STRUCTURALLY DAMPED PLATE AND WAVE EQUATIONS WITH RANDOM POINT FORCE IN ARBITRARY SPACE DIMENSIONS

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Abstract. In this paper we consider structurally damped plate and wave equations with point and distributed random forces. In order to treat space dimensions more than one, we work in the setting of L^q–spaces with (possibly small) $q \in (1,2)$. We establish existence, uniqueness and regularity of mild and weak solutions to the stochastic equations employing recent theory for stochastic evolution equations in UMD Banach spaces.

1. Introduction

Structurally damped plate and wave equations have been studied intensively in the deterministic case. In such equations the damping term has 'half of the order' of the leading elastic term, as it has been proposed in the seminal paper [29]. Point controls and feedbacks in elastic systems lead naturally to perturbations of damped equations by Dirac measures, cf. [22]. In this paper we investigate the situation when such point perturbations act randomly. For a one dimensional spatial domain S these problems have been treated in [24] by means of the well-established Hilbert space approach to stochastic partial differential equations, see e.g. [14]. However, it seems that in higher space dimensions $d \geq 2$ one cannot proceed in this way since the irregularity coming from the point measure and the stochastic terms cannot be balanced by the smoothing effect of the analytic semigroup for the damped plate equation. The point evaluation acts via duality on the state space so that it becomes even more singular if one works in the setting of $L^q(S)$ with $q > 2$. Thus it is natural to look for solutions in $L^q(S)$ with $q \in (1,2)$; in fact, we need $q \in (1, d/(d-1))$ for the plate equation and $q \in (1, 2d/(2d-1))$ for the wave equation.

Several authors have investigated stochastic partial differential equations on L^q spaces with $q \in [2,\infty)$ (see [6, 21] and the references therein), whereas in our case $q \in (1,2)$ it seems that the only known method is contained in the recent paper [28]. Our analysis is based on the theory developed in [28]. Stochastic damped wave equations have been treated in various papers during the last years, see e.g. [3, 4, 10, 13, 17, 30]. However, it seems that random forces acting at a single point in S have been studied only in [24] so far.

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To be concise, we will focus on the model

$$
(1.1)\quad \begin{cases} \ddot u(t,s) + \Delta^2 u(t,s) - \rho\Delta \dot u(t,s) = f(t,s,u(t,s),\dot u(t,s)) \\ \qquad + b(t,s,u(t,s),\dot u(t,s))\,\dfrac{\partial w_1(t,s)}{\partial t} + \Big[G(t,u(t,\cdot),\dot u(t,\cdot)) \\ \qquad + C(t,u(t,\cdot),\dot u(t,\cdot))\dfrac{\partial w_2(t)}{\partial t}\Big]\delta(s-s_0), \qquad t\in[0,T],\ s\in S, \\ u(0,s) = u_0(s),\ \dot u(0,s) = u_1(s), \qquad s\in S, \\ u(t,s) = \Delta u(t,s) = 0, \qquad t\in[0,T],\ s\in\partial S, \end{cases}
$$

on a bounded domain $S\subset\mathbb{R}^d$ of class C^4. Here $\rho>0$ is a constant and $\delta(\cdot - s_0)$ is the point mass at $s_0\in S$. The functions f,b,G,C are measurable, adapted and Lipschitz in a sense specified in Section 5. The process w_1 is a Gaussian process which is white in time and appropriately colored in space, as discussed in Section 5. The process w_2 is a standard one–dimensional Brownian motion which is independent of w_1. Note that w_2 drives the point loading whereas w_1 governs a distributed stochastic term.

In Theorem 5.1 we obtain a mild and weak solution $(u(t),\dot u(t))\in(W^{2,q}(S)\cap W^{1,q}_0(S))\times L^q(S)$ of (1.1), where u and $\dot u$ possess some additional regularity in time and in space if the initial data are regular enough. We also state a related result for the wave equation in Theorem 6.1. Our results can be generalized in various directions. For instance, in (1.1) one could replace the Dirichlet Laplacian by a more general elliptic operator. One can also allow for more general nonlinearities, see Remarks 5.2 and 5.8, and one could treat locally Lipschitz coefficients to some extend, see Remark 5.9. But for conciseness we will focus on the setting indicated above.

In Sections 2-4 we provide the necessary prerequisites for our main results. First, we briefly discuss the theory of stochastic integration developed in [27]. This theory is closely tied to the concept of Gauss functions and operators which is also presented in Section 2. Based on this material, in Section 3 we recall a theorem on existence, uniqueness and regularity of mild solutions of parabolic stochastic equations from [28]. In this theorem it is possible to consider deterministic and stochastic terms taking values in so–called extrapolation spaces which are larger than the state space. This fact is crucial for our approach since the Dirac functional $\delta(\cdot - s_0)$ lives in such extrapolation spaces. Moreover, one can use this flexibility to extend the class of admissible processes w_1, see the examples in Section 5.

The underlying deterministic equation is studied in Section 4, where we consider the problem

$$
(1.2)\quad \begin{aligned} &\ddot u(t) + \rho\mathcal{A}^{\frac12}\dot u(t) + \mathcal{A}u(t) = 0, \qquad t\ge 0, \\ &u(0) = u_0, \qquad \dot u(0) = u_1, \end{aligned}
$$

for a sectorial operator $\mathcal{A}$ on a Banach space E, see (4.3). (In (1.1) $\mathcal{A}$ is the square of the Dirichlet Laplacian on $E = L^q(S)$.) Using the operator matrix

$$
A = \begin{pmatrix} 0 & I \\ -\mathcal{A} & -\rho\mathcal{A}^{\frac12}\end{pmatrix} \qquad \text{with} \quad D(A) = D(\mathcal{A})\times D(\mathcal{A}^{\frac12})
$$

one can reformulate (1.2) as an abstract Cauchy problem on the state space $X = D(\mathcal{A}^{\frac12})\times E$. It is well known that A generates an analytic semigroup on X if $E = L^2(S)$, see [11], [22]. Recently, it has been shown in [9] that A also generates an analytic semigroup in the Banach space case. (See [12] and [19] for related results.) In view of the stochastic problem, it is crucial to determine the inter- and extrapolation spaces for this semigroup, see Proposition 4.1. It would be very interesting to extend these results to damping terms which are more general than $\rho\mathcal{A}^{\frac{1}{2}}$. In the Hilbert space case this is possible to some extend, see [11] and [22], but this approach makes heavy use of the Hilbert space structure. Let us explain the problem with an example, cf. [29]. In equation (1.1) it would be interesting to study also the clamped plate equation, where $u = \frac{\partial u}{\partial n} = 0$ on ∂S and n denotes the outer normal. If we let $\mathcal{A} = \Delta^2$ with the above boundary conditions, then $\mathcal{A}^{\frac{1}{2}}$ is not a differential operator anymore. Instead of $\mathcal{A}^{\frac{1}{2}}\dot{u}$, we would still like to have $\Delta\dot{u}$ as a damping term, but this does not lead to the algebraic structure of (1.2). Therefore, we do not know whether the corresponding operator matrix generates a strongly continuous semigroup if $q \neq 2$.

We will write $a \lesssim b$ if there exists a universal constant $C > 0$ such that $a \leq Cb$, and $a \eqsim b$ if $a \lesssim b \lesssim a$. If the constant C is allowed to depend on some parameter θ, we write $a \lesssim_\theta b$ and $a \eqsim_\theta b$ instead. Moreover, X always denotes a Banach space, $\mathcal{B}(X, Y)$ is the space of bounded linear operators from X to another Banach space Y, and we designate the norm in X and the operator norm by $\|\cdot\|$.

2. Preliminaries

Throughout this paper $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$. This space is used for the stochastic equations and stochastic integrals below. In Subsection 2.2 we recall the necessary definitions and facts from the theory of stochastic integration developed in [27]. As a preparation, we discuss Gauss operators and functions in the next subsection referring to [5, 16, 20] for proofs and more details. In the last subsection we describe a concept of Lipschitz continuity which is crucial for our work.

2.1. Gauss operators.

In this paper, $(\gamma_n)_{n\geq 1}$ always denotes a *Gaussian sequence*, i.e., a sequence of independent, standard, real-valued Gaussian random variables defined on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. A linear operator $R : \mathcal{H} \to X$ from a separable real Hilbert space $\mathcal{H}$ into a Banach space X is called *a Gauss operator* if for some (and then for every) orthonormal basis $(h_n)_{n\geq 1}$ of $\mathcal{H}$ the Gaussian sum $\sum_{n\geq 1} \gamma_n R h_n$ converges in $L^2(\tilde{\Omega}; X)$. In other papers R is sometimes called a radonifying operator. The space $\gamma(\mathcal{H}, X)$ of all Gauss operators from $\mathcal{H}$ to X is a Banach space with respect to the norm

$$\|R\|_{\gamma(\mathcal{H},X)} := \Big(\mathbb{E}\Big\|\sum_{n\geq 1} \gamma_n R h_n\Big\|^2\Big)^{\frac{1}{2}}.$$

This norm is independent of the orthonormal basis $(h_n)_{n\geq 1}$ and the Gaussian sequence $(\gamma_n)_{n\geq 1}$. It holds that $\|R\| \leq \|R\|_{\gamma(\mathcal{H},X)}$. Moreover, $\gamma(\mathcal{H}, X)$ is an operator ideal in the sense that if $S_1 : \tilde{\mathcal{H}} \to \mathcal{H}$ and $S_2 : X \to \tilde{X}$ are bounded operators, then $R \in \gamma(\mathcal{H}, X)$ implies $S_2 R S_1 \in \gamma(\tilde{\mathcal{H}}, \tilde{X})$ and

$$\|S_2 R S_1\|_{\gamma(\tilde{\mathcal{H}},\tilde{X})} \leq \|S_2\| \|R\|_{\gamma(\mathcal{H},X)} \|S_1\|. \tag{2.1}$$

If X is a Hilbert space, then $\gamma(\mathcal{H}, X)$ is isometrically isomorphic to the space $\mathcal{S}^2(\mathcal{H}, X)$ of Hilbert–Schmidt operators from $\mathcal{H}$ into X.

We are mainly interested in the case that $\mathcal{H} = L^2(M;H)$, where H is another separable real Hilbert space with inner product $[\cdot,\cdot]_H$ and (M,Σ,μ) is a σ-finite measure space. Let $\Phi : M \to \mathcal{B}(H,X)$. Assume that $\Phi^* x^* \in L^2(M;H)$ for all $x^* \in X^*$ and that there exists an $R \in \gamma(L^2(M;H),X)$ such that

$$\langle Rf, x^*\rangle = \int_M [f(t), \Phi^*(t)x^*]_H \, d\mu(t)$$

for all $f \in L^2(M;H)$ and $x^* \in X^*$. Then we say that R is *represented* by Φ. In this case Φ is called a *Gauss function*, and we write $\Phi \in \gamma(M;H,E)$ and

$$\|\Phi\|_{\gamma(M;H,X)} := \|R\|_{\gamma(L^2(M;H),X))}.$$

We write $\gamma(M;X)$ instead of $\gamma(M;\mathbb{R},X)$. If there is no danger of confusion we will identify R and Φ, cf. Subsection 2.3 in [27]. For a Hilbert space X, we have $\gamma(M;H,X) = L^2(M;\mathcal{S}^2(H,X))$ isometrically.

For the space $X = L^p(S)$, $p \in [1,\infty)$, the following square function estimate gives a useful way to verify that $\Phi : M \to \mathcal{B}(H,X)$ is a Gauss function, see [26, Proposition 6.1]:

$$\|\Phi\|_{\gamma(M;H,X)} \eqsim_p \Big\| \Big(\int_M \sum_{n\geq 1} |\Phi(t)h_n|^2 \, d\mu(t)\Big)^{\frac12} \Big\|_{L^p(S)}. \tag{2.2}$$

2.2. **Stochastic integration in UMD spaces.** We now discuss the stochastic integral for processes $\Phi : [0,T]\times\Omega \to \mathcal{B}(H,X)$ as it was introduced and investigated in [27]. Here X is a UMD Banach space and H is a separable real Hilbert space. The reader is referred to [8] and [27] concerning UMD spaces. But, for the present paper, it suffices to recall that the reflexive L^q, Sobelev, Bessel–potential and Besov spaces are UMD spaces.

We denote by $L^0(\Omega;E)$ the vector space of all equivalence classes of measurable functions from Ω to a Banach space E, and we endow $L^0(\Omega;E)$ with the convergence in probability. A process $\Phi : [0,T]\times\Omega \to \mathcal{B}(H,X)$ is called H*–strongly measurable* if Φh is strongly measurable in X for all $h \in H$, where we let $(\Phi h)(t,\omega) := \Phi(t,\omega)h$. The process Φ is called H*–strongly adapted* if the map $\omega \mapsto \Phi(t,\omega)h$ is $\mathcal{F}_t$–strongly measurable for all $t \in [0,T]$ and $h \in H$. We also set $\Phi_\omega(t) = \Phi(t,\omega)$.

An H*-cylindrical Brownian motion* is a family $W_H = (W_H(t))_{t\in[0,T]}$ of bounded linear operators from H to $L^2(\Omega)$ satisfying

(1) $W_H h = (W_H(t)h)_{t\in[0,T]}$ is a real-valued Brownian motion for each $h \in H$,
(2) $\mathbb{E}(W_H(s)g \cdot W_H(t)h) = (s\wedge t)\,[g,h]_H$ for all $s,t \in [0,T]$ and $g,h \in H$.

Let $0 \le a < b < T$, $A \subset \Omega$ be $\mathcal{F}_a$-measurable, $x \in X$, and $h \in H$. The stochastic integral of the indicator process $1_{(a,b]\times A} \otimes (h\otimes x)$ is then defined as

$$\int_0^T 1_{(a,b]\times A} \otimes (h\otimes x) \, dW_H := 1_A(W_H(b)h - W_H(a)h)x.$$

(Analogously, one defines the integral for the trivial process $1_{[0]\times A} \otimes (h\otimes x)$.) By linearity, this definition extends to adapted step processes $\Phi : [0,T]\times\Omega \to \mathcal{B}(H,X)$ whose values are finite rank operators. An H-strongly measurable and adapted process Φ is called *stochastically integrable* with respect to W_H if there exists a sequence of adapted step processes $\Phi_n : [0,T]\times\Omega \to \mathcal{B}(H,X)$ with values in the finite rank operators from H to X and a pathwise continuous process $\zeta : [0,T]\times\Omega \to X$ such that the following two conditions are satisfied:

(1) $\lim_{n\to\infty}\langle \Phi_n h, x^*\rangle = \langle \Phi h, x^*\rangle$ in measure on $[0,T]\times\Omega$ for all $h\in H$ and $x^*\in X^*$;

(2) $\displaystyle\lim_{n\to\infty}\int_0^\cdot \Phi_n(t)\,dW_H(t) = \zeta \quad \text{in } L^0(\Omega; C([0,T];X))$.

In this situation, ζ is uniquely determined as an element of $L^0(\Omega;C([0,T];X))$ and it is called the *stochastic integral* of Φ with respect to W_H. We write

$$\zeta = \int_0^\cdot \Phi\,dW_H = \int_0^\cdot \Phi(t)\,dW_H(t).$$

The process ζ is a continuous local martingale starting at zero, see [27, Theorem 5.5].

Proposition 2.1. [27, Theorems 5.9 and 5.12] *Let X be a UMD space. For an H-strongly measurable and adapted process $\Phi:[0,T]\times\Omega\to\mathcal{B}(H,X)$ the following assertions are equivalent.*

(1) *The process Φ is stochastically integrable with respect to W_H.*

(2) *For all $x^*\in X^*$ the process Φ^*x^* belongs to $L^0(\Omega;L^2(0,T;H))$, and there exists a pathwise continuous process $\zeta\in L^0(\Omega;C([0,T],X))$ such that for all $x^*\in X^*$ we have*

$$\langle\zeta,x^*\rangle = \int_0^\cdot \Phi^*x^*\,dW_H \qquad \text{in } L^0(\Omega;C([0,T]));$$

(3) *$\Phi_\omega\in\gamma(0,T;H,X)$ for a.e. $\omega\in\Omega$.*

In this situation we have $\zeta=\int_0^\cdot \Phi\,dW_H$ in $L^0(\Omega;C([0,T];X))$. Furthermore, for all $p\in(1,\infty)$,

$$\mathbb{E}\sup_{t\in[0,T]}\Big\|\int_0^t \Phi\,dW_H\Big\|^p \eqsim_{p,X} \mathbb{E}\|\Phi\|^p_{\gamma(0,T;H,X)}.$$

2.3. **L^2_γ-Lipschitz functions.** We now treat a class of Lipschitz functions which is needed in the existence result for the stochastic equation presented in the next section. See [28] for more details.

Let (M,Σ) be a countably generated measurable space and let μ be a finite measure on (M,μ). Then $L^2(M,\mu)$ is separable. We then define

$$L^2_\gamma(M,\mu;X) := \gamma(M,\mu;X)\cap L^2(M,\mu;X),$$

which is a Banach space endowed with the norm

$$\|\phi\|_{L^2_\gamma(M,\mu;X)} := \|\phi\|_{\gamma(M,\mu;X)} + \|\phi\|_{L^2(M,\mu;X)}.$$

Note that the simple functions are dense in $L^2_\gamma(M,\mu;X)$.

Let H be a separable real Hilbert space, let X_1 and X_2 be Banach spaces, and let $f: M\times X_1\to\mathcal{B}(H,X_2)$ be a function such that for all $x\in X_1$ we have $f(\cdot,x)\in\gamma(L^2(M,\mu;H),X_2)$. For simple functions $\phi:M\to X_1$ one easily checks that the map $s\mapsto f(s,\phi(s))$ belongs to $\gamma(L^2(M,\mu;H),X_2)$. We call f an *L^2_γ–Lipschitz function with respect to μ* if f is strongly continuous in the second variable and we have

$$\|f(\cdot,\phi_1)-f(\cdot,\phi_2)\|_{\gamma(L^2(M,\mu;H),X_2)} \le C\|\phi_1-\phi_2\|_{L^2_\gamma(M,\mu;X_1)} \tag{2.3}$$

for a constant $C\ge 0$ and all simple functions $\phi_1,\phi_2:M\to X_1$. In this case the mapping $\phi\mapsto f(\cdot,\phi(\cdot))$ extends uniquely to a Lipschitz mapping from $L^2_\gamma(M,\mu;X_1)$

into $\gamma(L^2(M,\mu;H),X_2)$. Its Lipschitz constant will be denoted by $L^\gamma_{\mu,f}$. Finally, if f is L^2_γ-Lipschitz with respect to all finite measures μ on (M,Σ) and

$$L^\gamma_f := \sup\{L^\gamma_{\mu,f} : \mu \text{ is a finite measure on } (M,\Sigma)\}$$

is finite, then we say that f is a L^2_γ*–Lipschitz function.*

In the next lemma we state a simpler sufficient condition for the L^2_γ–Lipschitz property. However, for this result one has to impose an additional restriction on the Banach space X_2 which we first introduce. Let $p \in [1,2]$, and let $(r_j)_{j\geq 1}$ be a Rademacher sequence, i.e., $(r_j)_{j\geq 1}$ is an independent, identically distributed sequence with $\mathbb{P}(r_1 = 1) = \mathbb{P}(r_1 = -1) = \frac{1}{2}$. A Banach space X has *type* p if there exists a constant $C_p \geq 0$ such that for all $x_1, \dots, x_n \in E$ we have

$$\Big(\mathbb{E}\Big\|\sum_{j=1}^n r_j\, x_j\Big\|^2\Big)^{\frac12} \leq C_p\Big(\sum_{j=1}^n \|x_j\|^p\Big)^{\frac1p}. \tag{2.4}$$

For more information on this concept we refer the reader to [16] and the references therein. We recall that every Banach space has type 1, that the spaces $L^p(S)$, $1 \leq p < \infty$, have type $\min\{p,2\}$ and that Hilbert spaces have type 2. The property has a certain ordering: If X has type p, then X has type $\tilde p$ for all $1 \leq \tilde p < p$ as well. Furthermore, every UMD space has nontrivial type, i.e., type p for some $p \in (1,2]$. But we will not need this fact. In type 2 spaces the L^2_γ–Lipschitz property can be checked using only the norm in $\gamma(H,X_2)$.

Lemma 2.2. [28, Lemma 5.2] *Let X_2 be a space with type 2. Let $f : M \times X_1 \to \gamma(H,X_2)$ be a function such that $f(\cdot,x)$ is strongly measurable for each $x \in X_1$. If there is a constant C such that*

$$\|f(s,x)\|_{\gamma(H,X_2)} \leq C(1 + \|x\|), \tag{2.5}$$

$$\|f(s,x) - f(s,y)\|_{\gamma(H,X_2)} \leq C\|x-y\| \tag{2.6}$$

for all $s \in M$ and $x,y \in X_1$, then f is a L^2_γ–Lipschitz function. We also have $L^\gamma_f \leq C_2 C$ for the constant C_2 from (2.4)*. Moreover, f satisfies*

$$\|f(\cdot,\phi)\|_{\gamma(L^2(M,\mu;H),X_2)} \leq C_2 C(1 + \|\phi\|_{L^2(M,\mu;X_1)}).$$

If f does not depend on M, one can check that (2.3) implies (2.5) and (2.6). Clearly, every L^2_γ-Lipschitz function $f : X_1 \to \gamma(H,X_2)$ is a Lipschitz function. The converse does not hold (see [25, Theorem 1]). The next example shows that standard substitution operators are L^2_γ–Lipschitz.

Example 2.3. [28, Example 5.5] Let $p \in [1,\infty)$, (M,Σ,μ) be a finite measure space, and $b : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous. Define the Nemytskii map $B : L^p(M) \to L^p(M)$ by $B(\varphi)(s) := b(\varphi(s))$. Then B is L^2_γ–Lipschitz with respect to μ.

3. The abstract stochastic evolution equation

Recall that $(\Omega,\mathcal{F},\mathbb{P})$ is a probability space with filtration $(\mathcal{F}_t)_{t\in[0,T]}$. Let H_1 and H_2 be separable real Hilbert spaces. Let X be a UMD Banach space and let Y be a Banach space. On the Banach space X we consider the problem

$$\text{(SE)}\qquad \begin{cases} dU(t) = (AU(t) + F(t,U(t)) + \Lambda_G G(t,U(t)))\,dt \\ \qquad\qquad + B(t,U(t))\,dW_{H_1}(t) + \Lambda_C C(t,U(t))\,dW_{H_2}(t), \ t\in[0,T], \\ U(0) = U_0. \end{cases}$$

We assume that A generates an analytic C_0–semigroup $(S(t))_{t\geq 0}$ on X. Thus, there are constants $M \geq 1$ and $w_0 \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{w_0 t}$ for $t \geq 0$. Let $w > w_0$. Our further assumptions make use of the fractional power scale associated to A, see e.g. [1]. For $a \in [0,1]$, we define the space $X_a = D((w-A)^a)$ with the norm $\|x\|_a = \|(w-A)^a x\|$. For $\theta \in [0,1]$, we further introduce the extrapolation space $X_{-\theta}$ which is the completion of X with respect to the norm $\|x\|_{-\theta} = \|(w-A)^{-\theta}x\|$. The operator A has a restriction (extension) to an operator on the space X_a (the space $X_{-\theta}$) which generates the analytic C_0–semigroup given by the restrictions (extensions) of $S(t)$ on the space X_a (the space $X_{-\theta}$). We usually denote the restrictions and extensions again by A and $S(t)$. Moreover, $(w-A)^\beta$ is an isomorphism from X_α to $X_{\alpha-\beta}$, where $-1 \leq \alpha - \beta \leq \alpha \leq 1$. Finally, X_α is continuously embedded into $X_{\alpha-\beta}$.

Going back to (SE), we now list the assumptions on the linear operators $\Lambda_j : Y \to X_{-\theta_j}$ for $j = G, C$ and on the functions

$$F : [0,T] \times \Omega \times X_a \to X, \qquad G : [0,T] \times \Omega \times X_a \to Y,$$
$$B : [0,T] \times \Omega \times X_a \to \mathcal{B}(H_1, X_{-\theta_B}), \qquad C : [0,T] \times \Omega \times X_a \to \mathcal{B}(H_2, Y).$$

Here the exponents $a, \theta_G, \theta_B, \theta_C$ belong to $[0,1]$, but in the next theorem we impose further restrictions. Moreover, the initial value $U_0 : \Omega \to X_a$ has to be strongly $\mathcal{F}_0$–measurable. The interval $[0,T]$ is endowed with the Borel σ–algebra $\mathcal{B}_{[0,T]}$.

(H1) A generates an analytic strongly continuous semigroup $(S(t))_{t\geq 0}$ on X.

(H2) The map $(t,\omega) \mapsto F(t,\omega,x) \in X$ is strongly measurable and adapted for each $x \in X_a$. The function F has linear growth and is Lipschitz continuous in x uniformly in $[0,T] \times \Omega$; i.e., there are constants $L_F, C_F \geq 0$ such that
$$\|F(t,\omega,x) - F(t,\omega,y)\|_X \leq L_F \|x-y\|_a,$$
$$\|F(t,\omega,x)\|_X \leq C_F(1 + \|x\|_a)$$
for all $t \in [0,T]$, $\omega \in \Omega$, and $x, y \in X_a$.

(H3) The map $(t,\omega) \mapsto G(t,\omega,x) \in Y$ is strongly measurable and adapted for all $x \in X_a$. The function $\Lambda_G G$ has linear growth and is Lipschitz continuous in x uniformly in $[0,T] \times \Omega$; i.e., there are constants $L_G, C_G \geq 0$ such that
$$\|\Lambda_G(G(t,\omega,x) - G(t,\omega,y))\|_{-\theta_G} \leq L_G \|x-y\|_a,$$
$$\|\Lambda_G G(t,\omega,x)\|_{-\theta_G} \leq C_F(1 + \|x\|_a)$$
for all $t \in [0,T]$, $\omega \in \Omega$, and $x, y \in X_a$.

(H4) The map $(t,\omega) \mapsto B(t,\omega,x) \in \mathcal{B}(H_1, X_{-\theta_B})$ is H_1-strongly measurable and adapted for all $x \in X_a$. The function B is L^2_γ-Lipschitz of linear growth uniformly in Ω; i.e., there are constants L_B^γ and C_B^γ such that
$$\|B(\cdot,\omega,\phi_1) - B(\cdot,\omega,\phi_2)\|_{\gamma((0,T),\mu;H_1,X_{-\theta_B})} \leq L_B^\gamma \|\phi_1 - \phi_2\|_{L^2_\gamma((0,T),\mu;X_a)},$$
$$\|B(\cdot,\omega,\phi_1)\|_{\gamma((0,T),\mu;H_1,X_{-\theta_B})} \leq C_B^\gamma(1 + \|\phi_1\|_{L^2_\gamma((0,T),\mu;X_a)}).$$
for all finite measures μ on $([0,T], \mathcal{B}_{[0,T]})$, for all $\omega \in \Omega$, and all $\phi_1, \phi_2 \in L^2_\gamma((0,T),\mu;X_a)$.

(H5) The map $(t,\omega) \mapsto \Lambda_C C(t,\omega,x) \in \mathcal{B}(H_2, X_{-\theta_C})$ is H_2-strongly measurable and adapted for all $x \in X_a$. The composition $\Lambda_C C$ is L^2_γ-Lipschitz of linear growth uniformly in Ω; i.e., there are constants L_C^γ and C_C^γ such that
$$\|\Lambda_C(C(\cdot,\omega,\phi_1) - C(\cdot,\omega,\phi_2))\|_{\gamma((0,T),\mu;H_2,X_{-\theta_C})} \leq L_C^\gamma \|\phi_1 - \phi_2\|_{L^2_\gamma((0,T),\mu;X_a)}$$

$$\|\Lambda_C C(\cdot,\omega,\phi_1)\|_{\gamma((0,T),\mu;H_2,X_{-\theta_C})} \leq C_C^\gamma(1+\|\phi_1\|_{L^2_\gamma((0,T),\mu;X_a)})$$

for all finite measures μ on $([0,T],\mathcal{B}_{[0,T]})$, for all $\omega\in\Omega$, and all $\phi_1,\phi_2\in L^2_\gamma((0,T),\mu;X_a)$.

For $p\in[1,\infty)$ and $\alpha\in(0,\frac12)$ we define $V^0_{\alpha,p}([0,T]\times\Omega;X)$ as the linear space of continuous adapted processes $\phi:[0,T]\times\Omega\to X$ such that

$$\|\phi(\cdot,\omega)\|_{C([0,T];X)} + \Big(\int_0^T \|s\mapsto (t-s)^{-\alpha}\phi(s,\omega)\|^p_{\gamma(L^2(0,t),X)}\,dt\Big)^{\frac1p} < \infty$$

for almost all $\omega\in\Omega$. In $V^0_{\alpha,p}([0,T]\times\Omega;X)$ we identify indistinguishable processes; i.e., processes ϕ_1 and ϕ_2 such that a.s. for all $t\in[0,T]$ we have $\phi_1(t)=\phi_2(t)$.

In order to introduce our solution concept, we recall some notation from [28]. For $\phi\in L^1(0,T;X_{-\theta})$ with $\theta\in[0,1)$, we write

$$S*\phi(t) = \int_0^t S(t-s)\phi(s)\,ds, \qquad t\in[0,T]. \tag{3.1}$$

Young's inequality and the regularity properties of $S(t)$ yield $S*\phi\in L^1(0,T;X)$. For $j=1,2$ and processes $\Phi:[0,T]\times\Omega\to\mathcal{B}(H_j,X_{-\theta})$ with $\theta\in[0,\frac12)$ which are H_j-strongly measurable and adapted and such that for all $t\in[0,T]$ the map

$$s\mapsto S(t-s)\Phi(s) \quad\text{belongs to}\quad \gamma(0,t;H_j,X),$$

almost surely, we set

$$S\diamond_j\Phi(t) = \int_0^t S(t-s)\Phi(s)\,dW_{H_j}(s). \tag{3.2}$$

This integral exists for each $t\in[0,T]$ due to Proposition 2.1.

Definition 3.1. *An X_a–valued process $(U(t))_{t\in[0,T]}$ is called a* mild solution *of* (SE) *if*

(i) *$U:[0,T]\times\Omega\to X_a$ is strongly measurable and adapted,*
(ii) *$F(\cdot,U)\in L^0(\Omega;L^1(0,T;X))$,*
(iii) *$\theta_G\in[0,1)$ and $\Lambda_G G(\cdot,U)\in L^0(\Omega;L^1(0,T;X_{-\theta_G}))$,*
(iv) *for all $t\in[0,T]$, $(s,\omega)\mapsto S(t-s)B(s,U(s))$ is H_1-strongly measurable and adapted and belongs to $\gamma(0,t;H_1,X)$ almost surely,*
(v) *for all $t\in[0,T]$, $(s,\omega)\mapsto S(t-s)\Lambda_C C(s,U(s))$ is H_2-strongly measurable and adapted and belongs to $\gamma(0,t;H_2,X)$ almost surely,*
(vi) *for all $t\in[0,T]$, the following equality holds a.s. in X:*

$$U(t)=S(t)U_0+S*F(\cdot,U)(t)+S*\Lambda_G G(\cdot,U)(t)+S\diamond_1 B(\cdot,U)(t)+S\diamond_2\Lambda_C C(\cdot,U)(t).$$

Definition 3.2. *An X_a–valued process $(U(t))_{t\in[0,T]}$ is called a* weak solution *of (SE) if*

(i) *U is strongly measurable and adapted and has paths in $L^1(0,T;X_a))$ a.s.,*
(ii) *$F(\cdot,U)\in L^0(\Omega;L^1(0,T;X))$,*
(iii) *$\theta_G\in[0,1)$ and $\Lambda_G G(\cdot,U)\in L^0(\Omega;L^1(0,T;X_{-\theta_G}))$,*
(iv) *$\theta_B\in[0,\frac12)$ and $B(\cdot,U):[0,T]\times\Omega\to\mathcal{B}(H_1,X_{-\theta_B})$ is H_1-strongly measurable with*

$$\int_0^T \|B(t,U(t))\|^2_{\mathcal{B}(H_1,X_{-\theta_B})}\,dt<\infty \textit{ almost surely,}$$

(v) *$\theta_C \in [0, \frac{1}{2})$ and $\Lambda_C C(\cdot, U)$ is H_2-strongly measurable with*

$$\int_0^T \|\Lambda_C C(t, U(t))\|^2_{\mathcal{B}(H_2, X_{-\theta_C})}\, dt < \infty \text{ almost surely,}$$

(vi) *for all $t \in [0, T]$ and $x^* \in D(A^*)$, we have*

$$\begin{aligned}\langle U(t),x^*\rangle - \langle u_0, x^*\rangle \\ &= \int_0^t \langle U(s), A^*x^*\rangle + \langle F(s, U(s)) + \Lambda_G G(s, U(s)), x^*\rangle\, ds \qquad (3.3)\\ &\quad + \int_0^t B^*(s, U(s))x^*\, dW_{H_1}(s) + \int_0^t (\Lambda_C C(s, U(s)))^* x^*\, dW_{H_2}(s),\end{aligned}$$

almost surely.

The following result and its proof are standard in stochastic evolution equations (cf. [14, Theorem 5.4]). Since our setting slightly differs from the existing literature, we include a short proof.

Proposition 3.3. *Let X be a UMD space. Let $\theta_G \in [0, 1)$, $\theta_B, \theta_C \in [0, \frac{1}{2})$ and let $a \in [0, \frac{1}{2})$. Then the following assertions hold.*

(1) *If $U \in L^0(\Omega; L^1(0, T; X_a))$ is a mild solution of (SE) such that Definition 3.2 (ii)–(v) hold, then U is a weak solution of (SE).*
(2) *If U is a weak solution of (SE) such that Definition 3.1 (iv)–(v) hold, then U is a mild solution of (SE).*

Proof. We will write $\tilde{F} = F + \Lambda_G G$, $H = H_1 \times H_2$, and $\tilde{B} = (B, \Lambda_C C)$.

(1) Let $t \in [0, T]$ and $x^* \in D(A^*)$. From the definition of a mild solution, Proposition 2.1 and the (stochastic) Fubini theorem we obtain that almost surely

$$\begin{aligned}&\int_0^t \langle U(s), A^*x^*\rangle\, ds\\ &= \int_0^t \langle u_0, S(s)^*A^*x^*\rangle\, ds + \int_0^t \int_r^t \langle \tilde{F}(r, U(r)), S(s - r)^*A^*x^*\rangle\, ds\, dr\\ &\quad + \int_0^t \int_r^t \tilde{B}^*(r, U(r))S^*(s - r)A^*x^*\, ds\, dW_H(r)\\ &= \langle S(t)u_0, x^*\rangle - \langle u_0, x^*\rangle + \int_0^t \langle S(t - r)\tilde{F}(r, U(r)), x^*\rangle\, dr - \int_0^t \langle \tilde{F}(r, U(r)), x^*\rangle\, dr\\ &\quad + \int_0^t \tilde{B}^*(r, U(r))S^*(t - r)x^*\, dW_H(r) - \int_0^t \tilde{B}^*(r, U(r))x^*\, dW_H(r)\\ &= \langle U(t), x^*\rangle - \langle u_0, x^*\rangle - \int_0^t \langle \tilde{F}(r, U(r)), x^*\rangle\, dr - \int_0^t \tilde{B}^*(r, U(r))x^*\, dW_H(r).\end{aligned}$$

This shows that U is a weak solution.

(2) Fix $t \in [0, T]$. Let $f \in C^1([0, t])$, $x^* \in D(A^*)$, $\varphi = f \otimes x^*$, and U be a mild solution. Itô's formula implies that

$$\begin{aligned}\langle U(t), \varphi(t)\rangle &= \langle u_0, \varphi(0)\rangle + \int_0^t \langle U(s), A^*\varphi(s)\rangle + \langle \tilde{F}(s, U(s)), \varphi(s)\rangle\, ds \qquad (3.4)\\ &\quad + \int_0^t \langle U(s), \varphi'(s)\rangle\, ds + \int_0^t \tilde{B}^*(s, U(s))\varphi(s)\, dW_H(s),\end{aligned}$$

almost surely. By linearity one can extend (3.4) to functions $\varphi : [0,t] \to D(A^*)$ of the form $\varphi = \sum_{n=1}^N f_n \otimes x_n^*$, with $f_n \in C^1([0,t])$ and $x_n^* \in D(A^*)$ for all $n = 1,\dots,N$. By density this extends to all $\varphi \in C^1([0,t];D(A^*))$. In particular, for $x^* \in D((A^*)^2)$ we can take $\varphi(s) = S^*(t-s)x^*$ and thus deduce

$$\langle U(t),x^*\rangle - \langle S(t)u_0,x^*\rangle = \int_0^t \langle S(t-s)\tilde F(s,U(s)),x^*\rangle\, ds + \int_0^t \tilde B^*(s,U(s))S^*(t-s)x^*\, dW_H(s),$$

almost surely. Since the integrals in Definition 3.1 exist by our assumptions, the Hahn-Banach theorem yields that U is a mild solution. □

We can now formulate the main abstract existence and uniqueness result which is a consequence of Theorems 7.1 and 7.2 in [28].

Theorem 3.4. *Let X be a UMD space with type $\tau \in [1,2]$ and suppose that (H1)-(H5) are satisfied. Assume that $0 \le a+\theta_G < \frac32 - \frac1\tau$ and $a+\max\{\theta_B,\theta_C\} < \frac12$. Let $U_0 : \Omega \to X_a$ be strongly $\mathcal{F}_0$–measurable. Then the following assertions hold.*

1. *If $\alpha \in (0,\frac12)$ and $p > 2$ are such that $a+\max\{\theta_B,\theta_C\} < \alpha - \frac1p$, then there exists a unique mild solution $U \in V^0_{\alpha,p}([0,T_0]\times\Omega;X_a)$ of* (SE).
2. *Let $\lambda \ge 0$ and $\delta \ge a$ satisfy $\lambda+\delta < \min\{1-\theta_G,\frac12-\theta_B,\frac12-\theta_C\}$. Then the mild solution U of* (SE) *has a version such that almost all paths satisfy $U - SU_0 \in C^\lambda([0,T];X_\delta)$.*

Proof. In [28] the problem

$$(\mathrm{SE}')\qquad \begin{cases} dU(t) = (AU(t) + \tilde F(t,U(t)))\,dt + \tilde B(t,U(t))\,dW_H(t), \ t\in[0,T],\\ U(0) = U_0\end{cases}$$

has been considered. Clearly, (SE) can be written as (SE′) if we take $\tilde F = F + \Lambda_G G$, $H = H_1\times H_2$ and $\tilde B = (B,\Lambda_C C)$. In this way the result follows immediately from Theorems 7.1 and 7.2 in [28]. □

Remark 3.5. There is a version of Theorem 3.4 for locally Lipschitz functions as well (see [28, Section 8]). Some of the results below remain true for locally Lipschitz coefficients. However, for the sake of simplicity we concentrate on the (global) Lipschitz case here.

4. Strongly damped second order equations

Before we turn to the equation (1.1), we have to treat a class of deterministic damped second order equations. We investigate the problem

$$(4.1)\qquad \begin{aligned} &\ddot u(t) + \rho\mathcal{A}^{\frac12}\dot u(t) + \mathcal{A}u(t) = 0, \qquad t\ge 0,\\ &u(0) = u_0, \qquad \dot u(0) = u_1,\end{aligned}$$

and, for $\alpha \in (\frac12,1]$, its variant

$$(4.2)\qquad \begin{aligned} &\ddot u(t) + \mathcal{A}^\alpha(\rho\dot u(t) + \mathcal{A}^{1-\alpha}u(t)) = 0, \qquad t\ge 0,\\ &u(0) = u_0, \qquad \dot u(0) = u_1,\end{aligned}$$

both on a Banach space E with norm $\|\cdot\|_0$. We assume that

$$\begin{aligned} &\mathcal{A} \text{ is invertible on } E,\ \overline{D(\mathcal{A})} = E,\ \lambda \in \rho(-\mathcal{A}) \text{ and } \|\lambda(\lambda I + \mathcal{A})^{-1}\|_{\mathcal{B}(E)} \le M \\ &\text{for all } \lambda \in \mathbb{C} \setminus \{0\} \text{ with } |\arg \lambda| \le \pi - \phi \text{ and some } \phi \in \Big(0, \frac{\pi}{2}\Big),\ M > 0. \\ &\text{Further, let } \alpha \in \Big[\frac{1}{2}, 1\Big],\ \rho > 0, \quad \text{and } \rho > 2\cos\frac{\pi - \phi}{2} \text{ if } \alpha = \frac{1}{2}. \end{aligned} \tag{4.3}$$

We denote by E_θ, $\theta \in [-1, 1]$, the fractional power spaces for $\mathcal{A}$ on E. Concerning (4.1) we look for solutions $u \in C^2(\mathbb{R}_+, E) \cap C^1(\mathbb{R}_+, E_{\frac{1}{2}}) \cap C(\mathbb{R}_+, E_1)$, whereas the solutions of (4.2) have to satisfy $u \in C^2(\mathbb{R}_+, E) \cap C^1(\mathbb{R}_+, E_{\frac{1}{2}})$ and $\rho\dot{u} + \mathcal{A}^{1-\alpha}u \in C(\mathbb{R}_+, E_\alpha)$.

In the recent paper [9], it was shown that the operator matrix

$$\begin{aligned} A &= \begin{pmatrix} 0 & I \\ -\mathcal{A} & -\rho\mathcal{A}^\alpha \end{pmatrix} \quad \text{with the domain} \\ D(\mathcal{A}) &= \{(\varphi, \psi) \in E_{\frac{3}{2}-\alpha} \times E_{\frac{1}{2}} : \mathcal{A}^{1-\alpha}\varphi + \rho\psi \in E_\alpha\} \end{aligned} \tag{4.4}$$

generates an analytic C_0–semigroup on $X = E_{\frac{1}{2}} \times E$, where the action of A is defined by $A(\varphi, \psi) = (\psi, -\mathcal{A}^\alpha(\mathcal{A}^{1-\alpha}\varphi + 2\rho\psi))$ if $\alpha > 1/2$. In the most important case $\alpha = 1/2$, we simply obtain $D(A) = E_1 \times E_{\frac{1}{2}}$ and the matrix in (4.4) is understood in the usual way. In (4.3) the constant $\rho > 0$ has to satisfy an additional lower bound if $\alpha = \frac{1}{2}$. This restriction cannot be avoided in view of Remark 1.1 of [9]. However, in the typical applications (as those discussed below) one can choose $\phi > 0$ arbitrarily small, so that (4.3) holds for all $\rho > 0$ in these applications. We recall that due to a result of Hörmander [18], A does not generate a C_0-semigroup if, say, $E = L^p(\mathbb{R}^d)$, $p \neq 2$, $\rho = 0$, and $\mathcal{A}$ is the negative Laplacian. Moreover, in the Hilbert space case and for a strictly positive self adjoint $\mathcal{A}$, Chen and Triggiani proved that A generates an analytic semigroup if one replaces the damping term $\rho\mathcal{A}^\alpha$ by a self adjoint operator B satisfying $\rho_1\mathcal{A}^\alpha \le B \le \rho_2\mathcal{A}^\alpha$ in form sense for some $\rho_2 > \rho_1 > 0$ and $\alpha \in [1/2, 1]$, see [11] and [22]. It would be very interesting to extend this result to the Banach space setting. In [11] it was also shown that for $\alpha < 1/2$, A does not generate an analytic semigroup.

In the next result we use the real interpolation spaces for a sectorial operator C on Banach space Y, see e.g. [23], [31]. Recall that

$$(Y, C)_{\gamma+\varepsilon,q} \hookrightarrow (Y, C)_{\gamma,1} \hookrightarrow D((w - C)^\gamma) \hookrightarrow (Y, C)_{\gamma,\infty} \hookrightarrow (Y, C)_{\gamma-\varepsilon,q} \tag{4.5}$$

for every $q \in [1, \infty]$ and $0 < \gamma - \varepsilon \le \gamma < \gamma + \varepsilon < 1$. Moreover, if $C_{-1} : X \to X_{-1}$ is the extrapolation of C, then $X_{\gamma-1}$ is isomorphic to the domain $D((w - C_{-1})^\gamma)$ in X_{-1}, see [1, Theorem V.1.3.8]. So the isomorphism (4.9) below implies that

$$E_{\frac{1}{2}-(1-\alpha)(\theta-\varepsilon)} \times E_{-\alpha(\theta-\varepsilon)} \hookrightarrow X_{-\theta} \hookrightarrow E_{\frac{1}{2}-(1-\alpha)(\theta+\varepsilon)} \times E_{-\alpha(\theta+\varepsilon)}, \tag{4.6}$$

for all $\varepsilon > 0$ with $0 < \theta - \varepsilon < \theta < \theta + \varepsilon \le 1/2$, and analogous embeddings hold in case of (4.10) and (4.12). We write $X \cong Y$ if X and Y are canonically isomorphic.

Proposition 4.1. *Assume that* (4.3) *holds on a Banach space E. Then the operator matrix A from* (4.4) *generates an analytic C_0–semigroup on $X = E_{\frac{1}{2}} \times E$. For all $(u_0, u_1) \in D(A)$ the problems* (4.1) *and* (4.2) *have unique solutions in the above specified sense. Moreover, we have*

$$X_\theta = E_{\frac{1}{2}+(1-\alpha)\theta} \times E_{\alpha\theta}, \tag{4.7}$$

$$X_{-\frac{1}{2}} \cong E_{\frac{\alpha}{2}} \times E_{-\frac{\alpha}{2}}\,, \tag{4.8}$$

$$(X_{-1}, X)_{1-\theta,q} \cong (E, E_1)_{\frac{1}{2}-(1-\alpha)\theta,q} \times (E_{-1}, E)_{1-\alpha\theta,q} \tag{4.9}$$

for all $\theta \in [0, 1/2]$ and $q \in [1, \infty]$.

If, additionally $\alpha = 1/2$, then

$$(X, D(A))_{\theta,q} = (E, E_1)_{\frac{1+\theta}{2},q} \times (E, E_1)_{\frac{\theta}{2},q}\,, \tag{4.10}$$

$$X_{-1} \cong E \times E_{-\frac{1}{2}}\,, \tag{4.11}$$

$$(X_{-1}, X)_{1-\theta,q} \cong (E, E_1)_{\frac{1-\theta}{2},q} \times (E_{-1}, E)_{1-\frac{\theta}{2},q} \tag{4.12}$$

for all $\theta \in (0, 1)$ and $q \in [1, \infty]$. Furthermore, if E is reflexive, then $X^* = (E^*)_{-\frac{1}{2}} \times E^*$ and

$$A^* = \begin{pmatrix} 0 & -\mathcal{A}^* \\ I & -\rho(\mathcal{A}^*)^{\frac{1}{2}} \end{pmatrix} \qquad \textit{with} \quad D(A^*) = E^* \times (E^*)_{\frac{1}{2}}\,, \tag{4.13}$$

where $(E^*)_\theta$ is the fractional power space for $\mathcal{A}^*$.

Proof. The generation property was shown in [9, Theorem 2.3]. It easily implies the unique solvability of (4.1) and (4.2). The equation (4.7) was also proved in [9, Theorem 2.3]. (We note that in [9] it was assumed that E is reflexive. However, this property is not needed in the parts of the proofs which are relevant to us.) Take $(\varphi, \psi) \in X_{\frac{1}{2}} = E_{1-\frac{\alpha}{2}} \times E_{\frac{\alpha}{2}}$. Due to [9, p.2316], we have

$$A^{-1} = \begin{pmatrix} -\rho\mathcal{A}^{\alpha-1} & -\mathcal{A}^{-1} \\ I & 0 \end{pmatrix}.$$

Using (4.7), we can estimate

$$\|A^{-1}(\varphi, \psi)\|_{X_{\frac{1}{2}}} \eqsim \|\mathcal{A}^{1-\frac{\alpha}{2}}(\rho\mathcal{A}^{\alpha-1}\varphi + \mathcal{A}^{-1}\psi)\|_0 + \|\varphi\|_{\frac{\alpha}{2}} \lesssim_\rho \|\varphi\|_{\frac{\alpha}{2}} + \|\psi\|_{-\frac{\alpha}{2}}\,,$$

where $\|\cdot\|_0$ denotes the norm on E. Conversely, we obtain

$$\begin{aligned} \|\varphi\|_{\frac{\alpha}{2}} + \|\psi\|_{-\frac{\alpha}{2}} &= \|\varphi\|_{\frac{\alpha}{2}} + \|\mathcal{A}^{-\frac{\alpha}{2}}\psi + \rho\mathcal{A}^{\frac{\alpha}{2}}\varphi - \rho\mathcal{A}^{\frac{\alpha}{2}}\varphi\|_0 \\ &\lesssim_\rho \|\varphi\|_{\frac{\alpha}{2}} + \|\mathcal{A}^{1-\frac{\alpha}{2}}(\rho\mathcal{A}^{\alpha-1}\varphi + \mathcal{A}^{-1}\psi)\|_0 \eqsim \|A^{-1}(\varphi, \psi)\|_{X_{\frac{1}{2}}}. \end{aligned}$$

The isomorphism (4.8) thus follows since $X_{-\frac{1}{2}}$ is isomorphic to the completion of $X_{\frac{1}{2}}$ with respect to the norm $\|A^{-1}(\varphi, \psi)\|_{X_{\frac{1}{2}}}$, cf. [1, Theorem V.1.3.8]. Notice that real interpolation respects cartesian products due to its definition via the K–functional. Furthermore, the reiteration theorem (see e.g. [23, Theorem 1.2.15]) implies that $(X_{-1}, X)_{1-\theta,q} = (X_{-\frac{1}{2}}, X)_{1-2\theta,q}$. The equality (4.9) is then a consequence of (4.8) and reiteration.

Let $\alpha = 1/2$. In this case we have $X_1 = E_1 \times E_{\frac{1}{2}}$. Take $(\phi, \psi) \in X$. We first show (4.10). We estimate as above

$$\|A^{-1}(\varphi, \psi)\|_X \eqsim \|\mathcal{A}^{\frac{1}{2}}(\rho\mathcal{A}^{-\frac{1}{2}}\varphi + \mathcal{A}^{-1}\psi)\|_0 + \|\varphi\|_0 \lesssim_\rho \|\varphi\|_0 + \|\psi\|_{-\frac{1}{2}},$$

$$\|\varphi\|_0 + \|\psi\|_{-\frac{1}{2}} = \|\mathcal{A}^{\frac{1}{2}}(\mathcal{A}^{-1}\psi + \rho\mathcal{A}^{-\frac{1}{2}}\varphi - \rho\mathcal{A}^{-\frac{1}{2}}\varphi)\|_0 + \|\varphi\|_0 \lesssim_\rho \|A^{-1}(\varphi, \psi)\|_X.$$

The formulas (4.10) and (4.12) can now be established as the isomorphism (4.9).

The last assertion follows easily from $(E_{\frac{1}{2}})^* = (E^*)_{-\frac{1}{2}}$ (see Theorem V.1.4.12 of [1]) and a straightforward calculation using that the operator matrix in (4.13) is invertible in X^*. □

5. The stochastically perturbed damped plate equation

In this section we prove existence, uniqueness and regularity results for the structurally damped plate equation with noise, given by

$$(5.1)\qquad \begin{cases} \ddot u(t,s) + \Delta^2 u(t,s) - \rho\Delta\dot u(t,s) = f(t,s,u(t,s),\dot u(t,s)) \\ \qquad\qquad + b(t,s,u(t,s),\dot u(t,s))\,\dfrac{\partial w_1(t,s)}{\partial t} + \Big[G(t,u(t,\cdot),\dot u(t,\cdot)) \\ \qquad\qquad + C(t,u(t,\cdot),\dot u(t,\cdot))\dfrac{\partial w_2(t)}{\partial t}\Big]\delta(s-s_0), \qquad t\in[0,T],\ s\in S, \\ u(0,s) = u_0(s),\ \dot u(0,s) = u_1(s), \qquad s\in S, \\ u(t,s) = \Delta u(t,s) = 0, \qquad t\in[0,T],\ s\in\partial S, \end{cases}$$

Using Proposition 4.1 and the theory from Section 3, we will reformulate problem (5.1) as an equation of the type (SE). We first list our assumptions and notations, where subsets M of $\mathbb{R}^n$ are endowed with the Borel σ–algebra $\mathcal{B}_M$.

(A0) Let $S\subset\mathbb{R}^d$ be a bounded domain with boundary ∂S of class C^4 and $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space with filtration $(\mathcal{F}_t)_{t\geq 0}$. The number $\rho>0$ and the point $s_0\in S$ are fixed, and δ denotes the usual point mass.

Let H_1 be a separable real Hilbert space, $H_2 = Y = \mathbb{R}$ and $E = L^q(S)$ for some $q\in(1,\infty)$. We identify Y with $\mathcal{B}(H_2,Y)$, where we interpret each $y\in Y$ as the operator $h\mapsto yh$. We further introduce the negative Dirichlet Laplacian on E by

$$\mathcal{B}\varphi = -\Delta\varphi, \quad D(\mathcal{B}) = W^{2,q}(S)\cap W_0^{1,q}(S).$$

We set $\mathcal{A} = \mathcal{B}^2$, so that $\mathcal{A}^{\frac12} = \mathcal{B}$. As in Section 4 we define the operator $(A, D(A))$ on $X := E_{\frac12}\times E$ by setting

$$A = \begin{pmatrix} 0 & I \\ -\mathcal{A} & -\rho\mathcal{A}^{\frac12}\end{pmatrix}, \qquad D(A) = E_1\times E_{\frac12}.$$

Since $\mathcal{A}$ is a sectorial operator of angle ϕ for all $\phi\in(0,\pi/2)$ (see e.g. [15, Theorem 8.2]), the assumption (4.3) is satisfied for the above $\mathcal{A}$ and $\rho>0$. So Hypothesis (H1) in Section 3 follows from Proposition 4.1, i.e., A generates an analytic C_0–semigroup $(S(t))_{t\geq0}$. We also recall that

$$(5.2)\qquad D(\mathcal{B}^\theta) = \begin{cases} H^{2\theta,q}(S), & \text{if } 0\leq 2\theta < \frac1q, \\ \{\varphi\in H^{2\theta,q}(S) : \varphi = 0 \text{ on } \partial S\}, & \text{if } \frac1q < 2\theta\leq 1. \end{cases}$$

(cf. [2, Corollary 2.2]). We also observe that equation (4.7) with $\alpha = \frac12$ gives

$$(5.3)\qquad X_\delta = E_{\frac12+\frac12\delta}\times E_{\frac12\delta}$$

for $\delta\in[0,\frac12]$. Combining this identity with (5.2), we deduce

$$X_\delta = \begin{cases} (H^{2+2\delta,q}(S)\cap W_0^{1,q}(S))\times H^{2\delta,q}(S), & \text{if } 2\delta\in(0,\frac1q), \\ \{(\varphi,\psi)\in H^{2+2\delta,q}(S)\times H^{2\delta,q}(S) : \varphi = \Delta\varphi = \psi = 0 \text{ on } \partial S\}, & \text{if } 2\delta\in(\frac1q,1). \end{cases}$$

We further make the following hypotheses.

(A1) The functions $f,b : [0,T]\times\Omega\times S\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are jointly measurable, adapted to $(\mathcal{F}_t)_{t\geq0}$, and Lipschitz functions and of linear growth in the fourth and fifth variable, uniformly in the other variables. The process w_2 is a standard real–valued Brownian motion with respect to $(\mathcal{F}_t)_{t\geq0}$. We set $W_{H_2}(t) := w_2(t)$ for $t\geq0$.

(A2) The maps $G, C : [0,T] \times \Omega \times X \to \mathbb{R}$ are jointly measurable, adapted to $(\mathcal{F}_t)_{t\geq 0}$, and Lipschitz and of linear growth in the third variable, uniformly in the other variables.

(A3) The process w_1 can be written in the form $i_1 W_{H_1}$, where $i_1 \in \mathcal{B}(H_1, L^r(S))$ for some $r \in [1,\infty]$ and W_{H_1} is a cylindrical Wiener process with respect to $(\mathcal{F}_t)_{t\geq 0}$ being independent of W_{H_2}.

Assumption (A3) has to be interpreted in the sense that

$$w_1(t,s) = \sum_{n\geq 1} (i_1 h_n)(s) W_{H_1}(t) h_n, \qquad t \in \mathbb{R}_+, s \in S,$$

where $(h_n)_{n\geq 1}$ is an orthonormal basis for H_1 and the sum converges in $L^r(S)$. In the examples below we will be more specific about i_1 and W_{H_1}. Typically, H_1 is $L^2(S)$ or the reproducing kernel Hilbert space and $i_1 i_1^*$ is the covariance operator of w_1, see e.g. [5], [7] and the references therein. It is also possible to assume that i_1 takes values in an extrapolation space of $L^r(S)$, but we do not consider this generalization here.

We now continue to reformulate (5.1) as a problem of the type (SE). We focus on the case $a = 0$ in (H2)–(H5), though we comment on possible extensions in some remarks below. We define $F : [0,T] \times \Omega \times X \to X$ by

$$F(t,\omega,x)(s) = \begin{pmatrix} 0 \\ f(t,\omega,s,x(s),\dot{x}(s)) \end{pmatrix}.$$

It straightforward to check that F satisfies (H2) because of (A1). Let $\theta_B \in [0, \frac{1}{2})$. The map $B : [0,T] \times \Omega \times X \to \mathcal{B}(H_1, X_{-\theta_B})$ is defined by

$$B(t,\omega,x)h(s) = \begin{pmatrix} 0 \\ b(t,\omega,s,x(s),\dot{x}(s))(i_1 h)(s) \end{pmatrix}. \tag{5.4}$$

It will be assumed that B satisfies (H4). Below, we discuss various classes of examples where this assumption holds. We further take a suitable $\theta_G = \theta_C \in (0, \frac{1}{2})$ and define $\Lambda_C = \Lambda_G = \Lambda \in \mathcal{B}(\mathbb{R}, X_{-\theta_C})$ by

$$\Lambda y(s) = \begin{pmatrix} 0 \\ \delta(s - s_0) y \end{pmatrix}.$$

We claim that for all $1 < q < \frac{d}{d-1}$ if $d \geq 2$ and all $1 < q < \infty$ if $d = 1$, there exists a $\theta_C \in (\frac{d}{2q'}, \frac{1}{2})$ such that Λ is well-defined. Indeed, let $\theta < \theta_C$. Due to equation (4.6) with $\alpha = \frac{1}{2}$, it holds

$$E_{\frac{1}{2}-\frac{1}{2}\theta} \times E_{-\frac{1}{2}\theta} \hookrightarrow D((-A)^{-\theta_C}) = X_{-\theta_C}.$$

So we have to find a $\theta \in [0, \frac{1}{2})$ with $\delta(\cdot - s_0) \in E_{-\frac{1}{2}\theta}$. Theorem V.1.4.12 of [1] implies that $E_{-\frac{1}{2}\theta} = (D((\mathcal{B}^*)^\theta))^*$. It thus remains to show that the point evaluation $\delta_{s_0} : D((\mathcal{B}^*)^\theta) \to \mathbb{R}$, $\delta_{s_0} f = f(s_0)$, defines a bounded linear map. Since $\mathcal{B}^*$ is the realization of the negative Dirichlet Laplacian on $L^{q'}(S)$, we deduce from (5.2) that $D((\mathcal{B}^*)^\theta)$ is a closed subset of $H^{2\theta,q'}(S)$. Sobolev's embedding theorem (cf. [31, Theorem 4.6.1]) yields $H^{2\theta,q'}(S) \hookrightarrow C(\overline{S})$ if $2\theta > \frac{d}{q'}$. So the claim follows.

Assertion (A2) now implies (H3) and (H5) since ΛG and ΛC factorize through the spaces $Y = \mathbb{R}$ and $\mathcal{B}(H_2, Y) = \mathbb{R}$, respectively. (Use the ideal property (2.1).)

Summing up, we have found spaces X, Y, H_1, H_2, maps $A, F, \Lambda_G, G, B, \Lambda_C, C$, and processes W_{H_1}, W_{H_2} for which we can formulate the equation (SE) from Section 3. In view of Theorem 3.4, this problem has a unique mild solution U which we call a *mild solution* of (5.1).

To justify this notion of a mild solution to (5.1), we need to define a weak solution of (5.1). To this aim, we assume that $D(\mathcal{A}^*) \hookrightarrow C(\overline{S})$. One easily checks that this embedding always holds for $d = 1, 2, 3, 4$ and for all $q < \frac{d}{d-4}$ if $d \geq 5$. Assume that f, G, b, C are as before. We say that a process $u : [0, T] \times \Omega \times S \to \mathbb{R}$ is a *weak solution* of (5.1) if it is measurable, $u(t, \cdot)$ is $\mathcal{F}_t \otimes \mathcal{B}_S$-measurable for all $t \in [0, T]$, $u \in W^{1,2}(0, T; L^q(S))$ a.s., and for all $\phi \in W^{2,q'}(S) \cap W_0^{1,q'}(S) = D(\mathcal{A}^*)$ we have

$$
\begin{aligned}
&\langle u(t,\cdot), \phi\rangle - \langle u_0, \phi\rangle - t\langle u_1, \phi\rangle + \int_0^t \int_0^\sigma \langle u(\tau,\cdot), \Delta^2\phi\rangle \, d\tau \, d\sigma \\
&\quad - \rho \int_0^t \langle u(\sigma,\cdot), \Delta\phi\rangle \, d\sigma + t\rho\langle u_0, \Delta\phi\rangle \\
&= \int_0^t \int_0^\sigma \Big(\langle f(\tau,\cdot,u(\tau,\cdot),\dot u(\tau,\cdot)), \phi\rangle + G(\tau, u(\tau,\cdot), \dot u(\tau,\cdot))\phi(s_0)\Big) \, d\tau \, d\sigma \\
&\quad + \int_0^t \int_0^\sigma \langle b(\tau,\cdot,u(\tau,\cdot),\dot u(\tau,\cdot)), \phi\rangle \, dw_1(\tau) \, d\sigma \\
&\quad + \int_0^t \int_0^\sigma C(\tau, u(\tau,\cdot), \dot u(\tau,\cdot))\phi(s_0) \, dw_2(\tau) \, d\sigma,
\end{aligned}
\tag{5.5}
$$

where $\langle \cdot, \cdot \rangle$ denotes the $(L^q(S), L^{q'}(S))$-duality.

We will show below that (5.1) has a unique mild U and a unique weak solution u satisfying $U = (u, \dot u)$. In the next theorem we use the notation introduced above.

Theorem 5.1. *Let $d \geq 1$ and $1 < q < \frac{d}{d-1}$. Assume that (A0)–(A3) hold, that B satisfies (H4), and that $u_0 : \Omega \to W^{2,q}(S) \cap W_0^{1,q}(S)$ and $u_1 : \Omega \to L^q(S)$ are $\mathcal{F}_0$-measurable. Then the following assertions hold.*

(1) *For all $\alpha \in (0, \frac{1}{2})$ and $p > 2$ with $\max\{\theta_B, \theta_C\} < \alpha - \frac{1}{p}$, there exists a unique mild solution U of (5.1) belonging to*

$$
V^0_{\alpha,p}([0,T] \times \Omega; (W^{2,q}(S) \cap W_0^{1,q}(S)) \times L^q(S)). \tag{5.6}
$$

There is a unique weak solution $u \in W^{1,2}(0, T; L^q(S))$ of (5.1) such that $(u, \dot u)$ belongs to the space in (5.6). Moreover, $U = (u, \dot u)$.

(2) *There exists a version of u with paths that satisfy $u \in C([0, T]; W^{2,q}(S) \cap W_0^{1,q}(S))$ and $\dot u \in C([0, T]; L^q(S))$.*

(3) *Let $\eta \in (0, \frac{1}{2}]$. If $u_0 : \Omega \to E_{\frac{1}{2}+\frac{1}{2}\eta}$ and $u_1 : \Omega \to E_{\frac{1}{2}\eta}$, then there exists a version of u with paths that satisfy $u \in C^\lambda([0, T]; E_{\frac{1}{2}+\frac{1}{2}\delta})$ and $\dot u \in C^\lambda([0, T]; E_{\frac{1}{2}\delta})$ for all $\delta, \lambda \geq 0$ with $\delta + \lambda < \min\{\eta, \frac{1}{2} - \theta_B, \frac{1}{2} - \theta_C\}$.*

Proof. We have already formulated (5.1) as (SE). Let $\alpha \in (0, \frac{1}{2})$ and $p > 2$ be such that $\max\{\theta_B, \theta_C\} < \alpha - \frac{1}{p}$. Set $U_0 = (u_0, u_1)$. Theorem 3.4(1) gives a unique mild solution $U \in V^0_{\alpha,p}([0, T] \times \Omega; X)$ of (SE′), where we set $\tilde F = F + \Lambda G$, $H = H_1 \times H_2$ and $\tilde B = (B, \Lambda C)$. It is given by

$$
U(t) = S(t)U_0 + \int_0^t S(t-s)\tilde F(s, U(s)) \, ds + \int_0^t S(t-s)\tilde B(s, U(s)) \, dW_H(s) \tag{5.7}
$$

almost surely. Write $U = (u, v)$. We show that $\dot{u} = v$ and that u is a weak solution of (5.1). Fix $t \in [0, T]$. Tt follows from Proposition 3.3 that U is a weak solution of (SE′). Hence, for all $x^* \in D(A^*)$, we have

$$
\begin{aligned}
\langle U(t), x^*\rangle - \langle U_0, x^*\rangle = \int_0^t \langle U(s), A^* x^*\rangle + \langle \tilde{F}(s, U(s)), x^*\rangle \, ds \\
+ \int_0^t \tilde{B}^*(s, U(s)) x^* \, dW_H(s),
\end{aligned} \tag{5.8}
$$

almost surely. In particular, for $x^* = (\phi, 0)$ with $\phi \in E^*$, the equations (5.8) and (4.13) yield that $\langle u(t, \cdot), \phi\rangle - \langle u_0, \phi\rangle = \int_0^t \langle v(\tau, \cdot), \phi\rangle \, d\tau$ almost surely. Therefore, $\dot{u} = v$ almost surely. Moreover, if we take $x^* = (0, \phi)$ with $\phi \in D(A^*)$ and use (4.13) again, we obtain

$$
\begin{aligned}
&\langle \dot{u}(t, \cdot), \phi\rangle - \langle u_1, \phi\rangle + \int_0^t \langle u(\tau, \cdot), \Delta^2 \phi\rangle \, d\tau - \rho \int_0^t \langle \dot{u}(\tau, \cdot), \Delta\phi\rangle \, d\tau \\
&= \int_0^t \Big(\langle f(\tau, \cdot, u(\tau, \cdot), \dot{u}(\tau, \cdot)), \phi\rangle + G(\tau, u(\tau, \cdot), \dot{u}(\tau, \cdot)) \phi(s_0) \Big) \, d\tau \\
&\quad + \int_0^t \langle b(\tau, \cdot, u(\tau, \cdot), \dot{u}(\tau, \cdot)), \phi\rangle \, dw_1(\tau) \\
&\quad + \int_0^t C(\tau, u(\tau, \cdot), \dot{u}(\tau, \cdot)) \phi(s_0) \, dw_2(\tau)
\end{aligned} \tag{5.9}
$$

almost surely. Now integration with respect to t yields the result.

To show that u is the unique weak solution, we show that every weak solution gives a mild solution $U = (u, \dot{u})$. The assumptions yield $u(0, \cdot) = u_0$ and $\dot{u}(0, \cdot) = u_1$ in $L^q(S)$. Fix $t \in [0, T]$ and $\phi \in D(\mathcal{A}^*)$. Equation (5.9) follows from (5.5) by differentiation with respect to t. We claim that (5.8) holds for all $x^* \in D(A^*)$. For $x^* = (\phi, 0)$ with $\phi \in E^*$ this is clear from (4.13) and $u(t, \cdot) - u_0 = \int_0^t \dot{u}(\tau, \cdot) \, d\tau$ almost surely. For $x^* = (0, \phi)$ with $\phi \in D(\mathcal{A}^*)$ one can check that (5.8) reduces to (5.9), using (4.13) again. By linearity and density we obtain (5.8) for all $x^* \in D(A^*)$. Now Proposition 3.3 implies that U is a mild solution of (5.1).

Theorem 3.4(2) shows that $U - S(u_0, u_1)$ has paths in $C^\lambda([0, T]; D((-A)^\delta))$ for all $\delta, \lambda \geq 0$ with $\lambda + \delta < \min\{1 - \theta_G, \frac{1}{2} - \theta_B, \frac{1}{2} - \theta_C\}$. By the assumption in (3) and equation (5.3) we have $(u_0, u_1) \in D((-A)^\eta)$. Therefore, $S(u_0, u_1) \in C^\lambda([0, T]; D((-A)^\delta))$ a.s. whenever $\lambda + \delta < \eta$. Now the result follows from (5.3). $\square$

Remark 5.2. We indicate an extension of the above result to the case where $C, G : [0, T] \times \Omega \times C(\overline{S}) \times E \to \mathbb{R}$ if $d \leq 3$. (Observe that in this case one can allow for point evaluations in the third coordinate of C and G.) First, we note that the identity (5.3) yields

$$X_a \hookrightarrow (W^{2+2\tilde{a},q}(S) \times W^{2\tilde{a},q}(S))$$

for all $\tilde{a} \in [0, a)$, where we must have $a \in [0, \frac{1}{2})$ in view of Theorem 3.4. Sobolev's embedding leads to $W^{2+2\tilde{a},q}(S) \hookrightarrow C(\overline{S})$ if

$$2 + 2\tilde{a} - \frac{d}{q} > 0 \iff q > \frac{d}{2 + 2\tilde{a}} . \tag{5.10}$$

If (5.10) holds for some $0 \leq \tilde{a} < a < \frac{1}{2}$ and $q \in (1, d/(d-1))$, then there is a version of Theorem 5.1 which is valid for C and G defined only for $\phi \in C(\overline{S})$ (provided that $a < \min\{\theta_B, \theta_C, \theta_G\} - \frac{1}{2}$). For $d = 1$ and $d = 2$ the condition (5.10) holds even for $a = \tilde{a} = 0$ and all $q > 1$. For $d = 3$ and each $1 < q < 3/2 = d/(d-1)$, we can find an arbitrarily small $\tilde{a} > 0$ satisfying (5.10). Therefore we can choose $a \in (\tilde{a}, \frac{1}{2} - \theta_B)$ if (H4) holds for some $\theta_B < \frac{1}{2}$. For $d \geq 4$, the inequality (5.10) contradicts $q < d/(d-1)$ and $\tilde{a} < 1/2$.

We now discuss the interplay between b and w_1 in several examples, where we specify H_1, i_1 and θ_B. Throughout the examples below W_{H_1} is a cylindrical Brownian process as in (A3). We start with the case when the Brownian motion is colored in space.

Example 5.3. Assume that the covariance $Q_1 \in \mathcal{B}(L^2(S))$ of w_1 is compact. Then there exist numbers $(\lambda_n)_{n\geq 1}$ in $\mathbb{R}_+$ and an orthonormal system $(e_n)_{n\geq 1}$ in $L^2(S)$ such that

$$Q_1 = \sum_{n\geq 1} \lambda_n e_n \otimes e_n \tag{5.11}$$

Assume that

$$\sum_{n\geq 1} \lambda_n \|e_n\|_\infty^2 < \infty. \tag{5.12}$$

Let $H_1 = L^2(S)$ and $i_1 : L^2(S) \to L^\infty(S)$ be given by $i_1 = \sum_{n\geq 1} \sqrt{\lambda_n} e_n \otimes e_n$. Then (H4) is satisfied with $a = \theta_B = 0$.

It will be clear from the proof that (5.12) can be replaced by

$$\Big(\sum_{n\geq 1} \lambda_n |e_n|^2\Big)^{\frac{1}{2}} \in L^\infty(S). \tag{5.13}$$

Remark 5.4. A symmetric and positive operator $Q \in \mathcal{B}(L^2(S))$ maps $L^2(S)$ continuously into $L^\infty(S)$ if only if (5.11) and (5.13) hold. Indeed, if Q satisfies (5.11) and (5.13), then the Cauchy-Schwarz inequality implies that

$$|\sqrt{Q}h(s)| = \Big|\sum_{n\geq 1} \sqrt{\lambda_n} e_n(s)[e_n, h]_{L^2(S)}\Big| \leq \Big(\sum_{n\geq 1} \lambda_n |e_n(s)|^2\Big)^{\frac{1}{2}} \|h\|_{L^2(S)}$$

for almost all $s \in S$ and all $h \in L^2(S)$. Conversely, if $\sqrt{Q} : L^2(S) \to L^\infty(S)$ is bounded, then $\sqrt{Q} \in \mathcal{B}(L^2(S))$ is compact (cf. e.g. [28, Lemma 2.1]). In particular, there exists an orthonormal basis $(e_n)_{n\geq 1}$ in $L^2(S)$ and $(\lambda_n)_{n\geq 1}$ in $\mathbb{R}_+$ such that (5.11) holds. Now, for almost all $s \in S$ we estimate

$$\begin{aligned}\Big(\sum_{n\geq 1} \lambda_n |e_n(s)|^2\Big)^{\frac{1}{2}} &= \sup_{\|\beta\|_{\ell^2}\leq 1} \Big|\sum_{n\geq 1} \sqrt{\lambda_n} e_n(s)\beta_n\Big| \\ &= \sup_{\|\beta\|_{\ell^2}\leq 1} \Big|\sqrt{Q}\Big(\sum_{n\geq 1} \beta_n e_n\Big)(s)\Big| \leq \|\sqrt{Q}\|_{\mathcal{B}(L^2(S),L^\infty(S))}.\end{aligned}$$

Proof of Example 5.3. Our assumptions imply that $i_1 \in \mathcal{B}(L^2(S), L^\infty(S))$. Equation (5.4) thus defines a map $B : [0,T]\times\Omega\times X \to \mathcal{B}(H_1, X)$. Moreover, the function $(t,\omega) \mapsto B(t,\omega,x)$ is H_1-strongly measurable and adapted in X, for each $x \in X$. We check the L^2_γ-Lipschitz property. Let μ be a finite measure on $[0,T]$. We have to show that

$$\|B(\cdot,\omega,\phi_1) - B(\cdot,\omega,\phi_2)\|_{\gamma(L^2((0,T),\mu;H,X)} \leq C\,\|\phi_1 - \phi_2\|_{L^2_\gamma((0,T),\mu;X)}$$

for all $\phi_1, \phi_2 \in L^2_\gamma((0,T),\mu;X)$ and some constants $C \geq 0$. We write $\phi_1 = (\phi_{11}, \phi_{12})$ and $\phi_2 = (\phi_{21}, \phi_{22})$ with $\phi_{i1} \in L^2_\gamma((0,T),\mu;E_{\frac12})$ and $\phi_{i2} \in L^2_\gamma((0,T),\mu;E)$ for $i = 1,2$. Recall that formula (2.2) says that

$$\|\Phi\|_{\gamma(L^2((0,T),\mu;H),L^q(S))} \eqsim_q \Big\|\Big(\int_0^T \sum_{n\geq 1} |\Phi(t)e_n|^2 \, d\mu(t)\Big)^{\frac12}\Big\|_{L^q(S)}$$

for all $\Phi \in \gamma(L^2((0,T),\mu;H),L^q(S))$. Using that b is Lipschitz, we thus obtain

$$\begin{aligned}
&\|B(\cdot,\omega,\phi_1) - B(\cdot,\omega,\phi_2)\|_{\gamma(L^2((0,T),\mu;H),X)}\\
&\quad\eqsim_q \Big\|\Big(\int_0^T |b(t,\omega,\phi_{11},\phi_{12}) - b(t,\omega,\phi_{21},\phi_{22})|^2 \, d\mu(t) \sum_{n\geq 1} |i_1 e_n|^2\Big)^{\frac12}\Big\|_E\\
&\quad\lesssim_{Q_1} \Big\|\Big(\int_0^T |b(t,\omega,\phi_{11},\phi_{12}) - b(t,\omega,\phi_{21},\phi_{22})|^2 \, d\mu(t)\Big)^{\frac12}\Big\|_E\\
&\quad\lesssim L_b \Big\|\Big(\int_0^T |\phi_{11} - \phi_{21}|^2 + |\phi_{12} - \phi_{22}|^2 \, d\mu(t)\Big)^{\frac12}\Big\|_E\\
&\quad\eqsim_q L_b \left(\|\phi_{11} - \phi_{21}\|_{\gamma(L^2((0,T),\mu),E)} + \|\phi_{12} - \phi_{22}\|_{\gamma(L^2((0,T),\mu),E)}\right)\\
&\quad\lesssim L_b \, \|\phi_1 - \phi_2\|_{\gamma(L^2((0,T),\mu),X)}
\end{aligned}$$

for all $\omega \in \Omega$. The other estimate in (H4) can be established in a similar way. □

We next consider an $L^r(S)$–valued Brownian motion w_1, where $r \in [1,\infty]$. In this case, we let H_1 be the reproducing kernel Hilbert space of the Gaussian random variable $w_1(1,\cdot)$ and let i_1 be the embedding of H_1 into $L^r(S)$. Then we have $i_1 \in \gamma(H_1, L^r(S))$ and $i_1 i_1^* \in \mathcal{B}(L^{r'}(S), L^r(S))$ is the covariance operator of $w_1(1,\cdot)$ (cf. [5], [7] and the references therein for details). Let W_{H_1} be a cylindrical Brownian motion such that $w_1 = i_1 W_{H_1}$.

Example 5.5. Assume that w_1 is an $L^r(S)$-valued Brownian motion with $r > d$. Then (H4) is satisfied for all $\theta_B \in (\frac{d}{2r}, \frac12)$ and $a = 0$.

Remark 5.6. If Q_1 is of the form (5.11), then a sufficient condition for w_1 to be L^r-valued is

$$\sum_{n\geq 1} \lambda_n \|e_n\|^2_{L^r(S)} < \infty,$$

or more generally

$$\Big(\sum_{n\geq 1} \lambda_n |e_n|^2\Big)^{\frac12} \in L^r(S). \tag{5.14}$$

Indeed, let $\tilde{i}_1 = \sum_{n\geq 1} \sqrt{\lambda_n} e_n \otimes e_n$ and let $W_{L^2(S)}$ be the cylindrical Brownian motion such that $w_1 = \tilde{i}_1 W_{L^2(S)}$. Lemma 2.1 of [28] then yields

$$\begin{aligned}
(\mathbb{E}\|w_1(t)\|^2_{L^r(S)})^{\frac12} &= \Big(\mathbb{E}\Big\|\sum_{n\geq 1} \tilde{i}_1 e_n W_{L^2(S)}(t) e_n\Big\|^2_{L^r(S)}\Big)^{\frac12}\\
&= \sqrt{t}\Big(\mathbb{E}\Big\|\sum_{n\geq 1} \gamma_n \sqrt{\lambda_n} e_n\Big\|^2_{L^r(S)}\Big)^{\frac12}\\
&\eqsim_r \sqrt{t}\Big\|\Big(\sum_{n\geq 1} \lambda_n |e_n|^2\Big)^{\frac12}\Big\|_{L^r(S)}.
\end{aligned}$$

Finally, we note that (5.14) is equivalent to $\sqrt{Q_1} \in \gamma(L^2(S), L^r(S))$, where $r \geq 2$, due to [28, Lemma 2.1].

Proof of Example 5.5. We use the same notation as in Example 5.3, but H_1 will be the reproducing kernel Hilbert space for $w_1(1,\cdot)$ and $(h_n)_{n\geq 1}$ is an orthonormal basis of H_1. Lemma 2.1 of [28] yields

$$\Big\| \Big(\sum_{n\geq 1} |i_1 h_n|^2 \Big)^{\frac{1}{2}} \Big\|_{L^r(S)} \eqsim_r \|i_1\|_{\gamma(H_1, L^r(S))} < \infty.$$

We set $\tilde{\theta}_B = \frac{d}{2r} < \theta_B$ and choose $\varepsilon > 0$ with $\tilde{\theta}_B + \varepsilon < \theta_B$. Since $E_{\frac{1}{2}-\frac{1}{2}(\tilde{\theta}_B+\varepsilon)} \times E_{-\frac{1}{2}(\tilde{\theta}_B+\varepsilon)} \hookrightarrow X_{-\theta_B}$ by (4.6), we can estimate

$$\begin{aligned} &\|B(\cdot,\omega,\phi_1) - B(\cdot,\omega,\phi_2)\|_{\gamma(L^2((0,T),\mu;H),X_{-\theta_B}))} \\ &\qquad \lesssim_{\theta_B,r,d,q} \|B_2(\cdot,\omega,\phi_1) - B_2(\cdot,\omega,\phi_2)\|_{\gamma(L^2((0,T),\mu;H),E_{-\frac{1}{2}(\tilde{\theta}_B+\varepsilon)})} \end{aligned}$$

for each $\omega \in \Omega$. Here B_2 is the second coordinate of B. The other one is zero.

Let $\frac{1}{v} = \frac{1}{q} + \frac{1}{r}$. We claim that $L^v(S) \hookrightarrow E_{-\frac{1}{2}(\tilde{\theta}_B+\varepsilon)}$. Indeed, let $\mathcal{B}_v$ denote the realization of the negative Dirichlet Laplacian in $L^v(S)$. Taking into account Theorem V.1.4.12 of [1], we have to show that

$$\|x\|_{L^q(S)} \lesssim_{q,v,\tilde{\theta}_B,\varepsilon} \|\mathcal{B}_v^{\tilde{\theta}_B+\varepsilon} x\|_{L^v(S)}$$

for all $x \in D(\mathcal{B}_v^{\tilde{\theta}+\varepsilon})$. From [31, Theorem 4.3.1.2] and (4.5) we deduce

$$\begin{aligned} \|x\|_{B^{2\tilde{\theta}_B}_{v,1}(S)} &\eqsim_{\tilde{\theta}_B,v} \|x\|_{(L^v(S),W^{2,v}(S))_{\tilde{\theta}_B,1}} \eqsim_v \|x\|_{(L^v(S),D(B_v))_{\tilde{\theta}_B,1}} \\ &\lesssim_{q,v,\tilde{\theta_B},\varepsilon} \|\mathcal{B}_v^{\tilde{\theta}_B+\varepsilon} x\|_{L^v(S)}, \end{aligned}$$

so that the claim follows from Sobolev's embedding (cf. [31, Theorem 4.6.1]). The claim, (2.2), Hölder's inequality and the Lipschitz continuity of b imply that

$$\begin{aligned} &\|B_2(\cdot,\omega,\phi_1) - B_2(\cdot,\omega,\phi_2)\|_{\gamma(L^2((0,T),\mu;H),E_{-\frac{1}{2}(\tilde{\theta}_B+\varepsilon)})} \\ &\lesssim_{\theta_B,r,d,q} \|B_2(\cdot,\omega,\phi_1) - B_2(\cdot,\omega,\phi_2)\|_{\gamma(L^2((0,T),\mu;H),L^v(S))} \\ &\eqsim_v \Big\| \Big(\int_0^T |b(t,\omega,\phi_{11},\phi_{12}) - b(t,\omega,\phi_{21},\phi_{22})|^2 \, d\mu(t) \sum_{n\geq 1} |i_1 h_n|^2 \Big)^{\frac{1}{2}} \Big\|_{L^v(S)} \\ &\leq \Big\| \Big(\int_0^T |b(t,\omega,\phi_{11},\phi_{12}) - b(t,\omega,\phi_{21},\phi_{22})|^2 \, d\mu(t) \Big)^{\frac{1}{2}} \Big\|_E \Big\| \Big(\sum_{n\geq 1} |i_1 h_n|^2 \Big)^{\frac{1}{2}} \Big\|_{L^r(S)} \\ &\lesssim_{w_1} L_b(\|\phi_{11} - \phi_{21}\|_{\gamma(L^2((0,T),\mu),E)} + \|\phi_{12} - \phi_{22}\|_{\gamma(L^2((0,T),\mu),E)}). \end{aligned}$$

The other estimate in (H4) can be established in a similar way. □

If $d = 1$ we can consider the space-time white noise situation, where the covariance operator $Q_1 : H_1 \to H_1$ is the identity. This is possible since in this case we can choose $q < d/(d-1)$ as large as needed.

Example 5.7. Let $Q_1 = I$ on $H_1 = L^2(S)$, $d = 1$, and $q \in (2,\infty)$. Then (H4) is satisfied for all $\theta_B > \frac{1}{4} + \frac{1}{2q}$ and $a = 0$.

Proof. Let $q \in (2,\infty)$ and $\frac{1}{2} > \theta_B > \frac{1}{4} + \frac{1}{2q}$. We take $\varepsilon > 0$ be such that $\theta_B - \varepsilon > \frac{1}{4} + \frac{1}{2q}$ and write $\theta_B - \varepsilon = \theta_1 + \theta_2$, where $\theta_1 > \frac{1}{4}$ and $\theta_2 > \frac{1}{2q}$. Since L^q with $q \in (2,\infty)$ has type 2, Lemma 2.2 says that B is L^2_γ–Lipschitz and of linear growth if $B(t,\omega,\cdot): X \to \gamma(H_1, X_{-\theta_B})$ is Lipschitz and of linear growth with a uniform constant.

We observe that $\mathcal{A}^{-\frac{\theta_1}{2}} \in \mathcal{B}(H_1, W^{2\theta_1,2}(S))$ and that the injection $i : W^{2\theta_1,2}(S) \to L^q(S)$ belongs to $\gamma(W^{2\theta_1,2}(S), L^q(S))$ because of [28, Corollary 2.2]. The right-ideal property (2.1) thus implies that

$$\|i\mathcal{A}^{-\frac{\theta_1}{2}}\|_{\gamma(H_1,L^q(S))} \leq \|i\|_{\gamma(W^{2\theta_1,2}(S),L^q(S))} \, \|\mathcal{A}^{-\frac{\theta_1}{2}}\|_{\mathcal{B}(H_1,W^{2\theta_1,2}(S))} < \infty.$$

For $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in X, we deduce from (4.6) and the right-ideal property that

$$\begin{aligned}
\|B(t,x) - B(t,y)\|_{\gamma(H_1,X_{-\theta_B})} &\lesssim_{\theta_B,q} \|b(t,\omega,x_1,x_2) - b(t,\omega,y_1,y_2))\|_{\gamma(H_1,E_{-\frac{1}{2}(\theta_B-\varepsilon)})} \\
&= \|i\mathcal{A}^{-\frac{\theta_1}{2}}\mathcal{A}^{-\frac{\theta_2}{2}}(b(t,\omega,x_1,x_2) - b(t,\omega,y_1,y_2))\|_{\gamma(H_1,L^q(S))} \\
&\leq \|i\mathcal{A}^{-\frac{\theta_1}{2}}\|_{\gamma(H_1,L^q(S))}\|\mathcal{A}^{-\frac{\theta_2}{2}}(b(t,\omega,x_1,x_2) - b(t,\omega,y_1,y_2))\|_{\mathcal{B}(H_1)}
\end{aligned}$$

for all $\omega \in \Omega$ and $t \geq 0$. As in the claim in the proof of Example 5.5 one can use Sobolev's embedding theorem to obtain

$$\begin{aligned}
\|\mathcal{A}^{-\frac{\theta_2}{2}}(b(t,\omega,x_1,x_2) - &b(t,\cdot,y_1,y_2))\|_{\mathcal{B}(H)} \\
&\leq \|\mathcal{A}^{-\frac{\theta_2}{2}}(b(t,\cdot,x_1,x_2) - b(t,\cdot,y_1,y_2))\|_{L^\infty(S)} \\
&\lesssim_{\theta_2,q} \|b(t,\cdot,x_1,x_2) - b(t,\cdot,y_1,y_2)\|_E \\
&\leq L_b\,(\|x_1 - y_1\|_E + \|x_2 - y_2\|_E) \lesssim L_b\|x-y\|_X.
\end{aligned}$$

Thus we have shown the Lipschitz estimate in Lemma 2.2. The other estimate in this lemma can be established in a similar way. □

Remark 5.8. It is clear from the proofs of Examples 5.3, 5.5 and 5.7 that (H4) also holds if b also depends on ∇u and $\nabla^2 u$ in an appropriate Lipschitz sense. The same is true for f, G and C in Theorem 5.1.

Remark 5.9. In the above examples one could allow f and b to be only locally Lipschitz in the third coordinate, i.e., the coordinate for $u(t,s)$. For this one needs to define the maps F, B, C, G on X_a for a suitable $a > 0$ such that the first component of X_a is embedded into $C(\overline{S})$. (See [28, Theorems 8.1 and 10.2] for details.) This gives the condition $2 + 2a - \frac{d}{q} > 0$. However, we can only take $a > 0$ such that $a + \theta_C < \frac{1}{2}$ and $a + \theta_B < \frac{1}{2}$. Since $\theta_C \in (\frac{d}{2q'}, \frac{1}{2})$ as explained before Theorem 5.1, we obtain the first condition $-1 + \frac{d}{2} < \frac{1}{2}$. This inequality holds for $d = 1, 2$.

For Example 5.3 there are no conditions on θ_B, so that $d = 1, 2$ are both allowed. For Example 5.5 we also need $\theta_B > \frac{d}{2r}$, and therefore $\frac{d}{2r} + \frac{d}{2q} < \frac{3}{2}$ must hold as well. This condition holds for $d = 1, 2$ and all $r > d$ and $1 < q < d/(d-1)$. For the Example 5.7 we have $d = 1$. There the condition reads $\theta_B > \frac{1}{4} + \frac{1}{2q}$. Therefore, we obtain $\frac{1}{4} + \frac{1}{q} < \frac{1}{2}$. This holds if and only if $q > 4$.

6. The damped wave equation

In this section we obtain existence, uniqueness and regularity results for a structurally damped wave equation. Since the proofs follow the line of arguments of the previous section, we omit the details. The equation is given by

$$(6.1)\quad \begin{cases} \ddot u(t,s) - \Delta u(t,s) - \rho(-\Delta)^{\frac12}\dot u(t,s) = f(t,s,u(t,s),\dot u(t,s)) \\ \qquad + b(t,s,u(t,s),\dot u(t,s))\dfrac{\partial w_1(t,s)}{\partial t} + \Big[G(t,u(t,\cdot),\dot u(t,\cdot)) \\ \qquad + C(t,u(t,\cdot),\dot u(t,\cdot))\dfrac{\partial w_2(t)}{\partial t}\Big]\delta(s-s_0), \quad t\in[0,T], s\in S, \\ u(0,s) = u_0(s), \ \dot u(0,s) = u_1(s), \quad s\in S, \\ u(t,s) = 0, \quad t\in[0,T], \ s\in\partial S, \end{cases}$$

where $S \subset \mathbb{R}^n$ has a C^2 boundary ∂S and $(-\Delta)^{\frac12}$ denotes the square root of the negative Dirichlet Laplacian. We reformulate this equation as (SE) in the same way as in Section 5.

Let $q \in (1,\infty)$ and $E = L^q(S)$. On E we define $(\mathcal{A}, D(\mathcal{A}))$ by

$$\mathcal{A}x = -\Delta x, \quad D(\mathcal{A}) = W^{2,q}(S) \cap W^{1,q}_0(S).$$

Let $X = E_{\frac12} \times E$ and define $(A, D(A))$ by

$$A = \begin{pmatrix} 0 & I \\ \mathcal{A} & -\rho\mathcal{A}^{\frac12} \end{pmatrix}, \quad D(A) = D(\mathcal{A}) \times D(\mathcal{A}^{\frac12}).$$

It follows from Proposition 4.1 that A generates an analytic semigroup $(S(t))_{t\geq 0}$.

We further assume that $\rho > 0$ and $s_0 \in S$ are fixed and that δ is the usual point evaluation. Moreover, f, b, C, G, w_1, w_2 shall satisfy the assumptions (A0)–(A3) in Section 5 for the above space X and the maps F, B and Λ are defined as in Section 5 for the above space X. Finally, it assumed that B fulfills hypothesis (H4). Noting that $\mathcal{A}$ is now of second order, one can see in the same way as in Section 5 that Λ is well-defined for all $1 < q < \frac{2d}{2d-1}$. A mild and weak solution are defined in a similar way as in Section 5. Finally, for $1 < q < 2$ we have

$$E_{\frac12+\frac12\delta} \times E_{\frac12\delta} = (W^{1+\delta,q}(S) \cap W^{1,q}_0(S)) \times W^{\delta,q}(S).$$

Theorem 6.1. *Let $1 < q < \frac{2d}{2d-1}$. Assume that $u_0 : \Omega \to W^{1,q}_0(S)$ and $u_1 : \Omega \to L^q(S)$ are $\mathcal{F}_0$-measurable. Let f, G, b, C, w_1 and w_2 be as above. The following assertions hold:*

(1) *For all $\alpha \in (0, \frac12)$ and $p > 2$ such that $a + \max\{\theta_B, \theta_C\} < \alpha - \frac1p$, there exists a unique mild solution U of (6.1) in*

$$V^0_{\alpha,p}([0,T]\times\Omega; W^{1,q}_0(S) \times L^q(S)).$$

There is a unique weak solution $u \in W^{1,2}(0,T;L^q(S))$ of (5.1) such that $(u, \dot u)$ belongs to the space in (5.6). Moreover, $U = (u, \dot u)$.

(2) *There exists a version of u with paths that satisfy $u \in C([0,T]; W^{1,q}_0(S))$ and $\dot u \in C([0,T]; L^q(S))$.*

(3) *Let $\eta \in (0, \frac12]$. If $u_0 : \Omega \to H^{1+\eta,q}(S) \cap W^{1,q}_0(S)$ and $u_1 : \Omega \to H^{\eta,q}(S)$, then there exists a version of u with paths that satisfy $u \in C^\lambda([0,T]; H^{1+\delta,q}(S) \cap W^{1,q}_0(S))$ and $\dot u \in C^\lambda([0,T]; H^{\delta,q}(S))$ for all $\delta, \lambda \geq 0$ such that $\delta + \lambda < \min\{\eta, \frac12 - \theta_B, \frac12 - \theta_C\}$.*

This theorem can be proved in the same way as Theorem 5.1. Let us give some examples for w_1. Example 5.3 works in exactly the same way for the wave equation. Example 5.5 has the following version for the wave equation.

Example 6.2. Assume that w_1 is an $L^r(S)$-valued Brownian motion with $r > 2d$. Then (H4) is satisfied for all $\theta_B \in (\frac{d}{r}, 1)$ and $a = 0$.

This assertion can be shown as in Example 5.5, we thus leave the details to reader.

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