ALMOST PERIODICITY OF PARABOLIC EVOLUTION EQUATIONS WITH INHOMOGENEOUS BOUNDARY VALUES

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Abstract. We show the existence and uniqueness of the (asymptotically) almost periodic solution to parabolic evolution equations with inhomogeneous boundary values on $\mathbb{R}$ and $\mathbb{R}_\pm$, if the data are (asymptotically) almost periodic. We assume that the underlying homogeneous problem satisfies the 'Acquistapace–Terreni' conditions and has an exponential dichotomy. If there is an exponential dichotomy only on half intervals $(-\infty, -T]$ and $[T, \infty)$, then we obtain a Fredholm alternative of the equation on $\mathbb{R}$ in the space of functions being asymptotically almost periodic on $\mathbb{R}_+$ and $\mathbb{R}_-$.

1. Introduction

In the present paper we study the almost periodicity of the solutions to the parabolic inhomogeneous boundary value problem

$$\begin{aligned} u'(t) &= A_m(t)u(t) + g(t), \qquad t \in \mathbb{R}, \\ B(t)u(t) &= h(t), \qquad t \in \mathbb{R}, \end{aligned} \tag{1.1}$$

for linear operators $A_m(t) : Z \to X$ and $B(t) : Z \to Y$ on Banach spaces $Z \hookrightarrow X$ and Y. Typically, $A_m(t)$ is an elliptic partial differential operator acting in, say, $X = L^p(\Omega)$, and $B(t)$ is a boundary operator mapping $Z = W_p^2(\Omega)$ into a 'boundary space' like $W_p^{1-1/p}(\partial\Omega)$, where $p \in (1, \infty)$, see Example 5.6. We want to show that the solutions $u : \mathbb{R} \to X$ of (1.1) inherit the (asymptotic) almost periodicity of the inhomogeneities $g : \mathbb{R} \to X$ and $h : \mathbb{R} \to Y$. Our basic assumptions say that $A_m(\cdot)$ and $B(\cdot)$ are (asymptotically) almost periodic in time and that the restrictions $A(t)$ of $A_m(t)$ to the kernels of $B(t)$ satisfy the 'Acquistapace–Terreni' conditions (2.1) and (2.2). In particular, the operators $A(t)$ are sectorial and they generate a parabolic evolution family $U(t, s)$, $t \geq s$, which solves the homogeneous problem (1.1) with $g = h = 0$, see Section 2.

If U has an exponential dichotomy on $\mathbb{R}$, then we show that for each almost periodic g and h there is a unique almost periodic solution of (1.1), see

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Proposition 5.2. Our main results concern the more complicated case that the evolution family U has exponential dichotomies on (possibly disjoint) time intervals $(-\infty, -T]$ and $[T, +\infty)$. Theorem 5.5 then gives a Fredholm alternative for (mild) solutions u of (1.1) in the space $AAP^{\pm}(\mathbb{R}, X)$ of continuous functions $u : \mathbb{R} \to X$ being asymptotically almost periodic on $\mathbb{R}_+$ and on $\mathbb{R}_-$. In fact we prove more detailed results on the Fredholm properties of (1.1), see Theorem 4.7, and we also treat the corresponding inhomogeneous initial/final value problems on $\mathbb{R}_\pm$, see Propositions 5.3 and 5.4.

One obtains exponential dichotomies on intervals $(-\infty, -T]$ and $[T, +\infty)$ in the asymptotically hyperbolic case where the operators $A_m(t)$ and $B(t)$ converge as $t \to \pm\infty$ and the resulting limit operators $A_{\pm\infty}$ have no spectrum on $i\mathbb{R}$, see [5], [26], [27]. It should be noted that if the limits at $+\infty$ and $-\infty$ differ, then the operators in (1.1) are asymptotically almost periodic only on $\mathbb{R}_+$ and $\mathbb{R}_-$ separately, so that the space $AAP^{\pm}(\mathbb{R}, X)$ seems to be a natural setting for our investigations. The asymptotically hyperbolic case can occur if one linearizes a nonlinear problem along a orbit connecting two hyperbolic equilibria, see e.g. [25], and also the references in [18], [22].

To establish our results, we develop a theory for the evolution equation

$$u'(t) = A_{\alpha-1}(t)u(t) + f(t), \qquad t \in \mathbb{R}, \tag{1.2}$$

in the continuous extrapolation spaces $X^t_{\alpha-1}$ for the operators $A(t)$ and $\alpha \in (0,1)$. We recall the definition of $X^t_{\alpha-1}$ in Section 2. Here, we just note that $X^t_{\alpha-1}$ contains X and that $A(t)$ can be extended to a generator $A_{\alpha-1}(t)$ in $X^t_{\alpha-1}$. To relate (1.2) with (1.1), we set $f(t) = g(t) + (\omega I - A_{\alpha-1}(t))D(t)h(t)$ for the solution operator $D(t) : \varphi \mapsto v$ of the corresponding abstract Dirichlet problem $(\omega I - A_m(t))v = 0$ and $B(t)v = \varphi$, where $\omega \in \mathbb{R}$ is large enough. Then (1.1) and (1.2) have the same classical solutions, see e.g. [9] and also Section 5.

It is known that the evolution family $U(t, s)$ can be extended to operators $U_{\alpha-1}(t, s) : X^s_{\alpha-1} \to X$, see Section 2. So we can define mild solutions of (1.2) as the functions $u \in C(\mathbb{R}, X)$ satisfying

$$u(t) = U(t, s)u(s) + \int_s^t U_{\alpha-1}(t, \tau)f(\tau)\, d\tau \tag{1.3}$$

for all $t \geq s$, where $f(\tau)$ belongs to $X^\tau_{\alpha-1}$. In our setting it can be shown that mild solutions essentially coincide with pointwise solutions of (1.2), see [22] and the comments at the beginning of Section 3. We prefer to work with the integral equation (1.3) in order to avoid difficulties with the differential equation (1.2) which lives in possibly time–varying extrapolation spaces.

When treating (1.3) or (1.2), it is crucial to identify suitable function spaces for the inhomogeneity f. To that purpose we consider the multiplication operator $A(\cdot)$ in the space $AAP^{\pm}(\mathbb{R}, X)$ endowed with the sup–norm. This space possesses the extrapolation spaces $AAP^{\pm}_{\alpha-1}$ corresponding to $A(\cdot)$. In Section 3 it is shown that the functions in these spaces can be characterized as limits of functions in $AAP^{\pm}(\mathbb{R},X)$. Moreover, if the operators $A(t)$ possess constant extrapolation spaces $X^t_{\alpha-1} \cong X_{\alpha-1}$, we have $AAP^{\pm}_{\alpha-1} = AAP^{\pm}(\mathbb{R},X_{\alpha-1})$. In so far the spaces $AAP^{\pm}_{\alpha-1}$ seem to be natural. For $f \in AAP^{\pm}_{\alpha-1}$ we then set $G_{\alpha-1}u = f$ if $u \in AAP^{\pm}(\mathbb{R},X)$ satisfies (1.3), thus defining a closed operator $G_{\alpha-1}$ in $AAP^{\pm}_{\alpha-1}$. Its Fredholm properties yield the desired Fredholm alternative for the mild solutions to (1.2) described in Theorems 4.7 and 4.9.

This paper combines three lines of research: Fredholm properties of evolution equations on the line, boundary value problems and extrapolation theory, and almost periodic equations in the context of exponential dichotomies. We are not aware of papers on Fredholm properties of inhomogeneous boundary value problems in the framework of almost periodic functions, but we want to recall related previous results. Our reformulation of a boundary value problem as an evolution equation in extrapolation spaces seems to go back to work in boundary control theory, see e.g. [7], [9], [13], [22] for more details and relevant references.

Almost periodicity of solutions of autonomous problems is a well studied subject, see e.g. [4]. These results for autonomous problems were partly extended to the case of time periodic $A(\cdot)$, see e.g. [6], [14], [15], [28]. The case of almost periodic $A(\cdot)$ was studied for special classes of parabolic problems in e.g. [16], [19], [24]. For the general case of almost periodic $A(\cdot)$ satisfying the Acquistapace–Terreni conditions we showed in [21] that for each almost periodic $f : \mathbb{R} \to X$ there is a unique mild solution u of (1.2) for $\alpha = 1$ provided that the underlying evolution family $U(t,s)$ has an exponential dichotomy on $\mathbb{R}$. (See [17] for the converse implication.) In [21] we also established similar results for asymptotically almost periodic functions $f : \mathbb{R}_+ \to X$.

In our previous paper [22] we treated the Fredholmity of parabolic boundary value problems and of the above operator $G_{\alpha-1}$ in the framework of bounded functions (see also [23] for corresponding perturbation theorems). There we generalized the approach of the works [10] and [11] which studied the case of homogeneous boundary values. In [18] one can find a detailed discussion of the literature and the background of such Fredholm theorems as well as a rather complete treatment of (1.2) for $\alpha = 1$, i.e., for f taking values in X (also in the non parabolic situation).

After discussing the above mentioned preparations in Section 2 and 3, we prove our main theorems on (1.2) in Section 4. In the last section we then treat (1.1) by means of the results on (1.2) and discuss an example arising in pde.

2. Notations, assumptions, and preliminaries

We denote by $D(A)$, $N(A)$, $R(A)$, $\sigma(A)$, $\rho(A)$ the domain, kernel, range, spectrum and resolvent set of a linear operator A. Moreover, we set $R(\lambda, A) := (\lambda I - A)^{-1} = (\lambda - A)^{-1}$ for $\lambda \in \rho(A)$, $\mathcal{L}(X,Y)$ is the space of bounded linear operators between Banach spaces X and Y, and $\mathcal{L}(X) := \mathcal{L}(X,X)$. By $c = c(\alpha,\dots)$ we designate a generic constant depending on quantities $\alpha, \cdots$. For an unbounded closed interval J, the space of bounded continuous functions $f : J \to X$ (vanishing at $\pm\infty$) is denoted by $C_b(J, X)$ (by $C_0(J, X)$).

We investigate linear operators $A(t)$, $t \in \mathbb{R}$, on a Banach space X subject to the following hypotheses introduced by P. Acquistapace and B. Terreni in [1] and [2]. There are constants $\omega \in \mathbb{R}$, $\theta \in (\pi/2, \pi)$, $K > 0$ and $\mu, \nu \in (0, 1]$ such that $\mu + \nu > 1$ and

$$\lambda \in \rho(A(t) - \omega), \qquad \|R(\lambda, A(t) - \omega)\| \leq \frac{K}{1 + |\lambda|}\,, \tag{2.1}$$

$$\|(A(t) - \omega)R(\lambda, A(t) - \omega)\,[R(\omega, A(t)) - R(\omega, A(s))]\| \leq K\frac{|t - s|^{\mu}}{|\lambda|^{\nu}} \tag{2.2}$$

for all $t, s \in \mathbb{R}$ and $\lambda \in \Sigma_\theta := \{\lambda \in \mathbb{C}\setminus\{0\} \text{ with } |\arg(\lambda)| \leq \theta\}$. (Observe that the domains $D(A(t))$ are not required to be dense.) Occasionally, we also consider operators $A(t)$ satisfying (2.1) and (2.2) for t and s in an interval J. In this case the results stated below hold in an analogous way.

The conditions (2.1) and (2.2) imply that the operators $A(\cdot)$ generate an evolution family $U(t, s)$ for $t, s \in \mathbb{R}$ with $t \geq s$. More precisely, for $t > s$, the map $(t, s) \mapsto U(t, s) \in \mathcal{L}(X)$ is continuous and continuously differentiable in t, $U(t, s)$ maps X into $D(A(t))$, and it holds $\partial_t U(t, s) = A(t)U(t, s)$. Moreover, $U(t, s)$ and $(t - s)A(t)U(t, s)$ are exponentially bounded. We further have

$$U(t, s)U(s, r) = U(t, r) \quad \text{and} \quad U(t, t) = I \quad \text{for} \quad t \geq s \geq r.$$

Finally, for $s \in \mathbb{R}$ and $x \in \overline{D(A(s))}$, the function $t \mapsto u(t) = U(t, s)x$ is continuous at $t = s$ and u is the unique solution in $C([s, \infty), X) \cap C^1((s, \infty), X)$ of the Cauchy problem

$$u'(t) = A(t)u(t), \quad t > s, \qquad u(s) = x.$$

These facts have been established in [1] and [2], see also [3], [20], [29].

We introduce the inter- and extrapolation spaces for $A(t)$. We refer to [3], [12], [20] for proofs and more details. Let A be a sectorial operator on X (i.e., (2.1) holds with $A(t)$ replaced by A) and $\alpha \in (0, 1)$. We define the new norm on $D(A)$ by

$$\|x\|_\alpha^A := \sup\nolimits_{r>0} \|r^\alpha (A - \omega)R(r, A - \omega)x\|,$$

and consider the continuous interpolation spaces $X_\alpha^A := \overline{D(A)}^{\|\cdot\|_\alpha^A}$ which are Banach spaces endowed with the norms $\|\cdot\|_\alpha^A$. For convenience we further write $X_0^A := X$, $\|x\|_0^A := \|x\|$, $X_1^A := D(A)$ and $\|x\|_1^A := \|(\omega - A)x\|$. We also need the closed subspace $\hat{X}^A := \overline{D(A)}$ of X. Moreover, we define the extrapolation space X_{-1}^A as the completion of $\hat{X}^A$ with respect to the norm $\|x\|_{-1}^A := \|R(\omega, A)x\|$. Then A has a unique continuous extension $A_{-1} : \hat{X}^A \to X_{-1}^A$. The operator A_{-1} satisfies (2.1) in X_{-1}^A, it is densely defined, it has the same spectrum as A, and it generates the semigroup $e^{tA_{-1}}$ on X_{-1}^A being the extension of e^{tA}. As above, we can then define the space $X^A_{\alpha-1} := (X_{-1})^{A_{-1}}_\alpha$ endowed with the norm

$$\|x\|^A_{\alpha-1} := \|x\|^{A_{-1}}_\alpha = \sup_{r>0} \|r^\alpha R(r, A_{-1} - \omega)x\|.$$

The restriction $A_{\alpha-1} : X^A_\alpha \to X^A_{\alpha-1}$ of A_{-1} is sectorial in $X^A_{\alpha-1}$ with the same type as A, it has the same spectrum as A, and the semigroup $e^{tA_{\alpha-1}}$ on $X^A_{\alpha-1}$ is the extension of e^{tA}. Observe that $\omega - A_{\alpha-1} : X^A_\alpha \to X^A_{\alpha-1}$ is an isometric isomorphism. We will frequently use the continuous embeddings

$$\begin{aligned} &D(A) \hookrightarrow X^A_\beta \hookrightarrow D((\omega - A)^\alpha) \hookrightarrow X^A_\alpha \hookrightarrow \hat{X}^A \subset X, \\ &X \hookrightarrow X^A_{\beta-1} \hookrightarrow D((\omega - A_{-1})^\alpha) \hookrightarrow X^A_{\alpha-1} \hookrightarrow X^A_{-1} \end{aligned} \tag{2.3}$$

for all $0 < \alpha < \beta < 1$, where the fractional powers are defined as usually. We note that $X^A_{\alpha-1}$, $0 \le \alpha < 1$, is the completion with respect to $\|\cdot\|^A_{\alpha-1}$ of each of the spaces in (2.3) which are contained in $X^A_{\alpha-1}$; in particular of X.

Given operators $A(t)$, $t \in \mathbb{R}$ which satisfy (2.1), we set

$$X^t_\alpha := X^{A(t)}_\alpha, \quad X^t_{\alpha-1} := X^{A(t)}_{\alpha-1}, \quad \hat{X}^t := \hat{X}^{A(t)}$$

for $0 \le \alpha \le 1$ and $t \in \mathbb{R}$, with the corresponding norms. Then the embeddings in (2.3) hold and the norms of the embeddings are uniformly bounded for $t \in \mathbb{R}$. For a closed interval J, we further define on $E = E(J) := C_b(J, X)$ the multiplication operator $A(\cdot)$ by

$$\begin{aligned} (A(\cdot)f)(t) &:= A(t)f(t) \qquad \text{for all } \ t \in J, \\ D(A(\cdot)) &:= \{f \in E : f(t) \in D(A(t)) \text{ for all } t \in J,\ A(\cdot)f \in E\}. \end{aligned}$$

It is clear that the operator $A(\cdot)$ is sectorial. We can thus introduce the spaces

$$E_\alpha := E^{A(\cdot)}_\alpha, \quad E_{\alpha-1} := E^{A(\cdot)}_{\alpha-1}, \quad \text{and} \quad \hat{E} := \overline{D(A(\cdot))}$$

for $\alpha \in [0, 1]$, where $E_0 := E$ and $E_1 := D(A(\cdot))$. We observe that $E_{-1} \subseteq \prod_{t\in J} X^t_{-1}$ and that the extrapolated operator $A(\cdot)_{-1} : \hat{E} \longrightarrow E_{-1}$ is given by $(A(\cdot)_{-1}f)(t) := A_{-1}(t)f(t)$ for $t \in J$ and $f \in E$. Further, $E_{\alpha-1}$ has the norm

$$\|f\|_{\alpha-1} := \sup_{r>0} \sup_{s\in J} \|r^\alpha R(r, A_{-1}(s) - \omega)f(s)\|,$$

and we have $f(t) \in X^t_{\alpha-1}$ for each $t \in J$ if $f \in E_{\alpha-1}$. Since $R(n, A_{\alpha-1}(\cdot))$ is the resolvent of the densely defined sectorial operator $A_{\alpha-1}(\cdot)$, we have $nR(n, A_{\alpha-1}(\cdot))f \to f$ in $E_{\alpha-1}$ as $n \to \infty$, for each $f \in E_{\alpha-1}$ and $0 \le \alpha < 1$.

The next lemma allows to extend the evolution family $U(t, s)$ to the extrapolated spaces $X^t_{\alpha-1}$, see Proposition 2.1 and Remark 3.12 of [22] for the proof.

Proposition 2.1. *Assume that* (2.1) *and* (2.2) *hold and let* $1 - \mu < \alpha < 1$. *Then the following assertions hold for* $s < t \le s + t_0$ *and* $t_0 > 0$.
(a) The operators $U(t, s)$ *have continuous extensions* $U_{\alpha-1}(t, s) : X^s_{\alpha-1} \to X$ *satisfying*

$$\|U_{\alpha-1}(t, s)\|_{\mathcal{L}(X^s_{\alpha-1}, X^t_\beta)} \le c(\alpha, t_0)(t - s)^{\alpha-\beta-1}, \tag{2.4}$$

and $U_{\alpha-1}(t,s)x = U_{\gamma-1}(t,s)x$ *for* $1-\mu < \gamma < \alpha < 1$, $\beta \in [0,1]$, *and* $x \in X^s_{\alpha-1}$.
(b) The map $\{(t,s) : t > s\} \ni (t,s) \longmapsto U_{\alpha-1}(t,s)f(s) \in X$ *is continuous for* $f \in E_{\alpha-1}$.

Exponential dichotomy is another important tool in our study, cf. [8], [20], [27]. We recall that an evolution family $U(\cdot,\cdot)$ has an *exponential dichotomy* on an interval J if there exists a family of projections $P(t) \in \mathcal{L}(X)$, $t \in J$, being strongly continuous with respect to t, and constants $\delta, N > 0$ such that

(a) $U(t,s)P(s) = P(t)U(t,s)$,
(b) $U(t,s) : Q(s)(X) \to Q(t)(X)$ is invertible with the inverse $\widetilde{U}(s,t)$,
(c) $\|U(t,s)P(s)\| \le Ne^{-\delta(t-s)}$ and $\|\widetilde{U}(s,t)Q(t)\| \le Ne^{-\delta(t-s)}$,

for all $s,t \in J$ with $s \le t$, where $Q(t) := I - P(t)$ is the 'unstable projection.' One further defines *Green's function* by

$$\Gamma(t,s) = \begin{cases} U(t,s)P(s), & t \ge s,\ t,s \in J, \\ -\widetilde{U}(t,s)Q(s), & t < s,\ t,s \in J. \end{cases}$$

In the parabolic case one easily obtains regularity results for Green's function and the dichotomy projections, see e.g. [27, Proposition 3.18]. For instance, if J is bounded from below, then we have $\|A(t)Q(t)\| \le c(\eta)$ for all $t > \eta + \inf J$ and each $\eta > 0$ since $A(t)Q(t) = A(t)U(t,t-\eta)\tilde{U}(t-\eta,t)Q(t)$. Similarly, it holds $\|A(t)Q(t)\| \le c$ for all $t \in J$ if J is unbounded from below. As a consequence $P(t) = I - Q(t)$ leaves invariant $\hat{X}^t$ and X^t_α for each $\alpha \in [0,1]$ and $t \in J \backslash \{\inf J\}$. In the next proposition (shown in Proposition 2.2 and Remark 3.12 of [22]) we state some properties of $\Gamma(t,s)$ and $Q(t)$ in extrapolation spaces. We use the convention $\pm\infty + r = \pm\infty$ for $r \in \mathbb{R}$, and we set $J' = J \backslash \{\sup J\}$, i.e., $J = J'$ if J is unbounded from above. Moreover, we write $U_0(t,s) := U(t,s)$, $P_0(t) := P(t)$, and $Q_0(t) := Q(t)$, where $X^t_0 = X$ by definition.

Proposition 2.2. *Assume that* (2.1) *and* (2.2) *hold and that* $U(t,s)$ *has an exponential dichotomy on an interval* J. *Let* $\eta > 0$ *and* $1 - \mu < \alpha \le 1$. *Then the operators* $P(t)$ *and* $Q(t)$ *have continuous extensions* $P_{\alpha-1}(t) : X^t_{\alpha-1} \to X^t_{\alpha-1}$ *and* $Q_{\alpha-1}(t) : X^t_{\alpha-1} \to X$, *respectively, for every* $t \in J'$; *which are uniformly bounded for* $t < \sup J - \eta$. *Moreover, the following assertions hold for* $t,s \in J'$ *with* $t \ge s$.

(a) $Q_{\alpha-1}(t)X^t_{\alpha-1} = Q(t)X$;
(b) $U_{\alpha-1}(t,s)P_{\alpha-1}(s) = P_{\alpha-1}(t)U_{\alpha-1}(t,s)$;
(c) $U_{\alpha-1}(t,s) : Q_{\alpha-1}(s)(X^s_{\alpha-1}) \to Q_{\alpha-1}(t)(X^t_{\alpha-1})$ *is invertible with the inverse* $\widetilde{U}_{\alpha-1}(s,t)$;
(d) $\|U_{\alpha-1}(t,s)P_{\alpha-1}(s)x\| \le N(\alpha,\eta)\max\{(t-s)^{\alpha-1},1\}e^{-\delta(t-s)}\|x\|^s_{\alpha-1}$ *for* $x \in X^s_{\alpha-1}$ *and* $s < t < \sup J - \eta$;
(e) $\|\widetilde{U}_{\alpha-1}(s,t)Q_{\alpha-1}(t)x\| \le N(\alpha,\eta)e^{-\delta(t-s)}\|x\|^t_{\alpha-1}$ *for* $x \in X^t_{\alpha-1}$ *and* $s \le t < \sup J - \eta$.

(f) *Let* $J_0 \subset J'$ *be a closed interval and* $f \in E_{\alpha-1}(J_0)$. *Then* $P(\cdot)f \in E_{\alpha-1}(J_0)$ *and* $Q(\cdot)f \in C_b(J_0, X)$.

Using this proposition, we define

$$\Gamma_{\alpha-1}(t,s) = \begin{cases} U_{\alpha-1}(t,s)P_{\alpha-1}(s), & t \geq s,\ t,s \in J, \\ -\widetilde{U}(t,s)Q_{\alpha-1}(s), & t < s,\ t,s \in J. \end{cases}$$

In some results we shall assume that $A(\cdot)$ is *asymptotically hyperbolic*, i.e., there are two operators $A_{-\infty} : D(A_{-\infty}) \to X$ and $A_{+\infty} : D(A_{+\infty}) \to X$ which satisfy (2.1) and

$$\lim_{t\to\pm\infty} R(\omega, A(t)) = R(\omega, A_{\pm\infty}) \qquad (\text{in } \mathcal{L}(X)); \tag{2.5}$$

$$\sigma(A_{+\infty}) \cap i\mathbb{R} = \sigma(A_{-\infty}) \cap i\mathbb{R} = \emptyset. \tag{2.6}$$

Under assumptions (2.1), (2.2), (2.5), (2.6), one can show that

$$\begin{aligned} &U(\cdot,\cdot) \text{ has exponential dichotomies on } [T', +\infty) \text{ and } (-\infty, -T'] \\ &\text{for some } T' \in \mathbb{R}. \text{ We fix a number } T \geq 0 \text{ such that } T > T'. \end{aligned} \tag{2.7}$$

See [27, Theorem 2.3], as well as [5] and [26] for earlier results under additional assumptions. We further need a result on embeddings of extrapolation spaces which we state in the more general setting of C_0-semigroups, see e.g. [12].

Lemma 2.3. *Let* A *be the generator of a* C_0*-semigroup* $T(\cdot)$ *on a Banach space* Z. *Let* Y *be an* $T(\cdot)$*-invariant closed subspace of* Z. *Endow* Y *with the norm of* Z *and consider the restriction* A_Y *of* A *to* Y. *Then the space* $Y_{-1}^{A_Y}$ *is canonically embedded into* Z_{-1}^{A} *as a closed subspace.*

Proof. The operator A_Y generates the semigroup of the restrictions $T_Y(t) \in \mathcal{L}(Y)$ of $T(t)$. By rescaling we may assume that $\|T_Y(t)\| \leq \|T(t)\| \leq ce^{-\epsilon t}$ for some $\epsilon > 0$ and all $t \geq 0$. Observe that then A and A_Y are invertible and that

$$A_Y^{-1} = \int_0^\infty T_Y(t)y\,dt = \int_0^\infty T(t)y\,dt = A^{-1}y$$

for each $y \in Y$. We mostly write A instead of A_Y, and we endow the extrapolation spaces of A and A_Y with the norm $\|x\|_{-1} = \|A_{-1}^{-1}x\|$. By definition, it holds

$$Y_{-1}^{A} = \{y = (y_n) + N_Y : (y_n) = (y_n)_{n\in\mathbb{N}} \subset Y \text{ is Cauchy for } \|\cdot\|_{-1}\},$$

where $N_Y = \{(y_n) \subset Y : y_n \to 0 \text{ for } \|\cdot\|_{-1}\}$. We identify $y \in Y$ with the element $(y)_{n\in\mathbb{N}} + N_Y$ of Y_{-1}^{A}, thus considering Y as a dense subspace of Y_{-1}^{A}. We define the operator

$$\Phi : Y_{-1}^{A} \longrightarrow Z_{-1}^{A}, \quad \Phi y = (y_n) + N_Z, \qquad \text{where } y_n \in Y,\ y_n \to y \text{ in } Y_{-1}^{A}.$$

If $(y_n), (\tilde{y}_n) \subset Y$ converge to y in Y_{-1}^{A}, then $y_n - \tilde{y}_n \to 0$ as $n \to \infty$ for $\|\cdot\|_{-1}$. Hence, $(y_n - \tilde{y}_n) \in N_Z$, and so Φ is well defined. Let $y \in Y_{-1}^{A}$ such that $\Phi y = 0$.

This means that $(y_n) \in N_Z$, and hence $y_n \to 0$ in $\|\cdot\|_{-1}$. Therefore $(y_n) \in N_Y$, and thus $y = 0$. It is clear that Φ is linear. It is also bounded since

$$\|\Phi y\|_{Z_{-1}^A} = \inf_{(z_n)\in N_Z} \|(y_n - z_n)\|_\infty \le \inf_{(z_n)\in N_Y} \|(y_n - z_n)\|_\infty = \|y\|_{Y_{-1}^A}.$$

We have shown that $Y_{-1}^A \hookrightarrow Z_{-1}^A$ with the canonical embedding Φ. To prove that the range $R(\Phi)$ is closed in Z_{-1}^A, we take $z_j = \Phi y_j \in R(\Phi) \subseteq Z_{-1}^A$ such that $z_j \to z$ in Z_{-1}^A as $j \to \infty$. Then $A_{-1}^{-1} z_j =: w_j$ converges in Z to $w := A_{-1}^{-1} z$. We further claim that

$$A_{-1}^{-1}\Phi = (A_Y)_{-1}^{-1}. \tag{2.8}$$

Indeed, for $x \in Y$ one has $A_{-1}^{-1}\Phi x = A^{-1}x = A_Y^{-1}x = (A_Y)_{-1}^{-1}x$. So assertion (2.8) follows from the density of Y in Y_{-1}^A. Equation (2.8) then yields

$$(A_Y)_{-1}^{-1} y_j = A_{-1}^{-1} z_j \to w \qquad (\text{in } Z).$$

Since Y is closed in Z and $(A_Y)_{-1}^{-1} y_j \in Y$, we obtain $(A_Y)_{-1}^{-1} y_j \to w$ in Y. As a consequence, y_j converges in Y_{-1}^A to $y := (A_Y)_{-1} w$. We conclude that $z_j = \Phi y_j \to \Phi y$ in Z_{-1}^A which means that $R(\Phi)$ is closed. $\square$

We further introduce the concept of almost periodicity, see e.g. [4], [19].

Definition 2.4. *Let Y be a Banach space. A continuous function $g : \mathbb{R} \to Y$ is called* almost periodic *if for every $\epsilon > 0$ there exist a set $P(\epsilon) \subseteq \mathbb{R}$ and a number $\ell(\epsilon) > 0$ such that each interval $(a, a + \ell(\epsilon))$, $a \in \mathbb{R}$, contains an* almost period *$\tau = \tau_\epsilon \in P(\epsilon)$ and the estimate $\|g(t+\tau) - g(t)\| \le \epsilon$ holds for all $t \in \mathbb{R}$ and $\tau \in P(\epsilon)$. The space of almost periodic functions is denoted by $AP(\mathbb{R}, Y)$.*

We recall that $AP(\mathbb{R}, Y)$ is a closed subspace of the space of bounded and uniformly continuous functions, see [19, Chapter 1]. For a closed unbounded interval J, we also define the space

$$AP(J, Y) := \{g : J \to Y : \exists\, \tilde{g} \in AP(\mathbb{R}, Y) \text{ s.t. } \tilde{g}|J = g\}$$

of almost periodic functions on J. We remark that the function $\tilde{g}$ in the above definition is uniquely determined, cf. [4, Proposition 4.7.1]. The following notion is more important for our investigations.

Definition 2.5. *Let $J = [t_0, \infty)$. A continuous function $g : J \to Y$ is called* asymptotically almost periodic *if for every $\epsilon > 0$ there exists a set $P(\epsilon) \subseteq J$ and numbers $s(\epsilon), \ell(\epsilon) > 0$ such that each interval $(a, a + \ell(\epsilon))$, $a \ge 0$, contains an* almost period *$\tau = \tau_\epsilon \in P(\epsilon)$ and the estimate $\|g(t+\tau) - g(t)\| \le \epsilon$ holds for all $t \ge s(\epsilon)$ and $\tau \in P(\epsilon)$. The space of asymptotically almost periodic functions is denoted by $AAP(J, Y)$.*

Due to e.g. [4, Theorem 4.7.5], these spaces are related by the equality

$$AAP([t_0, \infty), Y) = AP([t_0, \infty), Y) \oplus C_0([t_0, \infty), Y). \tag{2.9}$$

Analogously, we define the asymptotic almost periodicity on $J = (-\infty, t_0]$, and one also has

$$AAP((-\infty, t_0], Y) = AP((-\infty, t_0], Y) \oplus C_0((-\infty, t_0], Y). \tag{2.10}$$

Finally, we recall that $M(\cdot)f \in (A)AP(J, Y)$ if $f \in (A)AP(J, Y)$ and $M(\cdot) \in (A)AP(J, \mathcal{L}(Y))$. This follows from the above definitions if one takes into account that we can find common pseudo periods for f and M, cf. [19, p.6].

3. Amost periodicity of parabolic evolution equations

In this section of the paper we study the parabolic evolution equation

$$u'(t) = A_{\alpha-1}(t)u(t) + f(t), \qquad t \in J, \tag{3.1}$$

where J is an unbounded closed interval, $f \in E_{\alpha-1}(J)$ and $A(t), t \in \mathbb{R}$, are linear operators satisfying the assumptions (2.1) and (2.2). Let $U(t,s)$, $t \ge s$, be the evolution family generated by $A(t), t \in \mathbb{R}$, and be $U_{\alpha-1}(t,s)$, $t \ge s$, its extrapolated evolution family defined in Proposition 2.1 for each $\alpha \in (1-\mu, 1]$. A *mild solution* of (3.1) is a function $u \in C(J, X)$ satisfying

$$u(t) = U(t,s)u(s) + \int_s^t U_{\alpha-1}(t,\tau)f(\tau)\,d\tau, \qquad \forall\, t \ge s \text{ in } J. \tag{3.2}$$

In Proposition 2.6 of [22], we showed that a mild solution actually satisfies (3.1) pointwise in $X^t_{\beta-1}$ for each $\beta \in [0, \min\{\nu, \alpha\})$ and $t \in J$. Conversely, if $u \in C^1(J, X)$ solves (3.1) (and thus $u \in E_\alpha(J)$), then Proposition 2.1(iv) of [23] implies that

$$\partial_\tau^+ U(t,\tau)u(\tau) = -U_{\alpha-1}(t,\tau)A_{\alpha-1}(\tau)u(\tau) + U(t,\tau)u'(\tau) = U_{\alpha-1}(t,\tau)f(\tau)$$

in X for all $t > \tau$. As a result,

$$U(t, t-\varepsilon)u(t-\varepsilon) - U(t,s)u(s) = \int_s^{t-\varepsilon} U_{\alpha-1}(t,\tau)f(\tau)\,d\tau$$

for $t > t-\varepsilon > s$. Letting $\varepsilon \to 0$, we conclude that u is a mild solution of (3.1).

3.1. **Evolution equations on $\mathbb{R}$.**
In this subsection we study the almost periodicity of the solutions to (3.1) on $J = \mathbb{R}$ under the following assumptions.

- (H1) The operators $A(t)$, $t \in \mathbb{R}$, satisfy the assumptions (2.1) and (2.2).
- (H2) The evolution family U generated by $A(\cdot)$ has an exponential dichotomy on $\mathbb{R}$ with constants $N, \delta > 0$, projections $P(t)$, $t \in \mathbb{R}$, and Green's function Γ.
- (H3) $R(\omega, A(\cdot)) \in AP(J, \mathcal{L}(X))$.

It is not difficult to verify that then $R(\lambda, A(\cdot)) \in AP(J, \mathcal{L}(X))$ for $\lambda \in \omega + \Sigma_\theta \cup \{0\}$. We want to solve (3.2) for f belonging to the space $AP_{\alpha-1}(\mathbb{R})$ which is defined by

$$\begin{aligned} AP_{\alpha-1}(\mathbb{R}) &:= \{f \in E_{\alpha-1}(\mathbb{R}) : \exists\,(f_n) \in AP(\mathbb{R}, X) \text{ converging to } f \text{ in } E_{\alpha-1}(\mathbb{R})\} \\ &= \{f \in E_{-1}(\mathbb{R}) : \exists\,(f_n) \in AP(\mathbb{R}, X) \text{ converging to } f \text{ in } E_{\alpha-1}(\mathbb{R})\} \end{aligned}$$

for $\alpha \in [0,1]$. This space is endowed with the norm of $E_{\alpha-1}(\mathbb{R})$. Note that $AP_0(\mathbb{R}) = AP(\mathbb{R}, X)$.

We first characterize the space $AP_{\alpha-1}(\mathbb{R})$. On $F := AP(\mathbb{R}, X)$, we define the multiplication operator

$$\begin{aligned}(A(\cdot)v)(t) &:= A(t)v(t), \quad t \in \mathbb{R},\\ D(A(\cdot)) &:= \{v \in F : f(t) \in D(A(t)) \text{ for all } t \in \mathbb{R},\ A(\cdot)v \in F\}.\end{aligned}$$

Assumptions (H3) and (2.1) imply that the function $R(\lambda, A(\cdot))v$ belongs to F for every $v \in F$ and $\lambda \in \omega + \Sigma_\theta \cup \{0\}$. Therefore, the operator $A(\cdot)$ is sectorial on F with the resolvent $R(\lambda, A(\cdot))$. We can thus introduce the spaces $F_{\alpha-1} := F_{\alpha-1}^{A(\cdot)}$ for each $\alpha \in [0,1)$, where we set $F_0 := F$ and $F_1 := D(A(\cdot))$.

Proposition 3.1. *Let* (2.1) *and (H3) hold. We then have* $F_{\alpha-1} \cong AP_{\alpha-1}(\mathbb{R})$ *for each* $\alpha \in [0,1]$.

Proof. We first note that

$$\|f\|_{F_{\alpha-1}} = \|f\|_{E_{\alpha-1}} \quad \text{for all} \quad f \in F \text{ and } \alpha \in [0,1]. \tag{3.3}$$

The embedding $F_{-1} \hookrightarrow E_{-1}$ holds due to Lemma 2.3. Therefore we obtain

$$\begin{aligned}F_{\alpha-1} &= \{f \in F_{-1} : \exists\, f_n \in AP(\mathbb{R}, X),\ f_n \to f \ \text{ in } \ \|\cdot\|_{F_{\alpha-1}} = \|\cdot\|_{E_{\alpha-1}}\}\\ &\hookrightarrow \{f \in E_{-1} : \exists\, f_n \in AP(\mathbb{R}, X),\ f_n \to f \ \text{ in } \ \|\cdot\|_{F_{\alpha-1}} = \|\cdot\|_{E_{\alpha-1}}\}\\ &= AP_{\alpha-1}(\mathbb{R}).\end{aligned}$$

The asserted isomorphy now follows from (3.3). □

These spaces are much simpler in the case of constant extrapolation spaces.

Proposition 3.2. *Let* (2.1) *and (H3) hold. Assume that* $X_{\alpha-1}^t \cong X_{\alpha-1}^0 =: X_{\alpha-1}$ *for some* $\alpha \in [0,1]$ *and every* $t \in \mathbb{R}$ *with uniformly equivalent norms. Then it holds* $F_{\alpha-1} \cong AP_{\alpha-1}(\mathbb{R}) \cong AP(\mathbb{R}, X_{\alpha-1})$.

Proof. Due to the assumptions, the norms of $E_{\alpha-1}$ and of $C_b(\mathbb{R}, X_{\alpha-1})$ are equivalent on E, so that $E_{\alpha-1} \cong C_b(\mathbb{R}, X_{\alpha-1})$. Take $f \in AP(\mathbb{R}, X_{\alpha-1}) \hookrightarrow E_{\alpha-1}$ and the sequence $f_n := nR(n, A_{\alpha-1}(\cdot))f$ for $n > \omega$. We first show that $f_n \in AP(\mathbb{R}, X)$. For that purpose, let $x \in X_{\alpha-1}$ and take $x_k \in X$ converging to x in $X_{\alpha-1}$. Due to (H3), we have $nR(n, A(\cdot))x_k \in AP(\mathbb{R}, X)$. Since $R(n, A_{\alpha-1}(t))$ is bounded from $X_{\alpha-1}^t$ to X uniformly in t (see e.g. [23, (2.8)], we derive that $nR(n, A_{\alpha-1}(\cdot))x \in AP(\mathbb{R}, X)$. The same is true for functions $f = \phi(\cdot)x$, with scalar almost periodic function ϕ and $x \in X_{\alpha-1}$. Since the span of those functions is dense in $AP(\mathbb{R}, X_{\alpha-1})$ by [4, Theorem 4.5.7], it follows that $f_n \in AP(\mathbb{R}, X)$. Observing that $f_n \to f$ in $E_{\alpha-1}$, we conclude that $f \in AP_{\alpha-1}(\mathbb{R})$. For the converse, let $f \in AP_{\alpha-1}(\mathbb{R})$ and $AP(\mathbb{R}, X) \ni f_n \to f$ in $E_{\alpha-1} \cong C_b(\mathbb{R}, X_{\alpha-1})$. The continuous embedding $X \hookrightarrow X_{\alpha-1}$ implies that $f_n \in AP(\mathbb{R}, X_{\alpha-1})$, and hence $f \in AP(\mathbb{R}, X_{\alpha-1})$. □

We state the main result of this subsection.

Theorem 3.3. *Assume that (H1), (H2) and (H3) hold. Let* $f \in AP_{\alpha-1}(\mathbb{R})$ *for some* $\alpha \in (1-\mu, 1]$*. Then the evolution equation* (3.1) *has a unique mild solution* $u \in AP(\mathbb{R}, X)$ *given by*

$$u(t) = \int_{\mathbb{R}} \Gamma_{\alpha-1}(t,\tau) f(\tau)\, d\tau, \qquad t \in \mathbb{R}. \tag{3.4}$$

Proof. For $f \in E_{\alpha-1}$, one can show that the function u given by (3.4) is a bounded mild solution of (3.1), and that every bounded mild solution is given by (3.4). (See e.g. the remarks after Theorem 3.10 in [22].) This fact shows the uniqueness of bounded mild solutions to (3.1). Take a sequence $(f_n) \subset AP(\mathbb{R}, X)$ converging to f in $E_{\alpha-1}$. In Theorem 4.5 of [21] we have shown that the functions

$$u_n(t) = \int_{\mathbb{R}} \Gamma(t,\tau) f_n(\tau)\, d\tau, \qquad t \in \mathbb{R}, \tag{3.5}$$

belongs to $AP(\mathbb{R}, X)$. Proposition 2.2 further yields

$$\begin{aligned} \|u(t) - u_n(t)\| &\le \int_{\mathbb{R}} \|\Gamma_{\alpha-1}(t,\tau)\|_{\mathcal{L}(X^\tau_{\alpha-1}, X)} \|f_n(\tau) - f(\tau)\|^\tau_{\alpha-1}\, d\tau \\ &\le c\, \|f_n - f\|_{E_{\alpha-1}}, \qquad t \in \mathbb{R}. \end{aligned}$$

Therefore $u_n \to u$ in $C_b(\mathbb{R}, X)$ as $n \to \infty$, and so $u \in AP(\mathbb{R}, X)$. $\square$

3.2. **Forward evolution equations.** We investigate the parabolic initial value problem

$$\begin{aligned} u'(t) &= A_{\alpha-1}(t) u(t) + f(t), \qquad t \ge t_0, \\ u(t_0) &= x, \end{aligned} \tag{3.6}$$

under the following assumptions.

(H1') The operators $A(t)$, $t > a$, satisfy the assumptions (2.1) and (2.2) for $t, s > a$.

(H2') The evolution family U generated by $A(\cdot)$ has an exponential dichotomy on (a, ∞) with projections $P(t)$, $t > a$, constants $N, \delta > 0$, and Green's function Γ.

(H3') $R(\omega, A(\cdot)) \in AAP([t_0, \infty), \mathcal{L}(X))$ for some $t_0 > a$.

Let now $t_0 > a$, $1 - \mu < \alpha \le 1$, $x \in \overline{D(A(t_0))}$ and $f \in E_{\alpha-1}([t_0, \infty))$. Assume that (H1') and (H2') hold. Then a mild solution of (3.6) is a function $u \in C([t_0, \infty), X)$ being a mild solution of the evolution equation in the first line of (3.6) and satisfying $u(t_0) = x$. We have shown in [22, Proposition 2.7] that there is a bounded mild function u of (3.6) if and only if

$$Q(t_0)x = -\int_{t_0}^{\infty} \tilde{U}(t_0, s) Q_{\alpha-1}(s) f(s)\, ds. \tag{3.7}$$

In this case the mild solution of (3.6) is uniquely given by

$$u(t) = U(t, t_0)P(t_0)x + \int_{t_0}^{t} U_{\alpha-1}(t,s) P_{\alpha-1}(s) f(s)\, ds$$

$$
\begin{aligned}
&- \int_t^{+\infty} \tilde{U}_{\alpha-1}(t,s) Q_{\alpha-1}(s) f(s)\, ds \\
&= U(t,t_0)P(t_0)x + \int_{t_0}^{+\infty} \Gamma_{\alpha-1}(t,s) f(s)\, ds, \qquad t \geq t_0. \qquad (3.8)
\end{aligned}
$$

We want to study the asymptotic almost periodicity of this solution in the case of an asymptotically almost periodic f. For a closed unbounded interval $J \neq \mathbb{R}$, we introduce the space

$$
AAP_{\alpha-1}(J) := \{f \in E_{\alpha-1}(J) : \exists\, (f_n) \subseteq AAP(J,X), \;\; f_n \to f \text{ in } E_{\alpha-1}(J)\},
$$

endowed with the norm of $E_{\alpha-1}(J)$. We define the multiplication operator $A(\cdot)$ on $AAP(J,X)$ by

$$
\begin{aligned}
&(A(\cdot)v)(t) := A(t)v(t), \qquad t \in J, \\
&\quad D(A(\cdot)) := \{v \in AAP(J,X) : v(t) \in D(A(t))\; \forall\, t \in J,\; A(\cdot)v \in AAP(J,X)\}.
\end{aligned}
$$

Assumption (H3') and (2.1) imply that the function $R(\lambda, A(\cdot))v$ belongs to $AAP([t_0,\infty),X)$ for every $v \in AAP([t_0,\infty),X)$ and $\lambda \in \omega + \Sigma_\theta \cup \{0\}$. Therefore, the operator $A(\cdot)$ is sectorial on $AAP([t_0,\infty),X)$. We can thus introduce also the spaces $AAP([t_0,\infty),X)_{\alpha-1}^{A(\cdot)}$ for $\alpha \in [0,1]$. These spaces can be characterized as in the previous subsection.

Proposition 3.4. *Let* (2.1) *and (H3') hold. Then we have*

$$
AAP_{\alpha-1}([t_0,\infty)) \cong AAP([t_0,\infty),X)_{\alpha-1}^{A(\cdot)}.
$$

for each $\alpha \in [0,1]$. *If, in addition,* $X_{\alpha-1}^t \cong X_{\alpha-1}$ *with uniform equivalent norms for some* $1-\mu < \alpha \leq 1$ *and a Banach space* $X_{\alpha-1}$*, then we obtain*

$$
AAP_{\alpha-1}([t_0,\infty)) \cong AAP([t_0,\infty),X_{\alpha-1}).
$$

We can now prove the main result of this subsection.

Theorem 3.5. *Let* $1-\mu < \alpha \leq 1$. *Assume that (H1'), (H2'), and (H3') hold and that* $x \in \overline{D(A(t_0))}$ *and* $f \in AAP_{\alpha-1}([t_0,\infty))$ *satisfy* (3.7). *Then the unique bounded mild solution* u *of* (3.6) *is asymptotically almost periodic.*

Proof. Let $f \in AAP_{\alpha-1}([t_0,\infty))$ and $x \in X$ satisfy (3.7). Take a sequence $(f_n) \subset AAP([t_0,\infty),X)$ converging to f in $E_{\alpha-1}([t_0,\infty))$. Due to [21, Theorem 5.4], the functions

$$
u_n(t) = U(t,t_0)P(t_0)x + \int_{t_0}^{\infty} \Gamma(t,s) f_n(s)\, ds, \qquad t \geq t_0,\; n \in \mathbb{N}
$$

are asymptotically almost periodic in X (and they are mild solutions of (3.6) for the inhomogeneities f_n and the initial values $x_n = u_n(t_0)$). As in the proof of Theorem 3.3, we see that $u_n \to u$ in $C_b([t_0,\infty),X)$. So we conclude that $u \in AAP([t_0,\infty),X)$. ☐

3.3. **Backward evolution equations.** As a counterpart to the previous subsection, we now study the parabolic final value problem

$$\begin{aligned} u'(t) &= A_{\alpha-1}(t)u(t) + f(t), \qquad t \le t_0, \\ u(t_0) &= x. \end{aligned} \tag{3.9}$$

Mild solutions of (3.9) are defined as in the forward case. We make the following assumptions.

(H1") The operators $A(t)$, $t < b$, satisfy (2.1) and (2.2) for $t, s < b$.

(H2") The evolution family U has an exponential dichotomy on $(-\infty, b)$ with projections $P(t)$, $t < b$, constants $N, \delta > 0$, and Green's function Γ.

(H3") $R(\omega, A(\cdot)) \in AAP((-\infty, t_0], \mathcal{L}(X))$ for some $t_0 < b$.

Let $1 - \mu < \alpha \le 1$, $x \in X$, and $f \in E_{\alpha-1}((-\infty, t_0])$. We have shown in [22, Proposition 2.8] that there is a unique bounded mild solution $u \in C((-\infty, t_0], X)$ of (3.9) on $(-\infty, t_0]$ if and only if

$$P(t_0)x = \int_{-\infty}^{t_0} U_{\alpha-1}(t_0, s)P_{\alpha-1}(s)f(s)ds, \tag{3.10}$$

in which case u is given by

$$\begin{aligned} u(t) = \widetilde{U}(t, t_0)Q(t_0)x &+ \int_{-\infty}^{t} U_{\alpha-1}(t, s)P_{\alpha-1}(s)f(s)\,ds \\ &- \int_{t}^{t_0} \widetilde{U}_{\alpha-1}(t, s)Q_{\alpha-1}(s)f(s)\,ds \end{aligned} \tag{3.11}$$

for $t \le t_0$. As before, we obtain the asymptotic almost periodicity of this function if f belongs to $AAP_{\alpha-1}((-\infty, t_0])$. We note that the space $AAP_{\alpha-1}((-\infty, t_0])$ can de described as in Proposition 3.4.

Theorem 3.6. *Assume that (H1"), (H2"), and (H3") hold. Let $x \in X$ and $f \in AAP_{\alpha-1}((-\infty, t_0])$ satisfy* (3.10). *Then the unique bounded mild solution u of* (3.9) *given by* (3.11) *belongs to $AAP((-\infty, t_0], X)$.*

Proof. Let x and f be as in the assertion. Take a sequence (f_n) in $AAP((-\infty, t_0], X)$ converging to f in $E_{\alpha-1}((-\infty, t_0])$. Define the function

$$u_n(t) = \widetilde{U}(t, t_0)Q(t_0)x + \int_{-\infty}^{t_0} \Gamma_{\alpha-1}(t, s)Q_{\alpha-1}(s)f_n(s)\,ds$$

for $t \le t_0$ and $n \in \mathbb{N}$. Using the same arguments as in [21, Theorem 5.4], we can show that $u_n \in AAP((-\infty, t_0], X)$ for all $n \in \mathbb{N}$. Finally, as in Theorem 3.3 we see that $u_n \to u$ in $C_b((-\infty, t_0], X)$, so that $u \in AAP((-\infty, t_0], X)$. □

4. Fredholmity of almost periodic parabolic evolution equations on $\mathbb{R}$

Consider a family of operators $A(t)$, $t \in \mathbb{R}$, on X satisfying the hypotheses (2.1), (2.2), and (2.7). Again, $U(t, s)$ is the evolution family on X generated by $A(\cdot)$ and $U_{\alpha-1}(t, s)$ is its extrapolation on $X^s_{\alpha-1}$ for $1-\mu < \alpha \le 1$. Both families have exponential dichotomies on $(-\infty,-T]$ and $[T,+\infty)$ for some $T\geq 0$ with projections $P(\cdot)$ and $P_{\alpha-1}(\cdot)$, respectively. We further assume that

(H4) $R(\omega,A(\cdot))|[T,\infty)\in AAP([T,\infty),\mathcal{L}(X))$ and $R(\omega,A(\cdot))|(-\infty,-T]\in AAP((-\infty,-T],\mathcal{L}(X))$ for the number T from (2.7).

We will work on the space

$$AAP^{\pm}=AAP^{\pm}(\mathbb{R},X):=\{f\in C_b(\mathbb{R},X):f|\mathbb{R}_{\pm}\in AAP(\mathbb{R}_{\pm},X)\}.$$

of functions being asymptotically almost periodic on $\mathbb{R}_-$ and $\mathbb{R}_+$, separately. This space is endowed with the sup–norm. The following description of this space turns out to be crucial for our work.

Lemma 4.1. *Let* (2.1) *and (H4) hold. We then have* $AAP^{\pm}=\{f\in C_b(\mathbb{R},X):f|(-\infty,-a]\in AAP((-\infty,-a],X), f|[a,\infty)\in AAP([a,\infty),X)\}=:F^a$ *for each* $a\geq 0$.

Proof. Let $a\geq 0$ and $f\in C_b(\mathbb{R},X)$ such that

$$f^+:=f|[a,\infty)=g^++h^+\in C_0([a,\infty),X)\oplus AP([a,\infty),X);$$

$$f^-:=f|(-\infty,-a]=g^-+h^-\in C_0((-\infty,-a],X)\oplus AP((-\infty,-a],X).$$

It is clear that h^+ and h^- can be extended to functions in $AP(\mathbb{R}_+,X)$ and $AP(\mathbb{R}_-,X)$ respectively. The functions $\tilde{g}^{\pm}:=f|\mathbb{R}_{\pm}-h^{\pm}$ then belong to $C_0(\mathbb{R}_{\pm},X)$, i.e, $f|\mathbb{R}_{\pm}=\tilde{g}^{\pm}+h^{\pm}\in AAP(\mathbb{R}_{\pm},X)$. So we have shown the inclusion $F^a\subset AAP^{\pm}$. The other inclusion is clear. □

As in the previous sections we define the multiplication operator $A(\cdot)$ on $AAP^{\pm}(\mathbb{R},X)$ by

$$(A(\cdot)v)(t):=A(t)v(t),\qquad t\in\mathbb{R},$$

$$D(A(\cdot)):=\{v\in AAP^{\pm}(\mathbb{R},X):f(t)\in D(A(t))\ \forall\, t\in\mathbb{R},\ A(\cdot)v\in AAP^{\pm}\}.$$

Assumption (H4) shows that function $R(\lambda,A(\cdot))f$ belongs to $AAP^{\pm}$ for every $f\in AAP^{\pm}$ and $\lambda\in\omega+\Sigma_\theta\cup\{0\}$, and thus the operator $A(\cdot)$ is sectorial in $AAP^{\pm}$ with the resolvent $R(\lambda,A(\cdot))$. So we can define the extrapolation spaces

$$AAP^{\pm}_{\alpha-1}=AAP^{\pm}_{\alpha-1}(\mathbb{R}):=(AAP^{\pm}(\mathbb{R},X))^{A(\cdot)}_{\alpha-1}\qquad\text{for }\alpha\in[0,1],$$

which are characterized in the following proposition.

Proposition 4.2. *Let* (2.1) *and (H4) hold, and let* $\alpha\in[0,1]$. *Then we have*

$$AAP^{\pm}_{\alpha-1}\cong\{f\in E_{\alpha-1}(\mathbb{R}):f|[T,\infty)\in AAP_{\alpha-1}([T,\infty)),\\ f|(-\infty,-T]\in AAP_{\alpha-1}((-\infty,-T])\}.$$

Assume that, in addition, $X^t_{\alpha-1}\cong X_{\alpha-1}$ *with uniformly equivalent norms for some Banach space* $X_{\alpha-1}$ *and some* $\alpha\in[0,1]$. *Then we have*

$$AAP^{\pm}_{\alpha-1}\cong\{f\in C_b(\mathbb{R},X_{\alpha-1}):f|[T,\infty)\in AAP([T,\infty),X_{\alpha-1}),\\ f|(-\infty,-T]\in AAP((-\infty,-T],X_{\alpha-1})\}.$$

Proof. Due to Lemma 4.1 the space $AAP^{\pm}_{-1}$ is embedded into $E_{-1}(\mathbb{R})$. Let $f \in AAP^{\pm}_{\alpha-1}$. Then there are $f_n \in AAP^{\pm}$ converging to f in $E_{\alpha-1}$. The restrictions of f_n to $(-\infty, -T]$ and to $[T, +\infty)$ converge to the corresponding restrictions of f in $E_{\alpha-1}((-\infty, -T])$ and $E_{\alpha-1}([T, +\infty))$, respectively. Therefore the restrictions of f belong to $AAP_{\alpha-1}((-\infty, -T])$ and to $AAP_{\alpha-1}([T, +\infty))$, respectively, which shows the inclusion '$\subset$'. Let f belong to the space on the right side in the first assertion. The functions $f_n = nR(n, A_{\alpha-1}(\cdot))f$ then belong to $C_b(\mathbb{R}, X)$ for $n \geq \omega$, and their restrictions belong to $AAP((-\infty, -T], X)$ and to $AAP([T, +\infty), X)$ (since $R(n, A_{\alpha-1}(\cdot))$ is the resolvent of the respective multiplication operator $A_{\alpha-1}(\cdot)$). Lemma 4.1 thus yields $f_n \in AAP^{\pm}$. Since $f_n \to f$ in $E_{\alpha-1}$ as $n \to \infty$, the first assertion holds. The second assertion now follows from the results of the previous section. $\square$

As in [22], we define the operator $G_{\alpha-1}$ on $AAP^{\pm}_{\alpha-1}(\mathbb{R}, X)$ in the following way. A function $u \in AAP^{\pm}(\mathbb{R}, X)$ belongs to $D(G_{\alpha-1})$ and $G_{\alpha-1}u = f$ if there is a function $f \in AAP^{\pm}_{\alpha-1}$ such that (3.2) holds; i.e.,

$$u(t) = U(t,s)u(s) + \int_s^t U_{\alpha-1}(t,\tau)f(\tau)\,d\tau$$

for all $t, s \in \mathbb{R}$ with $t \geq s$. In particular, G_0 is defined on $AAP^{\pm}(\mathbb{R}, X)$ by (3.2), replacing $U_{\alpha-1}$ by U. To study the operator $G_{\alpha-1}$, we introduce the stable and unstable subspaces of $U_{\alpha-1}(\cdot,\cdot)$.

Definition 4.3. *Let $t_0 \in \mathbb{R}$. We define the* stable space at t_0 *by*

$$X_s(t_0) := \{x \in X^{t_0}_{\alpha-1} : \lim_{t\to+\infty} \|U_{\alpha-1}(t,t_0)x\| = 0\},$$

and the unstable space at t_0 *by*

$$X_u(t_0) := \{x \in X : \exists \text{ a mild solution } u \in C_0((-\infty, t_0], X) \text{ of (3.9) with } f = 0\}.$$

Observe that the function u in the definition of $X_u(t_0)$ satisfies $u(t) = U(t,s)u(s)$ for $s \leq t \leq t_0$ and $u(t_0) = x$, so that $X_u(t_0) \subset D(A(t_0))$. The following result was shown in [22, Lemma 3.2].

Lemma 4.4. *Assume that the assumptions* (2.1)*,* (2.2)*, and* (2.7) *are satisfied and that* $1 - \mu < \alpha \leq 1$*. Then the following assertions hold.*

(a) $X_s(t_0) = P_{\alpha-1}(t_0)X^{t_0}_{\alpha-1}$ *for* $t_0 \geq T$*;*
(b) $X_u(t_0) = Q(t_0)X$ *for* $t_0 \leq -T$*;*
(c) $U_{\alpha-1}(t,t_0)X_s(t_0) \subseteq X_s(t)$ *for* $t \geq t_0$ *in* $\mathbb{R}$*;*
(d) $U(t,t_0)X_u(t_0) = X_u(t)$ *for* $t \geq t_0$ *in* $\mathbb{R}$*;*
(e) $X_s(t_0)$ *is closed in* $X^{t_0}_{\alpha-1}$ *for* $t_0 \in \mathbb{R}$*.*

Finally, for technical purposes we introduce the space

$$F^T := \{f : C_b((-\infty, T], X) : f|(-\infty, -T] \in AAP((-\infty, -T], X)\}$$

and endow it with the sup norm. The corresponding extrapolation spaces $F^T_{\alpha-1}$ for $A(\cdot)$ are defined as above for $\alpha \in [0, 1]$.

The restrictions $G^+_{\alpha-1}$ and $G^-_{\alpha-1}$ of $G_{\alpha-1}$ to the halflines $[T,+\infty)$ and $(-\infty,T]$ are given in a similar way: A function $u \in AAP([T,+\infty),X)$ (resp., $u \in F^T$) belongs to $D(G^+_{\alpha-1})$ (resp., $D(G^-_{\alpha-1})$) if there is a function $f \in AAP_{\alpha-1}([T,+\infty))$ (resp., $f \in F^T_{\alpha-1}$) such that

$$u(t) = U(t,s)u(s) + \int_s^t U_{\alpha-1}(t,\sigma)f(\sigma)d\sigma$$

holds for all $t \geq s \geq T$ (resp., for all $s \leq t \leq T$). Then we set $G^+_{\alpha-1}u = f$ and $G^-_{\alpha-1}u = f$, respectively. The operators $G_{\alpha-1}$ and $G^\pm_{\alpha-1}$ are single valued and closed due to Remarks 2.5 and 3.12 of [22]. As in [10], [11] and [22], we obtain right inverses $R^+_{\alpha-1}$ and $R^-_{\alpha-1}$ on $AAP([T,+\infty),X)$ and on F^T for $G^+_{\alpha-1}$ and $G^-_{\alpha-1}$, respectively, by setting

$$(R^+_{\alpha-1}h)(t) = -\int_t^\infty \widetilde{U}_{\alpha-1}(t,s)Q_{\alpha-1}(s)h(s)\,ds + \int_T^t U_{\alpha-1}(t,s)P_{\alpha-1}(s)h(s)\,ds$$

for $h \in AAP_{\alpha-1}([T,+\infty),X)$ and $t \geq T$, and

$$(R^-_{\alpha-1}h)(t) = \begin{cases} \displaystyle\int_{-\infty}^T \Gamma_{\alpha-1}(t,s)h(s)\,ds, & t \leq -T, \\ \displaystyle\int_{-\infty}^{-T} U_{\alpha-1}(t,s)P_{\alpha-1}(s)h(s)\,ds + \int_{-T}^t U_{\alpha-1}(t,s)h(s)\,ds, & |t| \leq T, \end{cases}$$

for $h \in F^T_{\alpha-1}$.

Proposition 4.5. *Assume that the assumptions* (2.1), (2.2), (2.7) *and (H4) are satisfied and that* $1-\mu < \alpha \leq 1$. *Then the following assertions hold.*

(a) *The operator* $R^+_{\alpha-1} : AAP_{\alpha-1}([T,+\infty)) \to AAP([T,+\infty),X)$ *is bounded and* $G^+_{\alpha-1}R^+_{\alpha-1}h = h$ *for each* $h \in AAP_{\alpha-1}([T,+\infty))$.

(b) *The operator* $R^-_{\alpha-1} : F^T_{\alpha-1} \to F^T$ *is bounded and* $G^-_{\alpha-1}R^-_{\alpha-1}h = h$ *for each* $h \in F^T_{\alpha-1}$.

(c) *We have* $R^\pm_{\alpha-1}h(T) \in X^T_\varepsilon$ *for all* $0 \leq \varepsilon < \alpha$.

Proof. Let $h \in AAP_{\alpha-1}([T,+\infty))$. In Proposition 3.3 and Remark 3.12 of [22] it was shown that $R^+_{\alpha-1}h$ is a mild solution of the equation (3.6) for the inhomogeneity h and the initial value $x := -\int_T^\infty \tilde{U}(T,s)Q_{\alpha-1}(s)h(s)\,ds$ at $t_0 = T$. Since (3.7) holds for h and x, Theorem 3.5 gives the asymptotic almost periodicity of $R^+_{\alpha-1}h$. So the operator $R^+_{\alpha-1}$ maps $AAP_{\alpha-1}([T,+\infty))$ into $AAP([T,+\infty),X)$, and its boundedness follows from Proposition 2.2 d), e) as in the proof of [22, Proposition 3.3]. Assertion (a) is thus established. To show (b), let $h \in F^T_{\alpha-1}((-\infty,T])$. Proposition 3.3 and Remark 3.12 of [22] also yield that $R^-_{\alpha-1}h$ is a mild solution of the equation (3.9) with $t_0 = T$ and the inhomogeneity h. It is clear that $h|(-\infty,-T]$ satisfies (3.11) for $x := \int_{-\infty}^{-T} U_{\alpha-1}(-T,s)P_{\alpha-1}(s)f(s)\,ds$. Theorem 3.5 then implies that $R^-_{\alpha-1}h|(-\infty,-T] \in AAP((-\infty,-T],X)$ and consequently $R^-_{\alpha-1}$ maps $F^T_{\alpha-1}$ into F^T. The boundedness of $R^-_{\alpha-1}$ follows again from Proposition 2.2 d),e). The last assertion is a consequence of Propositions 2.1 a) and 2.2 d),e). $\square$

We can now describe the range and the kernel of $G_{\alpha-1}$.

Proposition 4.6. *Assume that* (2.1), (2.2), (2.7) *and (H4) are satisfied and that* $1-\mu < \alpha \le 1$. *For* $f \in AAP^{\pm}_{\alpha-1}$ *we set* $f^+ = f|[T,+\infty)$ *and* $f^- = f|(-\infty,T]$. *Then the following assertions hold for* $G_{\alpha-1}$.

(a) $N(G^+_{\alpha-1}) = \{u \in C_0([T,+\infty),X)) : u(t) = U(t,T)x \ (\forall\, t \ge T),\ x \in P(T)\hat{X}^T\}$;

(b) $N(G^-_{\alpha-1}) = \{u \in C_0((-\infty,T],X) : u(t) = U(t,s)u(s) \ (\forall\, s \le t \le T),\ u(T) \in X_u(T)\}$;

(c) $N(G_{\alpha-1}) = \{u \in C_0(\mathbb{R},X) : u(t) = U(t,s)u(s) \ (\forall\, t \ge s),\ u(T) \in P(T)X \cap X_u(T)\}$;

(d) $R(G_{\alpha-1}) = \{f \in AAP^{\pm}_{\alpha-1} : R^+_{\alpha-1}f^+(T) - R^-_{\alpha-1}f^-(T) \in P(T)X + X_u(T)\}$, *where for* $f \in R(G_{\alpha-1})$ *a function* $u \in D(G_{\alpha-1})$ *with* $G_{\alpha-1}u = f$ *is given by* (4.1) *below;*

(e) $\overline{R(G_{\alpha-1})} = \{f \in AAP^{\pm}_{\alpha-1} : R^+_{\alpha-1}f^+(T) - R^-_{\alpha-1}f^-(T) \in \overline{P(T)X + X_u(T)}\}$, *where the closure on the left (right) side is taken in* $AAP^{\pm}_{\alpha-1}$ *(in* X*).*

Proof. The assertions (a), (b) and (c) follow from Proposition 3.5 and Remark 3.12 of [22]. We note that $P(T)X \cap X_u(T) = P(T)\hat{X}^T \cap X_u(T)$ since $X_u(T) \subseteq D(A(T))$. To show (d), let $G_{\alpha-1}u = f \in AAP^{\pm}_{\alpha-1}(\mathbb{R})$ for some $u \in D(G_{\alpha-1})$. Then the functions $f^{\pm}$ belong to $R(G^+_{\alpha-1})$ and to $R(G^-_{\alpha-1})$, respectively, because of Proposition 4.2 and (3.2). Proposition 4.5 shows that the functions

$$v_+ = u|[T,+\infty) - R^+_{\alpha-1}f^+ \qquad \text{and} \qquad v_- = u|(-\infty,T] - R^-_{\alpha-1}f^-$$

are contained in the kernels of $G^+_{\alpha-1}$ and of $G^-_{\alpha-1}$, respectively. So we obtain

$$(R^+_{\alpha-1}f^+)(T) - (R^-_{\alpha-1}f^-)(T) = v_-(T) - v_+(T) \in X_u(T) + P(T)X$$

by (a) and (b). Conversely, let $f \in AAP^{\pm}_{\alpha-1}(\mathbb{R})$ with

$$(R^+_{\alpha-1}f^+)(T) - (R^-_{\alpha-1}f^-)(T) = y_s + y_u \in P(T)X + X_u(T).$$

Set $x_0 := (R^+_{\alpha-1}f^+)(T) - y_s = y_u + (R^-_{\alpha-1}f^-)(T)$ and

$$u(t) := \begin{cases} u_+(t) := -U(t,T)y_s + (R^+_{\alpha-1}f^+)(t), & t \ge T, \\ u_-(t) := \tilde{v}(t) + (R^-_{\alpha-1}f^-)(t), & t \le T, \end{cases} \tag{4.1}$$

where $\tilde{v} \in N(G^-_{\alpha-1})$ such that $\tilde{v}(T) = y_u$. Observe that $u_+(T) = u_-(T)$. From Proposition 4.5(c) we deduce $y_s \in P(T)\hat{X}^T$, so that $U(\cdot,T)y_s \in C_0([T,\infty),X)$. Proposition 4.5 shows that $R^+_{\alpha-1}f^+ \in AAP([T,\infty),X)$, and hence $u|[T,\infty) \in AAP([T,\infty),X)$. We also know from assertion (c) that $\tilde{v} \in C_0((-\infty,T],X)$ and from Proposition 4.5 that $R^-_{\alpha-1}f^- \in F^T$. Using also Lemma 4.1, we deduce that u belongs to $AAP^{\pm}(\mathbb{R},X)$. Finally, one can check as in the proof of Proposition 3.5 of [22] that $G_{\alpha-1}u = f$. The last assertion can be shown exactly as Proposition 3.5(e) of [22]. □

Using the above results, we are able to describe the Fredholm properties of the operator $G_{\alpha-1}$ in terms of properties of the subspaces $X_s(T)$ and $X_u(T)$. The proofs are similar to ones of Theorems 3.6 and 3.10 and Proposition 3.8 of [22] and therefore omitted. Recall that subspaces V and W of a Banach space E are called a *semi-Fredholm couple* if $V+W$ is closed and if at least one of the dimensions $\dim(V\cap W)$ and $\operatorname{codim}(V+W)$ is finite. The *index* of (V,W) is defined by $\operatorname{ind}(V,W) := \dim(V\cap W) - \operatorname{codim}(V+W)$. If the index is finite, then (V,W) is a *Fredholm couple.*

Theorem 4.7. *Assume that* (2.1), (2.2)*, and* (2.7) *are satisfied and that* $1-\mu < \alpha \le 1$*. Then the following assertions hold for* $G_{\alpha-1}$ *defined on* $AAP^{\pm}_{\alpha-1}(\mathbb{R})$*.*

(a) $R(G_{\alpha-1})$ *is closed in* $AAP^{\pm}_{\alpha-1}$ *if and only if* $P(T)X + X_u(T)$ *is closed in* X*.*

(b) *If* $G_{\alpha-1}$ *is injective, then* $P(T)X \cap X_u(T) = \{0\}$*. The converse is true if* $U(T,-T)_{|Q(-T)(X)}$ *is injective, in addition.*

(c) *If* $G_{\alpha-1}$ *is invertible, then* $P(T)X \oplus X_u(T) = X$*. The converse is true if* $U(T,-T)_{|Q(-T)(X)}$ *is injective in addition.*

(d) $\dim N(G_{\alpha-1}) = \dim(P(T)X \cap X_u(T)) + \dim N(U(T,-T)_{|Q(-T)(X)})$*.* *If* $R(G_{\alpha-1})$ *is closed in* $AAP^{\pm}_{\alpha-1}$*, then* $\operatorname{codim}(P(T)X + X_u(T)) = \operatorname{codim} R(G_{\alpha-1})$*. In particular,* $G_{\alpha-1}$ *is surjective if and only if* $P(T)X + X_u(T) = X$*.*

(e) *If* $G_{\alpha-1}$ *is a semi-Fredholm operator, then* $(P(T)X, X_u(T))$ *is a semi-Fredholm couple, and* $\operatorname{ind}(P(T)X, X_u(T)) \le \operatorname{ind} G_{\alpha-1}$*. If in addition the kernel of* $U(T,-T)_{|Q(-T)(X)}$ *is finite dimensional, then*

$$\operatorname{ind}(P(T)X, X_u(T)) = \operatorname{ind} G_{\alpha-1} - \dim N(U(T,-T)_{|Q(-T)(X)}). \tag{4.2}$$

Conversely, if $(P(T)X, X_u(T))$ *is a semi-Fredholm couple and the kernel of* $U(T,-T)_{|Q(-T)(X)}$ *is finite dimensional, then* $G_{\alpha-1}$ *is a semi-Fredholm operator and* (4.2) *holds.*

Proposition 4.8. *Assume that* (2.1), (2.2)*, and* (2.7) *hold and that* $1-\mu < \alpha \le 1$*. Then the closure of* $R(G_{\alpha-1})$ *is equal to the space*

$$\mathcal{F} := \{f \in AAP^{\pm}_{\alpha-1} : \int_{\mathbb{R}} \langle f(s), v(s)\rangle_{X^s_{\alpha-1}}\, ds = 0 \quad \text{for all } v \in \mathcal{V}\},$$

where $\mathcal{V}$ *is the space of those* $v \in L^1(\mathbb{R}, X^*)$ *such that* $v(s) = U_{\alpha-1}(t,s)^* v(t)$ *for all* $t \ge s$ *in* $\mathbb{R}$*.*

In the following Fredholm alternative, we restrict ourselves to the asymptotically hyperbolic case. The projections $Q_{\pm\infty}$ have finite rank if, for instance, the domains $D(A_{\pm\infty})$ are compactly embedded in X.

Theorem 4.9. *Assume that* (2.1), (2.2), (2.5) *and* (2.6) *are true, that* $\dim Q_{\pm\infty}X < \infty$*, and that* $1-\mu < \alpha \le 1$*. Let* $f \in AAP^{\pm}_{\alpha-1}$*. Then there* *is a mild solution* $u \in AAP^{\pm}(\mathbb{R}, X)$ *of* (3.1) *if and only if*

$$\int_{\mathbb{R}} \langle f(s), w(s) \rangle_{X^s_{\alpha-1}} \, ds = 0$$

for each $w \in L^1(\mathbb{R}, X^*)$ *with* $w(s) = U_{\alpha-1}(t,s)^* w(t)$ *for all* $t \geq s$ *in* $\mathbb{R}$. *The mild solutions* u *are given by*

$$\begin{aligned} u(t) &= v(t) - U(t,T)y_s + (R^+_{\alpha-1} f)(t), \qquad t \geq T, \\ u(t) &= v(t) + \tilde{v}(t) + (R^-_{\alpha-1} f)(t), \qquad t \leq T, \end{aligned}$$

where $R^{\pm}_{\alpha-1}$ *was defined before Proposition 4.5,* $(R^+_{\alpha-1} f)(T) - (R^-_{\alpha-1} f)(T) = y_s + y_u \in P(T)X + X_u(T)$, $\tilde{v} \in C_0((-\infty, T], X)$ *with* $\tilde{v}(T) = y_u$ *and* $\tilde{v}(t) = U(t,s)\tilde{v}(s)$ *for all* $T \geq t \geq s$, *and* $v \in C_0(\mathbb{R}, X)$ *with* $v(t) = U(t,s)v(s)$ *for all* $t \geq s$.

5. Non–autonomous parabolic boundary evolution equations

In this section we study the non–autonomous forward (resp. backward) parabolic boundary evolution equation

$$\begin{aligned} u'(t) &= A_m(t)u(t) + g(t), \qquad t \geq t_0 \quad (\text{resp. } t \leq t_0), \\ B(t)u(t) &= h(t), \qquad t \geq t_0 \quad (\text{resp. } t \leq t_0), \\ u(t_0) &= u_0, \end{aligned} \tag{5.1}$$

and their variant on the line

$$\begin{aligned} u'(t) &= A_m(t)u(t) + g(t), \qquad t \in \mathbb{R}, \\ B(t)u(t) &= h(t), \qquad t \in \mathbb{R}. \end{aligned} \tag{5.2}$$

Here $t_0 \in \mathbb{R}$, $u_0 \in X$, and the inhomogeneities g and h take values in Banach spaces X and Y, respectively. We assume that the following conditions hold.

- (A1) There are Banach spaces $Z \hookrightarrow X$ and Y such that the operators $A_m(t) \in \mathcal{L}(Z, X)$ and $B(t) \in \mathcal{L}(Z, Y)$ are uniformly bounded for $t \in \mathbb{R}$ and that $B(t) : Z \to Y$ is surjective for each $t \in \mathbb{R}$.
- (A2) The operators $A(t)u := A_m(t)u$ with domains $D(A(t)) := \{u \in Z : B(t)u = 0\}$, $t \in \mathbb{R}$, satisfy (2.1) and (2.2) with constants $\omega, \theta, K, L, \mu, \nu$. Moreover, the graph norm of $A(t)$ and the norm of Z are equivalent with constants being uniform in $t \in \mathbb{R}$.

In the typical applications $A_m(t)$ is a differential operator with 'maximal' domain not containing boundary conditions and $B(t)$ are boundary operators. Under the hypotheses (A1) and (A2), there is an evolution family $(U(t,s))_{t \geq s}$ solving the problem with homogeneous conditions $g = h = 0$. Moreover, by [13, Lemma 1.2] there exists the *Dirichlet map* $D(t)$ for $\omega - A_m(t)$; i.e., $v = D(t)y$ is the unique solution of the abstract boundary value problem

$$(\omega - A_m(t))v = 0, \qquad B(t)v = y,$$

for each $y \in Y$. (In [13] the density of Z in X was assumed, but this does not play a role in the cited Lemma 1.2.) Let $x \in X$ and $y \in Y$ be given. The problem

$$(\omega - A_m(t))v = x, \qquad B(t)v = y,$$

has the solution $v = R(\omega, A(t))x + D(t)y$. This solution is unique in Z since $\omega - A_m(t)$ is injective on $D(A(t)) = N(B(t))$. We further assume that

(A3) there is a $\beta \in (1-\mu, 1]$ such that $Z \hookrightarrow X^t_\beta$ for $t \in \mathbb{R}$ with uniformly bounded embedding constants and $\sup_{t\in\mathbb{R}} \|D(t)\|_{\mathcal{L}(Y,Z)} < \infty$.

Lemma 5.1. *Assume that assumptions (A1), (A2) and (A3) without* (2.2) *hold. For a closed unbounded interval J, let $A_m(\cdot) \in AP(J, \mathcal{L}(Z,X))$ and $B(\cdot) \in AP(J, \mathcal{L}(Z,Y))$. Then we have*

(a) $D(\cdot) \in AP(J, \mathcal{L}(Y,Z))$,

(b) $R(\omega, A(\cdot)) \in AP(J, \mathcal{L}(X,Z))$,

(c) $(\omega - A_{-1}(\cdot))D(\cdot)h \in AP_{\alpha-1}(J)$ for every $h \in AP(J,Y)$ and $\alpha \in (1-\mu, \beta)$.

The same results hold if one replaces throughout AP by AAP (if $J \neq \mathbb{R}$) or by $AAP^\pm$ (if $J = \mathbb{R}$).

Proof. (a) Let $y \in Y$ and $t, t+\tau \in J$. By the definition of $D(t)$, we have

$$\begin{aligned}(\omega - A_m(t))(D(t+\tau)y - D(t)y) &= (A_m(t+\tau) - A_m(t))D(t+\tau)y =: \varphi(t),\\ B(t)(D(t+\tau)y - D(t)y) &= -(B(t+\tau) - B(t))D(t+\tau)y =: \psi(t),\end{aligned}$$

and thus

$$D(t+\tau)y - D(t)y = R(\omega, A(t))\varphi(t) + D(t)\psi(t).$$

The assumptions now imply that

$$\begin{aligned}\|D(t+\tau)y - D(t)y\|_Z &\le c\,(\|\varphi(t)\|_X + \|\psi(t)\|_Y)\\ &\le c\,(\|A_m(t+\tau) - A_m(t)\|_{\mathcal{L}(Z,X)} + \|B(t+\tau) - B(t)\|_{\mathcal{L}(Z,Y)})\,\|y\|_Y.\end{aligned}$$

So the almost periodicity of $D(\cdot)$ follows from that of $A_m(\cdot)$ and $B(\cdot)$.

(b) For $x \in X$ and $t, t+\tau \in J$, set $y = R(\omega, A(t+\tau))x - R(\omega, A(t))x \in Z$. Then we obtain

$$\begin{aligned}(\omega - A_m(t))y &= (A_m(t+\tau) - A_m(t))R(\omega, A(t+\tau))x =: \varphi_1(t),\\ B(t)y &= (B(t) - B(t+\tau))R(\omega, A(t+\tau))x =: \psi_1(t).\end{aligned}$$

Hence $y = R(\omega, A(t))\varphi_1(t) + D(t)\psi_1(t)$, and assertion (b) can now be shown as in (a).

(c) Due to (a) and (b), the functions $D(\cdot)h$ and $f_n := nR(n, A(\cdot))D(\cdot)h$ are almost periodic in Z, and hence in X, for $n > \omega$. Then $A(\cdot)f_n = (n^2R(n, A(\cdot)) - n)D(\cdot)h$ belongs to $AP(J, X)$. Assumptions (2.1) and (A3) imply that f_n is uniformly bounded in the norm of E_β. Since $f_n \to D(\cdot)h$ in $C_b(J, X)$, we conclude by interpolation that $f_n \to D(\cdot)h$ in E_α. As a consequence, $(\omega - A(\cdot))f_n \to (\omega - A_{\alpha-1}(\cdot))D(\cdot)h$ in $E_{\alpha-1}$, whence (c) follows.

Similarly one establishes the assertions concerning AAP and $AAP^\pm$. $\square$

In order to apply the results from the previous sections to the boundary forward (resp. backward) evolution equation (5.1), we write it as the inhomogeneous Cauchy problem

$$\begin{aligned} &u'(t) = A_{-1}(t)u(t) + f(t), \qquad t \geq t_0 \quad (\text{resp. } t \leq t_0), \\ &u(t_0) = u_0, \end{aligned} \tag{5.3}$$

setting $f := g + (\omega - A_{-1}(\cdot))D(\cdot)h$. We also consider the evolution equation

$$u'(t) = A_{-1}(t)u(t) + f(t), \qquad t \in \mathbb{R}. \tag{5.4}$$

In the following we will have $f \in E_{\alpha-1}(J)$, where we fix the number $\alpha \in (1-\mu, \beta)$ from Lemma 5.1. We note that a function $u \in C^1(J, X)$ with $u(t) \in Z$ satisfies (5.1), resp. (5.2), if and only if it satisfies (5.3), resp. (5.4). These facts can be shown as in Proposition 4.2 of [9]. This motivates the following definition. We call a function $u \in C(J, X)$ a *mild solution* of (5.2) and (5.4) on J if the equation

$$u(t) = U(t,s)u(s) + \int_s^t U_{\alpha-1}(t,\sigma)[g(\sigma) + (\omega - A_{-1}(\sigma))D(\sigma)h(\sigma)]\, d\sigma \tag{5.5}$$

holds for all $t \geq s$ in J. The function u is called a *mild solution* of (5.1) (resp. (5.3) if in addition $u(t_0) = u_0$ and $J = [t_0, \infty)$ (resp. $J = (-\infty, t_0]$).

Theorems 3.3, 3.5 and 3.6 and Lemma 5.1 immediately imply three results on the existence of almost periodic mild solutions for (5.2) and (5.1).

Proposition 5.2. *Assume that (A1)–(A3) hold, that $A_m(\cdot) \in AP(\mathbb{R}, \mathcal{L}(Z, X))$ and $B(\cdot) \in AP(\mathbb{R}, \mathcal{L}(Z, Y))$, and that $U(t,s)$ has an exponential dichotomy on $\mathbb{R}$. Let $g \in AP(\mathbb{R}, X)$ and $h \in AP(\mathbb{R}, Y)$. Then there is a unique mild solution $u \in AP(\mathbb{R}, X)$ of the boundary equation (5.2) given by*

$$u(t) = \int_{\mathbb{R}} \Gamma_{\alpha-1}(t,s)[g(s) + (\omega - A_{-1}(s))D(s)h(s)]\, ds, \quad t \in \mathbb{R}.$$

Proposition 5.3. *Assume that assertions (A1)–(A3) hold, that $A_m(\cdot) \in AAP([a,\infty), \mathcal{L}(Z, X))$, and $B(\cdot) \in AAP([a,\infty), \mathcal{L}(Z, Y))$, and that $U(t,s)$ has an exponential dichotomy on $[a, \infty)$. Let $t_0 > a$, $g \in AAP([a,\infty), X)$, $h \in AAP([a,\infty), Y)$, and $u_0 \in \overline{D(A(t_0))}$. Then the mild solution u of the equation (5.1) belongs to $AAP([t_0, +\infty), X)$ if and only if*

$$Q(t_0)u_0 = -\int_{t_0}^{+\infty} \widetilde{U}_{\alpha-1}(t_0,s)Q_{\alpha-1}(s)[g(s) + (\omega - A_{-1}(s))D(s)h(s)]\, ds.$$

In this case u is given by

$$\begin{aligned} u(t) = U(t,t_0)P(t_0)u_0 &+ \int_{t_0}^t U_{\alpha-1}(t,s)P_{\alpha-1}(s)[g(s) + (\omega - A_{-1}(s))D(s)h(s)]\, ds \\ &- \int_t^\infty \tilde{U}_{\alpha-1}(t,s)Q_{\alpha-1}(s)[g(s) + (\omega - A_{-1}(s))D(s)h(s)]\, ds, \quad t \geq t_0. \end{aligned}$$

Proposition 5.4. *Assume that assertions (A1)–(A3) hold, that* $A_m(\cdot) \in AAP((-\infty, b], \mathcal{L}(Z,X))$ *and* $B(\cdot) \in AAP((-\infty, b], \mathcal{L}(Z,Y))$, *and that* $U(t,s)$ *has an exponential dichotomy on* $(-\infty, b]$. *Let* $t_0 < b$, $g \in AAP((-\infty, b], X)$, $h \in AAP((-\infty, b], Y)$, *and* $u_0 \in X$. *Then there is a mild solution* $u \in AAP((-\infty, t_0], X)$ *of the equation* (5.1) *if and only if*

$$P(t_0)u_0 = \int_{-\infty}^{t_0} U_{\alpha-1}(t_0, s)P_{\alpha-1}(s)[g(s) + (\omega - A_{-1}(s))D(s)h(s)]\, ds.$$

In this case u *is given by*

$$\begin{aligned} u(t) = \widetilde{U}(t, t_0)Q(t_0)u_0 - &\int_t^{t_0} \widetilde{U}_{\alpha-1}(t,s)Q_{\alpha-1}(s)[g(s) + (\omega - A_{-1}(s))D(s)h(s)]\, ds \\ &+ \int_{-\infty}^{t} U_{\alpha-1}(t,s)P_{\alpha-1}(s)[g(s) + (\omega - A_{-1}(s))D(s)h(s)]\, ds, \quad t \le t_0. \end{aligned}$$

Moreover, Theorem 4.9 implies the following Fredholm alternative for the mild solutions of (5.2), where we focus on the asymptotically hyperbolic case.

Theorem 5.5. *Assume that assumptions (A1)–(A3) hold and that* $A_m(t) \to A_m(\pm\infty)$ *in* $\mathcal{L}(Z,X)$ *and* $B(t) \to B(\pm\infty)$ *in* $\mathcal{L}(Z,Y)$ *as* $t \to \pm\infty$. *Set* $A_{\pm\infty} := A_m(\pm\infty)|N(B(\pm\infty))$. *We suppose that* $\sigma(A_{\pm\infty}) \cap i\mathbb{R} = \emptyset$ *and that the corresponding unstable projections* $Q_{\pm\infty}X$ *have finite rank. Let* $g \in AAP^{\pm}(\mathbb{R}, X)$ *and* $h \in AAP^{\pm}(\mathbb{R}, Y)$. *Then there is a mild solution* $u \in AAP^{\pm}(\mathbb{R}, X)$ *of* (5.2) *if and only if*

$$\int_{\mathbb{R}} \langle f(s), w(s)\rangle_{X^s_{\alpha-1}}\, ds = 0$$

for $f := g + (\omega - A_{-1}(\cdot))D(\cdot)h$ *and all* $w \in L^1(\mathbb{R}, X^*)$ *with* $w(s) = U_{\alpha-1}(t,s)^* w(t)$ *for all* $t \ge s$ *in* $\mathbb{R}$. *The mild solutions* u *are given by*

$$\begin{aligned} u(t) &= v(t) - U(t,T)y_s + (R^+_{\alpha-1}f^+)(t), \qquad t \ge T, \\ u(t) &= v(t) + \tilde{v}(t) + (R^-_{\alpha-1}f^-)(t), \qquad t \le T, \end{aligned}$$

where $R^{\pm}_{\alpha-1}$ *was defined before Proposition 4.5,* $f^+ = f|[T, +\infty)$, $f^- = f|(-\infty, -T]$, $(R^+_{\alpha-1}f^+)(T) - (R^-_{\alpha-1}f^-)(T) = y_s + y_u \in P(T)X + X_u(T)$, $\tilde{v} \in C_0((-\infty, T], X)$ *with* $\tilde{v}(T) = y_u$ *and* $\tilde{v}(t) = U(t,s)\tilde{v}(s)$ *for all* $T \ge t \ge s$, *and* $v \in C_0(\mathbb{R}, X)$ *with* $v(t) = U(t,s)v(s)$ *for all* $t \ge s$.

Proof. Observe that functions converging at $\pm\infty$ belong to $AAP^{\pm}$. So it remains to show that $R(\omega, A(t)) \to R(\omega, A_{\pm\infty})$ in $\mathcal{L}(X)$ as $t \to \pm\infty$. This can be established as Lemma 5.1(b). □

We conclude with a pde example. One could treat more general problems, in particular systems, cf. [11], and one could weaken the regularity assumptions; but we prefer to keep the example simple.

Example 5.6. We study the boundary value problem

$$\begin{aligned} \partial_t u(t,x) &= A(t,x,D)u(t,x) + g(t,x), \quad t\in\mathbb{R},\ x\in\Omega, \\ B(t,x,D)u(t,x) &= h(t,x), \quad t\in\mathbb{R},\ x\in\partial\Omega, \end{aligned} \tag{5.6}$$

on a bounded domain $\Omega\subseteq\mathbb{R}^n$ with boundary $\partial\Omega$ of class C^2, employing the differential expressions

$$\begin{aligned} A(t,x,D) &= \sum\nolimits_{k,l} a_{kl}(t,x)\partial_k\partial_l + \sum\nolimits_k a_k(t,x)\,\partial_k + a_0(t,x), \\ B(t,x,D) &= \sum\nolimits_k b_k(t,x)\,\partial_k + b_0(t,x), \end{aligned}$$

where $B(t)$ is understood in the sense of traces. We require that $a_{kl}=a_{lk}$ and b_k are real–valued, $a_{kl},a_k,a_0\in C_b^\mu(\mathbb{R},C(\overline{\Omega}))$, $b_k,b_0\in C_b^\mu(\mathbb{R},C^1(\partial\Omega))$,

$$\sum_{k,l=1}^n a_{kl}(t,x)\,\xi_k\,\xi_l \ge \eta\,|\xi|^2\,, \qquad \text{and} \qquad \sum_{k=1}^n b_k(t,x)\nu_k(x)\ge\beta$$

for constants $\mu\in(1/2,1)$, $\beta,\eta>0$ and all $\xi\in\mathbb{R}^n$, $k,l=1,\cdots,n$, $t\in\mathbb{R}$, $x\in\overline{\Omega}$ resp. $x\in\partial\Omega$. (C_b^μ is the space of bounded, globally Hölder continuous functions.) Let $p\in(1,\infty)$. We set $X=L_p(\Omega)$, $Z=W_p^2(\Omega)$, $Y=W_p^{1-1/p}(\Omega)$ (a Slobodeckij space), $A_m(t)u=A(t,\cdot,D)u$ and $B(t)u=B(t,\cdot,D)u$ for $u\in Z$ (in the sense of traces), and $A(t)=A_m(t)|N(B(t))$. The operators $A(t)$, $t\in\mathbb{R}$, satisfy (2.1) and (2.2), see [2], [3], [20], or [27, Example 2.9]. Thus $A(\cdot)$ generates an evolution family $U(\cdot,\cdot)$ on X. It is known that the graph norm of $A(t)$ is uniformly equivalent to the norm of Z, that $B(t):Z\to Y$ is surjective, that $X_\alpha^t=W_p^{2\alpha}(\Omega)$ with uniformly equivalent norms for $\alpha\in(1-\mu,1/2)$, and that the Dirichlet map $D(t):Z\to Y$ is uniformly bounded for $t\in\mathbb{R}$, see e.g. [3, Example IV.2.6.3].

Further let $g\in AAP^{\pm}(\mathbb{R},X)$ and $h\in AAP^{\pm}(\mathbb{R},Y)$. We define mild solutions of (5.6) again by (5.5). We further assume that

$$a_\alpha(t,\cdot)\to a_\alpha(\pm\infty,\cdot) \quad \text{in } C(\overline{\Omega}) \qquad \text{and} \qquad b_j(t,\cdot)\to b_j(\pm\infty,\cdot) \quad \text{in } C^1(\partial\Omega)$$

as $t\to\pm\infty$, where $\alpha=(k,l)$ or $\alpha=j$ for $k,l=1,\cdots,n$ and $j=0,\cdots,n$. As a result, $A_m(\cdot)\in AAP^{\pm}(\mathbb{R},\mathcal{L}(Z,X))$ and $B(\cdot)\in AAP^{\pm}(\mathbb{R},\mathcal{L}(Z,Y))$. We define the sectorial operators $A_{\pm\infty}$ in the same way as $A(t)$. As in [11, Example 5.1] one can check that (2.5) holds. Finally we assume that $i\mathbb{R}\subset\rho(A_{\pm\infty})$. Then the Fredholm alternative stated in Theorem 5.5 holds for mild solutions of (5.6) on $X=L^p(\Omega)$ for $g\in AAP^{\pm}(\mathbb{R},X)$ and $h\in AAP^{\pm}(\mathbb{R},Y)$.

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