# Higher-Order Corrections to Precision Observables in Kaon Decays

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# Chapter 1

# Introduction

The aim of particle physics is the discovery of the laws that govern the interactions of the fundamental particles. We also want to understand the structure of these interactions at the highest energies, which allows us to get a picture of what happened at the very early stages of the evolution of our universe.
Our present knowledge is contained in the Standard Model of particle physics. It is an extremely successful quantum field theory based on the gauge group $SU(3) \times SU(2) \times U(1)$, describing the strong and electroweak interactions of all particles observed so far.
Despite of the tremendous success of the Standard Model, there are some intriguing open questions, which are left unanswered. Among those are the following:

- What is the mechanism of electroweak symmetry breaking?
- What additional sources of $CP$ violation can explain the observed matter-antimatter asymmetry of the universe?
- What is the reason for the quantisation of the electric charge?
- How can we include the observed non-zero masses of neutrinos in the theory?
- How can we unify gravity as a quantum theory with the Standard Model gauge forces?

It is the common belief that the Standard Model is merely an effective theory, arising as some low energy limit of a more complete theory, which is able to answer at least part of the questions above. This opens a plethora of possibilities for model building, and we clearly need a guideline of which of these models may be correct. It is well possible that the experiments at the Large Hadron Collider (LHC) at CERN will discover new particles in the near future, which would give us the most direct access to the structure of the new interactions. A second possibility is to set stringent bounds on the new interactions by precision measurements of judiciously chosen observables, or conversely, to look for deviations from Standard Model predictions in the experimental data. This is the main goal of the flavour physics programme. Flavour physics is concerned with the investigation of particle interactions, in which the flavour quantum number of the participating particles is changed. An example with particularly rich phenomenological consequences is the decay of $B$-flavoured mesons.
The interest in these processes derives from the fact that only the weak interaction mediates such flavour changing transitions, and in addition it is the only known interaction that breaks the parity ($P$), charge ($C$) and combined charge-parity ($CP$) symmetries. The origin of $CP$

violation in the Standard Model lies solely in Yukawa-type interactions of the quark fields with the complex Higgs field. They are described by the Cabbibo-Kobayashi-Maskawa (CKM) matrix, which is parameterised by three angles and one complex phase. $CP$ violation in the Standard Model is then only an effect of this complex phase [1].
In order to find deviations from Standard Model predictions, it is most promising to study processes which are suppressed in the Standard Model, such that effects of new physics can lead to sizeable contributions. In this thesis we will focus on two prime candidates of this type: $CP$ violation in the neutral Kaon system, and the rare decay $K^+ \to \pi^+ \nu\bar{\nu}$.

## *CP* Violation in the Neutral Kaon System

Over the past fifty years, Kaon physics has contributed a lot to the understanding of particle interactions and the development of the Standard Model. Here we only want to mention the introduction of the concept of strangeness in 1953 by Gell-Mann [2], the discovery of parity violation as the solution to the "$\theta$ - $\tau$-puzzle" by Lee and Yang in 1956 [3], and the postulation of the charm quark in order to explain the small branching ratio of the decay $K_L \to \mu^+\mu^-$ by Glashow, Iliopoulos and Maiani in 1970 [4]. $K_L$ is the longer-lived mass eigenstate of the neutral Kaon system.
$CP$ violation was also discovered in Kaon decays by Christenson, Cronin, Fitch and Turlay in 1964, who observed the decay of a $K_L$ into two pions [5]. This decay is forbidden in the case of exact $CP$ symmetry, where the $K_L$ is a $CP$ eigenstate. $CP$ symmetry is violated by a small admixture of a $K_S$ component, an effect called "$CP$ violation in mixing" and parameterised by the real part of the parameter $\epsilon_K$. Direct $CP$ violation has been observed in the same decay mode [6]. It is traditionally described in terms of the parameter $\epsilon'$, whereas the interference induced $CP$ violation results in a non-zero imaginary part of $\epsilon_K$.
The parameter $\epsilon_K$ is measured with high accuracy: The value quoted by the Particle Data Group is $\epsilon_K = (2.229 \pm 0.012) \times 10^{-3} \times e^{i(43.5\pm0.7)^\circ}$ [7]. Whereas about a decade ago the numerical value of $\epsilon_K$ was used as an input to determine the Standard Model parameters, today the objective is a different one: The three angles and the phase of the CKM matrix have been determined with high precision by various measurements of observables related to $B$-meson decays, and we can now use $\epsilon_K$ to test the consistency of these values with the results obtained in the Kaon sector. For this reason $\epsilon_K$ is one of the most important ingredients of the global fit of the unitarity triangle, which summarises the results of the various measurements. It is therefore of vital importance to have a precise Standard Model prediction for its value. The improvement of this prediction constitutes one of the main parts of this thesis.
The transition matrix element describing the mixing of the two flavour eigenstates $|K^0\rangle$ and $|\bar{K}^0\rangle$, with strangeness $S = +1$ and $S = -1$, respectively, receives both short and long distance QCD corrections. The former can be calculated systematically in perturbation theory, while for the latter non-perturbative methods, such as lattice QCD, have to be used.
We describe the mixing in the Standard Model by an effective Hamiltonian of the form

$$\mathcal{H}_{\text{eff}}^{|\Delta S|=2} = \frac{G_F^2}{4\pi^2} M_W^2 \left[ \lambda_c^2 \eta_1 S\left(\frac{m_c^2}{M_W^2}\right) + \lambda_t^2 \eta_2 S\left(\frac{m_t^2}{M_W^2}\right) + 2\lambda_c\lambda_t \eta_3 S\left(\frac{m_c^2}{M_W^2}, \frac{m_t^2}{M_W^2}\right)\right] \times b(\mu)\tilde{Q}_{S2} + \text{h.c.}\,. \tag{1.1}$$

Here $G_F$ is the Fermi constant, $\lambda_i = V_{id}V_{is}^*$ comprises the CKM matrix elements, and $\tilde{Q}_{S2}$ is a local four-quark operator which induces the $|\Delta S| = 2$ transition. The short distance

contributions are contained in the coefficient in square brackets. The Inami-Lim functions $S(x)$ and $S(x,y)$ [8] describe the leading-order (LO) Standard Model contribution. The coefficients $\eta_1$, $\eta_2$, and $\eta_3$ contain the perturbative QCD corrections.
The matrix elements of the operator $Q_{S2}$ between external Kaon states constitute genuine long-distance effects that can be calculated by means of lattice QCD. The error of the prediction of the matrix elements has been reduced considerably in the recent past; the latest publication [9] quotes an error of only 4%. This development is a strong motivation to reconsider the perturbative QCD corrections, which are now dominating the theoretical error of the prediction.
The parameter $\eta_2$, comprising the top quark induced corrections, has been calculated at the next-to-leading order (NLO) in QCD already in 1990 by Buras, Jamin, and Weisz [10], leaving a remaining theoretical scale uncertainty of the order of one percent, which is clearly satisfactory. However, the corrections induced by the charm quark, $\eta_1$, and the mixed charm and top quark contribution, $\eta_3$, are harder to calculate and, due to the charm quark being present at rather low energies, plagued by larger theoretical uncertainties.
A first step beyond the LO calculations [11–15] has been performed by Herrlich and Nierste, who calculated the NLO QCD corrections to $\eta_1$ [16] and $\eta_3$ [17], resulting in a remaining theoretical scale uncertainty of 18% and 8%, respectively.
In light of the improvements on the long distance corrections mentioned above, these scale uncertainties should clearly be reduced. In this work we perform a major step towards this goal by calculating the next-to-next-to-leading order (NNLO) anomalous dimensions and performing the corresponding matching calculation for the charm-top contribution $\eta_3$.

## The Rare Decay $K^+ \to \pi^+ \nu\bar{\nu}$

The second main subject of this thesis is the rare decay $K^+ \to \pi^+ \nu\bar{\nu}$. It is induced by flavour-changing neutral currents (FCNC), which are forbidden at tree level in the Standard Model, and is dominated by short distance physics because of the quadratic GIM suppression of non-perturbative effects. The decay is induced by the effective Hamiltonian

$$\mathcal{H}_{\text{eff}} = \frac{4G_F}{\sqrt{2}} \frac{\alpha}{2\pi \sin^2\theta_W} \sum_{l=e,\mu,\tau} \left[ \lambda_c X^l \left( \frac{m_c^2}{M_W^2} \right) + \lambda_t X_t \left( \frac{m_t^2}{M_W^2} \right) \right] \tilde{Q}_\nu \,, \tag{1.2}$$

where $\alpha$ is the electromagnetic coupling and $\theta_W$ the weak mixing angle. $\tilde{Q}_\nu$ is to an excellent approximation the only operator contributing to the decay. The function $X_t(m_t^2/M_W^2)$ describes the top-quark contribution to the effective Hamiltonian. It is known up to NLO in QCD [8,18–20], yielding a remaining scale uncertainty of $\pm 1\%$. The leading term in the large top quark mass expansion of the electroweak two-loop corrections is also known and typically amounts to a per mil correction for the branching ratio, while the uncertainty related to unknown sub-leading electroweak contributions is conservatively estimated to be $\pm 2\%$ [21]. The function $X^l(m_c^2/M_W^2)$ parameterises the charm-quark contribution. The large scale uncertainty of the LO analysis [12] of $\pm 26\%$ was reduced by a NLO [19, 22] and a subsequent NNLO QCD calculation [23–25] to $\pm 2.5\%$, which leaves the QCD corrections in a satisfactory state.
Our contribution is the calculation of the LO and NLO logarithmic QED corrections as well as the electroweak matching corrections to the charm-quark contribution [26]. Only by the inclusion of the electroweak corrections it is possible to fix the renormalisation scheme of

the electroweak input parameters. The resulting increase of the charm-quark contribution of about 2% is of the same order of magnitude as the remaining scale uncertainty after inclusion of the NNLO QCD corrections, whereas now the scheme dependence related to the electroweak input parameters is completely negligible.
The hadronic matrix element of the low-energy effective Hamiltonian have been extracted with high accuracy from the full set of data on the well-measured $K_{\ell 3}$ decays by the use of isospin symmetry, including isospin breaking and long-distance QED radiative corrections [27–29]. In addition there are more theoretical uncertainties related to contributions of dimension-eight operators at the charm quark scale [30, 31] and to higher order electroweak corrections to the matrix elements [28]. They amount to an error of approximately 3% on the branching ratio. The role of theoretical and experimental accuracy for the decay $K^+ \to \pi^+ \nu\bar{\nu}$ is somewhat reversed with respect to the situation for $\epsilon_K$. The branching ratio is theoretically known with exceptional precision; the two decay modes $K^+ \to \pi^+ \nu\bar{\nu}$ and $K_L \to \pi^0 \nu\bar{\nu}$ are the only observables in the Kaon system with hadronic uncertainties below 5%. The theoretical error is now mainly of parametrical origin and will be further reduced in the future by better measurements of the input parameters. On the other hand, although seven events of the decay mode $K^+ \to \pi^+ \nu\bar{\nu}$ have been observed [32], we will have to wait for the results of experiments like NA62 at CERN, which aims at measuring the branching ratio with an accuracy of 10%, in order to exploit the potential of this decay in the search for new physics.

## Outline of this Thesis

This thesis is organised as follows. In Chapter 2 we give an introduction to the field theoretical concepts underlying our work. We start with a short introduction into quantum field theories and the Standard Model, with an emphasis on the renormalisation of QED and QCD. Afterwards we introduce effective theories and the weak effective Hamiltonian, and discuss the renormalisation and the renormalisation group running of the Wilson coefficients in detail. We also discuss the scheme dependence of the anomalous dimensions and Wilson coefficients. In Chapter 3 we consider more practical questions of our calculation, like the extraction of ultraviolet (UV) divergences and the calculation of the appearing integrals. In addition we provide a fairly general formula which relates the anomalous dimensions and Wilson coefficients in different renormalisation schemes; to this end we generalise results already present in the literature. The next two chapters are devoted to phenomenological applications: In Chapter 4 we perform a full renormalisation group analysis of the charm-top contribution $\eta_3$ to the parameter $\epsilon_K$, extending the known NLO QCD results to the NNLO level, and calculate the full two-loop QED and electroweak corrections to the branching ratio for the decay $K^+ \to \pi^+ \nu\bar{\nu}$ in Chapter 5. We summarise our results in Chapter 6.

# Part I

# Foundations

# Chapter 2

# Standard Model and Weak Effective Hamiltonian

We describe the physics of elementary particles by a quantum field theory. The reason is that any relativistic quantum theory will look at sufficiently low energy like a quantum field theory [33].
After introducing the Standard Model of particle physics in Section 2.1, we describe how to calculate S-matrix elements in perturbation theory in Section 2.2, where we also give an introduction to the general ideas of renormalisation. In Section 2.3 the formalism of the Weak Effective Hamiltonian is introduced, which allows the treatment of external mesons and provides a convenient method for the summation of large logarithms. The renormalisation of the appearing Wilson coefficients is treated in Section 2.4. We describe renormalisation group improved perturbation theory in Section 2.5 and the matching procedure in Section 2.6. In Section 2.7 we comment on the role of unphysical operators in our calculation. Finally, in Section 2.8 we derive general transformation properties of various quantities under a change of the renormalisation scheme.

## 2.1 Quantum Field Theory and the Standard Model

In this section we collect some basic facts about quantum field theories and the Standard Model. More details can be found for instance in [33, 34].
Any relativistic quantum field theory describing the interaction of elementary particles can be defined by a Lagrangian density which is composed of field operators in an Lorentz invariant way. In the case of a set of scalar fields it is given by

$$\mathcal{L} = \mathcal{L}[\phi(x), \partial_\mu \phi(x)] \,. \tag{2.1}$$

A solution to the theory is specified by describing the space of states and the way the operators act on this space. It can be reconstructed from the time-ordered Green's functions

$$G_N(x_1, \ldots, x_N) = \langle 0 | T\phi(x_1) \cdots \phi(x_N) | 0 \rangle \,, \tag{2.2}$$

which can be represented by a path integral

$$G_N(x_1, \ldots, x_N) = \mathcal{N}^{-1} \int [d\phi] e^{iS[\phi]} \phi(x_1) \cdots \phi(x_N) \,, \tag{2.3}$$

where $\mathcal{N}$ is a normalisation factor,

$$\mathcal{N} = \int [d\phi] e^{iS[\phi]} \,, \tag{2.4}$$

and the action $S$ is given by

$$S[\phi] = \int d^4\, x \mathcal{L}[\phi(x), \partial_\mu \phi(x)] \,. \tag{2.5}$$

The Standard Model is based on the concept of a quantum gauge field theory, which means that the action $S$ is invariant under space-time dependent transformations of the fields according to some representation of the gauge symmetry group $G$. The need for local invariance arises when we want to include massless spin-one particles into a Lorentz-invariant theory. We will now discuss the important case of $G = SU(N)$, which is the group of the special unitary $N \times N$ matrices. The fermion fields $\psi$ are assumed to transform according to the fundamental representation,

$$\psi_i(x) \to \psi_i'(x) = U_{ij}(x)\psi_j(x), \qquad SU(N) \ni U(x) = e^{-iT^a\theta^a(x)} \,, \tag{2.6}$$

where $\theta^a(x)$ is a real parameter which may depend on space-time, and the $T^a$ are the generators of the group $SU(N)$. They fulfill the following commutation relations:

$$\left[T^a, T^b\right] = if^{abc}T^c \,, \tag{2.7}$$

where the structure constants $f^{abc}$ define the Lie algebra of $SU(N)$. Here and in the following, we imply a summation over repeated indices. The Lagrangian will then be invariant if ordinary derivatives are replaced by covariant derivatives

$$\not{D}_{ij} = \not{\partial}\delta_{ij} - igT^a_{ij}\not{G}^a \,. \tag{2.8}$$

Here the fields $G^a$ describing the vector bosons associated with the interaction enter the game. The interaction with a fermion field of mass $m$ is given by

$$\mathcal{L}_{\rm f} = \bar{\psi}_i(i\not{D}_{ij} - m\delta_{ij})\psi_j \,. \tag{2.9}$$

The dynamics of the gauge fields is induced by

$$\mathcal{L}_{\rm g} = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} \tag{2.10}$$

with the field strength tensor

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + gf^{abc}G^b_\mu G^c_\nu \,. \tag{2.11}$$

## Quantisation

A powerful tool to solve a quantum field theory is the path integral. If the theory is gauge invariant, the integration involves field configurations which are related by the gauge transformation and yield the same action. The gauge variant propagator vanishes, which makes it impossible to formulate perturbation theory.

As shown by Faddeev and Popov [35], the problem can be solved by expressing the path integral as an integral over all fields satisfying some gauge condition times an integral over all gauge transformations. For a gauge-invariant Green's function, this yields an overall factor, which is cancelled by the corresponding factor in the normalisation, so the integral over the gauge transformations can be omitted. However, the gauge-variant Green's functions no longer vanish.
We impose a gauge condition of the general form $F^a[G,x] = f^a(x)$ and write a Green's function of any product of fields $X$ as

$$\langle 0|TX|0\rangle = \mathcal{N}\int [dG][d\bar{\psi}][d\psi]X\exp(iS_{\text{inv}})\Delta[G]\prod_{x,a}\delta(F^a(x)-f^a(x))\,. \tag{2.12}$$

Here $S_{\text{inv}}$ is the gauge invariant action and $\Delta[G]$ is the Jacobian that arises in transforming the integration variables to the set of fields satisfying the gauge condition and integrating over the gauge transformations. The normalisation factor is given by

$$\mathcal{N}^{-1} = \int [dG][d\bar{\psi}][d\psi]\exp(iS_{\text{inv}})\Delta[G]\prod_{x,a}\delta(F^a(x)-f^a(x))\,. \tag{2.13}$$

The Jacobian $\Delta[G]$ is a determinant and hence can be written as an integral:

$$\Delta[G] = \int [dc^a][d\bar{c}^a]\exp(i\mathcal{L}_{\text{gc}})\,. \tag{2.14}$$

Here we have introduced the anticommuting scalar ghost fields. For the choice $F^a[G,x] = \partial^\mu G^a_\mu(x)$, which we will use in the following, the *gauge compensating* Lagrangian is given by

$$\mathcal{L}_{\text{gc}} = \partial^\mu\bar{\eta}^a(\partial_\mu\eta^a + gf^{abc}\eta^b G^c_\mu)\,, \tag{2.15}$$

up to a total derivative.
A gauge invariant integral of the form (2.12) depends on the specific form of the gauge fixing functional only through an irrelevant overall factor. It turns out to be convenient to average over the functions $f^a$ with a Gaussian weight factor $\exp[-i\int d^4x(f^a)^2/(2\xi)]$. The total action is then given by

$$S = \int d^4\,x(\mathcal{L}_{\text{inv}} + \mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{gc}})\,, \tag{2.16}$$

where $\mathcal{L}_{\text{inv}}$ and $\mathcal{L}_{\text{gc}}$ have been defined above and $\mathcal{L}_{\text{gf}}$ is given by

$$\mathcal{L}_{\text{gf}} = -\frac{(\partial^\mu G^a_\mu(x))^2}{2\xi}\,. \tag{2.17}$$

Physical results are independent of the gauge parameter $\xi$.

## The Standard Model

The Standard Model of particle physics describes the strong interaction as based on the gauge group $SU(3)$. This means that quarks (the strongly interacting fundamental matter particles)

are described by a Lagrangian of the form $\sum \mathcal{L}_f + \mathcal{L}_g$, where the sum is over quark flavours, and the fields $\psi$ and $G$ correspond to the different quark flavours and the gluons, respectively. The electroweak sector is based on the gauge group $SU(2)_L \times U(1)_Y$. It exhibits a chiral structure, because the right- and left-handed fermion fields transform differently under the gauge group. In addition, the physical particle spectrum is not invariant under the full symmetry group, which is said to be spontaneously broken to the electromagnetic $U(1)_{em}$ via the Higgs mechanism. Thereby the $W$ and $Z$ gauge bosons acquire a mass, whereas the photon remains massless. In the Standard Model, spontaneous symmetry breaking is achieved by introducing an additional complex scalar field with an appropriately chosen potential energy term.
The mass terms for the fermions are generated by a Yukawa interaction between the fermion fields and the Higgs field, which acquires a non-vanishing vacuum expectation value. After diagonalising the mass terms, the interaction between the charged weak vector bosons $W^{\pm}$ and the quark fields is given by

$$\mathcal{L}_{\text{wc}} = -\frac{g_2}{\sqrt{2}} J^{\mu}_{cc}(x) W^{+}_{\mu}(x) + \text{h.c.} \tag{2.18}$$

in terms of the charged current

$$J^{\mu}_{cc}(x) = \bar{u}_{L,i}(x)\gamma^{\mu} V_{ij} d_{L,j}(x)\,. \tag{2.19}$$

The unitary Cabibbo-Kobayashi-Maskawa (CKM) matrix $V$ induces family-changing transitions via the coupling to the $W$ bosons. It contains four physical parameters (i.e. they cannot be removed by a redefinition of the quark fields, if the up- and down-type quarks have different masses): Namely, three angles and one complex phase, the latter being the only source of $CP$ violation in the Standard Model[1].
There are no flavour changing neutral currents (FCNC) in the Standard Model at tree level, and the unitarity of the CKM matrix implies the suppression of such processes induced by quantum effects; this is called the GIM mechanism.

## 2.2 Perturbation Theory and Renormalisation

In particle physics, we are interested in scattering processes, which are described by the $S$-matrix, given in terms of transition amplitudes of asymptotic in and out states,

$$S_{\beta\alpha} = \langle \beta, out | \alpha, in \rangle =: \langle \beta, in | S | \alpha, in \rangle\,. \tag{2.20}$$

The $S$-matrix can be related to the Green's functions of the theory via the reduction formula of Lehmann, Symanzik and Zimmermann [38]:

$$\langle p_1, \ldots, p_n, out | p_{n+1}, \ldots, p_{n+l}, in \rangle = \lim_{p_1^2 \to m_1^2} \cdots \lim_{p_{n+l}^2 \to m_{n+l}^2} \hat{z}^{\frac{n+l}{2}} \, \frac{\tilde{G}_{n+l}(-p_1, \ldots, -p_n, p_{n+1}, \ldots, p_{n+l},)}{\prod_{i=1}^{n+l} \tilde{G}_2(-p_i, p_i)}\,, \tag{2.21}$$

[1] Apart from the $CP$ violating $\theta$ term in QCD, which is, however, at most of the order $\mathcal{O}(10^{-9})$ [36,37].

where

$$\tilde{G}_N(p_1,\ldots,p_N) = \int d^4x_1 \ldots d^4x_N \exp[i(p_1x_1 + \cdots + p_Nx_N)]\, G_N(x_1,\ldots,x_N) \tag{2.22}$$

is the Fourier transform of the $N$-point Green's function and $i\hat{z}$ is the residue of $\tilde{G}_2(-p,p)$.

In an interacting theory, the Green's functions cannot be evaluated analytically in a closed form. One way to derive an approximate result is the use of perturbation theory. To this end we expand the kernel in equation (2.3) into a perturbation series in the coupling constants:

$$e^{iS[\phi]} = e^{iS_{\text{free}}[\phi]} \sum_n \frac{i^n}{n!} \left( \int d^4\, x \mathcal{L}_{\text{int}}[\phi(x), \partial_\mu \phi(x)] \right)^n , \tag{2.23}$$

where $\mathcal{L}_{\text{int}}$ is the interaction part of the Lagrangian, the free part being contained in $e^{iS_{\text{free}}}$.

## Renormalisation

The individual terms in the perturbation series involve integrals over four-momenta and are generally divergent. By the process of renormalisation it is nevertheless possible to obtain physically meaningful results.

The first step is to regularise the divergent integrals. We will exclusively use dimensional regularisation, which involves integration over an arbitrary (in fact, infinite) number of space-time dimensions $d = 4 - 2\epsilon$. After renormalisation the physical limit $d \to 4$ will be taken.

The renormalisation is performed by recursively subtracting all subdivergences of a Feynman diagram corresponding to one-particle irreducible (1PI) subdiagrams by suitably chosen counterterms. An explicit solution to this recursive method is given by Zimmermann's forest formula [39].

A key observation is that for a class of theories, which are termed *renormalisable* (by power counting), the subtraction terms can be absorbed into a redefinition of the parameters of the theory. This ensures the predictive power of the theory, as soon as these parameters have been determined by experiment. The Standard Model belongs to this class of theories.

Here, we will also be concerned with effective theories, which are not renormalisable in this strict sense. This means that new counterterms are generated in each order in perturbation theory. However, at each order in perturbation theory, only a finite number of new terms appear, which can be fixed by matching the effective theory to the Standard Model, thus preserving the predictive power of the theory.

For illustration, we will now describe the renormalisation of the QCD×QED Lagrangian, which constitutes an important part of the renormalisation program in the effective theory considered below. For the renormalisation of the electroweak sector, which is needed for computing the initial conditions of the Wilson coefficients, we refer the reader to [40].

The unrenormalised Lagrangian density for a coloured fermion with mass $m$ and electromag-

netic charge $Q_f$ reads:

$$\begin{aligned}\mathcal{L}_{0,\mathrm{QCD}\times\mathrm{QED}} = \quad & \overline{\psi}_{0i}(i\not{\partial} - m_0)\psi_{0i} + g_0\overline{\psi}_{0i}T^a_{ij}\not{G}^a_0\psi_{0j} + e_0\overline{\psi}_{0i}Q_f\not{A}_0\psi_{0i} \\ & - \frac{1}{4}(\partial_\mu G^a_{0\nu} - \partial_\nu G^a_{0\mu})(\partial^\mu G^{a\nu}_0 - \partial^\nu G^{a\mu}_0) - \frac{1}{2\xi_{g0}}\left(\partial_\mu G^{a\mu}_0\right)^2 \\ & - \frac{1}{4}(\partial_\mu A_{0\nu} - \partial_\nu A_{0\mu})(\partial^\mu A^\nu_0 - \partial^\nu A^\mu_0) - \frac{1}{2\xi_{a0}}\left(\partial_\mu A^\mu_0\right)^2 \\ & - \frac{g_0}{2}f^{abc}(\partial_\mu G^a_{0\nu} - \partial_\nu G^a_{0\mu})G^{b\mu}_0 G^{c\nu}_0 - \frac{g_0^2}{4}f^{abe}f^{cde}(G^a_{0\mu}G^b_{0\nu})G^{c\mu}_0 G^{d\nu}_0 \\ & + \overline{\eta}^a_0\partial_\mu\partial^\mu\eta^a_0 + g_0 f^{abc}(\partial_\mu\overline{\eta}^a_0)\eta^b_0 G^c_{0\mu}. \end{aligned} \tag{2.24}$$

Now we introduce renormalised fields and parameters:

$$\begin{aligned} G^a_{0\mu} &= Z_G^{1/2}G^a_\mu, & \eta^a_0 &= Z_\eta^{1/2}\eta^a, & \psi_0 &= Z_\psi^{1/2}\psi\,, \\ g_0 &= Z_g g\mu^\epsilon, & m_0 &= Z_m m, & \xi_{g0} &= Z_{\xi_g}\xi_g\,, \\ A^a_{0\mu} &= Z_A^{1/2}G^a_\mu, & e_0 &= Z_e e\mu^\epsilon, & \xi_{a0} &= Z_{\xi_a}\xi_a\,. \end{aligned} \tag{2.25}$$

Because in dimensional regularisation the Lagrangian is defined in $d$ dimensions, the coupling constants aquire a mass dimension. This dependence on an arbitrary mass scale can be factored out in order to keep the coupling constants dimensionless. In this way the scale $\mu$ enters the equations above.

We can now express the bare Lagrangian as a sum of the renormalised Lagrangian and a counterterm Lagrangian, $\mathcal{L}_0 = \mathcal{L} + \mathcal{L}_{\mathrm{ct}}$, where $\mathcal{L}$ has the same form as $\mathcal{L}_0$, but is expressed in terms of renormalised parameters and fields, and the counterterm Lagrangian is given by

$$\begin{aligned}\mathcal{L}_{\mathrm{ct}} = \quad & (Z_\psi - 1)\overline{\psi}_i i\not{\partial}\psi_i - (Z_\psi Z_m - 1)\overline{\psi}_i m\psi_i \\ & + (Z_g Z_G^{1/2}Z_\psi - 1)g\overline{\psi}_i T^a_{ij}\not{G}^a\psi_j + (Z_e Z_A^{1/2}Z_\psi - 1)e\overline{\psi}_i\not{A}\psi_j \\ & - \frac{1}{4}(Z_G - 1)(\partial_\mu G^a_\nu - \partial_\nu G^a_\mu)(\partial^\mu G^{a\nu} - \partial^\nu G^{a\mu}) - (Z_G - 1)\frac{1}{2\xi_g}\left(\partial_\mu G^{a\mu}\right)^2 \\ & - \frac{1}{4}(Z_A - 1)(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - (Z_A - 1)\frac{1}{2\xi_a}\left(\partial_\mu A^\mu\right)^2 \\ & - (Z_g Z_G^{3/2} - 1)\frac{g}{2}f^{abc}(\partial_\mu G^a_\nu - \partial_\nu G^a_\mu)G^{b\mu}G^{c\nu} \\ & - (Z_g^2 Z_G^2 - 1)\frac{g^2}{4}f^{abe}f^{cde}(G^a_\mu G^b_\nu)G^{c\mu}G^{d\nu} \\ & + (Z_\eta - 1)\overline{\eta}^a\partial_\mu\partial^\mu\eta^a + (Z_g Z_G^{1/2}Z_\eta - 1)gf^{abc}(\partial_\mu\overline{\eta}^a)\eta^b G^c_\mu\,. \end{aligned} \tag{2.26}$$

We now expand the renormalisation constants $Z$ in the strong and electromagnetic coupling constants:

$$Z = 1 + \left(\frac{e}{4\pi}\right)^2 Z^{(e)} + \left(\frac{e}{4\pi}\right)^2\left(\frac{g}{4\pi}\right)^2 Z^{(es)} + \sum_{k=1}^{\infty}\left(\frac{g}{4\pi}\right)^{2k} Z^{(k)}\,, \tag{2.27}$$

where we dropped higher powers of the electromagnetic coupling constant. It is possible to make the dependence of the renormalisation constants on $\epsilon$ explicit in the following way:

$$Z^{(k)} = \sum_{l\leq k} \left(\frac{1}{\epsilon}\right)^l Z^{(k,l)}\,, \quad Z^{(e)} = \sum_{l\leq 1} \left(\frac{1}{\epsilon}\right)^l Z^{(e,l)}\,, \quad Z^{(es)} = \sum_{l\leq 2} \left(\frac{1}{\epsilon}\right)^l Z^{(es,l)}\,. \tag{2.28}$$

The pole parts of the $Z$ factors must be chosen such that all divergences are cancelled. The remaining, non-pole parts can be chosen arbitrarily. In the minimal subtraction (MS) scheme these constant parts are set to zero, so that only the pole parts are subtracted. The modified minimal subtraction ($\overline{\text{MS}}$) scheme [41] is obtained from the MS scheme by redefining the parameter $\mu$:

$$\mu \to \mu \left(\frac{e^{\gamma_E}}{4\pi}\right)^{\frac{1}{2}}\,, \tag{2.29}$$

where $\gamma_E$ is Euler's constant. We will use this scheme exclusively in the following. The explicit forms of the renormalisation constants can be found in the literature [40, 42].

### Renormalisation Group

The renormalisation process involves a considerable amount of arbitrariness, because the renormalisation constants have to be chosen such that they cancel all divergences, but a finite subtraction is still possible. The choice of the finite subtractions is called a renormalisation scheme. It is important to note that results for physical observables do not depend on the renormalisation scheme, as the changes in the subtraction presription can be absorbed into a redefinition of the renormalised parameters. This invariance of the theory under a change of the renormalisation presription is described by the renormalisation group equations (RGE). Using the fact that the unrenormalised coupling constants are independent of the parameter $\mu$, we derive from Eq. (2.27) the following differential equations for the coupling constants:

$$\mu\frac{dg}{d\mu} = \beta(g(\mu), e(\mu), \epsilon)\,, \tag{2.30}$$

$$\mu\frac{de}{d\mu} = \beta_e(g(\mu), e(\mu), \epsilon)\,, \tag{2.31}$$

where the $\beta$ functions are given by

$$\beta(g(\mu), e(\mu), \epsilon) = -\epsilon g + \beta(g(\mu), e(\mu)) = -\epsilon g - g\frac{\mu}{Z_g}\frac{dZ_g}{d\mu}\,, \tag{2.32}$$

$$\beta_e(g(\mu), e(\mu), \epsilon) = -\epsilon e + \beta_e(g(\mu), e(\mu)) = -\epsilon e - e\frac{\mu}{Z_e}\frac{dZ_e}{d\mu}\,. \tag{2.33}$$

In a mass independent renormalisation scheme such as the $\overline{\text{MS}}$ scheme the only mass dependence of the renormalisation constants arises through the coupling constants. This observation leads to the following explicit form of the $\beta$ functions in terms of the $Z$ factors:

$$\beta(g, e) = -\beta_0\frac{g^3}{(4\pi)^2} - \beta_1\frac{g^5}{(4\pi)^4} - \beta_2\frac{g^7}{(4\pi)^6} - \beta_{se}\frac{g^3e^2}{(4\pi)^4} + \ldots \tag{2.34}$$

$$= 2Z_g^{(1,1)}\frac{g^3}{(4\pi)^2} + 4Z_g^{(2,1)}\frac{g^5}{(4\pi)^4} + 6Z_g^{(3,1)}\frac{g^7}{(4\pi)^6} + 2Z_g^{(es,1)}\frac{g^3e^2}{(4\pi)^4} + \ldots\,. \tag{2.35}$$

Explicit values for the coefficients are given in Appendix B. Similar equations can be found for the other parameters of the theory such as the quark masses.

## 2.3 Weak Effective Hamiltonian

In this thesis we will be calculating decay rates and transition matrix elements involving $K$-mesons. The former are characterised by widely separate energy scales, which are in our case the electroweak scale of the order $\mathcal{O}(M_W)$, the intermediate scale of the charm quark appearing in the LO Standard Model loop diagrams, of the order $\mathcal{O}(m_c)$, and the scale $\Lambda_{\mathrm{QCD}}$ of the order of the binding energy of the involved mesons.
These scale differences induce large logarithms in the perturbation series, making a fixed order calculation unreliable. Moreover, the strong coupling constant is scale-dependent, and at an energy of the order of $\Lambda_{\mathrm{QCD}}$, QCD becomes non-perturbative and an altogether different method of calculation has to be used.
The framework of effective field theory introduces a natural way to separate the different energy scales and allows for a convenient method to sum large logarithms to all orders in perturbation theory.
The basic idea of the effective field theory approach is contained in the decoupling theorem by Appelquist and Carrazone [43]. They consider Green's functions in a renormalisable theory in which some fields have much higher masses than other fields. The theorem then states that the Green's functions at momenta much lower than the heavy masses can be calculated in a theory with all heavy degrees of freedom removed, corrections being suppressed by a power of the momentum divided by a heavy mass. Provided that the low energy theory is still renormalisable, the only effect of the heavy fields is that they may change the values of the couplings in the low energy theory.
In the case of the weak interactions, removing the heavy gauge bosons leads to non-renormalisable four-fermion interactions. In fact, it is the non-decoupling property of the weak interactions which allows us to observe phenomena of high energy scales in decays of low mass particles like the Kaon. However, the arising divergences are still polynomial in the momenta and thus can be removed by local counterterms. Because of the smallness of the weak interactions, we need to consider contributions only of second order in the effective interactions, and in consequence only a finite number of counterterms is needed.
Strictly speaking, the Appelquist-Carrazone theorem is only valid in a physical renormalisation scheme like the momentum space subtraction scheme. In mass-independent renormalisation schemes like the $\overline{\mathrm{MS}}$ scheme, the decoupling has to be put in by hand. This is where the machinery of effective field theory comes into play [44, 45]: For each domain of validity, as defined by the different scales in the problem, we construct an effective theory containing operators that consist out of the fields in the effective theory and induce the interactions. They are multiplied by effective coupling constants, the so-called Wilson coefficients, which are chosen in such a way that the amplitudes in the theory just above the scale where we remove a heavy particle, and in the theory just below this scale are the same (up to powers of a momentum divided by a heavy mass). This is done by a matching calculation, which will be explained in Section 2.6.
In the following we list all the operators needed as building blocks for the weak effective Hamiltonian describing the $|\Delta S| = 1$ and $|\Delta S| = 2$ flavour-changing processes relevant for our work. It is obtained by integrating out all heavy particle fields with masses greater than

the electroweak scale characterised by $M_W$. We write it formally as

$$H_{\text{eff}} = C_i Z_{ij} Q_j + \left(\tilde{C}_i \tilde{Z}_{ik} + C_i C_j \hat{Z}_{ij,k}\right)\tilde{Q}_k \,. \tag{2.36}$$

Here the $Q_i$ denote dimension-six four-fermion operators, inducing $|\Delta S| = 1$ transitions, $C_i$ are the corresponding Wilson coefficients, and $Z_{ij}$ are the operator renormalisation constants, to be discussed in Section 2.4. The corresponding quantities for $|\Delta S| = 2$ transitions, induced by dimension-eight operators, are denoted by a tilde. Finally, the renormalisation constants $\hat{Z}_{ij,k}$ arise from the fact that the matrix elements of a double insertion of renormalised dimension-six operators need not be finite, and lead to a mixing of dimension-six into dimension-eight operators.

The operators which we need for our work can be divided into three classes: Physical operators, gauge-invariant operators that vanish by the QCD equations of motion (EOM), and evanescent operators, that vanish algebraically in four space-time dimensions. In the following, we introduce all the contributing operators in turn.

We start with the dimension-six operators, which we choose such that problems arising from the $\gamma_5$ matrix appearing in closed fermion loops in the framework of dimensional regularisation do not occur [46]. There are two current-current operators, which we choose as

$$\begin{aligned} Q_1^{qq'} &= (\overline{s}_L \gamma_\mu T^a q_L) \otimes (\overline{q}'_L \gamma^\mu T^a d_L) \,, \\ Q_2^{qq'} &= (\overline{s}_L \gamma_\mu q_L) \otimes (\overline{q}'_L \gamma^\mu d_L) \,, \end{aligned} \tag{2.37}$$

where $q_L = \frac{1}{2}(1-\gamma_5)q$ is the left-handed chiral quark field, and $q$ and $q'$ are either $u$ or $c$. We will later also need these operators in a certain linear combination, which can be written as

$$Q_\pm^{qq'} = \frac{1}{2}\left((\overline{s}_L^\alpha \gamma_\mu q_L^\alpha) \otimes (\overline{q}_L^{\prime\beta} \gamma^\mu d_L^\beta) \pm (\overline{s}_L^\alpha \gamma_\mu q_L^\beta) \otimes (\overline{q}_L^{\prime\beta} \gamma^\mu d_L^\alpha)\right) = \frac{1}{2}\left(1 \pm \frac{1}{N_c}\right) Q_2^{qq'} \pm Q_1^{qq'} \,, \tag{2.38}$$

where $\alpha$ and $\beta$ are colour indices, and $N_c$ is the number of colours. The advantage here is that the anomalous dimensions in the subspace of current-current operators are diagonal in this basis (see also Appendix C).

The QCD penguin operators are defined as

$$\begin{aligned} Q_3 &= (\overline{s}_L \gamma_\mu d_L) \otimes \sum\nolimits_q (\overline{q} \gamma^\mu q) \,, \\ Q_4 &= (\overline{s}_L \gamma_\mu T^a d_L) \otimes \sum\nolimits_q (\overline{q} \gamma^\mu T^a q) \,, \\ Q_5 &= (\overline{s}_L \gamma_{\mu_1\mu_2\mu_3} d_L) \otimes \sum\nolimits_q (\overline{q} \gamma^{\mu_1\mu_2\mu_3} q) \,, \\ Q_6 &= (\overline{s}_L \gamma_{\mu_1\mu_2\mu_3} T^a d_L) \otimes \sum\nolimits_q (\overline{q} \gamma^{\mu_1\mu_2\mu_3} T^a q) \,, \end{aligned} \tag{2.39}$$

where the sum extends over the light quark fields, and we have introduced the abbreviation $\gamma_{\mu_1\mu_2\mu_3} = \gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}$, etc. The semi-leptonic dimension-six operators relevant for the decay $K^+ \to \pi^+ \nu\bar{\nu}$ are given by

$$\begin{aligned} Q_{3q} &= \sum_{\ell=e,\mu,\tau} (\bar{s}_L \gamma_\mu q_L) \otimes (\bar{\nu}_{\ell L} \gamma^\mu \ell_L) \,, \\ Q_{4q} &= \sum_{\ell=e,\mu,\tau} (\bar{q}_L \gamma_\mu d_L) \otimes (\bar{\ell}_L \gamma^\mu \nu_{\ell L}) \,, \end{aligned} \tag{2.40}$$

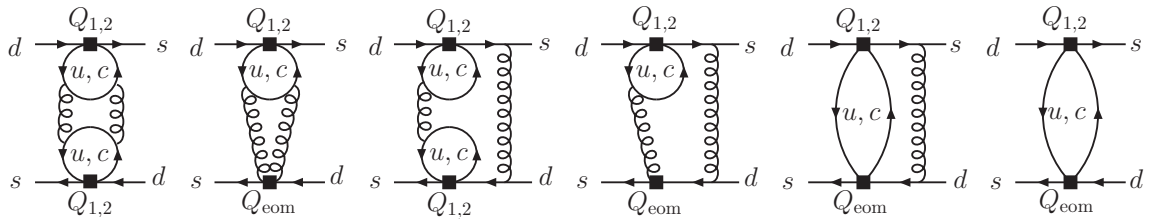


Figure 2.1: Sample three-loop diagrams with 1PI subdivergences that have to be subtracted by insertions of the EOM-vanishing operator. The corresponding 1PI one- and two-loop insertions of $Q_{\rm eom}$ are also shown.

inducing the quark-lepton-neutrino interactions, and the operators

$$\begin{aligned} Q_A &= \sum\nolimits_q \sum_{\ell=e,\mu,\tau} (-I_q^3)(\bar{q}\gamma_5\gamma_\mu q) \otimes (\bar{\nu}_{\ell L}\gamma^\mu \nu_{\ell L})\,, \\ Q_V &= \sum\nolimits_q \sum_{\ell=e,\mu,\tau} (I_q^3 - 2Q_q \sin^2\theta_W)(\bar{q}\gamma_\mu q) \otimes (\bar{\nu}_{\ell L}\gamma^\mu \nu_{\ell L})\,, \end{aligned} \tag{2.41}$$

describing the quark-neutrino interactions.
In order to subtract the divergences of all possible 1PI subdiagrams of the relevant Green's functions we need the following gauge-invariant EOM-vanishing operator

$$Q_{\rm eom} = \frac{1}{g}\bar{s}_L\gamma^\mu T^a d_L D^\nu G^a_{\mu\nu} + Q_4\,, \tag{2.42}$$

where $D_\mu$ denotes the covariant derivative, acting on the gluon field. Sample diagrams are shown in Figure 2.1.
The dimension-eight operators inducing the effective interactions below the charm quark scale can be chosen as

$$\tilde{Q}_7 = \frac{m_c^2}{g^2\mu^{2\epsilon}}(\bar{s}_L^\alpha\gamma_\mu d_L^\alpha) \otimes (\bar{s}_L^\beta\gamma^\mu d_L^\beta)\,, \tag{2.43}$$

for the $|\Delta S = 2|$ transition, where $\alpha$ and $\beta$ again denote colour indices, and

$$\tilde{Q}_\nu = \frac{m_c^2}{g^2\mu^{2\epsilon}} \sum_{\ell=e,\mu,\tau} (\bar{s}_L\gamma_\mu d_L) \otimes (\bar{\nu}_{\ell L}\gamma^\mu \nu_{\ell L})\,, \tag{2.44}$$

describing the $s \to d\nu\bar{\nu}$ transition. Note that we define the dimension-eight operators with two inverse powers of the strong coupling constant for reasons that will become clear in Section 2.4.

### Evanescent Operators

The use of dimensional regularisation in a theory involving fermions implies that also the Dirac algebra is infinite-dimensional. In order to remove all divergences of Green's functions calculated in $d$ dimensions, we also have to introduce a set of operators that vanish algebraically in four dimensions. These are called evanescent operators. At the one-loop level we

need

$$\begin{aligned}
E_1^{qq'(1)} &= (\overline{s}_L\gamma_{\mu_1\mu_2\mu_3}T^a q_L)\otimes(\overline{q}'_L\gamma^{\mu_1\mu_2\mu_3}T^a d_L) - 16Q_1^{qq'}\,,\\
E_2^{qq'(1)} &= (\overline{s}_L\gamma_{\mu_1\mu_2\mu_3}q_L)\otimes(\overline{q}'_L\gamma^{\mu_1\mu_2\mu_3}d_L) - 16Q_2^{qq'}\,,\\
E_3^{(1)} &= (\overline{s}_L\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5}d_L)\otimes\sum\nolimits_q(\overline{q}\gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5}q) + 64Q_3 - 20Q_5\,,\\
E_4^{(1)} &= (\overline{s}_L\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5}T^a d_L)\otimes\sum\nolimits_q(\overline{q}\gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5}T^a q) + 64Q_4 - 20Q_6\,,
\end{aligned} \tag{2.45}$$

for the four-quark operators, defined as in [46]. We define the evanescent operators needed for the renormalisation of the semileptonic operators as in [25]:

$$\begin{aligned}
E_{3q}^{(1)} &= \sum_{\ell=e,\mu,\tau}(\bar{s}_L\gamma_{\mu_1\mu_2\mu_3}q_L)(\bar{\nu}_{\ell L}\gamma^{\mu_1\mu_2\mu_3}\ell_L) - (16-4\epsilon)Q_{3q}\,,\\
E_{4q}^{(1)} &= \sum_{\ell=e,\mu,\tau}(\bar{q}_L\gamma_{\mu_1\mu_2\mu_3}d_L)(\bar{\ell}_L\gamma^{\mu_1\mu_2\mu_3}\nu_{\ell L}) - (16-4\epsilon)Q_{4q}\,.
\end{aligned} \tag{2.46}$$

At the two-loop level, the following four operators appear [46]:

$$\begin{aligned}
E_1^{qq'(2)} &= (\overline{s}_L\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5}T^a q_L)\otimes(\overline{q}'_L\gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5}T^a d_L) - 256Q_1^{qq'} - 20E_1^{qq'(1)}\,,\\
E_2^{qq'(2)} &= (\overline{s}_L\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5}q_L)\otimes(\overline{q}'_L\gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5}d_L) - 256Q_2^{qq'} - 20E_2^{qq'(1)}\,,\\
E_3^{(2)} &= (\overline{s}_L\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6\mu_7}d_L)\otimes\sum\nolimits_q(\overline{q}\gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6\mu_7}q) + 1280Q_3 - 336Q_5\,,\\
E_4^{(2)} &= (\overline{s}_L\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6\mu_7}T^a d_L)\otimes\sum\nolimits_q(\overline{q}\gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6\mu_7}T^a q) + 1280Q_4 - 336Q_6\,.
\end{aligned} \tag{2.47}$$

In the calculation of the mixing of dimension-six into dimension-eight operators, in addition to the Dirac structure $(\bar{s}_L^\alpha\gamma_\mu d_L^\alpha)\otimes(\bar{s}_L^\beta\gamma^\mu d_L^\beta)$ we will encounter the structure $(\bar{s}_L^\alpha\gamma_\mu d_L^\beta)\otimes(\bar{s}_L^\beta\gamma^\mu d_L^\alpha)$. It is related to the former structure by a Fierz transformation in four space-time dimensions. The difference of these structures is therefore evanescent, and we introduce another evanescent operator of the following form:

$$\tilde{E}_F = \frac{m_c^2}{g^2\mu^{2\epsilon}}(\bar{s}_L^\alpha\gamma_\mu d_L^\beta)\otimes(\bar{s}_L^\beta\gamma^\mu d_L^\alpha) - \tilde{Q}_7\,, \tag{2.48}$$

given simply by the difference of the two Dirac structures. We choose the remaining evanescent dimension-eight operators to be

$$\begin{aligned}
E_\nu^{(1)} &= \frac{m_c^2}{g^2\mu^{2\epsilon}}\sum_{\ell=e,\mu,\tau}(\bar{s}_L\gamma_{\mu_1\mu_2\mu_3}d_L)\otimes(\bar{\nu}_{\ell L}\gamma^{\mu_1\mu_2\mu_3}\nu_{\ell L}) - (16-4\epsilon)Q_\nu\,,\\
E_\nu^{(2)} &= \frac{m_c^2}{g^2\mu^{2\epsilon}}\sum_{\ell=e,\mu,\tau}(\bar{s}_L\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5}d_L)\otimes(\bar{\nu}_{\ell L}\gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5}\nu_{\ell L}) - 256Q_\nu\,,
\end{aligned} \tag{2.49}$$

in analogy to the choice in [25], and

$$\begin{aligned}
\tilde{E}_7^{(1)} &= \frac{m_c^2}{g^2\mu^{2\epsilon}}(\bar{s}_L^\alpha\gamma_{\mu_1\mu_2\mu_3}d_L^\alpha)\otimes(\bar{s}_L^\beta\gamma^{\mu_1\mu_2\mu_3}d_L^\beta) - 16\tilde{Q}_7\,,\\
\tilde{E}_8^{(1)} &= \frac{m_c^2}{g^2\mu^{2\epsilon}}(\bar{s}_L^\alpha\gamma_{\mu_1\mu_2\mu_3}d_L^\beta)\otimes(\bar{s}_L^\beta\gamma^{\mu_1\mu_2\mu_3}d_L^\alpha) - 16(\tilde{Q}_7+\tilde{E}_F)\,,
\end{aligned} \tag{2.50}$$

$$\begin{aligned}
\tilde{E}_7^{(2)} &= \frac{m_c^2}{g^2\mu^{2\epsilon}}(\bar{s}_L^\alpha \gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} d_L^\alpha) \otimes (\bar{s}_L^\beta \gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5} d_L^\beta) - 256\tilde{Q}_7\,, \\
\tilde{E}_8^{(2)} &= \frac{m_c^2}{g^2\mu^{2\epsilon}}(\bar{s}_L^\alpha \gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} d_L^\beta) \otimes (\bar{s}_L^\beta \gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5} d_L^\alpha) - 256(\tilde{Q}_7 + \tilde{E}_F)\,,
\end{aligned} \tag{2.51}$$

$$\begin{aligned}
\tilde{E}_7^{(3)} &= \frac{m_c^2}{g^2\mu^{2\epsilon}}(\bar{s}_L^\alpha \gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6\mu_7} d_L^\alpha) \otimes (\bar{s}_L^\beta \gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6\mu_7} d_L^\beta) - 4096\tilde{Q}_7\,, \\
\tilde{E}_8^{(3)} &= \frac{m_c^2}{g^2\mu^{2\epsilon}}(\bar{s}_L^\alpha \gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6\mu_7} d_L^\beta) \otimes (\bar{s}_L^\beta \gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6\mu_7} d_L^\alpha) - 4096(\tilde{Q}_7 + \tilde{E}_F)\,.
\end{aligned} \tag{2.52}$$

## 2.4 Renormalisation of the Wilson Coefficients

In an effective theory we need extra renormalisation constants, in addition to the QED×QCD counterterms, in order to render all Green's functions finite. In this section we will show explicitly how these can be extracted from operator matrix elements order by order in perturbation theory.
We write the Lagrangian up to the second order in $G_F$ formally as

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_{\text{QED}\times\text{QCD}} + \mathcal{L}^{\text{dim.6}} + \mathcal{L}^{\text{dim.8}} \\
&= \mathcal{L}_{\text{QED}\times\text{QCD}} + C_i Z_{ij} Q_j + (C_k C_{k'} \hat{Z}_{kk',l} + \tilde{C}_k \tilde{Z}_{kl})\tilde{Q}_l\,,
\end{aligned} \tag{2.53}$$

where the notation is the same as in Equation (2.36). Here the renormalisation factors $Z_{ij}$, $\tilde{Z}_{ij}$, and $\hat{Z}_{ij,k}$ of the Wilson coefficients are chosen in such a way that any renormalised effective amplitude, of the form

$$\mathcal{A}_{\text{eff}} = C_i(\mu) Z_{ij} \langle Z Q_j \rangle_R + (C_k C_{k'} \hat{Z}_{kk',l} + \tilde{C}_k \tilde{Z}_{kl}) \langle Z \tilde{Q}_l \rangle_R\,, \tag{2.54}$$

is finite. Here angle-brackets denote matrix elements between initial and final states $i$ and $f$, respectively, i.e. $\langle Q_j \rangle = \langle f|Q_j|i\rangle$. $Z$ denotes the wave function renormalisation of the fields in the operator, so that $\langle ZQ_i\rangle_R$ are the renormalised matrix elements of the bare operator $Q_i^{\text{bare}}$, where masses and gauge couplings are renormalised in the usual way.
As usual, we write the expansion in the coupling constants in the form

$$Z = \sum_{k=0}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^k Z^{(k)} + \frac{\alpha}{4\pi} Z^{(e)} + \frac{\alpha\alpha_s}{16\pi^2} Z^{(es)}\,. \tag{2.55}$$

We can also make the $\epsilon$-dependence explicit by expanding the coefficients in analogy to Equation (2.28).
According to the rules of the $\overline{\text{MS}}$ scheme, the $Z_{ij}$ are given by the pure pole part, unless the index $i$ corresponds to an evanescent operator, while the index $j$ does not. In this case a finite subtraction is performed; this will be explained in Section 2.7. An analoguous prescription is applied to $\tilde{Z}_{ij}$ and $\hat{Z}_{ij,k}$.
By convention, we define the dimension-eight operators $\tilde{Q}$ with an explicit factor of $m^2$ and two inverse powers of $g$, i.e. $\tilde{Q} = m^2/(g^2\mu^{2\epsilon}) \cdot \tilde{Q}'$. In this way, $\hat{Z} = \mathcal{O}(g^2)$ and there is no mixing at leading order. The explicit factor of $\mu$ arises from the definition of the renormalised coupling constant in Equation (2.25).

By calculating matrix elements of the effective operators with suitably chosen external quark and gluon states, the renormalisation constants can be determined order by order in perturbation theory from the requirement that the amplitude (2.54) be finite. Expanding the matrix elements and the renormalisation factors in the coupling constants, we find for the case of a single insertion of a dimension-six operator:

$$\begin{aligned}
Z_{ij}^{(1)}\langle Q_j\rangle_R^{(0)} &= -\langle ZQ_i\rangle_R^{(1)}\,,\\
Z_{ij}^{(2)}\langle Q_j\rangle_R^{(0)} &= -\langle ZQ_i\rangle_R^{(2)} - Z_{ij}^{(1)}\langle ZQ_j\rangle_R^{(1)}\,,\\
Z_{ij}^{(3)}\langle Q_j\rangle_R^{(0)} &= -\langle ZQ_i\rangle_R^{(3)} - Z_{ij}^{(2)}\langle ZQ_j\rangle_R^{(1)} - Z_{ij}^{(1)}\langle ZQ_j\rangle_R^{(2)}\,,\\
Z_{ij}^{(e)}\langle Q_j\rangle_R^{(0)} &= -\langle ZQ_i\rangle_R^{(e)}\,,\\
Z_{ij}^{(es)}\langle Q_j\rangle_R^{(0)} &= -\langle ZQ_i\rangle_R^{(es)} - Z_{ij}^{(e)}\langle ZQ_j\rangle_R^{(1)} - Z_{ij}^{(1)}\langle ZQ_j\rangle_R^{(e)}\,.
\end{aligned} \tag{2.56}$$

We can express these equations in terms of the bare quantities by writing $\langle ZQ_i\rangle_R = Z_i\langle Q_i\rangle_B$ (no sum over $i$). In this way we find

$$\begin{aligned}
Z_{ij}^{(1)}\langle Q_j\rangle_B^{(0)} &= -\langle Q_i\rangle_B^{(1)} - Z_i^{(1)}\langle Q_i\rangle_B^{(0)}\,,\\
Z_{ij}^{(2)}\langle Q_j\rangle_B^{(0)} &= -\langle Q_i\rangle_B^{(2)} - Z_i^{(1)}\langle Q_i\rangle_B^{(1)} - Z_{ij}^{(1)}\langle Q_j\rangle_B^{(1)}\\
&\quad - Z_i^{(2)}\langle Q_i\rangle_B^{(0)} - Z_{ij}^{(1)}Z_j^{(1)}\langle Q_j\rangle_B^{(0)}\,,\\
Z_{ij}^{(e)}\langle Q_j\rangle_B^{(0)} &= -\langle Q_i\rangle_B^{(e)} - Z_i^{(e)}\langle Q_i\rangle_B^{(0)}\,,\\
Z_{ij}^{(es)}\langle Q_j\rangle_B^{(0)} &= -\langle Q_i\rangle_B^{(es)} - Z_i^{(e)}\langle Q_i\rangle_B^{(1)} - Z_i^{(1)}\langle Q_i\rangle_B^{(e)} - Z_i^{(es)}\langle Q_i\rangle_B^{(0)}\\
&\quad - Z_{ij}^{(e)}Z_j^{(1)}\langle Q_j\rangle_B^{(0)} - Z_{ij}^{(e)}\langle Q_j\rangle_B^{(1)}\\
&\quad - Z_{ij}^{(1)}Z_j^{(e)}\langle Q_j\rangle_B^{(0)} - Z_{ij}^{(1)}\langle Q_j\rangle_B^{(e)}\,.
\end{aligned} \tag{2.57}$$

An implication of renormalisability is the locality of the counterterms. In a mass independent renormalisation scheme this means that the $Z$ factors depend on $\mu$ only through the coupling constants. This observation can be used as a check on the renormalisation of two- and three-loop matrix elements in the following way: In Equations (2.56) and (2.57) we have contributions from different two- and three-loop diagrams, as well as from one- and two-loop counterterm diagrams. If we denote the $1/\epsilon^l$-pole part of the sum of all $n$-loop contributions by $M^{(n,l)}$, we find the following pole structure:

$$Z_{ij}^{(k)}\langle Q_j\rangle_B^{(0)} = \sum_{n=0}^{k}\sum_{l=1}^{k}\left(\mu^{2\epsilon}\right)^n\frac{1}{\epsilon^l}M^{(n,l)}\,. \tag{2.58}$$

The $\mu$-independence of $Z$ then leads to the following conditions on the various contributions, providing a powerful check of our calculation:

$$\begin{aligned}
&3M^{(3,2)} + 2M^{(2,2)} + M^{(1,2)} = 0\,,\\
&3M^{(3,3)} + 2M^{(2,3)} + M^{(1,3)} = 0\,,\\
&9M^{(3,3)} + 4M^{(2,3)} + M^{(1,3)} = 0\,.
\end{aligned} \tag{2.59}$$

Clearly, the renormalisation factors $\tilde{Z}$ related to the mixing of dimension-eight into dimension-eight operators can be obtained in exactly the same way. The renormalisation constants $\hat{Z}_{ij,k}$, describing the mixing of dimension-six into dimension-eight operators, can be determined in an completely analoguous way by computing suitably chosen matrix elements of insertions of two local dimension-six operators. As above, we find after an expansion in the coupling constants:

$$\begin{aligned}
\hat{Z}^{(1)}_{kk',l}\langle\tilde{Q}_l\rangle^{(0)}_R &= -\langle ZQ_kZQ_{k'}\rangle^{(1)}_R,\\
\hat{Z}^{(2)}_{kk',l}\langle\tilde{Q}_l\rangle^{(0)}_R &= -\langle ZQ_kZQ_{k'}\rangle^{(2)}_R - Z^{(1)}_{km}\langle ZQ_mZQ_{k'}\rangle^{(1)}_R\\
&\quad - Z^{(1)}_{k'n}\langle ZQ_kZQ_n\rangle^{(1)}_R - \hat{Z}^{(1)}_{kk',l'}\langle Z\tilde{Q}_{l'}\rangle^{(1)}_R,\\
\hat{Z}^{(3)}_{kk',l}\langle\tilde{Q}_l\rangle^{(0)}_R &= -\langle ZQ_kZQ_{k'}\rangle^{(3)}_R - Z^{(2)}_{km}\langle ZQ_mZQ_{k'}\rangle^{(1)}_R\\
&\quad - Z^{(1)}_{km}\langle ZQ_mZQ_{k'}\rangle^{(2)}_R - Z^{(2)}_{k'n}\langle ZQ_kZQ_n\rangle^{(1)}_R\\
&\quad - Z^{(1)}_{k'n}\langle ZQ_kZQ_n\rangle^{(2)}_R - Z^{(1)}_{km}Z^{(1)}_{k'n}\langle ZQ_mZQ_n\rangle^{(1)}_R\\
&\quad - \hat{Z}^{(2)}_{kk',l'}\langle Z\tilde{Q}_{l'}\rangle^{(1)}_R - \hat{Z}^{(1)}_{kk',l'}\langle Z\tilde{Q}_{l'}\rangle^{(2)}_R,\\
\hat{Z}^{(es)}_{kk',l}\langle\tilde{Q}_l\rangle^{(0)}_R &= -\langle ZQ_kZQ_{k'}\rangle^{(es)}_R - Z^{(e)}_{km}\langle ZQ_mZQ_{k'}\rangle^{(1)}_R\\
&\quad - Z^{(e)}_{k'n}\langle ZQ_kZQ_n\rangle^{(1)}_R - \hat{Z}^{(1)}_{kk',l'}\langle Z\tilde{Q}_{l'}\rangle^{(e)}_R.
\end{aligned}\tag{2.60}$$

Again we express these equations in terms of bare matrix elements, and find

$$\begin{aligned}
\hat{Z}^{(1)}_{kk',l}\langle\tilde{Q}_l\rangle^{(0)}_B &= -\langle Q_kQ_{k'}\rangle^{(1)}_B,\\
\hat{Z}^{(2)}_{kk',l}\langle\tilde{Q}_l\rangle^{(0)}_B &= -\langle Q_kQ_{k'}\rangle^{(2)}_B - Z^{(1)}_{k}\langle Q_kQ_{k'}\rangle^{(1)}_B - Z^{(1)}_{k'}\langle Q_kQ_{k'}\rangle^{(1)}_B\\
&\quad - Z^{(1)}_{kn}\langle Q_nQ_{k'}\rangle^{(1)}_B - Z^{(1)}_{k'n}\langle Q_kQ_n\rangle^{(1)}_B\\
&\quad - \hat{Z}^{(1)}_{kk',l'}Z^{(1)}_{l'}\langle\tilde{Q}_{l'}\rangle^{(0)}_B - \hat{Z}^{(1)}_{kk',l'}\langle\tilde{Q}_{l'}\rangle^{(1)}_B,\\
\hat{Z}^{(3)}_{kk',l}\langle\tilde{Q}_l\rangle^{(0)}_B &= -\langle Q_kQ_{k'}\rangle^{(3)}_B - Z^{(1)}_{k}\langle Q_kQ_{k'}\rangle^{(2)}_B - Z^{(1)}_{k'}\langle Q_kQ_{k'}\rangle^{(2)}_B\\
&\quad - Z^{(2)}_{k}\langle Q_kQ_{k'}\rangle^{(1)}_B - Z^{(2)}_{k'}\langle Q_kQ_{k'}\rangle^{(1)}_B\\
&\quad - Z^{(1)}_{kn}\langle Q_nQ_{k'}\rangle^{(2)}_B - Z^{(1)}_{k'n}\langle Q_kQ_n\rangle^{(2)}_B\\
&\quad - Z^{(1)}_{kn}Z^{(1)}_{n}\langle Q_nQ_{k'}\rangle^{(1)}_B - Z^{(1)}_{k'n}Z^{(1)}_{n}\langle Q_kQ_n\rangle^{(1)}_B - Z^{(1)}_{km}Z^{(1)}_{k'n}\langle Q_mQ_n\rangle^{(1)}_B\\
&\quad - Z^{(1)}_{kn}Z^{(1)}_{k'}\langle Q_nQ_{k'}\rangle^{(1)}_B - Z^{(1)}_{k'n}Z^{(1)}_{k}\langle Q_kQ_n\rangle^{(1)}_B - Z^{(1)}_{k}Z^{(1)}_{k'}\langle Q_kQ_{k'}\rangle^{(1)}_B\\
&\quad - Z^{(2)}_{kn}\langle Q_nQ_{k'}\rangle^{(1)}_B - Z^{(2)}_{k'n}\langle Q_kQ_n\rangle^{(1)}_B\\
&\quad - \hat{Z}^{(2)}_{kk',l'}Z^{(1)}_{l'}\langle\tilde{Q}_{l'}\rangle^{(0)}_B - \hat{Z}^{(2)}_{kk',l'}\langle\tilde{Q}_{l'}\rangle^{(1)}_B\\
&\quad - \hat{Z}^{(1)}_{kk',l'}Z^{(2)}_{l'}\langle\tilde{Q}_{l'}\rangle^{(0)}_B - \hat{Z}^{(1)}_{kk',l'}Z^{(1)}_{l'}\langle\tilde{Q}_{l'}\rangle^{(1)}_B - \hat{Z}^{(1)}_{kk',l'}\langle\tilde{Q}_{l'}\rangle^{(2)}_B,
\end{aligned}$$

$$\begin{aligned}\hat{Z}^{(es)}_{kk',l}\langle\tilde{Q}_l\rangle^{(0)}_B = &-\langle Q_k Q_{k'}\rangle^{(es)}_B - Z^{(e)}_k\langle Q_k Q_{k'}\rangle^{(1)}_B - Z^{(e)}_{k'}\langle Q_k Q_{k'}\rangle^{(1)}_B \\ &- Z^{(e)}_{kn}\langle Q_n Q_{k'}\rangle^{(1)}_B - Z^{(e)}_{k'n}\langle Q_k Q_n\rangle^{(1)}_B \\ &- \hat{Z}^{(1)}_{kk',l'}Z^{(e)}_{l'}\langle\tilde{Q}_{l'}\rangle^{(0)}_B - \hat{Z}^{(1)}_{kk',l'}\langle\tilde{Q}_{l'}\rangle^{(e)}_B . \end{aligned} \tag{2.61}$$

Also in this case, we can use the $\mu$-independence of the renormalisation constants as a check on our calculation, exactly as above.

## 2.5 Renormalisation Group Equations

In this section we will derive the renormalisation group equations for the Wilson coefficients and provide an explicit perturbative solution up to NNLO. The renormalisation group equations provide a method of consistently summing certain powers of large logarithms, which appear in the calculation of weak decay amplitudes, to all orders in the strong coupling constant. We start with the equation for the Wilson coefficients of the dimension-six operators, and then proceed to the case of the Wilson coefficients of the dimension-eight operators, where we also have to take into account the effects of the mixing of dimension-six into dimension-eight operators.

### Single Operator Insertion

The renormalisation-scale independence of the bare Wilson coefficients $C^0 = C(\mu)Z(\mu)$ leads to the following renormalisation group equations:

$$\mu\frac{d}{d\mu}C(\mu) = \gamma^T C(\mu)\,. \tag{2.62}$$

Here the Wilson coefficients $C$ can be thought of as a column vector, and we have defined the *anomalous dimensions* of the Wilson coefficients as

$$\gamma = Z\mu\frac{d}{d\mu}Z^{-1}\,. \tag{2.63}$$

In a mass independent renormalisation scheme like the $\overline{\text{MS}}$ scheme the only dependence of $Z$ on $\mu$ arises through the coupling constants. We can therefore rewrite the derivatives with respect to $\mu$ in terms of the $\beta$ functions:

$$\gamma = Z\beta(g,e,\epsilon)\frac{d}{dg}Z^{-1} + Z\beta_e(g,e,\epsilon)\frac{d}{de}Z^{-1}\,. \tag{2.64}$$

In the remaining part of this work, we will always neglect the running of the QED coupling constant $\alpha$, and consequently set the $\epsilon$-independent part of $\beta_e$ in Equation (2.31) to zero. This is a very good approximation in the range $1\text{GeV} \leq \mu \leq M_W$.
We get an explicit expression for the anomalous dimensions order by order in perturbation theory by expanding $\gamma$ in the coupling constants,

$$\gamma = \sum_{k=1}^{\infty}\left(\frac{\alpha_s}{4\pi}\right)^k\gamma^{(k-1)} + \frac{\alpha}{4\pi}\gamma^{(e)} + \frac{\alpha\alpha_s}{16\pi^2}\gamma^{(es)}\,. \tag{2.65}$$

The coefficients can then be extracted by inserting the expansions of the $Z$ factors and the $\beta$ functions into the right side of Eq. (2.64). In this way we find up to terms of order $\epsilon$:

$$\gamma^{(0)} = 2Z^{(1,1)}\,, \tag{2.66}$$

$$\gamma^{(1)} = 4Z^{(2,1)} - 2Z^{(1,0)}Z^{(1,1)} - 2Z^{(1,1)}Z^{(1,0)} + 2\beta_0 Z^{(1,0)}\,, \tag{2.67}$$

$$\begin{aligned}\gamma^{(2)} = {} & 6Z^{(3,1)} - 4Z^{(2,1)}Z^{(1,0)} - 2Z^{(1,1)}Z^{(2,0)} - 4Z^{(2,0)}Z^{(1,1)} \\ & - 2Z^{(1,0)}Z^{(2,1)} + 2Z^{(1,1)}Z^{(1,0)}Z^{(1,0)} + 2Z^{(1,0)}Z^{(1,1)}Z^{(1,0)} \\ & + 2Z^{(1,0)}Z^{(1,0)}Z^{(1,1)} + 2\beta_1 Z^{(1,0)} + 4\beta_0 Z^{(2,0)} - 2\beta_0 Z^{(1,0)}Z^{(1,0)}\,,\end{aligned} \tag{2.68}$$

$$\gamma^{(e)} = 2Z^{(e,1)}\,, \tag{2.69}$$

$$\gamma^{(es)} = 4Z^{(es,1)} - 2Z^{(e,0)}Z^{(1,1)} - 2Z^{(e,1)}Z^{(1,0)} - 2Z^{(1,0)}Z^{(e,1)} - 2Z^{(1,1)}Z^{(e,0)}\,. \tag{2.70}$$

The finiteness of Eq. (2.63) requires that the pole parts in this equation cancel. This is equivalent to the conditions

$$Z^{(2,2)} = \frac{1}{2}Z^{(1,1)}Z^{(1,1)} - \frac{1}{2}\beta_0 Z^{(1,1)}\,, \tag{2.71}$$

$$Z^{(3,3)} = \frac{1}{6}Z^{(1,1)}Z^{(1,1)}Z^{(1,1)} - \frac{1}{2}\beta_0 Z^{(1,1)}Z^{(1,1)} + \frac{1}{3}\beta_0^2 Z^{(1,1)}\,, \tag{2.72}$$

$$Z^{(3,2)} = \frac{2}{3}Z^{(2,1)}Z^{(1,1)} + \frac{1}{3}Z^{(1,1)}Z^{(1,2)} - \frac{1}{3}Z^{(1,1)}Z^{(1,0)}Z^{(1,1)} \tag{2.73}$$

$$- \frac{1}{6}Z^{(1,0)}Z^{(1,1)}Z^{(1,1)} - \frac{1}{3}\beta_1 Z^{(1,1)} - \frac{2}{3}\beta_0 Z^{(2,1)} + \frac{1}{6}\beta_0 Z^{(1,0)}Z^{(1,1)}\,, \tag{2.74}$$

$$Z^{(es,2)} = \frac{1}{2}Z^{(e,1)}Z^{(1,1)} + \frac{1}{2}Z^{(1,1)}Z^{(e,1)}\,, \tag{2.75}$$

which provide an useful check of the renormalisation constants.
We can now find the running Wilson coefficients as solutions of the renormalisation group Equations (2.62). We write them in the form

$$C(\mu) = U(\mu, \mu_0, \alpha)C(\mu_0)\,, \tag{2.76}$$

where we have introduced the evolution matrix

$$U(\mu, \mu_0, \alpha) = T_g \exp\left[\int_{g(\mu_0)}^{g(\mu)} dg' \,\frac{\gamma^T(g', \alpha)}{\beta(g')}\right]\,, \tag{2.77}$$

describing the evolution of the Wilson coefficients from the high scale $\mu_0$ to the low scale $\mu$. Here $T_g$ denotes ordering in the coupling constant $g$, such that $g$ increases from right to left. Again, (2.77) can be evaluated perturbatively by an expansion in the coupling constants. We define the expansion coefficients of the evolution matrix as

$$\begin{aligned}U(\mu, \mu_0, \alpha) = {} & U^{(0)}(\mu, \mu_0) + \frac{\alpha_s(\mu)}{4\pi}U^{(1)}(\mu, \mu_0) + \left(\frac{\alpha_s(\mu, \mu_0)}{4\pi}\right)^2 U^{(2)}(\mu, \mu_0) \\ & + \frac{\alpha}{\alpha_s(\mu)}U_e^{(0)}(\mu, \mu_0) + \frac{\alpha}{4\pi}U_e^{(1)}(\mu, \mu_0)\,,\end{aligned} \tag{2.78}$$

and expand the Wilson coefficients

$$C(\mu) = C^{(0)}(\mu) + \frac{\alpha_s(\mu)}{4\pi} C^{(1)}(\mu) + \left(\frac{\alpha_s(\mu)}{4\pi}\right)^2 C^{(2)}(\mu) + \frac{\alpha}{\alpha_s(\mu)} C_e^{(0)}(\mu) + \frac{\alpha}{4\pi} C_e^{(1)}(\mu) \quad (2.79)$$

in the usual way. The leading order evolution matrix will sum terms proportional to $(\alpha_s \log)^n$, where $\log = \log \mu/\mu_0$, to all orders in the strong coupling constant. Accordingly, the next-to-leading order and next-to-next-to-leading order evolution matrix sums terms proportional to $\alpha_s(\alpha_s \log)^n$ and $\alpha_s^2(\alpha_s \log)^n$, respectively. This is called the *leading-logarithmic* (LL), *next-to-leading-logarithmic* (NLL) and *next-to-next-to-leading-logarithmic* (NNLL) approximation. On the other hand, the leading and next-to-leading QED contributions to $U$ sum all terms of the form $(\alpha \log)(\alpha_s \log)^n \simeq \alpha/\alpha_s(\alpha_s \log)^n$ and $\alpha(\alpha_s \log)^n$ (LL and NLL in QED). This pattern explains the inverse coupling constant factors in the expressions above. We will see how this works in practice in Chapters 4 and 5.
Using Equation (2.76) we can now easily find the individual contributions to the Wilson coefficients

$$C^{(0)}(\mu) = U^{(0)}(\mu, \mu_0) C^{(0)}(\mu_0)\,, \quad (2.80)$$

$$C^{(1)}(\mu) = \eta U^{(0)}(\mu, \mu_0) C^{(1)}(\mu_0) + U^{(1)}(\mu, \mu_0) C^{(0)}(\mu_0)\,, \quad (2.81)$$

$$C^{(2)}(\mu) = \eta^2 U^{(0)}(\mu, \mu_0) C^{(2)}(\mu_0) + \eta U^{(1)}(\mu, \mu_0) C^{(1)}(\mu_0) + U^{(2)}(\mu, \mu_0) C^{(0)}(\mu_0)\,, \quad (2.82)$$

$$C_e^{(0)}(\mu) = U_e^{(0)}(\mu, \mu_0) C^{(0)}(\mu_0), \quad (2.83)$$

$$C_e^{(1)}(\mu) = \eta U_e^{(0)}(\mu, \mu_0) C^{(1)}(\mu_0) + U^{(0)}(\mu, \mu_0) C_e^{(1)}(\mu_0) + U_e^{(1)}(\mu, \mu_0) C^{(0)}(\mu_0)\,, \quad (2.84)$$

where we have introduced the ratio $\eta = \alpha_s(\mu_0)/\alpha_s(\mu)$.
Now it only remains to find an explicit perturbative expression for the evolution matrix. We start with the case of pure QCD and switch off QED by setting $\alpha = 0$ for the moment. We will consider QED effects afterwards. Let us first decompose the evolution matrix $U(\mu, \mu_0) \equiv U(\mu, \mu_0, 0)$ as follows,

$$U(\mu, \mu_0) = K(\mu) U^{(0)}(\mu, \mu_0) K^{-1}(\mu_0)\,, \quad (2.85)$$

where $K(\mu)$ is given as an expansion in $\alpha_s$ by

$$K(\mu) = 1 + \frac{\alpha_s(\mu)}{4\pi} J^{(1)} + \left(\frac{\alpha_s(\mu)}{4\pi}\right)^2 J^{(2)}\,. \quad (2.86)$$

In Equation (2.85) we have introduced the leading order evolution matrix

$$U^{(0)}(\mu, \mu_0) = V \mathrm{diag}(\eta^{a_i}) V^{-1}\,. \quad (2.87)$$

The matrix $V$ and the so-called *magic numbers* $a_i$ are obtained by diagonalising the leading order anomalous dimension matrix $\gamma^{(0)T}$, in the following way:

$$\left(V^{-1} \gamma^{(0)T} V\right)_{ij} = 2\beta_0 a_i \delta_{ij}\,. \quad (2.88)$$

As a next step, we define the matrices $S$ and $G$ as follows:

$$S^{(i)} = V^{-1} J^{(i)} V; \quad G^{(i)} = V^{-1} \gamma^{(i)T} V\,. \quad (2.89)$$

Here the matrix kernels $S$ are given by

$$S_{ij}^{(1)} = \frac{\beta_1}{\beta_0} a_i \delta_{ij} - \frac{G_{ij}^{(1)}}{2\beta_0(1 + a_i - a_j)}\,, \tag{2.90}$$

$$S_{ij}^{(2)} = \frac{\beta_2}{2\beta_0} a_i \delta_{ij} + \sum_k \frac{1 + a_i - a_k}{2 + a_i - a_j} \left( S_{ik}^{(1)} S_{kj}^{(1)} - \frac{\beta_1}{\beta_0} \right) - \frac{G_{ij}^{(2)}}{2\beta_0(2 + a_i - a_j)}\,. \tag{2.91}$$

The derivation of these expressions is standard and can be found for instance in [47–50]. The evolution matrix up to the NNLO in QCD can now be computed solely in terms of the QCD $\beta$ function and the anomalous dimensions of the Wilson coefficients.
In order to include QED effects, we could proceed exactly as above. Let us nevertheless follow the approach proposed in [50], for the following reason: It turns out that the expressions analoguous to Equations (2.90) and (2.91) contain certain combinations of the magic numbers $a_i$ and $a_j$ that will lead to singularities in these expressions. Although these singularities are spurious and will cancel in the final expression for the evolution matrix, the alternative approach is more suitable for numerical calculations. Let us therefore decompose the evolution matrix, now with non-vanishing $\alpha$, as follows:

$$U(\mu, \mu_0, \alpha) = U(\mu, \mu_0) + \frac{\alpha}{4\pi} R(\mu, \mu_0)\,. \tag{2.92}$$

Next we separate the $\alpha$-dependent part of the anomalous dimension matrix and write

$$\gamma(\alpha_s, \alpha) = \gamma_s(\alpha_s) + \frac{\alpha}{4\pi} \Gamma(\alpha_s)\,, \tag{2.93}$$

where $\gamma_s(\alpha_s) = \gamma(\alpha_s, \alpha = 0)$ is the pure QCD part of the anomalous dimension matrix (2.65), and $\Gamma(\alpha_s)$ is given by

$$\Gamma(\alpha_s) = \gamma^{(e)} + \frac{\alpha_s}{4\pi} \gamma^{(es)}\,. \tag{2.94}$$

By inserting (2.94) into (2.77) and keeping only the term proportional to $\alpha$, we find the $\alpha$-dependent part of the evolution matrix

$$R(\mu, \mu_0) = \int_{g(\mu_0)}^{g(\mu)} dg' \frac{U(\mu, \mu') \Gamma^T(g') U(\mu', \mu_0)}{\beta(g')}\,. \tag{2.95}$$

This expression can now again be expanded in $\alpha_s$. To leading-log approximation we find

$$R^{(0)}(\mu, \mu_0) = -\frac{2\pi}{\beta_0} V K^{(0)}(\mu, \mu_0) V^{-1}\,, \tag{2.96}$$

where we have introduced

$$\Big( K^{(0)}(\mu, \mu_0) \Big)_{ij} = G_{e,ij} \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} d\alpha_s' \left( \frac{\alpha_s'}{\alpha_s(\mu)} \right)^{a_i} \frac{1}{{\alpha_s'}^2} \left( \frac{\alpha_s(\mu_0)}{\alpha_s'} \right)^{a_j}\,, \tag{2.97}$$

and $V$ and the magic numbers $a_i$ have already been defined in Equation (2.88). $G_e$ is given, in analogy to (2.89), by

$$G_e = V^{-1}\gamma_e^T V\,. \tag{2.98}$$

The expression (2.97) can now easily be integrated. It yields

$$\left(K^{(0)}(\mu,\mu_0)\right)_{ij} = \frac{G_{e,ij}}{a_i - a_j - 1}\left[\left(\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)}\right)^{a_j}\frac{1}{\alpha_s(\mu)} - \left(\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)}\right)^{a_i}\frac{1}{\alpha_s(\mu_0)}\right], \tag{2.99}$$

for the case $a_i \neq a_j + 1$, and

$$\left(K^{(0)}(\mu,\mu_0)\right)_{ij} = G_{e,ij}\frac{1}{\alpha_s(\mu)}\left(\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)}\right)^{a_j}\log\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\,, \tag{2.100}$$

otherwise. The advantage of this form is that it manifestly contains no spurious singularities. The procedure can be extended to the NLL level. To this end we keep the $\alpha_s$-dependent part in (2.95) and define

$$R(\mu,\mu_0) = -\frac{2\pi}{\beta_0}VK(\mu,\mu_0)V^{-1} = R^{(0)}(\mu,\mu_0) + R^{(1)}(\mu,\mu_0)\,. \tag{2.101}$$

Here we write

$$K(\mu,\mu_0) = K^{(0)}(\mu,\mu_0) + \frac{1}{4\pi}\sum_{i=1}^{3}K_i^{(1)}(\mu,\mu_0)\,, \tag{2.102}$$

the matrices $K_i^{(1)}$ parameterising the $\alpha_s$ effects. In addition we define

$$\Gamma^{(1)} = \gamma^{T(es)} - \frac{\beta_1}{\beta_0}\gamma^{T(e)}\,, \tag{2.103}$$

as well as

$$M^{(1)} = V^{-1}\left(\Gamma^{(1)} + \left[\gamma^{T(e)}, J\right]\right)V\,. \tag{2.104}$$

Then, an elementary integration yields the following explicit terms for the $K_i^{(1)}$:

$$\left(K_1^{(1)}(\mu,\mu_0)\right)_{ij} = \begin{cases} \frac{M_{ij}^{(1)}}{a_i - a_j}\left[\left(\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)}\right)^{a_j} - \left(\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)}\right)^{a_i}\right] & ,\text{for } i \neq j\,, \\ M_{ii}^{(1)}\left(\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)}\right)^{a_i}\log\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} & ,\text{for } i = j\,, \end{cases} \tag{2.105}$$

$$K_2^{(1)}(\mu,\mu_0) = -\alpha_s(\mu_0)K^{(0)}(\mu,\mu_0)S^{(1)}\,, \tag{2.106}$$

$$K_3^{(1)}(\mu,\mu_0) = \alpha_s(\mu)S^{(1)}K^{(0)}(\mu,\mu_0)\,. \tag{2.107}$$

They can again be expressed solely in terms of the $\beta$ function and the anomalous dimensions of the Wilson coefficients, now including also QED corrections.

In summary, we find the following explicit perturbative coefficients of the evolution matrix:

$$U^{(1)}(\mu,\mu_0) = J^{(1)}U^{(0)}(\mu,\mu_0) - \eta U^{(0)}(\mu,\mu_0)J^{(1)}\,, \tag{2.108}$$

$$\begin{aligned} U^{(2)}(\mu,\mu_0) &= J^{(2)}U^{(0)}(\mu,\mu_0) - \eta J^{(1)}U^{(0)}(\mu,\mu_0)J^{(1)} \\ &\quad - \eta^2 U^{(0)}(\mu,\mu_0)\left(J^{(2)} - (J^{(1)})^2\right)\,, \end{aligned} \tag{2.109}$$

$$U_e^{(0)}(\mu,\mu_0) = \frac{\alpha_s(\mu)}{4\pi}R^{(0)}(\mu,\mu_0)\,, \tag{2.110}$$

$$U_e^{(1)}(\mu,\mu_0) = R^{(1)}(\mu,\mu_0)\,. \tag{2.111}$$

### Double Operator Insertion

Let us now discuss the Wilson coefficients of the dimension-eight operators. Again from the $\mu$-independence of the bare Wilson coefficients, we derive the following renormalisation group equation:

$$\mu\frac{d}{d\mu}\tilde{C}_l(\mu) = \tilde{C}_{l'}(\mu)\tilde{\gamma}_{l'l} + C_k(\mu)C_n(\mu)\hat{\gamma}_{kn,l}\,. \tag{2.112}$$

Here, in addition to the anomalous dimension matrix $\tilde{\gamma}$, which describes the mixing among dimension-eight operators and can be treated by the methods of the last section, the anomalous dimension tensor $\gamma_{kn,l}$ of the double insertion comes into play [51]:

$$\hat{\gamma}_{kn,l} = -(\gamma_{kk'}\delta_{nn'} + \gamma_{nn'}\delta_{kk'})\hat{Z}_{k'n',l'}\tilde{Z}^{-1}_{l'l} - \left(\mu\frac{d}{d\mu}\hat{Z}_{kn,l'}\right)\tilde{Z}^{-1}_{l'l}\,. \tag{2.113}$$

It is related to the mixing of dimension-six into dimension-eight operators under renormalisation. As before, we can find a perturbative expansion in terms of the $\beta$ functions and the renormalisation constants. We write the anomalous dimension tensor in the form

$$\hat{\gamma}_{kn,l} = \sum_{k=1}^{\infty}\left(\frac{\alpha_s}{4\pi}\right)^k\hat{\gamma}^{(k-1)}_{kn,l} + \frac{\alpha}{4\pi}\hat{\gamma}^{(e)}_{kn,l} + \frac{\alpha\alpha_s}{16\pi^2}\hat{\gamma}^{(es)}_{kn,l}\,, \tag{2.114}$$

and insert the previously found expansions of the $Z$ factors and the $\beta$ functions into the right side of Eq. (2.113). In this way we find

$$\hat{\gamma}^{(0)}_{kn,l} = 2\hat{Z}^{(1,1)}_{kn,l}\,, \tag{2.115}$$

$$\begin{aligned} \hat{\gamma}^{(1)}_{kn,l} &= 4\hat{Z}^{(2,1)}_{kn,l} + 2\beta_0\hat{Z}^{(1,0)}_{kn,l} - 2\hat{Z}^{(1,0)}_{kn,l'}\tilde{Z}^{(1,1)}_{l'l} - 2\hat{Z}^{(1,1)}_{kn,l'}\tilde{Z}^{(1,0)}_{l'l} \\ &\quad - 2\left\{Z^{(1,0)}_{kk'}\delta_{nn'} + \delta_{kk'}Z^{(1,0)}_{nn'}\right\}\hat{Z}^{(1,1)}_{k'n',l} - 2\left\{Z^{(1,1)}_{kk'}\delta_{nn'} + \delta_{kk'}Z^{(1,1)}_{nn'}\right\}\hat{Z}^{(1,0)}_{k'n',l}\,, \end{aligned} \tag{2.116}$$

$$
\begin{aligned}
\hat{\gamma}^{(2)}_{kn,l} = {} & 6\hat{Z}^{(3,1)}_{kn,l} + 4\beta_0 \hat{Z}^{(2,0)}_{kn,l} + 2\beta_1 \hat{Z}^{(1,0)}_{kn,l} - 2\beta_0 \hat{Z}^{(1,0)}_{kn,l'} \tilde{Z}^{(1,0)}_{l'l} \\
& + 2\hat{Z}^{(1,0)}_{kn,l'} \tilde{Z}^{(1,0)}_{l'l''} \tilde{Z}^{(1,1)}_{l''l} + 2\hat{Z}^{(1,0)}_{kn,l'} \tilde{Z}^{(1,1)}_{l'l''} \tilde{Z}^{(1,0)}_{l''l} + 2\hat{Z}^{(1,1)}_{kn,l'} \tilde{Z}^{(1,0)}_{l'l''} \tilde{Z}^{(1,0)}_{l''l} \\
& - 2\hat{Z}^{(1,0)}_{kn,l'} \tilde{Z}^{(2,1)}_{l'l} - 2\hat{Z}^{(1,1)}_{kn,l'} \tilde{Z}^{(2,0)}_{l'l} - 4\hat{Z}^{(2,0)}_{kn,l'} \tilde{Z}^{(1,1)}_{l'l} - 4\hat{Z}^{(2,1)}_{kn,l'} \tilde{Z}^{(1,0)}_{l'l} \\
& + 2\left\{ Z^{(1,0)}_{kk''} Z^{(1,0)}_{k''k'} \delta_{nn'} + \delta_{kk'} Z^{(1,0)}_{nn''} Z^{(1,0)}_{n''n'} \right\} \hat{Z}^{(1,1)}_{k'n',l} \\
& + 2\left\{ Z^{(1,0)}_{kk''} Z^{(1,1)}_{k''k'} \delta_{nn'} + \delta_{kk'} Z^{(1,0)}_{nn''} Z^{(1,1)}_{n''n'} \right\} \hat{Z}^{(1,0)}_{k'n',l} \\
& + 2\left\{ Z^{(1,1)}_{kk''} Z^{(1,0)}_{k''k'} \delta_{nn'} + \delta_{kk'} Z^{(1,1)}_{nn''} Z^{(1,0)}_{n''n'} \right\} \hat{Z}^{(1,0)}_{k'n',l} \\
& + 2\left\{ Z^{(1,0)}_{kk'} \delta_{nn'} + \delta_{kk'} Z^{(1,0)}_{nn'} \right\} \hat{Z}^{(1,0)}_{k'n',l'} \tilde{Z}^{(1,1)}_{l'l} + 2\left\{ Z^{(1,0)}_{kk'} \delta_{nn'} + \delta_{kk'} Z^{(1,0)}_{nn'} \right\} \hat{Z}^{(1,1)}_{k'n',l'} \tilde{Z}^{(1,0)}_{l'l} \\
& + 2\left\{ Z^{(1,1)}_{kk'} \delta_{nn'} + \delta_{kk'} Z^{(1,1)}_{nn'} \right\} \hat{Z}^{(1,0)}_{k'n',l'} \tilde{Z}^{(1,0)}_{l'l} - 2\beta_0 \left\{ Z^{(1,0)}_{kk'} \delta_{nn'} + \delta_{kk'} Z^{(1,0)}_{nn'} \right\} \hat{Z}^{(1,0)}_{k'n',l} \\
& - 2\left\{ Z^{(1,0)}_{kk'} \delta_{nn'} + \delta_{kk'} Z^{(1,0)}_{nn'} \right\} \hat{Z}^{(2,1)}_{k'n',l} - 2\left\{ Z^{(1,1)}_{kk'} \delta_{nn'} + \delta_{kk'} Z^{(1,1)}_{nn'} \right\} \hat{Z}^{(2,0)}_{k'n',l} \\
& - 4\left\{ Z^{(2,0)}_{kk'} \delta_{nn'} + \delta_{kk'} Z^{(2,0)}_{nn'} \right\} \hat{Z}^{(1,1)}_{k'n',l} - 4\left\{ Z^{(2,1)}_{kk'} \delta_{nn'} + \delta_{kk'} Z^{(2,1)}_{nn'} \right\} \hat{Z}^{(1,0)}_{k'n',l} \,, \qquad (2.117)
\end{aligned}
$$

$$
\hat{\gamma}^{(e)}_{kn,l} = 0 \,, \qquad (2.118)
$$

$$
\begin{aligned}
\hat{\gamma}^{(es)}_{kn,l} = {} & 4\hat{Z}^{(es,1)}_{kn,l} - 2\hat{Z}^{(1,0)}_{kn,l'} \tilde{Z}^{(e,1)}_{l'l} - 2\hat{Z}^{(1,1)}_{kn,l'} \tilde{Z}^{(e,0)}_{l'l} \\
& - 2\left\{ Z^{(e,0)}_{kk'} \delta_{nn'} + \delta_{kk'} Z^{(e,0)}_{nn'} \right\} \hat{Z}^{(1,1)}_{k'n',l} - 2\left\{ Z^{(e,1)}_{kk'} \delta_{nn'} + \delta_{kk'} Z^{(e,1)}_{nn'} \right\} \hat{Z}^{(1,0)}_{k'n',l} \,. \qquad (2.119)
\end{aligned}
$$

Again, the finiteness of Equation (2.113) requires the cancellation of the pole parts and so implies the following relations between the various renormalisation constants, which can be used as a non-trivial check of the calculation[2]:

$$
2\hat{Z}^{(2,2)}_{kn,l} = \left\{ Z^{(1,1)}_{kk'} \delta_{nn'} + \delta_{kk'} Z^{(1,1)}_{nn'} \right\} \hat{Z}^{(1,1)}_{k'n',l} + \hat{Z}^{(1,1)}_{kn,l'} \tilde{Z}^{(1,1)}_{l'l} - \beta_0 \hat{Z}^{(1,1)}_{kn,l} \,, \qquad (2.120)
$$

$$
\begin{aligned}
3\hat{Z}^{(3,3)}_{kn,l} = {} & -2\beta_0 \hat{Z}^{(2,2)}_{kn,l} + 2\hat{Z}^{(2,2)}_{kn,l'} \tilde{Z}^{(1,1)}_{l'l} + \hat{Z}^{(1,1)}_{kn,l'} \tilde{Z}^{(2,2)}_{l'l} \\
& + \beta_0 \hat{Z}^{(1,1)}_{kn,l'} \tilde{Z}^{(1,1)}_{l'l} - \hat{Z}^{(1,1)}_{kn,l'} \tilde{Z}^{(1,1)}_{l'l''} \tilde{Z}^{(1,1)}_{l''l} \\
& - \left\{ Z^{(1,1)}_{kk'} \delta_{nn'} + \delta_{kk'} Z^{(1,1)}_{nn'} \right\} \hat{Z}^{(1,1)}_{k'n',l'} \tilde{Z}^{(1,1)}_{l'l} \\
& + \left\{ Z^{(1,1)}_{kk'} \delta_{nn'} + \delta_{kk'} Z^{(1,1)}_{nn'} \right\} \hat{Z}^{(2,2)}_{k'n',l} \,, \qquad (2.121)
\end{aligned}
$$

[2]Note that there may occur additional contributions from the renormalisation of the mass and coupling constant factors in front of the dimension-eight operators.

$$
\begin{aligned}
3\hat{Z}^{(3,2)}_{kn,l} = &-2\beta_0\hat{Z}^{(2,1)}_{kn,l} - \beta_1\hat{Z}^{(1,1)}_{kn,l} + \beta_0\hat{Z}^{(1,1)}_{kn,l'}\tilde{Z}^{(1,0)}_{l'l} + \beta_0\hat{Z}^{(1,0)}_{kn,l'}\tilde{Z}^{(1,1)}_{l'l} \\
&- \hat{Z}^{(1,0)}_{kn,l'}\tilde{Z}^{(1,1)}_{l'l''}\tilde{Z}^{(1,1)}_{l''l} - \hat{Z}^{(1,1)}_{kn,l'}\tilde{Z}^{(1,0)}_{l'l''}\tilde{Z}^{(1,1)}_{l''l} - \hat{Z}^{(1,1)}_{kn,l'}\tilde{Z}^{(1,1)}_{l'l''}\tilde{Z}^{(1,0)}_{l''l} \\
&+ \hat{Z}^{(1,0)}_{kn,l'}\tilde{Z}^{(2,2)}_{l'l} + \hat{Z}^{(1,1)}_{kn,l'}\tilde{Z}^{(2,1)}_{l'l} + 2\hat{Z}^{(2,1)}_{kn,l'}\tilde{Z}^{(1,1)}_{l'l} + 2\hat{Z}^{(2,2)}_{kn,l'}\tilde{Z}^{(1,0)}_{l'l} \\
&- \left\{Z^{(1,0)}_{kk''}Z^{(1,1)}_{k''k'}\delta_{nn'} + \delta_{kk'}Z^{(1,0)}_{nn''}Z^{(1,1)}_{n''n'}\right\}\hat{Z}^{(1,1)}_{k'n',l} \\
&- \left\{Z^{(1,1)}_{kk''}Z^{(1,0)}_{k''k'}\delta_{nn'} + \delta_{kk'}Z^{(1,1)}_{nn''}Z^{(1,0)}_{n''n'}\right\}\hat{Z}^{(1,1)}_{k'n',l} \\
&- \left\{Z^{(1,0)}_{kk'}\delta_{nn'} + \delta_{kk'}Z^{(1,0)}_{nn'}\right\}\hat{Z}^{(1,1)}_{k'n',l'}\tilde{Z}^{(1,1)}_{l'l} \\
&- \left\{Z^{(1,1)}_{kk'}\delta_{nn'} + \delta_{kk'}Z^{(1,1)}_{nn'}\right\}\hat{Z}^{(1,0)}_{k'n',l'}\tilde{Z}^{(1,1)}_{l'l} \\
&- \left\{Z^{(1,1)}_{kk'}\delta_{nn'} + \delta_{kk'}Z^{(1,1)}_{nn'}\right\}\hat{Z}^{(1,1)}_{k'n',l'}\tilde{Z}^{(1,0)}_{l'l} \\
&+ \beta_0\left\{Z^{(1,0)}_{kk'}\delta_{nn'} + \delta_{kk'}Z^{(1,0)}_{nn'}\right\}\hat{Z}^{(1,1)}_{k'n',l} + \left\{Z^{(1,0)}_{kk'}\delta_{nn'} + \delta_{kk'}Z^{(1,0)}_{nn'}\right\}\hat{Z}^{(2,2)}_{k'n',l} \\
&+ \left\{Z^{(1,1)}_{kk'}\delta_{nn'} + \delta_{kk'}Z^{(1,1)}_{nn'}\right\}\hat{Z}^{(2,1)}_{k'n',l} + 2\left\{Z^{(2,1)}_{kk'}\delta_{nn'} + \delta_{kk'}Z^{(2,1)}_{nn'}\right\}\hat{Z}^{(1,1)}_{k'n',l}\,, \quad (2.122)
\end{aligned}
$$

$$
2\hat{Z}^{(es,2)}_{kn,l} = \left\{Z^{(e,1)}_{kk'}\delta_{nn'} + \delta_{kk'}Z^{(e,1)}_{nn'}\right\}\hat{Z}^{(1,1)}_{k'n',l} + \hat{Z}^{(1,1)}_{kn,l'}\tilde{Z}^{(e,1)}_{l'l}\,. \quad (2.123)
$$

The general solution of the Equations (2.112) can be found by the method of separation of variables and reads [51]

$$
\begin{aligned}
\tilde{C}_n(\mu) = &\tilde{U}_{nn'}(\mu,\mu_0)\tilde{C}_{n'}(\mu_0) \\
&+ \int_{\mu_0}^{\mu} d\mu'' U_{ll'}(\mu'',\mu_0)C_{l'}(\mu_0)U_{kk'}(\mu'',\mu_0)C_{k'}(\mu_0)\gamma_{lk,m}(\mu'')\tilde{U}_{mn}(\mu,\mu'')\,, \quad (2.124)
\end{aligned}
$$

where the evolution matrices are defined as in Equation (2.77). We need not pursue the task of expressing (2.124) in terms of anomalous dimensions as an expansion in the coupling constants, because it turns out that in the cases of interest to us, the form of the renormalisation group equations (2.112) can be reduced to the form (2.62) for single insertions by an appropriate choice of the operator basis. We will see this in detail in Chapters 4 and 5.

## 2.6 The Matching Procedure

In the preceeding section we have shown how to compute the anomalous dimensions, which are the basic ingredients of the renormalisation group equations for the Wilson coefficients. With these differential equations, it is possible to sum large logarithms to all orders in perturbation theory.

In this section we describe how we can determine the initial conditions for the Wilson coefficients by a so-called matching calculation. Using the renormalisation group equations, we

are then able to find an expression for the Wilson coefficients at an arbitrary renormalisation scale $\mu$.
We can find the initial conditions for the Wilson coefficients in the effective theory by equating the amputated, connected, one light-particle irreducible Green's functions in the full and the effective theory at the scale $\mu_0$ where some particle is removed from the theory, for instance a quark of mass $M$. We then demand that both sides of the equation agree, up to terms suppressed by a power of the heavy mass $M$. This is called the matching condition.
Because the effective theory is obtained from the full theory by modifying the high energy behaviour only, the low energy behaviour is the same in the two theories and drops out in the matching condition. This has the important consequence that the matching calculation is automatically infrared finite.
Let us start with the case of single operator insertions. We write the result for the amplitude in the full theory as an expansion in the coupling constants in the following way:

$$\mathcal{A}_{\text{full}} = \left( A^{(0)} + \frac{\alpha_s(\mu_0)}{4\pi} A^{(1)} + \left( \frac{\alpha_s(\mu_0)}{4\pi} \right)^2 A^{(2)} + \frac{\alpha}{4\pi} A^{(e)} + \frac{\alpha \alpha_s(\mu_0)}{16\pi^2} A^{(es)} \right) \langle Q \rangle^{(0)} . \tag{2.125}$$

In this equation, the coefficients $A^{(i)}$ can be thought of as row vectors and the tree level matrix element $\langle Q \rangle^{(0)}$ as a column vector. Note that here and the remaining equations in this section, everything is expressed in terms of the bare quantities, meaning that counterterms are included implicitly. We can similarly expand the corresponding amplitude in the effective theory and express it likewise through the tree-level matrix elements:

$$\mathcal{A}_{\text{eff}} = C(\mu_0) \left( 1 + \frac{\alpha_s(\mu_0)}{4\pi} r^{(1)} + \left( \frac{\alpha_s(\mu_0)}{4\pi} \right)^2 r^{(2)} + \frac{\alpha}{4\pi} r^{(e)} + \frac{\alpha \alpha_s(\mu_0)}{16\pi^2} r^{(es)} \right) \langle Q \rangle^{(0)} . \tag{2.126}$$

Here the Wilson coefficient $C$ is represented by a row vector, and the coefficients $r^{(i)}$ as matrices encoding the higher order corrections. As a subtle point, note that also the coupling constants receive non-trivial matching corrections when we pass a flavour threshold.
The matching condition $\mathcal{A}_{\text{full}} = \mathcal{A}_{\text{eff}}$ now allows us to extract the initial conditions for the Wilson coefficients in the effective theory order by order [52]:

$$\begin{aligned} C(\mu_0) = A^{(0)} &+ \frac{\alpha_s(\mu_0)}{4\pi} \left( A^{(1)} - A^{(0)} r^{(1)} \right) + \frac{\alpha}{4\pi} \left( A^{(e)} - A^{(0)} r^{(e)} \right) \\ &+ \left( \frac{\alpha_s(\mu_0)}{4\pi} \right)^2 \left( A^{(2)} - A^{(1)} r^{(1)} - A^{(0)} \left[ r^{(2)} - r^{(1)} r^{(1)} \right] \right) \\ &+ \frac{\alpha \alpha_s(\mu_0)}{16\pi^2} \left( A^{(es)} - A^{(e)} r^{(1)} - A^{(0)} \left[ r^{(es)} - r^{(1)} r^{(e)} - r^{(e)} r^{(1)} \right] \right) . \end{aligned} \tag{2.127}$$

We proceed in a completely analogous way in the case of double operator insertions.
Let us write the expansion of the amplitude in the full theory as above,

$$\mathcal{A}_{\text{full}} = \left( A^{(0)} + \frac{\alpha_s(\mu_0)}{4\pi} A^{(1)} + \left( \frac{\alpha_s(\mu_0)}{4\pi} \right)^2 A^{(2)} + \frac{\alpha}{4\pi} A^{(e)} + \frac{\alpha \alpha_s(\mu_0)}{16\pi^2} A^{(es)} \right) \langle \tilde{Q} \rangle^{(0)} , \tag{2.128}$$

where we denote the dimension-eight operator by a tilde, as before. On the effective side, we now have to consider matrix elements of both single and double operator insertions:

$$\begin{aligned}\mathcal{A}_{\text{eff}} = C_k(\mu_0)C_{k'}(\mu_0)\bigg(d^{(1)}_{kk',l} + \frac{\alpha_s(\mu_0)}{4\pi}d^{(2)}_{kk',l} + \left(\frac{\alpha_s(\mu_0)}{4\pi}\right)^2 d^{(3)}_{kk',l} + \frac{\alpha}{4\pi}d^{(e)}_{kk',l} \\ + \tilde{C}_{l'}(\mu_0)\bigg(\delta_{l'l} + \frac{\alpha_s(\mu_0)}{4\pi}s^{(1)}_{l'l} + \left(\frac{\alpha_s(\mu_0)}{4\pi}\right)^2 s^{(2)}_{l'l} + \frac{\alpha}{4\pi}s^{(e)}_{l'l} + \frac{\alpha\alpha_s(\mu_0)}{16\pi^2}s^{(es)}_{l'l}\bigg)\langle\tilde{Q}_l\rangle^{(0)}\,.\end{aligned} \tag{2.129}$$

In this equation, the coefficients $d_{kk',l}$ encode the matrix elements of the double insertions of the dimension-six operators, whereas the $s_{ll'}$ encode the matrix elements of the dimension-eight operators $\tilde{Q}_l$, which are defined here *without* the factor $1/g^2$. It is instead moved into the Wilson coefficient $\tilde{C}_l$, which has the expansion

$$\tilde{C}_l = \frac{4\pi}{\alpha_s(\mu_0)}\tilde{C}^{(0)}_l + \tilde{C}^{(1)}_l + \frac{\alpha_s(\mu_0)}{4\pi}\tilde{C}^{(2)}_l + \frac{4\pi\alpha}{\alpha_s^2(\mu_0)}\tilde{C}^{(e)}_l + \frac{\alpha}{\alpha_s(\mu_0)}\tilde{C}^{(es)}_l\,. \tag{2.130}$$

The individual contributions are then extracted from the matching condition $\mathcal{A}_{\text{full}} = \mathcal{A}_{\text{eff}}$ as above.

## 2.7 The Role of Unphysical Operators

We have seen in the last section how to compute the initial conditions for the Wilson coefficients by a matching calculation. The renormalisation in the effective theory involves physical as well as evanescent operators; therefore the question arises whether the amplitude in the effective theory depends on the Wilson coefficients of the evanescent operators.
The answer to this question is as follows: The Wilson coefficients of evanescent operators indeed receive non-vanishing initial conditions. Furthermore, the matrix elements of evanescent operators generally have non-vanishing projections on physical matrix elements. Therefore the authors of [34, 53] have proposed to renormalise the evanescent operators by a finite amount in order to cancel exactly this component. This is natural, since these terms arise exclusively from poles in the integrals, multiplying $\epsilon$-dependent terms stemming from the Dirac algebra. The proposed renormalisation is therefore in accordance with the $\overline{\text{MS}}$ prescription.
The effect is that the Wilson coefficients of the evanescent operators multiply matrix elements that vanish in four dimension and hence do not affect the physical amplitude.
Yet for the contributions of evanescent operators to vanish at all scales, we must ensure that the evanescent operators do not mix into the physical operators via renormalisation [54]. This is equivalent to the requirement that in the anomalous dimension matrix

$$\gamma = \begin{pmatrix}\gamma_{QQ} & \gamma_{QE} \\ \gamma_{EQ} & \gamma_{EE}\end{pmatrix}, \tag{2.131}$$

where $Q$ corresponds to physical and $E$ to evanescent operators, the submatrix $\gamma_{EQ}$ vanishes, leading to an upper block-diagonal form of $\gamma$. This is indeed the case if we perform the finite renormalisation mentioned above, irrespective of the exact definition of the evanescent operators. An analoguous result holds for the case of double insertions of operators. The proof has been given in [51, 54] and will not be repeated here.

Let us finally discuss two other classes of unphysical operators, which are present in the effective theory: Operators vanishing by the equation of motion, and BRST-exact operators. The role of these operators has been examined in detail in [55–61], and we summarise the arguments in the remaining part of this section.
We start with those operators vanishing by the equations of motion. For instance, consider the operator $Q_{\text{eom}}$ introduced in Equation (2.42), which vanishes after applying the equation of motion for the gluon field. In principle, also terms stemming from the gauge-compensating and gauge-fixing parts $\mathcal{L}_{\text{gc}}$ and $\mathcal{L}_{\text{gf}}$ contribute to this operator, but they can be shown to appear only in a combination which is the BRST [62] variation of some other operator and thus correspond to a BRST-exact operator. Furthermore, on-shell Green's functions of a single insertion of an EOM-vanishing or BRST-exact operator vanish, such that they do not contribute to the matching and play no role in the determination of the Wilson coefficient of the physical operators.
Again, this property is maintained by the renormalisation group evolution, because BRST-exact operators mix only among themselves and into EOM-vanishing operators, while the latter mix only among themselves. This in turn leads to the following block-triangular form of the anomalous dimension matrix:

$$\gamma = \begin{pmatrix} \gamma_{\mathcal{P}\mathcal{P}} & \gamma_{\mathcal{P}\mathcal{B}} & \gamma_{\mathcal{P}\mathcal{E}} \\ 0 & \gamma_{\mathcal{B}\mathcal{B}} & \gamma_{\mathcal{B}\mathcal{E}} \\ 0 & 0 & \gamma_{\mathcal{P}\mathcal{E}} \end{pmatrix} , \tag{2.132}$$

where now $\mathcal{P}$ corresponds to physical and evanescent, $\mathcal{B}$ to BRST-exact, and $\mathcal{E}$ to EOM-vanishing operators.
The situation is changed if we consider double insertions of operators, because now the EOM-vanishing operators lead to non-trivial contact terms, thus yielding a non-vanishing contribution to the Green's function. The same is true for BRST-exact operators which may involve EOM-vanishing operators as counterterms. However, as explained in [55], these extra terms are local and proportional to dimension-eight operators, which we include in our calculation anyway.
In short, the discussion above amounts to the fact that we can safely neglect EOM-vanishing and BRST-exact operators in our analysis.
As a last comment, however, we remark that our treatment of infrared (IR) divergences in extracting the UV poles requires the use of EOM-vanishing operators in intermediate steps of the calculation; we will discuss this in more detail in Section 3.1.

## 2.8 Renormalisation Scheme Dependence

It is important to examine the dependence of the Wilson coefficients and anomalous dimensions on the renormalisation scheme: We have to ensure that theoretical predictions of physical observables do not depend on the choice of a renormalisation scheme.
A dependence on the renormalisation scheme arises beyond the leading order in perturbation theory, because the renormalisation of all parameters, operators, and fields is fixed only to the extent that all UV divergences be removed. This requirement does not determine the finite parts of the counterterms, and many choices are possible.
For instance, the choice of a basis of local operators, describing the interactions in the effective theory, is not unique. As we will see in Section 3.4, different choices of the operator basis are related by a change of the renormalisation scheme.

In order to ensure that physical observables and the related decay amplitudes are independent of this choice, it is sufficient to examine the scheme independence of the effective Hamiltonian; any dependence on the chosen scheme has to cancel between the different contributions. This scheme independence has been shown previously [23, 51, 53, 54] and the proofs will not be repeated here. However, although physical observables are scheme independent, quantities like the Wilson coefficients and anomalous dimensions are not, and it is useful to have explicit transformation formulas at hand: In this way we can compare parts of our results with the literature, where a different operator basis has been used. Also it turns out that, whereas the calculation of the operator matrix elements is most easily performed in the basis chosen by us, the solution of the renormalisation group equations is simplified by transforming to a different basis.

In the remaining part of this section, we derive the transformation properties of the anomalous dimensions for an arbitrary change of the renormalisation scheme. This generalises the results already obtained in the literature.

Suppose we perform the following change of scheme for the Wilson coefficients

$$C_i \to C_i' = C_j \rho^{-1}_{ji} \,, \tag{2.133}$$

$$\tilde{C}_k \to \tilde{C}_k' = \tilde{C}_j \tilde{\rho}^{-1}_{jk} - C_l C_m \hat{\rho}_{lm,k} \,. \tag{2.134}$$

As usual, we have denoted Wilson coefficients belonging to dimension-eight operators with a tilde and those belonging to dimension-six operators without superscript. Furthermore, we introduced the parameters $\rho$, $\tilde{\rho}$ and $\hat{\rho}$, which parameterise the finite transformations:

$$\rho_{ij} = \delta_{ij} - \frac{\alpha_s}{4\pi}\rho^{(1)}_{ij} - \left(\frac{\alpha_s}{4\pi}\right)^2 \left(\rho^{(2)}_{ij} - \rho^{(1)}_{ik}\,\rho^{(1)}_{kj}\right) + \mathcal{O}(\alpha_s^3) \,, \tag{2.135}$$

$$\tilde{\rho}_{ij} = \delta_{ij} - \frac{\alpha_s}{4\pi}\tilde{\rho}^{(1)}_{ij} - \left(\frac{\alpha_s}{4\pi}\right)^2 \left(\tilde{\rho}^{(2)}_{ij} - \tilde{\rho}^{(1)}_{ik}\,\tilde{\rho}^{(1)}_{kj}\right) + \mathcal{O}(\alpha_s^3) \,, \tag{2.136}$$

$$\hat{\rho}_{lm,k} = \frac{\alpha_s}{4\pi}\hat{\rho}^{(1)}_{lm,k} + \left(\frac{\alpha_s}{4\pi}\right)^2 \hat{\rho}^{(2)}_{lm,k} + \mathcal{O}(\alpha_s^3) \,. \tag{2.137}$$

Then, in order for the effective Hamiltonian of the form

$$H_{\text{eff}} = C_i Z_{ij} Q_j + \left(\tilde{C}_i \tilde{Z}_{ik} + C_i C_j \hat{Z}_{ij,k}\right)\tilde{Q}_k \tag{2.138}$$

to stay invariant, the renormalisation constants must transform as

$$Z_{ij} \to Z_{ij}' = \rho_{ik} Z_{kj} \,, \tag{2.139}$$

$$\tilde{Z}_{ij} \to \tilde{Z}_{ij}' = \tilde{\rho}_{ik} \tilde{Z}_{kj} \,, \tag{2.140}$$

$$\hat{Z}_{ij,k} \to \hat{Z}_{ij,k}' = \rho_{il}\rho_{jm}\hat{Z}_{lm,k} + \rho_{il}\rho_{jm}\hat{\rho}_{lm,p}\tilde{\rho}_{pq}\tilde{Z}_{qk} \,. \tag{2.141}$$

The transformation of the anomalous dimensions can now be obtained by inserting the transformed renormalisation constants into the defining equations for the anomalous dimension matrix (2.63) and the anomalous dimension tensor (2.113), respectively. In this way we get

the well-known results for the case of single insertions [23, 47, 63]:

$$\gamma'^{(0)} = \gamma^{(0)}\,, \tag{2.142}$$

$$\gamma'^{(1)} = \gamma^{(1)} - [\rho^{(1)}, \gamma^{(0)}] - 2\beta_0\rho^{(1)}\,, \tag{2.143}$$

$$\begin{aligned}\gamma'^{(2)} = {} & \gamma^{(2)} - [\rho^{(2)}, \gamma^{(0)}] - [\rho^{(1)}, \gamma^{(1)}] + \rho^{(1)}[\rho^{(1)}, \gamma^{(0)}] \\ & - 4\beta_0\rho^{(2)} - 2\beta_1\rho^{(1)} + 2\beta_0\rho^{(1)}\rho^{(1)}\,.\end{aligned} \tag{2.144}$$

The general transformation law for the anomalous dimension tensor for double insertion has never been given explicitly before and reads[3]:

$$\hat\gamma'^{(0)}_{ij,k} = \gamma^{(0)}_{ij,k}\,, \tag{2.145}$$

$$\begin{aligned}\hat\gamma'^{(1)}_{ij,k} = {} & \gamma^{(1)}_{ij,k} + \hat\rho^{(1)}_{ij,l}\tilde\gamma^{(0)}_{lk} + 2\hat\rho^{(1)}_{ij,k}\beta_0 + \hat\gamma^{(0)}_{ij,l}\tilde\rho^{(1)}_{lk} \\ & - \left\{\gamma^{(0)}_{il}\delta_{jm} + \delta_{il}\gamma^{(0)}_{jm}\right\}\hat\rho^{(1)}_{lm,k} - \left\{\rho^{(1)}_{il}\delta_{jm} + \delta_{il}\rho^{(1)}_{jm}\right\}\hat\gamma^{(0)}_{lm,k}\,,\end{aligned} \tag{2.146}$$

$$\begin{aligned}\hat\gamma'^{(2)}_{ij,k} = {} & \gamma^{(2)}_{ij,k} + \hat\rho^{(2)}_{ij,l}\tilde\gamma^{(0)}_{lk} + 4\hat\rho^{(2)}_{ij,k}\beta_0 + \hat\rho^{(1)}_{ij,l}\tilde\gamma^{(1)}_{lk} + 2\hat\rho^{(1)}_{ij,k}\beta_1 + \hat\rho^{(1)}_{ij,l}\tilde\gamma^{(0)}_{lm}\tilde\rho^{(1)}_{mk} - \hat\rho^{(1)}_{ij,l}\tilde\rho^{(1)}_{lm}\tilde\gamma^{(0)}_{mk} \\ & + \hat\gamma^{(1)}_{ij,l}\tilde\rho^{(1)}_{lk} + \hat\gamma^{(0)}_{ij,l}\tilde\rho^{(2)}_{lk} + \rho^{(1)}_{il}\gamma^{(0)}_{jm}\hat\rho^{(1)}_{lm,l} + \gamma^{(0)}_{il}\rho^{(1)}_{jm}\hat\rho^{(1)}_{lm,l} + \rho^{(1)}_{il}\rho^{(1)}_{jm}\hat\gamma^{(0)}_{lm,l} \\ & - \left\{\rho^{(2)}_{il}\delta_{jm} + \delta_{il}\rho^{(2)}_{jm}\right\}\hat\gamma^{(0)}_{lm,k} - \left\{\rho^{(1)}_{il}\delta_{jm} + \delta_{il}\rho^{(1)}_{jm}\right\}\hat\gamma^{(1)}_{lm,k} \\ & - \left\{\rho^{(1)}_{il}\delta_{jm} + \delta_{il}\rho^{(1)}_{jm}\right\}\hat\gamma^{(0)}_{lm,n}\tilde\rho^{(1)}_{nk} - \left\{\rho^{(1)}_{il}\delta_{jm} + \delta_{il}\rho^{(1)}_{jm}\right\}\hat\rho^{(1)}_{lm,n}\tilde\gamma^{(0)}_{nk} \\ & - 2\left\{\rho^{(1)}_{il}\delta_{jm} + \delta_{il}\rho^{(1)}_{jm}\right\}\hat\rho^{(1)}_{lm,k}\beta_0 - \left\{\rho^{(1)}_{ii'}\rho^{(1)}_{i'l}\delta_{jm} + \delta_{il}\rho^{(1)}_{jj'}\rho^{(1)}_{j'm}\right\}\hat\gamma^{(0)}_{lm,k} \\ & - \left\{\rho^{(1)}_{ii'}\gamma^{(0)}_{i'l}\delta_{jm} + \delta_{il}\rho^{(1)}_{jj'}\gamma^{(0)}_{j'm}\right\}\hat\rho^{(1)}_{lm,k} - \left\{\gamma^{(1)}_{il}\delta_{jm} + \delta_{il}\gamma^{(1)}_{jm}\right\}\hat\rho^{(1)}_{lm,k} \\ & - \left\{\gamma^{(0)}_{il}\delta_{jm} + \delta_{il}\gamma^{(0)}_{jm}\right\}\hat\rho^{(2)}_{lm,k}\,.\end{aligned} \tag{2.147}$$

[3]Note that also here additional finite contributions arise if we include the factor of $m_c^2/g^2$ the definition of the dimension-eight operators.

# Chapter 3

# Method of Calculation

After having set up the theoretical framework in the last chapter, we now proceed to more practical questions concerning the calculation of the anomalous dimensions. In Section 3.1 we describe how we extract the UV divergent parts of one-particle irreducible Feynman diagrams. In Section 3.2 we shortly describe the evaluation of the integrals and in particular the reduction of tensor to scalar integrals. In Section 3.3 we collect the formulas for the physical parts of the anomalous dimension tensor, and Section 3.4 is devoted to the effects of a change of the operator basis.

## 3.1 Infrared Rearrangement

As we have seen in the previous chapter, in the framework of dimensional regularisation and the $\overline{\text{MS}}$ scheme the anomalous dimensions are simply related to the renormalisation constants of various operator insertions. One of the main difficulties in the determination of the renormalisation constants is the separation of IR and UV divergences. In our calculation we follow the approach suggested by Chetyrkin, Misiak and Münz [64].
We start with the following exact decomposition of a scalar propagator for a particle of (possibly vanishing) mass $m$

$$\frac{1}{(p+q)^2-m^2} = \frac{1}{p^2-M^2} + \frac{m^2-q^2-2pq-M^2}{p^2-M^2}\frac{1}{(p+q)^2-m^2}\,, \tag{3.1}$$

where $p$ is a linear combination of integration momenta and $q$ is a linear combination of external momenta. The mass parameter $M$ regularises the IR divergences.
The decomposition is chosen in such a way that the first term on the right side of Eq. (3.1) depends only on the artificial mass $M$ and the integration momenta, whereas the contribution of the second term to the UV degree of divergence is reduced. Moreover, because the last factor of the second term has the same form as the original propagator, the procedure can be applied recursively.
Because the dimensionality of the relevant operators in our effective theory is bounded from above, after a certain number of replacements, substituting any of the propagators by the last term in Eq. (3.1) would render a considered diagram finite. We may then drop this last term.
Because the decomposition (3.1) is exact, the final result for the UV divergent part of the considered diagram is actually independent of $M$. UV divergent terms stemming from expres-

sions with a $M^2$ in a propagator numerator are exactly cancelled by contributions originating from integrals with no $M^2$ in the numerator. This observation leads to a further simplification of the procedure [64]: The UV divergent part of integrals with a $M^2$ in the numerator are local after the subtraction of all subdivergences. Thus, instead of calculating them, we can just replace them by local counterterms proportional to $M^2$. The final result will be the same as if the full propagators had been used.

Because the dimension of the counterterms proportional to $M^2$ must be at least two units less than the maximal dimension of the operators in the considered effective Hamiltonian, only the following, gauge-variant counterterms contribute in our case:

$$M^2 G^a_\mu G^{a\mu} \quad \text{and} \quad \frac{M^2}{g} \bar{s}_L \not{G} d_L \,. \tag{3.2}$$

As a side effect of the described procedure, we have to consider contributions of non-physical operators, such as the EOM-vanishing operator (2.42), in intermediate steps of the calculation. Whereas non-physical operators are generally needed in order to subtract divergences of 1PI subdiagrams, the projections of their matrix elements on the physical operators vanish[1]. However, the procedure above gives the correct overall UV divergence only after subtraction of all subdivergences. The subtraction of subdivergences is performed, in the spirit of the forest formula, by the inclusion of counterterm diagrams. The subdivergences of a diagram as well as the corresponding (sub)divergences of the counterterm diagrams depend on the finite parts of certain lower loop diagrams, which are not necessarily correct on account of the method described above, and the matrix elements of EOM-vanishing operators have to be retained in intermediate steps. However, these incorrect contributions exactly cancel in their sum, if the calculation is performed in precisely the same way, yielding the correct overall divergences.

We remark that also the finite renormalisation constants needed for the renormalisation of evanescent operators are given correctly by our method, because they derive only from explicitly $\epsilon$-dependent terms stemming from the Dirac algebra that multiply UV divergent pieces of the integrals.

## 3.2 Calculation of the Integrals

The very large number ($\mathcal{O}(10\,000)$) of Feynman diagrams appearing in our calculation requires an automated setup of computer algebra for their evaluation. We use qgraf [65] for generating the diagrams; the evaluation of the integrals is performed using the program packages MATAD/q2e/exp [66, 67], where MATAD is written in FORM [68] and based on the Integration-By-Parts algorithm [69, 70]. In addition, we have written our own FORM routine in order to evaluate two-loop diagrams with an arbitrary number of (possibly vanishing) masses, using the algorithm described in [71].

All calculations have been checked using an completely independent setup, based on Feynarts [72] and Mathematica.

[1] Note that there is an important qualification in the case of double insertions, as discussed in Section 2.7. We will come back to this issue in Chapter 4.

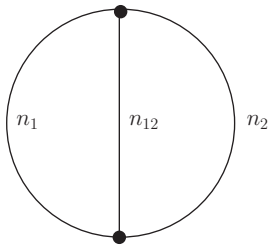


Figure 3.1: Graphical representation of the integral $T^{22}_{n_1 n_2 n_{12}}$.

### Tensor Decomposition

Here we describe a method [64] to reduce tensor integrals to scalar ones which can be computed using the programs mentioned above. A general euclidean tensor vacuum integral up to three loops with one common mass $M$ can be written as

$$T^{a_1 \ldots a_l}_{n_1 \ldots n_l n_{11} \ldots n_{1l} \ldots n_{l-1,l}} = M^{-l\,d - \sum a_i + 2\sum n_i} \pi^{-l\,d/2} \int \frac{\prod_i \; d^d p_i \, p_{i\mu_{i,1}} \cdots p_{i\mu_{i,a_i}}}{\prod_i (p_i^2 + M^2)^{n_i} \prod_{i<j} [(p_i - p_j)^2 + M^2]^{n_{ij}}} \tag{3.3}$$

and is proportional to a linear combination of products of metric tensors.
An example is

$$T^{22}_{n_1 n_2 n_{12}} = F_1 g_{\mu_{1,1}\mu_{1,2}} g_{\mu_{2,1}\mu_{2,2}} + F_2 (g_{\mu_{1,1}\mu_{2,1}} g_{\mu_{1,2}\mu_{2,2}} + g_{\mu_{1,1}\mu_{2,2}} g_{\mu_{1,2}\mu_{2,1}}) \,, \tag{3.4}$$

where only two coefficient functions appear because of an obvious symmetry.
In general, these coefficient functions can be determined by contracting the tensor integral (3.3) with all products of metric tensors. In the example above, we find the following two equations

$$\begin{aligned} d^2 F_1 + 2d F_2 &= X_1 \,, \\ d F_1 + d(d+1) F_2 &= X_2 \,, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} X_1 &= m^{-2\,d-4+2\sum n_i} \pi^{-d} \int \frac{d^d p \, d^d k \; p^2 \, k^2}{(p^2+m^2)^{n_1} (k^2+m^2)^{n_2} [(p-k)^2+m^2]^{n_3}} \,, \\ X_2 &= m^{-2\,d-4+2\sum n_i} \pi^{-d} \int \frac{d^d p \, d^d k \; (p \cdot k)^2}{(p^2+m^2)^{n_1} (k^2+m^2)^{n_2} [(p-k)^2+m^2]^{n_3}} \,, \end{aligned} \tag{3.6}$$

and we can solve for $F_1$ and $F_2$ by a matrix inversion:

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} d^2 & 2d \\ d & d(d+1) \end{pmatrix}^{-1} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} . \tag{3.7}$$

This method easily generalises to the case of more metric tensors, where the matrix is most easily inverted by substituting $d = 4 - 2\epsilon$ and expanding up to an appropriate power in $\epsilon$.

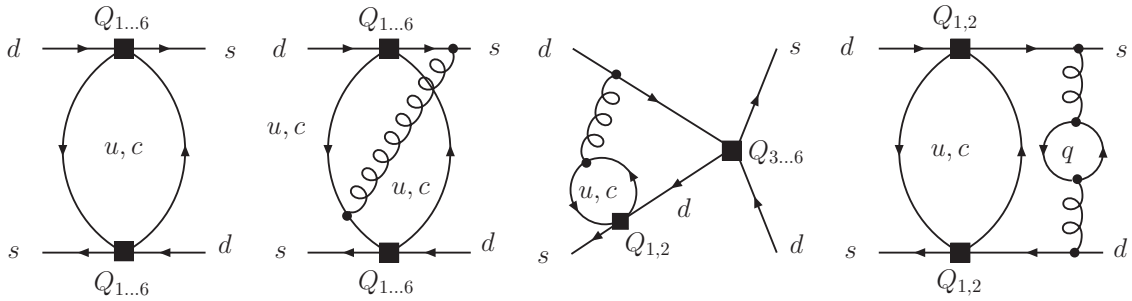


Figure 3.2: Sample one-, two-, and three-loop diagrams contributing to the anomalous dimensions relevant for $\eta_3$.

## 3.3 The Anomalous Dimensions

In this section we collect the formulas for the physical parts of the anomalous dimensions in terms of the renormalisation constants, as derived in Section 2.5. The renormalisation constants are determined by calculating various one-, two-, and three-loop insertions of the dimension-six operators (see Figure 3.2 and the diagrams in Chapter 5). We have verified that all renormalisation constants are local. As an additional check of our results, we have performed all calculations in a generalised $R_\xi$ gauge and verified that the anomalous dimensions are independent of the gauge parameters $\xi_g$ and $\xi_a$, introduced in Equation (2.24). We have also checked explicitly that the mixing of evanescent into physical operators vanishes up to NLO.
The non-vanishing contributions to the physical part of the anomalous dimension tensor are given by

$$\hat{\gamma}^{(0)}_{kn,l} = 2\hat{Z}^{(1,1)}_{kn,l}\,, \tag{3.8}$$

$$\hat{\gamma}^{(1)}_{kn,l} = 4\hat{Z}^{(2,1)}_{kn,l} - 2\hat{Z}^{(1,1)}_{kn,l'}\tilde{Z}^{(1,0)}_{l'l} - 2\left\{Z^{(1,1)}_{kk'}\delta_{nn'} + \delta_{kk'}Z^{(1,1)}_{nn'}\right\}\hat{Z}^{(1,0)}_{k'n',l}\,, \tag{3.9}$$

$$\begin{aligned}\hat{\gamma}^{(2)}_{kn,l} = {} & 6\hat{Z}^{(3,1)}_{kn,l} - 4\hat{Z}^{(2,1)}_{kn,l'}\tilde{Z}^{(1,0)}_{l'l} - 2\hat{Z}^{(1,1)}_{kn,l'}\tilde{Z}^{(2,0)}_{l'l} \\ & - 2\left\{Z^{(1,1)}_{kk'}\delta_{nn'} + \delta_{kk'}Z^{(1,1)}_{nn'}\right\}\hat{Z}^{(2,0)}_{k'n',l} - 4\left\{Z^{(2,1)}_{kk'}\delta_{nn'} + \delta_{kk'}Z^{(2,1)}_{nn'}\right\}\hat{Z}^{(1,0)}_{k'n',l}\,,\end{aligned} \tag{3.10}$$

$$\gamma^{(es)}_{kn,l} = 4\hat{Z}^{(es,1)}_{kn,l} - 2\hat{Z}^{(1,1)}_{kn,l'}\tilde{Z}^{(e,0)}_{l'l} - 2\left\{Z^{(e,1)}_{kk'}\delta_{nn'} + \delta_{kk'}Z^{(e,1)}_{nn'}\right\}\hat{Z}^{(1,0)}_{k'n',l}\,, \tag{3.11}$$

where the indices $k$, $n$ and $l$ correspond to physical operators.

## 3.4 Change of the Operator Basis

In this section we will examine how the Wilson coefficients and the anomalous dimensions transform under a change of the operator basis. This is important for two reasons: In order to find a compact form for the renormalisation group equations for double operator insertions,

it turns out to be useful to transform to a diagonal operator basis in the subspace of current-current operators. Moreover, we have to transform our results in order to compare them with results that are available in the literature and have been calculated using a different operator basis.

In $d = 4$ dimensions, a change of $n$ dimension-six operators $Q$ and $m$ dimension-eight operators $\tilde{Q}$ is simply given by a linear transformation

$$Q_i \to Q'_i = R_{ij} Q_j \,, \qquad \tilde{Q}_i \to \tilde{Q}'_i = \tilde{R}_{ij} \tilde{Q}_j \,, \tag{3.12}$$

described by matrices $R \in \mathrm{GL}(n)$, $\tilde{R} \in \mathrm{GL}(m)$. Under this transformation the renormalisation constants transform according to

$$Z'_{ij} = R_{ik} Z_{kl} R^{-1}_{lj} \,, \qquad \tilde{Z}'_{ij} = \tilde{R}_{ik} \tilde{Z}_{kl} \tilde{R}^{-1}_{lj} \,, \qquad \hat{Z}'_{kn,l} = R_{kk'} R_{nn'} \hat{Z}_{k'n',l'} \tilde{R}^{-1}_{l'l} \,. \tag{3.13}$$

In general, the situation is more complicated because of the presence of evanescent operators which receive a finite renormalisation, as explained in Section 2.7. Now, in addition to the transformation (3.12) which only mixes the physical operators among themselves, more transformations are possible, which mix physical and evanescent operators. The transformation law of the renormalisation constants is still given by the formula (3.13). But this rule may now lead to additional finite contributions to the renormalisation constants for the physical operators. In order to restore the standard $\overline{\mathrm{MS}}$ definition of the renormalisation constants, which is crucial for our treatment of the renormalisation group, we have to subtract these extra contributions by a finite renormalisation.

We can write a general transformation among all dimension-six operators as

$$\begin{pmatrix} Q' \\ E' \end{pmatrix} = \begin{pmatrix} R & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \epsilon U + \epsilon^2 V & 1 \end{pmatrix} \begin{pmatrix} 1 & W \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Q \\ E \end{pmatrix} , \tag{3.14}$$

where the matrices $R$ and $M$ parameterise a linear transformation among the physical and evanescent operators, respectively, $W$ parameterises the addition of multiples of evanescent operators to the physical operators, and $U$ and $V$ parameterise the addition of multiples of $\epsilon$ and $\epsilon^2$ times physical operators to the evanescent operators. We apply a transformation of the same form to the dimension-eight operators, where we denote the corresponding matrices by a tilde, as usual.

The finite renormalisation constants can now be determined by observing that an effective amplitude of the form $C_i Z_{ij} \langle Q_j \rangle + (\tilde{C}_l \tilde{Z}_{lk} + C_i C_j \hat{Z}_{ij,k}) \langle \tilde{Q}_k \rangle$ must be invariant under the basis transformation and be renormalised according to the $\overline{\mathrm{MS}}$ prescription.

Let us start with the anomalous dimension matrices for the mixing of dimension-six into dimension-six operators. The finite renormalisation induced by the change (3.14) is given by [23, 25]

$$\begin{aligned} Z'^{(1,0)}_{QQ} &= R \left[ W Z^{(1,0)}_{EQ} - \left( Z^{(1,1)}_{QE} + W Z^{(1,1)}_{EE} - \frac{1}{2} \gamma^{(0)} W \right) U \right] R^{-1} \,, \\ Z'^{(2,0)}_{QQ} &= -R \left( Z^{(2,1)}_{QE} U + Z^{(2,2)}_{QE} V - \frac{1}{2} Z^{(1,1)}_{QE} V \gamma^{(0)} \right) R^{-1} \,, \end{aligned} \tag{3.15}$$

where

$$Z^{(2,2)}_{QE} = \frac{1}{2} \left( Z^{(1,1)}_{QE} Z^{(1,1)}_{EE} + \frac{1}{2} \gamma^{(0)} Z^{(1,1)}_{QE} - \beta_0 Z^{(1,1)}_{QE} \right) . \tag{3.16}$$

We have set $W$ to zero in the second line of Equation (3.15) as these terms are not needed in the following.
We find the transformation law for the anomalous dimension matrices in a straightforward manner using Equations (2.142) to (2.144):

$$\begin{aligned}
\gamma'^{(0)} &= R\gamma^{(0)}R^{-1}\,,\\
\gamma'^{(1)} &= R\gamma^{(1)}R^{-1} - \left[Z'^{(1,0)}_{QQ}, \gamma'^{(0)}\right] - 2\beta_0 Z'^{(1,0)}_{QQ}\,,\\
\gamma'^{(2)} &= R\gamma^{(2)}R^{-1} - \left[Z'^{(2,0)}_{QQ}, \gamma'^{(0)}\right] - \left[Z'^{(1,0)}_{QQ}, \gamma'^{(1)}\right] + \left[Z'^{(1,0)}_{QQ}, \gamma'^{(0)}\right] Z'^{(1,0)}_{QQ}\\
&\quad - 4\beta_0 Z'^{(2,0)}_{QQ} - 2\beta_1 Z'^{(1,0)}_{QQ} + 2\beta_0 \left(Z'^{(1,0)}_{QQ}\right)^2\,.
\end{aligned} \tag{3.17}$$

The Wilson coefficients change according to

$$C'(\mu) = \left[1 + \frac{\alpha_s(\mu)}{4\pi} Z'^{(1,0)}_{QQ} + \left(\frac{\alpha_s(\mu)}{4\pi}\right)^2 Z'^{(2,0)}_{QQ}\right]^T \left(R^{-1}\right)^T C(\mu)\,. \tag{3.18}$$

The explicit forms of the finite renormalisation constants as well as of the matrices $R$, $U$, $V$, $M$ and $Z$ for the transformation to the diagonal basis are collected in Appendix C.
Now we consider the mixing of dimension-six into dimension-eight operators. Clearly, the transformation law of the anomalous dimension matrix describing the mixing among the dimension-eight operators themselves is given by a formula completely analoguous to (3.17). The transformation of the anomalous dimension tensor, describing the mixing of dimension-six into dimension-eight operators, and of the dimension-eight Wilson coefficients is slightly more complicated. It can, however, be derived by the same method as above.
In addition to the finite renormalisation constants (3.15), we now get extra finite contributions to $\hat{Z}$. Again, we set $W$ to zero in the two-loop contribution, as it is not needed for our purposes, and find

$$\begin{aligned}
\hat{Z}'^{(1,0)}_{ij,k} &= R_{im}R_{jn}\Big(\hat{Z}^{(1,1)}_{mn,l}\tilde{W}_{ll'}\tilde{U}_{l'p} - \hat{Z}^{(1,1)}_{mn,l}\tilde{U}_{lp} + W_{ml}\hat{Z}^{(1,0)}_{ln,p} + W_{nl}\hat{Z}^{(1,0)}_{ml,p}\\
&\qquad - W_{il}\hat{Z}^{(1,1)}_{ln,m}\tilde{U}_{mp} - W_{nl}\hat{Z}^{(1,1)}_{il,m}\tilde{U}_{mp}\Big)\tilde{R}^{-1}_{pk}\,,
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
\hat{Z}'^{(2,0)}_{ij,k} &= R_{im}R_{jn}\Big(-\hat{Z}^{(2,1)}_{mn,l}\tilde{U}_{lp} - \hat{Z}^{(2,2)}_{mn,l}\tilde{V}_{lp} - \hat{Z}^{(1,1)}_{mn,l}\tilde{V}_{ll'}\tilde{Z}^{(1,1)}_{l'p}\\
&\qquad - Z^{(1,1)}_{ml}V_{ll'}\hat{Z}^{(1,1)}_{l'n,p} - Z^{(1,1)}_{nl}V_{ll'}\hat{Z}^{(1,1)}_{ml',p}\Big)\tilde{R}^{-1}_{pk}\,.
\end{aligned} \tag{3.20}$$

Here the indices $i$, $j$, and $k$ correspond to physical operators only. These expressions have never been given explicitly in the literature before.
The anomalous dimension tensor then transforms according to

$$\gamma'^{(0)}_{ij,k} = R_{im}R_{jn}\gamma^{(0)}_{mn,l}\tilde{R}^{-1}_{lk}\,, \tag{3.21}$$

$$\begin{aligned}\gamma'^{(1)}_{ij,k} &= R_{im}R_{jn}\gamma^{(1)}_{mn,l}\tilde{R}^{-1}_{lk} + \hat{Z}'^{(1,0)}_{ij,l}\tilde{\gamma}'^{(0)}_{lk} + 2\hat{Z}'^{(1,0)}_{ij,k}\beta_0 + \hat{\gamma}'^{(0)}_{ij,l}\tilde{Z}'^{(1,0)}_{lk} \\ &\quad - \left\{\gamma'^{(0)}_{il}\delta_{jm} + \delta_{il}\gamma'^{(0)}_{jm}\right\}\hat{Z}'^{(1,0)}_{lm,k} - \left\{Z'^{(1,0)}_{il}\delta_{jm} + \delta_{il}Z'^{(1,0)}_{jm}\right\}\hat{\gamma}'^{(0)}_{lm,k}\,, \end{aligned} \tag{3.22}$$

$$\begin{aligned}\gamma'^{(2)}_{ij,k} &= R_{im}R_{jn}\gamma^{(2)}_{mn,l}\tilde{R}^{-1}_{lk} + \hat{Z}'^{(2,0)}_{ij,l}\tilde{\gamma}'^{(0)}_{lk} + 4\hat{Z}'^{(2,0)}_{ij,k}\beta_0 + \hat{Z}'^{(1,0)}_{ij,l}\tilde{\gamma}'^{(1)}_{lk} + 2\hat{Z}'^{(1,0)}_{ij,k}\beta_1 \\ &\quad + \hat{Z}'^{(1,0)}_{ij,l}\tilde{\gamma}'^{(0)}_{lm}\tilde{Z}'^{(1,0)}_{mk} - \hat{Z}'^{(1,0)}_{ij,l}\tilde{Z}'^{(1,0)}_{lm}\tilde{\gamma}'^{(0)}_{mk} + \hat{\gamma}'^{(1)}_{ij,l}\tilde{Z}'^{(1,0)}_{lk} + \hat{\gamma}'^{(0)}_{ij,l}\tilde{Z}'^{(2,0)}_{lk} \\ &\quad + Z'^{(1,0)}_{il}\gamma'^{(0)}_{jm}\hat{Z}'^{(1,0)}_{lm,l} + \gamma'^{(0)}_{il}Z'^{(1,0)}_{jm}\hat{Z}'^{(1,0)}_{lm,l} + Z'^{(1,0)}_{il}Z'^{(1,0)}_{jm}\hat{\gamma}'^{(0)}_{lm,l} \\ &\quad - \left\{Z'^{(2,0)}_{il}\delta_{jm} + \delta_{il}Z'^{(2,0)}_{jm}\right\}\hat{\gamma}'^{(0)}_{lm,k} - \left\{Z'^{(1,0)}_{il}\delta_{jm} + \delta_{il}Z'^{(1,0)}_{jm}\right\}\hat{\gamma}'^{(1)}_{lm,k} \\ &\quad - \left\{Z'^{(1,0)}_{il}\delta_{jm} + \delta_{il}Z'^{(1,0)}_{jm}\right\}\hat{\gamma}'^{(0)}_{lm,n}\tilde{Z}'^{(1,0)}_{nk} - \left\{Z'^{(1,0)}_{il}\delta_{jm} + \delta_{il}Z'^{(1,0)}_{jm}\right\}\hat{Z}'^{(1,0)}_{lm,n}\tilde{\gamma}'^{(0)}_{nk} \\ &\quad - 2\left\{Z'^{(1,0)}_{il}\delta_{jm} + \delta_{il}Z'^{(1,0)}_{jm}\right\}\hat{Z}'^{(1,0)}_{lm,k}\beta_0 - \left\{Z'^{(1,0)}_{ii'}Z'^{(1,0)}_{i'l}\delta_{jm} + \delta_{il}Z'^{(1,0)}_{jj'}Z'^{(1,0)}_{j'm}\right\}\hat{\gamma}'^{(0)}_{lm,k} \\ &\quad - \left\{Z'^{(1,0)}_{ii'}\gamma'^{(0)}_{i'l}\delta_{jm} + \delta_{il}Z'^{(1,0)}_{jj'}\gamma'^{(0)}_{j'm}\right\}\hat{Z}'^{(1,0)}_{lm,k} - \left\{\gamma'^{(1)}_{il}\delta_{jm} + \delta_{il}\gamma'^{(1)}_{jm}\right\}\hat{Z}'^{(1,0)}_{lm,k} \\ &\quad - \left\{\gamma'^{(0)}_{il}\delta_{jm} + \delta_{il}\gamma'^{(0)}_{jm}\right\}\hat{Z}'^{(2,0)}_{lm,k}\,, \end{aligned} \tag{3.23}$$

as can be derived easily from Equations (2.145) to (2.147). A special case of these formulas has been derived in Reference [51] up to NLO; the remaining results are new. Using the definition (2.134), we see that the dimension-eight Wilson coefficients transform as

$$\begin{aligned}\tilde{C}'_k(\mu) &= \tilde{C}_i(\mu)\tilde{R}^{-1}_{ij}\left[\delta_{jk} + \frac{\alpha_s(\mu)}{4\pi}\tilde{Z}'^{(1,0)}_{jk} + \left(\frac{\alpha_s(\mu)}{4\pi}\right)^2\tilde{Z}'^{(2,0)}_{jk}\right] \\ &\quad - C_i(\mu)R^{-1}_{im}C_j(\mu)R^{-1}_{jn}\left[\frac{\alpha_s(\mu)}{4\pi}\hat{Z}'^{(1,0)}_{mn,k} + \left(\frac{\alpha_s(\mu)}{4\pi}\right)^2\hat{Z}'^{(2,0)}_{mn,k}\right]. \end{aligned} \tag{3.24}$$

# Part II

# Applications

# Chapter 4

# NNLO QCD Corrections to the Parameter $\epsilon_K$

We will now apply the formalism developed in the preceeding chapters to the calculation of precision observables in Kaon physics. In this chapter we calculate the NNLO QCD corrections to the charm-top contribution to the parameter $\epsilon_K$.
In Section 4.1 we give a short overview of the formalism of meson mixing and $CP$ violation in the neutral Kaon system and introduce the relevant effective Hamiltonian. We continue with an overview of the NNLO calculation in Section 4.2 and proceed to the calculation of the charm-top contribution $\eta_3$ in Section 4.3. The renormalisation group evolution below the charm quark scale is described in Section 4.4.
In this and the following chapter operators are expressed in terms of bare quark fields, for instance $d_L = Z_\psi^{1/2} d_L^{\text{ren}}$, so that no wave function renormalisation constants appear explicitly.

## 4.1 Effective Hamiltonian for Neutral Kaon Mixing

The time-dependent mixing of the two weak interaction eigenstates $|K^0\rangle$ and $|\bar{K}^0\rangle$ of the neutral $K$ meson is described in the Wigner-Weisskopf [73] approximation by

$$i\frac{d}{dt}\begin{pmatrix}|K^0(t)\rangle\\|\bar{K}^0(t)\rangle\end{pmatrix} = \left(M - i\frac{\Gamma}{2}\right)\begin{pmatrix}|K^0(t)\rangle\\|\bar{K}^0(t)\rangle\end{pmatrix}, \tag{4.1}$$

where the effective Hamiltonian $H = M - i\Gamma/2$ has been split into the Hermitian $2\times 2$ mass matrix $M$ and decay matrix $\Gamma$.
The weak interaction eigenstates do not coincide with the mass eigenstates, which can be found by diagonalising $M - i\Gamma/2$. We write them as

$$|K_S\rangle = p|K^0\rangle + q|\bar{K}^0\rangle\,, \qquad |K_L\rangle = p|K^0\rangle - q|\bar{K}^0\rangle\,, \tag{4.2}$$

the subscripts $S$ and $L$ referring to the short and long lifetimes of the two eigenstates, respectively. Here $p$ and $q$ are two complex coefficients which fulfill the relation $|p|^2 + |q|^2 = 1$. $K^0 - \bar{K}^0$ mixing corresponds to the off-diagonal element $M_{12}$ of the mass matrix $M$, given by

$$2m_K M_{12}^* = \langle \bar{K}^0|\mathcal{H}_{\text{eff}}^{|\Delta S|=2}|K^0\rangle\,, \tag{4.3}$$

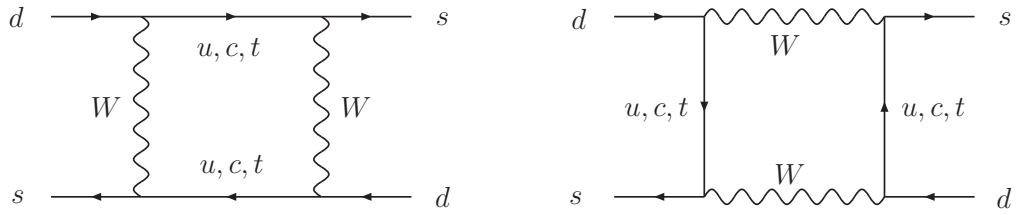


Figure 4.1: Leading order Standard Model contribution to $K$–$\bar{K}$ mixing. The electroweak would-be Goldstone bosons have to be taken into account, too.

$m_K$ being the average mass of the neutral $K$ mesons. The effective Hamiltonian valid at energy scales below the charm threshold $\mu_c = \mathcal{O}(m_c)$ is given by

$$\mathcal{H}_{\text{eff}}^{|\Delta S|=2} = \frac{G_F^2}{4\pi^2} M_W^2 \left[\lambda_c^2 \eta_1 S(x_c) + \lambda_t^2 \eta_2 S(x_t) + 2\lambda_c \lambda_t \eta_3 S(x_c, x_t)\right] b(\mu) \tilde{Q}_{S2} + \text{h.c.}\,. \tag{4.4}$$

Here

$$\tilde{Q}_{S2} = (\overline{s}_L \gamma_\mu d_L) \otimes (\overline{s}_L \gamma^\mu d_L) \tag{4.5}$$

is the only contributing dimension-eight operator at this scale, $G_F$ is the Fermi constant and $M_W$ the $W$-boson mass. The coefficient functions in square brackets are scale independent apart from the the common factor $b(\mu)$. Its $\mu$-dependence has to cancel against the corresponding scale dependence of the hadronic matrix element of $\tilde{Q}_{S2}$, which is parameterised by the so-called bag parameter $B_K(\mu)$:

$$\langle K^0 | \tilde{Q}_{S2} | \bar{K}^0 \rangle = \frac{2}{3} f_K^2 m_K^2 \frac{\hat{B}_K}{b(\mu)}\,, \tag{4.6}$$

where $f_K$ the Kaon decay constant, and we have introduced the scale invariant combination $\hat{B}_K = b(\mu) B_K(\mu)$.
The leading order contribution to the $|\Delta S| = 2$ transition in the Standard Model is shown in Figure 4.1. The different quark contributions are proportional to different products of CKM factors $\lambda_i = V_{is}^* V_{id}$, where the index $i$ runs over the up-type quarks $i = u, c, t$. The unitarity of the CKM matrix allows us to eliminate the factor $\lambda_u = -\lambda_c - \lambda_t$, such that the connected and truncated Green's functions in the Standard Model can be divided into

$$\tilde{G} = \lambda_c^2 \tilde{G}^c + \lambda_t^2 \tilde{G}^t + 2\lambda_c \lambda_t \tilde{G}^{ct}\,, \tag{4.7}$$

where the tilde denotes Fourier transformation. If we expand the Green's functions in $\alpha_s$ as

$$\tilde{G}^j = \tilde{G}^{j(0)} + \frac{\alpha_s}{4\pi} \tilde{G}^{j(1)} + \mathcal{O}(\alpha_s^2)\,, \tag{4.8}$$

the leading order Standard Model contribution is given by

$$i\tilde{G}^{ct} = \frac{G_F^2}{4\pi^2} M_W^2 S(x_c, x_t) \langle \tilde{Q}_{\text{S2}} \rangle^{(0)}\,, \tag{4.9}$$

$$i\tilde{G}^{j} = \frac{G_F^2}{4\pi^2} M_W^2 S(x_j) \langle \tilde{Q}_{\text{S2}} \rangle^{(0)}\,, \quad j = c, t\,, \tag{4.10}$$

in terms of the Inami-Lim [8] functions

$$S(x_j, x_k) = \tilde{S}(x_j, x_k) - \tilde{S}(x_j, 0) - \tilde{S}(x_k, 0) + \tilde{S}(0, 0)\,, \tag{4.11}$$

$$S(x_j) = S(x_j, x_j)\,. \tag{4.12}$$

Here the results of the box diagrams $\tilde{S}(x_j, x_k)$ with internal quarks $j$ and $k$ are given by

$$S(x_c) = x_c + \mathcal{O}(x_c^2)\,, \tag{4.13}$$

$$S(x_t) = \frac{4x_t - 11x_t^2 + x_t^3}{4(1-x_t)^2} - \frac{3x_t^3 \log x_t}{2(1-x_t)^3}\,, \tag{4.14}$$

$$S(x_c, x_t) = -x_c \log x_c + x_c F(x_t) + \mathcal{O}(x_c^2 \log x_c)\,, \tag{4.15}$$

where we have defined the mass ratio $x_i \equiv m_i^2/M_W^2$ and the up-quark mass has been set to zero. The function $F$ is defined as

$$F(x_t) = \frac{x_t^2 - 8x_t + 4}{4(1-x_t)^2} \log x_t - \frac{3x_t}{4(1-x_t)}\,. \tag{4.16}$$

In Equations (4.13) to (4.15) we have kept only terms linear in the small ratio $x_c$, but all orders in $x_t$.

The parameters $\eta_1$, $\eta_2$, and $\eta_3$ contain the higher order corrections to the leading order Inami-Lim functions. We will have a closer look at the three individual terms in the following.

However, it is helpful to get a clear picture of the structure of the effective Hamiltonian (4.4) first. Let us therefore recall the two main reasons for the effective field theory approach. First, an amplitude with external quark states makes not much sense from a physical point of view. We rather would like to evaluate matrix elements, between external hadron states, of an effective Hamiltonian defined at an energy scale below the charm quark threshold, which contains all effects of the physics at high energy scales.

Second, this effective Hamiltonian, when calculated to some higher order in the coupling constant $\alpha_s$ in fixed-order perturbation theory, contains products of the coupling constant and large logarithms of the form $\alpha_s(\mu_c)\log(\mu_c^2/\mu_W^2) = \mathcal{O}(1)$, which invalidate the use of fixed-order perturbation theory. Therefore we use renormalisation-group improved perturbation theory, which sums these logarithms to all orders by means of the renormalisation group equations.

It is useful to think of renormalisation-group improved perturbation theory as a reordering of the perturbation series, such that we consider terms of the form $(\alpha_s \log)^n$, $n \in \mathbb{N}$, as belonging to the same order in the expansion. In this framework, the sum of these terms over all $n$ is called the LL approximation (see also Section 2.5). Correspondingly the sum of all terms proportional to $\alpha_s(\alpha_s \log)^n$ and $\alpha_s^2(\alpha_s \log)^n$ constitute the NLL and the NNLL approximation, respectively. If there appears a logarithm already at the leading order $\mathcal{O}(\alpha_s^0)$ in fixed-order perturbation theory, the leading-log approximation corresponds to a sum of terms proportional to $(\alpha_s \log)^n \log \simeq (\alpha_s \log)^n/\alpha_s$, and so on. With this understanding we will now examine the effective Hamiltonian (4.4).

Let us start with the LO charm-top contribution $S(x_c, x_t)$. It contains a logarithm of the form $\log(m_c^2/M_W^2)$, which we rewrite as

$$\log \frac{m_c^2}{M_W^2} = \log \frac{\mu_c^2}{M_W^2} - \log \frac{\mu_c^2}{m_c^2}\,. \tag{4.17}$$

We identify the term proportional to the large logarithm $\log(\mu_c^2/M_W^2)$ as the first term in the LL approximation, where these contributions are summed to all orders in $\alpha_s$. By contrast, the term containing the small logarithm $\log(\mu_c^2/m_c^2)$ and the non-logarithmic part already belong to the NLL approximation. Similarly, including NLO $\alpha_s$ corrections to $S(x_c, x_t)$ will produce terms proportional to $\alpha_s \log^2(\mu_c^2/M_W^2)$, belonging to LL, $\alpha_s \log(\mu_c^2/M_W^2)$, belonging to the NLL, and $\alpha_s \log^0(\mu_c^2/M_W^2)$, belonging to the NNLL approximation. We see that the LO Inami-Lim function actually contains LL and NLL contributions. We also see already here that for the NNLL analysis we have to perform only a two-loop matching calculation.
The situation is different for the top contribution. Here only heavy particles contribute to the LO Standard Model amplitude, and correspondingly no large logarithms appear in the Inami-Lim function $S(x_t)$. Nevertheless, when we compute the NLO $\alpha_s$ corrections, terms proportional to $\alpha_s \log(\mu_c^2/M_W^2)$ will emerge, and these together with the leading order constant terms will be summed by the renormalisation group in a LL analysis, whereas at the NLL level we also sum the constant $\alpha_s$ terms.
Let us finally examine the pure charm-quark contribution. Because of the presence of the charm quark, we would naively expect a large logarithm $\log x_c$ in $S(x_c)$, as in the case of the mixed charm-top contribution. The absence of this term is caused by the GIM mechanism. Accordingly, the structure of this term is similar to $S(x_t)$. In particular, a NNLL analysis requires a three-loop matching calculation.

## 4.2 Outline of the NNLL Analysis

The calculation consists of three main steps. First we have to construct an effective theory by integrating out the heavy particles at the electroweak scale. We match the Standard Model Green's functions to those in the effective theory, thus obtaining the initial condition for the Wilson coefficients in the effective theory. In a second step, these Wilson coefficients are evolved down to the charm-quark scale using the renormalisation group equations. In the third and last step, we match the four-flavour theory onto an effective theory where also the charm quark has been removed as a dynamical degree of freedom. Now only the single operator (4.5) contributes, and we then perform the renormalisation group running down to the scales where the hadronic matrix elements are computed.
In the remaining part of this section we specify the effective theories which we need in our analysis by displaying the corresponding effective Lagrangians explicitly.
We obtain the effective Lagrangian valid between the electroweak and the bottom-quark scale by removing the top quark and the $W$ boson as dynamical degrees of freedom from the Standard Model Lagrangian. It reads in terms of the renormalised Wilson coefficients

$$\begin{aligned}\mathcal{L}_{\text{eff}}^{|\Delta S|=2} = &-\frac{4G_F}{\sqrt{2}} \sum_{i=1}^{6} C_i \left[ \sum_{j=1}^{2} Z_{ij} \sum_{k,l=u,c} V_{ks}^* V_{ld} Q_j^{kl} - \lambda_t \sum_{j=3}^{6} Z_{ij} Q_j \right] \\ &- \frac{G_F^2}{4\pi^2} \lambda_t^2 \tilde{C}_{\text{S2}}^t \tilde{Z}_{\text{S2}} \tilde{Q}_{\text{S2}} - 8 G_F^2 \lambda_c \lambda_t \left[ \sum_{k=1}^{2} \sum_{l=1}^{6} C_k C_l \hat{Z}_{kl,7} + \tilde{C}_7 \tilde{Z}_{77} \right] \tilde{Q}_7 \,. \end{aligned} \tag{4.18}$$

Here the first line represents the $|\Delta S| = 1$ part of the effective Lagrangian, whereas the second line contains the $|\Delta S| = 2$ contributions. The first term in the second line is related to a single insertion of $\tilde{Q}_{\text{S2}}$, induced by the top quark contribution to the SM amplitude.

The remaining terms arise from the mixing of insertions of two $|\Delta S| = 1$ operators into the operator $\tilde{Q}_7$. There is no $\lambda_c^2$ contribution to the Wilson coefficient of a $|\Delta S| = 2$ operator because of the GIM mechanism.
We will also need the Green's functions in this effective theory to the second order in $G_F$. They are given by

$$\left\langle T \exp\left[ i \int d^d x \mathcal{L}_{\text{eff}}^{|\Delta S|=2}(x) \right] \right\rangle \Bigg|_{|\Delta S|=2} = -i \left\langle H^c + H^t + H^{ct} \right\rangle + \mathcal{O}(G_F^3)\,, \tag{4.19}$$

where

$$H^c(x) = \lambda_c^2 2 G_F^2 \sum_{i,i',j,j'=1}^{2} C_i C_j \underbrace{Z_{ii'} Z_{jj'} \mathcal{O}'_{i'j'}(x)}_{=:\mathcal{O}_{ij}(x)}\,, \tag{4.20}$$

$$H^t(x) = \lambda_t^2 \frac{G_F^2}{4\pi^2} \tilde{C}^t_{\text{S2}} \tilde{Z}_{\text{S2}} \tilde{Q}_{\text{S2}}(x)\,, \tag{4.21}$$

$$\begin{aligned} H^{ct}(x) = {} & \lambda_c \lambda_t 8 G_F^2 \\ & \times \left[ \sum_{i=1}^{2} \sum_{j=1}^{6} C_i C_j \underbrace{\left( \sum_{i'=1}^{2} \sum_{j'=1}^{6} Z_{ii'} Z_{jj'} \mathcal{R}'_{i'j'}(x) + \hat{Z}_{ij,7} \tilde{Q}_7(x) \right)}_{=:\mathcal{R}_{ij}(x)} + \tilde{C}_7 \tilde{Z}_{77} \tilde{Q}_7(x) \right]. \end{aligned} \tag{4.22}$$

Here we have defined the following structures involving two $|\Delta S| = 1$ operators,

$$\mathcal{O}'_{ij}(x) = \frac{-i}{2} \int d^d y T\Big[ Q_i^{cc}(x) Q_j^{cc}(y) - Q_i^{cu}(x) Q_j^{uc}(y) - Q_i^{uc}(x) Q_j^{cu}(y) + Q_i^{uu}(x) Q_j^{uu}(y) \Big]\,, \tag{4.23}$$

$$\mathcal{R}'_{ij}(x) = \begin{cases} \frac{-i}{2} \int d^d y T\Big[ 2 Q_i^{uu}(x) Q_j^{uu}(y) - Q_i^{uc}(x) Q_j^{cu}(y) - Q_i^{cu}(x) Q_j^{uc}(y) \Big]\,, j = 1, 2\,, \\ \frac{-i}{2} \int d^d y T\Big[ \Big( Q_i^{uu} - Q_i^{cc} \Big)(x) Q_j(y) + Q_j(x) \Big( Q_i^{uu} - Q_i^{cc} \Big)(y) \Big]\,, j = 3 \ldots 6\,, \end{cases} \tag{4.24}$$

where the special form of the linear combinations is an implication of the GIM mechanism, which we already have put in explicitly. Note that for the sake of readability we have not displayed terms proportional to unphysical operators in Equations (4.18) and (4.20) to (4.24), which are needed for the renormalisation as discussed in Chapter 2.
The effective theory valid between the bottom- and the charm-quark scale looks exactly the same. The only difference is induced by the presence of penguin operators, which explicitly depend on all light quark fields.
Below the charm-quark scale, the charm quark is removed as a dynamical degree of freedom. As a consequence, the $|\Delta S| = 1$ operators can now be dropped from the effective Lagrangian, because the matrix elements of double insertions of these operators are at most proportional to $m_s^2$, and we have already neglected such contributions by setting the external momenta to zero. The effective Lagrangian is thus given by

$$\mathcal{L}_{\text{eff}}^{|\Delta S|=2} = -\frac{G_F^2}{4\pi^2} \left[ \lambda_c^2 \tilde{C}^c_{S2}(\mu) + \lambda_t^2 \tilde{C}^t_{S2}(\mu) + \lambda_c \lambda_t \tilde{C}^{ct}_{S2}(\mu) \right] \tilde{Z}_{S2} \tilde{Q}_{S2} \tag{4.25}$$

and now only contains the $|\Delta S| = 2$ operator $\tilde{Q}_{S2}$.
We now have all the tools at hand which are needed for a NNLL analysis, and we proceed to the calculation of $\eta_3$ in the next section.

## 4.3 Calculation of $\eta_3$

In this section we give the details of the calculation of $\eta_3$ in the NNLL approximation. We start with the determination of the initial conditions for the Wilson coefficients at the electroweak scale. Afterwards we use the renormalisation group equations to evolve them down to the charm-quark scale. Finally we determine the charm-top contribution to the Wilson coefficient of the operator $\tilde{Q}_{S2}$ by a matching calculation at the charm-quark scale.
It has been observed in the NLO calculation that the contribution of the penguin operators to $\eta_3$ is small, typically of the order of one percent [17]. We expect the same effect at the NNLO level and we will therefore neglect the penguin contributions to the NNLO corrections (but we will keep them in the NLO part). To be specific, we set the parts of the NNLO anomalous dimensions related to the mixing of penguin operators among themselves and into the dimension-eight operators to zero, and similarly, we drop the corresponding terms of the initial conditions.
For the same reason, we can neglect the contribution of the operator $Q_{\text{eom}}$, which receives the same initial conditions as $Q_4$. The mixing of this operator into $\tilde{Q}_7$ vanishes up to NLO. It is also not needed for the NNLO matching, because it multiplies a vanishing renormalisation factor. Therefore no additional contributions related to keeping $Q_{\text{eom}}$ in the operator basis arise.

### 4.3.1 Initial Conditions at the Electroweak Scale

The initial conditions for the Wilson coefficients of the dimension-six operators are available in the literature. In our basis, where we can use a naive anticommuting $\gamma_5$, the results up to second order in $\alpha_s$ for $C_1$ and $C_2$ and to first order in $\alpha_s$ for $C_3, \ldots, C_6$ read [46, 74]:

$$\begin{aligned}
C_1(\mu) &= (15 + 6\,L_W)\,\frac{\alpha_s(\mu)}{4\pi} + \left(\frac{\alpha_s(\mu)}{4\pi}\right)^2 \left(\frac{7987}{72} + \frac{17}{3}\pi^2 - \widetilde{T}_0(x_t) + \frac{475}{6}L_W + 17\,L_W^2\right), \\
C_2(\mu) &= 1 + \left(\frac{\alpha_s(\mu)}{4\pi}\right)^2 \left(\frac{127}{18} + \frac{4}{3}\pi^2 + \frac{46}{3}L_W + 4\,L_W^2\right), \\
C_3(\mu) &= 0\,, \qquad C_4(\mu) = \frac{\alpha_s(\mu)}{4\pi}\left(\widetilde{E}_0(x_t) + \frac{2}{3}L_W\right), \qquad C_5(\mu) = 0\,, \qquad C_6(\mu) = 0\,,
\end{aligned} \tag{4.26}$$

where $x_t = m_t^2/M_W^2$, and we have introduced the abbreviation $L_W = \log(\mu^2/M_W^2)$, which we will also use in the remaining part of this chapter. The one-loop Inami-Lim function $\widetilde{E}_0(x_t)$ is [8]

$$\widetilde{E}_0(x_t) = \frac{8 - 42x_t + 35x_t^2 - 7x_t^3}{12(x_t-1)^3} - \frac{4 - 16x_t + 9x_t^2}{6(x_t-1)^4}\ln x_t\,, \tag{4.27}$$

while the one-loop function $\widetilde{T}_0(x_t)$ is given by [74]

$$\begin{aligned}\widetilde{T}_0(x_t) = \frac{112}{9} + 32x_t + \left(\frac{20}{3} + 16x_t\right)\ln x_t \\ - (8+16x_t)\sqrt{4x_t-1}\,\mathrm{Cl}_2\left(2\arcsin\left(\frac{1}{2\sqrt{x_t}}\right)\right),\end{aligned} \tag{4.28}$$

with $\mathrm{Cl}_2(x) = \mathrm{Im}[\mathrm{Li}_2(e^{ix})]$ and $\mathrm{Li}_2(x) = -\int_0^x dt\,\ln(1-t)/t$.
We will also need the initial conditions for the Wilson coefficients $C_\pm$ corresponding to the operators defined in Equation (2.38). They can be obtained from Equation (4.26) by the basis transformation described in Appendix C and read:

$$\begin{aligned}C_\pm^{(0)}(\mu) = 1\,, \quad C_\pm^{(1)}(\mu) = \pm\frac{1}{2}\left(1\mp\frac{1}{3}\right)(11+6\,L_W)\,, \\ C_\pm^{(2)}(\mu) = -\frac{1}{3600}\,(135677\mp 124095) + \frac{1}{18}\,(7\pm 51)\,\pi^2 \mp \frac{1}{2}\left(1\mp\frac{1}{3}\right)T(x_t) \\ - \frac{5}{36}\,(11\mp 249)\,L_W + \frac{1}{6}\,(7\pm 51)\,L_W^2\,.\end{aligned} \tag{4.29}$$

In addition we need the initial conditions for the Wilson coefficients of the evanescent dimension-six operators to order $\alpha_s$. The only contributing, non-zero coefficient is

$$C_{E_1^{(1)}}(\mu) = \frac{\alpha_s(\mu)}{4\pi}\left(\frac{3}{2} + \log\frac{\mu^2}{M_W^2}\right) + \mathcal{O}(\alpha_s^2)\,. \tag{4.30}$$

With these ingredients, we can now calculate the initial conditions for the Wilson coefficients of the dimension-eight operators. To this end we have to match the Green's functions in the Standard Model and the effective five-flavour theory. The matching condition can be derived from (4.7) and (4.22), which yields

$$2i\tilde{G}^{ct} = \frac{1}{\lambda_c\lambda_t}\left\langle H^{ct}\right\rangle = 8G_F^2\left(\sum_{i,j} C_iC_j\langle\mathcal{R}_{ij}\rangle + \tilde{C}_7\tilde{Z}_{77}\langle\tilde{Q}_7\rangle\right). \tag{4.31}$$

In order to determine the initial conditions, we have to compute the finite parts of Feynman diagrams of the type shown in Figures 4.1 and 4.2. To this end, we perform a Taylor expansion in the charm-quark mass of all propagators corresponding to a charm-quark field. The constant terms cancel because of the GIM mechanism, whereas the terms proportional to $m_c^2$ give the leading non-vanishing contribution we are interested in. This procedure leads to massless vacuum integrals on the right side of the equation, such that only terms proportional to tree-level matrix elements remain. Some of these terms multiply divergent renormalisation constants and correspond to infrared divergences in the effective theory. They exactly cancel the corresponding infrared divergent terms on the left side of the equation, leaving us with a finite result. Note that in this way also the initial conditions of evanescent operators enter the matching conditions. On the other hand, the initial condition for $C_4$ is actually not needed, because it multiplies a vanishing renormalisation constant.
Expanding the dimension-eight Wilson coefficient as

$$\tilde{C}_7(\mu) = \frac{4\pi}{\alpha_s(\mu)}\tilde{C}_7^{(0)}(\mu) + \tilde{C}_7^{(1)}(\mu) + \frac{\alpha_s(\mu)}{4\pi}\tilde{C}_7^{(2)}(\mu)\,, \tag{4.32}$$

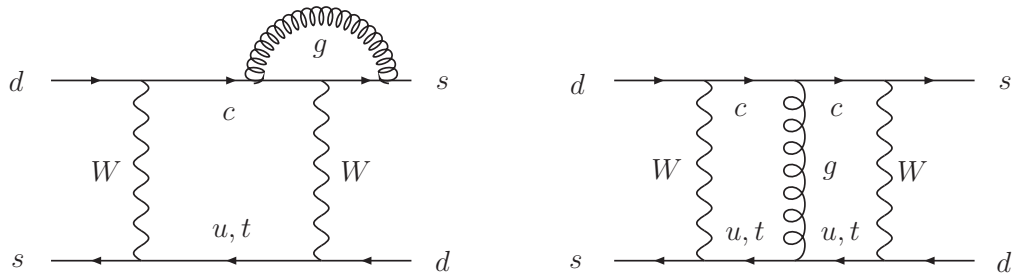


Figure 4.2: Sample two-loop Feynman diagrams contributing to the matching at the electroweak scale.

we get the following results:

$$\begin{aligned}
&\tilde{C}_7^{(0)}(\mu) = 0\,, \qquad \tilde{C}_7^{(1)}(\mu) = F(x_t) + \frac{3}{2} - L_W\,, \\
&\tilde{C}^{(0)}_{\tilde{E}_7^{(1)}}(\mu) = 0\,, \qquad \tilde{C}^{(1)}_{\tilde{E}_7^{(1)}}(\mu) = \frac{3-5x_t}{8(1-x_t)} - \frac{\log x_t}{4(x_t-1)^2} + \frac{1}{4}L_W\,,
\end{aligned} \tag{4.33}$$

$$\begin{aligned}
\tilde{C}_7^{(2)}(\mu) = {} & -7L_W + \left( \frac{-47x_t^2 + 31x_t - 56}{6(x_t-1)^2} + \frac{\left(5x_t^3 - 21x_t^2 + 60x_t - 20\right)\log x_t}{2(x_t-1)^3} \right) L_W \\
& - \frac{\left(-12x_t^5 + 34x_t^4 + 9x_t^3 + 33x_t^2 + 116x_t - 36\right)\log^2 x_t}{12(x_t-1)^3} \\
& + \frac{2\pi^2\left(6x_t^4 + x_t^3 - 65x_t^2 - 8\right)}{36x_t^2} + \frac{-3\left(24x_t^4 + 11x_t^3 + 168x_t^2 + 21x_t - 32\right)}{36(x_t-1)^2 x_t} \\
& + \frac{12\left(6x_t^6 - 11x_t^5 - 8x_t^4 - 29x_t^3 + 23x_t^2 - 16x_t + 8\right)\mathrm{Li}_2(1-x_t)}{36(x_t-1)^2 x_t^2} \\
& + \frac{\left(-12x_t^5 + 27x_t^4 + 23x_t^3 + 150x_t^2 - 108x_t + 16\right)\log x_t}{6(x_t-1)^3 x_t}
\end{aligned} \tag{4.34}$$

The first line in Equation (4.33) agrees with the results obtained already by Herrlich and Nierste in [17] after the corresponding change of scheme. The remaining results are new.

### 4.3.2 Structure of the Renormalisation Group Equations

Having determined the initial conditions for the Wilson coefficients, the next step consists of the renormalisation group evolution to lower scales. The renormalisation group equations relevant for the Wilson coefficient $\tilde{C}_7$ are given in their general form in Equation (2.112):

$$\mu\frac{d}{d\mu}\tilde{C}_7(\mu) = \tilde{C}_7(\mu)\tilde{\gamma}_{77} + \sum_{k=1}^{2}\sum_{n=1}^{6} C_k(\mu)C_n(\mu)\hat{\gamma}_{kn,7}\,, \tag{4.35}$$

with the anomalous dimension tensor given by Equation (2.113), corresponding to the definition (4.22). Writing

$$\hat{\gamma}_7^T = \begin{pmatrix} \hat{\gamma}_{11,7} & \hat{\gamma}_{12,7} & \hat{\gamma}_{13,7} & \hat{\gamma}_{14,7} & \hat{\gamma}_{15,7} & \hat{\gamma}_{16,7} \\ \hat{\gamma}_{21,7} & \hat{\gamma}_{22,7} & \hat{\gamma}_{23,7} & \hat{\gamma}_{24,7} & \hat{\gamma}_{25,7} & \hat{\gamma}_{26,7} \end{pmatrix}, \tag{4.36}$$

we find explicitly

$$\hat{\gamma}_7^{T(0)} = \begin{pmatrix} -\frac{11}{9} & -\frac{2}{3} & 0 & 0 & -64 & \frac{8}{3} \\ -\frac{2}{3} & -2 & 0 & 0 & -48 & -16 \end{pmatrix}, \tag{4.37}$$

$$\hat{\gamma}_7^{T(1)} = \begin{pmatrix} -\frac{1234}{27} & \frac{197}{9} & -\frac{44}{3} & -\frac{125}{9} & \frac{2024}{3} & -\frac{532}{9} \\ \frac{197}{9} & -\frac{118}{3} & -56 & -\frac{182}{3} & -160 & \frac{416}{3} \end{pmatrix}, \tag{4.38}$$

$$\hat{\gamma}_7^{T(2)} = \begin{pmatrix} -\frac{7350917}{5832} + \frac{10001}{243}N_f + \frac{748}{27}\zeta_3 & -\frac{375335}{1944} + \frac{1082}{81}N_f + \frac{442}{9}\zeta_3 & * & * & * & * \\ -\frac{375335}{1944} + \frac{1082}{81}N_f + \frac{442}{9}\zeta_3 & -\frac{332533}{324} + \frac{542}{27}N_f + \frac{1072}{3}\zeta_3 & * & * & * & * \end{pmatrix}. \tag{4.39}$$

The LO and NLO result agrees with the literature [17] after the corresponding change of the operator basis, described in Section 3.4. As discussed at the beginning of Section 4.3, we have not calculated the entries corresponding to the penguin operators at NNLO. We marked them with a star and set them to zero in the following.

We will now adapt the Equations (4.35) to the special case under consideration, as described in [17], and thereby achieve a considerable simplification.

The first step consists of changing the basis $(Q_1, Q_2)$ of the subspace of current-current operators, as given in Equation (2.37) to the basis $(Q_+, Q_-)$, given in Equation (2.38). This basis is chosen such that the corresponding anomalous dimension matrix is diagonal. Now, beyond the leading order, the anomalous dimensions depend on the choice of the evanescent operators. A judicious choice of the evanescent operators ensures the diagonal form of the anomalous dimension up to the NNLO; it has been given explicitly in [25]. We have collected the definition of the evanescent operators as well as the explicit formulas needed to transform the anomalous dimensions to the new basis in Appendix C.

Equation (4.35) then splits into two independent equations

$$\mu\frac{d}{d\mu}\tilde{C}_7^{\pm}(\mu) = \tilde{C}_7^{\pm}(\mu)\tilde{\gamma}_{77} + \sum_{k=1}^{6} C_{\pm}(\mu)C_k(\mu)\hat{\gamma}_{\pm k,7}\,, \tag{4.40}$$

if we accordingly decompose the Wilson coefficient

$$\tilde{C}_7(\mu) = \tilde{C}_7^{+}(\mu) + \tilde{C}_7^{-}(\mu)\,, \tag{4.41}$$

where this decomposition is completely arbitrary and preserved by the renormalisation group evolution. For instance, we may choose $\tilde{C}_7^{+}(\mu) = \tilde{C}_7(\mu)$ and $\tilde{C}_7^{-}(\mu) = 0$.

As the second step, we observe that these two equations, together with the renormalisation group equations for the operators $Q_1, \ldots, Q_6$, are equivalent to the following system of eight equations [17]

$$\mu\frac{d}{d\mu}D = \gamma^T D\,, \tag{4.42}$$

where the anomalous dimension matrix and the Wilson coefficients are now given by

$$\gamma^T = \begin{pmatrix} \gamma_Q^T & 0 & 0 \\ \tilde{\gamma}_{+,7}^T & \tilde{\gamma}_{77} - \gamma_+ & 0 \\ \tilde{\gamma}_{-,7}^T & 0 & \tilde{\gamma}_{77} - \gamma_- \end{pmatrix}, \qquad D(\mu) = \begin{pmatrix} C(\mu) \\ \tilde{C}_7^{+}(\mu)/C_+(\mu) \\ \tilde{C}_7^{-}(\mu)/C_-(\mu) \end{pmatrix}. \tag{4.43}$$

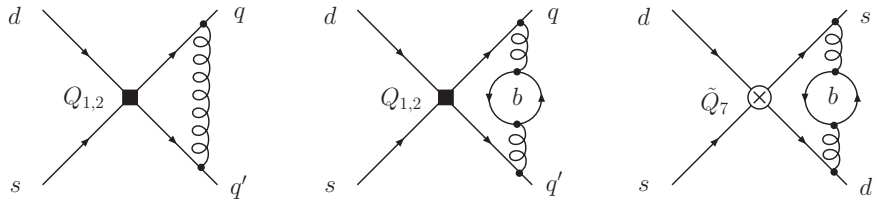


Figure 4.3: Feynman diagrams relevant for the threshold corrections at the bottom quark scale. The one-loop diagram of $Q_1$ and $Q_2$ is the same in both theories, whereas at the two-loop level they receive non-trivial corrections from virtual bottom quarks. The same applies to insertions the operator $\tilde{Q}_7$.

The advantage of this expression is that is has the form of a renormalisation group equation for a single operator insertion, and the explicit solution constructed in Section 2.5 can be used.
The anomalous dimension matrix $\gamma_Q$ of the operators $Q_1, \ldots, Q_6$ and the anomalous dimensions of the operators $Q_+$ and $Q_-$ can be found in Appendix B. The anomalous dimension for the double insertion of either $Q_+$ or $Q_-$ and one of the operators $Q_1, \ldots, Q_6$ is denoted as

$$\tilde{\gamma}^T_{\pm,7} = (\tilde{\gamma}_{\pm1,7}, \tilde{\gamma}_{\pm2,7}, \tilde{\gamma}_{\pm3,7}, \tilde{\gamma}_{\pm4,7}, \tilde{\gamma}_{\pm5,7}, \tilde{\gamma}_{\pm6,7}) \,, \tag{4.44}$$

and we find

$$\tilde{\gamma}^{T(0)}_{+,7} = \left(-\frac{5}{3}, -2, 0, 0, -96, -8\right) \,, \qquad \tilde{\gamma}^{T(0)}_{-,7} = (1, 0, 0, 0, 48, -8) \,, \tag{4.45}$$

$$\begin{aligned} \tilde{\gamma}^{T(1)}_{+,7} &= \left(-37, -9, -52, -\frac{163}{3}, 248, \frac{92}{3}\right) \,, \\ \tilde{\gamma}^{T(1)}_{-,7} &= \left(54, -33, -4, -\frac{19}{3}, -664, \frac{356}{3}\right) \,, \end{aligned} \tag{4.46}$$

$$\begin{aligned} \tilde{\gamma}^{T(1)}_{+,7} &= \left(-\frac{71410531}{48600} + \frac{2141}{54} N_f + \frac{544}{9}\,\zeta_3, -\frac{10461817}{16200} + \frac{484}{27} N_f + \frac{862}{3}\,\zeta_3, 0, 0, 0, 0\right) \,, \\ \tilde{\gamma}^{T(1)}_{-,7} &= \left(\frac{8545111}{6075} - \frac{4595}{162} N_f - \frac{34}{3}\,\zeta_3, -\frac{1140557}{3240} - \frac{284}{81} N_f + 70\,\zeta_3, 0, 0, 0, 0\right) \,, \end{aligned} \tag{4.47}$$

at LO, NLO, and NNLO, respectively. These expressions can be obtained from Equations (4.37) to (4.39) by the basis transformation described in Section 3.4.

### 4.3.3 Threshold Corrections at the Bottom-Quark Scale

When we pass the bottom-quark threshold, we must perform a proper matching between the effective theories with five and four flavours. In contrast to the NLO calculation, where only the penguin operators were affected, at the NNLO level also the matching of the current-current and the dimension-eight operators is non-trivial. The source of such contributions are virtual bottom quarks in two-loop matrix elements of the form shown in Figure 4.3.
The threshold correction is computed as usual by matching the Green's functions in the two theories at the scale $\mu_b = \mathcal{O}(m_b)$, where $m_b$ is the bottom-quark mass.
Let us write the equality of the corresponding amplitudes at the matching scale $\mu_f$ in a general form as

$$C_{f-1}(\mu_f)\langle Q_{f-1}\rangle(\mu_f) = C_f(\mu_f)\langle Q_f\rangle(\mu_f) \,, \tag{4.48}$$

the variables with subscripts $f$ and $f-1$ belonging to the $f$- and $f-1$-flavour theory. At the bottom-quark scale, we have $f=5$. We parameterise the matrix elements of the operators in the following way as an expansion in the coupling constant defined in the corresponding $f$-flavour theory:

$$\langle Q_f\rangle(\mu_f) = \langle Q_f\rangle^{(0)}\left(1+\frac{\alpha_s^{(f)}(\mu_f)}{4\pi}r_f^{(1)}(\mu_f)+\left(\frac{\alpha_s^{(f)}(\mu_f)}{4\pi}\right)^2 r_f^{(2)}(\mu_f)\right). \tag{4.49}$$

An additional subtlety arises here, because the strong coupling constant also gets a non-trivial matching correction at a flavour threshold. Up to the NLO approximation we have the relation [75–77]

$$\alpha_s^{(f-1)}(\mu_f) = \alpha_s^{(f)}(\mu_f)\left(1-\frac{\alpha_s^{(f)}(\mu_f)}{4\pi}\frac{2}{3}\log\frac{\mu_f^2}{m_f^2}\right), \tag{4.50}$$

which we use to express all quantities in terms of the coupling constant $\alpha_s^{(f-1)}(\mu_f)$, which is appropriate for the renormalisation group running in the effective theory with $f-1$ flavours. Here $m_f = m_f(m_f)$ is the $\overline{\text{MS}}$ mass of the quark which is integrated out.
We now introduce the discontinuities

$$\delta C^{(k)}(\mu_f) = C_f^{(k)}(\mu_f) - C_{f-1}^{(k)}(\mu_f), \quad \delta r^{(k)}(\mu_f) = r_f^{(k)}(\mu_f) - r_{f-1}^{(k)}(\mu_f), \tag{4.51}$$

of the Wilson coefficients and the matrix elements, respectively, and find for the solution of Equation (4.48):

$$\begin{aligned}
\delta C^{(0)}(\mu_f) &= 0, \qquad \delta C^{(1)}(\mu_f) = -C_f^{(0)}(\mu_f)\delta r^{(1)}(\mu_f),\\
\delta C^{(2)}(\mu_f) &= -C_f^{(1)}(\mu_f)\left(\delta r^{(1)}(\mu_f)+\frac{2}{3}\log\frac{\mu_f^2}{m_f^2}\right)\\
&\quad - C_f^{(0)}(\mu_f)\left(\delta r^{(2)}(\mu_f) - \delta r^{(1)}(\mu_f) r_{f-1}^{(1)}(\mu_f) + \frac{2}{3}\log\frac{\mu_f^2}{m_f^2} r_f^{(1)}(\mu_f)\right).
\end{aligned} \tag{4.52}$$

At NLO, only the matrix elements of the penguin operators get non-vanishing contributions. They can be obtained from

$$\delta r_Q^{(1)}(\mu_b) = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&-\frac{2}{3}\log\frac{\mu_b^2}{m_b^2}&0&0\\ 0&0&0&0&0&0\\ 0&0&0&4-\frac{20}{3}\log\frac{\mu_b^2}{m_b^2}&0&0 \end{pmatrix}, \tag{4.53}$$

where $\delta r_Q$ denotes the difference of the matrix elements in the subspace of dimension-six operators. At NNLO, we neglect the contributions of the penguin operators, but now the

current-current as well as the dimension-eight operators receive non-vanishing corrections. We find

$$\delta\tilde{C}_7^{(2)}(\mu_b) = -\frac{2}{3}\log^2\left(\frac{\mu_b^2}{m_b^2}\right) - \frac{32}{9}\log\left(\frac{\mu_b^2}{m_b^2}\right) - \frac{2\zeta_2}{3} + \frac{1}{54}\,. \tag{4.54}$$

The matching corrections for $C_\pm$ can be extracted from [25]:

$$\delta C_\pm^{(2)}(\mu_b) = -\left(\frac{2}{3}\log\frac{\mu_b^2}{m_b^2}C_\pm^{(1)}(\mu_b) \mp \left(1 \mp \frac{1}{3}\right)\left(\frac{59}{36} + \frac{1}{3}\log\frac{\mu_b^2}{m_b^2} + \log^2\frac{\mu_b^2}{m_b^2}\right) C_\pm^{(0)}(\mu_b)\right). \tag{4.55}$$

### 4.3.4 Matching at the Charm-Quark Scale

At the scale $\mu_c = \mathcal{O}(m_c)$ the charm quark is removed from the theory as a dynamical degree of freedom, and the effective Lagrangian is now given by Equation (4.25). Equating the Green's functions in both theories at the charm-quark scale leads to the matching condition

$$\sum_{i=+,-}\sum_{j=1}^{6} C_i(\mu_c)C_j(\mu_c)\langle\mathcal{R}_{ij}\rangle(\mu_c) + \tilde{C}_7(\mu_c)\tilde{Z}_{77}\langle\tilde{Q}_7\rangle(\mu_c) = \frac{1}{32\pi^2}\tilde{C}_{S2}^{ct}(\mu_c)\tilde{Z}_{S2}\langle\tilde{Q}_{S2}\rangle(\mu_c)\,, \tag{4.56}$$

which we use to determine the Wilson coefficient $\tilde{C}_{S2}^{ct}(\mu)$. To proceed, we parameterise the matrix elements in the following way:

$$\langle\tilde{Q}_7\rangle = r_7\langle\tilde{Q}_7\rangle^{(0)}\,, \quad \langle\tilde{Q}_{S2}\rangle = r_{S2}\langle\tilde{Q}_{S2}\rangle^{(0)}\,, \quad \text{and} \quad \langle\mathcal{R}_{ij}\rangle(\mu_c) = \frac{m_c^2(\mu_c)}{32\pi^2}r_{ij,S2}\langle\tilde{Q}_{S2}\rangle^{(0)}\,. \tag{4.57}$$

If we take into account the explicit factor of $m_c^2/g_s^2$ in the definition of $\tilde{Q}_7$ and expand the Wilson coefficient $\tilde{C}_{S2}^{ct}$ as

$$\tilde{C}_{S2}^{ct}(\mu) = \frac{4\pi}{\alpha_s(\mu)}\tilde{C}_{S2}^{ct(0)}(\mu) + \tilde{C}_{S2}^{ct(1)}(\mu) + \frac{\alpha_s(\mu)}{4\pi}\tilde{C}_{S2}^{ct(2)}(\mu)\,, \tag{4.58}$$

we find the following contributions to the matching:

$$\tilde{C}_{S2}^{ct(0)}(\mu_c) = 2m_c^2(\mu_c)\tilde{C}_7^{(0)}(\mu_c)\,, \tag{4.59}$$

$$\begin{aligned}\tilde{C}_{S2}^{ct(1)}(\mu_c) = {}& 2m_c^2(\mu_c)\left[\tilde{C}_7^{(0)}(\mu_c)(r_7^{(1)} - r_{S2}^{(1)}) + \tilde{C}_7^{(1)}(\mu_c)\right] \\ & + m_c^2(\mu_c)C_i^{(0)}(\mu_c)C_j^{(0)}(\mu_c)r_{ij,S2}^{(0)}\,,\end{aligned} \tag{4.60}$$

$$\begin{aligned}\tilde{C}_{S2}^{ct(2)}(\mu_c) = {}& 2m_c^2(\mu_c)\Big[\tilde{C}_7^{(0)}(\mu_c)(r_7^{(2)} - r_{S2}^{(2)} - (r_7^{(1)} - r_{S2}^{(1)})r_{S2}^{(1)}) \\ & + \tilde{C}_7^{(1)}(\mu_c)(r_7^{(1)} - r_{S2}^{(1)}) + \tilde{C}_7^{(2)}(\mu_c)\Big] + m_c^2(\mu_c)\Big[C_i^{(0)}(\mu_c)C_j^{(0)}(\mu_c) \\ & (r_{ij,S2}^{(1)} - r_{ij,S2}^{(0)}r_{S2}^{(1)}) + C_i^{(0)}(\mu_c)C_j^{(1)}(\mu_c)r_{ij,S2}^{(0)} + C_i^{(1)}(\mu_c)C_j^{(0)}(\mu_c)r_{ij,S2}^{(0)}\Big]\,,\end{aligned} \tag{4.61}$$

In order to evaluate these equations, we have to compute the finite parts of one- and two-loop Feynman diagrams of the type shown in Figure 3.2. In this way we find for $r_{ij,S2}$ at LO:

$$r_{ij,S2}^{(0),T}(\mu_c) = \begin{pmatrix} \frac{11}{9}\log\left(\frac{\mu_c^2}{m_c^2}\right) - \frac{11}{6} & \frac{2}{3}\log\left(\frac{\mu_c^2}{m_c^2}\right) - 1 \\ \frac{2}{3}\log\left(\frac{\mu_c^2}{m_c^2}\right) - 1 & 2\log\left(\frac{\mu_c^2}{m_c^2}\right) - 3 \\ 0 & 0 \\ 0 & 0 \\ 64\log\left(\frac{\mu_c^2}{m_c^2}\right) + 192 & 48\log\left(\frac{\mu_c^2}{m_c^2}\right) + 144 \\ -\frac{8}{3}\log\left(\frac{\mu_c^2}{m_c^2}\right) - 8 & 16\log\left(\frac{\mu_c^2}{m_c^2}\right) + 48 \end{pmatrix}. \tag{4.62}$$

This result agrees with the one obtained in [17] after the appropriate change of scheme. A NLO matching calculation yields

$$\begin{aligned} &r_{ij,S2}^{(1),T}(\mu_c) \\ &= \begin{pmatrix} \frac{23}{3}\log^2\left(\frac{\mu_c^2}{m_c^2}\right) - \frac{14}{27}\log\left(\frac{\mu_c^2}{m_c^2}\right) - \frac{19}{27}\zeta_2 - \frac{2579}{108} & -\log^2\left(\frac{\mu_c^2}{m_c^2}\right) - \frac{356}{9}\log\left(\frac{\mu_c^2}{m_c^2}\right) - \frac{10}{9}\zeta_2 - \frac{1499}{36} \\ -\log^2\left(\frac{\mu_c^2}{m_c^2}\right) - \frac{356}{9}\log\left(\frac{\mu_c^2}{m_c^2}\right) - \frac{10}{9}\zeta_2 - \frac{1499}{36} & 6\log^2\left(\frac{\mu_c^2}{m_c^2}\right) + \frac{76}{3}\log\left(\frac{\mu_c^2}{m_c^2}\right) + \frac{8}{3}\zeta_2 + \frac{515}{6} \\ \frac{4}{3}\log^2\left(\frac{\mu_c^2}{m_c^2}\right) + \frac{68}{3}\log\left(\frac{\mu_c^2}{m_c^2}\right) + \frac{47}{3} & -8\log^2\left(\frac{\mu_c^2}{m_c^2}\right) + 8\log\left(\frac{\mu_c^2}{m_c^2}\right) + 14 \\ -\frac{59}{9}\log^2\left(\frac{\mu_c^2}{m_c^2}\right) - \frac{229}{9}\log\left(\frac{\mu_c^2}{m_c^2}\right) - \frac{955}{36} & -\frac{26}{3}\log^2\left(\frac{\mu_c^2}{m_c^2}\right) + \frac{26}{3}\log\left(\frac{\mu_c^2}{m_c^2}\right) + \frac{19}{6} \\ \frac{1096}{3}\log^2\left(\frac{\mu_c^2}{m_c^2}\right) + \frac{1096}{3}\log\left(\frac{\mu_c^2}{m_c^2}\right) - \frac{3554}{3} & -32\log^2\left(\frac{\mu_c^2}{m_c^2}\right) - 896\log\left(\frac{\mu_c^2}{m_c^2}\right) - 4376 \\ -\frac{884}{9}\log^2\left(\frac{\mu_c^2}{m_c^2}\right) - \frac{4988}{9}\log\left(\frac{\mu_c^2}{m_c^2}\right) - \frac{12959}{9} & \frac{112}{3}\log^2\left(\frac{\mu_c^2}{m_c^2}\right) + \frac{688}{3}\log\left(\frac{\mu_c^2}{m_c^2}\right) - \frac{2900}{3} \end{pmatrix}. \end{aligned} \tag{4.63}$$

In these results, the index $i$ corresponds to the operators $Q_1$ and $Q_2$, and we still have to perform the transformation to the diagonal basis. We find

$$r_{\pm j,S2}^{(0),T}(\mu_c) = \begin{pmatrix} \frac{5}{3}\log\left(\frac{\mu^2}{m_c^2}\right) - \frac{5}{2} & \frac{3}{2} - \log\left(\frac{\mu^2}{m_c^2}\right) \\ 2\log\left(\frac{\mu^2}{m_c^2}\right) - 3 & 0 \\ 0 & 0 \\ 0 & 0 \\ 96\log\left(\frac{\mu^2}{m_c^2}\right) + 288 & -48\log\left(\frac{\mu^2}{m_c^2}\right) - 144 \\ 8\log\left(\frac{\mu^2}{m_c^2}\right) + 24 & 8\log\left(\frac{\mu^2}{m_c^2}\right) + 24 \end{pmatrix} \tag{4.64}$$

at LO, and

$$r^{(1),T}_{\pm j,S2}(\mu_c) = \begin{pmatrix} 7\log^2\left(\frac{\mu^2}{m_c^2}\right) - \frac{299}{9}\log\left(\frac{\mu^2}{m_c^2}\right) - \frac{13}{9}\zeta_2 - \frac{1517}{36} & -8\log^2\left(\frac{\mu^2}{m_c^2}\right) - 13\log\left(\frac{\mu^2}{m_c^2}\right) + \frac{1}{3}\zeta_2 + \frac{21}{2} \\ 3\log^2\left(\frac{\mu^2}{m_c^2}\right) - \frac{94}{3}\log\left(\frac{\mu^2}{m_c^2}\right) + \frac{2}{3}\zeta_2 + \frac{343}{12} & 3\log^2\left(\frac{\mu^2}{m_c^2}\right) + \frac{146}{3}\log\left(\frac{\mu^2}{m_c^2}\right) + 2\zeta_2 + \frac{277}{4} \\ -4\log^2\left(\frac{\mu^2}{m_c^2}\right) + 28\log\left(\frac{\mu^2}{m_c^2}\right) + 25 & -4\log^2\left(\frac{\mu^2}{m_c^2}\right) - 20\log\left(\frac{\mu^2}{m_c^2}\right) - 11 \\ -\frac{37}{3}\log^2\left(\frac{\mu^2}{m_c^2}\right) - \frac{59}{3}\log\left(\frac{\mu^2}{m_c^2}\right) - \frac{293}{12} & \frac{11}{3}\log^2\left(\frac{\mu^2}{m_c^2}\right) + \frac{85}{3}\log\left(\frac{\mu^2}{m_c^2}\right) + \frac{331}{12} \\ 344\log^2\left(\frac{\mu^2}{m_c^2}\right) - \frac{1816}{3}\log\left(\frac{\mu^2}{m_c^2}\right) - 5222 & -376\log^2\left(\frac{\mu^2}{m_c^2}\right) - \frac{2024}{3}\log\left(\frac{\mu^2}{m_c^2}\right) - 306 \\ -\frac{220}{3}\log^2\left(\frac{\mu^2}{m_c^2}\right) - \frac{3988}{9}\log\left(\frac{\mu^2}{m_c^2}\right) - \frac{6629}{3} & \frac{332}{3}\log^2\left(\frac{\mu^2}{m_c^2}\right) + \frac{5764}{9}\log\left(\frac{\mu^2}{m_c^2}\right) + 1147 \end{pmatrix}. \tag{4.65}$$

at NLO. This result completes the matching onto the three-flavour theory. Now only a single operator contributes, and the renormalisation group evolution below the charm quark scale is the same for the top, the charm, and the charm-top contribution.

## 4.4 RGE below the Charm Threshold

The effective Hamiltonian valid below the charm-quark threshold contains only the single operator $\tilde{Q}_{S2}$. The renormalisation group evolution is therefore the same for the three Wilson coefficients $\tilde{C}^j_{S2}$, where $j = c, t, ct$, and described by the evolution matrix corresponding to the anomalous dimension of $\tilde{Q}_{S2}$:

$$\tilde{C}^j_{S2}(\mu) = U(\mu, \mu_c)\tilde{C}^j_{S2}(\mu_c)\,. \tag{4.66}$$

By comparing (4.4) and (4.25), we see that we can express the coefficients $\eta_1$, $\eta_2$, $\eta_3$ as

$$\eta_1 = \frac{1}{m_c^2(\mu_c)}\tilde{C}^{(c)}_{S2}(\mu_c)\left[\alpha_s(\mu_c)\right]^{a_+^{[3]}} K_+^{-1}(\mu_c)\,, \tag{4.67a}$$

$$\eta_2 = \frac{1}{M_W^2 S(x_t(\mu_c))}\tilde{C}^{(t)}_{S2}(\mu_c)\left[\alpha_s(\mu_c)\right]^{a_+^{[3]}} K_+^{-1}(\mu_c)\,, \tag{4.67b}$$

$$\eta_3 = \frac{1}{2M_W^2 S(x_c(\mu_c), x_t(\mu_{tW}))}\tilde{C}^{(ct)}_{S2}(\mu_c)\left[\alpha_s(\mu_c)\right]^{a_+^{[3]}} K_+^{-1}(\mu_c)\,. \tag{4.67c}$$

The $\mu$-dependence present in (4.67) is absorbed into $b(\mu)$, which equals

$$b(\mu) = \left[\alpha_s(\mu)\right]^{-a_+^{[3]}} K_+(\mu)\,, \tag{4.68}$$

where

$$K_+(\mu) = \left(1 + J_+^{(1)[3]}\frac{\alpha_s(\mu)}{4\pi} + J_+^{(2)[3]}\left(\frac{\alpha_s(\mu)}{4\pi}\right)^2\right), \tag{4.69}$$

and the exponent $a_+$ is defined in Equation (2.88).

# Chapter 5

# Electroweak Corrections to $K^+ \to \pi^+ \nu\bar{\nu}$

## 5.1 Introduction

The rare decay $K^+ \to \pi^+ \nu\bar{\nu}$ is both theoretically very clean and highly sensitive to short-distance physics and thus plays an outstanding role among flavour-changing neutral current processes both in the Standard Model and its extensions [78–80]. Together with the process $K_L \to \pi^0 \nu\bar{\nu}$ it provides a critical test for the CKM mechanism of CP violation, while it probes operators generated by new physics at energy scales of several TeV [81].
In the Standard Model the decay $K^+ \to \pi^+ \nu\bar{\nu}$ proceeds through $Z$-penguin and electroweak box diagrams of $\mathcal{O}(G_F^2)$ (see Figure 5.1) which exhibit a power-like GIM mechanism. This implies that non-perturbative effects are severely suppressed and, related to this, that the low-energy effective Hamiltonian [19, 22]

$$\mathcal{H}_{\text{eff}} = \frac{4G_F}{\sqrt{2}} \frac{\alpha}{2\pi \sin^2 \theta_W} \sum_{l=e,\mu,\tau} \Big( \lambda_c X^l(x_c) + \lambda_t X(x_t) \Big) (\bar{s}_L \gamma_\mu d_L)(\bar{\nu}_{lL} \gamma^\mu \nu_{lL}) \tag{5.1}$$

involves to an excellent approximation only a single effective operator. Here $G_F$ is the Fermi constant, $\alpha$ the electromagnetic coupling and $\theta_W$ the weak mixing angle. The sum is over all lepton flavours, $\lambda_i = V_{is}^* V_{id}$ comprises the CKM factors and $f_L$ represents left-handed fermion fields.
The function $X(x_t)$, where $x_t = m_t^2(\mu_t)/M_W^2$ and $m_t^2(\mu_t)$ is the top quark $\overline{\text{MS}}$ mass, describes the matching contributions of internal top quarks to the operator in Equation (5.1), where the matching is carried out at the scale $\mu_t = \mathcal{O}(m_t)$. Sample diagrams are shown in Figure 5.1. The energy scales involved are of the order of the electroweak scale or higher, while both the QCD and QED anomalous dimensions of the corresponding operator vanish. Hence $X(x_t)$ can be calculated within fixed-order perturbation theory. The relevant $Z$-penguin and electroweak box diagrams are known through NLO in QCD [8, 18–20]. The inclusion of these $\mathcal{O}(\alpha_s)$ corrections allowed to reduce the $\pm 6\%$ uncertainty related to the top quark matching scale $\mu_t = \mathcal{O}(m_t)$ present in the leading order (LO) formula down to $\pm 1\%$. The leading term in the large top quark mass expansion of the electroweak two-loop corrections typically amounts to a per mil correction for the branching ratio if the $\overline{\text{MS}}$ definition of $\alpha$ and $\sin^2 \theta_W$ is used, while the uncertainty related to unknown sub-leading electroweak contributions is conservatively estimated to be $\pm 2\%$ [21].

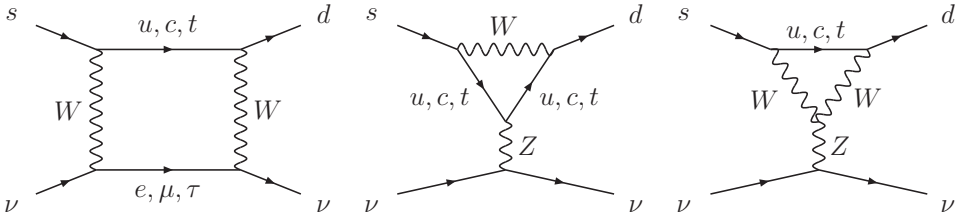


Figure 5.1: Examples of leading-order diagrams contributing to the decay $K^+ \to \pi^+ \nu\bar{\nu}$ in the Standard Model.

The function $X^l(x_c)$, relevant only for $K^+ \to \pi^+ \nu\bar{\nu}$, depends on the charm quark $\overline{\text{MS}}$ mass through the parameter $x_c$, conventionally defined as

$$x_c = \frac{m_c^2(\mu_c)}{M_W^2}\,. \tag{5.2}$$

As now both high-energy and low-energy scales, namely $\mu_W = \mathcal{O}(M_W)$ and $\mu_c = \mathcal{O}(m_c)$ are involved, a complete renormalisation group analysis of $X^\ell(x_c)$ is required. In this manner, large logarithms $\ln(\mu_c^2/\mu_W^2)$ are summed to all orders in $\alpha_s$. At LO such an analysis has been performed in [12]. The large scale uncertainty due to $\mu_c$ of $\pm 26\%$ in this result was reduced by a NLO [19, 22] and a subsequent NNLO calculation [23–25] to $\pm 2.5\%$. While the QCD part of the calculation has reached a high level of sophistication no QED or electroweak corrections have been included so far. We close this gap by calculating the LO and NLO logarithmic QED corrections as well as fixing the scheme of the input parameters in $\sin^2\theta_W$ and $\alpha$ by an electroweak matching calculation. The latter point can be exemplified by noting that the charm quark contribution is mediated by a double insertion of two dimension-six operators. This results in a contribution of $\mathcal{O}(G_F^2)$ – the second power of $G_F$ resides in $x_c$ – plus electroweak corrections. Yet the leading result of Equation (5.1) can only approximate the electroweak corrections for a specific choice of the renormalisation scheme for the prefactor of the charm quark contribution, expressed as $\alpha/\sin^2\theta_W$. While it is expected that using $\overline{\text{MS}}$ parameters renormalised at the electroweak scale would approximate the electroweak corrections best [82] only an explicit calculation can provide a definite result. In this work we normalise all dimension-six operators to $G_F$. Thus, we replace the parameter $x_c$ in Equation (5.2) with the unfamiliar definition

$$x_c = \sqrt{2}\frac{\sin^2\theta_W}{\pi\alpha} G_F m_c^2(\mu_c)\,, \tag{5.3}$$

which only at tree level equals the familiar ratio $m_c^2(\mu_c)/M_W^2$.

The hadronic matrix element of the low-energy effective Hamiltonian can be extracted from the well-measured $K_{l3}$ decays, including isospin breaking and long-distance QED radiative corrections [27–29]. After summation over the three neutrino flavours the resulting branching

ratio for $K^+ \to \pi^+ \nu\bar{\nu}$ can be written as[1] [19, 22, 31]

$$B\left(K^+ \to \pi^+ \nu\bar{\nu}(\gamma)\right) = \kappa_+(1+\Delta_{\mathrm{EM}})\left[\left(\frac{\mathrm{Im}\lambda_t}{\lambda^5}X(x_t)\right)^2 + \left(\frac{\mathrm{Re}\lambda_c}{\lambda}\left(P_c(X)+\delta P_{c,u}\right) + \frac{\mathrm{Re}\lambda_t}{\lambda^5}X(x_t)\right)^2\right]. \quad (5.4)$$

The parameter

$$P_c(X) = \frac{1}{\lambda^4}\left(\frac{2}{3}X^e(x_c) + \frac{1}{3}X^\tau(x_c)\right) \quad (5.5)$$

describes the short-distance contribution of the charm quark, where $\lambda = |V_{us}|$. The charm quark contribution of dimension-eight operators at the charm quark scale $\mu_c$ [30] combined with long distance contributions were calculated in Reference [31] to be

$$\delta P_{c,u} = 0.04 \pm 0.02\,. \quad (5.6)$$

The quoted error on this value can in principle be reduced with the help of lattice QCD [83]. The remaining long distance corrections are factored out into the following two parameters: $\kappa_+$ contains higher-order electroweak corrections to the low energy matrix elements, and $\Delta_{\mathrm{EM}}$ denotes long distance QED corrections. A detailed analysis of these contributions to NLO and partially NNLO in chiral perturbation theory has been performed by Mescia and Smith in [28], who found the numerical values $\kappa_+ = (0.5173 \pm 0.0025) \times 10^{-10}(\lambda/0.225)^8$ and $\Delta_{\mathrm{EM}} = -0.3\%$.

## 5.2 Electroweak Corrections in the Charm Sector

The charm quark contribution involves several different scales and the corresponding large logarithms have to be summed using renormalisation group improved perturbation theory. Keeping terms to $\mathcal{O}(\alpha_s)$ and $\mathcal{O}(\alpha/\alpha_s)$ the expansion of the parameter $P_c(X)$ reads

$$P_c(X) = \frac{4\pi}{\alpha_s(\mu_c)}P_c^{(0)}(X) + P_c^{(1)}(X) + \frac{\alpha_s(\mu_c)}{4\pi}P_c^{(2)}(X) + \frac{4\pi\alpha}{\alpha_s^2(\mu_c)}P_c^{(e)}(X) + \frac{\alpha}{\alpha_s(\mu_c)}P_c^{(es)}(X). \quad (5.7)$$

The LO term $P_c^{(0)}(X)$, the NLO term $P_c^{(1)}(X)$, and the NNLO term $P_c^{(2)}(X)$ have been calculated in [12], in [19, 22], and in [25] respectively. The main goal of the this paper is to present the electroweak corrections $P_c^{(e)}(X)$ and $P_c^{(es)}(X)$.
The calculation is performed in two steps. First, at the scale $\mu_W = \mathcal{O}(M_W)$ the Standard Model is matched to an effective theory where the top quark, the $W$ boson, and the $Z$ boson are integrated out, but the charm quark is still a dynamical degree of freedom. Second, at the scale $\mu_c = \mathcal{O}(m_c)$ the charm quark is integrated out and the effective Hamiltonian in Equation (5.1) is obtained.

[1] We have omitted a term which arises from the implicit sum over lepton flavours in $P_c$ because it amounts to only 0.2% of the branching fraction.

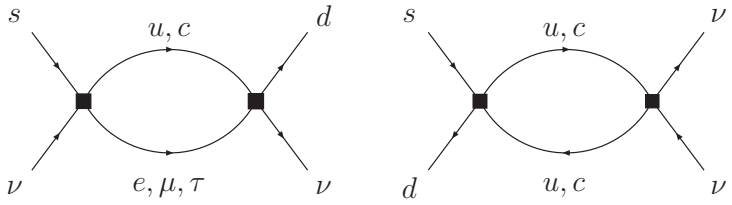


Figure 5.2: Leading-order diagrams for the mixing of various dimension-six operators into $\tilde{Q}_\nu$ (see text for details).

After integrating out the particles at the electroweak scale the effective Hamiltonian containing the dimension-six operators takes the following form:

$$\mathcal{H}_{\text{eff}}^{\text{dim.6}} = \frac{4G_F}{\sqrt{2}}\Bigg( C_W(\mu) \sum_{q=u,c} (V_{qs} Q_{3q} + V_{qd}^\star Q_{4q}) \\ + \lambda_c \sum_{j=\pm} C_j(\mu)(Q_j^c - Q_j^u) + \frac{1}{2} C_A(\mu) Q_A + \frac{1}{2} C_V(\mu) Q_V \Bigg). \tag{5.8}$$

Here we kept only operators relevant for the decay $K^+ \to \pi^+ \nu\bar{\nu}$. They can be found in Equations (2.38), (2.40) and (2.41). The evanescent operators are not displayed explicitly in Equation (5.8), but clearly have to be included in the sum. They are given in Equations (2.46) and (2.49).
These operators mix via double insertions into the operator given in Equation (5.1). Traditionally one distinguishes the box contribution which comprises double insertions of the semileptonic operators $Q_{3q}$ and $Q_{4q}$ (see Figure 5.2, left side) and the penguin contribution which comprises double insertions of the current-current-type operators $Q_\pm$ and the operators $Q_A$ and $Q_V$ (Figure 5.2, right side). The relevant dimension-eight part of the effective Hamiltonian can then be written as

$$\mathcal{H}_{\text{eff}}^{\text{charm}} = \left(2G_F^2 \lambda_c C_\nu^B(\mu) + G_F^2 \lambda_c C_\nu^P(\mu)\right) \tilde{Q}_\nu, \tag{5.9}$$

where the operator $\tilde{Q}_\nu$ is defined in Equation (2.44), while $C_\nu^B$ and $C_\nu^P$ denote the box and penguin contribution, respectively.
The renormalisation group analysis proceeds in several steps. The initial conditions for the renormalisation group equations, which govern the running of the Wilson coefficients, are calculated in Sec. 5.2.1. The anomalous dimensions are computed in Sec. 5.2.2. After integrating out the bottom and the charm quark, the theory is matched onto the low energy effective Hamiltonian of Equation (5.1). The relevant results are collected in Sec. 5.2.3. In Sec. 5.2.4 the pieces are put together to give the final result for $P_c(X)$.
We have computed all Feynman diagrams in this chapter using self-written `FORM` [68] routines and independently using `Mathematica`. All the QCD corrections relevant to a NNLO analysis of $P_c(X)$ are given in [25] and references therein.

### 5.2.1 Initial Conditions

The Wilson coefficients are found by matching the one light-particle irreducible Green's functions in the full and the effective theory at the electroweak scale $\mu_W^2 \sim M_W^2$. We use the

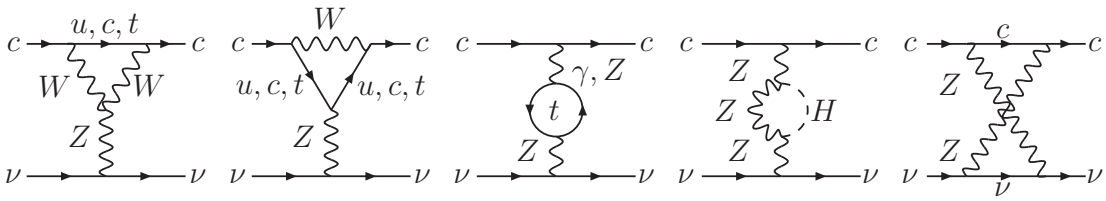


Figure 5.3: Feynman diagrams contributing to the NLO matching for $C_A$.

$\overline{\text{MS}}$ scheme for both theories and remark that a finite field redefinition for the light particles ensures the correct normalisation of the kinetic term in the effective theory. In the box sector only $C_W$ and in the penguin sector only $C_\pm$ and $C_{A/V}$ receive electroweak corrections at the order considered here (see Figure 5.3). We expand the Wilson coefficients in powers of the coupling constants

$$C(\mu) = C^{(0)}(\mu) + \frac{\alpha_s(\mu)}{4\pi} C^{(1)}(\mu) + \frac{\alpha}{\alpha_s} C^{(e)}(\mu) + \frac{\alpha}{4\pi} C^{(es)}(\mu) \tag{5.10}$$

and use a similar expansion for any quantity in the following, unless explicitly stated otherwise.

We normalise the Wilson coefficients $C_W$, $C_\pm$, and $C_{A/V}$ to the muon decay constant $G_F$ [84]. In this way most of the radiative corrections cancel, including all terms dependent on $m_t$ and $M_H$ in the case of $C_W$ and $C_\pm$. All our matching calculations have been performed in the generalised $R_\xi$ gauge for the photon field and in the case of $C_A$ also for the $W$ and $Z$ fields as a check of our results.
At the one-loop level a neutrino-photon Green's function is generated which contributes to $C_V$ via the equations of motion. Yet $Q_V$ does not mix into $Q_\nu$ and the Wilson coefficient $C_V$ is not needed.
For the relevant electroweak corrections we find

$$C_\pm^{(0)}(\mu_W) = 1\,, \qquad C_\pm^{(e)}(\mu_W) = 0\,, \qquad C_\pm^{(es)}(\mu_W) = -\frac{22}{9} - \frac{4}{3}\ln\frac{\mu_W^2}{M_Z^2}\,, \tag{5.11}$$

in agreement with Reference [85, 86],

$$C_W^{(0)}(\mu_W) = 1\,, \qquad C_W^{(e)}(\mu_W) \;\; = 0\,, \qquad C_W^{(es)}(\mu_W) = -\frac{11}{3} - 2\ln\frac{\mu_W^2}{M_Z^2}\,, \tag{5.12}$$

and

$$\begin{aligned} C_A^{(0)}(\mu_W) &= 1\,, \qquad C_A^{(e)}(\mu_W) = 0\,, \\ C_A^{(es)}(\mu_W) &= \frac{3m_t^2}{4s_w^2 M_W^2} + \frac{11s_w^2 - 6}{4s_w^2 c_w^2} - \frac{3}{4}\frac{M_W^2 - c_w^2 M_H^2}{(M_H^2 - M_W^2)s_w^4}\ln\frac{M_W^2}{M_Z^2} \\ &\quad + \frac{3M_H^4}{4(M_H^2 - M_W^2)(M_W^2 - c_w^2 M_H^2)}\ln\frac{M_H^2}{M_Z^2}\,. \end{aligned} \tag{5.13}$$

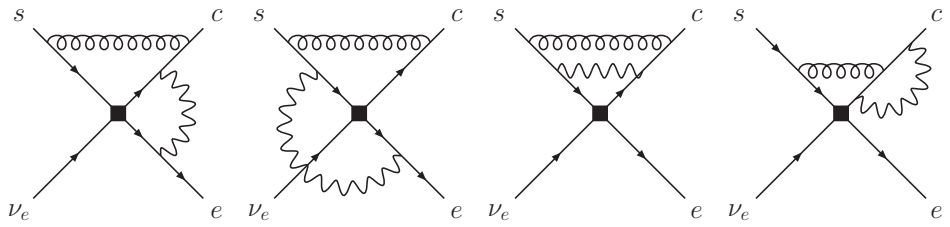


Figure 5.4: Sample two-loop diagrams contributing to the self mixing of $Q_{3c}$. Wavy lines denote photons, curly lines denote gluons.

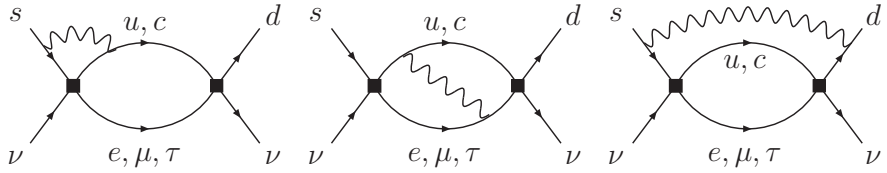


Figure 5.5: Sample diagrams for the NLO mixing of $Q_{3q}$ and $Q_{4q}$ into $\tilde{Q}_\nu$.

### 5.2.2 Anomalous Dimensions and RGE

The mixing of dimension-six into dimension-eight operators through double insertions leads in general to inhomogeneous renormalisation group equations [51]. In the box sector they are given by

$$\mu\frac{d}{d\mu}C_\nu^B(\mu) = \gamma_\nu C_\nu^B(\mu) + 4\gamma_\nu^B C_W(\mu)C_W(\mu)\,, \tag{5.14}$$

$$\mu\frac{d}{d\mu}C_W(\mu) = \gamma_W C_W(\mu)\,, \tag{5.15}$$

where $\gamma_W$ is the anomalous dimension of $Q_{3q}$, $\gamma_\nu$ encodes the running of $\tilde{Q}_\nu$, which stems solely from the running mass and coupling constant which in our definition multiply the $\tilde{Q}_\nu$ operator, and $\gamma_\nu^B$ is the anomalous dimension tensor of the mixing of the operators $Q_{3q}$ and $Q_{4q}$ into $\tilde{Q}_\nu$.
$\gamma_\nu$ is given in terms of the QCD $\beta$-function and the anomalous dimension of the charm quark mass by

$$\gamma_\nu^{(k)} = 2(\gamma_m^{(k)} - \beta_k). \tag{5.16}$$

The explicit values are given in Appendix B.
The remaining anomalous dimensions can be calculated from the pole parts of one- and two-loop diagrams, some of which are shown in Figs. 5.4 and 5.5, using the methods described in Chapter 2. We find the following values:

$$\gamma_\nu^{B(0)} = -8\,, \qquad \gamma_\nu^{B(e)} = 0\,, \qquad \gamma_\nu^{B(es)} = -\frac{316}{9}\,, \tag{5.17}$$

$$\gamma_W^{(0)} = 0\,, \qquad \gamma_W^{(e)} = -4\,, \qquad \gamma_W^{(es)} = 4\,. \tag{5.18}$$

$\gamma_\nu^{B(0)}$ is known for a long time (see [22] and references therein), and $\gamma_W^{(e)}$ and $\gamma_W^{(es)}$ have already been calculated in [84].

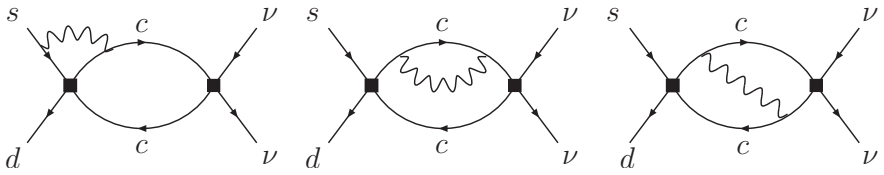


Figure 5.6: Sample diagrams for the NLO mixing of $Q_A$ and $Q_\pm$ into $\tilde{Q}_\nu$.

In order to solve the renormalisation group equations we perform a trick similar to the one used in Chapter 4, so that we can use the renormalisation group equations for single insertions also in our case. To this end, we rewrite Equation (5.15) as

$$\mu\frac{d}{d\mu}C_W^2(\mu) = 2\gamma_W^T C_W^2(\mu). \tag{5.19}$$

Then we can combine both equations (5.14) and (5.15) into a linear equation

$$\mu\frac{d}{d\mu}C_B(\mu) = \gamma_B^T C_B(\mu), \tag{5.20}$$

where

$$C_B(\mu) = \begin{pmatrix} 4C_W^2(\mu) \\ C_\nu^B(\mu) \end{pmatrix} \quad \text{and} \quad \gamma_B^T = \begin{pmatrix} 2\gamma_W & 0 \\ \gamma_\nu^B & \gamma_\nu \end{pmatrix}. \tag{5.21}$$

The renormalisation group equations for the penguin sector are given by

$$\mu\frac{d}{d\mu}C_\nu^P(\mu) = \gamma_\nu C_\nu^P(\mu) + 4\sum_{i=\pm}\gamma_{i,\nu}^P C_i(\mu)C_A(\mu), \tag{5.22}$$

$$\mu\frac{d}{d\mu}C_\pm(\mu) = \gamma_\pm^T C_\pm(\mu). \tag{5.23}$$

The anomalous dimension tensor $\gamma_{\pm,\nu}^P$ governs the mixing of the double insertion of $Q_\pm$ and $Q_A$ into $\tilde{Q}_\nu$ (see Figure 5.6), while $\gamma_\pm$ describes the self-mixing of $Q_\pm$ and was computed in [87]. The anomalous dimensions read:

$$\gamma_{\pm,\nu}^{P(0)} = 2(1\pm 3), \qquad \gamma_{\pm,\nu}^{P(e)} = 0, \qquad \gamma_{\pm,\nu}^{P(es)} = \frac{52}{3}(1\pm 3). \tag{5.24}$$

We have defined the matrix $\gamma_\pm^P$ as

$$\gamma_{\pm,\nu}^{P(k)} = -\frac{1}{2}\gamma_{\pm,\nu}^{A(k)} - \left(\frac{1}{2} - \frac{4}{3}\sin^2\theta_W\right)\gamma_{\pm,\nu}^{V(k)}, \tag{5.25}$$

with the superscripts $A$ and $V$ denoting the contributions stemming from double insertion of $(Q_\pm^q, Q_A^q)$ and $(Q_\pm^q, Q_V^q)$, respectively. The LO result agrees with [22, 25]. The other contributions are new.
The anomalous dimension of $Q_A$ vanishes and the renormalisation group equations in the penguin sector is the linear equation

$$\mu\frac{d}{d\mu}C_P(\mu) = \gamma_P^T C_P(\mu), \tag{5.26}$$

where

$$C_P(\mu) = \begin{pmatrix} 4C_+(\mu)C_A \\ 4C_-(\mu)C_A \\ C_\nu^P(\mu) \end{pmatrix}, \qquad \gamma_P^T = \begin{pmatrix} & & \gamma_{+,\nu}^P \\ & \gamma_\pm^T & \gamma_{-,\nu}^P \\ 0 & 0 & \gamma_\nu \end{pmatrix}. \tag{5.27}$$

The renormalisation group equations for single insertions can be solved explicitly using the method described in Chapter 2.

### 5.2.3 Below $\mu_c$

At $\mu_c$, i.e. the scale of the charm quark mass, the charm quark is integrated out and removed as a degree of freedom. All necessary matrix elements are given in [22, 25] – no new contributions arise to the orders considered here. There are some new terms stemming from the expansion of $m_c(\mu_c)$ about $m_c(m_c)$ in these expressions, though, and we collect these results for convenience.
The matching in the box sector leads to the following matrix elements:

$$r_\tau^{B(1)}(\mu_c) = 5 + \frac{4x_\tau}{1-x_\tau}\ln x_\tau + 4\ln\frac{\mu_c^2}{m_c^2}, \tag{5.28}$$

where $x_\tau = m_\tau^2/m_c^2$ and $m_c = m_c(\mu_c)$. Neglecting the lepton masses for the electron and muon, the above formula yields

$$r_{e,\mu}^{B(1)}(\mu_c) = 5 + 4\ln\frac{\mu_c^2}{m_c^2}. \tag{5.29}$$

We have defined the matrix elements for lepton flavour $l$ by

$$\langle Q_l^B(\mu_c)\rangle = \frac{\alpha_s(\mu_c)}{4\pi} r_l^{B(1)}(\mu_c)\langle\tilde{Q}_\nu\rangle^{(0)}, \tag{5.30}$$

where $\langle Q_l^B(\mu_c)\rangle$ denotes the double insertion of the operators in the box sector. In the penguin sector we find

$$r_\pm^{P(1)}(\mu_c) = (1 \pm 3)\left(1 - \ln\frac{\mu_c^2}{m_c^2}\right), \tag{5.31}$$

where

$$\langle Q_\pm^P(\mu_c)\rangle = \frac{\alpha_s(\mu_c)}{4\pi} r_\pm^{P(1)}(\mu_c)\langle\tilde{Q}_\nu\rangle^{(0)}, \tag{5.32}$$

and $\langle Q_\pm^P(\mu_c)\rangle$ denotes the double insertion of the operators in the penguin sector.

### 5.2.4 Final Analytic Expression for $P_c(X)$

Now all that remains to do is to combine all relevant terms and compute the box and penguin contributions to the function $X^l(x_c)$ defined in Equation (5.1). Here we closely follow [25]. Let us start with the box contribution. We expand the result as

$$C_B^l(\mu_c) = \kappa_c \frac{x_c(m_c)}{16}\left(\frac{4\pi}{\alpha_s(\mu_c)} C_B^{l(0)}(\mu_c) + \frac{4\pi\alpha}{\alpha_s(\mu_c)^2} C_B^{l(e)}(\mu_c) + \frac{\alpha}{\alpha_s(\mu_c)} C_B^{l(es)}(\mu_c)\right) \tag{5.33}$$

and express the running charm quark mass $m_c(\mu_c)$ in terms of initial condition $m_c(m_c)$,

$$x_c(\mu_c) = \kappa_c \left(1 + \frac{\alpha_s(\mu_c)}{4\pi}\xi_c^{(1)} + \frac{\alpha}{\alpha_s(\mu_c)}\xi_c^{(e)} + \frac{\alpha}{4\pi}\xi_c^{(es)}\right) x_c(m_c), \tag{5.34}$$

where we defined $\kappa_c = \eta_c^{(\gamma_m^{(0)}/\beta_0)}$ and $\eta_c = \alpha_s(\mu_c)/\alpha_s(m_c)$, with the individual contributions

$$\begin{aligned}
\xi_c^{(1)} &= \left(\frac{\gamma_m^{(1)}}{\beta_0} - \frac{\gamma_m^{(0)}\beta_1}{\beta_0^2}\right)(1-\eta_c^{-1}), \qquad \xi_c^{(e)} = \frac{\gamma_m^{(e)}}{\beta_0}(\eta_c - 1), \\
\xi_c^{(es)} &= \left(\frac{\gamma_m^{(es)}}{\beta_0} - \frac{\beta_{es}\gamma_m^{(0)}}{\beta_0^2} - \frac{\beta_1\gamma_m^{(e)}}{\beta_0^2}\right)\ln\eta_c + \frac{\gamma_m^{(e)}}{\beta_0}\left(\frac{\gamma_m^{(0)}\beta_1}{\beta_0^2} - \frac{\gamma_m^{(1)}}{\beta_0}\right)(1-\eta_c^{-1})(1-\eta_c).
\end{aligned} \tag{5.35}$$

We find the following expansion coefficients for $C_B^l$:

$$\begin{aligned}
C_B^{l(0)}(\mu_c) &= C_\nu^{B(0)}(\mu_c), \\
C_B^{l(e)}(\mu_c) &= C_\nu^{B(e)}(\mu_c) + C_\nu^{B(0)}(\mu_c)\xi_c^{(e)} + 4C_W^{(0)}(\mu_c)^2\rho_l^{B(e)}(\mu_c), \\
C_B^{l(es)}(\mu_c) &= C_\nu^{B(es)}(\mu_c) + C_\nu^{B(e)}(\mu_c)\xi_c^{(1)} + C_\nu^{B(1)}(\mu_c)\xi_c^{(e)} + C_\nu^{B(0)}(\mu_c)\xi_c^{(es)} \\
&\quad + 4C_W^{(0)}(\mu_c)^2\rho_l^{B(es)}(\mu_c) + 4C_W^{(0)}(\mu_c)^2\rho_l^{B(e)}(\mu_c)\xi_c^{(1)} \\
&\quad + 8C_W^{(0)}(\mu_c)C_W^{(e)}(\mu_c)\rho_l^{B(1)}(\mu_c) + 4C_W^{(0)}(\mu_c)^2\rho_l^{B(1)}(\mu_c)\xi_c^{(e)}.
\end{aligned} \tag{5.36}$$

We obtain the parameters $\rho_l^B$ by inserting the expansion of $m_c(\mu_c)$ into the expressions for $r_l^B$ (see Sec. 5.2.3):

$$\begin{aligned}
\rho_\tau^{B(1)} &= r_\tau^{B(1)}(m_c) + \frac{4}{x_\tau - \kappa_c}\left(\kappa_c\ln\kappa_c - \frac{x_\tau(1-\kappa_c)}{1-x_\tau}\ln x_\tau\right), \\
\rho_\tau^{B(e)} &= 0, \qquad \rho_\tau^{B(es)} = -\frac{4\kappa_c\xi_c^{(e)}\left[\kappa_c - x_\tau\left(1 - \ln\frac{x_\tau}{\kappa_c}\right)\right]}{(\kappa_c - x_\tau)^2}.
\end{aligned} \tag{5.37}$$

The corresponding expressions for the electron and the muon, where we can neglect the masses, are given by

$$\rho_{e,\mu}^{B(1)} = r_{e,\mu}^{B(1)}(m_c) - 4\ln\kappa_c, \qquad \rho_{e,\mu}^{B(e)} = 0, \qquad \rho_{e,\mu}^{B(es)} = -4\xi_c^{(e)}. \tag{5.38}$$

The penguin contribution to the function $X^l(x_c)$ can be obtained in the same way. Expanding the Wilson coefficients $C_P(\mu_c)$ as

$$C_P(\mu_c) = \kappa_c\frac{x_c(m_c)}{32}\left(\frac{4\pi}{\alpha_s(\mu_c)}C_P^{(0)}(\mu_c) + \frac{4\pi\alpha}{\alpha_s(\mu_c)^2}C_P^{(e)}(\mu_c) + \frac{\alpha}{\alpha_s(\mu_c)}C_P^{(es)}(\mu_c)\right), \tag{5.39}$$

| | | | | | |
|---|---|---|---|---|---|
| $M_W$ | $(80.403 \pm 0.029)$ GeV | [88] | $\alpha_s(M_Z)$ | $0.1176 \pm 0.0020$ | [88] |
| $M_Z$ | $(91.1876 \pm 0.0021)$ GeV | [88] | $\alpha(M_Z)$ | $1/127.9$ | [88] |
| $M_t$ | $(172.6 \pm 1.4)$ GeV | [89] | $\sin^2\theta_W^{\overline{\rm MS}}$ | $0.23122 \pm 0.00015$ | [88] |
| $m_b(m_b)$ | $(4.164 \pm 0.025)$ GeV | [90] | $G_F$ | $1.166\,37 \times 10^{-5}\text{GeV}^{-2}$ | [88] |
| $m_c(m_c)$ | $(1.286 \pm 0.013)$ GeV | [90] | $\lambda$ | $0.2255 \pm 0.0007$ | [91] |
| $M_H$ | $(155 \pm 40)$ GeV | – | $\lvert V_{cb}\rvert$ | $(4.15 \pm 0.09) \times 10^{-2}$ | [92] |
| $m_\tau$ | $(1776.99^{+0.29}_{-0.26})$ MeV | [88] | $\bar{\rho}$ | $0.141^{+0.029}_{-0.017}$ | [92] |
| | | | $\bar{\eta}$ | $0.343 \pm 0.016$ | [92] |

Table 5.1: Input parameters used in our numerical analysis.

we find the following contributions:

$$
\begin{aligned}
C_P^{(0)}(\mu_c) &= C_\nu^{P(0)}(\mu_c),\\
C_P^{(e)}(\mu_c) &= C_\nu^{P(e)}(\mu_c) + C_\nu^{P(0)}(\mu_c)\xi_c^{(e)} + 4C_A^{(0)}(\mu_c)\sum_{i=\pm} C_i^{(0)}(\mu_c)\rho_i^{P(e)}(\mu_c),\\
C_P^{(es)}(\mu_c) &= C_\nu^{P(es)}(\mu_c) + C_\nu^{P(e)}(\mu_c)\xi_c^{(1)} + C_\nu^{P(1)}(\mu_c)\xi_c^{(e)} + C_\nu^{P(0)}(\mu_c)\xi_c^{(es)}\\
&\quad + 4\sum_{i=\pm}\left(\rho_i^{P(es)}(\mu_c) + \rho_i^{P(e)}(\mu_c)\xi_c^{(1)} + \rho_i^{P(1)}(\mu_c)\xi_c^{(e)}\right) C_i^{(0)}(\mu_c)C_A^{(0)}(\mu_c)\\
&\quad + 4\sum_{i=\pm}\rho_i^{P(1)}(\mu_c)\left(C_i^{(e)}(\mu_c)C_A^{(0)}(\mu_c) + C_i^{(0)}(\mu_c)C_A^{(e)}(\mu_c)\right)\\
&\quad + 4\sum_{i=\pm}\rho_i^{P(e)}(\mu_c)C_i^{(1)}(\mu_c)C_A^{(0)}(\mu_c).
\end{aligned}
\tag{5.40}
$$

Again we obtain the parameters $\rho_i^P$ by inserting the expansion of $m_c(\mu_c)$ into the expressions for $r_i^P$:

$$
\rho_\pm^{P(1)} = r_\pm^{P(1)}(m_c) + (1 \pm 3)\ln\kappa_c, \qquad \rho_\pm^{P(e)} = 0, \qquad \rho_\pm^{P(es)} = (1 \pm 3)\xi_c^{(e)}. \tag{5.41}
$$

The final result for $X^l$ is then

$$
X^l(x_c) = C_P(\mu_c) + C_B^l(\mu_c). \tag{5.42}
$$

The corresponding expressions for $C_P(\mu_c)$ and $C_B^l(\mu_c)$ can be found in Equations (5.33) and (5.39), respectively. Equation (5.5) then yields the contribution to the branching fraction.

## 5.3 Final Results and Numerical Discussion

Having all necessary ingredients at hand we will discuss the numerical implications of our results, where we use the input parameters given in Table 5.1. Our numerical procedure follows closely the one of Reference [25]. In particular we use the numerical solution of the renormalisation group equations of the program `RunDec` [93] to compute $\alpha_s(\mu_c)$ from $\alpha_s(M_Z)$ and neglect all terms proportional to $\beta_{es}$. We have checked numerically that this is indeed justified.

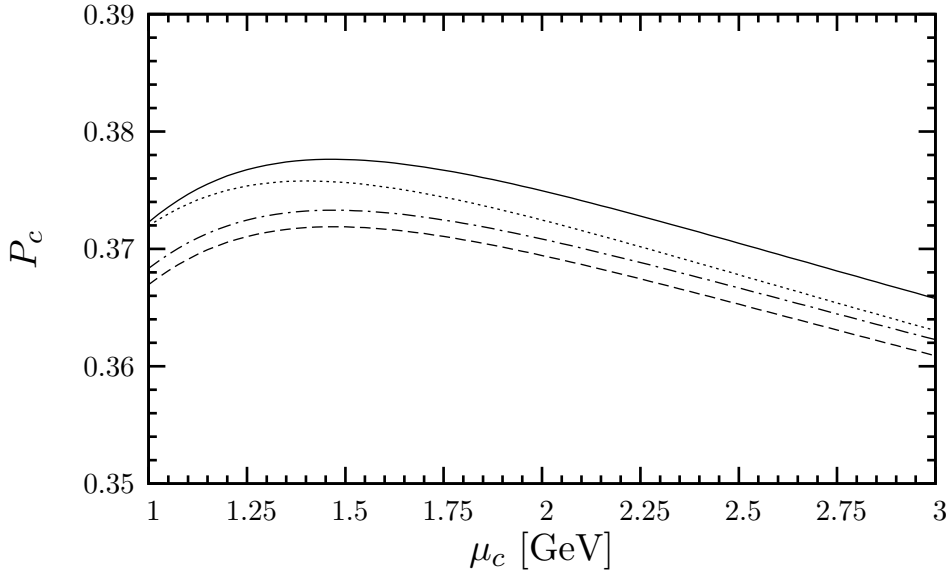


Figure 5.7: $P_c(X)$ as a function of $\mu_c$ at NNLO QCD (dashed dotted line), including LO QED (dotted line), and NLO electroweak corrections (solid line). The dashed line shows $P_c(X)$ at NNLO QCD where the definition $x_c = m_c^2/M_W^2$ is used.

The dependence of $P_c(X)$ on the parameter $\mu_c$ can be seen in Figure 5.7. We use central values for all relevant input parameters of Table 5.1 and fix $\mu_b = 5\,\text{GeV}$ and $\mu_W = 80\,\text{GeV}$. The dashed line shows $P_c(X)$ as a function of $\mu_c$ including the NNLO QCD corrections, as computed in [25] where the parameter $x_c$ equals $m_c^2/M_W^2$. The dashed-dotted line shows the same quantity, but using our improved definition of $x_c$, see Equation (5.3). We observe that this line is shifted by about 0.5% compared to $P_c(X)$ using the conventional definition of $x_c$. The dotted and the solid lines show the results including LO QED and the NLO electroweak corrections, respectively. We see that including the full electroweak corrections, $P_c(X)$ is increased by another 1.5% as compared to the pure NNLO QCD result with the improved definition of $x_c$. Also the cancellation of the scheme dependence between the LO QED and the NLO electroweak contribution is clearly visible.
The explicit analytic expression for $P_c(X)$ including the complete NNLO corrections is so complicated and long that we derive an approximate formula. Setting $\lambda = 0.2255$ and $m_t(m_t) = 163.0\text{GeV}$ we derive an approximate formula for $P_c(X)$ that summarises the dominant parametric and theoretical uncertainties due to $m_c(m_c)$, $\alpha_s(M_Z)$, $\mu_c$, $\mu_W$, and $\mu_b$. It reads

$$\begin{aligned} P_c(X) = {} & 0.38049 \left(\frac{m_c(m_c)}{1.30\text{GeV}}\right)^{0.5081} \left(\frac{\alpha_s(M_Z)}{0.1176}\right)^{1.0192} \left(1 + \sum_{i,j} \kappa_{ij} L_{m_c}^i L_{\alpha_s}^j\right) \\ & \pm 0.008707 \left(\frac{m_c(m_c)}{1.30\text{GeV}}\right)^{0.5276} \left(\frac{\alpha_s(M_Z)}{0.1176}\right)^{1.8970} \left(1 + \sum_{i,j} \epsilon_{ij} L_{m_c}^i L_{\alpha_s}^j\right) , \end{aligned} \tag{5.43}$$

where

$$L_{m_c} = \ln\left(\frac{m_c(m_c)}{1.30\text{GeV}}\right) , \qquad L_{\alpha_s} = \ln\left(\frac{\alpha_s(M_Z)}{0.1176}\right) , \tag{5.44}$$

| | | | | |
|---|---|---|---|---|
| $\kappa_{10} = 1.6624$ | $\kappa_{01} = -2.3537$ | $\kappa_{11} = -1.5862$ | $\kappa_{20} = 1.5036$ | $\kappa_{02} = -4.3477$ |
| $\epsilon_{10} = -0.3537$ | $\epsilon_{01} = 0.6003$ | $\epsilon_{11} = -4.7652$ | $\epsilon_{20} = 1.0253$ | $\epsilon_{02} = 0.8866$ |

Table 5.2: The coefficients $\kappa_{ij}$ and $\epsilon_{ij}$ arising in the approximate formula for $P_c(X)$.

and the sum includes the expansion coefficients $\kappa_{ij}$ and $\epsilon_{ij}$ given in Table 5.2. The above formula approximates the central value of the full NNLO QCD result plus electroweak corrections with an accuracy of $\pm 0.05\%$ in the ranges $1.15\,\text{GeV} \leq m_c(m_c) \leq 1.45\,\text{GeV}$, $0.114 \leq \alpha_s(M_Z) \leq 0.122$, while the scale uncertainty for varying $1.0\,\text{GeV} \leq \mu_c \leq 3.0\,\text{GeV}$, $40\,\text{GeV} \leq \mu_W \leq 160\,\text{GeV}$, and $2.5\,\text{GeV} \leq \mu_b \leq 10.0\,\text{GeV}$ is correct up to $\pm 2.3\%$ in Equation (5.43). The uncertainties due to $m_t(m_t)$, and the different methods of computing $\alpha_s(\mu_c)$ from $\alpha_s(M_Z)$, which are not quantified above, are all below $\pm 0.2\%$. For $\lambda = 0.2255$ we find $P_c(X) = 0.372 \pm 0.015$, where 42% of the error are related to the remaining theoretical uncertainty and 58% to the uncertainties in $m_c$ and $\alpha_s$. In the future one could utilise the correlation of $m_c$ and $\alpha_s$ in Reference [90] to further reduce the parametric uncertainty.
Finally we provide an updated number for the branching ratio:

$$B\left(K^+ \to \pi^+ \nu\bar{\nu}(\gamma)\right) = (8.51\,^{+0.57}_{-0.62} \pm 0.20 \pm 0.36) \times 10^{-11}. \tag{5.45}$$

The first error stems from the uncertainties in the CKM parameters. The second error is related to the uncertainties in $m_c$, $m_t$, and $\alpha_s$, where all three quantities contribute in equal shares. The dependence on $M_H$ is completely negligible (below one per mil). The last error quantifies the remaining theoretical uncertainty. Here the main contributions stem from the uncertainty in $\delta P_{c,u}$ and $X_t$, where we used an error of 2%. In detail, the contributions to the theory error are ($\kappa_\nu^+ : 6\%$, $X_t : 38\%$, $P_c : 17\%$, $\delta P_{c,u} : 39\%$), respectively. All errors have been added in quadrature.

# Chapter 6

# Conclusions

Kaon decays play an important role in flavour physics. The parameter $\epsilon_K$, describing indirect $CP$ violation in the decay $K_L \to \pi\pi$, is one of the main ingredients of the global fit of the unitarity triangle, and the only one related to the Kaon sector. The rare decay $K^+ \to \pi^+\nu\bar{\nu}$, together with its $CP$-violating counterpart $K_L \to \pi^0\nu\bar{\nu}$, offers one of the cleanest windows to new physics.

Here, in contrast to other interesting observables in flavour physics, hadronic uncertainties are well under control. In case of the rare decays, the hadronic matrix elements can be extracted with very high precision from the set of $K_{\ell 3}$ decays [28], whereas in case of $\epsilon_K$ the determination of the matrix elements using lattice QCD is improving rapidly [9]. This situation motivates the extension of the existing perturbative calculations.

In this thesis we have calculated higher order corrections to the parameter $\epsilon_K$ and the branching ratio for $K^+ \to \pi^+\nu\bar{\nu}$.

In Chapter 4 we have computed all ingredients necessary for an analysis of the mixed charm-top contribution $\eta_3$ to $\epsilon_K$ in renormalisation-group improved perturbation theory at the NNLO level. In particular, we have calculated the three-loop anomalous dimension tensor describing the mixing of the relevant dimension-six operators into the operator $\tilde{Q}_{S2}$. We have determined the initial condition of the corresponding Wilson coefficient by a matching calculation at the electroweak scale and computed the renormalisation group evolution down to the charm-quark scale, taking into account the corrections at the bottom-quark threshold. Finally, we have performed a matching calculation onto the effective three-flavour theory, where the charm quark is integrated out.

Our results are in complete agreement with the previously unconfirmed NLO results in the literature [17] and hence constitute the first check by an independent calculation. Furthermore, we have extended the calculation to the NNLO for the first time.

As a check of our results, we have explicitly verified the locality of the UV divergences, the gauge-parameter independence of the anomalous dimensions and the Wilson coefficients, and the vanishing of the mixing of evanescent into physical operators.

In Chapter 5 we have calculated the $\mathcal{O}(\alpha)$ and $\mathcal{O}(\alpha\alpha_s)$ anomalous dimensions and the electroweak matching corrections of the charm quark contribution relevant for the rare decay $K^+ \to \pi^+\nu\bar{\nu}$. The parametric dependence of the relevant parameter $P_c(X)$ and its theoretical uncertainty is summarised in an approximate but very accurate formula.

$P_c(X)$ is increased by up to 2% as compared to the previously known results [25]. This change is of the same order of magnitude as the remaining scale uncertainties after the NNLO QCD

calculation. Together with the recently achieved very precise determination of the hadronic matrix elements and further improvements on the long-distance contribution of the charm quark [83], the theoretical prediction of the branching ratio $B(K^+ \to \pi^+\nu\bar{\nu})$ will reach an exceptional degree of precision, with the uncertainties mainly due to the CKM parameters. The latter errors will be reduced in the coming years by the $B$-physics experiments.
Until now seven events of the decay $K^+ \to \pi^+\nu\bar{\nu}$ have been observed [32]. A precise measurement of the branching ratio at future experiments will provide a unique test of the flavour sector of the SM and its extensions.
In the future we will pursue two main projects related to the work described above. On the one hand, we will finalise the NNLO matching calculation for the pure charm-quark contribution $\eta_1$ to $\epsilon_K$ [94]. This is the last piece needed for a complete NNLL analysis of $\eta_1$ and, together with the NNLL analysis of $\eta_3$, will significantly reduce the remaining scale uncertainty, which is presently dominating the error of the theoretical prediction. On the other hand, we will compute the full two-loop electroweak matching corrections for the top-quark contribution to $K^+ \to \pi^+\nu\bar{\nu}$ [95], which is also of great relevance for the decay $K_L \to \pi^0\nu\bar{\nu}$. This calculation will further reduce the theoretical uncertainties of the Standard Model prediction of the branching ratio, leaving these two decay modes among the cleanest observables in flavour physics.
To summarise, the results obtained in this thesis represent a major step towards a highly improved theoretical prediction of two very important precision observables in Kaon physics, namely, $\epsilon_K$ and the branching ratio of the decay mode $K^+ \to \pi^+\nu\bar{\nu}$. Both these observables play an outstanding role in our quest for the fundamental laws of nature.

# Appendix A

# Renormalisation Constants for Double Insertions

In this chapter we give the explicit values of all $Z$ factors needed for the renormalisation of the mixing of two dimension-six operators into the dimension-eight operators. We display only the $1/\epsilon$-pole part; the higher pole parts can be obtained using Equations (2.71) and (2.120) to (2.123). Entries which are not needed for our calculation are indicated by a star. In order to get more compact expressions, we set $N_c = 3$ and $N_f = 5$. In the matrices related to double insertions, the component $(\hat{Z}_{\tilde{Q}})_{ij}$ denotes an insertion of the two dimension-six operators $Q_i$ and $Q_j$, given in the basis

$$(Q_1, \ldots, Q_{15}) = (Q_1, \ldots, Q_6, E_1^{(1)}, \ldots, E_4^{(1)}, E_1^{(2)}, \ldots, E_4^{(2)}, Q_{\text{eom}})\,, \tag{A.1}$$

mixing into the dimension-eight operator $\tilde{Q}$. We have set to zero all contributions which are not proportional to $\lambda_c\lambda_t$ as they do not contribute to $\eta_3$.

The $1/\epsilon$-pole part of the mixing of the $Q_{1,\ldots,6}$ into $\tilde{Q}_7$ and $\tilde{E}_F$ is given by

$$\hat{Z}^{(1,1)}_{\tilde{Q}_7} = \begin{pmatrix} -\frac{11}{18} & -\frac{1}{3} & 0 & 0 & -16 & \frac{2}{3} \\ -\frac{1}{3} & -1 & 0 & 0 & -12 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -16 & -12 & 0 & 0 & 0 & 0 \\ \frac{2}{3} & -4 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{Z}^{(1,1)}_{\tilde{E}_F} = \begin{pmatrix} -\frac{7}{12} & -\frac{1}{2} & 0 & 0 & 0 & 1 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -6 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{A.2}$$

All other pole parts vanish. The remaining $Z$ factors are given by

$$\hat{Z}^{(1,1)}_{\tilde{E}_7^{(1)}} = \begin{pmatrix}
\frac{1}{144} & -\frac{1}{24} & 0 & 0 & -\frac{8}{3} & -\frac{1}{18} & -\frac{1}{4} & \frac{3}{2} & 320 & \frac{20}{3} & -\frac{11}{9} & \frac{22}{3} & 6720 & 140 & 0 \\
-\frac{1}{24} & \frac{1}{4} & 0 & 0 & -2 & \frac{1}{3} & \frac{3}{2} & -9 & 240 & -40 & \frac{22}{3} & -44 & 5040 & -840 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{8}{3} & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{18} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{4} & \frac{3}{2} & 0 & 0 & 0 & 0 & \frac{16}{9} & -\frac{32}{3} & -10240 & -\frac{640}{3} & * & * & * & * & 0 \\
\frac{3}{2} & -9 & 0 & 0 & 0 & 0 & -\frac{32}{3} & 64 & -7680 & 1280 & * & * & * & * & 0 \\
320 & 240 & 0 & 0 & 0 & 0 & -10240 & -7680 & 0 & 0 & * & * & 0 & 0 & 0 \\
\frac{20}{3} & -40 & 0 & 0 & 0 & 0 & -\frac{640}{3} & 1280 & 0 & 0 & * & * & 0 & 0 & 0 \\
-\frac{11}{9} & \frac{22}{3} & 0 & 0 & 0 & 0 & * & * & * & * & * & * & * & * & 0 \\
\frac{22}{3} & -44 & 0 & 0 & 0 & 0 & * & * & * & * & * & * & * & * & 0 \\
6720 & 5040 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
140 & -840 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \tag{A.3}$$

$$\hat{Z}^{(1,1)}_{\tilde{E}_8^{(1)}} = \begin{pmatrix}
\frac{7}{48} & \frac{1}{8} & 0 & 0 & 0 & \frac{1}{6} & -\frac{21}{4} & -\frac{9}{2} & 0 & -20 & -\frac{77}{3} & -22 & 0 & -420 & 0 \\
\frac{1}{8} & 0 & 0 & 0 & 0 & -1 & -\frac{9}{2} & 0 & 0 & 120 & -22 & 0 & 0 & 2520 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{6} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{21}{4} & -\frac{9}{2} & 0 & 0 & 0 & 0 & \frac{112}{3} & 32 & 0 & 640 & * & * & * & * & 0 \\
-\frac{9}{2} & 0 & 0 & 0 & 0 & 0 & 32 & 0 & 0 & -3840 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 \\
-20 & 120 & 0 & 0 & 0 & 0 & 640 & -3840 & 0 & 0 & * & * & 0 & 0 & 0 \\
-\frac{77}{3} & -22 & 0 & 0 & 0 & 0 & * & * & * & * & * & * & * & * & 0 \\
-22 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
-420 & 2520 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \tag{A.4}$$

$$\hat{Z}^{(1,1)}_{\tilde{E}_7^{(2)}} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{144} & -\frac{1}{24} & -16 & -\frac{1}{3} & -\frac{1}{18} & \frac{1}{3} & 336 & 7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{24} & \frac{1}{4} & -12 & 2 & \frac{1}{3} & -2 & 252 & -42 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{144} & -\frac{1}{24} & 0 & 0 & 0 & 0 & -\frac{5}{36} & \frac{5}{6} & 512 & \frac{32}{3} & * & * & * & * & 0 \\
-\frac{1}{24} & \frac{1}{4} & 0 & 0 & 0 & 0 & \frac{5}{6} & -5 & 384 & -64 & * & * & * & * & 0 \\
-16 & -12 & 0 & 0 & 0 & 0 & 512 & 384 & 0 & 0 & * & * & 0 & 0 & 0 \\
-\frac{1}{3} & 2 & 0 & 0 & 0 & 0 & \frac{32}{3} & -64 & 0 & 0 & * & * & 0 & 0 & 0 \\
-\frac{1}{18} & \frac{1}{3} & 0 & 0 & 0 & 0 & * & * & * & * & * & * & * & * & 0 \\
\frac{1}{3} & -2 & 0 & 0 & 0 & 0 & * & * & * & * & * & * & * & * & 0 \\
336 & 252 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
7 & -42 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad \text{(A.5)}$$

$$\hat{Z}^{(1,1)}_{\tilde{E}_8^{(2)}} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \frac{7}{48} & \frac{1}{8} & 0 & 1 & -\frac{7}{6} & -1 & 0 & -21 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 & -6 & -1 & 0 & 0 & 126 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{7}{48} & \frac{1}{8} & 0 & 0 & 0 & 0 & -\frac{35}{12} & -\frac{5}{2} & 0 & -32 & * & * & * & * & 0 \\
\frac{1}{8} & 0 & 0 & 0 & 0 & 0 & -\frac{5}{2} & 0 & 0 & 192 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 \\
1 & -6 & 0 & 0 & 0 & 0 & -32 & 192 & 0 & 0 & * & * & 0 & 0 & 0 \\
-\frac{7}{6} & -1 & 0 & 0 & 0 & 0 & * & * & * & * & * & * & * & * & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
-21 & 126 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad \text{(A.6)}$$

$$\hat{Z}^{(1,1)}_{\tilde{E}^{(3)}_7} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{144} & -\frac{1}{24} & -40 & -\frac{5}{6} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{24} & \frac{1}{4} & -30 & 5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{144} & -\frac{1}{24} & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{24} & \frac{1}{4} & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 \\
\frac{1}{144} & -\frac{1}{24} & 0 & 0 & 0 & 0 & * & * & * & * & * & * & * & * & 0 \\
-\frac{1}{24} & \frac{1}{4} & 0 & 0 & 0 & 0 & * & * & * & * & * & * & * & * & 0 \\
-40 & -30 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
-\frac{5}{6} & 5 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \tag{A.7}$$

$$\hat{Z}^{(1,1)}_{\tilde{E}^{(3)}_8} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7}{48} & \frac{1}{8} & 0 & \frac{5}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 & -15 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{7}{48} & \frac{1}{8} & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 \\
\frac{7}{48} & \frac{1}{8} & 0 & 0 & 0 & 0 & * & * & * & * & * & * & * & * & 0 \\
\frac{1}{8} & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
\frac{5}{2} & -15 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. \tag{A.8}$$

The finite renormalisation of the evanescent operators is given by

$$\hat{Z}^{(1,0)}_{\tilde{Q}_7} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \frac{44}{3} & 8 & -2560 & \frac{320}{3} & 880 & 480 & -197120 & \frac{24640}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 8 & 24 & -1920 & -640 & 480 & 1440 & -147840 & -49280 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 320 & 240 & 0 & 0 & 5376 & 4032 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{40}{3} & 80 & 0 & 0 & -224 & 1344 & 0 & 0 & 0 \\
\frac{44}{3} & 8 & 0 & 0 & 320 & -\frac{40}{3} & \frac{1936}{3} & 352 & 92160 & -3840 & * & * & * & * & 0 \\
8 & 24 & 0 & 0 & 240 & 80 & 352 & 1056 & 69120 & 23040 & * & * & * & * & 0 \\
-2560 & -1920 & 0 & 0 & 0 & 0 & 92160 & 69120 & 0 & 0 & * & * & 0 & 0 & 0 \\
\frac{320}{3} & -640 & 0 & 0 & 0 & 0 & -3840 & 23040 & 0 & 0 & * & * & 0 & 0 & 0 \\
880 & 480 & 0 & 0 & 5376 & -224 & * & * & * & * & * & * & * & * & 0 \\
480 & 1440 & 0 & 0 & 4032 & 1344 & * & * & * & * & * & * & * & * & 0 \\
-197120 & -147840 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
\frac{24640}{3} & -49280 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. \tag{A.9}$$

The two-loop contributions read

$$\hat{Z}^{(2,1)}_{\tilde{Q}_7} = \begin{pmatrix}
-\frac{85}{108} & \frac{743}{36} & -\frac{11}{6} & -\frac{125}{72} & 295 & -\frac{1543}{6} & \frac{3874}{9} & \frac{2236}{3} & \frac{276640}{3} & -\frac{121880}{9} & * & * & * & * & 0 \\
\frac{743}{36} & -\frac{47}{6} & -7 & -\frac{91}{12} & -132 & -374 & \frac{2164}{3} & 112 & 63040 & -\frac{113360}{3} & * & * & * & * & 0 \\
-\frac{11}{6} & -7 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
-\frac{125}{72} & -\frac{91}{12} & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
295 & -132 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
-\frac{1543}{6} & -374 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
\frac{3874}{9} & \frac{2164}{3} & * & * & * & * & * & * & * & * & * & * & * & * & * \\
\frac{2236}{3} & 112 & * & * & * & * & * & * & * & * & * & * & * & * & * \\
\frac{276640}{3} & 63040 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
-\frac{121880}{9} & -\frac{113360}{3} & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0
\end{pmatrix}, \tag{A.10}$$

$$\hat{Z}^{(2,1)}_{\tilde{E}_F} = \begin{pmatrix}
\frac{505}{72} & \frac{371}{24} & -\frac{11}{4} & \frac{35}{48} & \frac{105}{2} & \frac{141}{4} & 535 & 598 & -10160 & 6140 & * & * & * & * & 0 \\
\frac{371}{24} & \frac{93}{4} & -\frac{21}{2} & -\frac{35}{8} & -558 & -189 & 562 & 780 & -42720 & -28200 & * & * & * & * & 0 \\
-\frac{11}{4} & -\frac{21}{2} & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
\frac{35}{48} & -\frac{35}{8} & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
\frac{105}{2} & -558 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
\frac{141}{4} & -189 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
535 & 562 & * & * & * & * & * & * & * & * & * & * & * & * & * \\
598 & 780 & * & * & * & * & * & * & * & * & * & * & * & * & * \\
-10160 & -42720 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
6140 & -28200 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0
\end{pmatrix}, \tag{A.11}$$

$$\hat{Z}^{(2,0)}_{\tilde{Q}_7} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \frac{35863}{81} & \frac{7762}{27} & \frac{174752}{3} & \frac{41168}{3} & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{8086}{27} & -\frac{188}{9} & 92576 & 28304 & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
\frac{35863}{81} & \frac{8086}{27} & * & * & * & * & * & * & * & * & * & * & * & * & 0 \\
\frac{7762}{27} & -\frac{188}{9} & * & * & * & * & * & * & * & * & * & * & * & * & 0 \\
\frac{174752}{3} & 92576 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
\frac{41168}{3} & 28304 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & 0 \\
* & * & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad \text{(A.12)}$$

$$\hat{Z}^{(2,1)}_{\tilde{E}^{(1)}_7} = \begin{pmatrix}
-\frac{1063}{1728} & \frac{397}{288} & \frac{13}{72} & -\frac{89}{864} & \frac{1511}{36} & -\frac{22237}{432} & \frac{1915}{216} & \frac{1093}{72} & \frac{62387}{9} & -\frac{75391}{108} & * & * & * & * & 0 \\
\frac{397}{288} & -\frac{217}{48} & \frac{5}{12} & -\frac{55}{144} & \frac{469}{6} & -\frac{2585}{72} & \frac{517}{72} & -\frac{661}{12} & \frac{13862}{3} & \frac{1519}{18} & * & * & * & * & 0 \\
\frac{13}{72} & \frac{5}{12} & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
-\frac{89}{864} & -\frac{55}{144} & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
\frac{1511}{36} & \frac{469}{6} & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
-\frac{22237}{432} & -\frac{2585}{72} & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
\frac{1915}{216} & \frac{517}{72} & * & * & * & * & * & * & * & * & * & * & * & * & * \\
\frac{1093}{72} & -\frac{661}{12} & * & * & * & * & * & * & * & * & * & * & * & * & * \\
\frac{62387}{9} & \frac{13862}{3} & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
-\frac{75391}{108} & \frac{1519}{18} & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0
\end{pmatrix}, \quad \text{(A.13)}$$

$$\hat{Z}^{(2,1)}_{\tilde{E}_8^{(1)}} = \begin{pmatrix}
\frac{199}{64} & \frac{379}{96} & -\frac{13}{24} & \frac{25}{288} & -\frac{383}{12} & \frac{1277}{144} & -\frac{1559}{72} & -\frac{1117}{24} & -\frac{6067}{3} & \frac{2375}{4} & * & * & * & * & 0 \\
\frac{379}{96} & \frac{97}{16} & -\frac{5}{4} & -\frac{25}{48} & -\frac{229}{2} & -\frac{1079}{24} & -\frac{541}{24} & -\frac{3}{4} & -2662 & -\frac{7605}{2} & * & * & * & * & 0 \\
-\frac{13}{24} & -\frac{5}{4} & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
\frac{25}{288} & -\frac{25}{48} & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
-\frac{383}{12} & -\frac{229}{2} & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
\frac{1277}{144} & -\frac{1079}{24} & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
-\frac{1559}{72} & -\frac{541}{24} & * & * & * & * & * & * & * & * & * & * & * & * & * \\
-\frac{1117}{24} & -\frac{3}{4} & * & * & * & * & * & * & * & * & * & * & * & * & * \\
-\frac{6067}{3} & -2662 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
\frac{2375}{4} & -\frac{7605}{2} & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0
\end{pmatrix}, \tag{A.14}$$

$$\hat{Z}^{(2,1)}_{\tilde{E}_7^{(2)}} = \begin{pmatrix}
-\frac{1}{6912} & -\frac{11}{288} & 0 & 0 & -\frac{17}{36} & \frac{1031}{864} & -\frac{1907}{1728} & \frac{269}{288} & \frac{9277}{36} & -\frac{58487}{432} & * & * & * & * & 0 \\
-\frac{11}{288} & \frac{1}{24} & 0 & 0 & -\frac{11}{12} & \frac{121}{144} & \frac{341}{288} & -\frac{161}{48} & \frac{2531}{6} & -\frac{7645}{72} & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
-\frac{17}{36} & -\frac{11}{12} & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
\frac{1031}{864} & \frac{121}{144} & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
-\frac{1907}{1728} & \frac{341}{288} & * & * & * & * & * & * & * & * & * & * & * & * & * \\
\frac{269}{288} & -\frac{161}{48} & * & * & * & * & * & * & * & * & * & * & * & * & * \\
\frac{9277}{36} & \frac{2531}{6} & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
-\frac{58487}{432} & -\frac{7645}{72} & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0
\end{pmatrix}, \tag{A.15}$$

$$\hat{Z}^{(2,1)}_{\tilde{E}_8^{(2)}} = \begin{pmatrix}
-\frac{5}{768} & -\frac{5}{96} & 0 & 0 & \frac{17}{12} & -\frac{55}{288} & \frac{2443}{576} & \frac{571}{96} & -\frac{239}{4} & \frac{3095}{144} & * & * & * & * & 0 \\
-\frac{5}{96} & -\frac{1}{8} & 0 & 0 & \frac{11}{4} & \frac{55}{48} & \frac{499}{96} & \frac{105}{16} & -\frac{579}{2} & -\frac{1619}{24} & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
\frac{17}{12} & \frac{11}{4} & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
-\frac{55}{288} & \frac{55}{48} & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
\frac{2443}{576} & \frac{499}{96} & * & * & * & * & * & * & * & * & * & * & * & * & * \\
\frac{571}{96} & \frac{105}{16} & * & * & * & * & * & * & * & * & * & * & * & * & * \\
-\frac{239}{4} & -\frac{579}{2} & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
\frac{3095}{144} & -\frac{1619}{24} & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0
\end{pmatrix}, \quad \text{(A.16)}$$

$$\hat{Z}^{(2,1)}_{\tilde{E}_7^{(3)}} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{6912} & -\frac{11}{288} & -\frac{101}{72} & \frac{3703}{864} & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{11}{288} & \frac{1}{24} & -\frac{43}{12} & \frac{473}{144} & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
-\frac{1}{6912} & -\frac{11}{288} & * & * & * & * & * & * & * & * & * & * & * & * & * \\
-\frac{11}{288} & \frac{1}{24} & * & * & * & * & * & * & * & * & * & * & * & * & * \\
-\frac{101}{72} & -\frac{43}{12} & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
\frac{3703}{864} & \frac{473}{144} & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0
\end{pmatrix}, \quad \text{(A.17)}$$

$$\hat{Z}^{(2,1)}_{\tilde{E}^{(3)}_8} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & -\frac{5}{768} & -\frac{5}{96} & \frac{101}{24} & -\frac{215}{288} & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{5}{96} & -\frac{1}{8} & \frac{43}{4} & \frac{215}{48} & * & * & * & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
-\frac{5}{768} & -\frac{5}{96} & * & * & * & * & * & * & * & * & * & * & * & * & * \\
-\frac{5}{96} & -\frac{1}{8} & * & * & * & * & * & * & * & * & * & * & * & * & * \\
\frac{101}{24} & \frac{43}{4} & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
-\frac{215}{288} & \frac{215}{48} & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\
* & * & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0
\end{pmatrix}. \tag{A.18}$$

Of the three-loop insertions, we computed only the part related to an insertion of an operator belonging to the current-current subspace:

$$\hat{Z}^{(3,1)}_{\tilde{Q}_7} = \begin{pmatrix}
\frac{7000225}{34992} + \frac{374}{81}\zeta_3 & \frac{2248069}{11664} + \frac{221}{27}\zeta_3 \\
\frac{2248069}{11664} + \frac{221}{27}\zeta_3 & -\frac{388957}{1944} + \frac{536}{9}\zeta_3
\end{pmatrix}. \tag{A.19}$$

We have also calculated the self mixing of the dimension-eight operators, in the basis given by $\left(\tilde{Q}_7, \tilde{E}_F, \tilde{E}^{(1)}_7, \tilde{E}^{(1)}_8, \tilde{E}^{(2)}_7, \tilde{E}^{(2)}_8\right)$:

$$\tilde{Z}^{(1,1)} = \begin{pmatrix}
2 & 3 & -\frac{1}{6} & \frac{1}{2} & 0 & 0 \\
0 & -4 & \frac{5}{12} & \frac{1}{12} & 0 & 0 \\
0 & 0 & \frac{13}{3} & -13 & -\frac{1}{6} & \frac{1}{2} \\
0 & 0 & -5 & -\frac{59}{3} & \frac{1}{4} & \frac{7}{12} \\
0 & 0 & \frac{1568}{3} & -288 & -\frac{67}{3} & 3 \\
0 & 0 & 96 & -\frac{1888}{3} & -9 & \frac{41}{3}
\end{pmatrix}, \qquad \tilde{Z}^{(1,0)} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 \\
-48 & 0 & 0 & 0 & 0 & 0 \\
48 & 0 & 0 & 0 & 0 & 0 \\
-3840 & 0 & 0 & 0 & 0 & 0 \\
3840 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}; \tag{A.20}$$

$$\tilde{Z}^{(2,1)} = \begin{pmatrix}
\frac{457}{36} & \frac{349}{24} & \frac{223}{216} & \frac{389}{72} & -\frac{77}{1152} & -\frac{35}{384} \\
-\frac{53}{3} & \frac{46}{9} & -\frac{137}{108} & \frac{275}{108} & -\frac{43}{1152} & \frac{73}{1152} \\
1000 & 748 & \frac{15097}{216} & -\frac{5425}{72} & \frac{29}{108} & \frac{277}{36} \\
152 & 1092 & -\frac{8333}{72} & -\frac{10571}{216} & \frac{157}{72} & \frac{1471}{216} \\
* & * & * & * & * & * \\
* & * & * & * & * & *
\end{pmatrix}, \tag{A.21}$$

$$\tilde{Z}^{(2,0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{197}{18} & 0 & 0 & 0 & 0 & 0 \\ -\frac{752}{3} & 0 & 0 & 0 & 0 & 0 \\ \frac{824}{3} & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \tag{A.22}$$

The $1/\epsilon$-pole of the mixing of $\tilde{Q}_7$ into $\tilde{Q}_7$ reads [96]

$$\tilde{Z}^{(3,1)}_{\tilde{Q}_7\tilde{Q}_7} = \frac{8725}{36} - \frac{1408}{9}\zeta_3 \,. \tag{A.23}$$

Finally, we provide the mixing of the relevant dimension-six operators into $\tilde{Q}_\nu$ to order $\mathcal{O}(\alpha\alpha_s)$. Again, only the $1/\epsilon$-pole part is needed for computing the anomalous dimensions:

$$\hat{Z}^{(es,1)}_{Q_{3q}Q_{4q},\tilde{Q}_\nu} = -\frac{52}{9}\,, \quad \hat{Z}^{(es,1)}_{Q_\pm Q_A,\tilde{Q}_\nu} = \frac{13}{3}(1 \pm 3)\,. \tag{A.24}$$

The $Z$ factors needed for the renormalisation of the dimension-six operators can be found in the literature [23].

# Appendix B

# Anomalous Dimensions

In this appendix we collect various quantities needed for the renormalisation group analysis. We start with the explicit expressions for the QCD $\beta$ function and the anomalous dimension of the up-type quark mass [97–100]:

$$\begin{aligned}
&\beta_0 = 11 - \frac{2}{3} f\,, \qquad \beta_1 = 102 - \frac{38}{3} f\,, \qquad \beta_2 = \frac{2857}{2} - \frac{5033}{18} f + \frac{325}{54} f^2\,, \\
&\beta_e = 0\,, \qquad \beta_{es} = -\frac{8}{9}\left(f_u + \frac{f_d}{4}\right),
\end{aligned} \tag{B.1}$$

where $f_u$ and $f_d$ denote the number of up- and down-type quark flavours, and $f = f_u + f_d$, and

$$\begin{aligned}
&\gamma_m^{(0)} = 8\,, \qquad \gamma_m^{(1)} = \frac{404}{3} - \frac{40}{9} f\,, \qquad \gamma_m^{(2)} = 2498 - \left(\frac{4432}{27} + \frac{320}{3}\zeta_3\right) f - \frac{280}{81} f^2\,, \\
&\gamma_m^{(e)} = \frac{8}{3}\,, \qquad \gamma_m^{(es)} = \frac{32}{9}\,.
\end{aligned} \tag{B.2}$$

We continue with the explicit expressions for the anomalous dimensions describing the mixing of the dimension-six operators in the basis $(Q_1, \ldots, Q_6)$ for the physical operators and the definition of evanescent operators given by (2.45) and (2.47) [23,46,63,101]. It is given in the LO approximation by

$$\gamma_Q^{(0)} = \begin{pmatrix}
-4 & \frac{8}{3} & 0 & -\frac{2}{9} & 0 & 0 \\
12 & 0 & 0 & \frac{4}{3} & 0 & 0 \\
0 & 0 & 0 & -\frac{52}{3} & 0 & 2 \\
0 & 0 & -\frac{49}{9} & -\frac{160}{9} + \frac{4}{3} N_f & \frac{4}{9} & \frac{5}{6} \\
0 & 0 & 0 & -\frac{256}{3} & 0 & 20 \\
0 & 0 & -\frac{256}{9} & -\frac{544}{9} + \frac{40}{3} N_f & \frac{40}{9} & -\frac{2}{3}
\end{pmatrix}. \tag{B.3}$$

The NLO expression reads

$$\gamma_Q^{(1)} = \begin{pmatrix} -\frac{145}{3}+\frac{16}{9}N_f & -26+\frac{40}{27}N_f & -\frac{1412}{243} & -\frac{1369}{243} & \frac{134}{243} & -\frac{35}{162} \\ -45+\frac{20}{3}N_f & -\frac{28}{3} & -\frac{416}{81} & \frac{1280}{81} & \frac{56}{81} & \frac{35}{27} \\ 0 & 0 & -\frac{4468}{81} & -\frac{29129}{81}-\frac{52}{9}N_f & \frac{400}{81} & \frac{3493}{108}-\frac{2}{9}N_f \\ 0 & 0 & -\frac{13678}{243}+\frac{368}{81}N_f & -\frac{79409}{243}+\frac{1334}{81}N_f & \frac{509}{486}-\frac{8}{81}N_f & \frac{13499}{648}-\frac{5}{27}N_f \\ 0 & 0 & -\frac{244480}{81}-\frac{160}{9}N_f & -\frac{29648}{81}-\frac{2200}{9}N_f & \frac{23116}{81}+\frac{16}{9}N_f & \frac{3886}{27}+\frac{148}{9}N_f \\ 0 & 0 & \frac{77600}{243}-\frac{1264}{81}N_f & -\frac{28808}{243}+\frac{164}{81}N_f & -\frac{20324}{243}+\frac{400}{81}N_f & -\frac{21211}{162}+\frac{622}{27}N_f \end{pmatrix}, \tag{B.4}$$

and at the NNLO we have

$$\gamma_Q^{(2)} = \begin{pmatrix} -\frac{1927}{2}+\frac{257}{9}N_f+\frac{40}{9}N_f^2+\left(224+\frac{160}{3}N_f\right)\zeta_3 & \frac{475}{9}+\frac{362}{27}N_f-\frac{40}{27}N_f^2-\left(\frac{896}{3}+\frac{320}{9}N_f\right)\zeta_3 \\ \frac{307}{2}+\frac{361}{3}N_f-\frac{20}{3}N_f^2-\left(1344+160N_f\right)\zeta_3 & \frac{1298}{3}-\frac{76}{3}N_f-224\zeta_3 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{269107}{13122}-\frac{2288}{729}N_f-\frac{1360}{81}\zeta_3 & -\frac{2425817}{13122}+\frac{30815}{4374}N_f-\frac{776}{81}\zeta_3 \\ \frac{69797}{2187}+\frac{904}{243}N_f+\frac{2720}{27}\zeta_3 & \frac{1457549}{8748}-\frac{22067}{729}N_f-\frac{2768}{27}\zeta_3 \\ -\frac{4203068}{2187}+\frac{14012}{243}N_f-\frac{608}{27}\zeta_3 & -\frac{18422762}{2187}+\frac{888605}{2916}N_f+\frac{272}{27}N_f^2+\left(\frac{39824}{27}+160N_f\right)\zeta_3 \\ -\frac{5875184}{6561}+\frac{217892}{2187}N_f+\frac{472}{81}N_f^2+\left(\frac{27520}{81}+\frac{1360}{9}N_f\right)\zeta_3 & -\frac{70274587}{13122}+\frac{8860733}{17496}N_f-\frac{4010}{729}N_f^2+\left(\frac{16592}{81}+\frac{2512}{27}N_f\right)\zeta_3 \\ -\frac{194951552}{2187}+\frac{358672}{81}N_f-\frac{2144}{81}N_f^2+\frac{87040}{27}\zeta_3 & -\frac{130500332}{2187}-\frac{2949616}{729}N_f+\frac{3088}{27}N_f^2+\left(\frac{238016}{27}+640N_f\right)\zeta_3 \\ \frac{162733912}{6561}-\frac{2535466}{2187}N_f+\frac{17920}{243}N_f^2+\left(\frac{174208}{81}+\frac{12160}{9}N_f\right)\zeta_3 & \frac{13286236}{6561}-\frac{1826023}{4374}N_f-\frac{159548}{729}N_f^2-\left(\frac{24832}{81}+\frac{9440}{27}N_f\right)\zeta_3 \end{pmatrix}$$

$$\begin{pmatrix} -\frac{343783}{52488}+\frac{392}{729}N_f+\frac{124}{81}\zeta_3 & -\frac{37573}{69984}+\frac{35}{972}N_f+\frac{100}{27}\zeta_3 \\ -\frac{37889}{8748}-\frac{28}{243}N_f-\frac{248}{27}\zeta_3 & \frac{366919}{11664}-\frac{35}{162}N_f-\frac{110}{9}\zeta_3 \\ \frac{674281}{4374}-\frac{1352}{243}N_f-\frac{496}{27}\zeta_3 & \frac{9284531}{11664}-\frac{2798}{81}N_f-\frac{26}{27}N_f^2-\left(\frac{1921}{9}+20N_f\right)\zeta_3 \\ \frac{2951809}{52488}-\frac{31175}{8748}N_f-\frac{52}{81}N_f^2-\left(\frac{3154}{81}+\frac{136}{9}N_f\right)\zeta_3 & \frac{3227801}{8748}-\frac{105293}{11664}N_f-\frac{65}{54}N_f^2+\left(\frac{200}{27}-\frac{220}{9}N_f\right)\zeta_3 \\ \frac{14732222}{2187}-\frac{27428}{81}N_f+\frac{272}{81}N_f^2-\frac{13984}{27}\zeta_3 & \frac{16521659}{2916}+\frac{8081}{54}N_f-\frac{316}{27}N_f^2-\left(\frac{22420}{9}+200N_f\right)\zeta_3 \\ -\frac{22191107}{13122}+\frac{395783}{4374}N_f-\frac{1720}{243}N_f^2-\left(\frac{33832}{81}+\frac{1360}{9}N_f\right)\zeta_3 & -\frac{32043361}{8748}+\frac{3353393}{5832}N_f-\frac{533}{81}N_f^2+\left(\frac{9248}{27}-\frac{1120}{9}N_f\right)\zeta_3 \end{pmatrix}. \tag{B.5}$$

We also need the anomalous dimensions for the subspace of the current-current operators $Q_1$ and $Q_2$ in the diagonal basis. They have been given explicitly in [25] for the choice of

evanescent operators as in appendix C and read:

$$\begin{aligned}
\gamma_{\pm}^{(0)} &= \pm 6\left(1 \mp \frac{1}{3}\right), \\
\gamma_{\pm}^{(1)} &= \left(-\frac{21}{2} \pm \frac{2}{3}N_f\right)\left(1 \mp \frac{1}{3}\right), \\
\gamma_{\pm}^{(2)} &= \frac{1}{300}\left(349049 \pm 201485\right) - \frac{1}{1350}\left(115577 \mp 9795\right) N_f \\
&\quad \mp \frac{130}{27}\left(1 \mp \frac{1}{3}\right) N_f^2 \mp \left(672 + 80\left(1 \mp \frac{1}{3}\right) N_f\right) \zeta_3 .
\end{aligned} \tag{B.6}$$

In addition we need the anomalous dimensions of the operator $\tilde{Q}_{S2}$, which are given for our choice of evanescent operators by

$$\begin{aligned}
\gamma_{77}^{(0)} &= 4, \\
\gamma_{77}^{(1)} &= \frac{20}{9}N_f - \frac{109}{3}, \\
\gamma_{77}^{(2)} &= \frac{2879}{6} + \frac{43}{9}N_f - \frac{20}{9}N_f^2 - \left(\frac{160}{3}N_f + 672\right)\zeta_3 .
\end{aligned} \tag{B.7}$$

# Appendix C

# Change of the Operator Basis

In this appendix we describe the change of the operator basis. We are concerned with two cases: The transformation to the "traditional" operator basis in order to compare our results with the literature, and the transformation of the subspace of current-current operators to the diagonal basis. In the first case we only need the NLO transformation formula, whereas in the second case a NNLO transformation is necessary.

## C.1 Transformation to the Traditional Operator Basis

The calculation of the NLO QCD corrections to $\eta_3$ in [17] has been performed in a different basis for the physical operators than the one chosen by us. It is given by

$$\begin{aligned}
Q'^{qq'}_1 &= (\bar{s}^\alpha_L \gamma_\mu q^\alpha_L) \otimes (\bar{q'}^\beta_L \gamma^\mu d^\beta_L)\,, \\
Q'^{qq'}_2 &= (\bar{s}^\alpha_L \gamma_\mu q^\beta_L) \otimes (\bar{q'}^\beta_L \gamma^\mu d^\alpha_L)\,, \\
Q'_3 &= (\bar{s}^\alpha_L \gamma_\mu d^\alpha_L) \otimes \sum\nolimits_q (\bar{q'}^\beta_L \gamma^\mu q^\beta_L)\,, \\
Q'_4 &= (\bar{s}^\alpha_L \gamma_\mu d^\beta_L) \otimes \sum\nolimits_q (\bar{q'}^\beta_L \gamma^\mu q^\alpha_L)\,, \\
Q'_5 &= (\bar{s}^\alpha_L \gamma_\mu d^\alpha_L) \otimes \sum\nolimits_q (\bar{q'}^\beta_R \gamma^\mu q^\beta_R)\,, \\
Q'_6 &= (\bar{s}^\alpha_L \gamma_\mu d^\beta_L) \otimes \sum\nolimits_q (\bar{q'}^\beta_R \gamma^\mu q^\alpha_R)\,.
\end{aligned} \tag{C.1}$$

Note that we have expressed the operators in terms of left- and right-handed fermion fields, in contrast to the definition used in [17]. The evanescent operators chosen in [17] are equivalent

to the following set of operators:

$$
\begin{aligned}
E'^{qq'(1)}_{\ 1} &= (\bar{s}^\alpha_L \gamma_{\mu_1\mu_2\mu_3} q^\alpha_L) \otimes (\bar{q'}^\beta_L \gamma^{\mu_1\mu_2\mu_3} d^\beta_L) - (16-4\epsilon) Q'^{qq'}_{\ 1}\,, \\
E'^{qq'(1)}_{\ 2} &= (\bar{s}^\alpha_L \gamma_{\mu_1\mu_2\mu_3} q^\beta_L) \otimes (\bar{q'}^\beta_L \gamma^{\mu_1\mu_2\mu_3} d^\alpha_L) - (16-4\epsilon) Q'^{qq'}_{\ 2}\,, \\
E'^{(1)}_{\ 3} &= (\bar{s}^\alpha_L \gamma_{\mu_1\mu_2\mu_3} d^\alpha_L) \otimes \sum\nolimits_q (\bar{q'}^\beta_L \gamma^{\mu_1\mu_2\mu_3} q^\beta_L) - (16-4\epsilon) Q'_3\,, \\
E'^{(1)}_{\ 4} &= (\bar{s}^\alpha_L \gamma_{\mu_1\mu_2\mu_3} d^\beta_L) \otimes \sum\nolimits_q (\bar{q'}^\beta_L \gamma^{\mu_1\mu_2\mu_3} q^\alpha_L) - (16-4\epsilon) Q'_4\,, \\
E'^{(1)}_{\ 5} &= (\bar{s}^\alpha_L \gamma_{\mu_1\mu_2\mu_3} d^\alpha_L) \otimes \sum\nolimits_q (\bar{q'}^\beta_R \gamma^{\mu_1\mu_2\mu_3} q^\beta_R) - (4+4\epsilon) Q'_5\,, \\
E'^{(1)}_{\ 6} &= (\bar{s}^\alpha_L \gamma_{\mu_1\mu_2\mu_3} d^\beta_L) \otimes \sum\nolimits_q (\bar{q'}^\beta_R \gamma^{\mu_1\mu_2\mu_3} q^\alpha_R) - (4+4\epsilon) Q'_6\,.
\end{aligned}
\tag{C.2}
$$

It turns out that in order to transform from our operator basis to the traditional one the following four evanescent operators must be introduced at the one-loop level in addition to the evanescent operators given in Equation (2.45) (see Reference [23]):

$$
\begin{aligned}
E^{(1)}_5 &= (\overline{s}_L \gamma_\mu d_L) \otimes \sum\nolimits_q (\overline{q} \gamma^\mu \gamma_5 q) - \frac{5}{3} Q_3 + \frac{1}{6} Q_5\,, \\
E^{(1)}_6 &= (\overline{s}_L \gamma_\mu T^a d_L) \otimes \sum\nolimits_q (\overline{q} \gamma^\mu \gamma_5 T^a q) - \frac{5}{3} Q_4 + \frac{1}{6} Q_6\,, \\
E^{(1)}_7 &= (\overline{s}_L \gamma_{\mu_1\mu_2\mu_3} d_L) \otimes \sum\nolimits_q (\overline{q} \gamma^{\mu_1\mu_2\mu_3} \gamma_5 q) - \frac{32}{3} Q_3 + \frac{5}{3} Q_5\,, \\
E^{(1)}_8 &= (\overline{s}_L \gamma_{\mu_1\mu_2\mu_3} T^a d_L) \otimes \sum\nolimits_q (\overline{q} \gamma^{\mu_1\mu_2\mu_3} \gamma_5 T^a q) - \frac{32}{3} Q_4 + \frac{5}{3} Q_6\,,
\end{aligned}
\tag{C.3}
$$

The transformation matrices $R$, $M$, $W$, and $U$ representing the basis transformation according to Equation (3.14), as well as the finite renormalisation induced by this transformation, can be found in [23]. The parts of the transformation matrices relevant to us are given by

$$
R = \begin{pmatrix}
2 & \frac{1}{3} & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{3} & 0 & \frac{1}{12} & 0 \\
0 & 0 & -\frac{1}{9} & -\frac{2}{3} & \frac{1}{36} & \frac{1}{6} \\
0 & 0 & \frac{4}{3} & 0 & -\frac{1}{12} & 0 \\
0 & 0 & \frac{4}{9} & \frac{8}{3} & -\frac{1}{36} & -\frac{1}{6}
\end{pmatrix}, \quad
M = \begin{pmatrix}
2 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 8 & 0 & -\frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & \frac{8}{3} & 16 & -\frac{1}{6} & -1 \\
0 & 0 & 0 & 0 & -2 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & -\frac{2}{3} & -4 & \frac{1}{6} & 1
\end{pmatrix},
\tag{C.4}
$$

$$
W = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -6 & 0 & 0
\end{pmatrix}, \quad
U = \begin{pmatrix}
4 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & -112 & 0 & 16 & 0 \\
0 & 0 & 0 & -112 & 0 & 16 \\
0 & 0 & -\frac{10}{9} & 0 & \frac{1}{9} & 0 \\
0 & 0 & 0 & -\frac{10}{9} & 0 & \frac{1}{9} \\
0 & 0 & -\frac{136}{9} & 0 & \frac{10}{9} & 0 \\
0 & 0 & 0 & -\frac{136}{9} & 0 & \frac{10}{9}
\end{pmatrix},
\tag{C.5}
$$

whereas the matrix $V$ vanishes. They correspond to the bases

$$Q' = (Q'_1, \ldots, Q'_6)\,, \quad E' = \left(E'^{(1)}_1, \ldots, E'^{(1)}_6\right), \tag{C.6}$$

and

$$Q = (Q_1, \ldots, Q_6)\,, \quad E = \left(E^{(1)}_1, \ldots, E^{(1)}_8\right) \tag{C.7}$$

in the notation of (3.14). The one-loop contribution to the finite renormalisation in the dimension-six sector is given by

$$Z'^{(1,0)}_{QQ} = \begin{pmatrix} -\frac{7}{3} & -1 & 0 & 0 & 0 & 0 \\ -2 & \frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{178}{27} & -\frac{34}{9} & -\frac{164}{27} & \frac{20}{9} \\ 0 & 0 & 1-\frac{N_f}{9} & \frac{N_f}{3}-\frac{25}{3} & -\frac{N_f}{9}-2 & \frac{N_f}{3}+6 \\ 0 & 0 & -\frac{160}{27} & \frac{16}{9} & \frac{146}{27} & -\frac{2}{9} \\ 0 & 0 & \frac{N_f}{9}-2 & 6-\frac{N_f}{3} & \frac{N_f}{9}+3 & -\frac{N_f}{3}-\frac{11}{3} \end{pmatrix}. \tag{C.8}$$

The finite renormalisation relevant for the mixing of dimension-six into dimension-eight operators has never been calculated before. We find, in the notation of (4.24):

$$\hat{Z}'^{(1,0)}_{QQ,\tilde{Q}_7} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -24 & -8 \\ -8 & -8 \\ 24 & 8 \\ 8 & 8 \end{pmatrix}. \tag{C.9}$$

## C.2 Transformation to the Diagonal Operator Basis

Here we describe the change from our operator basis in the current-current subspace to the so-called diagonal basis, as defined in equation (2.38). The following definition of the evanescent operators ensures a diagonal anomalous dimension matrix through NNLO [25]:

$$\begin{aligned} E_1^{qq'} &= (\bar{s}_L\gamma_{\mu_1\mu_2\mu_3}T^a q_L)(\bar{q}'_L\gamma^{\mu_1\mu_2\mu_3}T^a d_L) - \left(16-4\epsilon-4\epsilon^2\right)Q_1^{qq'}\,, \\ E_2^{qq'} &= (\bar{s}_L\gamma_{\mu_1\mu_2\mu_3}q_L)(\bar{q}'_L\gamma^{\mu_1\mu_2\mu_3}d_L) - \left(16-4\epsilon-4\epsilon^2\right)Q_2^{qq'}\,, \\ E_3^{qq'} &= (\bar{s}_L\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5}T^a q_L)(\bar{q}'_L\gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5}T^a d_L) - \left(256-224\epsilon-\frac{5712}{25}\epsilon^2\right)Q_1^{qq'}\,, \\ E_4^{qq'} &= (\bar{s}_L\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5}q_L)(\bar{q}'_L\gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5}d_L) - \left(256-224\epsilon-\frac{10032}{25}\epsilon^2\right)Q_2^{qq'}\,. \end{aligned} \tag{C.10}$$

The transformation matrices $R$, $M$, $U$, and $V$ in the notation of (3.14) are now given by [25][1]

$$R = \begin{pmatrix} 1 & \frac{2}{3} \\ -1 & \frac{1}{3} \end{pmatrix}, \; M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 20 & 0 & 1 & 0 \\ 0 & 20 & 0 & 1 \end{pmatrix}, \; U = \begin{pmatrix} 4 & 0 \\ 0 & 4 \\ 144 & 0 \\ 0 & 144 \end{pmatrix}, \; V = \begin{pmatrix} 4 & 0 \\ 0 & 4 \\ \frac{3712}{25} & 0 \\ 0 & \frac{8032}{25} \end{pmatrix}, \tag{C.11}$$

and the matrix $W$ vanishes. These matrices correspond to the following bases of operators (the roles of the primed and unprimed set of operators is reversed with respect to Reference [25]):

$$Q' = (Q_+, Q_-)\,, \quad E' = \left(E_1^{qq'}, E_2^{qq'}, E_3^{qq'}, E_4^{qq'}\right), \tag{C.12}$$

and

$$Q = (Q_1, Q_2)\,, \quad E = \left(E_1^{(1)}, E_2^{(1)}, E_1^{(2)}, E_2^{(2)}\right). \tag{C.13}$$

The finite renormalisation constants for the dimension-six sector can be extracted from [25] and read up to the NNLO:

$$Z'^{(1,0)}_{QQ} = \begin{pmatrix} -\frac{7}{3} & 2 \\ 1 & \frac{2}{3} \end{pmatrix}, \quad Z'^{(2,0)}_{QQ} = \begin{pmatrix} -\frac{23833}{900} - \frac{35N_f}{54} & \frac{27}{2} + \frac{5N_f}{9} \\ \frac{11}{4} + \frac{5N_f}{18} & -\frac{11609}{450} + \frac{5N_f}{27} \end{pmatrix}. \tag{C.14}$$

Again, additional contributions to the mixing of dimension-six into dimension-eight operators arise. In the case at hand, the finite renormalisation needed for the transformation at NLO level vanishes. For the NNLO contribution we find:

$$\hat{Z}'^{(2,0)}_{QQ,\tilde{Q}_7} = \begin{pmatrix} \frac{53}{18} & -\frac{1}{2} \\ \frac{7}{3} & -1 \\ 0 & 0 \\ 0 & 0 \\ 160 & -32 \\ \frac{4}{3} & -\frac{20}{3} \end{pmatrix}. \tag{C.15}$$

[1] Note that there are some typos in Ref. [25].

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