

# Characterization of mass-stationarity by Bernoulli and Cox transports

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## Abstract

Consider a random measure  $\xi$  on a locally compact Abelian group  $G$  acting on some random element  $X$ . Mass-stationarity – introduced in [6] – means (informally) that the origin is a typical location for  $(X, \xi)$  in the mass of  $\xi$ . It is an intrinsic characterization of Palm versions w.r.t stationary random measures. In this paper we show that mass-stationarity w.r.t. discrete  $\xi$  is characterized by distributional invariance under shifts of the origin by certain mass-preserving transports involving a Bernoulli randomization of the group-identity and an allocation rule. We also show that mass-stationarity w.r.t. a general  $\xi$  is characterized by mass-stationarity w.r.t. a Cox process driven by  $\xi$ .

## 1 Introduction

Let  $\xi$  be a random measure on a locally compact Abelian group  $G$ . *Mass-stationarity* is a formalization of the intuitive idea that the origin is a typical location in the mass of  $\xi$ , just like stationarity means that the origin is a typical location in the space  $G$ . The formal definition is given in Section 2 below. Actually, we will consider  $\xi$  jointly with a random element  $X$  which  $G$  acts on, for instance a random field indexed by  $G$ . Then  $(X, \xi)$  is *mass-stationary* if the origin is a typical location for  $(X, \xi)$  in the mass of  $\xi$ .

The word ‘typical’ needs some explanation. If  $\xi$  is finite and  $S$  is a random element in  $G$  with conditional distribution  $\xi/\xi(G)$  given  $\xi$ , then we say that  $S$  is a *typical* location in the *mass* of  $\xi$ . We also say that the *origin* is a typical location in the mass of the *shifted* measure  $\xi(\cdot - S)$ . Further, if  $S$  has conditional distribution  $\xi/\xi(G)$  given  $(X, \xi)$ , then we say that  $S$  is a *typical* location for  $(X, \xi)$  in the mass of  $\xi$ , and also that the origin is a *typical* location in the mass of  $\xi(\cdot - S)$  for the pair  $(X, \xi)$  shifted by  $S$ . In this introduction we use the term ‘typical’ even for infinite  $\xi$  in order to explain informally the basic ideas of the paper.

Mass-stationarity was introduced in [6] as an extension to random measures of *point-stationarity*, which in turn was introduced in [8] for simple point processes in  $\mathbb{R}^d$  having a

point at the origin. Point-stationarity formalizes the intuitive idea that the point at the origin is a typical point of the point process (think of the Poisson process on the line with an extra point added at the origin: shifting the origin to the  $n^{\text{th}}$  point on the right – or to the  $n^{\text{th}}$  point on the left – does not change the fact that the inter-point distances are i.i.d. exponential). The definition in [8] involved an external randomization, but in [1] (and in [2] for the group case) it is shown that point-stationarity can be defined as ‘distributional invariance under shifts of the origin by preserving allocation rules’: an *allocation rule*  $\tau$  is a map taking each location  $s \in G$  to another location  $\tau(s) \in G$  depending on  $\xi(\cdot - s)$ , and  $\tau$  is *preserving* if the image of  $\xi$  under  $\tau$  is  $\xi$  itself. In fact, [1] and [2] show that ‘matchings’ suffice for the definition: an allocation rule  $\tau$  is a *matching* if  $\tau$  is its own inverse.

In [8] it was shown that point-stationarity is an intrinsic characterization of Palm versions of stationary point processes, and the same is proved in [6] for mass-stationarity and random measures. In this paper we will derive further characterizations of mass-stationarity.

The term ‘Bernoulli transport’ refers to a randomized allocation rule that allows staying at a location  $s$  with a probability  $p(s)$  depending on  $\xi(\cdot - s)$ , and otherwise chooses another location according to a (non-randomized) allocation rule. This makes it possible to preserve discrete point-masses even if there are point-masses of different sizes. In Section 3 we show that mass-stationarity of discrete random measures can be reduced to distributional invariance of  $\xi$  under shifts of the origin by preserving Bernoulli transports, Theorem 3.2. A similar result holds for random pairs  $(X, \xi)$ .

A Cox process  $\zeta$  is a Poisson process with a random intensity measure  $\xi$ . Such a process can be thought of as a collection of points scattered independently over the space  $G$  according to the mass distribution of  $\xi$ , so these points are at typical locations in the mass of  $\xi$ . Thus if  $\xi$  is mass-stationary and we add a point at the origin to the Cox process to obtain  $\zeta^0 := \zeta + \delta_0$ , then also the points of  $\zeta^0$  are at typical locations in the mass of  $\xi$ . In fact, one might expect that the new point at the origin is a typical point of  $\zeta^0$ , in other words that  $\zeta^0$  is point-stationary, and even that the pair  $(\xi, \zeta^0)$  is point-stationary. Actually, one might expect that the pair  $(\xi, \zeta^0)$  is point-stationary *if and only if*  $\xi$  is mass-stationary. In Section 4 we show that this is indeed the case. In fact, the result extends to random pairs  $(X, \xi)$ , Theorem 4.1.

The term ‘Cox transport’ refers to applying an allocation rule to a Cox process driven by a general random measure (think of the mass of the random measure being represented by the points of the Cox process). In particular, mass-stationarity of  $\xi$  then reduces to point-stationarity with respect to  $\zeta^0$ , Theorem 4.1. Also, it follows that mass-stationarity is characterized by applying preserving Bernoulli transports to the Cox process, Corollary 4.3. Finally, for diffuse random measures mass-stationarity is characterized by applying matchings to the Cox process, Corollary 4.4.

## 2 Transports and mass-stationarity

We consider a topological Abelian group  $G$  that is assumed to be a locally compact, second countable Hausdorff space with Borel  $\sigma$ -field  $\mathcal{G}$  and Haar measure  $\lambda$ . Let  $M$  denote the set of all locally finite measures on  $G$  equipped with the cylindrical  $\sigma$ -field  $\mathcal{M}$ . Let  $(\Omega, \mathcal{F}, \mathbb{P})$

be a  $\sigma$ -finite measure space. Although  $\mathbb{P}$  need not be a probability measure, we still use a probabilistic language. A *random measure* is a random element  $\xi$  in  $M$ . We use the kernel notation  $\xi(\omega, \cdot) := \xi(\omega)(\cdot)$ ,  $\omega \in \Omega$ . We equip  $(M, \mathcal{M})$  with a *measurable flow*  $\theta_s : M \rightarrow M$ ,  $s \in G$ , defined by  $\theta_s \mu(B) := \mu(B + s)$ , where  $B \in \mathcal{G}$  and  $B + s := \{t + s : t \in B\}$ . Then  $(\mu, s) \mapsto \theta_s \mu$  is a measurable mapping,  $\theta_0$  is the identity on  $M$ , and we have the flow property

$$\theta_s \circ \theta_t = \theta_{s+t}, \quad s, t \in G. \quad (2.1)$$

Here 0 denotes the neutral element in  $G$  and  $\circ$  denotes composition. Together with  $\xi$  we consider a random element  $X$  in a measurable space  $(W, \mathcal{W})$ . We assume that this space is equipped with a measurable flow  $\theta_s : W \rightarrow W$ ,  $s \in G$ , having the properties listed above. (Denoting this flow again by  $\theta_s$ ,  $s \in G$ , will cause no risk of ambiguity.)

Next we adapt some terminology from [6] to the setting established above. This makes some of the definitions more cumbersome. However, the present setting is closer to chapter 11 of [4] and chapter 9 of [9] and will allow for a more convenient formulation of our main results in Section 4. In the remainder of this paper we consider a pair  $(X, \xi)$  as introduced above such that  $\mathbb{P}((X, \xi) \in \cdot)$  is  $\sigma$ -finite and  $\mathbb{P}(\xi(G) = 0) = 0$ . We call  $(X, \xi)$  *stationary* if  $\mathbb{P}(\theta_s(X, \xi) \in \cdot) = \mathbb{P}((X, \xi) \in \cdot)$  for all  $s \in G$ . Here we define  $\theta_s(w, \mu) := (\theta_s w, \theta_s \mu)$  for  $s \in G$  and  $(w, \mu) \in W \times M$ . If  $(X, \xi)$  is stationary, then we also call  $\mathbb{P}((X, \xi) \in \cdot)$  *invariant*. In this case the measure

$$\mathbb{P}_{X, \xi}(A) := \lambda(B)^{-1} \iint \mathbf{1}_A(\theta_s(X(\omega), \xi(\omega))) \mathbf{1}_B(s) \xi(\omega, ds) \mathbb{P}(d\omega), \quad A \in \mathcal{W} \otimes \mathcal{M}, \quad (2.2)$$

is called the *Palm measure* of  $(X, \xi)$  (with respect to  $\mathbb{P}$ ), see [7]. Here  $B \in \mathcal{G}$  has  $0 < \lambda(B) < \infty$ . This measure is  $\sigma$ -finite. As the definition (2.2) is independent of  $B$ , we can use a monotone class argument to conclude the *refined Campbell theorem*

$$\iint f(\theta_s(X(\omega), \xi(\omega)), s) \xi(\omega, ds) \mathbb{P}(d\omega) = \iint f(x, \mu, s) ds \mathbb{P}_{X, \xi}(d(x, \mu))$$

for all measurable  $f : W \times M \times G \rightarrow [0, \infty)$ , where  $ds$  refers to integration with respect to the Haar measure  $\lambda$ . Using a standard convention in probability theory, we write this as

$$\mathbb{E}_{\mathbb{P}} \left[ \int f(\theta_s(X, \xi), s) \xi(ds) \right] = \mathbb{E}_{\mathbb{P}_{X, \xi}} \left[ \int f(X, \xi, s) ds \right], \quad (2.3)$$

where  $\mathbb{E}_{\mathbb{P}}$  and  $\mathbb{E}_{\mathbb{P}_{X, \xi}}$  denote integration with respect to  $\mathbb{P}$  and  $\mathbb{P}_{X, \xi}$ , respectively.

Next we define *mass-stationarity* of  $(X, \xi)$ . Let  $C \in \mathcal{G}$  be a relatively compact set having  $\lambda(C) > 0$  and  $\lambda(\partial C) = 0$ , where  $\partial C$  denotes the boundary of  $C$ . Let  $U, V$  be random elements in  $G$ , possibly obtained by extending  $(\Omega, \mathcal{F}, \mathbb{P})$ . Assume that  $(X, \xi)$  and  $U$  are independent,  $U$  has the uniform distribution on  $C$  (w.r.t. Haar measure), and that the conditional distribution of  $V$  given  $(X, \xi, U)$  is uniform in the mass of  $\xi$  on  $C - U$ . Then  $(X, \xi)$  is called *mass-stationary* if

$$(\theta_V(X, \xi), U + V) \stackrel{d}{=} (X, \xi, U) \quad (2.4)$$

holds for all such  $C$ . In this case we call the distribution  $\mathbb{P}((X, \xi) \in \cdot)$  mass-stationary. By Theorem 6.3 in [6] this is equivalent to the validity of the Mecke equation

$$\mathbb{E}_{\mathbb{P}} \left[ \int g(\theta_s(X, \xi), -s) \xi(ds) \right] = \mathbb{E}_{\mathbb{P}} \left[ \int g(X, \xi, s) \xi(ds) \right] \quad (2.5)$$

for all measurable  $g : W \times M \times G \rightarrow [0, \infty)$ .

**Remark 2.1.** The random element  $X$  is *stationary* if  $\mathbb{P}(\theta_s X \in \cdot) = \mathbb{P}(X \in \cdot)$  for all  $s \in G$  and if this measure is  $\sigma$ -finite. Mass-stationarity generalizes this concept. Indeed, assuming (2.5) for  $\xi = \lambda$  we easily get that  $\mathbb{P}(\theta_s X \in \cdot) = \mathbb{P}(X \in \cdot)$  for  $\lambda$ -a.e.  $s \in G$ . Assuming that  $W$  is a metric space with Borel  $\sigma$ -field  $\mathcal{W}$  and that  $s \mapsto \theta_s X$  is  $\mathbb{P}$ -a.e. continuous, we obtain stationarity of  $X$ .

**Remark 2.2.** By definition, mass-stationarity of  $(X, \xi)$  is equivalent to mass-stationarity of  $((X, \xi), \xi)$ .

For the next definitions it is convenient to abbreviate  $\Omega' := W \times M$  and  $\mathcal{F}' := \mathcal{W} \otimes \mathcal{M}$ . A *weighted transport-kernel* is a kernel  $T$  from  $\Omega' \times G$  to  $G$  such that  $T(\omega', s, \cdot)$  is locally finite for all  $(\omega', s) \in \Omega' \times G$ . If  $T$  is Markovian, then it is called *transport-kernel*. A weighted transport-kernel is *invariant* if  $T(\theta_s \omega', 0, B - s) = T(\omega', s, B)$  for all  $(\omega', s) \in \Omega' \times G$  and  $B \in \mathcal{G}$ . An *allocation rule* is a measurable mapping  $\tau : \Omega' \times G \rightarrow G$  which is *covariant*, i.e. which has  $\tau(\theta_s \omega', 0) = \tau(\omega', s) - s$  for all  $\omega', s$ . A weighted transport-kernel  $T$  is *mass-preserving* if

$$\int T(w, \mu, s, \cdot) \mu(ds) = \mu(\cdot) \quad (2.6)$$

holds for all  $(w, \mu) \in \Omega'$ . An allocation rule is *mass-preserving* if

$$\int \mathbf{1}\{\tau(w, \mu, s) \in \cdot\} \mu(ds) = \mu(\cdot) \quad (2.7)$$

holds for all  $(w, \mu) \in \Omega'$ . If these relations hold almost everywhere w.r.t. some measure  $\mathbb{Q}$  on  $\Omega'$ , then we say that  $T$  (resp.  $\tau$ ) is  $\mathbb{Q}$ -a.e. mass-preserving.

**Remark 2.3.** Let  $T$  be a locally finite kernel from  $W \times M \times G$  to  $G$ . Assume that there is some  $A \in \mathcal{W} \otimes \mathcal{M}$  such that

$$\int T(w, \mu, s, \cdot) \mu(ds) = \mu(\cdot), \quad (2.8)$$

holds for all  $(w, \mu) \in A$ . Then we can redefine  $T$  on  $((W \times M) \setminus A) \times G$  by  $T(w, \mu, s, \cdot) := \delta_s$ , to obtain a kernel  $T$  satisfying (2.8) for all  $(w, \mu) \in W \times M$ . If  $A$  is *invariant* (i.e.  $\theta_s A = A$ ,  $s \in G$ ) and  $T$  is invariant, then the modified  $T$  is an invariant kernel too. A similar remark applies to allocation rules.

By Theorem 7.2 in [6]  $(X, \xi)$  is mass-stationary, iff

$$\mathbb{E}_{\mathbb{P}} \left[ \int \mathbf{1}\{\theta_t(X, \xi) \in A\} T(X, \xi, 0, dt) \right] = \mathbb{P}((X, \xi) \in A), \quad A \in \mathcal{F}', \quad (2.9)$$

holds for all invariant mass-preserving weighted transport-kernels  $T$ .

A measure  $\mu \in M$  is *discrete* if

$$\mu = \sum_{s:\mu\{s\}>0} \mu\{s\}\delta_s$$

and *diffuse* if  $\mu\{s\} = 0$  for all  $s \in G$ . Lemma 2.2 in [3] shows that any  $\mu \in M$  can be measurably and uniquely written as the sum of a discrete measure  $\mu^d$  and a diffuse measure  $\mu^c$ . The proof of this result shows that the mapping  $\mu \mapsto (\mu^d, \mu^c)$  is covariant in the obvious sense. Therefore the characterization (2.5) of mass-stationarity together with  $\xi = \xi^d + \xi^c$  implies the following result.

**Proposition 2.4.** *If  $(X, \xi^d)$  and  $(X, \xi^c)$  are both mass-stationary, then  $(X, \xi)$  is mass-stationary.*

### 3 Bernoulli transports

A *Bernoulli transport-kernel* is a transport-kernel  $T$  of the form

$$T(w, \mu, s, \cdot) = p(w, \mu, s)\delta_s + (1 - p(w, \mu, s))\delta_{\tau(w, \mu, s)}, \quad (w, \mu, s) \in W \times M \times G, \quad (3.1)$$

where  $p : W \times M \times G \rightarrow [0, 1]$  is measurable and  $\tau : W \times M \times G \rightarrow G$  is a measurable mapping. Invariance of Bernoulli transport-kernels can easily be characterized as follows.

**Lemma 3.1.** *Let  $T$  be a Bernoulli transport-kernel as in (3.1) such that for all  $(w, \mu) \in W \times M$  it holds that  $p(w, \mu, s) = 1$  iff  $\tau(w, \mu, s) = s$ . Then  $T$  is invariant iff  $\tau$  is covariant and  $p(w, \mu, s) = p(\theta_s(w, \mu), 0)$  for all  $(w, \mu, s) \in W \times M \times G$ .*

Recall that  $(X, \xi)$  is a random pair such that  $\mathbb{P}((X, \xi) \in \cdot)$  is  $\sigma$ -finite and  $\mathbb{P}(\xi(G) = 0) = 0$ . We will show that the validity of (2.9) for all invariant Bernoulli transport-kernels is sufficient for mass-stationarity of  $(X, \xi)$ . The *support* of a measure  $\mu \in M$  is denoted by  $\text{supp } \mu$ . Here we need to make the weak assumption, that  $(W, \mathcal{W})$  is a *Borel space*, i.e. Borel isomorphic to a Borel subset of  $[0, 1]$ , see e.g. Appendix A1 in [4].

**Theorem 3.2.** *Assume that  $(W, \mathcal{W})$  is a Borel space, that  $\mathbb{P}(0 \notin \text{supp } \xi) = 0$ , and that  $\mathbb{P}(\xi \neq \xi^d) = 0$ . Assume also that (2.9) holds for all invariant mass-preserving Bernoulli transport-kernels  $T$ . Then  $(X, \xi)$  is mass-stationary.*

Our proof of Theorem 3.2 requires the following generalization of a result in [2]. A proof can be found in [5]. A *matching* is an allocation rule  $\tau$  such that the following holds for all  $(w, \mu) \in W \times M$ :  $\tau(w, \mu, s) \in \text{supp } \mu$  and  $\tau(w, \mu, \tau(w, \mu, s)) = s$  for all  $s \in \text{supp } \mu$ , and  $\tau(w, \mu, s) = s$  for all  $s \notin \text{supp } \mu$ .

**Lemma 3.3.** *Assume that  $(W, \mathcal{W})$  is a Borel space. Then there exist invariant matchings  $\tau_k$ ,  $k \in \mathbb{N}$ , such that for all  $(w, \mu) \in W \times M$  with  $\text{supp } \mu$  locally finite and  $0 \in \text{supp } \mu$*

$$\{0\} \cup \{t \in \text{supp } \mu : \theta_t(w, \mu) \neq (w, \mu)\} \subset \{\tau_k(w, \mu, 0) : k \in \mathbb{N}\}. \quad (3.2)$$

For  $n \in \mathbb{N}$  and  $\mu \in M$  we define  $\mu_n \in M$  by

$$\mu_n(B) := \int_B \mathbf{1}\{1/n \leq \mu\{s\} \leq n\} \mu(ds), \quad B \in \mathcal{G}.$$

Then  $1/n \leq \mu_n\{s\} \leq n$ ,  $s \in \text{supp } \mu_n$ , and  $\text{supp } \mu_n$  is locally finite. We will use the following version of Lemma 3.3.

**Lemma 3.4.** *Assume that  $(W, \mathcal{W})$  is a Borel space and let  $n \in \mathbb{N}$ . Then there exist invariant matchings  $\tau_k$ ,  $k \in \mathbb{N}$ , such that for all  $(w, \mu) \in W \times M$  with  $0 \in \text{supp } \mu_n$*

$$\{0\} \cup \{t \in \text{supp } \mu_n : \theta_t(w, \mu) \neq (w, \mu)\} \subset \{\tau_k(w, \mu, 0) : k \in \mathbb{N}\}. \quad (3.3)$$

Furthermore, the  $\tau_k$  can be chosen such that the following holds for all  $(w, \mu) \in W \times M$ . If  $s \notin \text{supp } \mu_n$  then  $\tau_k(w, \mu, s) = s$  and if  $s \in \text{supp } \mu_n$  then  $\tau_k(w, \mu, s) \in \text{supp } \mu_n$ .

*Proof.* We apply Lemma 3.3 with  $W$  replaced by  $W \times M$ . This gives matchings  $\tau'_k$ ,  $k \in \mathbb{N}$ , such that for all  $(w, \mu, \nu) \in W \times M \times M$  with  $\text{supp } \nu$  locally finite and  $0 \in \text{supp } \nu$

$$\{0\} \cup \{t \in \text{supp } \nu : \theta_t(w, \mu, \nu) \neq (w, \mu, \nu)\} \subset \{\tau'_k((w, \mu), \nu, 0) : k \in \mathbb{N}\}.$$

For any  $k \in \mathbb{N}$  we define a mapping  $\tau_k : W \times M \times G \rightarrow G$  by  $\tau_k(w, \mu) := \tau'_k((w, \mu), \mu_n)$ . Then (3.3) holds. (Note that  $\theta_t(w, \mu, \mu_n) = (w, \mu, \mu_n)$  iff  $\theta_t(w, \mu) = (w, \mu)$ .) It is now easy to see that the  $\tau_k$  are invariant matchings with the properties stated in the lemma.  $\square$

*Proof of Theorem 3.2.* It is convenient (and no restriction of generality) to assume that  $(\Omega, \mathcal{F}) = (W \times M, \mathcal{W} \otimes \mathcal{M})$ ,  $\mathbb{P} = \mathbb{P}((X, \xi) \in \cdot)$ , and that  $(X, \xi)$  is the identity on  $W \times M$ . We will prove the Mecke equation (2.5). Satz 2.5 in [7] (see also Section 2 in [6]) shows that  $\mathbb{P}$  is the Palm measure of  $(X, \xi)$  w.r.t. a  $\sigma$ -finite invariant measure on  $\Omega$ . By Theorem 7.3 in [6] this is equivalent to mass-stationarity of  $(X, \xi)$ .

In the sequel we fix  $n \in \mathbb{N}$ . Let  $\tau$  be an invariant matching with the properties listed after (3.3). Define a Bernoulli transport-kernel  $T$  by

$$T(w, \mu, s, \cdot) := \frac{\mu\{s\}}{\mu\{s\} + \mu\{\tau(s)\}} \delta_s + \frac{\mu\{\tau(s)\}}{\mu\{s\} + \mu\{\tau(s)\}} \delta_{\tau(s)} \quad (3.4)$$

if  $s \in \text{supp } \mu_n$ , and  $T(w, \mu, s, \cdot) := \delta_s$ , otherwise. Here and below we skip the argument  $(w, \mu)$  whenever possible. This transport-kernel is of the form (3.1) with

$$p(s) := \mathbf{1}\{\tau(s) \neq s\} \frac{\mu\{s\}}{\mu\{s\} + \mu\{\tau(s)\}} + \mathbf{1}\{\tau(s) = s\}, \quad (3.5)$$

where we recall that  $\tau(s) = s$  for  $s \notin \text{supp } \mu_n$ . We have

$$p(\theta_s, 0) = \mathbf{1}\{\tau(\theta_s, 0) \neq 0\} \frac{\theta_s \mu\{0\}}{\theta_s \mu\{0\} + \theta_s \mu\{\tau(\theta_s, 0)\}} + \mathbf{1}\{\tau(\theta_s, 0) = 0\}.$$

Since  $\tau(\theta_s, 0) = \tau(s) - s$  and  $\theta_s \mu\{t\} = \mu\{t + s\}$ ,  $t \in G$ , we obtain that  $p(\theta_s, 0) = p(s)$ . Lemma 3.1 implies that  $T$  is invariant.

We next prove that  $T$  is mass-preserving, i.e.

$$\int T(w, \mu, s, \{t\}) \mu(ds) = \mu\{t\}, \quad t \in G, w \in W, \mu \in M. \quad (3.6)$$

Fix  $w \in W$  and  $\mu \in M$ , and take  $t \in G$ . Assume first that  $t \notin \text{supp } \mu_n$ . Then  $\tau(t) = t$  and  $T(t, \{t\}) = 1$ . Let  $s \in G \setminus \{t\}$ . If  $s \notin \text{supp } \mu_n$  then  $\tau(s) = s$  and  $T(s, \{t\}) = 0$ . If  $s \in \text{supp } \mu_n$ , then  $T(s, \{t\}) > 0$  is only possible if  $\tau(s) = t$ , i.e.  $\tau(t) = s$ . As this would contradict  $\tau(t) = t$ , we again get  $T(s, \{t\}) = 0$ . Hence  $T(s, \{t\}) = \mathbf{1}\{s = t\}$ , implying (3.6) for  $t \notin \text{supp } \mu_n$ .

Assume now that  $t \in \text{supp } \mu_n$ . Then  $T(s, \{t\}) = 0$  for  $s \notin \text{supp } \mu_n$ . (Otherwise we would obtain that  $\tau(s) = t \neq s$ .) For  $s \in \text{supp } \mu_n$  we can have  $T(s, \{t\}) > 0$  only if  $s = t$  or  $\tau(s) = t$ . The latter equality implies  $\tau(t) = s$ . If  $\tau(t) = t$  then  $T(s, \{t\}) = 0$  for all  $s \in \text{supp } \mu_n \setminus \{t\}$  and thus (3.6) holds. The only non-trivial case is  $\tau(t) \neq t$ . Then the left-hand side of (3.6) equals

$$\begin{aligned} & \mu\{t\}T(t, \{t\}) + \mu\{\tau(t)\}T(\tau(t), \{t\}) \\ &= \mu\{t\} \frac{\mu\{t\}}{\mu\{t\} + \mu\{\tau(t)\}} + \mu\{\tau(t)\} \frac{\mu\{t\}}{\mu\{t\} + \mu\{\tau(t)\}} = \mu\{t\}, \end{aligned}$$

where we have again used that  $\tau(\tau(t)) = t$ .

We have established that  $T$  is an invariant mass-preserving Bernoulli transport-kernel and will now head towards (2.5). Let us define the *mass-shift*  $\theta_\tau : \Omega \rightarrow \Omega$  by  $\theta_\tau(\omega) := \theta_{\tau(\omega, 0)}(\omega)$ . (We also define the random measure  $\theta_\tau \xi$  by  $\theta_\tau \xi(\omega) := \theta_{\tau(\omega, 0)} \xi(\omega)$ ; the random measure  $\theta_\tau \xi_n = (\theta_\tau \xi)_n$  is defined in the same way.) A quick consequence of the matching property of  $\tau$  is

$$\tau(\theta_\tau, 0) = -\tau(0). \quad (3.7)$$

In particular we have

$$\mathbf{1}_A(\theta_\tau) = \mathbf{1}_A, \quad (3.8)$$

where  $A := \{\tau(0) \neq 0\}$ . Note that  $A \subset \{0 \in \text{supp } \xi_n, \tau(0) \in \text{supp } \xi_n\}$ . Let  $f : \Omega \rightarrow [0, \infty)$  be measurable with  $\mathbb{E}_\mathbb{P}[f] < \infty$ . Let  $B \in \mathcal{G}$  and define  $g(\omega, s) := f(\omega) \mathbf{1}\{s \in B\}$ . By assumption and the facts established above we can apply (2.9) for our specific  $T$ , to obtain

$$\begin{aligned} \mathbb{E}_\mathbb{P}[\mathbf{1}_A g(\theta_0, \tau(0)) \xi\{\tau(0)\}] &= \mathbb{E}_\mathbb{P} \left[ \int \mathbf{1}_A(\theta_s) g(\theta_s, \tau(\theta_s, 0)) \xi(\theta_s, \{\tau(\theta_s, 0)\}) T(0, ds) \right] \\ &= \mathbb{E}_\mathbb{P} [\mathbf{1}_A g(\theta_0, \tau(0)) \xi\{\tau(0)\} p(0)] + \mathbb{E}_\mathbb{P} [\mathbf{1}_A g(\theta_\tau, -\tau(0)) \xi(\theta_\tau, \{-\tau(0)\}) (1 - p(0))], \end{aligned}$$

where we have used (3.8) and (3.7) for the second equality. (We suppress the dependence on  $(X, \xi)$  in the notation; for instance we use  $\theta_s$  as a shorthand for  $\theta_s(X, \xi)$ .) Recalling the definition of  $p$  and using  $\theta_\tau \xi\{-\tau(0)\} = \xi\{0\}$ , we get

$$\begin{aligned} & \mathbb{E}_\mathbb{P}[\mathbf{1}_A g(\theta_0, \tau(0)) \xi\{\tau(0)\}] \\ &= \mathbb{E}_\mathbb{P} \left[ \mathbf{1}_A g(\theta_0, \tau(0)) \frac{\xi\{0\} \xi\{\tau(0)\}}{\xi\{0\} + \xi\{\tau(0)\}} \right] + \mathbb{E}_\mathbb{P} \left[ \mathbf{1}_A g(\theta_\tau, -\tau(0)) \frac{\xi\{0\} \xi\{\tau(0)\}}{\xi\{0\} + \xi\{\tau(0)\}} \right]. \end{aligned}$$

Since for  $0 \in \text{supp } \xi_n$  and  $\tau(0) \in \text{supp } \xi_n$

$$g(\theta_0, \tau(0)) \frac{\xi\{0\}\xi\{\tau(0)\}}{\xi\{0\} + \xi\{\tau(0)\}} \leq f \frac{n^3}{2}, \quad g(\theta_0, \tau(0))\xi\{\tau(0)\} \leq fn,$$

and  $\mathbb{E}_{\mathbb{P}}[f] < \infty$ , we get by subtraction

$$\mathbb{E}_{\mathbb{P}} \left[ \mathbf{1}_{Ag}(\theta_0, \tau(0)) \frac{\xi\{\tau(0)\}\xi\{\tau(0)\}}{\xi\{0\} + \xi\{\tau(0)\}} \right] = \mathbb{E}_{\mathbb{P}} \left[ \mathbf{1}_{Ag}(\theta_\tau, -\tau(0)) \frac{\xi\{\tau(0)\}\xi\{0\}}{\xi\{0\} + \xi\{\tau(0)\}} \right]. \quad (3.9)$$

Consider the function  $\tilde{g} : \Omega \times G \rightarrow [0, \infty)$  given by

$$\tilde{g}(s) := \mathbf{1}\{0 \in \text{supp } \xi_n, s \in \text{supp } \xi_n\} \frac{\xi\{0\} + \xi\{s\}}{\xi\{s\}}.$$

We have

$$\begin{aligned} \tilde{g}(\theta_\tau, -\tau(0)) &= \mathbf{1}\{0 \in \text{supp } \theta_\tau \xi_n, -\tau(0) \in \text{supp } \theta_\tau \xi_n\} \frac{\theta_\tau \xi\{0\} + \theta_\tau \xi\{-\tau(0)\}}{\theta_\tau \xi\{-\tau(0)\}} \\ &= \mathbf{1}\{\tau(0) \in \text{supp } \xi_n, 0 \in \text{supp } \xi_n\} \frac{\xi\{\tau(0)\} + \xi\{0\}}{\xi\{0\}}. \end{aligned}$$

Since  $\tilde{g}(\theta_0, \tau(0)) \leq 2n^2$  and  $\tilde{g}(\theta_\tau, -\tau(0)) \leq 2n^2$ , we can apply (3.9) with  $g \cdot \tilde{g}$  instead of  $g$ . Together with monotone convergence this gives for all measurable  $g : \Omega \times G \rightarrow [0, \infty)$ :

$$\mathbb{E}_{\mathbb{P}} [\mathbf{1}\{\tau(0) \neq 0\} g(\theta_0, \tau(0)) \xi_n\{\tau(0)\}] = \mathbb{E}_{\mathbb{P}} [\mathbf{1}\{\tau(0) \neq 0\} g(\theta_\tau, -\tau(0)) \xi_n\{\tau(0)\}]. \quad (3.10)$$

We now apply Lemma 3.4. If  $0 \in \text{supp } \xi_n$ , then (3.3) yields that

$$\int h(t) \mathbf{1}\{\theta_t(X, \xi) \neq (X, \xi)\} \xi_n(dt) = \sum_{k \in \mathbb{N}} h_k(X, \xi, \tau_k(0)) h(\tau_k(0)) \xi_n\{\tau_k(0)\} \quad (3.11)$$

for all measurable  $h : W \times G \rightarrow [0, \infty)$ , where

$$h_k(t) := \mathbf{1}\{\theta_t(X, \xi) \neq (X, \xi)\} \mathbf{1}\{\tau_l(0) \neq t \text{ for } 1 \leq l \leq k-1\}.$$

We claim that

$$h_k(\theta_{\tau_k}(X, \xi), -\tau_k(0)) = h_k(X, \xi, \tau_k(0)), \quad k \in \mathbb{N}. \quad (3.12)$$

Indeed, for  $k \geq 2$  and  $l \leq k-1$  we have by covariance of  $\tau_l$  that  $\tau_l(\theta_{\tau_k}, 0) = -\tau_k(0)$  iff  $\tau_l(\tau_k(0)) = 0$ . By the matching property of  $\tau_l$  this is in turn equivalent to  $\tau_k(0) = \tau_l(0)$ . From (3.11), (3.10) and (3.12) we obtain

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} \left[ \int \mathbf{1}\{0 \in \text{supp } \xi_n, \theta_t(X, \xi) \neq (X, \xi)\} g(X, \xi, t) \xi_n(dt) \right] \\ = \sum_{k \in \mathbb{N}} \mathbb{E}_{\mathbb{P}} [h_k(X, \xi, \tau_k(0)) g(X, \xi, \tau_k(0)) \xi_n\{\tau_k(0)\}] \\ = \sum_{k \in \mathbb{N}} \mathbb{E}_{\mathbb{P}} [h_k(X, \xi, \tau_k(0)) g(\theta_{\tau(k)}(X, \xi), -\tau_k(0)) \xi_n\{\tau_k(0)\}]. \end{aligned}$$



Using (3.12) again we arrive at

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} \left[ \int \mathbf{1}\{0 \in \text{supp } \xi_n, \theta_t(X, \xi) \neq (X, \xi)\} g(X, \xi, t) \xi_n(dt) \right] \\ = \mathbb{E}_{\mathbb{P}} \int \mathbf{1}\{0 \in \text{supp } \xi_n, \theta_t(X, \xi) \neq (X, \xi)\} g(\theta_t(X, \xi), -t) \xi_n(dt). \end{aligned} \quad (3.13)$$

Let  $t \in \text{supp } \xi_n$  be such that  $\theta_t(X, \xi) = (X, \xi)$ . Then  $\xi_n = \theta_{-t}\xi_n$  and

$$\xi_n\{t\} = \theta_t\xi_n\{0\} = \theta_{-t}\xi_n\{0\} = \xi\{-t\}.$$

Therefore,

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} \left[ \int \mathbf{1}\{0 \in \text{supp } \xi_n, \theta_t(X, \xi) = (X, \xi)\} g(X, \xi, t) \xi_n(dt) \right] \\ = \mathbb{E}_{\mathbb{P}} \left[ \int \mathbf{1}\{0 \in \text{supp } \xi_n, \theta_t(X, \xi) = (X, \xi)\} g(\theta_t(X, \xi), -t) \xi_n(dt) \right]. \end{aligned}$$

Adding this to (3.13) and taking the limit as  $n \rightarrow \infty$ , yields (2.5) and hence the assertion of the theorem.  $\square$

**Remark 3.5.** The last part of the preceding proof (starting with (3.12)) coincides with the second half of the proof of Theorem 1.1 in [2]. But it does also close a gap in the latter proof in that it is using Lemma 3.3 instead of the (slightly) weaker Theorem 3.6 in [2]. This theorem is not sufficient for the conclusion made in [2].

The definitions of the previous section apply in particular in the case where  $W$  is a singleton. In this case we can identify  $W \times M$  with  $M$  and abbreviate the set of all mass-preserving invariant weighted transport-kernels as  $\mathbf{T}$  and the set of all mass-preserving allocation rules as  $\mathbf{A}$ . Moreover, the set of all mass-preserving invariant Bernoulli transport-kernels (a subset of  $\mathbf{T}$ ) is denoted by  $\mathbf{T}_b$ , while the set of all invariant matchings (a subset of  $\mathbf{A}$ ) is denoted by  $\mathbf{A}_m$ .

The proof of Theorem 3.2 yields the following result without a Borel assumption on the space  $W$ .

**Proposition 3.6.** *Assume that  $\mathbb{P}(0 \notin \text{supp } \xi) = 0$  and  $\mathbb{P}(\xi \neq \xi^d) = 0$ . Assume further that*

$$\mathbb{E}_{\mathbb{P}} \left[ \int \mathbf{1}\{\theta_t(X, \xi) \in A\} T(\xi, 0, dt) \right] = \mathbb{P}((X, \xi) \in A), \quad A \in \mathcal{W} \otimes \mathcal{M}, \quad (3.14)$$

holds for all  $T \in \mathbf{T}_b$ . Then, for all measurable  $g : W \times M \times G \rightarrow [0, \infty)$ ,

$$\mathbb{E}_{\mathbb{P}} \left[ \int \mathbf{1}\{\theta_t\xi \neq \xi\} g(\theta_t(X, \xi), -t) \xi(dt) \right] = \mathbb{E}_{\mathbb{P}} \left[ \int \mathbf{1}\{\theta_t\xi \neq \xi\} g(X, \xi, t) \xi(dt) \right]. \quad (3.15)$$

*Proof:* Let  $n \in \mathbb{N}$ . We apply Lemma 3.4 in the case where  $W$  is a singleton. We can then proceed as in the proof of Theorem 3.2, to obtain as at (3.13)

$$\mathbb{E}_{\mathbb{P}} \left[ \int \mathbf{1}\{\theta_t\xi \neq \xi\} g(X, \xi, t) \xi'_n(dt) \right] = \mathbb{E}_{\mathbb{P}} \left[ \int \mathbf{1}\{\theta_t\xi \neq \xi\} g(\theta_t(X, \xi), -t) \xi'_n(dt) \right].$$

where  $\xi'_n(dt) = \mathbf{1}\{0 \in \text{supp } \xi_n\} \xi_n(dt)$ . Letting  $n \rightarrow \infty$  gives the assertion.  $\square$

Let  $N \subset M$  be the set of all discrete measures  $\mu$  on  $G$  having  $\mu\{s\} \in \{0, 1\}$  for all  $s \in G$ . Strengthening the assumptions of Proposition 3.6, we can use a simplified version of the proof of Theorem 3.2 to get the following result. We refer here also to Theorem 1.1 in [2].

**Proposition 3.7.** *Assume  $\mathbb{P}(0 \notin \text{supp } \xi) = 0$ ,  $\mathbb{P}(\xi \notin N) = 0$ , and that*

$$\mathbb{P}(\theta_\tau(X, \xi) \in A) = \mathbb{P}((X, \xi) \in A), \quad A \in \mathcal{W} \otimes \mathcal{M}, \quad (3.16)$$

*holds for all  $\tau \in \mathbf{A}_m$ , where  $\theta_\tau : W \times M \rightarrow \Omega$  is defined by  $\theta_\tau(w, \mu) := \theta_{\tau(\mu, 0)}(w, \mu)$ . Then (3.15) holds for all measurable  $g : W \times M \times G \rightarrow [0, \infty)$ .*

A measure  $\mu \in M$  is called *periodic* if  $\theta_t \mu = \mu$  for some  $t \neq 0$ . A measure  $\mathbb{Q}$  on  $M$  is called *aperiodic* if it is supported by the set of all measures  $\mu \in M$  that are not periodic. Since the Mecke equation (2.5) implies mass-stationarity, Propositions 3.6 and 3.7 give the following result.

**Proposition 3.8.** *Assume that  $\mathbb{P}(0 \notin \text{supp } \xi) = 0$  and  $\mathbb{P}(\xi \neq \xi^d) = 0$ . Assume further that  $\mathbb{P}(\xi \in \cdot)$  is aperiodic. If either (3.14) holds for all  $T \in \mathbf{T}_b$  or  $\mathbb{P}(\xi \notin N) = 0$  and (3.16) holds for all  $\tau \in \mathbf{A}_m$ , then  $(X, \xi)$  is mass-stationary.*

**Remark 3.9.** Assume that  $\mathbb{P}(0 \notin \text{supp } \xi) = 0$  and  $\mathbb{P}(\xi \neq \xi^d) = 0$ . If (3.14) holds for all  $T \in \mathbf{T}_b$  we conjecture that  $(X, \xi)$  is mass-stationary without the additional aperiodicity assumption.

**Remark 3.10.** Let  $\mathbb{P}$  satisfy the assumptions of Proposition 3.8 and assume in addition that  $\mathbb{P}(\xi \notin N) = 0$ . If  $\mathbb{P}(\xi \in \cdot)$  is not aperiodic, then Proposition 3.8 does not apply. However, we might assume that (3.16) holds for all  $\tau \in \mathbf{A}$ . We believe that this implies mass-stationarity of  $(X, \xi)$ . In case  $G = \mathbb{R}^d$  this was established in Theorem 4.1 in [1].

**Remark 3.11.** Let the assumptions of Proposition 3.7 be satisfied. Example 7.1 in [6] shows that invariance of  $\mathbb{P}((X, \xi) \in \cdot)$  under mass-preserving allocation rules (in the sense of (3.16)) is not enough to imply mass-stationarity of  $(X, \xi)$ . Therefore Theorem 3.2 does not only solve Problem 7.3 in [6] for discrete random measures (up to the fact that in case of periodicities we have to allow the weighted transport-kernels to depend on  $X$ ) but is also the natural (and minimal) extension of Theorem 1.1 in [2] to discrete random measures.

## 4 Cox transports

For any  $\alpha \in M$  we let  $\Pi_\alpha$  denote the distribution of a Poisson process with intensity measure  $\alpha$ . It is convenient to consider  $\Pi_\alpha$  as a probability measure on  $M$ . It is concentrated on those  $\mu \in M$  having locally finite support and  $\mu\{s\} \in \mathbb{N}_0$ ,  $s \in G$ . We consider a *Cox process* (see e.g. [4]) driven by  $(X, \xi)$ , i.e. a random measure  $\zeta$  on  $G$  satisfying

$$\mathbb{P}((X, \xi, \zeta) \in \cdot) = \mathbb{E}_{\mathbb{P}} \left[ \int \mathbf{1}\{(X, \xi, \mu) \in \cdot\} \Pi_\xi(d\mu) \right]. \quad (4.1)$$

Possibly extending  $(\Omega, \mathcal{F}, \mathbb{P})$ , the existence of  $\zeta$  can be assumed without loss of generality. Let  $\zeta^0 := \zeta + \delta_0$  and define  $\Pi_\alpha^0 := \int \mathbf{1}\{\mu + \delta_0 \in \cdot\} \Pi_\alpha(d\mu)$ ,  $\alpha \in M$ .

**Theorem 4.1.** *Assume that  $\mathbb{P}(X \in \cdot)$  is  $\sigma$ -finite. Then  $(X, \xi)$  is mass-stationary iff  $(X, \zeta^0)$  is mass-stationary. In this case even  $((X, \xi), \zeta^0)$  is mass-stationary.*

We will prove this theorem later in this section.

**Remark 4.2.** Assume that  $\mathbb{P}(X \in \cdot)$  is  $\sigma$ -finite and that  $(X, \xi)$  is mass-stationary. Then Theorem 4.1 and (2.9) imply

$$\mathbb{E}_{\mathbb{P}} \left[ \iint \mathbf{1}_A(\theta_s X, \theta_s \xi, \theta_s \mu) T(X, \xi, \mu, 0, ds) \Pi_{\xi}^0(d\mu) \right] = \mathbb{E}_{\mathbb{P}} \left[ \int \mathbf{1}_A(X, \xi, \mu) \Pi_{\xi}^0(d\mu) \right] \quad (4.2)$$

for all  $A \in \mathcal{W} \otimes \mathcal{M} \otimes \mathcal{M}$  and all mass-preserving invariant weighted transport-kernels  $T$  from  $(W \times M) \times M \times G$  to  $G$ .

Combining Theorem 4.1 with Proposition 3.8 gives the following characterization of mass-stationarity via Bernoulli transport-kernels. Recall the definitions of the sets  $\mathbf{T}$ ,  $\mathbf{T}_b$ ,  $\mathbf{A}$ , and  $\mathbf{A}_m$  given before Remark 2.3.

**Corollary 4.3.** *Assume that  $\mathbb{P}(X \in \cdot)$  is  $\sigma$ -finite. Then  $(X, \xi)$  is mass-stationary iff*

$$\mathbb{E}_{\mathbb{P}} \left[ \iint \mathbf{1}_A(\theta_s X, \theta_s \mu) T(\mu, 0, ds) \Pi_{\xi}^0(d\mu) \right] = \mathbb{E}_{\mathbb{P}} \left[ \int \mathbf{1}_A(X, \mu) \Pi_{\xi}^0(d\mu) \right] \quad (4.3)$$

holds for all  $A \in \mathcal{W} \otimes \mathcal{M}$  and all  $T \in \mathbf{T}_b$ .

*Proof:* If  $(X, \xi)$  is mass-stationary then (4.3) follows as a special case of (4.2). Conversely, assume that (4.3) holds. The properties of a Poisson process imply

$$\Pi_{\alpha}^0(\{\mu \in M : \theta_s \mu = \mu \text{ for some } s \in \text{supp } \mu \setminus \{0\}\}) = 0, \quad \alpha \in M. \quad (4.4)$$

It follows that  $\mathbb{P}(\zeta^0 \in \cdot)$  is aperiodic. Hence we obtain from Proposition 3.8 that  $(X, \zeta^0)$  is mass-stationary. Theorem 4.1 yields mass-stationarity of  $(X, \xi)$ .  $\square$

For diffuse random measures the condition (4.3) can be simplified as follows.

**Corollary 4.4.** *Assume that  $\mathbb{P}(X \in \cdot)$  is  $\sigma$ -finite and that  $\mathbb{P}(\xi \neq \xi^c) = 0$ . Then  $(X, \xi)$  is mass-stationary iff*

$$\mathbb{E}_{\mathbb{P}} \left[ \int \mathbf{1}_A(\theta_{\tau(\mu, 0)} X, \theta_{\tau(\mu, 0)} \mu) \Pi_{\xi}^0(d\mu) \right] = \mathbb{E}_{\mathbb{P}} \left[ \int \mathbf{1}_A(X, \mu) \Pi_{\xi}^0(d\mu) \right], \quad A \in \mathcal{W} \otimes \mathcal{M}, \quad (4.5)$$

holds for all  $\tau \in \mathbf{A}_m$ .

*Proof:* Using the second part of Proposition 3.8, the result can be proved as Corollary 4.3.  $\square$

**Remark 4.5.** Equation (4.3) can be written as

$$\mathbb{E}_{\mathbb{P}} \left[ \int \mathbf{1}_A(\theta_s X, \theta_s \zeta^0) T(\zeta^0, 0, ds) \right] = \mathbb{P}((X, \zeta^0) \in A), \quad A \in \mathcal{W} \otimes \mathcal{M}. \quad (4.6)$$

The point here is that the random measure  $\xi$  is not entering this equation explicitly, but only implicitly, as random intensity measure of  $\zeta$ .

**Remark 4.6.** Assume that  $\mathbb{P}(X \in \cdot)$  is  $\sigma$ -finite. Let  $T$  be a transport-kernel. Define another transport-kernel  $T'$  by

$$T'(w, \alpha, s, \cdot) := \int T(w, \mu + \delta_s, s, \cdot) \Pi_\alpha(d\mu). \quad (4.7)$$

Then (4.8) below implies invariance of  $T'$ , while (4.9) easily implies that  $T'$  is mass-preserving. If  $(X, \xi)$  is mass-stationary, then Remark 4.2 yields

$$\mathbb{E}_{\mathbb{P}} \left[ \int \mathbf{1}\{\theta_s(X, \xi) \in \cdot\} T'(X, \xi, 0, ds) \right] = \mathbb{P}((X, \xi) \in \cdot).$$

We do not know whether the validity of this equation for all such *Cox transport-kernels*  $T'$  is enough to imply mass-stationarity of  $(X, \xi)$ . We refer here also to Problem 7.3 in [6].

**Remark 4.7.** Take  $\tau \in \mathbf{A}$ , and let  $V := \tau(\zeta^0, 0)$ . Then (4.5) can be written as

$$(\theta_V X, \theta_V \zeta^0) \stackrel{d}{=} (X, \zeta^0).$$

**Remark 4.8.** Assuming that  $\mathbb{P}(\xi \in \cdot)$  is  $\sigma$ -finite is stronger than only assuming that  $\mathbb{P}((X, \xi) \in \cdot)$  is  $\sigma$ -finite. If, for instance,  $X$  is a constant,  $\mathbb{P}(X \in \cdot)$  can only be  $\sigma$ -finite, if  $\mathbb{P}$  is a finite measure. We do not know, whether the results of this section remain true in the more general case, where only  $\mathbb{P}((X, \xi) \in \cdot)$  is  $\sigma$ -finite.

*Proof of Theorem 4.1:* First we recall that

$$\int \mathbf{1}\{\mu \in \cdot\} \Pi_{\theta_s \alpha}(d\mu) = \int \mathbf{1}\{\theta_s \mu \in \cdot\} \Pi_\alpha(d\mu), \quad \alpha \in M, s \in G, \quad (4.8)$$

and

$$\iint \mathbf{1}\{(\mu, s) \in \cdot\} \mu(ds) \Pi_\alpha(d\mu) = \iint \mathbf{1}\{(\mu + \delta_s, s) \in \cdot\} \alpha(ds) \Pi_\alpha(d\mu), \quad \alpha \in M. \quad (4.9)$$

The first equation comes directly from the definition of  $\Pi_\alpha$ , while the second is from [7].

Assume now that  $(X, \xi)$  is mass-stationary. By Theorem 6.3 in [6] there is a stationary  $\sigma$ -finite measure  $\mathbb{Q}$  on  $W \times M$  such that

$$\mathbb{P}((X, \xi) \in \cdot) = \lambda(B)^{-1} \iint \mathbf{1}_A(\theta_s(w, \mu)) \mathbf{1}_B(s) \mu(ds) \mathbb{Q}(d(w, \mu)), \quad A \in \mathcal{W} \otimes \mathcal{M}, \quad (4.10)$$

where  $0 < \lambda(B) < \infty$ . This means that  $\mathbb{P}((X, \xi) \in \cdot)$  is the Palm measure of the projection from  $W \times M$  onto  $M$  with respect to  $\mathbb{Q}$ , cf. (2.2).

Consider the measurable space  $(\Omega^*, \mathcal{F}^*) := (\Omega \times M \times M, \mathcal{F} \otimes \mathcal{M} \otimes \mathcal{M})$  equipped with the measurable flow  $\theta_s^*(w, \alpha, \mu) := (\theta_s w, \theta_s \alpha, \theta_s \mu)$ . Define a measure  $\mathbb{Q}^*$  on  $(\Omega^*, \mathcal{F}^*)$  by

$$\mathbb{Q}^* := \iint \mathbf{1}\{(w, \alpha, \mu) \in \cdot\} \Pi_\alpha(d\mu) \mathbb{Q}(d(w, \alpha)). \quad (4.11)$$

Since  $\mathbb{Q}$  is  $\sigma$ -finite, so is  $\mathbb{Q}^*$ . Using (4.8), we get for any measurable  $f : \Omega^* \rightarrow [0, \infty)$

$$\begin{aligned} \int f(\theta_s^*(w, \alpha, \mu)) \mathbb{Q}^*(d(w, \alpha, \mu)) &= \iint f(\theta_s w, \theta_s \alpha, \mu) \Pi_{\theta_s \alpha}(d\mu) \mathbb{Q}(d(w, \alpha)) \\ &= \iint f(w, \alpha, \mu) \Pi_\alpha(d\mu) \mathbb{Q}(d(w, \alpha)), \end{aligned}$$

where the second equality comes from stationarity of  $\mathbb{Q}$ . Hence  $\mathbb{Q}^*$  is invariant under the flow  $\{\theta_s^* : s \in G\}$ .

Denote by  $(X^*, \xi^*, \zeta^*)$  the identity on  $\Omega^*$ . Our next aim is to compute the Palm measure of  $((X^*, \xi^*), \zeta^*)$  w.r.t.  $\mathbb{Q}^*$ . Using (4.8) and (4.9), we obtain for all measurable  $f : \Omega^* \times G \rightarrow [0, \infty)$  that

$$\begin{aligned} \iint f(\theta_s(w, \alpha), \theta_s \mu, s) \mu(ds) \mathbb{Q}^*(d(w, \alpha, \mu)) \\ &= \iiint f(\theta_s(w, \alpha), \theta_s \mu, s) \mu(ds) \Pi_\alpha(d\mu) \mathbb{Q}(d(w, \alpha)) \\ &= \iiint f(\theta_s(w, \alpha), \theta_s(\mu + \delta_s), s) \alpha(ds) \Pi_\alpha(d\mu) \mathbb{Q}(d(w, \alpha)) \\ &= \iiint f(\theta_s(w, \alpha), \mu + \delta_0, s) \Pi_{\theta_s \alpha}(d\mu) \alpha(ds) \mathbb{Q}(d(w, \alpha)) \\ &= \iiint f(\theta_s(w, \alpha), \mu + \delta_0, s) \Pi_\alpha(d\mu) ds \mathbb{P}((X, \xi) \in d(w, \alpha)), \end{aligned}$$

where the final equality is due to (4.10) and the refined Campbell theorem (2.3) for the pair  $(\mathbb{Q}, \mathbb{P}((X, \xi) \in \cdot))$ . Therefore

$$\iint \mathbf{1}\{(w, \alpha, \mu) \in \cdot\} \Pi_\alpha^0(d\mu) \mathbb{P}((X, \xi) \in d(w, \alpha)) = \mathbb{P}((X, \xi, \zeta^0) \in \cdot) \quad (4.12)$$

is the Palm measure of  $((X^*, \xi^*), \zeta^*)$  w.r.t.  $\mathbb{Q}^*$ . Theorem 6.3 in [6] implies that  $((X, \xi), \zeta^0)$  is mass-stationary and that

$$\mathbb{E}_{\mathbb{P}} \left[ \int g(\theta_s(X, \xi), \theta_s \zeta^0, -s) \zeta^0(ds) \right] = \mathbb{E}_{\mathbb{P}} \left[ \int g(X, \xi, \zeta^0, s) \zeta^0(ds) \right] \quad (4.13)$$

for any measurable  $g : W \times M \times M \times G \rightarrow [0, \infty)$ . In particular we have

$$\mathbb{E}_{\mathbb{P}} \left[ \int g(\theta_s(X, \zeta^0), -s) \zeta^0(ds) \right] = \mathbb{E}_{\mathbb{P}} \left[ \int g(X, \zeta^0, s) \zeta^0(ds) \right] \quad (4.14)$$

for any measurable  $g : W \times M \times G \rightarrow [0, \infty)$ . As  $\sigma$ -finiteness of  $\mathbb{P}(X \in \cdot)$  entails the same property of  $\mathbb{P}((X, \zeta^0) \in \cdot)$ , we conclude that  $(X, \zeta^0)$  is mass-stationary.

To prove the other implication, we assume that  $(X, \zeta^0)$  is mass-stationary. Since mass-stationarity is equivalent to the Mecke equation (4.14), we have

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} \left[ \iint f(\theta_s X, \theta_s \mu + \delta_{-s}, -s) (\mu + \delta_0)(ds) \Pi_\xi(d\mu) \right] \\ = \mathbb{E}_{\mathbb{P}} \left[ \iint f(X, \mu + \delta_0, s) (\mu + \delta_0)(ds) \Pi_\xi(d\mu) \right] \end{aligned}$$

for all measurable  $f : W \times M \times G \rightarrow [0, \infty)$ . If  $\mathbb{E}_{\mathbb{P}}[\int f(X, \mu + \delta_0, 0) \Pi_{\xi}(d\mu)] < \infty$ , we obtain

$$\mathbb{E}_{\mathbb{P}} \left[ \iint f(\theta_s X, \theta_s \mu + \delta_{-s}, -s) \mu(ds) \Pi_{\xi}(d\mu) \right] = \mathbb{E}_{\mathbb{P}} \left[ \iint f(X, \mu + \delta_0, s) \mu(ds) \Pi_{\xi}(d\mu) \right].$$

Since  $\mathbb{P}(X \in \cdot)$  is  $\sigma$ -finite, this remains true for any measurable  $f : W \times M \times G \rightarrow [0, \infty)$ . Using (4.9) and then (4.8) we get

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} \left[ \iint f(\theta_s X, \mu + \delta_{-s} + \delta_0, -s) \Pi_{\theta_s \xi}(d\mu) \xi(ds) \right] \\ = \mathbb{E}_{\mathbb{P}} \left[ \iint f(X, \mu + \delta_s + \delta_0, s) \Pi_{\xi}(d\mu) \xi(ds) \right]. \end{aligned}$$

We apply this with  $f(w, \mu, s) := \mathbf{1}\{\mu\{s\} \geq 1, \mu\{0\} \geq 1\} f_1(w, \mu - \delta_s - \delta_0, s)$  for a measurable function  $f_1 : W \times M \times G \rightarrow [0, \infty)$ . It follows that

$$\mathbb{E}_{\mathbb{P}} \left[ \iint f_1(\theta_s X, \mu, -s) \Pi_{\theta_s \xi}(d\mu) \xi(ds) \right] = \mathbb{E}_{\mathbb{P}} \left[ \iint f_1(X, \mu, s) \Pi_{\xi}(d\mu) \xi(ds) \right]. \quad (4.15)$$

Take  $B \in \mathcal{G}$  and measurable functions  $h_1 : W \rightarrow \mathbb{R}$  and  $h : M \rightarrow \mathbb{R}$ . Equation (4.15) implies

$$\mathbb{E}_{\mathbb{P}} \left[ \int h_1(\theta_s X) h^*(\theta_s \xi) \mathbf{1}_B(-s) \xi(ds) \right] = \mathbb{E}_{\mathbb{P}}[h_1(X) \xi(B) h^*(\xi)], \quad (4.16)$$

where the measurable function  $h^* : M \rightarrow [0, \infty]$  is defined by

$$h^*(\alpha) := \int h(\mu) \Pi_{\alpha}(d\mu). \quad (4.17)$$

Our next aim is to show that the class of measurable functions defined by (4.17) is rich enough, to conclude from (4.16) that

$$\mathbb{E}_{\mathbb{P}} \left[ \int h_1(\theta_s X) g(\theta_s \xi) \mathbf{1}_B(-s) \xi(ds) \right] = \mathbb{E}_{\mathbb{P}}[h_1(X) \xi(B) g(\xi)] \quad (4.18)$$

holds for all measurable  $g : M \rightarrow \mathbb{R}$ . For  $n \in \mathbb{N}$  and  $\mu \in M$  we define a measure  $\mu^{(n)}$  on  $G^n$  by

$$\mu^{(n)}(C) := \int \cdots \int \mathbf{1}_C(s_1, \dots, s_n) \mu_{s_1, \dots, s_{n-1}}(ds_n) \cdots \mu_{s_1}(ds_2) \mu(ds_1),$$

where, for  $1 \leq k \leq n-1$ , the measure  $\mu_{s_1, \dots, s_k}$  on  $G$  is defined by

$$\mu_{s_1, \dots, s_k} := \mathbf{1}\{\mu - \delta_{s_1} - \dots - \delta_{s_k}(\{s_1, \dots, s_k\}) \geq 0\} (\mu - \delta_{s_1} - \dots - \delta_{s_k}).$$

A well-known property of a Poisson process (following from (4.9) and induction) is

$$\int \mu^{(n)}(C) \Pi_{\alpha}(d\mu) = \alpha^n(C), \quad C \in \mathcal{G}^{\otimes n}, \alpha \in M.$$

For  $k, i_1, \dots, i_k \in \mathbb{N}$  and relatively compact sets  $B_1, \dots, B_k \in \mathcal{G}$  this gives

$$\int \mu^{(i_1+\dots+i_k)}(B_1^{i_1} \times \dots \times B_k^{i_k}) \Pi_\alpha(d\mu) = \alpha(B_1)^{i_1} \dots \alpha(B_k)^{i_k}. \quad (4.19)$$

Now we consider the measurable function

$$h(\mu) := c_0 + \sum_{i_1, \dots, i_k \in \mathbb{N}} c_{i_1, \dots, i_k} \mu^{(i_1+\dots+i_k)}(B_1^{i_1} \times \dots \times B_k^{i_k}), \quad (4.20)$$

where  $c_0 \in \mathbb{R}$  and the numbers  $c_{i_1, \dots, i_k} \in \mathbb{R}$  satisfy

$$\sum_{i_1, \dots, i_k \in \mathbb{N}} |c_{i_1, \dots, i_k}| x_1^{i_1} \dots x_k^{i_k} < \infty$$

for all  $x_1, \dots, x_k \geq 0$ . Let the *entire* function  $f : \mathbb{R}^k \rightarrow \mathbb{R}$  be given by

$$f(x_1, \dots, x_k) := c_0 + \sum_{i_1, \dots, i_k \in \mathbb{N}} c_{i_1, \dots, i_k} x_1^{i_1} \dots x_k^{i_k}.$$

Then (4.19) and dominated convergence implies that

$$h^*(\alpha) = f(\alpha(B_1), \dots, \alpha(B_k)), \quad \alpha \in M, \quad (4.21)$$

where we recall the definition (4.17) of  $h^*$ . Let  $B \in \mathcal{G}$  be relatively compact and  $c > 0$ . Consider the function  $f(x_1, \dots, x_{k+1}) := f(x_1, \dots, x_k) e^{-cx_{k+1}}$ ,  $x_1, \dots, x_{k+1} \in \mathbb{R}$ , where  $f$  is as in (4.21). Define  $\tilde{h}$  as in (4.20) with  $(B_1, \dots, B_k)$  replaced by  $(B_1, \dots, B_k, B)$  and with the appropriate coefficients  $c_{i_1, \dots, i_{k+1}} \in \mathbb{R}$ . Then  $\tilde{h}^*(\alpha) = f(\alpha(B_1), \dots, \alpha(B_k)) e^{-c\alpha(B)}$  and we get from (4.16) that

$$\mathbb{E}_{\mathbb{P}} \left[ \int h_1(\theta_s X) h(\theta_s \xi) \mathbf{1}_B(-s) e^{-c\xi(B+s)} \xi(ds) \right] = \mathbb{E}_{\mathbb{P}} [h_1(X) h(\xi) \xi(B) e^{-c\xi(B)}] \quad (4.22)$$

holds for all  $c > 0$  and all functions  $h$  in the class  $\mathcal{H}$  of bounded measurable functions of the form (4.21). Assume that  $\mathbb{E}_{\mathbb{P}}[|h_1(X)|] < \infty$ . Applying (4.22) with  $h \equiv 1$  and  $h_1$  replaced with  $|h_1|$ , yields

$$\mathbb{E}_{\mathbb{P}} \left[ \int |h_1(\theta_s X)| \mathbf{1}_B(-s) e^{-c\xi(B+s)} \xi(ds) \right] = \mathbb{E}_{\mathbb{P}} [|h_1(X)| \xi(B) e^{-c\xi(B)}] < \infty.$$

Therefore the class of all bounded measurable functions  $h$  satisfying (4.22) is a vector space containing the constant functions and being closed under monotone bounded convergence. Since  $\mathcal{H}$  is stable under multiplication and generates the  $\sigma$ -field  $\mathcal{M}$ , we can apply a well-known functional version of the monotone class theorem to obtain that (4.22) holds for any bounded measurable function  $h$ . Assume that  $h \geq 0$ . Since  $\mathbb{P}(X \in \cdot)$  is  $\sigma$ -finite, (4.22) remains true for any measurable  $h_1 : W \rightarrow [0, \infty)$ . Moreover, for  $c \rightarrow 0$  we get from monotone convergence the desired equation (4.18), and in particular

$$\mathbb{E}_{\mathbb{P}} \left[ \int \mathbf{1}_A(\theta_s X, \theta_s \xi, -s) \xi(ds) \right] = \mathbb{E}_{\mathbb{P}} \left[ \int \mathbf{1}_A(X, \xi, s) \xi(ds) \right], \quad (4.23)$$

for all  $A \in \mathcal{W} \otimes \mathcal{M} \otimes \mathcal{M}$  that are of product form. The measure on the right-hand side of (4.23) is finite on product sets of the form  $C \times \{\alpha \in M : \alpha(B) \leq k\} \times B$ , where  $\mathbb{Q}(X \in C) < \infty$ ,  $B \in \mathcal{G}$  is compact, and  $k \in \mathbb{N}$ . Since  $W \times M \times G$  is the monotone union of countably many such sets, (4.23) extends to all  $A \in \mathcal{W} \otimes \mathcal{M} \otimes \mathcal{M}$ . This is equivalent to the Mecke equation (2.5) and hence to mass-stationarity of  $(X, \xi)$ .  $\square$

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