\Omega \times \cdot)$ is a σ-finite measure, then we may *disintegrate* $\mathbb{P}_\zeta$, to get another form of (3.24). For simplicity we even assume that the intensity γ_ξ of $\xi := \zeta(\cdot \times S')$ is finite. Assuming also $\gamma_\xi > 0$ we can define the *mark distribution* $\mathbb{W}$ of ζ by $\mathbb{W} := \gamma_\xi^{-1}\mathbb{P}_\zeta(\Omega \times \cdot)$. There exists a stochastic kernel $(z, A) \mapsto \mathbb{P}_\zeta^z(A)$ from S' to Ω satisfying

$$\mathbb{P}_\zeta(d(\omega, z)) = \gamma_\xi \mathbb{P}_\zeta^z(d\omega)\mathbb{W}(dz).$$

Therefore (3.24) can be written as

$$\mathbb{E}_\mathbb{P} \iint f(\theta_g^{-1}, g, z)\lambda_x(dg)\zeta(d(x, z)) = \gamma_\xi \iiint f(\omega, g, z)\lambda(dg)\mathbb{P}_\zeta^z(d\omega)\mathbb{W}(dz).$$

As might be expected, the Palm measure is invariant under G_c, see [15].

Proposition 3.13. *For all $h \in G_c$ we have $\mathbb{P}_\xi \circ \theta_h = \mathbb{P}_\xi$.*

Proof. Let $h \in G_c$. By (2.8) and (2.7) we have for all $x \in S$ that

$$\lambda_{hx} = \int \mathbf{1}\{hg_x g \in \cdot\}\lambda_c(dg) = \int \mathbf{1}\{hg_x gh^{-1} \in \cdot\}\lambda_c(dg) = \int \mathbf{1}\{hgh^{-1} \in \cdot\}\lambda_x(dg),$$

where the second equality comes from the right-invariance of λ_c. Using this fact, we get from the definition (3.9) of $\mathbb{P}_\xi$ and invariance of ξ for all $A \in \mathcal{F}$ that

$$\begin{aligned}
\mathbb{P}_\xi(\theta_h A) &= \mathbb{E}_\mathbb{P} \iint \mathbf{1}\{\theta_g^{-1} \in \theta_h A\} w(hx)\lambda_{hx}(dg)\xi \circ \theta_h^{-1}(dx) \\
&= \mathbb{E}_\mathbb{P} \iint \mathbf{1}\{\theta_g^{-1} \circ \theta_h^{-1} \in A\} w(hx)\lambda_x(dg)\xi \circ \theta_h^{-1}(dx) \\
&= \mathbb{E}_\mathbb{P} \iint \mathbf{1}\{\theta_g^{-1} \in A\} w(hx)\lambda_x(dg)\xi(dx),
\end{aligned}$$

where we have used stationarity (3.7) for the last equation. By invariance of μ we have $\int w(hy)\mu(dy) = 1$. Since the definition (3.9) is independent of the choice of w, we conclude the assertion. ☐

4 Transport-kernels and an exchange formula

We first adapt the terminology from [9] to our present more general setting. A *transport-kernel* (on S) is a kernel T from $\Omega \times S$ to S which is Markovian, i.e. which has $T(\omega, x, S) = 1$ for all $(\omega, x) \in \Omega \times S$. We think of $T(\omega, x, B)$ as redistributing a (potential) unit mass at x within S. A *weighted transport-kernel* is is a kernel T from $\Omega \times S$ to S such that $T(\omega, x, \cdot)$ is locally finite for all $(\omega, x) \in \Omega S$. A weighted transport-kernel T is called *invariant* if

$$T(\theta_g \omega, gx, gB) = T(\omega, x, B), \quad g \in G,\ x \in S,\ \omega \in \Omega,\ B \in \mathcal{S}. \tag{4.1}$$

Quite often we use the short-hand notation $T(x, \cdot) := T(\theta_e, x, \cdot)$. If ξ is an invariant random measure on S and $\eta := \int T(\omega, x, \cdot)\xi(\omega, dx)$ is locally finite for each $\omega \in \Omega$, then η is again an invariant random measure. Our interpretation is, that T transports ξ to η in an invariant way.

Let ξ and η be two invariant random measures on S. A weighted transport-kernel T on S is called (ξ,η)-*balancing* if

$$\int T(\omega,x,\cdot)\xi(\omega,dx)=\eta(\omega,\cdot) \tag{4.2}$$

holds for all $\omega\in\Omega$. In case $\xi=\eta$ we also say that T is ξ-*preserving*. If $\mathbb{Q}$ is a measure on $(\Omega,\mathcal{F})$ such that (4.2) holds for $\mathbb{Q}$-a.e. $\omega\in\Omega$ then we say that T is $\mathbb{Q}$-a.e. (ξ,η)-balancing.

We nox fix a σ-finite stationary measure $\mathbb{P}$ on $(\Omega,\mathcal{F})$. The next result is a fundamental transport property of Palm measures. It generalizes Theorem 4.2 in [9]. We use the function $\Delta^*: S\to(0,\infty)$ defined by

$$\Delta^*(x):=\Delta(g_x^{-1}),\quad x\in S, \tag{4.3}$$

where $g_x\in G_x$. This definition is independent of the choice of g_x. Indeed, if $g,h\in G_x$ then $g^{-1}h\in G_c$ so that (2.6) implies $1=\Delta(g^{-1}h)$, i.e. $\Delta(g^{-1})=\Delta(h^{-1})$. The representation $\Delta^*(x)=\int\Delta(g^{-1})\lambda_x(dg)$ shows that Δ^* is measurable.

Theorem 4.1. *Consider two invariant random measures ξ and η on S and let T and T^* be invariant weighted transport-kernels satisfying*

$$\iint \mathbf{1}\{(x,y)\in\cdot\}T(\omega,x,dy)\xi(\omega,dx)=\iint \mathbf{1}\{(x,y)\in\cdot\}T^*(\omega,y,dx)\eta(\omega,dy) \tag{4.4}$$

for $\mathbb{P}$-a.e. $\omega\in\Omega$. Then we have for any measurable function $f:\Omega\times G\to\mathbb{R}_+$ that

$$\mathbb{E}_{\mathbb{P}_\xi}\iint f(\theta_g^{-1},g^{-1})\Delta^*(x)\lambda_x(dg)T(c,dx)=\mathbb{E}_{\mathbb{P}_\eta}\iint f(\theta_e,g)\lambda_x(dg)T^*(c,dx). \tag{4.5}$$

The following special case of the previous theorem has been proved in [9] for Abelian and in [8] for general groups.

Proposition 4.2. *Consider two invariant random measures ξ and η on G and let T and T^* be invariant weighted transport-kernels satisfying*

$$\iint \mathbf{1}\{(g,h)\in\cdot\}T(\omega,g,dh)\xi(\omega,dg)=\iint \mathbf{1}\{(g,h)\in\cdot\}T^*(\omega,h,dg)\eta(\omega,dh) \tag{4.6}$$

for $\mathbb{P}$-a.e. $\omega\in\Omega$. Then we have for any measurable function $f:\Omega\times G\to\mathbb{R}_+$ that

$$\mathbb{E}_{\mathbb{P}_\xi}\int f(\theta_g^{-1},g^{-1})\Delta(g^{-1})T(e,dg)=\mathbb{E}_{\mathbb{P}_\eta}\int f(\theta_e,g)T^*(e,dg). \tag{4.7}$$

Proof of Theorem 4.1. Define a kernel T' from $\Omega\times G$ to G by

$$T'(g,B'):=\int\lambda_x(B')T(gc,dx),\quad g\in G,\ B'\in\mathcal{G}. \tag{4.8}$$

Lemma 3.2 implies that T' is a weighted transport-kernel on G. From (2.8) and invariance of T we have for all $g,h\in G$ and $B'\in\mathcal{G}$ that

$$T'(\theta_h,hg,hB')=\int\lambda_x(hB')T(\theta_h,hgc,dx)=\int\lambda_{hx}(hB')T(gc,dx)=\int\lambda_x(B')T(gc,dx).$$

Hence T' is invariant. Define the invariant weighted transport-kernel T'^* on G in terms of T^* as T' in terms of T. Defining ξ' by (3.5), we have

$$\iint \mathbf{1}\{(g,h) \in \cdot\} T'(g,dh)\xi'(dg) = \iiiint \mathbf{1}\{(g,h) \in \cdot\} \lambda_x(dh) T(gc,dx) \lambda_y(dg) \xi(dy)$$
$$= \iiiint \mathbf{1}\{(g,h) \in \cdot\} \lambda_y(dg) \lambda_x(dh) T(y,dx) \xi(dy),$$

where we have also used Fubini's theorem. Applying assumption (4.4), we obtain that

$$\iint \mathbf{1}\{(g,h) \in \cdot\} T'(g,dh)\xi'(dg) = \iint \mathbf{1}\{(g,h) \in \cdot\} T'^*(h,dg)\eta'(dh)$$

holds $\mathbb{P}$-a.e., where $\eta' := \int \lambda_x(\cdot)\eta(dx)$. Hence we can apply Proposition 4.2. In view of Lemma 3.9 this yields the assertion. ☐

Corollary 4.3. *Let the assumptions of Theorem 4.1 be satisfied. Assume moreover that ξ and η have finite intensities and that $\mathbb{P}(\hat{\xi} = 0) = \mathbb{P}(\hat{\eta} = 0) = 0$. Then we have for any measurable function $f : \Omega \times G \to \mathbb{R}_+$ that*

$$\hat{\xi}\, \mathbb{E}_{\mathbb{P}^*_\xi}\Big[\iint f(\theta_g^{-1}, g^{-1}) \Delta^*(x) \lambda_x(dg) T(c,dx) \Big| \mathcal{I} \Big] = \hat{\eta}\, \mathbb{E}_{\mathbb{P}^*_\eta}\Big[\iint f(\theta_e, g) \lambda_x(dg) T^*(c,dx) \Big| \mathcal{I} \Big],$$

$\mathbb{P}$-a.e. for any choice of the conditional expectations.

Proof. Define the random variables $X := \iint f(\theta_g^{-1}, g^{-1}) \Delta^*(x) \lambda_x(dg) T(c,dx)$ and $X' := \iint f(\theta_e, g) \lambda_x(dg) T^*(c,dx)$ and let $A \in \mathcal{I}$. Due to (3.21) we have $\mathbb{P}^*_\xi = \mathbb{P}^*_\eta$ on $\mathcal{I}$. Hence we have to show that

$$\mathbb{E}_{\mathbb{P}^*_\xi} \mathbf{1}_A \hat{\xi} X = \mathbb{E}_{\mathbb{P}^*_\eta} \mathbf{1}_A \hat{\eta} X'.$$

By (3.22) this amounts to $\mathbb{E}_{\mathbb{P}_\xi} \mathbf{1}_A X = \mathbb{E}_{\mathbb{P}_\eta} \mathbf{1}_A X'$, i.e. to a consequence of (4.5). ☐

5 Transport properties of Palm measures

In this section we fix a stationary σ-finite measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$. The following fundamental invariance property of Palm measures has recently been established in [9] and [8].

Proposition 5.1. *Consider two invariant random measures ξ and η on G and an invariant weighted transport-kernel T on G. Then T is $\mathbb{P}$-a.e. (ξ, η)-balancing iff*

$$\mathbb{E}_{\mathbb{P}_\xi} \int f(\theta_g^{-1}) \Delta(g^{-1}) T(e,dg) = \mathbb{E}_{\mathbb{P}_\eta} f \tag{5.1}$$

holds for all measurable $f : \Omega \to \mathbb{R}_+$.

The previous proposition generalizes as follows:

Theorem 5.2. *Consider two invariant random measures ξ and η on S and an invariant weighted transport-kernel T. Then T is $\mathbb{P}$-a.e. (ξ, η)-balancing iff*

$$\mathbb{E}_{\mathbb{P}_\xi} \iint f(\theta_g^{-1}) \Delta^*(x) \lambda_x(dg) T(c, dx) = \mathbb{E}_{\mathbb{P}_\eta} f \tag{5.2}$$

holds for all measurable $f : \Omega \to \mathbb{R}_+$.

Proof. Assume first that T is $\mathbb{P}$-a.e. (ξ, η)-balancing. Lemma 5.3 below shows that there exists a invariant transport-kernel T^* satisfying (4.4) for $\mathbb{P}$-a.e. $\omega \in \Omega$. Applying (4.5) to a function not depending on the second argument, yields (5.2).

Let us now assume that (5.2) holds. Defining the invariant weighted transport-kernel T' by (4.8), (5.2) can be written as

$$\mathbb{E}_{\mathbb{P}_\xi} \int f(\theta_g^{-1}) \Delta(g^{-1}) T'(e, dg) = \mathbb{E}_{\mathbb{P}_\eta} f.$$

Hence we get from Lemma 3.9 and Proposition 5.1 that

$$\int T'(g, B') \lambda_x(dg) \xi(dx) = \int \lambda_x(B') \eta(dx)$$

holds $\mathbb{P}$-a.e. for any $B' \in \mathcal{G}$. Applying this with $B' := \pi_c^{-1} B$ for $B \in \mathcal{S}$ easily yields that (4.2) holds $\mathbb{P}$-a.e. $\square$

The above proof has used the following lemma:

Lemma 5.3. *Assume that T is a $\mathbb{P}$-a.e. (ξ, η)-balancing invariant weighted transport-kernel. Then there exists an invariant transport-kernel T^* on S such that (4.4) holds for $\mathbb{P}$-a.e. $\omega \in \Omega$.*

Proof. Consider the following measure W on $\Omega \times S \times S$:

$$W := \iiint \mathbf{1}\{(\omega, x, y) \in \cdot\} T(\omega, x, dy) \xi(\omega, dx) \mathbb{P}(d\omega).$$

Stationarity of $\mathbb{P}$, (3.2), and (4.1) easily imply that

$$\int \mathbf{1}\{(\theta_g \omega, gx, gy) \in \cdot\} W(d(\omega, x, y)) = W, \quad g \in G.$$

Moreover, as T is a $\mathbb{P}$-a.e. (ξ, η)-balancing, we have

$$W' := \int \mathbf{1}\{(\omega, y) \in \cdot\} W(d(\omega, x, y)) = \iint \mathbf{1}\{(\omega, y) \in \cdot\} \eta(\omega, dy) \mathbb{P}(d\omega). \tag{5.3}$$

This is a σ-finite measure on $\Omega \times S$. We can now apply Theorem 3.5 in [6] to obtain an invariant transport-kernel T^* satisfying

$$W = \iint \mathbf{1}\{(\omega, x, y) \in \cdot\} T^*(\omega, y, dx) W'(d(\omega, y)).$$

(In fact the theorem yields an invariant kernel T', satisfying this equation. But in our specific situation we have $T'(\omega, y, G) = 1$ for W'-a.e. (ω, y), so that T' can be modified in an obvious way to yield the desired T^*.) Recalling the definition of W and the second equation in (5.3) we get that

$$\iiint \mathbf{1}\{(\omega, x, y) \in \cdot\} T(\omega, x, dy)\xi(\omega, dx)\mathbb{P}(d\omega)$$
$$= \iiint \mathbf{1}\{(\omega, x, y) \in \cdot\} T^*(\omega, y, dx)\eta(\omega, dy)\mathbb{P}(d\omega)$$

and hence the assertion of the lemma. $\square$

6 Existence of balancing weighted transport-kernels

We fix a stationary σ-finite measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$. Our aim is to establish a necessary and sufficient condition for the existence of a balancing invariant weighted transport-kernel T satisfying

$$\int \Delta^*(x)T(c, dx) = 1. \tag{6.1}$$

Theorem 6.1. *Assume that ξ and η are invariant random measures on S with positive and finite intensities. Then there exists a $\mathbb{P}$-a.e. (ξ, η)-balancing invariant weighted transport-kernel satisfying* (6.1) *iff*

$$\mathbb{E}_{\mathbb{P}}[\xi(B)|\mathcal{I}] = \mathbb{E}_{\mathbb{P}}[\eta(B)|\mathcal{I}] \quad \mathbb{P} - a.e. \tag{6.2}$$

for some $B \in \mathcal{S}$ satisfying $0 < \mu(B) < \infty$.

The following consequence of Theorem 6.1 has been proved in [9] for Abelian and in [8] for general groups.

Proposition 6.2. *Assume that ξ and η are invariant random measures on G with positive and finite intensities. Then there exists a $\mathbb{P}$-a.e. (ξ, η)-balancing invariant weighted transport-kernel satisfying $\int \Delta(g^{-1})T(e, dg) = 1$ iff* (6.2) *holds for some $B \in \mathcal{G}$ satisfying $0 < \lambda(B) < \infty$.*

Proof of Theorem 6.1: Let $B \in \mathcal{S}$ satisfy $0 < \mu(B) < \infty$ and take $A \in \mathcal{I}$. Applying the refined Campbell theorem (3.14) with $f(\theta_e, g) := \mathbf{1}_A \mathbf{1}\{gc \in B\}$ yields that

$$\mu(B)\mathbb{P}_\xi(A) = \mathbb{E}_{\mathbb{P}}[\mathbf{1}_A \xi(B)], \quad \mu(B)\mathbb{P}_\eta(A) = \mathbb{E}_{\mathbb{P}}[\mathbf{1}_A \eta(B)]. \tag{6.3}$$

Assume now that T is a $\mathbb{P}$-a.e. (ξ, η)-balancing invariant weighted transport-kernel satisfying (6.1). Then Theorem 5.2 implies for all $A \in \mathcal{I}$ the equality $\mathbb{P}_\xi(A) = \mathbb{P}_\eta(A)$. Thus (6.3) implies $\mathbb{E}_{\mathbb{P}}\mathbf{1}_A\xi(B) = \mathbb{E}_{\mathbb{P}}\mathbf{1}_A\eta(B)$ and hence (6.2).

Let us now assume that (6.2) holds for some $B \in \mathcal{S}$ satisfying $0 < \mu(B) < \infty$. From (6.3) and conditioning we obtain that $\mathbb{P}_\xi = \mathbb{P}_\eta$ on $\mathcal{I}$. By Lemma 3.9 and Proposition 6.2 there is an invariant weighted transport-kernel T' on G such that

$$\int \Delta(g^{-1})T'(e, dg) = 1 \tag{6.4}$$

and

$$\iint T'(g,B')\lambda_x(dg)\xi(dx) = \int \lambda_x(B')\eta(dx) \quad \mathbb{P}-\text{a.e.} \tag{6.5}$$

for all $B' \in \mathcal{G}$. Define

$$T(\omega,x,B) := \int T'(\omega,g,\pi_c^{-1}B)\lambda_x(dg), \quad x \in S,\, \omega \in \Omega,\, B \in \mathcal{S}. \tag{6.6}$$

Applying (6.5) with $B' := \pi_c^{-1}B$ for $B \in \mathcal{S}$, gives

$$\int T(x,B)\xi(dx) = \eta(B) \quad \mathbb{P}-\text{a.e.} \tag{6.7}$$

It remains to show that T is an invariant weighted transport-kernel satisfying (6.1). By definition of T' and Δ^*,

$$\int \Delta^*(x)T(c,dx) = \iint \Delta^*(hc)T'(g,dh)\lambda_c(dg) = \iint \Delta(h^{-1})T'(g,dh)\lambda_c(dg).$$

Using invariance of T', we get

$$\int \Delta^*(x)T(c,dx) = \iint \Delta((gh)^{-1})T'(e,dh)\lambda_c(dg) = \iint \Delta(h^{-1})T'(e,dh)\lambda_c(dg) = 1,$$

where we have used (6.4) and that $\Delta(g^{-1}) = 1$ for $g \in G_c$. Now we take a compact $B \subset S$. Since $g \mapsto \Delta(g^{-1})$ is bounded away from 0 on the compact set $\pi_c^{-1}B = \cup_{x\in B}G_x$, the mapping Δ^* has the same property on B. Therefore (6.1) implies that $T(c,B)$ must be finite. Hence T is a weighted transport-kernel. To show invariance, we take $h \in G$, $x \in S$, and $B \in \mathcal{S}$. Then

$$T(\theta_h,hx,hB) = \int T'(\theta_h,g,\pi_c^{-1}(hB))\lambda_{hx}(dg) = \int T'(\theta_h,hg,h\pi_c^{-1}B)\lambda_x(dg),$$

where we have used (2.8) and $\pi_c^{-1}(hB) = h\pi_c^{-1}B$. Therefore invariance of T is implied by the same property of T'. $\square$

7 Mecke's characterization of Palm measures

In contrast to the previous sections we do not fix a stationary measure on $(\Omega,\mathcal{F})$. Instead we consider here a measure $\mathbb{Q}$ on $(\Omega,\mathcal{F})$ as a candidate for a Palm measure of a given invariant random measure w.r.t. some stationary measure $\mathbb{P}$ on $(\Omega,\mathcal{F})$. In case of an invariant random measure on an Abelian group (and within a canonical framework) the following fundamental characterization theorem was proved in [12]. A proof of the present more general version (close to Mecke's original proof) can be found in [8].

Proposition 7.1. *Let ξ be an invariant random measure on G. The measure $\mathbb{Q}$ is a Palm measure of ξ with respect to some σ-finite stationary measure iff $\mathbb{Q}$ is σ-finite, $\mathbb{Q}(\xi(G)=0)=0$, and*

$$\mathbb{E}_{\mathbb{Q}} \int f(\theta_g^{-1},g^{-1})\Delta(g^{-1})\xi(dg) = \mathbb{E}_{\mathbb{Q}} \int f(\theta_e,g)\xi(dg) \tag{7.1}$$

holds for all measurable $f : \Omega \times G \to \mathbb{R}_+$.

In the canonical framework of Example 3.4 and for finite intensities the following extension of Proposition 7.1 has been established in [15].

Theorem 7.2. *Let ξ be an invariant random measure on S. The measure $\mathbb{Q}$ is a Palm measure of ξ with respect to some σ-finite stationary measure iff $\mathbb{Q}$ is σ-finite and invariant under G_c, $\mathbb{Q}(\xi(S) = 0) = 0$, and*

$$\mathbb{E}_{\mathbb{Q}} \iint f(\theta_g^{-1}, g^{-1}c)\Delta^*(x)\lambda_x(dg)\xi(dx) = \mathbb{E}_{\mathbb{Q}} \int f(\theta_e, x)\xi(dx) \tag{7.2}$$

holds for all measurable $f : \Omega \times S \to \mathbb{R}_+$.

Proof. If $\mathbb{Q}$ is a Palm measure of ξ, then $\mathbb{Q}$ is σ-finite by Theorem 3.10 and invariant under G_c by Proposition 3.13. Equation $\mathbb{Q}(\xi(S) = 0) = 0$ holds by (3.10), while the Mecke equation (7.2) is a special case of (4.5).

Let us now conversely assume the stated conditions. We define a random measure ξ' on G by (3.5) and consider a measurable function $f : \Omega \times G \to \mathbb{R}_+$. Applying (7.2) to the function $(\omega, x) \mapsto \int f(\omega, h)\lambda_x(dh)$, we obtain

$$\mathbb{E}_{\mathbb{Q}} \int f(\theta_e, g)\xi'(dg) = \mathbb{E}_{\mathbb{Q}} \iiint f(\theta_g^{-1}, h)\lambda_{g^{-1}c}(dh)\Delta(g^{-1})\lambda_x(dg)\xi(dx).$$

Equation (2.8) and Fubini's theorem imply

$$\mathbb{E}_{\mathbb{Q}} \int f(\theta_e, g)\xi'(dg) = \int \mathbb{E}_{\mathbb{Q}} \left[\iint f(\theta_g^{-1}, g^{-1}h)\Delta(g^{-1})\lambda_x(dg)\xi(dx)\right] \lambda_c(dh). \tag{7.3}$$

By invariance of $\mathbb{Q}$ under G_c the expectation occuring in the above right-hand side equals

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}} \iint f(\theta_g^{-1} \circ \theta_h, g^{-1}h)\Delta(g^{-1})\lambda_x(dg)\xi \circ \theta_h(dx) \\
&\qquad\qquad =\mathbb{E}_{\mathbb{Q}} \iint f(\theta_{hg}^{-1} \circ \theta_h, (hg)^{-1}h)\Delta((hg)^{-1})\lambda_x(dg)\xi(dx) \\
&\qquad\qquad =\mathbb{E}_{\mathbb{Q}} \iint f(\theta_g^{-1}, g^{-1})\Delta(g^{-1})\lambda_x(dg)\xi(dx),
\end{aligned}$$

where we have used the properties of the modular function and (2.6). Inserting this into (7.3), yields

$$\mathbb{E}_{\mathbb{Q}} \int f(\theta_e, g)\xi'(dg) = \mathbb{E}_{\mathbb{Q}} \int f(\theta_g^{-1}, g^{-1})\Delta(g^{-1})\xi'(dg).$$

Since $\mathbb{Q}(\xi'(G) = 0) = 0$, Proposition 7.1 implies that there is a σ-finite stationary measure $\mathbb{P}$ such that $\mathbb{Q} = \mathbb{P}_{\xi'}$. A reference to Lemma 3.9 concludes the proof of the theorem. ☐

Remark 7.3. Let ξ be an invariant random measure on S such that $\mathbb{Q}(\xi(S) = 0) = 0$. The inversion formula (3.18) shows that there is at most one stationary σ-finite measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ such that $\mathbb{Q} = \mathbb{P}_\xi$.

8 Stationary partitions

A stationary partition is decomposing a random subset Z of S in pairwise disjoint sets. This partition is assumed to be consistent with the flow $\{\theta_g : g \in G\}$. Motivated by recent work in [3] and [2] on *allocations* to a simple point process, stationary partitions were introduced and studied (in case $S = G = \mathbb{R}^d$) in [7]. Stationary tessellations of classical stochastic geometry (see e.g. [17, 16]) are a special case.

Let η be an invariant simple point process on S. It is convenient to assume that $\eta(\omega) \in \mathbf{N}_s$ for all $\omega \in \Omega$ and to identify η with its support. A *stationary partition* (based on η) is a pair (Z, π) consisting of a *measurable* set $Z : \Omega \to \mathcal{S}$ and a mapping $\pi : \Omega \times S \to S$ such that both Z and π are *covariant* and such that $\pi(x) \in \eta$ whenever $x \in Z$. We also assume that $\{Z = \emptyset\} = \{\eta = \emptyset\}$. Measurability of Z just means that $(\omega, x) \mapsto \mathbf{1}\{x \in Z(\omega)\}$ is measurable, while covariance of Z means that

$$Z(\theta_g \omega) = gZ(\omega), \quad \omega \in \Omega,\, g \in G. \tag{8.1}$$

Covariance of π is defined by

$$\pi(\theta_g \omega, gx) = g\pi(\omega, x), \quad \omega \in \Omega,\, x \in S,\, g \in G. \tag{8.2}$$

For convenience we also assume that $\pi(x) = x$, $x \in S$, whenever $\eta = \emptyset$. Define

$$C(\omega, x) := \{y \in Z(\omega) : \pi(\omega, y) = x\}, \quad \omega \in \Omega,\, x \in S. \tag{8.3}$$

Note that $C(x) = \emptyset$ whenever $x \notin \eta \neq \emptyset$. The system $\{C(x) : x \in \eta\}$ forms a partition of Z into measurable sets provided that $\eta \neq \emptyset$. Equations (8.1) and (8.2) imply the following covariance property:

$$C(\theta_g \omega, gx) = gC(\omega, x), \quad \omega \in \Omega,\, x \in S,\, g \in G. \tag{8.4}$$

Although we do not make any topological or geometrical assumptions we refer to $C(x)$ as *cell* with (generalized) *centre* $x \in \eta$. We do not assume that $x \in C(x)$ and some of the cells might be empty.

We now fix a σ-finite stationary measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$. The following theorem generalizes Theorem 7.1 in [7] from the case $S = G = \mathbb{R}^d$ to homogeneous spaces. The former case is also touched by Lemma 16 in [2].

Theorem 8.1. *Let (Z, π) be a stationary partition. Then we have for any measurable $f, \tilde{f} : \Omega \to \mathbb{R}_+$,*

$$\mathbb{E}_{\mathbb{P}} \mathbf{1}\{c \in Z\} \Delta^*(\pi(c)) \tilde{f} \int f(\theta_g^{-1}) \lambda_{\pi(c)}(dg) = \mathbb{E}_{\mathbb{P}_\eta} f \int \tilde{f}(\theta_g^{-1}) \mathbf{1}\{gc \in C(c)\} \lambda(dg). \tag{8.5}$$

Proof. Consider the random measure $\xi := \mu(Z \cap \cdot)$. By covariance (8.1) of Z and invariance of μ we have $\xi(\theta_g, gB) = \int \mathbf{1}\{x \in gB \cap gZ\} \mu(dx) = \xi(B)$ for all $g \in G$ and $B \in \mathcal{S}$. Hence ξ is invariant. An equally simple calculation shows that the Palm measure of ξ is given by

$$\mathbb{P}_\xi(A) = \mathbb{P}(A \cap \{c \in Z\}), \quad A \in \mathcal{F}. \tag{8.6}$$

Define weighted transport-kernels T and T^* by $T(x,\cdot) := \delta_{\pi(x)}$ and $T^*(x,\cdot) := \mu(C(x)\cap\cdot)$. Since $\pi(x) \in \eta$ whenever $x \in Z$, it is straigthforward to check that (4.4) holds even for all $\omega \in \Omega$. By (8.2), T is invariant. Invariance of T^* follows from (8.4) and invarance of μ:

$$T^*(\theta_g, gx, gB) = \mu(C(\theta_g, gx) \cap gB) = \mu(gC(x) \cap gB) = T^*(x, B).$$

Theorem 4.1 implies that (4.5) holds. Applying this formula to the measurable function $(\omega, g) \mapsto f(\omega)\tilde{f}(\theta_g^{-1}\omega)$ and taking into account (8.6) as well as the definitions of T and T^*, yields the assertion (8.5). ☐

Putting $\tilde{f} \equiv 1$ in (8.5) yields:

Corollary 8.2. *Let (Z, π) be a stationary partition. Then we have for any measurable $f : \Omega \to \mathbb{R}_+$,*

$$\mathbb{E}_{\mathbb{P}} \mathbf{1}\{c \in Z\} \int f(\theta_g^{-1})\Delta^*(\pi(c))\lambda_{\pi(c)}(dg) = \mathbb{E}_{\mathbb{P}_\eta} \mu(C(c)) f. \tag{8.7}$$

Under additional assumptions on f and $\tilde{f}$, Theorem 8.1 can be simplified as follows.

Theorem 8.3. *Let (Z, π) be a stationary partition and $f, \tilde{f} : \Omega \times S \to \mathbb{R}_+$ be invariant and measurable. Then*

$$\mathbb{E}_{\mathbb{P}} \mathbf{1}\{c \in Z\}\Delta^*(\pi(c))\tilde{f}(c)f(\pi(c)) = \mathbb{E}_{\mathbb{P}_\eta} f \int_{C(c)} \tilde{f}(x)\mu(dx). \tag{8.8}$$

Proof. Apply (8.5) with f (resp. $\tilde{f}$) replaced by $f(\theta_e, c)$ (resp. $\tilde{f}(\theta_e, c)$). ☐

Corollary 8.4. *Let (Z, π) be a stationary partition and $f : \Omega \times S \to \mathbb{R}_+$ be invariant and measurable. Then*

$$\mathbb{E}_{\mathbb{P}} \mathbf{1}\{c \in Z\}\Delta^*(\pi(c))f(\pi(c)) = \mathbb{E}_{\mathbb{P}_\eta} f \mu(C(c)). \tag{8.9}$$

The special case $f \equiv 1$ of (8.9) gives

$$\mathbb{E}_{\mathbb{P}_\eta} \mu(C(c)) = \mathbb{E}_{\mathbb{P}} \mathbf{1}\{c \in Z\}\Delta^*(\pi(c)) \tag{8.10}$$

If η has a positive and finite intensity γ_η and G is unimodular, this yields the formula

$$\mathbb{E}_{\mathbb{P}^0_\eta} \mu(C(c)) = \mathbb{P}(c \in Z)\gamma_\eta^{-1}. \tag{8.11}$$

Define

$$V(x) := \{y \in S : \pi(y) = \pi(x)\}$$

as the cell containing $x \in S$.

Corollary 8.5. *Let (Z, π) be a stationary partition. Then we have for any $\beta \geq 0$ that*

$$\mathbb{E}_{\mathbb{P}} \mathbf{1}\{c \in Z\}\Delta^*(\pi(c))\mu(V(c))^\beta = \mathbb{E}_{\mathbb{P}_\eta} \mu(C(c))^{\beta+1}. \tag{8.12}$$

Proof. Define $f(\omega, x) := \mu(C(\omega, x))^\beta$. Equation (8.4) and invariance of μ imply that f is invariant. Hence we can apply (8.9). It remains to note that $C(\pi(c)) = V(c)$. $\square$

A stationary partition (Z, π) is called *proper*, if

$$\mathbb{P}_\eta(\mu(C(c) = 0) = \mathbb{P}_\eta(\mu(C(c) = \infty) = 0. \tag{8.13}$$

For unimodal groups the second equation is implied by (8.10). The following two results can be proved as in Section 5 of [7].

Proposition 8.6. *Let (Z, π) be a stationary and proper partition. Then we have for any measurable $f : \Omega \to \mathbb{R}_+$,*

$$\mathbb{E}_\mathbb{P}\mathbf{1}\{c \in Z\}\Delta^*(\pi(c))\mu(V(c))^{-1} f(\theta^{-1}_{\pi(c)}) = \mathbb{E}_{\mathbb{P}_\eta} f, \tag{8.14}$$

where $\theta^{-1}_{\pi(c)} : \Omega \to \Omega$ is defined by $\theta^{-1}_{\pi(c)}(\omega) := \theta^{-1}_{\pi(\omega,c)}\omega$.

Corollary 8.7. *Let (Z, π) be a stationary and proper partition. Then* (8.12) *holds for all $\beta \in \mathbb{R}$. If the intensity γ_η of η is finite, then we have in particular,*

$$\gamma_\eta = \mathbb{E}_\mathbb{P}\mathbf{1}\{c \in Z\}\Delta^*(\pi(c))\mu(V(c))^{-1}. \tag{8.15}$$

From now on we assume that η has a finite intensity and $\mathbb{P}(\hat{\eta} = 0) = 0$. Just for simplicity we also assume that $\mathbb{P}$ (and hence also $\mathbb{P}^*_\eta$) is a probability measure. We first note the following consequence of the proof of Theorem 8.1 and Corollary 4.3:

Corollary 8.8. *Under the hypothesis of Theorem 8.1 we have for any measurable $f, \tilde{f} : \Omega \to \mathbb{R}_+$,*

$$\mathbb{E}_\mathbb{P}\Big[\mathbf{1}\{c \in Z\}\Delta^*(\pi(c))\tilde{f}\int f(\theta_g^{-1})\lambda_{\pi(c)}(dg)\Big|\mathcal{I}\Big] = \hat{\eta}\,\mathbb{E}_{\mathbb{P}^*_\eta}\Big[f\int \tilde{f}(\theta_g^{-1})\mathbf{1}\{gc \in C(c)\}\lambda(dg)\Big|\mathcal{I}\Big], \tag{8.16}$$

$\mathbb{P}$-a.e. for any choice of the conditional expectations. In particular,

$$\hat{\eta}^{-1}\mathbb{E}_\mathbb{P}\Big[\mathbf{1}\{c \in Z\}\Delta^*(\pi(c))\int f(\theta_g^{-1})\lambda_{\pi(c)}(dg)\Big|\mathcal{I}\Big] = \mathbb{E}_{\mathbb{P}^*_\eta}[f\mu(C(c))|\mathcal{I}]. \tag{8.17}$$

Let $\alpha > 0$. Essentially following [2] (dealing with the case $S = G = \mathbb{R}^d$), we call a stationary partition (Z, π) (based on η) *α-balanced*, if

$$\mathbb{P}(\mu(C(x)) = \alpha\hat{\eta}^{-1} \text{ for all } x \in \eta) = 1. \tag{8.18}$$

The significance of α-balanced stationary partitions is due to the following theorem. The result extends Theorem 13 in [3] and Theorem 9.1 in [7] (both dealing with $\alpha = 1$) from $\mathbb{R}^d$ to general homogeneous spaces.

Theorem 8.9. *Let $\alpha > 0$. A stationary partition (Z, π) is α-balanced iff*

$$\mathbb{P}^*_\eta = \alpha^{-1}\mathbb{E}_\mathbb{P}\mathbf{1}\{c \in Z\}\Delta^*(\pi(c))\int \mathbf{1}\{\theta_g^{-1} \in \cdot\}\lambda_{\pi(c)}(dg). \tag{8.19}$$

Proof. If (Z, π) is a α-balanced stationary partition, then (8.19) follows from (8.17).

Assume now that (8.19) holds. Since $\mathbb{P}^*_\eta$ has the invariant density $\hat{\eta}$ with respect to $\mathbb{P}_\eta$, (8.19) implies that

$$\mathbb{P}_{\alpha\eta} = \mathbb{E}_{\mathbb{P}} \mathbf{1}\{c \in Z\} \Delta^*(\pi(c)) \hat{\eta} \int \mathbf{1}\{\theta_g^{-1} \in \cdot\} \lambda_{\pi(c)}(dg), \tag{8.20}$$

where we have also used that $\mathbb{P}_{\alpha\eta} = \alpha \mathbb{P}_\eta$. Using the invariant weighted transport-kernels $T(x, \cdot) := \hat{\eta} \mathbf{1}\{x \in Z\} \delta_{\pi(x)}$ this reads,

$$\mathbb{P}_{\alpha\eta} = \mathbb{E}_{\mathbb{P}} \iint \mathbf{1}\{\theta_g^{-1} \in \cdot\} \Delta^*(x) \lambda_x(dg) T(c, dx).$$

Since $\mathbb{P}$ is the Palm measure of μ, we get from Theorem 5.2 that T is $\mathbb{P}$-a.e. $(\mu, \alpha\eta)$-balancing. Therefore we have $\mathbb{P}$-a.e. that

$$\int \mathbf{1}\{x \in Z, \pi(x) \in \cdot\} \mu(dx) = \alpha \hat{\eta}^{-1} \eta(\cdot).$$

This is just saying that (Z, π) is α-balanced. □

Remark 8.10. If a α-balanced stationary partition (Z, π) is given, then (8.19) provides an explicit method for constructing the modified Palm probability measure $\mathbb{P}^*_\eta$ by a *shift-coupling* with the stationary measure $\mathbb{P}$. In case $S = G$ is a unimodal group, (8.19) simplifies to

$$\mathbb{P}^*_\eta = \alpha^{-1} \mathbb{E}_{\mathbb{P}} \mathbf{1}\{e \in Z\} \mathbf{1}\{\theta_{\pi(e)}^{-1} \in \cdot\}. \tag{8.21}$$

Since $\alpha = \mathbb{P}(e \in Z)$, this means that $\mathbb{P}^*_\eta = \mathbb{P}(\theta_{\pi(e)}^{-1} \in \cdot | e \in Z)$.

The actual construction of α-balanced partitions is an interesting topic in its own right. Triggered by [10], the case $S = G = \mathbb{R}^d$ was discussed in [3, 2]. Among many other things it was shown there that α-balanced partitions do actually exist for any $\alpha \leq 1$. The occurence of the sample intensity $\hat{\eta}$ in (8.18) is explained by the spatial ergodic theorem, see Proposition 9.1 in [7]. The paper [3] has also results on discrete groups in case $\alpha = 1$. In case of a general homogeneous space with a diffuse invariant measure μ it might be conjectured that α-balanced partitions exist for all $\alpha \leq 1$. For a special case we will establish this in the final section of this paper.

9 Allocations to isotropic flat processes

In this section we will consider balanced stationary partitions in the case where S is the affine Grassmannian $A(d, k)$ of all k-dimensional affine subspaces (k-flats) of $\mathbb{R}^d$. Any $F \in A(d, k)$ can be uniquely written as $F = L + x$, where L is an element of the linear Grassmannian $L(d, k)$ of all k-dimensional linear subspaces of $\mathbb{R}^d$ and x is in the orthogonal complement $L^\perp$ of L. We let G_d denote the goup of *rigid motions* on $\mathbb{R}^d$. Any $g \in G_d$ can be written as the composition of a rotation $\vartheta \in SO_d$ and a translation t_x, $x \in \mathbb{R}^d$. Here SO_d is the special orthogonal group and $t_x : \mathbb{R}^d \to \mathbb{R}^d$ is defined by $t_x(y) := x + y$. This representation can be used to introduce a natural topology on G_d, making G_d a locally compact second countable topological group. Moreover, a left- and rightinvariant Haar measure λ on G is given by

$$\lambda = \int_{SO_d} \int_{\mathbb{R}^d} \mathbf{1}\{t_x \circ \vartheta \in \cdot\} \lambda^d(dx)\nu(d\vartheta),$$

where λ^d is Lebesgue measure and ν is the normalized Haar measure on SO_d, see [16] for more detail. Note that $\lambda(B \times SO_d) = 1$ whenever $\lambda^d(B) = 1$. The group G_d is acting on $A(d,k)$ by

$$gF := \{gx : x \in F\}, \quad g \in G_d,\ F \in A(d,k).$$

Under this action $A(d,k)$ becomes a homogeneous space. We fix a subspace $L_0 \in L(d,k)$ and define an invariant measure μ on $A(d,k)$ by

$$\mu := \int \mathbf{1}\{gL_0 \in \cdot\}\lambda(dg).$$

Let $\mathbb{P}$ be a stationary probability measure on $(\Omega, \mathcal{F})$ and η an invariant simple point process on $A(d,k)$. Then η is an *isotropic* and stationary process of k-flats, see [16]. The Palm measure $\mathbb{P}_\xi$ is defined by (3.9), with L_0 serving as the reference point c.

Theorem 9.1. *Assume that η has a finite intensity and a positive sample intensity. Let $\alpha \in (0,1]$. Then there exists an α-balanced stationary partition (Z,π) based on η.*

Proof. Our proof is based on some of the arguments used by the authors of [2] to prove their Theorems 1 and 3. (These theorems deal with the special case $S = \mathbb{R}^d$, but make stronger assertions.) To do so, we need an invariant metric ρ on $A(d,k)$, i.e. a metric generating the topology on $A(d,k)$ and having $\rho(gF, gF') = \rho(F,F')$ for all $F, F' \in A(d,k)$ and all $g \in G_d$. If $F = x + L$ and $F' = x' + L'$ for $L, L' \in G(d,k)$, $x \perp F$, and $x' \perp F'$, we can take $\rho(F,F')$ as the sum of the Euclidean distance between x and x' and a SO_d-invariant distance between L and L'. We can now use the *site-optimal Gale-Shapley algorithm* as in [2] to construct a stationary partition (Z,π) with the following properties

$$0 < \mu(C(F)) \leq \alpha\hat{\eta}^{-1}, \quad F \in \eta, \tag{9.1}$$

and

$$\{\eta' \neq \emptyset\} \cap \{Z \neq A(d,k)\} = \emptyset, \tag{9.2}$$

where the invariant point process η' is given by

$$\eta' := \{F \in \eta : \mu(C(F)) < \alpha\hat{\eta}^{-1}\}. \tag{9.3}$$

By (8.17) we have $\mathbb{P}$-almost surely that

$$\mathbb{P}(L_0 \in Z|\mathcal{I}) = \mathbb{E}_{\mathbb{P}^*_\eta}[\hat{\eta} \cdot \mu(C(L_0))|\mathcal{I}], \tag{9.4}$$

while (9.2) entails $Z = A(d,k)$ on the invariant event $\{\eta' \neq \emptyset\}$. Hence (9.4) implies that

$$\mathbb{E}_{\mathbb{P}^*_\eta}[\hat{\eta} \cdot \mu(C(L_0)) = \alpha|\mathcal{I}] = 1 \quad \mathbb{P} - \text{a.s. on } \{\eta' \neq \emptyset\}. \tag{9.5}$$

On the other hand, (9.1) implies

$$\mathbb{P}^*_\eta(\hat{\eta}\cdot\mu(C(L_0))=\alpha|\mathcal{I})=1 \quad \mathbb{P}-\text{a.s. on } \{\eta'=\emptyset\}. \tag{9.6}$$

This yields

$$\mathbb{P}^*_\eta(\hat{\eta}\cdot\mu(C(L_0))=\alpha)=1 \tag{9.7}$$

in case $\alpha=1$. Assume $\alpha<1$. Then (9.4) and (9.1) imply that $\mathbb{P}(L_0\in Z)\le\alpha<1$, so that $\mathbb{P}(Z\ne A(d,k))=1$. Therefore (9.2) yields $\mathbb{P}(\eta'=\emptyset)=1$, so that (9.6) implies (9.7) also in case $\alpha<1$.

To finish the proof, we use the definition (3.20) of $\mathbb{P}^*_\eta$ together with invariance of μ and (8.4) to conclude

$$0=\mathbb{E}_{\mathbb{P}}\hat{\eta}\int_B \mathbf{1}\{\hat{\eta}\cdot\mu(C(F))\ne\alpha\}\eta(dF)$$

for all measurable B having $0<\mu(B)<\infty$. This yields $\mathbb{P}$-a.s. that

$$\int \mathbf{1}\{\hat{\eta}\cdot\mu(C(F))\ne\alpha\}\eta(dF)=0,$$

as desired. □

Remark 9.2. The stationary partition (Z,π) used in the proof of Theorem 9.1 has an interesting property of *stability* (see [2]) expressed in terms of the distance ρ. Each centre in η desires to have the flats in its cell as close as possible, while each flat in $A(d,k)$ would like to have its centre as close as possible. A pair $(E,F)\in A(d,k)\times\eta$ is called *unstable* if it has the following two properties. The first property is that $\rho(E,F)<\rho(E,\pi(E))$ or $E\notin Z$, meaning that E prefers F to its actual centre $\pi(E)$. (In particular $E\notin C(F)$.) The second property is that $\rho(E,F)<\rho(E',F)$ for some $E'\in C(F)$ or $\mu(C(F))<\alpha$, meaning that F would like to have E in its cell. It can be proved as in [2] that unstable pairs do not exist almost surely.

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